Bulletin
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

TOWARDS TEMPERED ANABELIAN BEHAVIOUR OF BERKOVICH ANNULI

Sylvain Gaulhiac

Tome 151
Fascicule 1

2023

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

pages 1-36

Le *Bulletin de la Société Mathématique de France* est un périodique trimestriel
de la Société Mathématique de France.

Fascicule 1, tome 151, mars 2023

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

AMS
P.O. Box 6248
Providence RI 02940
USA
`www.ams.org`

Tarifs

Vente au numéro : 43 € ($ 64)
Abonnement électronique : 135 € ($ 202),
avec supplément papier : Europe 179 €, hors Europe 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat : Bulletin de la SMF

Bulletin de la Société Mathématique de France
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Tél : (33) 1 44 27 67 99 • Fax : (33) 1 40 46 90 96
`bulletin@smf.emath.fr` • `smf.emath.fr`

ISSN 0037-9484 (print) 2102-622X (electronic)

Directeur de la publication : Fabien DURAND

Bull. Soc. Math. France
151 (1), 2023, p. 1-36

TOWARDS TEMPERED ANABELIAN BEHAVIOUR OF BERKOVICH ANNULI

by Sylvain Gaulhiac

Abstract. — This work brings to light some partial *anabelian behaviours* of analytic annuli in the context of Berkovich geometry. More specifically, if k is a valued non-archimedean complete field of mixed characteristic that is algebraically closed, and $\mathcal{C}_1$, $\mathcal{C}_2$ are two k-analytic annuli with isomorphic tempered fundamental group, we show that the lengths of $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot be too far from each other. When they are finite, we show that the absolute value of their difference is bounded above with a bound depending only on the residual characteristic p.

Résumé (*Vers un comportement anabélien tempéré des couronnes de Berkovich*). — Ce travail met en lumière, partiellement, un comportement anabélien des couronnes dans le cadre de la géométrie analytique de Berkovich. Plus précisément, si k est un corps non-archimédien complet algébriquement clos de caractéristique mixte, et $\mathcal{C}_1$, $\mathcal{C}_2$ deux couronnes k-analytiques ayant des groupes fondamentaux tempérés isomorphes, nous montrons que les longueurs de ces deux couronnes ne peuvent être trop éloignées l'une de l'autre. Quand ces longueurs sont finies, nous prouvons que la valeur absolue de leur différence est bornée par une expression ne dépendant que de la caractéristique résiduelle p.

Texte reçu le 22 novembre 2021, modifié le 17 mai 2022, accepté le 9 septembre 2023.

Sylvain Gaulhiac, University of Alberta, Edmonton, Canada • *E-mail :* `gaulhiac@ualberta.ca`

Mathematical subject classification (2010). — 14H30, 12J25.

Key words and phrases. — Anabelian geometry, Berkovich spaces, tempered fundamental group, analytic curves, resolution of non-singularities.

 0037-9484/2023/1/$ 5.00
 doi:10.24033/bsmf.2862

1. Introduction

Anabelian geometry is concerned with the following question.

> *To what extent is a geometric object determined by its fundamental group?*

It is within the framework of algebraic geometry that Grothendieck gave the first conjectures of anabelian geometry in a famous letter to Faltings in 1983, where the fundamental group is nothing other than the étale one. Some deep results for hyperbolic curves have been obtained by Tamagawa and Mochizuki, answering certain conjectures of Grothendieck. However, almost no results are known for higher dimensions.

In the context of Berkovich analytic geometry, it is possible to define several "fundamental groups" classifying, for instance, *topological*, *finite étale* or *étale* (in the sense of [5]) coverings. However, the group that seems to best capture anabelian behaviours of analytic spaces over non-archimedian fields is the *tempered fundamental group*, introduced by Yves André in [2]. This group classifies *tempered coverings*, defined as étale coverings that become topological after a finite étale base change. Both finite étale and topological coverings are examples of tempered coverings.

In [1], Yves André obtained for the first time some results of anabelian nature related to the tempered fundamental group. A few years later, a huge step was taken in this direction with some results of Shinichi Mochizuki ([14]) followed by Emmanuel Lepage ([9] and [10]). These results relate the fundamental tempered group of the analytification of an algebraic hyperbolic curve to the dual graph of its stable reduction. If X is a hyperbolic curve defined over some non-archimedian complete field k, the homotopy type of its analytification X^{an} can be described in terms of the stable model $\mathscr{X}$ of X. More precisely, if $\mathscr{X}_s$ stands for the special fibre of $\mathscr{X}$, the *dual graph of the stable reduction* of X, denoted $\mathbb{G}_X$, is the finite graph whose vertices are the irreducible components of $\mathscr{X}_s$, and whose edges correspond to the nodes (singularities in ordinary double points) between irreducible components. If $\overline{X}$ denotes the normal compactification of X, a cusp of X is an element of $\overline{X} \setminus X$. Let us denote by $\mathbb{G}_X^{\mathfrak{c}}$ the graph obtained from $\mathbb{G}_X$, adding one open edge to each cusp of X, called the *extended dual graph of the stable reduction* of X. There exists a canonical topological embedding $\mathbb{G}_X^{\mathfrak{c}} \hookrightarrow X^{\mathrm{an}}$, which admits a topologically proper deformation retraction $X^{\mathrm{an}} \twoheadrightarrow \mathbb{G}_X^{\mathfrak{c}}$, and, thus, X^{an} and $\mathbb{G}_X^{\mathfrak{c}}$ have the same homotopy type.

Using the language of *semi-graphs of anabelioids* and *temperoids* introduced in high generality in [13] and [14], Mochizuki proves in [14] that the fundamental tempered group of the analytification of a hyperbolic curve determines the dual graph of its stable reduction:

THEOREM 1.1 ([14], Corollary 3.11). — *Let X_1 and X_2 be two hyperbolic curves over $\mathbb{C}_p$. Any outer isomorphism of groups $\varphi : \pi_1^{\mathrm{temp}}(X_1^{\mathrm{an}}) \xrightarrow{\sim} \pi_1^{\mathrm{temp}}(X_2^{\mathrm{an}})$ determines, functorially in φ, a unique isomorphism of graphs: $\overline{\varphi} : \mathbb{G}^{\mathtt{c}}_{X_1} \xrightarrow{\sim} \mathbb{G}^{\mathtt{c}}_{X_2}$.*

Mochizuki shows more precisely that it is possible to reconstruct the graph of the stable reduction $\mathbb{G}_X$ of a hyperbolic curve X from a (p')-version $\pi_1^{\mathrm{temp},\,(p')}(X^{\mathrm{an}})$ of the tempered fundamental group.

A few years later, Emmanuel Lepage refined this result. He proved that the knowledge of the tempered fundamental group of the analytification of a hyperbolic curve X enables us to not only reconstruct the graph $\mathbb{G}_X$, but also, in some cases, its canonical metric. This metric is such that the length of an edge corresponding to a node is the width of the annulus corresponding to the generic fibre of the formal completion on this node. It is, however, necessary to restrict ourselves to *Mumford curves*, which are defined as proper algebraic curves X over $\mathbb{C}_p$, such that the normalized irreducible components of the stable reduction are isomorphic to $\mathbb{P}^1$. This is equivalent to saying in Berkovich language that the analytification X^{an} is locally isomorphic to open subsets of $\mathbb{P}^{1,\mathrm{an}}$, or that X^{an} does not contains any point of genus > 0.

THEOREM 1.2 ([10]). — *Let X_1 and X_2 be two hyperbolic Mumford curves over $\mathbb{C}_p$ and $\varphi : \pi_1^{\mathrm{temp}}(X_1^{\mathrm{an}}) \xrightarrow{\sim} \pi_1^{\mathrm{temp}}(X_2^{\mathrm{an}})$ an isomorphism of groups. Then the isomorphism of graphs $\overline{\varphi} : \mathbb{G}_{X_1} \xrightarrow{\sim} \mathbb{G}_{X_2}$ is an isomorphism of metric graphs.*

These two results deal with analytic curves that are of *algebraic nature*, that is, analytifications of algebraic curves. Yet the theory of Berkovich analytic spaces is rich enough to contain many curves that are of *analytic nature* without coming from algebraic curves. The most important examples of such curves, which are still very simple to define, are *disks* and *annuli*. In the wake of Mochizuki's and Lepage's results, one wonders whether similar anabelian results exist for more general analytic curves without imposing any algebraic nature. The generalisation of Mochizuki's results for such analytic curves was carried out in the article [7], whereas the investigation about some analogue of Lepage's result is partially answered in this present article.

Reconstruction of the analytic skeleton. For a quasi-smooth analytic curve X, the good analogue of the extended dual graph of the stable reduction is the *analytic skeleton* $S^{\mathrm{an}}(X)$, defined in 2.5. When the skeleton meets all the connected components of X, there exists a canonical topological embedding $S^{\mathrm{an}}(X) \hookrightarrow X$, which admits a topologically proper deformation retraction $X \twoheadrightarrow S^{\mathrm{an}}(X)$. Therefore, X and $S^{\mathrm{an}}(X)$ have the same homotopy type. The restriction $S^{\mathrm{an}}(X)^{\natural}$ obtained from the skeleton by removing non-relatively compact edges is called the *truncated skeleton* of X (see 2.8), and is the analogue of the dual graph of the stable reduction. Let k be a complete algebraically closed non-archimedean field of residual exponent p. In [7], 3.29, a certain of class of

k-analytic curves is defined, called *k-analytically hyperbolic.* Their interest lies in the fact that for a k-analytically hyperbolic curve X it is possible to reconstruct its truncated skeleton $S^{\mathrm{an}}(X)^{\natural}$ from the tempered group $\pi_1^{\mathrm{temp}}(X)$, or even from a *prime-to-p* version $\pi_1^{\mathrm{temp},\,(p')}(X)$, obtained by taking the projective limit of all quotients of $\pi_1^{\mathrm{temp}}(X)$ admitting a normal torsion-free subgroup of finite index prime to p. The reconstruction of $S^{\mathrm{an}}(X)^{\natural}$ from this group is given by the following:

- the vertices correspond to the conjugacy classes of maximal compact subgroups of $\pi_1^{\mathrm{temp},\,(p')}(X)$;
- the edges correspond to the conjugacy classes of non-trivial intersections of two maximal compact subgroups of $\pi_1^{\mathrm{temp},\,(p')}(X)$.

The condition for a quasi-smooth k-analytic curve to be analytically hyperbolic is stated in terms of non-emptiness of the sets of nodes of the skeleton and some combinatorial hyperbolic condition at each of these nodes. However, the analytical hyperbolicity may not be enough to recover all the skeleton. In order to recover also the non-relatively compact edges of $S^{\mathrm{an}}(X)$ is defined in [7], 3.55, a sub-class of k-analytically hyperbolic curves called *k-analytically anabelian.* A k-analytically anabelian curve is a k-analytically hyperbolic curve satisfying a technical condition called *ascendance vicinale*, which enables us to reconstruct open edges of the skeleton:

THEOREM 1.3 ([7], 3.56). — *Let X_1 and X_2 be two k-analytically anabelian curves. Any group isomorphism $\varphi : \pi_1^{\mathrm{temp}}(X_1) \xrightarrow{\sim} \pi_1^{\mathrm{temp}}(X_2)$ induces (functorially in φ) an isomorphism of semi-graphs between the analytic skeletons: $S^{\mathrm{an}}(X_1) \xrightarrow{\sim} S^{\mathrm{an}}(X_2)$.*

Anabelianity of length? This present article concentrates more on the potential anabelianity of lengths of edges of the skeleton of a k-analytic curve, inspired from the result of Lepage cited above. There is a natural way to define the length of an analytic annulus (see 2.19), invariant by automorphisms, which makes the skeleton $S^{\mathrm{an}}(X)$ of a quasi-smooth k-analytic curve X a *metric* graph. The question that naturally arises is the following:

> *Does the tempered fundamental group $\pi_1^{\mathrm{temp}}(X)$ of a k-analytically anabelian curve X determine $S^{\mathrm{an}}(X)$ as a metric graph?*

Before tackling the general case, it seems *a priori* simpler to study first the case of a k-analytic annulus, even if this latter is not a k-analytically anabelian curve. The (p')-tempered group $\pi_1^{\mathrm{temp},\,(p')}(\mathcal{C})$ of an annulus is always isomorphic to the p'-profinite completion $\widehat{\mathbb{Z}}^{(p')}$ of $\mathbb{Z}$, but its total tempered group $\pi_1^{\mathrm{temp}}(\mathcal{C})$ depends on its length whenever k has mixed characteristic. The new question arising is the following:

> *Does the tempered group $\pi_1^{\mathrm{temp}}(\mathcal{C})$ of a k-analytic annulus $\mathcal{C}$ determine its length?*

In order to investigate this question, one is tempted to follow the scheme of proof that Lepage develops in [10]. An idea would be to start from an "ovoid" μ_p-covering of the annulus totally split at the middle of the skeleton, which would be analytically anabelian. Then knowing how to compute the length of any cycle would be enough to know the length of the annulus (by a limit argument). Yet one quickly faces problems of analytic nature that do not appear with Mumford curves: problems of detection of μ_{p^h}-torsors with trivial $\mathbb{Z}/p^h\mathbb{Z}$-cochain. Indeed, if $Y \to X$ is a μ_n-torsor, associating to some edge e of $S^{\mathrm{an}}(X)$, the growth rate of any analytic function defining locally this torsor over e leads to a harmonic cochain on the graph $S^{\mathrm{an}}(X)$ with values in $\mathbb{Z}/n\mathbb{Z}$. This growth rate corresponds to the degree of the strictly dominant monomial (see remark 2.13) of the corresponding analytic function. Therefore, when X is a quasi-smooth k-analytic curve, we show in Lemma 3.4 that there exists a cochain morphism $\theta : H^1(X, \mu_n) \to \mathrm{Harm}(S^{\mathrm{an}}(X), \mathbb{Z}/n\mathbb{Z})$, for any $n \in \mathbb{N}^\times$. However, when $n = p^h$ with $h > 1$, it seems difficult to detect the kernel of θ from $\pi_1^{\mathrm{temp}}(X)$, which makes the hoped for scheme of proof illusory. Nevertheless, the detection of $\ker(\theta)$ when $n = p$ is possible in some cases.

THEOREM 0. — *Let X be a k-analytic curve satisfying one of the two following conditions:*

1. *X is an annulus;*
2. *X is a k-analytically hyperbolic curve of finite skeleton without a bridge, without boundary or any point of genus > 0, with only annular cusps and at least a finite-annular one, such that there is never strictly more than one cusp coming from each node.*

Then the set of μ_p-torsors of X with trivial $\mathbb{Z}/p\mathbb{Z}$-cochain, $H^1(X, \mu_p) \cap \ker(\theta)$, is completely determined by $\pi_1^{\mathrm{temp}}(X)$.

This result uses *resolution of non-singularities* (section 4) coupled with a characterisation of non-triviality of cochains in terms of minimality of splitting radius at rigid points (Proposition 3.8). This characterisation can be re-phrased set-theoretically with the splitting sets of torsors (corollary 3.10), which can themselves be characterised from the tempered group by means of solvability (Proposition 4.7).

As for the initial question about the potential anabelianity of lengths of annuli, we found a partial answer, using the solvability of annuli (Proposition 4.6) doubled with some considerations of splitting sets of μ_p-torsors.

THEOREM 1. — *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two k-analytic annuli whose tempered fundamental groups $\pi_1^{\mathrm{temp}}(\mathcal{C}_1)$ and $\pi_1^{\mathrm{temp}}(\mathcal{C}_2)$ are isomorphic. Then $\mathcal{C}_1$ has finite length if and only if $\mathcal{C}_2$ has finite length. In this case:*

$$|\ell(\mathcal{C}_1) - \ell(\mathcal{C}_2)| < \frac{2p}{p-1}.$$

We also have $d\left(\frac{p-1}{p}\ell(\mathcal{C}_1), p\mathbb{N}^\times\right) > 1$ *if and only if* $d\left(\frac{p-1}{p}\ell(\mathcal{C}_2), p\mathbb{N}^\times\right) > 1$*, and in this case:*

$$|\ell(\mathcal{C}_1) - \ell(\mathcal{C}_2)| < \frac{p}{p-1}.$$

2. Berkovich analytic curves

In this whole text, k will denote a complete algebraically closed non-archimedean field of mixed characteristic $(0,p)$, i.e. $\mathrm{char}(k) = 0$ and $\mathrm{char}(\widetilde{k}) = p > 0$, where $\widetilde{k}$ is the residue field of k. Let us assume that the absolute value on k is normalized such that $|p| = p^{-1}$. The field $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ with the usual p-adic absolute value is an example of such field.

2.1. Points and skeleton of an analytic curve. — A k-analytic curve is defined as a separated k-analytic space of pure dimension 1. We send the reader to the foundational text [4] for references about analytic space, [3] for the cohomology on analytic spaces, and [6] for a precise and systematic study of analytic curves.

We recall here some properties about analytic curves that will be important in this text.

Any k-analytic curve endowed with the Berkovich topology, has very nice topological properties; they are locally compact, locally arcwise connected and locally contractible, which makes it possible to apply to it the usual theory of universal topological covering. Moreover, k-analytic curves are *real graphs*, with potentially infinite branching, as stated by the following proposition.

Proposition 2.1 ([6], 3.5.1). — *Let X be a non-empty connected k-analytic curve. The following statements are equivalent:*

i) *the topological space X is contractible;*
ii) *the topological space X is simply connected;*
iii) *for any pair (x, y) of points of X, there exists a unique closed subspace of X homeomorphic to a compact interval with extremities x and y.*

Moreover, any point of a k-analytic curve admits a basis of neighbourhood, which is real trees*, i.e. satisfies the equivalent properties above.*

Remark 2.2. — Any real tree can be endowed with a topology called the *topology of real tree*, which might be different from its initial topology. The Berkovich topology on an open subset of an analytic curve that is a tree is coarser than the topology of a real tree on this tree.

Points of a k-analytic curve. Let X be a k-analytic curve and $x \in X$. If $\mathscr{H}(x)$ denotes the completed residual field of x, it is possible to associate to the complete extension $\mathscr{H}(x)/k$ two transcendental values:

$$f_x = \mathrm{degtr}_{\widetilde{k}} \widetilde{\mathscr{H}(x)},$$
$$e_x = \mathrm{rang}_{\mathbb{Q}} \left(|\mathscr{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q} \right),$$

which satisfy $f_x + e_x \leqslant 1$ (in accordance with the Abhyankar inequality). The points of X can be classified into four types according to the following transcendental values:

Definition 2.3. — A point $x \in X$ is

1. of *type* 1, if $\mathscr{H}(x) = k$ (in this case $f_x = e_x = 0$),
2. of *type* 2, if $f_x = 1$,
3. of *type* 3, if $e_x = 1$,
4. of *type* 4, if $f_x = e_x = 0$, but x is not of type 1.

For $i \in \{1, 2, 3, 4\}$, let $X_{[i]}$ be the subset of X consisting of type-i points.

This definition of type-1 points holds here since we assumed that k is algebraically closed. In general, a point $x \in X$ is of type 1 if $\mathscr{H}(x) \subseteq \widehat{\overline{k}}$, where $\widehat{\overline{k}}$ denotes the completion of an algebraic closure of k. Since k is algebraically closed, type-1 points are exactly the *rigid* points, i.e. the points $x \in X$ such that the extension $\mathscr{H}(x)/k$ is finite. Since k is by assumption non-trivially valued, $X_{[2]}$ is dense in X.

Preservation of type of points by finite morphisms. If $f : X' \to X$ is a finite morphism of k-analytic curves, for any $i \in \{1, 2, 3, 4\}$, a point $x' \in X'$ is of type i, if and only if $f(x)$ is of type i.

A specificity of Berkovich geometry compared to rigid geometry is the existence of a *boundary* that is embodied in the space. It is possible to define two boundaries of a k-analytic space: the *analytic boundary* $\Gamma(X)$ and the *Shilov boundary* $\partial^{\mathrm{an}} X$. However, specifically in the dimension 1 case, i.e. for analytic curves, these two notions coincide, which allows us to speak without any ambiguity about the *boundary of* X $\partial^{\mathrm{an}} X \subseteq X$, potentially empty.

Description of the k-analytic affine line. $\mathbb{A}_k^{1,\mathrm{an}}$. The analytification $\mathbb{A}_k^{1,\mathrm{an}}$ of the (algebraic) affine line $\mathbb{A}_k^1$ is the smooth, without boundary and connected k-analytic curve whose points are the multiplicative semi-norms on the polynomial ring $k[T]$ extending the absolute value of k. We shall give an explicit description of $\mathbb{A}_k^{1,\mathrm{an}}$. For $r \geqslant 0$ and $a \in k$, let $B(a, r) = \{x \in k, |x - a| \leqslant r\}$ be the closed ball (which is also open since k is non-archimedean!) of k, centred in a and of radius r.

- Any element $a \in k$ determines a multiplicative semi-norm on $k[T]$, the *evaluation at* a, given by $P \in k[T] \mapsto |P(a)|$. It defines an element of $\mathbb{A}_k^{1,\mathrm{an}}$ denoted η_a, or $\eta_{a,0}$. Then $\mathscr{H}(\eta_a) = k[T]/(T - a) \simeq k$, so η_a is a rigid point.

- Let $a \in k$ et $r > 0$. Consider the map:
$$P \in k[T] \mapsto \sup_{b \in B(a,r)} |P(b)| = \sup_{b \in B(a,r)} |P|_{\eta_b}.$$
It actually defines an element of $\mathbb{A}_k^{1,\mathrm{an}}$, denoted $\eta_{a,r}$ and given by:
$$|P(\eta_{a,r})| = \max_{0 \leqslant i \leqslant n}(|\alpha_i| r^i) \text{ if } P = \sum_{i=0}^{n} \alpha_i (T-a)^i.$$
One can verify that $\eta_{a,r}$ only depends on $B(a,r)$ (i.e. $\eta_{a,r} = \eta_{b,r}$ as soon as $b \in B(a,r)$). There are two cases.
When $r \in |k^\times|$, $\widetilde{\mathscr{H}(\eta_{a,r})} = \widetilde{k}(T)$, and $|\mathscr{H}(\eta_{a,r})^\times| = |k^\times|$, such that $\eta_{a,r}$ is a type-2 point.
When $r \notin |k^\times|$, $\widetilde{\mathscr{H}(\eta_{a,r})} = \widetilde{k}$, and $|\mathscr{H}(\eta_{a,r})^\times|$ is the group generated by $|k^\times|$ and r, so $\eta_{a,r}$ is a type-3 point.
- If $\mathscr{B} = (B_n)_{n \in \mathbb{N}}$ is a decreasing sequence of non-empty closed balls (i.e. $B_{n+1} \subseteq B_n$, for $n \in \mathbb{N}$) of k. Let $|\cdot|_{B_n}$ be the unique point of $\mathbb{A}_k^{1,\mathrm{an}}$ determined by B_n (i.e. $|\cdot|_{B_n} = \eta_{a_n,r_n}$ as soon as $B_n = B(a_n, r_n)$). Then the map:
$$P \in k[T] \mapsto \inf_{n \in \mathbb{N}} |P|_{B_n}$$
defines an element $|\cdot|_{\mathscr{B}}$ of $\mathbb{A}_k^{1,\mathrm{an}}$.
If $\bigcap_n B_n$ is a point $a \in k$, then $|\cdot|_{\mathscr{B}}$ corresponds exactly to η_a. If $\bigcap_n B_n$ is a closed ball centred in $a \in k$ and of radius $r \in \mathbb{R}_+^*$, then $|\cdot|_{\mathscr{B}}$ corresponds to $\eta_{a,r}$. It is also possible that $\bigcap_n B_n$ is empty; in this case, $|\cdot|_{\mathscr{B}}$ is a type-4 point.

This description is exhaustive; all the points of $\mathbb{A}_k^{1,\mathrm{an}}$ can be described in this way.

Remark 2.4. — Points of type 4 exist if and only if k is not *spherically complete*. A valued field is *spherically complete* when it does not admit any *immediate* extension. The field $\mathbb{C}_p$ is not spherically complete, therefore there exist in $\mathbb{A}_{\mathbb{C}_p}^{1,\mathrm{an}}$ some type-4 points.

The k-analytic projective line. $\mathbb{P}_k^{1,\mathrm{an}}$ is the analytification of the algebraic k-projective line $\mathbb{P}_k^1$. It is a proper (compact and without boundary) quasi-smooth connected curve. It admits a rigid point ∞ such that there exists a natural isomorphism an k-analytic curve:
$$\rho : \mathbb{P}_k^{1,\mathrm{an}} \setminus \{\infty\} \xrightarrow{\sim} \mathbb{A}_k^{1,\mathrm{an}}.$$

The k-analytic affine and projective curves are trees (see 2.1), so for each pair (x,y) of points, there exists a unique closed subspace homeomorphic to a compact interval (a segment) with extremities x and y. If a et b are in k, the segment joining the rigid points η_a and η_b is:

$$[\eta_a, \eta_b] = \{\eta_{a,r}\}_{0\leqslant r\leqslant |b-a|} \cup \{\eta_{b,s}\}_{0\leqslant r\leqslant |b-a|}.$$

The segment joining η_a and ∞ is $[\eta_a, \infty] = \{\eta_{a,r}\}_{0\leqslant r\leqslant \infty}$, with $\infty = \eta_{a,\infty}$.

The type of points of $\mathbb{P}_k^{1,\mathrm{an}}$ (or $\mathbb{A}_k^{1,\mathrm{an}}$) can be read on the tree:

- type-2 points are the branching points of the tree;
- type-3 points are the points where nothing special happens (their valence is 2);
- type-1 or type-4 points are the unibranched points on the tree, the "leaves".

Analytic skeleton of an analytic curve. The following notion of *analytic skeleton* of a k-analytic curve is the analogue in the analytic world of the dual graph of the special fibre of the stable model of an algebraic k-curve.

A *k-analytic disk* is a k-analytic curve isomorphic to the analytic domain of $\mathbb{P}_k^{1,\mathrm{an}}$ defined by the condition $|T| \in I$, where I is an interval of the form $[0, r[$ or $[0, r]$ for some $r > 0$, or $I = [0, +\infty[$.

Definition 2.5 (Analytic skeleton). — The *analytic skeleton* of a quasi-smooth k-analytic curve X, denoted $S^{\mathrm{an}}(X)$, is the subset of X consisting of points that do not belong to any open k-analytic disk.

Proposition 2.6 (see [6], 1.6.13, 5.1.11). — *Let X be a quasi-smooth k-analytic curve.*

- *The analytic skeleton $S^{\mathrm{an}}(X)$ is a locally finite graph contained in $X_{[2,3]}$ and containing the boundary $\partial^{\mathrm{an}}X$ of X.*
- *If $S^{\mathrm{an}}(X)$ meets all the connected components of X, there exists a canonical deformation retraction $r_X : X \to S^{\mathrm{an}}(X)$. In particular, X and $S^{\mathrm{an}}(X)$ have the same homotopy type.*

Remark 2.7. — In order to be coherent with the terminology of [14], in [7] we used the term *semi-graphs* for graphs with potentially "open" edges, i.e. edges that are either not abutting to any vertex or with only one extremity abutting to one vertex. However, we will not make this terminological distinction in this text to avoid some unnecessary heaviness and speak only about *graphs* instead of semi-graphs.

Definition 2.8 (Truncated skeleton). — Let X be a quasi-smooth connected k-analytic curve with non-empty skeleton $S^{\mathrm{an}}(X)$ and $r_X : X \to S^{\mathrm{an}}(X)$ the canonical retraction. The *truncated skeleton* of X, denoted $S^{\mathrm{an}}(X)^{\natural}$, is the subgraph of $S^{\mathrm{an}}(X)$ obtained from $S^{\mathrm{an}}(X)$ by removing the edges e, such that $r_X^{-1}(e)$ is not relatively compact in X.

REMARK 2.9. — The edges e of $S^{\mathrm{an}}(X)$ such that $r_X^{-1}(e)$ is not relatively compact in X are exactly the "open" edges of $S^{\mathrm{an}}(X)$. So in the terminology of [7], $S^{\mathrm{an}}(X)^{\natural}$ is actually the biggest sub-semi-graph of the semi-graph $S^{\mathrm{an}}(X)$, which is a graph.

DEFINITION 2.10 (Nodes of the analytic skeleton). — If x is a point of a k-analytic curve, its *genus*, denoted $g(x)$, is defined as being 0 if x is of type $1, 3$ or 4 and equals the genus of the residual curve (see [7], 3.1.4) $\mathscr{C}_x$ of x, when it is of type 2. A point $x \in S^{\mathrm{an}}(X)$ is a *node* of $S^{\mathrm{an}}(x)$ if it satisfies one of the following conditions:

- x is a branching point of $S^{\mathrm{an}}(X)$ (i.e. x is a vertex of $S^{\mathrm{an}}(x)$ and at least three different branches of $S^{\mathrm{an}}(X)$ abut to x);
- $x \in \partial^{\mathrm{an}} X$;
- $g(x) > 0$.

2.2. Analytic annuli: functions, length and torsors. — We are going to define and study the basic properties of k-analytic annuli, which are central in this text.

If $I = [a, b]$ is a compact interval of $\mathbb{R}_{>0}$ (possibly reduced to one point), let $\mathcal{C}_I$ be the k-analytic curve defined as an k-affinoid space by:

$$\mathcal{C}_I = \mathscr{M}\left(k\{b^{-1}T, aU\}/(TU-1)\right).$$

If $I \subset J$ are compact intervals of $\mathbb{R}_{>0}$, there is a natural morphism $\mathcal{C}_I \to \mathcal{C}_J$, which makes $\mathcal{C}_I$ an analytic domain of $\mathcal{C}_J$. If I is an arbitrary interval of $\mathbb{R}_{>0}$, we can define

$$\mathcal{C}_I = \varinjlim_{J \subset I} \mathcal{C}_J \subset \mathbb{G}_m^{\mathrm{an}},$$

where J describes all compact intervals of $\mathbb{R}_{>0}$ containing I. It would have been equivalent to define $\mathcal{C}_I$ as the analytic domain of $\mathbb{P}_k^{1,\mathrm{an}}$ defined by the condition $|T| \in I$.

- *Analytic functions.* The k-algebra of analytic functions on $\mathcal{C}_I$ is given by:
$$\mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I) = \left\{ \sum_{i \in \mathbb{Z}} a_i T^i, a_i \in k, \lim_{|i| \to +\infty} |a_i| r^i = 0, \forall r \in I \right\}.$$
- *Boundary.* If $s < r \in \mathbb{R}_+^*$, $\partial^{\mathrm{an}} \mathcal{C}_{\{r\}} = \{\eta_{0,r}\}$, whereas $\partial^{\mathrm{an}} \mathcal{C}_{[s,r]} = \{\eta_{0,s}, \eta_{0,r}\}$.

DEFINITION 2.11. — A k-analytic *annulus* is defined as a k-analytic curve isomorphic to $\mathcal{C}_I$ for some interval I of $\mathbb{R}_{>0}$. Annuli are quasi-smooth curves.

PROPOSITION 2.12 (Condition of invertibility of an analytic function, [6] 3.6.6.1 et 3.6.6.2). — *Let I be an interval of $\mathbb{R}_{>0}$ and $f = \sum_{i \in \mathbb{Z}} a_i T^i \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ an* *analytic function on $\mathcal{C}_I$. The function f is invertible if and only if there exists an integer $i_0 \in \mathbb{Z}$ (necessarily unique), such that $|a_{i_0}|r^{i_0} > \max_{i \neq i_0} |a_i|r^i$ for all $r \in I$.*

REMARK 2.13. — For an analytic function $f = \sum_{i \in \mathbb{Z}} a_i T^i \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$, we will say that f admits a *strictly dominant monomial* $a_{i_0}T^{i_0}$, when there exists an integer $i_0 \in \mathbb{Z}$ such that $|a_{i_0}|r^{i_0} > \max_{i \neq i_0} |a_i|r^i$ for all $r \in I$. Such a strictly dominant monomial is unique, and i_0 is the *degree* of this monomial. The last proposition says that $f \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ is invertible if and only if it admits a strictly dominant monomial, $a_{i_0}T^{i_0}$, in which case, f is written as $f = a_{i_0}T^{i_0}(1+u)$ with $u \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ of norm < 1 on $\mathcal{C}_I$.

Let $f \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)^\times$ be an invertible function on $\mathcal{C}_I$, such that the degree i_0 of its strictly dominant monomial is different from 0. Let $\varphi_f : \mathcal{C}_I \to \mathbb{A}_k^{1,\mathrm{an}}$ be the morphism induced by f, and Λ the map from $\mathbb{R}_{>0}$ to itself defined by $r \mapsto |a_{i_0}|r^{i_0}$.

PROPOSITION 2.14 ([6], 3.6.8). — *The map Λ induces a homeomorphism from I to the interval $\Lambda(I)$ of $\mathbb{R}_{>0}$, and φ_f induces a finite and flat morphism of degree $|i_0|$ from $\mathcal{C}_I$ to $\mathcal{C}_{\Lambda(I)}$.*

DEFINITION 2.15 (Coordinate functions). — If $\mathcal{C}$ is a k-analytic annulus, a function $f \in \mathscr{O}_{\mathcal{C}}(\mathcal{C})$ is a *coordinate function* when it induces an isomorphism of k-analytic curves $\mathcal{C} \xrightarrow{\sim} \mathcal{C}_I$, for some interval I of $\mathbb{R}_{>0}$.

COROLLARY 2.16 (Characterisation of coordinate functions, [6], 3.6.11.3 et 3.6.12.3). — *An analytic function $f \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ is a coordinate function if and only if f admits a strictly dominant monomial of degree $i_0 \in \{-1, 1\}$. If this is the case, f is invertible in $\mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ and induces an analytic isomorphism $\mathcal{C}_I \simeq \mathcal{C}_{|a_{i_0}|I^{i_0}}$.*

One can directly deduce the following corollary.

COROLLARY 2.17. — *Let I and I' be two intervals of $\mathbb{R}_{>0}$; then $C_{I'}$ is isomorphic to $\mathcal{C}_I$ if and only if $I' \in |k^\times|\, I^{\pm 1}$.*

REMARK 2.18 (Algebraic characterisation of coordinate functions of annuli ([6], 3.6.13.1)). — Define $\mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)^{\circ\circ}$ as the subset of $\mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ consisting of functions of norms strictly lower than 1 on $\mathcal{C}_I$. We saw that a function $f \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ is invertible if and only if it admits a strictly dominant monomial, $a_{i_0}T^{i_0}$. In this case, it can be written as $f = a_{i_0}T^{i_0}(1+u)$ with $u \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)^{\circ\circ}$, and $|f|$ equals $|a_{i_0}| \cdot |T|^{i_0}$ on $\mathcal{C}_I$. Consequently, the group

$$\mathscr{Z}_I := \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)^\times / k^\times \cdot (1 + \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)^{\circ\circ})$$

is isomorphic to $\mathbb{Z}$; such an isomorphism is given by the degree of the strictly dominant monomial. From Corollary 2.16, a function $f \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)$ is a coordinate function of $\mathcal{C}_I$ if and only if it is invertible, and its class in $\mathscr{Z}_I$ is a generator of $\mathscr{Z}_I$.

Therefore, if $\mathcal{C}$ is any k-analytic annulus and $f \in \mathscr{O}_{\mathcal{C}}(\mathcal{C})$, f is a coordinate function if and only if it is invertible and is sent to a generator of the free abelian group of rank 1:

$$\mathscr{Z}(\mathcal{C}) := \mathscr{O}_{\mathcal{C}}(\mathcal{C})^\times / k^\times \cdot (1 + \mathscr{O}_{\mathcal{C}}(\mathcal{C})^{\circ\circ}).$$

Definition 2.19 (Length of an analytic annulus). —
If I is an interval of $\mathbb{R}_{>0}$, the length of the annulus $\mathcal{C}_I$ is defined as:

$$\ell(\mathcal{C}_I) = \log_p \left(\frac{\sup I}{\inf I} \right),$$

with $\ell(\mathcal{C}_I) = +\infty$ whenever $\inf I = 0$ or $\sup I = +\infty$.

The *length* of a general k-analytic annulus $\mathcal{C}$, denoted $\ell(\mathcal{C})$, is defined as the length of $\mathcal{C}_I$ for any interval I of $\mathbb{R}_{>0}$, such that $\mathcal{C}$ is isomorphic to $\mathcal{C}_I$. From Corollary 2.17, we can see that this definition does not depend on the choice of such I.

There exists a natural distance on the set of type-2 and type-3 points of $\mathbb{P}^{1,\mathrm{an}}_k$, which is consistent with this definition of the length of an annulus. However, we will not define it in this text.

Kummer torsors of an annulus. If X is a k-analytic space and $\ell \in \mathbb{N}^\times$ an integer (in general, it is necessary to ask that ℓ is not 0 in k, but it is obviously the case here since $\mathrm{char}(k) = 0$), the *Kummer exact sequence* on $X_{\text{ét}}$:

$$1 \longrightarrow \mu_\ell \longrightarrow \mathbb{G}_m \overset{z \mapsto z^\ell}{\longrightarrow} \mathbb{G}_m \longrightarrow 1$$

induces an injective morphism

$$\mathscr{O}_X(X)^\times / (\mathscr{O}_X(X)^\times)^\ell \overset{\iota}{\hookrightarrow} H^1(X_{\text{ét}}, \mu_\ell),$$

whose image will be denoted $\mathsf{Kum}_\ell(X)$. It is known ([3]) that any locally constant étale sheaf on $X_{\text{ét}}$ is representable. Consequently, $H^1(X_{\text{ét}}, \mu_\ell)$ classifies all the *analytic* étale μ_ℓ-torsors on X up to isomorphism. If $f \in \mathscr{O}_X(X)^\times$, its image (f) in $H^1(X_{\text{ét}}, \mu_\ell)$ by ι corresponds to $\mathscr{M}(\mathscr{O}_X[T]/(T^\ell - f))$. The elements of $\mathsf{Kum}_\ell(X)$ seen as analytic étale μ_ℓ-torsors will be called *Kummer μ_ℓ-torsors.*

Example 2.20. — If I is a non-empty interval of $\mathbb{R}_{>0}$, the (invertible) function $T^\ell \in \mathscr{O}_{\mathcal{C}_I}(\mathcal{C}_I)^\times$ induces a Kummer μ_ℓ-torsor $\mathcal{C}_I \to \mathcal{C}_{I^\ell}$, identifying $\mathcal{C}_I$ with $\mathscr{M}(\mathscr{O}_{\mathcal{C}_{I^\ell}}[T]/(T^\ell - S))$, where S is the standard coordinate of $\mathcal{C}_{I^\ell}$.

PROPOSITION 2.21. — *Let $\mathcal{C}$ be a k-analytic annulus and $\ell \in \mathbb{N}^\times$ an integer* prime to the residual characteristic p.

1. *The group $\mathsf{Kum}_\ell(\mathcal{C})$ is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$. This isomorphism is non-canonical as soon as $\ell \geqslant 3$ but becomes canonical when one fixes an orientation of $\mathcal{C}$.*
2. *Any connected component of a Kummer μ_ℓ-torsor of $\mathcal{C}$ is a k-analytic annulus.*
3. *Any μ_ℓ-torsor of $\mathcal{C}$ is Kummer, which leads to an isomorphism:*

$$H^1(\mathcal{C}_{\text{ét}}, \mu_\ell) \simeq \mathsf{Kum}_\ell(\mathcal{C}) \simeq \mathbb{Z}/\ell\mathbb{Z}.$$

Proof. — The proof of the two first points can be found in [6], 3.6.30 and 3.6.31. The facts that k is algebraically closed and that ℓ is prime to p are necessary for the first point, since it implies that the subgroup $k^\times \cdot (1 + \mathscr{O}_\mathcal{C}(\mathcal{C})^{\circ\circ})$ of $\mathscr{O}_\mathcal{C}(\mathcal{C})^\times$ is ℓ-divisible. In terms of the group $\mathscr{Z}(\mathcal{C})$ defined in remark 2.18, this means that $\ell\mathscr{Z}(\mathcal{C}) \simeq \left(\mathscr{O}_\mathcal{C}(\mathcal{C})^\times\right)^\ell / k^\times \cdot (1 + \mathscr{O}_\mathcal{C}(\mathcal{C})^{\circ\circ})$. Therefore, there is a canonical isomorphism:

$$\mathrm{Kum}_\ell(\mathcal{C}) \simeq \mathscr{O}_\mathcal{C}(\mathcal{C})^\times / \left(\mathscr{O}_\mathcal{C}(\mathcal{C})^\times\right)^\ell \simeq \mathscr{Z}(\mathcal{C})/\ell\mathscr{Z}(\mathcal{C}) \simeq \mathbb{Z}/\ell\mathbb{Z}.$$

The proof of the last point comes from [3], 6.3.5, where Berkovich shows that any connected tame finite étale covering of a compact annulus is Kummer, and that any μ_ℓ-torsor is tame since ℓ is assumed to be prime to p. It is easy to extend it to the case when $\mathcal{C}$ is not compact since it is then identified with the colimit of its compact subannuli. □

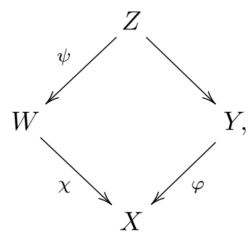
2.3. Tempered fundamental group. — Let X be a quasi-smooth strictly k-analytic space (not necessarily a curve). As defined in [5], an *étale covering* of X is a morphism $\varphi : Y \to X$, such that X admits an open covering $X = \bigcup_{i\in I} U_i$, such that for each $i \in I$:

$$\varphi^{-1}(U_i) = \coprod_{j\in J_i} Y_{i,j},$$

where each $Y_{i,j} \to U_i$ is finite étale, with potentially infinite index sets. If X is connected, an étale covering $\varphi : Y \to X$ is *Galois* when Y is connected, and the action of the automorphism group $G = \mathrm{Aut}(\varphi)$ is simply transitive.

For instance, finite étale coverings, as well as topological coverings (for the Berkovich topology), are étale coverings and are surjective. In [2], Yves André defined the notion of *tempered covering*, defined as follows.

DEFINITION 2.22. — An étale covering $\varphi : Y \to X$ is *tempered* if it is the quotient of the composition of a topological covering and of a finite étale covering, i.e. if there exists a commutative diagram of k-analytic spaces:



where χ is a finite étale covering and ψ a topological covering. It is equivalent to saying that φ becomes topological after pull-back by some (surjective) finite étale covering. Let $\mathrm{Cov}^{\mathrm{temp}}(X)$ be the category of tempered coverings of X.

If $x \in X$ is a geometric point, consider the fibre functor

$$F_x : \mathrm{Cov}^{\mathrm{temp}}(X) \to \mathrm{Set},$$

which maps a covering $Y \to X$ to the fibre Y_x. The *tempered fundamental group pointed at* x is defined as the automorphism group of the fibre functor in x:

$$\pi_1^{\mathrm{temp}}(X, x) := \mathrm{Aut}(F_x).$$

The group $\pi_1^{\mathrm{temp}}(X, x)$ becomes a topological group by considering the basis of open subgroups consisting of the stabilizers $(\mathrm{Stab}_{F_x(Y)}(y))_{Y \in \mathrm{Cov}^{\mathrm{temp}}(X),\ y \in F_x(Y)}$. It is a prodiscrete topological group.

If x and x' are two different geometric points, the functors F_x and $F_{x'}$ are (non-canonically) isomorphic, and any automorphism of F_x induces an inner automorphism of $\pi_1^{\mathrm{temp}}(X, x)$. Thus, one can consider the *tempered fundamental group* $\pi_1^{\mathrm{temp}}(X)$, defined up to a unique outer isomorphism.

If $\pi_1^{\mathrm{alg}}(X, x)$ (or $\pi_1^{\mathrm{top}}(X, x)$) denotes the group classifying pointed finite étale (or topological) coverings of X, the natural morphism $\pi_1^{\mathrm{temp}}(X, x) \to \pi_1^{\mathrm{top}}(X, x)$ is always surjective, and the natural morphism $\pi_1^{\mathrm{temp}}(X, x) \to \pi_1^{\mathrm{alg}}(X, x)$ has a dense image, such that $\pi_1^{\mathrm{alg}}(X, x)$ can be identified with the profinite completion of $\pi_1^{\mathrm{temp}}(X, x)$:

$$\pi_1^{\mathrm{alg}}(X, x) = \widehat{\pi_1^{\mathrm{temp}}(X, x)}.$$

In dimension 1, when X is a k-analytic curve, the morphism $\pi_1^{\mathrm{temp}}(X, x) \to \pi_1^{\mathrm{alg}}(X, x)$ is injective (these results can be found in [2], 2.1.6). As a consequence, the affine and projective lines $\mathbb{A}_k^{1,\mathrm{an}}$ and $\mathbb{P}_k^{1,\mathrm{an}}$ do not admit any non-trivial tempered coverings:

$$\pi_1^{\mathrm{temp}}(\mathbb{P}_k^{1,\mathrm{an}}) \simeq \pi_1^{\mathrm{temp}}(\mathbb{A}_k^{1,\mathrm{an}}) \simeq 0.$$

DEFINITION 2.23 (Moderate tempered coverings). — Let $\mathrm{Cov}^{\mathrm{temp},\,(p')}(X)$ be the full subcategory of $\mathrm{Cov}^{\mathrm{temp}}(X)$ consisting of tempered coverings that are quotients of a topological covering and a Galois finite étale covering of *degree prime to p*. In the same way as for the tempered case, it is possible to consider a classifying group defined as the automorphism group of a geometric fibre functor and well defined up to a unique outer automorphism; this group $\pi_1^{\mathrm{temp},\,(p')}(X)$ is called the *moderately tempered group* of X. It is naturally a topological pro-discrete group.

REMARK 2.24. — When X is a k-analytic curve, the group $\pi_1^{\mathrm{temp},\,(p')}(X)$ can be constructed group-theoretically from $\pi_1^{\mathrm{temp}}(X)$ as the projective limit of quotients of $\pi_1^{\mathrm{temp}}(X)$ admitting a torsion-free normal subgroup of a finite index prime to p.

2.4. Verticial, vicinal and cuspidal subgroups. — We recall here some notions and terminology of [7] about k-analytically hyperbolic curves. Let X be a quasi-smooth connected k-analytic curve with non-empty skeleton $S^{\mathrm{an}}(X)$ and $r_X : X \twoheadrightarrow S^{\mathrm{an}}(X)$ be the canonical retraction. Let Σ_X be the set of vertices of $S^{\mathrm{an}}(X)$ (it is the set of *nodes* of $S^{\mathrm{an}}(X)$ in the language of [7]).

An edge e of $S^{\mathrm{an}}(X)$ can be of two different types.

- It is a *vicinal* edge whenever the connected component of $X \setminus \Sigma_X$ associated to e, i.e. $r_X^{-1}(\mathring{e})$, is relatively compact in X, which is the same as asking that each of the two extremities of e abuts to a vertex (it is a "closed" edge).
- It is a *cusp* whenever the associated connected component of $X \setminus \Sigma_X$ is non-relatively compact in X; in other words when it is either a isolated edge, or only one of its extremities abuts to a vertex.

REMARK 2.25. — The connected component of $X \setminus \Sigma_X$ associated to a vicinal edge is always a k-analytic annulus. However, this might not always be the case for cusps (see [7], Remark 2.18). A cusp whose associated connected component of $X \setminus \Sigma_X$ is an annulus will be called *annular*.

Recall, from [7], that a finite étale covering $\varphi : Y \to X$ of a quasi-smooth connected curve X is called *moderate*, if for any $y \in Y$, the degree $[\mathscr{H}(y)^{\mathrm{gal}} : \mathscr{H}(\varphi(y))]$ is prime to p, where $\mathscr{H}(y)^{\mathrm{gal}}$ stands for the Galois closure of the extension $\mathscr{H}(y)/\mathscr{H}(\varphi(y))$. The category of moderate covering of X is a Galois category whose fundamental group is denoted $\pi_1^{\mathrm{t}}(X)$, the moderate fundamental group of X, which is a profinite group.

Let e be an edge of $S^{\mathrm{an}}(X)$ and $\mathcal{C}_e$ the associated connected component of $X \setminus \Sigma_X$. Let $\pi_e = \pi_1^{\mathrm{t}}(\mathcal{C}_e)$ be the moderate fundamental group of $\mathcal{C}_e$. If v is a vertex of $S^{\mathrm{an}}(X)$, the *star* centred in v, denoted $\mathrm{St}(v, X)$, is defined by

$$\mathrm{St(v,X)} = \{v\} \sqcup \bigsqcup_e \mathcal{C}_e,$$

where the disjoint union is taken over all edges e of $S^{\mathrm{an}}(X)$ abutting to v. Let $\pi_v = \pi_1^{\mathrm{t}}(\mathrm{St}(v, X))$ be the moderate fundamental group of St(v, X).

We saw in [7] that if X is k-analytically hyperbolic, for any component c of $S^{\mathrm{an}}(X)$ (vertex or edge), there is a natural embedding $\pi_c \hookrightarrow \pi_1^{\mathrm{temp},\,(p')}(X)$. This comes from the fact that the semi-graph of anabelioids $\mathcal{G}(X, \Sigma_X)$ is of *injective type* and that there is a natural isomorphism $\pi_1^{\mathrm{temp}}(\mathcal{G}(X, \Sigma_X)) \simeq \pi_1^{\mathrm{temp},\,(p')}(X)$ (see [7], Corollary 3.36).

Definition 2.26. — If X is a k-analytically hyperbolic curve, a compact subgroup of $\pi_1^{\mathrm{temp},\,(p')}(X)$ is called:

- *vicinal* if it is of the form π_e for some vicinal edge e of $S^{\mathrm{an}}(X)$;
- *cuspidal* if it is of the form π_e for some cusp e of $S^{\mathrm{an}}(X)$;
- *verticial* if it is of the form π_v for some vertex v of $S^{\mathrm{an}}(X)$.

Remark 2.27. — The Kummer nature of moderate coverings of an annulus implies that vicinal subgroups and cuspidal subgroups are always isomorphic to $\widehat{\mathbb{Z}}^{(p')}$, even for non-annular cusps. However, a compact subgroup of $\pi_1^{\mathrm{temp},\,(p')}(X)$ cannot be vicinal and cuspidal at the same time . Verticial subgroups are always isomorphic to the *prime-to-p*-profinite completion of the fundamental group of a hyperbolic Riemann surface (see [7], Corollary 3.23 and proof of 3.30).

For a k-analytically hyperbolic curve X, verticial and vicinal subgroups can be characterised directly from the group $\pi_1^{\mathrm{temp},\,(p')}(X)$; verticial subgroups correspond exactly to (conjugacy classes of) maximal compact subgroups, whereas vicinal subgroups correspond to (conjugacy classes of) non-trivial intersections of two maximal compact subgroups. Therefore, one can reconstruct the truncated skeleton $S^{\mathrm{an}}(X)^{\natural}$ from the tempered group $\pi_1^{\mathrm{temp},\,(p')}(X)$ (so also from $\pi_1^{\mathrm{temp}}(X)$, since the first one can be deduced from the second one taking a suited inverse limit; see 2.24).

3. Harmonic cochains and torsors

3.1. Splitting conditions of μ_p-torsors. —

Lemma 3.1. — *Let ξ and ξ' be two distinct p^{th}-roots of unity in k (recall that k is assumed to be algebraically closed). Then $|\xi - \xi'| = p^{-\frac{1}{p-1}}$.*

Proof. — Write $\Phi_p = \dfrac{X^p - 1}{X - 1} = \displaystyle\sum_{i=0}^{p-1} X^i = \prod_{\xi \in \mu'_p} X - \xi \in \mathbb{Q}[X]$, for the p^{th} cyclotomic polynomial, where μ'_p stands for the set of the $p-1$ primitive p^{th}-roots of unity in k. The evaluation at 1 gives $p = \prod_{\xi \in \mu'_p} 1 - \xi$. For ξ describing μ'_p, all the $1-\xi$ have the same norm since they are on the same $\mathrm{Aut}(k/\mathbb{Q}_p)$-conjugacy class. Therefore, $|1-\xi| = p^{-\frac{1}{p-1}}$, and we obtain the result since multiplication by any p^{th}-root of unity is an isometry of k. □

An étale covering $\varphi : Y \to X$ between two k-analytic curves *totally splits* over a point $x \in X$, if for any $y \in \varphi^{-1}(\{x\})$, the extension $\mathscr{H}(x) \to \mathscr{H}(y)$ is an isomorphism. When φ is of degree n, φ totally splits over x if and only if the fibre $\varphi^{-1}(\{x\})$ has exactly n elements, which is the same as saying that locally, over a neighbourhood of x, φ is a topological covering (see [2], III, 1.2.1).

The following proposition, which characterizes the splitting sets of the μ_{p^h}-torsor given by the function $\sqrt[p^h]{1+T}$, will be of paramount importance in this article.

Proposition 3.2. — *If $h \in \mathbb{N}^\times$, the étale covering $\mathbb{G}_m^{\mathrm{an}} \xrightarrow{z \mapsto z^{p^h}} \mathbb{G}_m^{\mathrm{an}}$ totally splits over a point $\eta_{z_0,r}$ satisfying $r < |z_0| =: \alpha$ if and only if: $r < \alpha p^{-h-\frac{p}{p-1}}$. More precisely, the inverse image of $\eta_{z_0,r}$ contains:*

- *only one element when $r \in [\alpha p^{-\frac{p}{p-1}}, \alpha]$;*
- *p^i elements when $r \in [\alpha p^{-i-\frac{p}{p-1}}, \alpha p^{-i-\frac{1}{p-1}}[$, with $1 \leqslant i \leqslant h-1$;*
- *p^h elements when $r \in [0, \alpha p^{-h-\frac{1}{p-1}}[$.*

Proof. — Let $f : \mathbb{G}_m^{\mathrm{an}} \to \mathbb{G}_m^{\mathrm{an}}$ be the covering given by $f(z) = z^p$. Let $z_1 \in k^\times$ and $\rho \in \mathbb{R}_{\geqslant 0}$ satisfying $\rho < |z_1|$ (such that $\eta_{z_1,\rho} \notin]0,\infty[$). In order to compute $f(\eta_{z_1,\rho})$, notice that for any polynomial $P \in k[T]$:

$$\begin{aligned}|P\left(f(\eta_{z_1,\rho})\right)| &= |(P\circ f)(\eta_{z_1,\rho})| = |P(T^p)(\eta_{z_1,\rho})| \\ &= \sup_{x\in \mathrm{B}(z_1,\rho)} |P\circ f(x)| \\ &= \sup_{y \in f(\mathrm{B}(z_1,\rho))} |P(y)|.\end{aligned}$$

As k is algebraically closed, there exists $\widehat{\rho} > 0$, such that $f(\mathrm{B}(z_1,\rho)) = \mathrm{B}(z_1^p, \widehat{\rho})$, which gives $f(\eta_{z_1,\rho}) = \eta_{z_1^p,\widehat{\rho}}$.

In order to compute $\widehat{\rho}$, notice that $\widehat{\rho} = |(T-z_1^p)(f(\eta_{z_1,\rho}))| = |(T^p - z_1^p)(\eta_{z_1,\rho})|$, and:

$$T^p - z_1^p = \sum_{i=1}^{p} \binom{p}{i} z_1^{p-i}(T-z_1)^i = \sum_{i=1}^{p} \gamma_i (T-z_1)^i,$$

where $\gamma_i = \binom{p}{i} z_1^{p-i}$, with:

$$|\gamma_i| = \begin{cases} 1 & \text{if } i = p \\ p^{-1}|z_1|^{p-i} & \text{if } 1 \leqslant i \leqslant p-1 \end{cases}.$$

Therefore,

$$\widehat{\rho} = |(T - z_1^p)(f(\eta_{z_1,\rho}))| = \max_{1\leqslant i\leqslant p}\{|\gamma_i|\rho^i\} = \max\{\rho^p, (p^{-1}\rho^i|z_1|^{p-i})_{1\leqslant i\leqslant p-1}\}.$$

Since we assumed $\rho < |z_1|$, we get $\widehat{\rho} = \max\{\rho^p, p^{-1}\rho|z_1|^{p-1}\}$, that is to say:

$$\widehat{\rho} = \begin{cases} p^{-1}\rho|z_1|^{p-1} & \text{if } \rho \leqslant |z_1|p^{-\frac{1}{p-1}} \\ \rho^p & \text{if } \rho \geqslant |z_1|p^{-\frac{1}{p-1}} \end{cases}.$$

Consequently:

$$f(\eta_{z_1,\rho}) = \begin{cases} \eta_{z_1^p, p^{-1}\rho|z_1|^{p-1}} & \text{if } \rho \leqslant |z_1|p^{-\frac{1}{p-1}} \\ \eta_{z_1^p,\rho^p} & \text{if } \rho \geqslant |z_1|p^{-\frac{1}{p-1}} \end{cases}.$$

Let us try to find the preimages by f of $\eta_{z_0,r}$, where $0 \leqslant r < \alpha := |z_0|$. Define:

$$\widetilde{r} = \begin{cases} rp\alpha^{-\frac{p-1}{p}} & \text{if } r \leqslant \alpha p^{-\frac{p}{p-1}} \\ r^{\frac{1}{p}} & \text{if } r \geqslant \alpha p^{-\frac{p}{p-1}} \end{cases}.$$

From the above, if $\widetilde{z_0}$ is a p^{th}-root of z_0, then:

$$\eta_{\widetilde{z_0},\widetilde{r}} \in f^{-1}\left(\{\eta_{z_0,r}\}\right),$$

and $f^{-1}\left(\{\eta_{z_0,r}\}\right)$ consists of all conjugates $\eta_{\xi\widetilde{z_0},\widetilde{r}}$ of $\eta_{\widetilde{z_0},\widetilde{r}}$, for $\xi \in \mu_p$. Therefore:

$$f^{-1}\left(\{\eta_{z_0,r}\}\right) = \begin{cases} \{\eta_{\xi\widetilde{z_0}, rp\alpha^{-\frac{p-1}{p}}}\}_{\xi\in\mu_p} & \text{if } r \leqslant \alpha p^{-\frac{p}{p-1}} \\ \{\eta_{\xi\widetilde{z_0}, r^{\frac{1}{p}}}\}_{\xi\in\mu_p} & \text{if } r \geqslant \alpha p^{-\frac{p}{p-1}} \end{cases}.$$

Since $|\widetilde{z_0}| = \alpha^{\frac{1}{p}}$, we have $|\xi\widetilde{z_0} - \xi'\widetilde{z_0}| = \alpha^{\frac{1}{p}}p^{-\frac{1}{p-1}}$ as soon as $\xi \neq \xi' \in \mu_p$, from Lemma 3.1. Thus, $f^{-1}\left(\{\eta_{z_0,r}\}\right)$ has a unique element if $r \geqslant \alpha p^{-\frac{p}{p-1}}$, p otherwise.

For the general case, with $h \geqslant 1$, a recursive reasoning on h leads to the conclusion. □

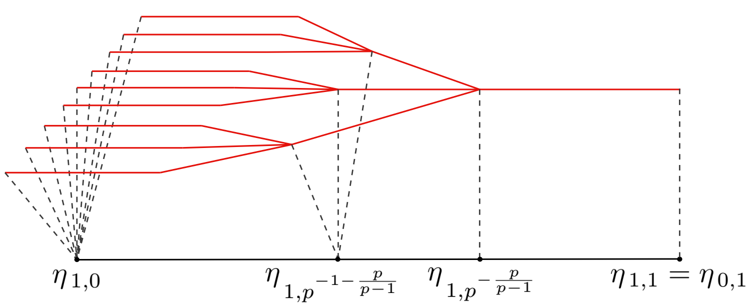


FIGURE 3.1. Covering $\mathbb{G}_m^{\text{an}} \xrightarrow{z\mapsto z^9} \mathbb{G}_m^{\text{an}}$ with $p = 3$, $h = 2$ and $z_0 = 1$.

3.2. Cochain morphism. — We shall define the important notion of $\mathbb{Z}/n\mathbb{Z}$-*cochain* associated to a μ_n-torsor. This is exactly from a close look at the behaviours of such cochains of torsors that it will be possible, in Section 5, to extract some information about lengths of annuli.

Definition 3.3 (Harmonic cochains). — Let Γ be a locally finite graph and A an abelian group. A *harmonic A-cochain on* Γ is map c : {oriented edges of Γ} $\to A$ satisfying the following two conditions:

1. if e and e' correspond to the same edge with its two different orientations: $c(e') = -c(e)$;
2. if x is a vertex of Γ:

$$\sum_{\text{edges oriented towards } x} c(e) = 0_A.$$

The set of harmonic A-cochains of Γ forms an abelian group denoted $\mathrm{Harm}(\Gamma, A)$. In the following, we will simply write A-cochains or cochains when A is explicit.

Let X be a non-empty k-analytic curve with skeleton $\mathbb{G} = S^{\mathrm{an}}(X)$ and truncated skeleton $\mathbb{G}^{\natural} = S^{\mathrm{an}}(X)^{\natural}$.

Lemma 3.4. — *Let $n \in \mathbb{N}^{\times}$.*

- *There exists a natural morphism:*

$$H^1(X, \mu_n) \xrightarrow{\theta} \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/n\mathbb{Z}).$$

- *In the case where X has a finite skeleton, does not have any point of genus > 0, is without boundary and has only annular cusps, the image of θ contains* $\mathrm{Harm}(\mathbb{G}^{\natural}, \mathbb{Z}/n\mathbb{Z})$ *(seen as a subgroup of* $\mathrm{Harm}(\mathbb{G}, \mathbb{Z}/n\mathbb{Z})$ *by prolongation of cochains by* 0 *on all cuspidal edges of* $\mathbb{G}$*).*

Proof. — The exact Kummer sequence gives the following exact sequence:

$$1 \to \mathscr{O}_X(X)^{\times} / \left(\mathscr{O}_X(X)^{\times}\right)^n \to H^1(X, \mu_n) \to {}_nH^1(X, \mathbb{G}_m) \to 1,$$

where ${}_nH^1(X, \mathbb{G}_m)$ denotes the n-torsion subgroup of $H^1(X, \mathbb{G}_m)$.

Moreover, $H^1(X, \mathbb{G}_m) = H^1_{\mathrm{top}}(X, \mathbb{G}_m)$; any étale $\mathbb{G}_m$-torsor is topological, this comes from [3] (4.1.10).

Let $h \in H^1(X, \mu_n)$ and $\overline{h}$ be its image in ${}_nH^1(X, \mathbb{G}_m)$. Thus, if $x \in X$, there exists an open neighbourhood $\mathscr{U}$ of x in X, such that $\overline{h}$ is trivial on $\mathscr{U}$. Then $h_{|\mathscr{U}}$ comes from a function $f \in \mathscr{O}_{\mathscr{U}}(\mathscr{U})^{\times}$ defined modulo n-th powers. There is a natural morphism $\mathscr{O}_{\mathscr{U}}(\mathscr{U})^{\times} \xrightarrow{\theta_{\mathscr{U}}} \mathrm{Harm}(S^{\mathrm{an}}(\mathscr{U}), \mathbb{Z})$, which factorises through:

$$\mathscr{O}_{\mathscr{U}}(\mathscr{U})^{\times} / \left(\mathscr{O}_{\mathscr{U}}(\mathscr{U})^{\times}\right)^n \to \mathrm{Harm}\left(S^{\mathrm{an}}(\mathscr{U}), \mathbb{Z}/n\mathbb{Z}\right).$$

This morphism $\theta_{\mathscr{U}}$ can be constructed in the following way: if e is an oriented edge of $S^{\mathrm{an}}(\mathscr{U})$ and r is the canonical retraction of $\mathscr{U}$ on its skeleton, then $r^{-1}(e)$ is isomorphic to some open annulus of $\mathbb{P}_k^{1,\mathrm{an}}$ defined by the condition $\{1 < |T| < \lambda\}$, where the beginning of the edge corresponds to 1, whereas the end corresponds to λ.

Let $\tau : \{z \in \mathbb{P}_k^{1,\mathrm{an}}, 1 < |T(z)| < \lambda\} \xrightarrow{\sim} r^{-1}(e)$ be such an isomorphism and $\alpha \in \mathscr{O}_{\mathscr{U}}(\mathscr{U})^{\times}$. There exists a unique $m \in \mathbb{Z}$, such that $\alpha \circ \tau$ is written $z \mapsto z^m g(z)$ with g of constant norm. This comes from the characterisation of invertibility of analytic functions on an annulus, and m is the degree of the unique strictly dominant monomial of $\alpha \circ \tau$. It is enough to write $\theta_{\mathscr{U}}(\alpha)(e) = m$; this defines an element of $\mathrm{Harm}(S^{\mathrm{an}}(\mathscr{U}), \mathbb{Z})$.

We have $S^{\mathrm{an}}(X) \cap \mathscr{U} \subseteq S^{\mathrm{an}}(\mathscr{U})$, but the inclusion can be *a priori* strict. However, we shall show that the support of $\theta_{\mathscr{U}}(f)$ (i.e. the set of edges e of $S^{\mathrm{an}}(\mathscr{U})$, such that $\theta_{\mathscr{U}}(f)(e) \neq 0$) is included in $S^{\mathrm{an}}(X) \cap \mathscr{U}$. Let e be an oriented edge of $S^{\mathrm{an}}(\mathscr{U})$ not included in $S^{\mathrm{an}}(X)$. If $y \in e$, y belongs to an open disk $\mathcal{D}$ of X. Then there exists a closed disk $\mathcal{D}_0 \subsetneq \mathcal{D}$ containing y in its interior. As $\mathcal{D}_0$ is a closed disk, its Picard group $\mathrm{Pic}\,(\mathcal{D}_0)$ is trivial. Therefore, the μ_n-torsor $h_{|\mathcal{D}_0}$ is given by a function $f_0 \in \mathscr{O}_{\mathcal{D}_0}(\mathcal{D}_0)^{\times}$. Moreover, any invertible function on a closed disk has a constant norm, and hence the cochain associated to f_0 at a neighbourhood of y is trivial. In particular, $\theta_{\mathscr{U} \cap \mathcal{D}_0}(f_0)$ is the trivial cochain on $S^{\mathrm{an}}(\mathscr{U} \cap \mathcal{D}_0)$. Moreover, all these local constructions are compatible between each other: $\theta_{\mathscr{U} \cap \mathcal{D}_0}(f_0) = \theta_{\mathscr{U} \cap \mathcal{D}_0}(f)$. Thus $\theta_{\mathscr{U}}(f)(e) = 0$, so the support of $\theta_{\mathscr{U}}(f)$ is included in $S^{\mathrm{an}}(X) \cap \mathscr{U}$.

These local constructions $x \mapsto \theta_{\mathscr{U}}(f)$ can be glued together to finally give a morphism: $H^1(X, \mu_n) \to \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/n\mathbb{Z})$.

For the second point, let us first explain how to embed X in the analytification of a Mumford k-curve. Let X' be a proper k-analytic curve obtained from X by prolongation of each cusp by a disk. Then X' is the analytification $\mathscr{X}^{\mathrm{an}}$ of a Mumford k-curve $\mathscr{X}$. Moreover $\mathbb{G}^{\natural} = S^{\mathrm{an}}(X')$; the annular cusps of X no longer appear in the skeleton of X' since they are prolonged by disks.

As $\mathbb{G}^{\natural} = S^{\mathrm{an}}(X')$, from [10] we have a morphism $\overline{\theta} : H^1(X', \mu_n) \to \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/n\mathbb{Z})$, whose image exactly equals $\mathrm{Harm}(\mathbb{G}^{\natural}, \mathbb{Z}/n\mathbb{Z})$. If ι denotes the embedding of X in X', there is a commutative diagram:

$$\begin{array}{ccc} H^1(X', \mu_n) & \xrightarrow{\iota^*} & H^1(X, \mu_n)\ , \\ & {\scriptstyle\overline{\theta}} \searrow \quad \swarrow {\scriptstyle\theta} & \\ & \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/n\mathbb{Z}) & \end{array}$$

which is enough to conclude that $\mathrm{Harm}(\mathbb{G}^{\natural}, \mathbb{Z}/n\mathbb{Z}) \subseteq \mathrm{im}(\theta)$. □

Remark 3.5. — As the morphism $H^1(X,\mu_n) \xrightarrow{\theta} \mathrm{Harm}(\mathbb{G},\mathbb{Z}/n\mathbb{Z})$ exists for any n, we will now consider θ as a map:

$$\theta : \bigsqcup_{n\in\mathbb{N}^\times} H^1(X,\mu_n) \to \bigsqcup_{n\in\mathbb{N}^\times} \mathrm{Harm}(\mathbb{G},\mathbb{Z}/n\mathbb{Z}),$$

which induces for each $n \in \mathbb{N}^\times$ a morphism $H^1(X,\mu_n) \to \mathrm{Harm}(\mathbb{G},\mathbb{Z}/n\mathbb{Z})$.

3.3. Cochains and minimality of the splitting radius on an annulus. — Let $\mathcal{C}$ be a k-analytic annulus of finite length, $\alpha \in k$ and $\eta_{\alpha,0} = \eta_\alpha$ (sometimes simply denoted α) the associated rigid point of $\mathbb{P}^{1,\mathrm{an}}_k$. We shall show that μ_p-torsors with non-trivial cochains modulo p on $\mathcal{C}$ satisfy a minimality condition that enables us to distinguish them from trivial cochain torsors.

Definition 3.6. — If X is a k-analytic curve and $f \in H^1(X,\mu_n)$, let $\mathscr{D}(f)$ denote the set of points of X over which the analytic torsor defined by f totally splits.

Definition 3.7 (Splitting radius of a torsor on a rigid point). — Assume $\mathcal{C}$ is the subannulus of $\mathbb{P}^{1,\mathrm{an}}_k$ defined by $|T| \in I$, where I is an interval of $\mathbb{R}_{>0}$. If $\eta_\alpha \in \mathcal{C}$, for any torsor $f \in H^1(\mathcal{C},\mu_p)$, let $\varrho_f(\alpha)$ be the *splitting radius of f in* α, defined by:

$$\varrho_f(\alpha) = \sup\left\{r \in]0,|\alpha|[, \eta_{\alpha,r} \in \mathscr{D}(f)\right\}.$$

The following proposition shows how one can detect the triviality of the $\mathbb{Z}/p\mathbb{Z}$-cochain $\theta(f)$ with this notion.

Proposition 3.8. — *Fix the rigid point $\eta_\alpha \in \mathcal{C}$. Then $\varrho_f(\alpha)$ is minimal exactly when the cochain of $f \in H^1(\mathcal{C},\mu_p)$ is non-trivial modulo p, i.e. when $f \notin \ker(\theta)$.*

More precisely:

- *$f \notin \ker(\theta)$ if and only if $\varrho_f(\alpha) = |\alpha|\, p^{-\frac{p}{p-1}}$;*
- *$f \in \ker(\theta)$ if and only if $\varrho_f(\alpha) > |\alpha|\, p^{-\frac{p}{p-1}}$.*

Proof. — The exact Kummer sequence gives a morphism

$$\mathscr{O}_{\mathcal{C}}(\mathcal{C})^\times/(\mathscr{O}_{\mathcal{C}}(\mathcal{C})^\times)^p \hookrightarrow H^1(\mathcal{C},\mu_p),$$

which becomes an isomorphism when one restricts it to any compact subannulus because of the triviality of the Picard group of any k-affinoid subspace. Up to restricting $\mathcal{C}$, one can assume f is given by a function $g \in \mathscr{O}_{\mathcal{C}}(\mathcal{C})^\times$, which means that the associated analytic torsor is defined by $\mathscr{O}_{\mathcal{C}}(\mathcal{C})[S]/(S^p - g)$.

Studying the splitting radius of f along the interval $[\eta_\alpha, \eta_{\alpha,|\alpha|}]$ amounts to making a change of coordinate function $t := T - \alpha$ and studying the convergence of $\sqrt[p]{g(t+\alpha)}$.

- Assume $f \notin \ker(\theta)$. There exists $n \in \mathbb{Z} \setminus \{0\}$, prime to p, such that g has growth rate n, i.e. n is the degree of the strictly dominant monomial: $g(T) = a_n T^n (1 + u(T))$, with $|u| < 1$ on $\mathcal{C}$. After normalisation (k is algebraically closed), one can assume $a_n = 1$.

  The series defining $\sqrt[n]{1+T}$ has a convergence radius equal to 1 since n is prime to p. Therefore, there exists a function $v(T)$ of norm < 1 on $\mathcal{C}$, such that $(1+v)^n = 1 + u$, so $g(T) = (T(1+v))^n$. As $T(1+v)$ is a coordinate function, we can assume $g(T) = T^n$. Since n is prime to p, the two μ_p-torsors given by functions T^n et T have the same sets of total splitting, so we can assume $g(T) = T$. Then the result is given by Proposition 3.2.
- Assume $f \in \ker(\theta)$. This means that the degree of the strictly dominant monomial of g (the growth rate) is 0 modulo p; there exists $m \in \mathbb{Z}$, such that $g(T) = a_0 T^{mp}(1 + u(T))$, with $|u| < 1$ on $\mathcal{C}$.

  Up to division by T^{mp} (it is the class of g modulo $(\mathscr{O}(\mathcal{C})^\times)^p$ that determines the torsor f), we can take $m = 0$. Let us write

$$g(T) = \sum_{k \in \mathbb{Z}} a_k T^k.$$

Thus, for all $r \in I$ and $k \in \mathbb{Z} \setminus \{0\}$, $|a_k| r^k < |a_0|$. Up to normalisation and restriction to a subannulus, we can assume that $a_0 = 1$ and that the extremities of the interval I (open or closed), are $1 - \varepsilon$ and 1, for some $\varepsilon \in]0,1[$. In this case, for all $k \in \mathbb{N}^\times$, we have $|a_k| < 1$ and $|a_{-k}| < (1-\varepsilon)^k$. For all $i \geqslant 0$ and $k \in \mathbb{Z}$, let us write $\binom{k}{i} = \frac{k(k-1)\dots(k-i+1)}{i!}$. Using the generalised binomial expansion, we write:

$$\begin{aligned} g(t+\alpha) &= \sum_{k \in \mathbb{Z}} a_k (t+\alpha)^k \\ &= \sum_{k \in \mathbb{Z}} a_k \left(\sum_{i \geqslant 0} \binom{k}{i} \alpha^{k-i} t^i \right) \\ &= \sum_{i \geqslant 0} \underbrace{\left(\sum_{k \in \mathbb{Z}} a_k \binom{k}{i} \alpha^{k-i} \right)}_{A_i} t^i = \sum_{i \geqslant 0} A_i t^i. \end{aligned}$$

We have $|\alpha| \leqslant 1$ since $\eta_\alpha \in \mathcal{C}$, which implies $|A_0| = |a_0| = 1$. Writing $v(t) = \sum_{i \geqslant 1} A_i t^i$, Proposition 3.2 states that the torsor f splits totally on $\eta_{\alpha,r}$ as soon as $|v(\eta_{0,r})| < p^{-\frac{p}{p-1}}$. Consequently, $\varrho_f(\alpha) \geqslant r$ if $|A_i| r^i < p^{-\frac{p}{p-1}}$, for all $i \geqslant 1$. Therefore:

$$\varrho_f(\alpha) \geqslant \inf_{i \geqslant 1} \left\{ \sqrt[i]{|A_i|^{-1}} p^{-\frac{1}{i}\left(\frac{p}{p-1}\right)} \right\}.$$

Moreover, for all $k \in \mathbb{Z} \setminus \{0\}$, $|a_k \alpha^k| < 1$. Then, all $i \geqslant 1$ satisfy $|A_i| < |\alpha|^{-i}$, so:

$$\sqrt[i]{|A_i|^{-1}} p^{-\frac{1}{i}\left(\frac{p}{p-1}\right)} > |\alpha|\, p^{-\frac{1}{i}\left(\frac{p}{p-1}\right)}.$$

We deduce:

$$\varrho_f(\alpha) \geqslant \min\left\{|A_1|^{-1} p^{-\frac{p}{p-1}}, |\alpha|\, p^{-\frac{1}{2}\left(\frac{p}{p-1}\right)}\right\} > |\alpha|\, p^{-\frac{p}{p-1}}. \qquad \square$$

Remark 3.9. — If $h > 1$, it is no longer true that $\varrho_f(\alpha)$ is minimal if and only if $f \in H^1(\mathcal{C}, \mu_{p^h})$ has a non-trivial $\mathbb{Z}/p^h\mathbb{Z}$-cochain, i.e. when $f \notin \ker(\theta)$. It is not difficult to show that if f has a cochain *prime to* p, then:

$$\varrho_f(\alpha) = |\alpha|\, p^{-h-\frac{1}{p-1}}.$$

However, if $f' \in H^1(\mathcal{C}, \mu_{p^h})$ is the element corresponding to the function $T^p \in \mathscr{O}_{\mathcal{C}}(\mathcal{C})^\times$, then its $\mathbb{Z}/p^h\mathbb{Z}$-cochain $\theta(f')$ is non-trivial since it equals p, but one can show that its splitting radius on α satisfies:

$$\varrho_{f'}(\alpha) \geqslant |\alpha|\, p^{1-h-\frac{1}{p-1}} = p\varrho_f(\alpha) > \varrho_f(\alpha),$$

implying that $\varrho_{f'}(\alpha)$ is not minimal even though $f' \notin \ker(\theta)$. Moreover, if the annulus $\mathcal{C}$ is, for instance, given by the condition $|T| \in]1-\varepsilon, 1[$ with $\varepsilon > 0$, the torsor $g \in H^1(\mathcal{C}, \mu_{p^h})$ given by the function $1 + T \in \mathscr{O}_{\mathcal{C}}(\mathcal{C})^\times$ has a trivial cochain, so it belongs to $\ker(\theta)$, but its splitting radius on a rigid point $\alpha = \eta_\alpha$ is:

$$\varrho_g(\alpha) = p^{-h-\frac{1}{p-1}}.$$

Consequently, as soon as $|\alpha| \in]\frac{1}{p}, 1[$, we have $\varrho_g(\alpha) < \varrho_{f'}(\alpha)$.

Corollary 3.10. — *If $f \in H^1(\mathcal{C}, \mu_p)$, the triviality of the cochain corresponding to f can be detected set-theoretically from the splitting sets of the different μ_p-torsors on $\mathcal{C}$:*

- $f \notin \ker(\theta) \iff \mathscr{D}(f)_{[2]} \subseteq \bigcap_{f' \in H^1(\mathcal{C}, \mu_p)} \mathscr{D}(f')_{[2]}$,
- $f \in \ker(\theta) \iff \exists f' \in H^1(\mathcal{C}, \mu_p), \mathscr{D}(f)_{[2]} \nsubseteq \mathscr{D}(f')_{[2]}$
  $\iff \forall f' \in H^1(\mathcal{C}, \mu_p) \setminus \ker(\theta), \mathscr{D}(f)_{[2]} \nsubseteq \mathscr{D}(f')_{[2]}$.

Proof. — It is a direct consequence of 3.8 coupled with the density of $X_{[2]}$ in X. $\square$

3.4. Characterisation of μ_p-torsors with trivial cochain. — So far, this study, which gives a set-theoretic characterisation of μ_p-torsors with trivial $\mathbb{Z}/p\mathbb{Z}$-cochain, has only dealt with k-analytic annuli. In order to extend these considerations, we will need a definition and a few restrictions.

Definition 3.11. — An edge e of a graph $\mathbb{H}$ is a *bridge* if an only if the map $\pi_0(\mathbb{H}\setminus\{e\})\to\pi_0(\mathbb{H})$ is not injective, which happens when the edge e "separates" several connected components of $\mathbb{H}\setminus\{e\}$. The graph is said to be *without a bridge* when none of its edges is a bridge.

Proposition 3.12. — *Let X be a curve as considered in the second part of Lemma 3.4, without boundary or points of genus >0, of finite skeleton, with only annular cusps. Assume, moreover, that $\mathbb{G}=S^{\mathrm{an}}(X)$ is without a bridge, and that there is never strictly more than one cusp coming from each node.*

If $f\in H^1(X,\mu_p)$, then $f\in\ker(\theta)$ if and only if, for any vicinal edge e of $S^{\mathrm{an}}(X)$ of associated annulus $\mathcal{C}_e$, there exists $f_e\in H^1(X,\mu_p)$, such that:

$$\big(\mathscr{D}(f)_{[2]}\setminus\mathscr{D}(f_e)_{[2]}\big)\cap\mathcal{C}_{e\,[2]}\neq\emptyset.$$

Proof. — The assumption that there is never strictly more than one cusp coming from a node implies that a cochain $c\in\mathrm{Harm}(\mathbb{G},\mathbb{Z}/p\mathbb{Z})$ is trivial if and only if it is trivial on all vicinal edges of $\mathbb{G}$.

- Assume that for any vicinal edge e of $\mathbb{G}$ of corresponding annulus $\mathcal{C}_e$, there exists $f_e\in H^1(X,\mu_p)$, such that: $\big(\mathscr{D}(f)_{[2]}\setminus\mathscr{D}(f_e)_{[2]}\big)\cap\mathcal{C}_{e\,[2]}\neq\emptyset$. Let $f_{e\,|\,\mathcal{C}_e}$ and $f_{|\,\mathcal{C}_e}$ be the restrictions of f_e and f to $\mathcal{C}_e$. Then we have $\mathscr{D}(f_{|\,\mathcal{C}_e})_{[2]}\nsubseteq\mathscr{D}(f_{e\,|\,\mathcal{C}_e})_{[2]}$. With Corollary 3.10, this implies $\theta(f)(e)=0$. It is true for any vicinal edge e, so $\theta(f)$ is the trivial cochain.
- Let $f\in H^1(X,\mu_p)$, e a vicinal edge of annulus $\mathcal{C}_e$, and assume $f\in\ker(\theta)$. From 3.8, as $\theta(f)(e)=0$, we have $\mathscr{D}(f_{|\,\mathcal{C}_e})_{[2]}\nsubseteq\mathscr{D}(g_e)_{[2]}$, for all $g_e\in H^1(\mathcal{C}_e,\mu_p)$ of a non-trivial cochain.

  It remains to show that there exists $f_e\in H^1(X,\mu_p)$ with a non-trivial cochain at e.

  From the assumption, the edge e is not a bridge of $\mathbb{G}$, so it is also not a bridge of $\mathbb{G}^\natural$. Thus, the evaluation at e:

  $$\mathrm{Harm}(\mathbb{G}^\natural,\mathbb{Z}/p\mathbb{Z})\xrightarrow{\mathrm{ev_e}}\mathbb{Z}/p\mathbb{Z}$$

  is non-zero. Let us choose $c_e\in\mathrm{ev}_e^{-1}\left(\mathbb{Z}/p\mathbb{Z}\setminus\{0\}\right)$, i.e. such that $c_e(e)\neq 0$. From Lemma 3.4, the image of θ contains $\mathrm{Harm}(\mathbb{G}^\natural,\mathbb{Z}/p\mathbb{Z})$. It is enough to take $f_e\in\theta^{-1}(\{c_e\})$; we have $\mathscr{D}(f_{|\,\mathcal{C}_e})_{[2]}\nsubseteq\mathscr{D}(f_{e\,|\,\mathcal{C}_e})_{[2]}$, which can be written as:

  $$\big(\mathscr{D}(f)_{[2]}\setminus\mathscr{D}(f_e)_{[2]}\big)\cap\mathcal{C}_{e\,[2]}\neq\emptyset.\qquad\square$$

4. Resolution of non-singularities

In algebraic geometry, resolution of non-singularities consists in knowing whether a hyperbolic curve $\mathscr{X}$ admits a finite cover $\mathscr{Y}$, whose stable reduction has some irreducible components above the smooth locus of the stable (or semi-stable) reduction of $\mathscr{X}$. Such techniques happen to be useful in anabelian geometry; see, for instance, [15]. If X_0 is a geometrically connected hyperbolic curve over a finite extension K of $\mathbb{Q}_p$, such that $X_{0,\overline{\mathbb{Q}}_p}$ satisfies such resolution of non-singularities, then any section of $\pi_1^{\mathrm{alg}}(X_0) \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ has its image in a decomposition group of a unique valuation point.

In [11], Lepage shows that any Mumford curve over $\overline{\mathbb{Q}}_p$ satisfies a resolution of non-singularities and applies this result to the anabelian study of the tempered group of such curves. He shows, for instance, that if X_1 and X_2 are two Mumford curves over $\overline{\mathbb{Q}}_p$, whose analytifications have isomorphic tempered fundamental groups, then X_1^{an} are X_2^{an} are naturally homeomorphic ([11], Theorem 3.9).

4.1. Definition and properties of solvable points. — Within the framework of this article, we shall give an *ad hoc* definition of a *solvable point* and *resolution of non-singularities* in order to stay in the language of analytic geometry without entering in the considerations of a (semi-)stable model.

Definition 4.1 (Solvable point). — Let X be a k-analytic quasi-smooth curve. We will say that a point $x \in X$ *satisfies the resolution of non-singularities*, equivalently *is solvable*, when there exists a finite étale covering Y of X and a node y of $S^{\mathrm{an}}(Y)$ above x. This amounts to "singularising" x to some node of some finite étale covering of X, whence the terminology. The set of *solvable points* is denoted X_{res}.

Remark 4.2. — One always has $X_{\mathrm{res}} \subseteq X_{[2]}$. We will say that X *satisfies resolution of non-singularities* when $X_{\mathrm{res}} = X_{[2]}$. In [11] (Theorem 2.6), Lepage shows that the analytification of any Mumford curve over $\overline{\mathbb{Q}}_p$ satisfies the resolution of non-singularities.

Definition 4.3. — If $f \in H^1(X, \mu_n)$, define $\mathscr{D}(f)_{\mathrm{res}} := \mathscr{D}(f) \cap X_{\mathrm{res}}$ as the set of solvable points of X over which the analytic torsor defined by f totally splits.

Resolution of non-singularities has a specific anabelian flavour; from the tempered group $\pi_1^{\mathrm{temp}}(X)$, it is possible to determine the set of solvable points, as well as the set of solvable points belonging to an annulus defined by a vicinal edge, to the skeleton itself or to the splitting sets of analytic torsors on X.

Properties. If X is a k-analytically hyperbolic curve, the tempered fundamental group $\pi_1^{\mathrm{temp}}(X)$ enables us to determine:

- *the set X_{res} of solvable points of X;*
- *if e is a vicinal edge of annulus $\mathcal{C}_e$, the set $\mathcal{C}_e \cap X_{\mathrm{res}}$;*
- *the set $S^{\mathrm{an}}(X)_{\mathrm{res}} := S^{\mathrm{an}}(X) \cap X_{\mathrm{res}}$ of solvable points belonging to the skeleton;*
- *if f is a μ_n-torsor of X, the set $\mathscr{D}(f)_{\mathrm{res}}$.*

More precisely, we have the following.

1. The decomposition groups D_x of solvable points of X in $\pi_1^{\mathrm{temp}}(X)$ correspond exactly to the maximal compact subgroups D of $\pi_1^{\mathrm{temp}}(X)$, such that there exists a open finite index subgroup H of $\pi_1^{\mathrm{temp}}(X)$, such that the image of $D \cap H$ by the natural morphism $H \to H^{(p')}$ is non-commutative.
2. Let e be a vicinal edge of $S^{\mathrm{an}}(X)$ and $\mathcal{C}_e$ the associated annulus. If D_x is a decomposition group of a point $x \in X_{\mathrm{res}}$, then $x \in \mathcal{C}_e$ if and only if the image $D_x^{(p')}$ of D_x by the morphism $\pi_1^{\mathrm{temp}}(X) \to \pi_1^{\mathrm{temp},\,(p')}(X)$ is open in some vicinal subgroup π_e associated to e.
3. Let $x \in X_{\mathrm{res}}$ be a solvable point. Let $D_x^{(p')}$ be a decomposition group of x in $\pi_1^{\mathrm{temp},\,(p')}(X)$, and Y a finite étale covering, such that there exists a node y of $S^{\mathrm{an}}(Y)$ above x, which amounts to considering an open subgroup H of $\pi_1^{\mathrm{temp}}(X)$ of finite index, such that $\pi_y = D_x^{(p')} \cap H^{(p')}$ is non-commutative. Let ι be the morphism $H^{(p')} \to \pi_1^{\mathrm{temp},\,(p')}$. Then $\iota(\pi_y)$ is an open subgroup of $D_x^{(p')}$. There are three possibilities.
   - *Case* 1: $x \notin S^{\mathrm{an}}(X)$, it is the case when $\iota(\pi_y)$ is trivial.
   - *Case* 2: x is a vertex of the skeleton; this is the case when $\iota(\pi_y)$ is not commutative. In this case, $D_x^{(p')} = \pi_x$ is the only verticial subgroup containing $\iota(\pi_y)$, and from Lemma 3.51 of [7], it is also the commensurator of $\iota(\pi_y)$ in $\pi_1^{\mathrm{temp},\,(p')}(X)$.
   - *Case* 3: x belongs to an edge e of $S^{\mathrm{an}}(X)$; this is the case when $\iota(\pi_y)$ is non-trivial and commutative. In this case, $D_x^{(p')} = \pi_e$, and π_e is the only vicinal or cuspidal subgroup (according to the nature of the edge e) containing $\iota(\pi_y)$; it also equals the commensurator of $\iota(\pi_y)$ in $\pi_1^{\mathrm{temp},\,(p')}(X)$.
4. Let $f \in H^1(X, \mu_n)$ and D_x be a decomposition group in $\pi_1^{\mathrm{temp}}(X)$ of a point $x \in X_{\mathrm{res}}$. Then the knowledge of $\pi_1^{\mathrm{temp}}(X)$, of f (considered as a morphism from $\pi_1^{\mathrm{temp}}(X)$ to $\mathbb{Z}/n\mathbb{Z}$) and of D_x is enough to determine whether $x \in \mathscr{D}(f)$.

The point (1), which appears in [11], is a consequence of ([12], Prop. 10): if D is a compact subgroup of $\pi_1^{\mathrm{temp}}(X)$, there exists $x \in X$ and a decomposition subgroup D_x of x in $\pi_1^{\mathrm{temp}}(X)$, such that $D \subseteq D_x$. Therefore, decomposition subgroups in $\pi_1^{\mathrm{temp}}(X)$ of points of X are exactly the maximal compact subgroups of $\pi_1^{\mathrm{temp}}(X)$. The image $D_x^{(p')}$ of D_x by the morphism $\pi_1^{\mathrm{temp}}(X) \to \pi_1^{\mathrm{temp},\,(p')}(X)$ is trivial if x does not belong to $S^{\mathrm{an}}(X)$, non-trivial and commutative (in fact, isomorphic to $\widehat{\mathbb{Z}}^{(p')})$) if x belongs to an edge of $S^{\mathrm{an}}(X)$, and non-commutative if x is a vertex of $S^{\mathrm{an}}(X)$.

The point (2) comes from the following fact: if $Y \xrightarrow{f} X$ is a finite étale Galois covering of group G with a node y of $S^{\mathrm{an}}(Y)$ resolving $x \in X_{\mathrm{rés}}$, there exists a canonical retraction $S^{\mathrm{an}}(Y)/G \twoheadrightarrow f^{-1}(S^{\mathrm{an}}(X))/G \simeq S^{\mathrm{an}}(X)$, and the subgraph $f^{-1}(S^{\mathrm{an}}(X)) \subseteq S^{\mathrm{an}}(Y)$ is such that $f^{-1}(S^{\mathrm{an}}(X)) \cap Y_{\mathrm{res}}$ is determined by the data of $\pi_1^{\mathrm{temp}}(X)$ and of an open subgroup of finite index $H \subseteq \pi_1^{\mathrm{temp}}(X)$ defining the covering f.

The point (3) can be interpreted as a consequence of Lemmas 3.6 and 3.8 of [11].

For the point (4), one needs to bear in mind the following fact: if $Y \xrightarrow{f} X$ is a finite étale Galois covering given by an open subgroup $H \subseteq \pi_1^{\mathrm{temp}}(X)$, the data of the morphism $\iota : H^{(p')} \to \pi_1^{\mathrm{temp},\,(p')}(X)$ enables us to know the preimage by f of a fixed node $x \in S^{\mathrm{an}}(X)$. In particular, when f is a μ_n-torsor, the data of ι enables us to know whether f totally splits over x (cf. [10], prop. 7). Now, if $x \in X_{\mathrm{res}}$, if $Z \xrightarrow{g} X$ is a finite étale Galois covering of group G with a node $z \in S^{\mathrm{an}}(Z)$, which resolves x, and if $f \in H^1(X, \mu_n)$ corresponds to the analytic torsor $Y \to X$, the pull-back $Y \times_X Z \to Z$ inherits a natural action of $\mu_n \times G$. Triviality of f over x can be read on the action of $G \times \mu_n$ over $f^{-1}(x)$, i.e. over the G-orbit of z.

4.2. Resolution of non-singularities for annuli and more general curves. — We are going to show the resolution of non-singularities of annuli using some arguments inspired by [11].

Let $\mathcal{D} \subseteq \mathbb{P}_k^{1,\mathrm{an}}$ be a disk centred at 0. For $n \geqslant 1$, let $c_{\mathrm{can},n} \in H^1(\mathbb{G}_k^{\mathrm{an}}, \mu_{p^n})$ be the μ_{p^n}-torsor $\mathbb{G}_k^{\mathrm{an}} \to \mathbb{G}_k^{\mathrm{an}}$ given by the multiplication by p^n. Any invertible function $f \in \mathcal{O}_{\mathcal{D}}(\mathcal{D})^\times$ defines a morphism $f : \mathcal{D} \to \mathbb{G}_k^{\mathrm{an}}$. Let $c_n := f^* c_{\mathrm{can},n} \in H^1(\mathcal{D}, \mu_{p^n})$ be the pull-back of $c_{\mathrm{can},n}$ along f. Let $Y_n \to \mathcal{D}$ be the corresponding torsor. If $\alpha \in k$ defines a k-point of $\mathcal{D}$, the *ramification index of f at α* is given by: $e_\alpha(f) = \inf\{k \geqslant 1, a_k \neq 0\}$ with $f(T) = \sum_{k \geqslant 0} a_k (T-\alpha)^k$.

Let $r_0(c_n) = \inf\{r > 0, Y_n \text{ is not split at } \eta_{0,r_0(c_n)}\}$. When $r_0(c_n)$ exists, let $x_n = \eta_{0,r_0(c_n)}$ and $y_n \in Y_n$ be above x_n. Here is a result of Lepage about the Artin–Schreier nature of the extension $\widetilde{\mathscr{H}(x_n)} \to \widetilde{\mathscr{H}(y_n)}$ and the asymptotic behaviour of $r_0(c_n)$ when $n \to \infty$.

Lemma 4.4 ([11], 2.3). — *There exists $\kappa \in |k^\times|$, such that for n big enough:*

- $r_0(c_n) = \kappa p^{-\frac{n}{e_0(f)}}$;
- $[\widetilde{\mathscr{H}(y_n)} : \widetilde{\mathscr{H}(x_n)}] = [\mathscr{H}(y_n) : \mathscr{H}(x_n)] = p$;
- *$\widetilde{\mathscr{H}(y_n)}$ is isomorphic to the Artin–Schreier extension $\widetilde{k}(X)[T]/(T^p - T - X^{e_0(f)})$ after identification $\widetilde{\mathscr{H}(x_n)} \simeq \widetilde{k}(X)$.*

REMARK 4.5. — If $e_0(f) = p^m d$ with d prime to p, it is known that the genus of the Artin–Schreier curve given by $T^p - T = X^{e_0(f)}$ is $g = (d-1)(p-1)/2$ (see [8], §2.2, eq.(8)). In particular, when $e_0(f)$ is not a power of p, for n big enough, the residual curve $\mathscr{C}_{y_n}$ has genus $\geqslant 1$; in other words, $g(y_n) > 0$. It this situation, from the definition of the skeleton and its nodes, y_n is a node of $S^{\mathrm{an}}(Y_n)$, so $x_n \in \mathcal{D}_{\mathrm{res}}$.

PROPOSITION 4.6. — *Any k-analytic annulus $\mathcal{C}$ satisfies resolution of non-singularities $\mathcal{C}_{[2]} = \mathcal{C}_{\mathrm{res}}$.*

Proof. — Because of Corollary 2.17, one can assume without loss of generality that $\mathcal{C}$ is isomorphic to $\mathcal{C}_I \subseteq \mathbb{A}_k^{1,\mathrm{an}}$ (notation of Section 1.2) for some interval I, such that $\mathcal{C}_I \subseteq \mathcal{D}$, where $\mathcal{D}$ is the open unit disk centered at 0. In other words, if T is the standard parameter of $\mathbb{A}_k^{1,\mathrm{an}}$, one can assume that there exists a non-empty interval $I \subseteq (0,1)$, such that $\mathcal{C}$ is isomorphic to the analytic domain of $\mathbb{A}_k^{1,\mathrm{an}}$ defined by the condition $\{|T| \in I\}$.

Let $x \in \mathcal{C}_{[2]}$. There exist $\alpha \in k$ with $|\alpha| \in I$ and $r \in |k^\times|$, such that $x \in \eta_{\alpha,r}$.

Let $f \in \mathcal{O}_{\mathcal{D}}(\mathcal{D})^\times$, such that the ramification index $e_0(f)$ is not a power of p. One can, for instance, take $f(T) = 1 + T^N$ with $N \geqslant 1$ prime to p; this function is invertible on $\mathcal{D}$ by Proposition 2.12, and the ramification index is N by definition. Let n be big enough such that Lemma 4.4 (related to f) applies, and $\kappa p^{-\frac{n}{e_0(f)}} < r$. The coordinate function $\frac{\kappa}{r} p^{-\frac{n}{e_0(f)}}(T-\alpha)$ induces an isomorphism of annuli $\iota : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ with $C' \subseteq \mathcal{D}$ and $\iota(x) = \eta_{0,r_0(c_n)} = x_n$. If $j : \mathcal{C}' \hookrightarrow \mathcal{D}$ is the natural inclusion, consider $\iota^* j^* c_n \in H^1(\mathcal{C}, \mu_{p^n})$, ie. the pull-back along ι of the restriction to $\mathcal{C}'$ of the torsor $c_n \in H^1(\mathcal{D}, \mu_{p^n})$ associated to f. Let $Z_n \to \mathcal{C}$ be the corresponding analytic μ_{p^n}-torsor of $\mathcal{C}$. As $e_0(f)$ is not a power of p, from Remark 4.5, any point $z_n \in Z_n$ above x is a node of $S^{\mathrm{an}}(Z_n)$, so we get $x \in \mathcal{C}_{\mathrm{res}}$. ☐

Let X be a quasi-smooth k-analytic curve of finite skeleton, without boundary or points of genus > 0, with only annular cusps. We have seen that X can be considered as a non-empty open subset of the analytification X' of a Mumford k-curve. One cannot *a priori* conclude that X satisfies resolution of non-singularities directly using Theorem 2.6 of [11] when the curve is not defined over $\overline{\mathbb{Q}}_p$. However, the proof of Lepage can easily be adapted to give the following result.

PROPOSITION 4.7. — *Let $X \subset \mathscr{X}^{\mathrm{an}}$ be a k-analytic curve that is a non-empty, open subset of the analytification of a Mumford curve $\mathscr{X}$ minus a (non-empty) disk. Then X satisfies the resolution of non-singularities.*

Proof. — Since type-2 points are dense in X, X satisfying resolution of non-singularities would imply $\overline{X_{\mathrm{res}}} = \overline{X}$, where the overline stands for the closure in $\mathscr{X}^{\mathrm{an}}$. As explained in ([11], Proposition 2.1), the converse is true, and X satisfies the resolution of non-singularities if and only if $X(k) \subset \overline{X_{\mathrm{res}}}$.

Let $x \in X(k)$. The assumptions on X ensure that its universal topological covering Ω can be embedded in a disk $\mathcal{D}$, such that $\Omega \subset \mathcal{D} \subset \mathbb{P}_k^{1,\mathrm{an}}$. Let $z \in \Omega(k)$ be a point over x. Let $f \in \mathcal{O}_{\mathcal{D}}(\mathcal{D})^{\times}$, such that the ramification index $e_z(f)$ of f at z is not a power of p (such an invertible function always exists). Let $c_n = f^* c_{\mathrm{can},n} \in H^1(\mathcal{D}, \mu_{p^n})$ be the pull-back of the canonical μ_{p^n}-torsor along f, and $Y_n \to \mathcal{D}$ the corresponding analytic torsor, considered also as a torsor on Ω by restriction. From Lemma 4.4, for each n big enough, there exists a node $y_n \in Y_n$ of $S^{\mathrm{an}}(Y_n)$ over a point $z_n \in \Omega$, such that the sequence $(z_n)_n$ converges to z, with $g(y_n) \geqslant 1$. In particular, $z_n \in \Omega_{\mathrm{res}}$, and Ω satisfies the resolution of non-singularities since $\Omega(k) \subset \overline{\Omega_{\mathrm{res}}}$. Let x_n be the image of z_n in X. We want to prove that $x_n \in X_{\mathrm{res}}$, which would be enough to conclude, given the remark at the beginning of this proof and the fact that $(x_n)_n$ converges to x.

The idea is to "periodise" f on Ω, such that it descends to a finite cover of X without changing the behaviours of the corresponding torsors around z_n much. Up to multiplying by a constant, one can assume $f(z_0) = 1$, for some $z_0 \in \Omega(k)$. Consider the group $\Gamma = \pi_1^{\mathrm{top}}(X) = \mathrm{Gal}(\Omega/X)$. For any normal subgroup $\Gamma' \subseteq \Gamma$ of finite index, the product $f_{\Gamma'}(z) = \displaystyle\prod_{\gamma \in \Gamma'} \frac{f(\gamma(z))}{f(\gamma(z_0))}$ converges uniformly on every affinoid domain of Ω, defining an element of $\mathcal{O}_{\Omega}(\Omega)$ that descends to the finite topological covering Ω/Γ' of X.

Let $\varepsilon > 0$, such that the canonical μ_{p^n}-torsor $c_{\mathrm{can},n}$ splits over the open disk $\mathcal{D}(1, \varepsilon)$ (from 3.2, one can take, for instance, $\varepsilon = p^{-n-\frac{1}{p-1}}$). Being a free group, Γ is residually finite, and there exists a decreasing sequence $(\Gamma_m)_{m \geqslant 0}$ of normal subgroups of finite index satisfying $\bigcap_{m \geqslant 0} \Gamma_m = \{1\}$. The sequence $(f_{\Gamma_m})_{m \geqslant 0}$ converges uniformly to f on every affinoid domain of Ω, so there exists m such that $|f_{\Gamma_m}/f - 1|_{z_n} < \varepsilon$. Let $c = f_{\Gamma_m}^* c_{\mathrm{can},n} \in H^1(\mathcal{D}, \mu_{p^n})$ be the pull-back of $c_{\mathrm{can},n}$ along f_{Γ_m}, and $Y \to \Omega$ the corresponding analytic torsor restricted to Ω. The condition $|f_{\Gamma_m}/f - 1|_{z_n} < \varepsilon$ implies that c_n and c are isomorphic over z_n, so there exists a point $y \in Y$ above z_n of genus $g(y) \geq 1$. Moreover, f_{Γ_m} descending to the finite topological cover $X' = \Omega/\Gamma_m$ implies that there exists a μ_{p^n}-torsor $c' : Y' \to X'$, whose pull-back along the topological cover $\pi : \Omega \to X'$ gives c, i.e. $c = \pi^* c'$. Since $q : Y \to Y'$ is a topological cover, the image $y' = q(y)$ of y has a genus $g(y') \geqslant 1$, so y' is a node of $S^{\mathrm{an}}(Y')$. Since y' is situated above x_n, and the composition $Y' \to X' \to X$ is finite, x_n is a solvable point. □

Remark 4.8. — Let X be a quasi-smooth k-analytic curve of finite skeleton, without boundary or points of genus > 0, with only annular cusps, and $\iota : X \hookrightarrow \mathscr{X}^{\mathrm{an}}$ the embedding of X into the analytification of a Mumford curve obtained by prolongation of each cusp by a disk. It is easy to see that ι embeds X into $\mathscr{X}^{\mathrm{an}}$ minus a non-empty disk if and only if X has at least one finite-annular cusp, i.e. a cusp such that the corresponding component of $X \setminus \Sigma_X$ is an annulus of finite length, where Σ_X stands for the set of nodes of $S^{\mathrm{an}}(X)$.

4.3. Anabelianity of the triviality of $\boldsymbol{\mu_p}$-torsors. — We are now up to proving some tempered anabelianity of the triviality of cochains associated to μ_p-torsors on a curve X: either when X is an annulus or a k-analytically hyperbolic curve that is some open subset of the analytification of a Mumford k-curve.

THEOREM 4.9. — *Let X be a k-analytic curve satisfying one of the two following conditions:*

1. *X is an annulus;*
2. *X is a k-analytically hyperbolic curve of finite skeleton without a bridge, without boundary nor any point of genus > 0, with only annular cusps and at least a finite-annular one, such that there is never strictly more than one cusp coming from each node.*

Then the set of μ_p-torsors of X of a trivial $\mathbb{Z}/p\mathbb{Z}$-cochain, i.e. the set $H^1(X, \mu_p) \cap \ker(\theta)$, is completely determined by $\pi_1^{\mathrm{temp}}(X)$.

Proof. — Let us concentrate on the second case, when the curve is k-analytically hyperbolic. The case of an annulus is treated in exactly the same way, inspired from Corollary 3.10 and Proposition 4.6, rather than Propositions 3.12 and 4.7.

From 3.12, an element $f \in H^1(X, \mu_p)$ belongs to $\ker(\theta)$ if and only if, for any vicinal edge e of $S^{\mathrm{an}}(X)$ of associated annulus $\mathcal{C}_e$, there exists $f_e \in H^1(X, \mu_p)$ such that:

$$\left(\mathscr{D}(f)_{[2]} \setminus \mathscr{D}(f_e)_{[2]}\right) \cap \mathcal{C}_{e\,[2]} \neq \emptyset,$$

and one can always choose f_e, such that $\theta(f_e)(e) \neq 0$. In this case, as soon as α is a rigid point of $\mathcal{C}_e$, the threshold point $x_\alpha \in]\alpha, r(\alpha)[$ situated at a distance $\frac{p}{p-1}$ of $r(\alpha)$ is split by f but not by f_e; this comes from Proposition 3.8. Therefore, $x_\alpha \in \mathscr{D}(f) \setminus \mathscr{D}(f_e)$, and as such points are solvable by Proposition 4.7 (x_α is a type-2 point), we obtain:

$$x_\alpha \in \mathscr{D}(f)_{\mathrm{res}} \setminus \mathscr{D}(f_e)_{\mathrm{res}}.$$

In other words, we have $f \in \ker(\theta)$ if and only if there exists $f_e \in H^1(X, \mu_p)$, such that

$$(\mathscr{D}(f)_{\mathrm{res}} \setminus \mathscr{D}(f_e)_{\mathrm{res}}) \cap \mathcal{C}_e \neq \emptyset. \tag{1}$$

From the properties of solvable points presented in 4.1, the sets $\mathscr{D}(f)_{\mathrm{res}}$, $\mathscr{D}(f_e)_{\mathrm{res}}$ and $\mathcal{C}_e \cap X_{\mathrm{res}}$ are determined by the tempered group $\pi_1^{\mathrm{temp}}(X)$, so the condition (1) above can be detected from the tempered group, hence the result. □

REMARK 4.10. — The second condition on the curve X implies, from Theorem 3.63 of [7], that X is k-analytically anabelian and not only hyperbolic.

COROLLARY 4.11. — *Let us stay within the framework of theorem 4.9. Let* $h \in \mathbb{N}^\times$*, and* $\mathrm{mod}(p) : \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/p^h\mathbb{Z}) \to \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/p\mathbb{Z})$ *be the reduction modulo* p *of the* $\mathbb{Z}/p^h\mathbb{Z}$*-cochains. Then it is possible to characterize from the tempered group* $\pi_1^{\mathrm{temp}}(X)$ *the kernel of the composed morphism*

$$\mathrm{mod}(p) \circ \theta : H^1(X, \mu_{p^h}) \to \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/p\mathbb{Z}).$$

Proof. — We have a commutative diagram:

$$\begin{array}{ccc} H^1(X, \mu_{p^h}) & \xrightarrow{\theta} & \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/p^h\mathbb{Z}) \\ \downarrow & & \downarrow \scriptstyle{\mathrm{mod}(p)} \\ H^1(X, \mu_p) & \xrightarrow{\theta} & \mathrm{Harm}(\mathbb{G}, \mathbb{Z}/p\mathbb{Z}) \end{array},$$

where the first vertical arrow is induced by the exact sequence

$$1 \to \mu_{p^{h-1}} \to \mu_{p^h} \xrightarrow{\pi} \mu_p \to 1.$$

With the identification $H^1(X, \mu_{p^i}) \simeq \mathrm{Hom}(\pi_1^{\mathrm{temp}}(X), \mu_{p^i})$, this morphism is nothing other than

$$\mathrm{Hom}(\pi_1^{\mathrm{temp}}(X), \mu_{p^h}) \xrightarrow{\pi_*} \mathrm{Hom}(\pi_1^{\mathrm{temp}}(X), \mu_p),$$

so it only depends on the tempered group $\pi_1^{\mathrm{temp}}(X)$. The conclusion follows from 4.9 and the commutativity of the diagram. □

5. Partial anabelianity of lengths of annuli

We shall show how all these set-theoretical considerations about the intersection of the skeleton of an annulus with the splitting sets of its μ_p-torsors enable us to extract some information about the length of the annulus, before giving an anabelian interpretation.

5.1. Lengths and splitting sets. — The following lemma enables us to determine whether the length of an annulus is $> \frac{2p}{p-1}$ from the knowledge of its μ_p-torsors of trivial cochains.

LEMMA 5.1. — *A k-analytic annulus $\mathcal{C}$ has a length strictly greater than* $\frac{2p}{p-1}$ *if and only if any μ_p-torsor of trivial cochains on $\mathcal{C}$ totally splits over a non-empty portion of its analytic skeleton:*

$$\ell(\mathcal{C}) > \frac{2p}{p-1} \iff \forall f \in H^1(\mathcal{C}, \mu_p) \cap \ker(\theta), \mathscr{D}(f)_{[2]} \cap S^{\mathrm{an}}(\mathcal{C}) \neq \emptyset.$$

Proof. — Assume $\ell(\mathcal{C}) > \frac{2p}{p-1}$ and consider $f \in H^1(\mathcal{C}, \mu_p) \cap \ker(\theta)$. As in the proof of Proposition 3.8, up to restricting $\mathcal{C}$ (but only slightly, in order to keep the condition on the length), one can assume that $\mathcal{C}$ is the subannulus of $\mathbb{P}_k^{1,\mathrm{an}}$ given by the condition $T \in]1-\varepsilon, 1[$ with $1-\epsilon < p^{-\frac{2p}{p-1}}$, and that the μ_p-torsor f is defined by a function $g \in \mathscr{O}_{\mathcal{C}}(\mathcal{C})^\times$ written as:

$$g(T) = 1 + \underbrace{\sum_{k \in \mathbb{Z}\setminus\{0\}} a_k T^k}_{u(T)},$$

with, for all $k \in \mathbb{N}^\times$, $|a_k| < 1$ and $|a_{-k}| < (1-\varepsilon)^k$. The skeleton of $\mathcal{C}$ is the interval $]\eta_{0,1-\varepsilon}, \eta_{0,1}[$, and the corresponding analytic torsor totally splits over a point $\eta_{0,r} \in S^{\mathrm{an}}(\mathcal{C})$ as soon as $|u(\eta_{0,r})| < p^{-\frac{p}{p-1}}$. Let $k \in \mathbb{N}^\times$:

- if $r < p^{-\frac{p}{p-1}}$, $|a_k r^k| < p^{-\frac{kp}{p-1}} \leqslant p^{-\frac{p}{p-1}}$;
- if $r > (1-\varepsilon)\, p^{\frac{p}{p-1}}$, $|a_{-k} r^{-k}| < (1-\varepsilon)^k (1-\varepsilon)^{-k} p^{-\frac{kp}{p-1}} = p^{-\frac{kp}{p-1}} < p^{-\frac{p}{p-1}}$.

But we have $1-\varepsilon < p^{-\frac{2p}{p-1}}$ (from the assumption on the length of $\mathcal{C}$), and hence:

$$r_1 = (1-\varepsilon)\, p^{\frac{p}{p-1}} < p^{-\frac{p}{p-1}} = r_2.$$

Consequently, the torsor f totally splits over the non-empty interval $]\eta_{0,r_1}, \eta_{0,r_2}[$ of the skeleton. From the density of $S^{\mathrm{an}}(\mathcal{C})_{[2]}$ in $S^{\mathrm{an}}(\mathcal{C})$, we obtain that $\mathscr{D}(f)_{[2]} \cap S^{\mathrm{an}}(\mathcal{C}) \neq \emptyset$.

Reciprocally, if $\ell(\mathcal{C}) \leqslant \frac{2p}{p-1}$, one can check that the torsor given by the function $g(T) = 1 + T + (1-\varepsilon)T^{-1}$ never totally splits over any point of $S^{\mathrm{an}}(\mathcal{C})$. □

It is actually possible to reduce by half the previous bound from a finer condition requiring us to look at the set of μ_p-torsors, which totally split *over a neighbourhood of a fixed extremity.* We need the following definition.

Definition 5.2. — Let $\mathcal{C}$ be a non-empty k-analytic annulus. Its skeleton $S^{\mathrm{an}}(\mathcal{C})$ is an interval (open or closed), and let ω be one of its extremities. Let $H^1_\omega(\mathcal{C}, \mu_p)$ be the subgroup of $H^1(\mathcal{C}, \mu_p)$ of μ_p-torsors, which totally split *over a neighbourhood of* ω, i.e. which totally splits over a subinterval of $S^{\mathrm{an}}(\mathcal{C})$ of non-empty interior, and whose complementarity in $S^{\mathrm{an}}(\mathcal{C})$ is an interval that does not admit ω as an extremity.

Lemma 5.3. — *A k-analytic annulus $\mathcal{C}$ has a length strictly greater than $\frac{p}{p-1}$ if and only if, for any extremity ω of $S^{\mathrm{an}}(\mathcal{C})$:*

$$\mathrm{Card}\left(\bigcap_{f \in H^1_\omega(\mathcal{C}, \mu_p)} \mathscr{D}(f)_{[2]} \cap S^{\mathrm{an}}(\mathcal{C}) \right) \geqslant 2.$$

Proof. — Assume $\ell(\mathcal{C}) > \frac{p}{p-1}$ and consider $f \in H^1_\omega(\mathcal{C}, \mu_p)$. Up to restriction of $\mathcal{C}$ (but only slightly, such that the condition on the length still holds), one can assume that $\mathcal{C}$ is the subannulus of $\mathbb{P}^{1,\mathrm{an}}_k$ given by the condition $T \in]1-\varepsilon, 1]$ with $1-\varepsilon < p^{-\frac{p}{p-1}}$. Let $\mathcal{D}_0$ be the closed k-analytic disk of $\mathbb{P}^{1,\mathrm{an}}_k$ centred in 0 and of radius 1, i.e. defined by the condition $|T| \in [0,1]$. The annulus $\mathcal{C}$ is then a k-analytic subspace of $\mathcal{D}_0$. From the assumption on f, it is possible to extend f into a torsor $\widetilde{f} \in H^1(\mathcal{D}_0, \mu_p)$ of $\mathcal{D}_0$, trivial over $\mathcal{D}_0 \setminus \mathcal{C}$. Since $\mathrm{Pic}(\mathcal{D}_0)$ is trivial ($\mathcal{D}_0$ is a k-affinoid space), $\widetilde{f}$ is given by a function $g \in \mathscr{O}_{\mathcal{D}_0}(\mathcal{D}_0)^\times$ written as

$$g(T) = 1 + \underbrace{\sum_{k \in \mathbb{N}^\times} a_k T^k}_{v(T)},$$

with $|a_k| < 1$ for all $k \in \mathbb{N}^\times$.

The skeleton of $\mathcal{C}$ is the interval $]\eta_{0,1-\varepsilon}, \eta_{0,1}]$, and the torsor $f = \widetilde{f}_{|\mathcal{C}}$ totally splits over the point $\eta_{0,r} \in S^{\mathrm{an}}(\mathcal{C})$ as soon as $|v(\eta_{0,r})| < p^{-\frac{p}{p-1}}$.

For all $k \in \mathbb{N}^\times$ and $r \in\,]1-\varepsilon, p^{-\frac{p}{p-1}}[$, we have $|a_k r^k| < p^{-\frac{p}{p-1}}$, so $|v(\eta_{0,r})| < p^{-\frac{p}{p-1}}$. Thus, the interval $]\eta_{0,1-\varepsilon}, \eta_{0,p^{-\frac{p}{p-1}}}[$ belongs to $\mathscr{D}(f)$. As the reasoning is independent of the choice of $f \in H^1_\omega(\mathcal{C}, \mu_p)$, one obtains:

$$]\eta_{0,1-\varepsilon}, \eta_{0,p^{-\frac{p}{p-1}}}[\; \subseteq \bigcap_{f \in H^1_\omega(\mathcal{C},\mu_p)} \mathscr{D}(f) \cap S^{\mathrm{an}}(\mathcal{C}),$$

and the conclusion follows from the density of type-2 points in $\mathcal{C}$.

Reciprocally, consider an annulus $\mathcal{C}$ of length $\ell(\mathcal{C}) \leqslant \frac{p}{p-1}$ and assume there exist two distinct points $x_1, x_2 \in \bigcap_{f \in H^1_\omega(\mathcal{C},\mu_p)} \mathscr{D}(f)_{[2]} \cap S^{\mathrm{an}}(\mathcal{C})$. Let $y \in\,]x_1, x_2[$ be a type-2 point. Let I be the connected component of $S^{\mathrm{an}}(C) \setminus \{y\}$, which does not abut to ω, and $\mathcal{C}_I$ the subannulus of $\mathcal{C}$ of skeleton I. Up to exchanging x_1 and x_2, one can assume $x_2 \in I$. As the annulus $\mathcal{C}_I$ has a length $< \frac{p}{p-1}$, there exists $h \in H^1(\mathcal{C}_I, \mu_p)$, such that $\mathscr{D}(h) \cap S^{\mathrm{an}}(\mathcal{C}_I) =]y, x_2[$. Therefore, h can be extended into a torsor $\widetilde{h} \in H^1(\mathcal{C}, \mu_p)$ of $\mathcal{C}$, such that $\mathscr{D}(\widetilde{h}) \cap S^{\mathrm{an}}(\mathcal{C}) =]x_1, y[$ (or $[x_1, y[$, according to whether $\mathcal{C}$ is open or closed in x_1). Then $\widetilde{h} \in H^1_\omega(\mathcal{C}, \mu_p)$, which leads to a contradiction since $x_2 \notin \mathscr{D}(\widetilde{h})$. □

Corollary 5.4. — *From the tempered fundamental group of a k-analytic annulus, it is possible to determine whether the length of the latter is strictly greater than $\frac{p}{p-1}$.*

Proof. — We showed in 4.6 that all type-2 points of $\mathcal{C}$ are solvable, so there is the equality:

$$\bigcap_{f\in H^1_\omega(\mathcal{C},\mu_p)} \mathscr{D}(f)_{[2]}\cap S^{\mathrm{an}}(\mathcal{C}) = \bigcap_{f\in H^1_\omega(\mathcal{C},\mu_p)} \mathscr{D}(f)_{\mathrm{res}}\cap S^{\mathrm{an}}(\mathcal{C})_{\mathrm{res}}.$$

From the properties of solvable points presented in 4.1, the tempered group $\pi_1^{\mathrm{temp}}(\mathcal{C})$ characterises the sets $\mathscr{D}(f)_{\mathrm{res}}$ and $S^{\mathrm{an}}(\mathcal{C})_{\mathrm{res}}$. Moreover, a torsor $f\in H^1(\mathcal{C},\mu_p)$ belongs to $H^1_\omega(\mathcal{C},\mu_p)$ if and only if it totally splits over the set of type-2 points of a non-empty neighbourhood of ω in $S^{\mathrm{an}}(\mathcal{C})$. But all the type-2 points of $S^{\mathrm{an}}(\mathcal{C})$ are solvable, so $H^1_\omega(\mathcal{C},\mu_p)$ is itself characterized by the tempered group. The result follows from Lemma 5.3. □

5.2. Results on lengths of annuli. — We are now in a position to state our result of partial anabelianity of lengths of annuli. Even if we are not yet in a position to know whether the fundamental group of an annulus determines its length, the following result shows that the lengths of two annuli that have isomorphic tempered fundamental groups cannot be too far from each other. When the lengths are finite, we give an explicit bound, depending only on the residual characteristic p, for the absolute value of the difference of these lengths.

THEOREM 5.5. — *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two k-analytic annuli whose tempered fundamental groups $\pi_1^{\mathrm{temp}}(\mathcal{C}_1)$ and $\pi_1^{\mathrm{temp}}(\mathcal{C}_2)$ are isomorphic. Then $\mathcal{C}_1$ has finite length if and only if $\mathcal{C}_2$ has finite length. In this case:*

$$|\ell(\mathcal{C}_1)-\ell(\mathcal{C}_2)|<\frac{2p}{p-1}.$$

We also have $d\left(\frac{p-1}{p}\ell(\mathcal{C}_1),p\mathbb{N}^\times\right)>1$ if and only if $d\left(\frac{p-1}{p}\ell(\mathcal{C}_2),p\mathbb{N}^\times\right)>1$, and in this case:

$$|\ell(\mathcal{C}_1)-\ell(\mathcal{C}_2)|<\frac{p}{p-1}.$$

Proof. — Let $n\in\mathbb{N}^\times$ prime to p, and $i\in\{1,2\}$. We know that all μ_n-torsors of $\mathcal{C}_i$ are Kummer. Thus, annuli defined by torsors coming from $H^1(\mathcal{C}_i,\mu_n)$ have length $\frac{\ell(\mathcal{C}_i)}{n}$ (with potentially $\ell(\mathcal{C}_i)=+\infty$).

Moreover, all the μ_n-torsors of $\mathcal{C}_i$ can be "read" on the tempered group $\pi_1^{\mathrm{temp}}(\mathcal{C}_i)$ since $H^1(\mathcal{C}_i,\mu_n)\simeq \mathrm{Hom}(\pi_1^{\mathrm{temp}}(\mathcal{C}_i),\mu_n)$. From Corollary 5.4 it is then possible, from $\pi_1^{\mathrm{temp}}(\mathcal{C}_i)$, to know whether $\frac{\ell(\mathcal{C}_i)}{n}>\frac{p}{p-1}$, and to find the smallest integer j step by step, such that:

$$\frac{\ell(\mathcal{C}_i)}{n^{j+1}}\leqslant\frac{p}{p-1}<\frac{\ell(\mathcal{C}_i)}{n^j},\qquad\text{i.e. such that}\qquad n^j\frac{p}{p-1}<\ell(\mathcal{C}_i)\leqslant n^{j+1}\frac{p}{p-1}.$$

But the tempered groups of these two annuli are isomorphic, so such a j will be the same for $\mathcal{C}_1$ and $\mathcal{C}_2$. In particular, for any $N \in \mathbb{N}^\times$ *prime to* p:

$$N\frac{p}{p-1} < \ell(\mathcal{C}_1) \iff N\frac{p}{p-1} < \ell(\mathcal{C}_2),$$

which leads to the conclusion. □

Acknowledgements. — I would like to express my strong gratitude to Emmanuel Lepage for his numerous ideas that appeared to be essential in this work, as well as Antoine Ducros for his detailed rereading and the time spent discussing the subtleties of splitting radius.

BIBLIOGRAPHY

[1] Y. André – "On a geometric description of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a p-adic avatar of $\widehat{\mathrm{gt}}$", *Duke Mathematical Journal* **119** (2003), no. 1, p. 1–39.

[2] Y. André, F. Katō & N. Tsuzuki – *Period mappings and differential equations: From* $\mathbb{C}$ *to* $\mathbb{C}_p$, Citeseer, 2003.

[3] V. G. Berkovich – "Étale cohomology for non-archimedean analytic spaces", *Publications Mathématiques de l'IHÉS* **78** (1993), p. 5–161.

[4] ———, *Spectral theory and analytic geometry over non-archimedean fields*, no. 33, American Mathematical Soc., 2012.

[5] A. J. De Jong – "Étale fundamental groups of non-archimedean analytic spaces", *Compositio mathematica* **97** (1995), no. 1-2, p. 89–118.

[6] A. Ducros – "La structure des courbes analytiques", *Manuscript available at www. math. jussieu. fr/ ducros* (2014).

[7] S. Gaulhiac – "Reconstruction anabélienne du squelette des courbes analytiques", *Annales de l'Institut Fourier* (2021).

[8] A. G. Lauder & D. Wan – "Computing zeta functions of artin–schreier curves over finite fields", *LMS journal of computation and mathematics* **5** (2002), p. 34–55.

[9] E. Lepage – "Tempered fundamental group and metric graph of a mumford curve", *Publications of the Research Institute for Mathematical Sciences* **46** (2010), no. 4, p. 849–897.

[10] ———, "Tempered fundamental group and graph of the stable reduction", in *The Arithmetic of Fundamental Groups*, Springer, 2012, p. 209–224.

[11] ———, "Resolution of nonsingularities for mumford curves", *Publications of the Research Institute for Mathematical Sciences* **49** (2013), no. 4, p. 861–891.

[12] ———, "Tempered fundamental group", in *Geometric and differential Galois theories*, vol. 27, Séminaires et Congrès, SMF, 2013, p. 93–113.

[13] S. Mochizuki – "The geometry of anabelioids", *Publications of the Research Institute for Mathematical Sciences* **40** (2004), no. 3, p. 819–881.

[14] ———, "Semi-graphs of anabelioids", *Publications of the Research Institute for Mathematical Sciences* **42** (2006), no. 1, p. 221–322.

[15] F. Pop & J. Stix – "Arithmetic in the fundamental group of a p-adic curve. on the p-adic section conjecture for curves", *Journal für die reine und angewandte Mathematik (Crelles Journal)* **2017** (2017), no. 725, p. 1–40.