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# SYSTOLES IN TRANSLATION SURFACES

by Corentin Boissy & Slavyana Geninska

Abstract. — For a translation surface, we define the relative systole to be the length of the shortest saddle connection. We give a characterization of the maxima of the systole function on a stratum and give a family of examples providing local but non-global maxima on each stratum of genus at least 3. We further study the relation between the (local) maxima of the systole function and the number of shortest saddle connections.

Résumé (*Systoles dans les surfaces de translation*). — Pour une surface de translation, nous définissons la systole relative comme étant la longueur d'une plus petite connexion de selles. Nous donnons une caractérisation des maxima de la fonction systole sur une strate et donnons une famille d'exemples qui sont des maxima locaux mais non globaux sur chaque strate de genre au moins trois. Nous étudions de plus des relations entre les maxima (locaux) de la fonction systole et le nombre de plus petites connexions de selles.

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## 1. Introduction

This paper deals with flat metrics defined by Abelian differentials on compact Riemann surfaces (*translation surfaces*). Such flat metrics have conical singularities of angle $(k+1)2\pi$, where $k$ is the order of the zero of the corresponding Abelian differential. A stratum of the moduli space of the Abelian differentials corresponds to translation surfaces that share the same combinatorics of zeroes, possibly including marked points.

A saddle connection on a translation surface is a geodesic joining two singularities (possibly the same) and with no singularity in its interior. A sequence of area 1 translation surfaces in a stratum leaves any compact set, if and only if, the length of the shortest saddle connection tends to zero. The set of translation surfaces with short saddle connections and compactification issues of strata are related to dynamics and counting problems on translation surfaces and have been widely studied in the last 30 years (see, for instance, [9, 5, 4]).

In this paper, we are interested in the opposite problem: we study surfaces that are as far as possible from the boundary and that would represent the "core" of a stratum. For a translation surface, we define the *relative systole* $\mathrm{Sys}(S)$ to be the length of the shortest saddle connection of $S$. Our primary goal is to study global and local maxima of the function Sys when restricted to area 1 translation surfaces. Note that our definition is different from the "true systole", i.e., the shortest closed curve that was studied by Judge and Parlier in [8]. In the rest of the paper, for simplicity, if not mentioned otherwise, the term "systole" will mean "relative systole".

This kind of question also appears in other contexts. The maxima of the systole function for moduli spaces of hyperbolic surfaces, where the systole is the length of the shortest closed geodesic, have been studied by various authors, for instance, Bavard [2], Schmutz Schaller [13], or more recently Balacheff, Makover, and Parlier [1]. A related question is the maximal number of geodesics realizing the systole, the so-called kissing number, see, for instance, Schmutz Schaller [14], and Fanoni and Parlier [7].

In the context of area 1 translation surfaces, while the characterization of global maxima for Sys seems to have been known for some time in the mathematics community, the existence of local maxima was unknown. We provide explicit examples of local maxima that are not global in each stratum with genus $g=2$ with marked points or $g\geq 3$. We also study the relation between the (locally) maximal values of the function Sys and the (locally) maximal number of shortest saddle connections.

The paper is organized as follows. In Section 2, we give some general background on translation surfaces. In Section 3, we study global maxima of the function Sys for area 1 translation surfaces. We prove the following theorem (see Theorem 3.3):

Theorem. — *Let $S$ be a genus $g \geq 1$ translation surface of area 1 and $r > 0$ singularities or marked points. Then,*

$$\mathrm{Sys}(S) \leq \left( \frac{\sqrt{3}}{2}(2g - 2 + r) \right)^{-\frac{1}{2}}.$$

*The equality is obtained if and only if $S$ is built with equilateral triangles whose sides are saddle connections of length* $\mathrm{Sys}(S)$. *Such a surface exists in any connected component of any stratum.*

This result was independently proven recently by Judge and Parlier [8] for surfaces with one singularity; the authors are interested in the shortest closed curves, but their proof should work in any strata in our context.

In Section 4, we study the local maxima of the function Sys that are not global. With the help of explicit examples we prove the following result, which is Theorem 4.7 in the text.

Theorem. — *Each stratum of area 1 surfaces with genus $g = 2$ with marked points or $g \geq 3$ contains local maxima of the function* Sys *that are not global.*

The examples are obtained by considering surfaces that decompose into equilateral triangles and regular hexagons, with some further conditions (see Theorem 4.1 for a precise statement).

In Section 5, we study the relation between (locally) maximal values of the function Sys and the (locally) maximal number of shortest saddle connections. We call a surface *rigid* if it corresponds to a local maximum of the number of shortest saddle connections. While the connection is clear for global maxima (see Proposition 5.1), the situation is more complex for the local maxima. The examples that we provide for local maxima of the function $Sys$ are rigid. Even more, a surface that is a local maximum of the function $Sys$ and that decomposes into equilateral triangles and regular hexagons must be rigid (Proposition 5.2). However, rigid surfaces are not necessarily local maxima (see Proposition 5.3).

## 2. Background

A *translation surface* is a (real, compact, connected) genus $g$ surface $S$ with a translation atlas, i.e., a triple $(S, \mathcal{U}, \Sigma)$, such that $\Sigma$ is a finite subset of $S$ (whose elements are called *singularities*) and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $S \setminus \Sigma$ whose transition maps are translations of $\mathbb{C} \simeq \mathbb{R}^2$. We will require that, for each $s \in \Sigma$, there is a neighborhood of $s$ isometric to a Euclidean cone, whose total angle is a multiple of $2\pi$. One can show that the holomorphic structure on $S \setminus \Sigma$ extends to $S$ and that the holomorphic 1-form $\omega = dz_i$ extends to a holomorphic $1-$form on $X$, where $\Sigma$ corresponds to the zeroes of $\omega$ and maybe some marked points. We usually call $\omega$ an *Abelian differential.*

A zero of $\omega$ of order $k$ corresponds to a singularity of angle $(k+1)2\pi$. By a slight abuse of notation, we authorize the order of a zero to be 0; in this case, it corresponds to a regular marked point. A saddle connection is a geodesic segment joining two singularities (possibly the same) and with no singularity in its interior. Integrating $\omega$ along the saddle connection we get a complex number. Considered as a planar vector, this complex number represents the affine holonomy vector of the saddle connection. In particular, its Euclidean length is the modulus of its holonomy vector.

For $g \geq 1$, we define the moduli space of Abelian differentials $\mathcal{H}_g$ as the moduli space of pairs $(X, \omega)$, where $X$ is a genus $g$ (compact, connected) Riemann surface, and $\omega$ a nonzero holomorphic $1-$form defined on $X$. The term moduli space means that we identify the points $(X, \omega)$ and $(X', \omega')$ if there exists an analytic isomorphism $f : X \to X'$ such that $f^*\omega' = \omega$. The group $\mathrm{SL}(2,\mathbb{R})$ naturally acts on the moduli space of translation surfaces by post composition on the charts defining the translation structures.

One can also see a translation surface obtained as a polygon (or a finite union of polygons) whose sides come in pairs, and for each pair, the corresponding segments are parallel and of the same length. These parallel sides are glued together by translation, and we assume that this identification preserves the natural orientation of the polygons. In this context, two translation surfaces are identified in the moduli space of Abelian differentials if and only if the corresponding polygons can be obtained from each other by cutting and gluing and preserving the identifications. Also, the $\mathrm{SL}(2,\mathbb{R})$ action in this representation is just the natural linear action on the polygons.

The moduli space of Abelian differentials is stratified by the combinatorics of the zeroes; we will denote by $\mathcal{H}(k_1, \dots, k_r)$ the stratum of $\mathcal{H}_g$, where $\sum_i k_i = 2g-2$, consisting of (classes of) pairs $(X, \omega)$, such that $\omega$ has exactly $r$ zeroes of order $k_1, \dots, k_r$. This space is (Hausdorff) complex analytic (see, for instance, [11, 15, 16]). We often restrict ourselves to the subset $\mathcal{H}_1(k_1, \dots, k_r)$ of *area 1* surfaces. Local coordinates for a stratum of Abelian differentials are obtained by integrating the holomorphic 1-form along a basis of the relative homology $H_1(S, \Sigma; \mathbb{Z})$, where $\Sigma$ denotes the set of conical singularities of $S$.

## 3. Maximal systole

We recall that the systole $\mathrm{Sys}(S)$ of a translation surface $S$ is the length of the shortest saddle connection of $S$. The aim of this section is to prove Theorem 3.3, which characterizes translation surfaces of area 1 with maximal systoles. One key tool is Delaunay triangulation.

Let $S$ be a translation surface. A *Delaunay triangulation* $S$ is a triangulation of $S$, such that the vertices are singularities, the 1-cells (the sides of the triangles) are saddle connections and, for a 2-cell (triangle) $T$ of the triangulation, the circumcircle of any representative $\tilde{T}$ of the universal covering does not have any singularity in its interior.

In Section 4 of [12], Masur and Smillie prove the existence of Delaunay triangulations for every translation surface $S$.

Lemma 3.1. — *All shortest saddle connections of $S$ are 1-cells in every Delaunay triangulation of $S$.*

*Proof.* — Let $\sigma$ be a saddle connection that is not included in a Delaunay triangulation $\mathcal{T}$. Denote by $P, Q$ the extremities of $\sigma$. Let $T \in \mathcal{T}$ be the triangle in $\mathcal{T}$ with $P$ as a vertex and containing a subsegment of $\sigma$. Let $P', P''$ be the other vertices of $T$ (see Figure 3.1).

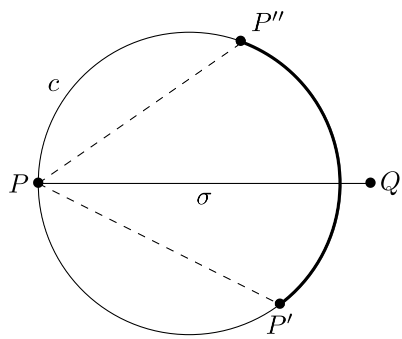


Figure 3.1. Illustration of Lemma 3.1

Consider the circumcircle $c$ of $T$ and the open arc of $P'P''$ which does not contain $P$. Each chord of $c$ joining $P$ to an element of this arc is of length strictly greater than $\min(d(P,P'), d(P,P'')) \geq \mathrm{Sys}(S)$. One of these chords is in the direction of $\sigma$, and since there is no singularity in the interior of $c$, this chord is a subsegment of $\sigma$. Therefore, $\sigma$ is not a shortest saddle connection. □

The first statement of the following lemma is needed for the proof of the next theorem. The second statement will be useful for Theorem 4.7.

Lemma 3.2. — *Let $\mathcal{C} \subset \mathcal{H}(k_1, \ldots, k_r)$ be a connected component of a stratum of abelian differentials with $k_1, \ldots, k_r \geq 0$.*

1. *There exists in $\mathcal{C}$ a surface $S$ that decomposes into equilateral triangles whose sides are saddle connections.*
2. *Furthermore, for each $i, j$ we can find such a surface with a side of an equilateral triangle being a saddle connection joining a singularity of degree $k_i$ to a singularity of degree $k_j$, with the convention that the two singularities are different, if $i \neq j$ and equal if $i = j$.*

*Proof.* — We first prove (1). By Lemma 18 in [10] there exists in each connected component of each stratum a surface with a horizontal one-cylinder decomposition. Up to a shear transformation that creates a vertical saddle connection, such surface can be described as a rectangle with the two vertical sides identified that correspond to a saddle connection, and each horizontal side decomposes into horizontal saddle connections (each one appearing on the top and on the bottom). We can freely change the lengths of these saddle connections, and hence we can assume that they are all of length 1 and get a square tiled surface with singularities in each corner of the squares. Now we rotate the vertical one until it makes an angle of $\pi/3$ with the horizontal ones (see Figure 3.2), this gives the surface $S$ required.

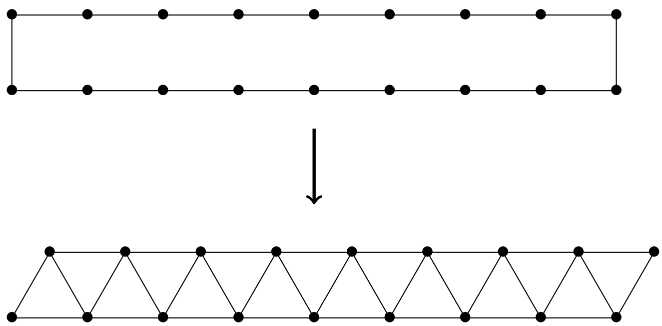

FIGURE 3.2. Surface with an equilateral triangle decomposition

The proof of (2) is a small variation of the above proof: observe first that each singularity appears both on the top line and on the bottom line of the cylinder. Recall that $\mathrm{SL}(2,\mathbb{R})$ acts on the connected component of the stratum by linear action on the polygons. Then applying the matrix $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and suitably cutting and pasting we obtain a new rectangle. For a suitable $n$, there is a vertical length 1 saddle connection joining the singularity of degree $k_i$ to the singularity of degree $k_j$, and the above argument finishes the proof. □

THEOREM 3.3. — *Let $S$ be a genus $g \geq 1$ translation surface of area 1 and $r > 0$ singularities or marked points. Then,*

$$\mathrm{Sys}(S) \leq \left(\frac{\sqrt{3}}{2}(2g-2+r)\right)^{-\frac{1}{2}}.$$

*The equality is obtained if and only if $S$ is built with equilateral triangles whose sides are saddle connections of length* $\mathrm{Sys}(S)$. *Such a surface exists in any connected component of any stratum.*

*Proof.* — For simplicity, instead of looking at a translation surface of area 1 and trying to determine the longest systole possible, we suppose that $S$ has a systole of length 1 and try to minimize the area $\mathcal{A}(S)$.

We consider a Delaunay triangulation of $S$ given by saddle connections. By Lemma 3.1 all shortest saddle connections of $S$ are 1-cells in this triangulation. Note that some triangles in the Delaunay triangulation might have a small area.

We consider the Voronoi diagram of $S$. This is a partitioning of $S$ into cells. Each cell contains exactly one singularity and is the set of points of $S$ that are closer to that singularity than to any other. The boundary of each cell consists of points that are equidistant to at least two singularities, in the sense that there are at least two different distance realizing geodesics of equal length connecting the point with a singularity (see Section 4 of [12] for reference).

The boundaries of the cells of the Voronoi diagram are parts of the orthogonal bisectors of the saddle connection in the Delaunay triangulation. Even though the triangulation is not unique, the Voronoi diagram is.

We can compute $\mathcal{A}(S)$ as the sum of the areas of the triangles with one of the vertices a singularity and its opposite side a side of the Voronoi cell containing the singularity (see Figure 3.3). The height of such a triangle is a half of a saddle connection, and hence, its length is greater than or equal to $\frac{1}{2}$. Therefore, $\mathcal{A}(S)$ is greater than or equal to one half of the sum of the lengths of all the sides of the Voronoi cells.

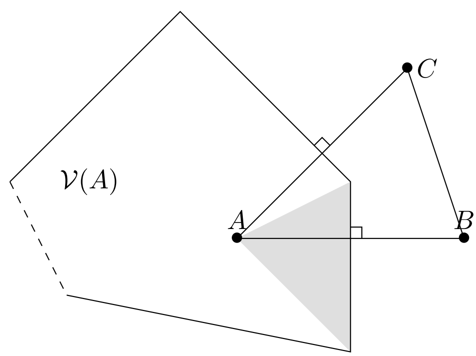


FIGURE 3.3. A Delaunay triangle $T = \triangle ABC$, with the Voronoi cell $\mathcal{V}(A)$ containing $A$. The area of the gray triangle is at least $\frac{1}{2}$ times the length of corresponding side of the Voronoi cell.

For each triangle $T$ in the Delaunay triangulation, we consider the sum $\sigma(T)$ of the signed distances from the circumcenter of $T$ to its sides. The sum of the lengths of all the sides of the Voronoi cells equals the sum of $\sigma(T)$ of all $T$ in the triangulation. We want to bound from below $\sigma(T)$ for each triangle $T$.

By Carnot's theorem[1] $\sigma(T)$ is equal to the sum of the inradius and the circumradius of $T$ (see, for instance, [6]). Hence, by Lemma 3.4, $\sigma(T) \geq \frac{\sqrt{3}}{2}$ with equality if and only if $T$ is equilateral of side 1.

The number of triangles in the triangulation is $2(2g-2+r)$. Hence $\mathcal{A}(S) \geq \frac{\sqrt{3}}{2}(2g-2+r)$ if the systole is of length 1. Thus, for a translation surface of area 1, we have that the systole is at most $\left(\frac{\sqrt{3}}{2}(2g-2+r)\right)^{-\frac{1}{2}}$ and can be obtained only if $S$ is built with equilateral triangles whose sides are saddle connections of length $\mathrm{Sys}(S)$.

We conclude by using the first statement of Lemma 3.2. □

Lemma 3.4. — *Let $T$ be a nondegenerate Euclidean triangle with sides of length at least 1. Then, the sum of the circumradius and the inradius of $T$ is at least $\frac{\sqrt{3}}{2}$, with equality if and only if $T$ is equilateral of side 1.*

*Proof.* — Denote by $R$ the circumradius and by $r$ the inradius of $T$. First, we note that when we shrink $T$, we decrease the sum $R+r$. So without loss of generality, we can assume that at least one of the sides of $T$ is of length 1. So for the triangle $T = \triangle ABC$ with $1 = AB \leq BC \leq AC$, we take a point $D$ on the side $BC$ so that $BD = AB$. Note that $AD \geq 1$. For the inradius $\tilde{r}$ and the circumradius $\tilde{R}$ of the isosceles $\triangle ABD$, we can see that $\tilde{r} \leq r$ and $\tilde{R} \leq R$. Indeed, the circumcenter of $\triangle ABD$ is nearer to $AB$ than the circumcenter of $\triangle ABC$ and therefore $\tilde{R} \leq R$. To obtain that $\tilde{r} \leq r$ we note that the incenter of $\triangle ABD$ is nearer to $B$ than that of $\triangle ABC$.

For a triangle with sides $1, 1$ and $x$, we can find the inradius and the circumradius with the help of the lengths of the sides:

$$\tilde{R}(x) = \sqrt{\frac{1}{4-x^2}}, \qquad \tilde{r}(x) = \frac{x}{2}\sqrt{\frac{2-x}{2+x}},$$

with $x \in [1,2)$. For the sum $(\tilde{R}+\tilde{r})(x)$ and its derivative, we obtain

$$(\tilde{R}+\tilde{r})(x) = \frac{2+2x-x^2}{2\sqrt{4-x^2}}, \quad (\tilde{R}+\tilde{r})'(x) = \frac{8-6x+x^3}{2\sqrt{(4-x^2)^3}}.$$

Since $8-6x+x^3 = x(1-x)^2 + 2(2-x)^2 + x > 0$ for $x \in [1,2)$, we have that $(\tilde{R}+\tilde{r})(x)$ is strictly increasing in the interval $[1,2)$ and hence obtains its minimum for $x = 1$. Therefore, $R + r \geq \tilde{R} + \tilde{r} \geq \frac{\sqrt{3}}{2}$ with equality exactly when the triangle $T$ is equilateral with side 1. □

## 4. Locally maximal systole

The question is if there exists local but not global maxima in any given stratum $\mathcal{H}_1(k_1, \ldots, k_r)$ of translation surfaces of area 1. Note that such maxima are

1. Lazare Carnot 1753–1823.

never strict since rotating a translation surface preserves the systole. We denote by $\mathbb{P}\mathcal{H}(k_1,\ldots,k_r)$ the moduli space of translation surfaces in $\mathcal{H}(k_1,\ldots,k_r)$ up to rotation and scaling. The systole function is well defined in $\mathbb{P}\mathcal{H}(k_1,\ldots,k_r)$: for $[S]\in\mathbb{P}\mathcal{H}(k_1,\ldots,k_r)$, we define $\mathrm{Sys}([S])$ to be $\mathrm{Sys}(S)$, where $S$ is any area 1 representative of $[S]$.

In this section, we show examples of local maxima of the function Sys that are not global and prove that such examples are realized in all but a finite number of strata.

We need to first, for technical reasons, define a distance around a point in $\mathcal{H}(k_1,\ldots,k_r)$ and in $\mathbb{P}\mathcal{H}(k_1,\ldots,k_r)$. Let $S_0\in\mathcal{H}(k_1,\ldots,k_r)$. Fix a basis of the relative homology given by saddle connections that determines local coordinates $(v_1,\ldots,v_k)$ around $S_0$. Then for $S$ in a sufficiently small neighborhood of $S_0$, we define $d(S,S_0)=\max_i\{|v_i-v_{i_0}|\}$.

We will identify a sufficiently small neighborhood of an element $[S_0]\in\mathbb{P}\mathcal{H}(k_1,\ldots,k_r)$ with the subset of representatives in $\mathcal{H}(k_1,\ldots,k_r)$ normalized in the following way:

1. The first coordinate $v_1$ is in $]0,+\infty[$.
2. The length of the shortest saddle connection is 1.

Then, the distance to $[S_0]$ is the distance in $\mathcal{H}(k_1,\ldots,k_r)$ following this identification.

Theorem 4.1. — *Let $S_{reg}$ be a translation surface in $\mathcal{H}_1(k_1,\ldots,k_r)$, such that when cut along its saddle connections of length* $\mathrm{Sys}(S_{reg})$, *it decomposes to equilateral triangles and regular hexagons so that:*

- *The set of the equilateral triangles without the vertices is connected.*
- *The boundary of each polygon is contained in the boundary of the set of triangles.*

*Then,* $\mathrm{Sys}(S_{reg})$ *is a local maximum in $\mathcal{H}_1(k_1,\ldots,k_r)$ and even a strict local maximum in $\mathbb{P}\mathcal{H}(k_1,\ldots,k_r)$.*

Remark 4.2. — The second condition of the above statement is equivalent to the hexagons being neither adjacent nor self-adjacent.

The idea of the proof is the following: when deforming $[S_{reg}]$ a little following the normalization described above, the area of each triangle does not decrease, and the area of each hexagon might decrease, but this will be compensated by an increase coming from at least one triangle.

The next two lemmas are estimations of the variation of areas of hexagons and triangles that are deformed in our context.

Lemma 4.3. — *Let $H_{reg}$ be the regular hexagon of sides of length* 1. *There exists a positive constant $c$ such that for every $\varepsilon>0$ small enough and every convex hexagon $H=A_1A_2\ldots A_6$ with sides of lengths in the interval* $[1,1+\varepsilon]$

*and diagonals* $A_1A_3$*,* $A_3A_5$ *and* $A_5A_1$ *of lengths in the interval* $[\sqrt{3}-\varepsilon, \sqrt{3}+\varepsilon]$*, we have* $\mathrm{Area}(H) \geq \mathrm{Area}(H_{reg}) - c\varepsilon^2$*.*

*Proof.* — We consider the convex hexagon $H' = A_1A_2'A_3A_4'A_5A_6'$ such that all of its sides are of length 1 (see Figure 4.1 ). We see that $\mathrm{Area}(H) \geq \mathrm{Area}(H')$.

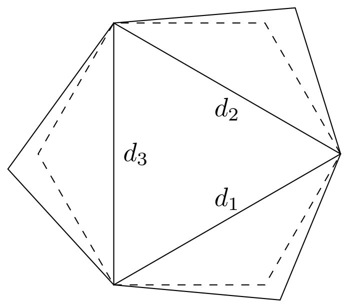


FIGURE 4.1. The hexagon $H$ and the new hexagon $H'$ of side 1 (dashed line).

We note the lengths of the diagonals $A_1A_3$, $A_3A_5$ and $A_5A_1$ by $d_1$, $d_2$ and $d_3$, respectively. The area of the hexagon $H'$ depends smoothly on $(d_1, d_2, d_3)$ and admits a local minimum at the point $(\sqrt{3}, \sqrt{3}, \sqrt{3})$ (which corresponds to the regular hexagon).

Therefore, by the Taylor–Young formula we obtain

$$Area(H) = Area(H_{reg}) + o(||(d_1 - \sqrt{3}, d_2 - \sqrt{3}, d_3 - \sqrt{3})||^2).$$

Since for $i \in \{1,2,3\}$ we have $d_i \in [\sqrt{3}-\varepsilon, \sqrt{3}+\varepsilon]$, and there exists a constant $c \in \mathbb{R}$ such that

$$\mathrm{Area}(H) \geq \mathrm{Area}(H_{reg}) - c\varepsilon^2. \qquad \square$$

LEMMA 4.4. — *Let* $T_{reg}$ *be an equilateral triangle with sides of length 1. There exists a positive constant* $c \in \mathbb{R}$ *such that for every* $\varepsilon > 0$ *small enough and every triangle* $T$ *with one of its sides of length* $1+\varepsilon$ *and the other sides of lengths in the interval* $[1, 1+\varepsilon]$*, we have that* $\mathrm{Area}(T) > \mathrm{Area}(T_{reg}) + c\varepsilon$*.*

*Proof.* — Let $T = \triangle ABC$ and $d(A,B) = 1+\varepsilon$ and let $C'$ be such that $d(A,C') = d(B,C') = 1$. We have $\mathrm{Area}(\triangle ABC) \geq \mathrm{Area}(\triangle ABC')$.

By Heron's formula, the area of $\triangle ABC'$ is:

$$\mathrm{Area}(\triangle ABC') = \frac{1}{4}\sqrt{(3+\varepsilon)(1-\varepsilon)(1+\varepsilon)^2} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{6}\varepsilon + o(\varepsilon).$$

Therefore, there exists a constant $c > 0$ such that for all $\varepsilon$ small enough, we have $\mathrm{Area}(T) > \mathrm{Area}(T_{reg}) + c\varepsilon$. $\square$

Lemma 4.5. — *Let $ABC$ be a nondegenerate triangle of sides of length $l_1 = BC$, $l_2 = AC$, and $l_3 = AB$. For $\varepsilon$ small enough, let $A'B'C'$ be a triangle with sides of lengths $l_1', l_2', l_3'$ such that for each $i \in \{1,2,3\}$, $|l_i - l_i'| \le \varepsilon$. We assume further that $d(A,A') \le \varepsilon$, $d(B,B') \le \varepsilon$, and $C$ and $C'$ are in the same half-plane determined by $AB$. Then there is a constant $J > 1$ only depending on $l_1, l_2, l_3$ such that $d(C,C') \le J\varepsilon$.*

*Proof.* — We consider first the translation $\tau$ of $\mathbb{R}^2$ of direction $\overrightarrow{A'A}$. We remark that $\tau(A') = A$ and $d(B', \tau(B')) < 2\varepsilon$. Then we consider the rotation $\rho$ with center $A$ and of angle $\measuredangle BA\tau(B')$. We note $X'' = \rho(\tau(X'))$ where $X \in \{A', B', C'\}$ (see Figure 4.2). We remark that $A$, $B$, and $B''$ are on the same line and that

$$d(\tau(C'), C'') = \frac{d(\tau(A'), \tau(C'))}{d(\tau(A'), \tau(B'))} d(\tau(B'), B'').$$

Since $d(\tau(B'), B'') \le d(\tau(B'), B) + d(B, B'') < 2\varepsilon + \varepsilon$, we obtain for $\varepsilon$ small enough a constant $J_1 = J_1(l_2, l_3)$ such that

$$d(\tau(C'), C'') < J_1\varepsilon.$$

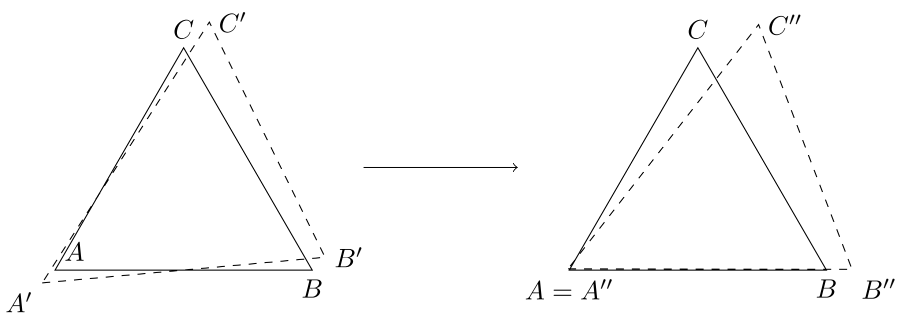


Figure 4.2. The triangles $ABC$, $A'B'C'$, and $A''B''C''$.

We want to bound $d(C, C'')$. For $(M,t)$ in a neighborhood of $(C, l_3)$, we consider the triangle $AMN_t$ where $N_t$ is in the ray $AB$ with $d(A, N_t) = t$ and we define $\phi(M,t) = (d(M,A), d(M,N_t), d(A,N_t))$. The map $\phi$ is smooth, and its Jacobian derivative at $(C, l_3)$ is invertible. Hence it defines a locally invertible map and $\phi^{-1}$ is smooth. This implies that there is a constant $J_2 = J_2(l_1, l_2, l_3)$ such that, for $\varepsilon$ small enough,

$$d(C, C'') < J_2\varepsilon.$$

Combining this with the above estimations, we obtain $d(C, C') < (J_1 + J_2 + 1)\varepsilon$. □

*Proof of Theorem 4.1.* — We show directly that $\mathrm{Sys}([S_{reg}])$ is a strict local maximum in the projective stratum and replace $S_{reg}$ by a surface, still denoted $S_{reg}$, with the shortest saddle connections of length 1.

First, we remark that removing all shortest saddle connections of $S_{reg}$ gives a union of topological disks. Hence, we can find a basis of the relative homology that consists of shortest saddle connections $(\gamma_1,\dots,\gamma_k)$ and we can assume that $\gamma_1$ is horizontal and oriented from left to right. We use this basis to fix local coordinates of the stratum $\mathcal{H}(k_1,\dots,k_r)$ and define a distance in a neighborhood of $S_{reg}$. Recall that we identify a neighborhood of element $[S_{reg}]\in\mathbb{P}\mathcal{H}(k_1,\dots,k_r)$ with a subset $\mathcal{U}$ of $\mathcal{H}(k_1,\dots,k_r)$ satisfying the following conditions: the shortest saddle connection is of length 1, and $\gamma_1$ stays horizontal. For $S\in\mathcal{U}$, we call a *short saddle connection* any saddle connection that corresponds to a shortest saddle connection of $S_{reg}$.

Let $\varepsilon>0$ be small enough and $S\in\mathcal{U}$ be such that $\varepsilon=d(S,S_{reg})$. Let us define $\rho(S)=Max_\gamma(l(\gamma)-1)$ where the maximum is taken on all short saddle connections of $S$. By hypothesis, $\rho(S)\geq 0$.

In a more general setting, in Section 5.2 we prove that we have $\rho(S)=0$ if and only if $S=S_{reg}$. However, in the current proof we need a stronger result (see the claim below).

We observe that since any short saddle connection $\gamma$ is a linear combination of $\{\gamma_1,\dots,\gamma_k\}$ in the relative homology group, then its corresponding affine holonomy $v_\gamma$ satisfies $|v_\gamma-v_{\gamma,reg}|\leq K\varepsilon$. Since there are only a finite number of short saddle connections, $K$ can be made universal for all short saddle connections. In particular, $\rho(S)\leq K\varepsilon$.

We have the following facts:

1. The sides of each hexagon $H$ in $S$ corresponding to a regular hexagon $H_{reg}$ in the decomposition of $S_{reg}$ are short saddle connections. By the above observation, we can apply Lemma 4.3 to $H$ for $\varepsilon'=2K\varepsilon$. Hence there is a constant $c_1$, such that
$$Area(H)\geq Area(H_{reg})-c_1\varepsilon^2.$$
2. By Lemma 4.4, there exists at least one equilateral triangle $T_{reg}$ in the decomposition of $S_{reg}$, such that for the corresponding triangle $T$ in $S$ we have $Area(T)\geq Area(T_{reg})+c_2\rho(S)$, where $c_2$ is a positive constant. Furthermore, the area of each triangle in $S_{reg}$ is not greater than that of the corresponding triangle in $S$. Summing up the corresponding contributions of the triangles, we obtain
$$Area(\cup T)\geq Area(\cup T_{reg})+c_2\rho(S).$$

Claim. — *There is a constant $D$, such that for $\varepsilon=d(S,S_{reg})$ small enough, $\varepsilon<D\rho(S)$. In other words: the lengths of short saddle connections control the distance from $S$ to $S_{reg}$.*

Summing up all contributions, assuming the claim, we see that the area of $S$ is greater than that of $S_{reg}$ for $\varepsilon > 0$ small enough. Hence, $S_{reg}$ is a local maximum of Sys which is nonglobal since the surface $S_{reg}$ is not built with equilateral triangles whose sides are saddle connections.

Now we prove the claim. Recall that we assume that $\gamma_1$ does not change direction. Let $\delta = \rho(S)$.

Let $\gamma \in \{\gamma_2, \ldots, \gamma_k\}$ be a saddle connection in the fixed basis. By hypothesis, there is a sequence of pairwise distinct equilateral triangles $T_1, \ldots, T_l$ (whose sides are length 1 saddle connections) that form a "path" from $\gamma_1$ to $\gamma$, i.e., such that

1. $\gamma_1$ is a side of $T_1$,
2. for each $i \in \{1, \ldots, l-1\}$, $T_i$ and $T_{i+1}$ are adjacent,
3. $\gamma$ is a side of $T_l$.

Observe that $l$ is bounded from above by the total number $N$ of triangles in the decomposition of $S_{reg}$. Denote by $v_{reg}$ the affine holonomy of $\gamma$ in $S_{reg}$ and by $v$ the affine holonomy of $\gamma$ in $S$. We will use Lemma 4.5 to bound $|v - v_{reg}|$.

Using the developing map (see Figure 4.3), we can view the triangles $(T_i)_i$ as a sequence of adjacent equilateral triangles of the plane, although in this case the triangles might intersect. We deform the surface $S_{reg}$ to obtain the surface $S$. The triangles $(T_i)_i$ persist but are no longer necessarily equilateral. Again, we can view them as a sequence of adjacent triangles $(T_i')_i$ in the plane.

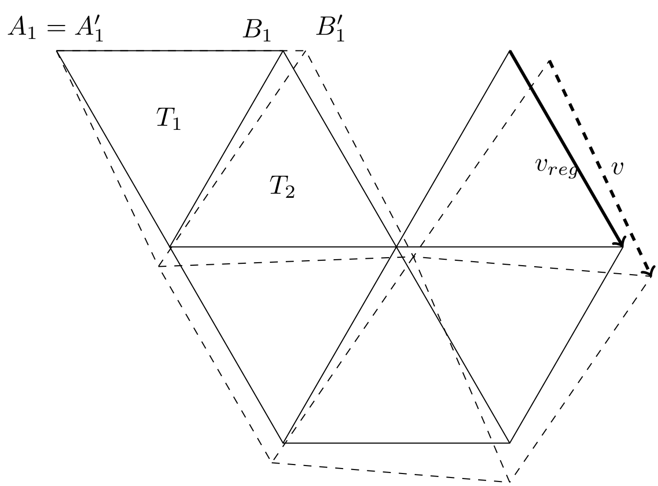


FIGURE 4.3. A sequence of adjacent equilateral triangles and their perturbation

Denote by $T_1 = A_1B_1C_1$ and $T_1' = A_1'B_1'C_1'$. We can assume that $A_1 = A_1'$ is the vertex that is not in $T_2$ or $T_2'$, and $B_2, B_2'$ are such that the segments $A_1B_1$ and $A_1'B_1'$ are horizontal (see Figure 4.3). More generally, for $i > 1$, denote the triangle $T_i$ by $A_iB_iC_i$ in such a way that $A_iB_i$ is a side of the previous triangle and that $B_iC_i$ is a side of the next triangle, and we denote the vertices of $T_i'$ analogously. Using Lemma 4.5 we see that $d(C_1, C_1') < J\delta$ (recall that since $\rho(S) < Kd(S, S_{reg}) = K\varepsilon$, we can assume $\delta$ to be arbitrarily small). Since $d(B_1, B_1') < \delta < J\delta$ we can apply Lemma 4.5 to the triangles $T_2$ and $T_2'$ for the constant $J\delta$ and we get $d(C_2, C_2') < J^2\delta$. Since $l$ is bounded from above by $N$ and $\delta$ can be chosen arbitrarily small, we get $d(C_l, C_l') < J^l\delta$ and $d(B_l, B_l') < J^{l-1}\delta$. Finally, observe that $v$ is given by the difference of the coordinates of $B_l'$ and $C_l'$, and therefore:

$$|v - v_{reg}| < (J^l + J^{l-1})\delta < 2.J^N\delta.$$

This concludes the proof of the claim and of the theorem. □

Example 4.6. — The surfaces given in Figure 4.4 are examples (with one hexagon) of local maxima that are nonglobal in the strata $\mathcal{H}(2, 0^k)$ and $\mathcal{H}(1, 1, 0^k)$, for $k \geq 1$.

The above examples will be used in the next theorem to build examples in most strata.

Theorem 4.7. — *Let $\mathcal{H}$ be a stratum of area 1 and genus $g \geq 2$ surfaces. We assume that $\mathcal{H}$ is neither $\mathcal{H}(1,1)$ nor $\mathcal{H}(2)$. Then $\mathcal{H}$ contains local maxima of the function* Sys *that are not global.*

We first prove the following lemma.

Lemma 4.8. — *We consider the stratum $\mathcal{H} = \mathcal{H}(m_1, \dots, m_r, x, y)$ with $m_1, \dots, m_r, x, y \geq 0$. We assume that there exists a surface $S_1 \in \mathcal{H}$ that satisfies the hypothesis of Theorem 4.1 and such that there is a shortest saddle connection $\gamma_1$ joining a singularity of degree $x$ to a distinct singularity of degree $y$. Then:*

a) *For any $n_1, \dots, n_k, p, q \geq 0$ with $p + q + \sum_i n_i$ even, there exists a local but nonglobal maximum of Sys in the stratum $\mathcal{H}(m_1, \dots, m_r, p + a + 1, q + a + 1, n_1, \dots, n_k)$.*
b) *For any $n_1, \dots, n_k, p \geq 0$ with $p + \sum_i n_i$ even, there exists a local but nonglobal maximum of Sys in the stratum $\mathcal{H}(m_1, \dots, m_r, p + x + y + 2, n_1, \dots, n_k)$.*

*Proof.* — By Lemma 3.2, there is a surface $S_2$ that decomposes into equilateral triangles whose sides are saddle connections in $\mathcal{H}(p, q, n_1, \dots, n_k)$, and with a shortest saddle connection $\gamma_2$ joining a singularity of degree $p$ to a (distinct) singularity of degree $q$.

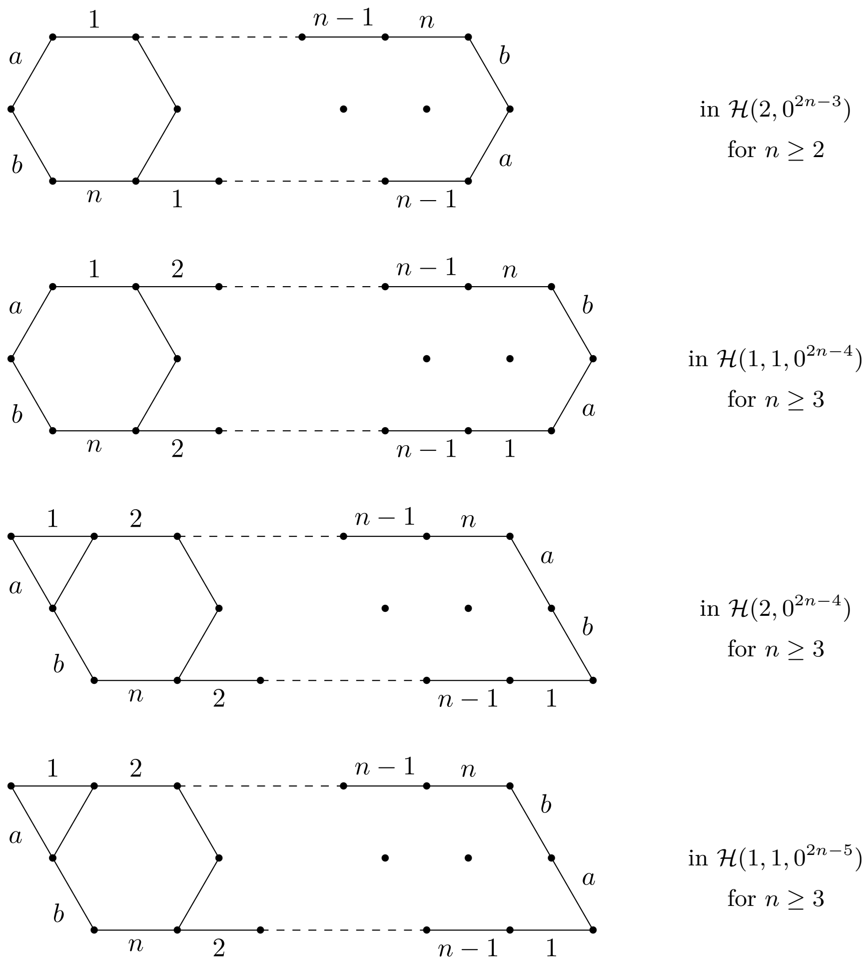


Figure 4.4. Examples of local but nonglobal maxima

We can assume that $\gamma_1, \gamma_2$ are vertical and of the same length. Now we glue the two surfaces by the following classical surgery: cut the two surfaces along $\gamma_1$ and $\gamma_2$ and glue the left-hand side of $\gamma_1$ with the right-hand side of $\gamma_2$ and the right-hand side of $\gamma_1$ with the left-hand side of $\gamma_2$. We get a surface in $\mathcal{H}(m_1, \dots, m_r, p+a+1, q+a+1, n_1, \dots, n_k)$ that satisfies the hypothesis of Theorem 4.1 and, hence, is a local but nonglobal maximum for $Sys$. This proves Case a).

The proof of Case b) is the same by considering a surface $S_2$ in $\mathcal{H}(p, n_1, \dots, n_k)$ with a shortest saddle connection joining a singularity of degree $p$ to itself. $\square$

*Proof of Theorem 4.7.* — Recall that examples of local but nonglobal maxima of $Sys$ in the strata $\mathcal{H}(2, 0^k)$ and $\mathcal{H}(1, 1, 0^k)$, for $k \geq 1$, were already constructed in Example 4.6. It remains to construct examples in all strata of genus at least 3.

We start from the example $S_{0,2} \in \mathcal{H}(2, 0)$ given in Example 4.6. There is a saddle connection joining the two singularities.

- By Case b) of Lemma 4.8, there is a local maximum in any stratum of the form $\mathcal{H}(p + 4, n_1, \ldots, n_k)$ with $p \geq 0$, $k \geq 0$, and $n_1, \ldots, n_k \geq 0$.
- By Case a) of Lemma 4.8, there is a local maximum in any stratum of the form $\mathcal{H}(p + 3, q + 1, n_1, \ldots, n_k)$ with $p, q \geq 0$, $k \geq 0$, and $n_1, \ldots, n_k \geq 0$.

There remains to construct examples in strata with singularities of degree at most 2. Now we consider $S_{2,0,0} \in \mathcal{H}(2, 0, 0)$ given in Example 4.6. There is a saddle connection joining the two marked points.

- By Case b) of Lemma 4.8, there is a local maximum in any stratum of the form $\mathcal{H}(2, 2, n_1, \ldots, n_k)$ with $k \geq 0$, and $n_1, \ldots, n_k \geq 0$.

Now we consider $S_{1,1,0,0} \in \mathcal{H}(1, 1, 0, 0)$ given in Example 4.6. There is a saddle connection joining the two marked points.

- By Case b) of Lemma 4.8, there is a local maximum in any stratum of the form $\mathcal{H}(1, 1, 2, n_1, \ldots, n_k)$ with $k \geq 0$, and $n_1, \ldots, n_k \geq 0$.
- By Case a) of Lemma 4.8, there is a local maximum in any stratum of the form $\mathcal{H}(1, 1, 1, 1, n_1, \ldots, n_k)$ with $k \geq 0$, and $n_1, \ldots, n_k \geq 0$.

Finally, we have produced examples in all strata of genus $g \geq 2$ except $\mathcal{H}(2)$ and $\mathcal{H}(1, 1)$. □

Remark 4.9. — We remark that with these constructions we cannot build local maxima in $\mathcal{H}(2)$ and in $\mathcal{H}(1, 1)$. Indeed, for $\mathcal{H}(2)$, we need one hexagon and two triangles, and there is only one possibility that provides a surface in $\mathcal{H}(2)$. However, in this case, the hexagon is self-adjacent (see Section 5 for a proof that it is not a local maximum). For $\mathcal{H}(1, 1)$, we need one hexagon and four triangles, and by checking all the possibilities, we see that we cannot build the required example.

In a following paper [3], we prove that in these strata (and more generally in any hyperelliptic connected components of strata) any local maximum is a global maximum.

## 5. Number of shortest saddle connections

In this section, we explore the relations between the (locally) maximal values of the function Sys and the (locally) maximal number of short saddle connections.

**5.1. Maximal number.** — In the case of global maxima, the relation is clear, as is shown in the next proposition.

Proposition 5.1. — *The greatest number of shortest saddle connections of a surface in $\mathcal{H}(k_1,\dots,k_r)$ is equal to $\sum_{i=1}^{r} 3(k_i+1)$, and this number is realized if and only if the surface is a global maximum for the function* Sys *in* $\mathbb{P}\mathcal{H}(k_1,\dots,k_r)$.

*Proof.* — Let $S$ be a surface in $\mathcal{H}(k_1,\dots,k_r)$. We consider two shortest saddle connections $\gamma_1$ and $\gamma_2$ in $S$ starting at the same singularity.

Let us assume that the conical angle between $\gamma_1$ and $\gamma_2$ is less than $\frac{\pi}{3}$. Then

- Either the not common ends of $\gamma_1$ and $\gamma_2$ can be connected by a saddle connection, and as a consequence, this saddle connection is shorter than $\gamma_1$ and $\gamma_2$.
- Or there is a saddle connection between $\gamma_1$ and $\gamma_2$ (starting at the same singularity) that is shorter than them.

In both cases we have a contradiction, and hence, the maximal number of shortest saddle connections starting at a singularity of order $k_i$ is $6(k_i+1)$. This gives us that the total number of shortest saddle connections cannot exceed $\sum_{i=1}^{r} 3(k_i+1)$.

This number is the number of 1-cells in the Delaunay triangulation. Hence, by Lemma 3.1, the surface has this number of shortest saddle connections, if, and only if, its Delaunay triangulation is given by equilateral triangles. By Theorem 3.3 this situation corresponds precisely to the global maxima of the function Sys. □

**5.2. Locally maximal number: rigid surfaces.** — For a given translation surface, we would like to find a path joining this surface to a global maximum for the function Sys. Following the above proposition, a greedy algorithm could be to try to increase the number of shortest saddle connections until we reach a surface with the maximal number. Unfortunately, this algorithm does not always work.

We call a surface $S$ in $\mathcal{H}(k_1,\dots,k_r)$ *rigid*, if there exists a punctured neighborhood of $[S] \in \mathbb{P}\mathcal{H}(k_1,\dots,k_r)$, where all surfaces have a strictly smaller number of shortest saddle connections. As explained above, the global maxima of the systole function are rigid surfaces.

An example of a rigid surface is every surface $S$ that, when cut along its shortest saddle connections, decomposes into equilateral triangles and polygons with no singularities in the interior, satisfying the following conditions:

- The set of the equilateral triangles without the vertices is connected.
- The boundary of each polygon is contained in the boundary of the set of triangles.

Indeed, when deforming such a surface in such a way that the initial shortest saddle connections stay of the same length, the set of triangles is isometrically preserved, and therefore, the set of polygons is also isometrically preserved. In particular, the examples of Theorem 4.1 are rigid surfaces.

We give another family of examples. Consider a surface $S$ as above, but instead of having one, it has two connected components of triangles. We further assume that there is a polygon $\mathcal{P}$, such that the sum of the affine holonomy of the set of saddle connections of its boundary associated to each component of the triangles is nonzero when orienting the saddle connections according to the natural orientation of the $\partial\mathcal{P}$. Indeed, as above, when deforming such a surface in such a way that the initial shortest saddle connections stay of the same length, then each connected component of the triangles is isometrically preserved, and the condition on the holonomy implies that the boundary $\mathcal{P}$ is unchanged, which rigidifies the whole surface. If, further, the polygons are regular hexagons, we can adapt the proof of Theorem 4.1 to show that these are also local but nonglobal maxima.

The examples given in Figure 5.1 show that it is not sufficient to be decomposed into equilateral triangles and regular hexagons in order to be a local maximum. In Figure 5.1, the shortest saddle connections remain of length 1 and, hence, the area of the triangles does not change, but the hexagon is deformed, and therefore, its area decreases. The first example has one connected component of triangles, but the hexagon is self-adjacent. The second one has two connected components of triangles. Note that the example in $\mathcal{H}(0,0,0)$ can be easily modified to give a surface with true singularities (see Remark 5.4).

More generally, we have the following proposition:

Proposition 5.2. — *Let $S$ be a translation surface such that, when cut along its saddle connections of shortest length, it decomposes into equilateral triangles and regular hexagons. If the function* Sys *admits a local maximum at $[S] \in \mathbb{P}\mathcal{H}(k_1,\dots,k_r)$, then $S$ is rigid.*

*Proof.* — We assume that $S$ is nonrigid and deform the surface so that we keep all shortest saddle connections of the same length 1. This deformation does not change the metric on each triangle. Therefore, it must change the metric on at least one hexagon, otherwise the metric would be globally unchanged, and the transformation would be just a rotation. In particular, the area of the deformed hexagons must strictly decrease, while the area of the triangles (and the unchanged hexagons) remains the same. Hence the area of the surface decreases and thus $\mathrm{Sys}([S])$ increases. □

An interesting question is whether the converse of the above proposition is true. We can also ask if, in general, any local maximum for Sys comes from a rigid surface. Note that, in general, rigid surfaces do not necessarily give local maxima, as shown in the following example.

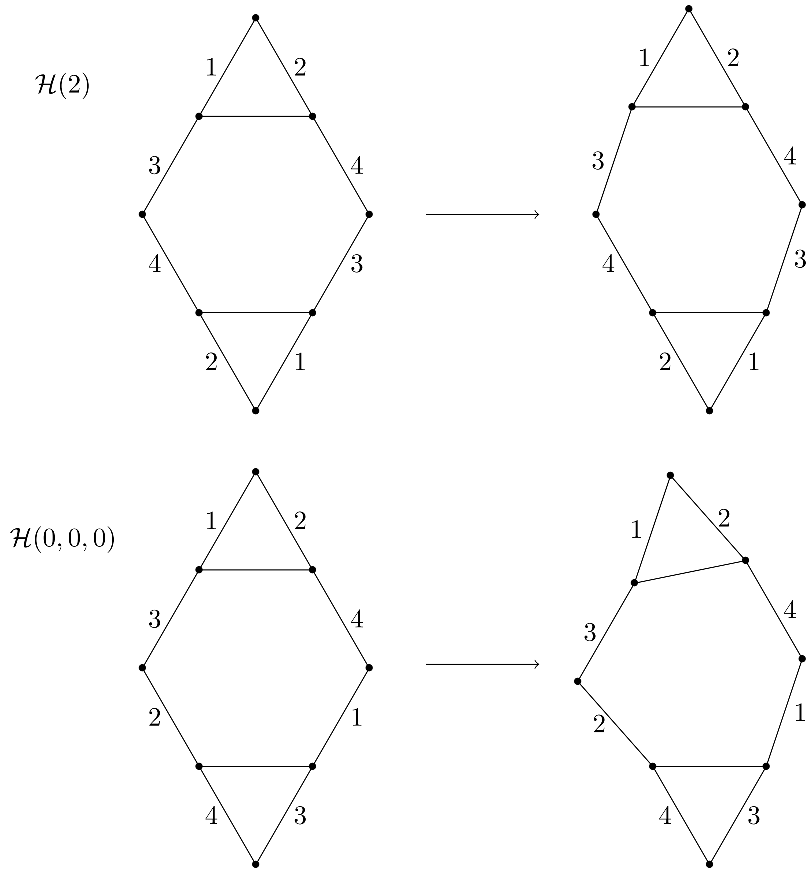


FIGURE 5.1. Examples of nonrigid surfaces in $\mathcal{H}(0,0,0)$ and $\mathcal{H}(2)$

PROPOSITION 5.3. — *The translation surface given by Figure 5.2 is rigid but it is not a local maximum for the function* Sys *in* $\mathbb{P}\mathcal{H}$ *for* $n \geq 3$.

REMARK 5.4. — Note that the translation surface given in Figure 5.2 contains marked points in the set of singularities. We can easily make them true singularities by operations analogous to the ones described in the proof of Lemma 4.8.

*Proof.* — The fact that the surface is rigid is clear: when cut along the shortest saddle connections, it decomposes into equilateral triangles and a non self-adjacent polygon with no singularities in the interior in such a way that the set of triangles is connected.

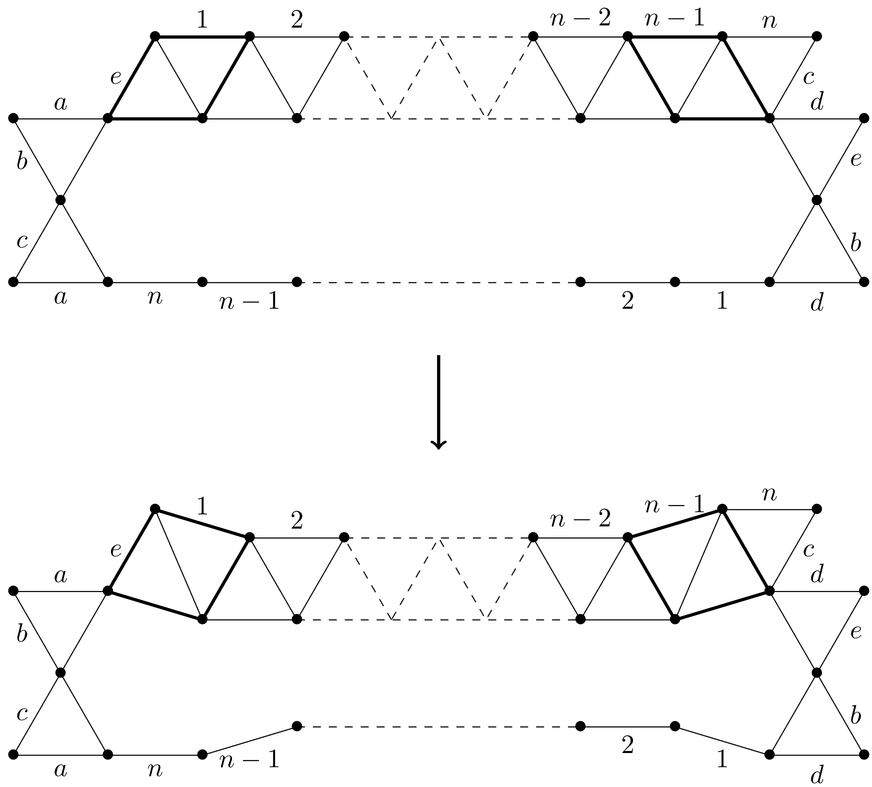


FIGURE 5.2. Example of a rigid surface that is not a local maximum

Now, we deform the surface as shown in Figure 5.2: the only short saddle connections that change are the horizontal ones in the parallelograms drawn with thick sides (see the labels "1" and "$n-1$") and their diagonals. The affine holonomy of the saddle connection corresponding to the label "1" is changed by adding $-i\varepsilon$, and similarly, we add $i\varepsilon$ to the one corresponding to the label "$n-1$".

Since all short saddle connections are kept to be of length at least one, we need to check that the area of the surface decreases.

1. The area of each thick parallelogram increases exactly by the area of the gray parallelogram in Figure 5.3, which is less than $\varepsilon$, and the two thick parallelograms in Figure 5.2 are disjoints for $n \geq 3$.
2. The area of the polygon decreases by $(n-1)\varepsilon + (n-2)\varepsilon = (2n-3)\varepsilon$.

Hence, the total area decreases if $n \geq 3$. □

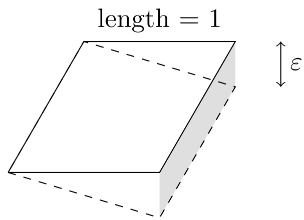


FIGURE 5.3. Comparing of the areas of the two parallelograms

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