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CENTRAL EXTENSIONS OF PREORDERED GROUPS

by Marino Gran & Aline Michel

Abstract. — We prove that the category of preordered groups contains two full reflective subcategories that give rise to some interesting Galois theories. The first one is the category of so-called commutative objects, which are precisely the preordered groups whose group law is commutative. The second one is the category of abelian objects, which turns out to be the category of monomorphisms in the category of abelian groups. We give a precise description of the reflector to this subcategory and we prove that it induces an admissible Galois structure and then a natural notion of *categorical central extension*. We then characterize the central extensions of preordered groups in purely algebraic terms; these are shown to be the central extensions of groups having the additional property that their restriction to positive cones is a special Schreier surjection of monoids.

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Résumé (*Extensions centrales de groupes préordonnés*). — Nous prouvons que la catégorie des groupes préordonnés contient deux sous-catégories pleines réflexives qui donnent lieu à certaines théories de Galois intéressantes. La première est la catégorie des objets commutatifs, qui sont précisément les groupes préordonnés dont la loi de groupe est commutative. La seconde est la catégorie des objets abéliens, qui s'avère être la catégorie des monomorphismes dans la catégorie des groupes abéliens. Nous donnons une description précise du réflecteur vers cette sous-catégorie, et nous prouvons qu'il induit une structure galoisienne admissible et donc une notion naturelle d'extension centrale catégorique. Nous caractérisons ensuite les extensions centrales de groupes préordonnés en termes purement algébriques : on montre qu'elles sont données par les extensions centrales de groupes ayant la propriété additionnelle que leur restriction aux cônes positifs est une surjection spéciale de Schreier de monoïdes.

Introduction

A *preordered group* $(G, \leq)$ is a group $G = (G, +, 0)$ endowed with a preorder relation $\leq$ that is compatible with the addition $+$ of the group G, in the sense that, if $a \leq c$ and $b \leq d$, then $a+b \leq c+d$ (for $a, b, c, d \in G$). Preordered groups and monotone group homomorphisms form a category, denoted by $\mathsf{PreOrdGrp}$. This category is actually isomorphic to another category, whose objects are given by pairs (G, P_G), where G is a group and P_G a submonoid of G closed under conjugation in G. This submonoid P_G is usually called the *positive cone* of G, and an object (G, P_G) can be depicted as

$$P_G \rightarrowtail G,$$

where the arrow represents the inclusion of P_G in G. An arrow between two such objects (G, P_G) and (H, P_H) is given by a pair $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ of monoid morphisms making the diagram

(1)
$$\begin{tikzcd} P_G \arrow[r, "\bar{f}"] \arrow[d, tail] & P_H \arrow[d, tail] \\ G \arrow[r, "f"'] & H \end{tikzcd}$$

commute, so that $f\colon G \to H$ is a group homomorphism that "restricts" to the positive cones, in the sense that $f(P_G) \subseteq P_H$. It is this second equivalent definition of the category $\mathsf{PreOrdGrp}$ of preordered groups that we shall use throughout this article. As shown in [4] the category $\mathsf{PreOrdGrp}$ is both complete and cocomplete and is a *normal* category in the sense of [14], which is a pointed regular category where every regular epimorphism is a normal epimorphism (i.e. a cokernel).

In this article, we prove that the lattice of normal subobjects on any preordered group (G, P_G) is *modular* (Proposition 2.5), and this implies that any reflective subcategory of $\mathsf{PreOrdGrp}$ that is also closed in it under subobjects and regular quotients is *admissible* from the point of view of categorical Galois theory [7] (see Proposition 2.3). In particular, the full subcategory $\mathsf{PreOrdAb}$ of preordered *abelian* groups satisfies this property, giving rise to the adjunction

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{PreOrdAb}, \arrow[l, bend left, hook, "V"] \end{tikzcd} \tag{2}$$

where V is the inclusion functor, and its left adjoint C sends a preordered group (G, P_G) to the preordered abelian group $(G/[G,G], \eta_G(P_G))$, where $\eta_G \colon G \twoheadrightarrow G/[G,G]$ is the quotient of G by its derived subgroup $[G,G]$. Preordered abelian groups turn out to be precisely the *commutative objects* (in the sense of [2]) of the category $\mathsf{PreOrdGrp}$. A characterization of the normal extensions of preordered groups with respect to this adjunction is given in Theorem 2.9; these are precisely the normal epimorphisms $(f, \bar{f}) \colon (G, P_G) \longrightarrow\!\!\!\!\!\rightarrow (H, P_H)$ such that $f \colon G \longrightarrow\!\!\!\!\!\rightarrow H$ is a central extension of groups and, moreover, $a - b + c \in P_G$ whenever $a, b, c \in P_G$ are such that $\eta_G(a) = \eta_G(b)$ and $f(b) = f(c)$.

We then turn our attention to the composite adjunction

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{PreOrdAb} \arrow[l, bend left, hook, "V"] \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathsf{Mono(Ab)}, \arrow[l, bend left, hook, "W"] \end{tikzcd}$$

where $\mathsf{Mono(Ab)}$ is the category of monomorphisms in the category Ab of abelian groups, W is the inclusion functor and A its left adjoint (described in detail in Section 3). We prove that $\mathsf{Mono(Ab)}$ is the category of *abelian objects* in $\mathsf{PreOrdGrp}$ (Corollary 3.8), and we characterize the normal epimorphisms

$$\begin{tikzcd} P_G \arrow[r, two heads, "\bar{f}"] \arrow[d, tail] & P_H \arrow[d, tail] \\ G \arrow[r, two heads, "f"'] & H \end{tikzcd} \tag{3}$$

in $\mathsf{PreOrdGrp}$ (i.e. both f and $\bar{f}$ are surjective) that are *central extensions* in the sense of categorical Galois theory [10] for this composite adjunction. By using the results established in [15] we show in Theorem 6.3 that these are characterized by the fact that the surjective morphism f is a central extension of groups and $\bar{f}$ a special homogeneous surjection (or, equivalently, a special Schreier surjection) in the sense of [3]. This result opens the way to the possibility of studying the non-abelian homology of preordered groups by using the approach adopted in [5] (which is itself based on the one in [9]), which we leave for future work.

1. Categorical Galois structures and central extensions

We first recall some definitions and results of categorical Galois theory that will be needed for our work. For this section, we mainly follow [7, 8, 10].

Definition 1.1. — A *Galois structure* is a system $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$ in which

- $\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{F} \arrow[l, bend left, "U"] \end{tikzcd}$ is an adjunction, with unit η and counit ϵ;
- $\mathscr{E}$ and $\mathscr{Z}$ are classes of morphisms in $\mathscr{C}$ and $\mathscr{F}$, respectively,

such that

- $\mathscr{C}$ and $\mathscr{F}$ admit all pullbacks along morphisms from $\mathscr{E}$ and $\mathscr{Z}$, respectively;
- $\mathscr{E}$ and $\mathscr{Z}$ are closed under composition, contain all isomorphisms and are pullback-stable;
- $F(\mathscr{E}) \subseteq \mathscr{Z}$;
- $U(\mathscr{Z}) \subseteq \mathscr{E}$.

Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$ be a Galois structure. For any object B in $\mathscr{C}$, let $\mathscr{E}(B)$ denote the full subcategory of the slice category $\mathscr{C} \downarrow B$ determined by the morphisms $A \xrightarrow{f} B$ in the class $\mathscr{E}$. Objects in this subcategory are called *extensions* of B and are denoted by (A, f). Let $p\colon E \to B$ be any arrow in $\mathscr{C}$. Then $p^*\colon \mathscr{E}(B) \to \mathscr{E}(E)$ is the change-of-base functor associating, with any object $f\colon A \to B$ in $\mathscr{E}(B)$, the object $p^*(f) = \pi_1\colon E \times_B A \to E$ as in the following pullback diagram:

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B. \end{tikzcd}$$

It is well known that a Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$ induces, for any object $B \in \mathscr{C}$, an adjunction between the categories of extensions as in the diagram

$$\begin{tikzcd} \mathscr{E}(B) \arrow[r, bend left, "F^B"] \arrow[r, phantom, "\perp"] & \mathscr{Z}(F(B)) \arrow[l, bend left, "U^B"] \end{tikzcd} \tag{4}$$

with unit and counit denoted by η^B and ϵ^B, respectively. The left adjoint $F^B\colon \mathscr{E}(B) \to \mathscr{Z}(F(B))$ is defined, for any $(A, f) \in \mathscr{E}(B)$, by $F^B(A, f) = (F(A), F(f)) (\in \mathscr{Z}(F(B)))$, while the right adjoint $U^B\colon \mathscr{Z}(F(B)) \to \mathscr{E}(B)$

sends any $(X,\phi) \in \mathscr{X}(F(B))$ to the pullback $\eta_B^*(U(\phi))$ of $U(\phi)$ along η_B:

$$\begin{tikzcd} B \times_{UF(B)} U(X) \arrow[r] \arrow[d, "\eta_B^*(U(\phi))"'] & U(X) \arrow[d, "U(\phi)"] \\ B \arrow[r, "\eta_B"'] & UF(B). \end{tikzcd}$$

Definition 1.2. — A Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$ is *admissible* when, for any $B \in \mathscr{C}$, the counit morphism ϵ^B of the adjunction (4) is an isomorphism.

There is an equivalent way to define the admissibility of a Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$, which corresponds to the equivalence (1) $\Leftrightarrow$ (2) in the following proposition. Under an additional condition on the Galois structure, we also obtain the equivalence with (3). This extra condition is that the counit ϵ of the adjunction $F \dashv U$ is an isomorphism. In this paper, we shall always be in such a situation, so that it will be possible to use this last equivalent definition of the admissibility.

Proposition 1.3. — *Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$ be a Galois structure such that the counit ϵ of the adjunction $F \dashv U$ is an isomorphism. Then, the following conditions are equivalent:*

(1) Γ *is admissible;*
(2) *for any $B \in \mathscr{C}$, the functor $U^B \colon \mathscr{X}(F(B)) \to \mathscr{E}(B)$ is fully faithful;*
(3) *F preserves all pullbacks of the form*

$$\begin{tikzcd} B \times_{UF(B)} U(X) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & U(X) \arrow[d, "U(\phi)"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

where $\phi \in \mathscr{Z}$.

From now on, we assume that an admissible Galois structure $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$ as in Proposition 1.3 has been fixed. Let us then recall the notions of (Γ-)*trivial*, (Γ-)*central* and (Γ-)*normal extensions*.

Definition 1.4. — A morphism $f \colon A \to B$ in $\mathscr{E}$ is a *(Γ-)trivial extension* when the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UF(A) \arrow[d, "UF(f)"] \\ B \arrow[r, "\eta_B"'] & UF(B) \end{tikzcd}$$

is a pullback. Equivalently, $f \colon A \to B$ in $\mathscr{E}$ is a *(Γ-)trivial extension* when f lies in the image of the functor $U^B \colon \mathscr{X}(F(B)) \to \mathscr{E}(B)$.

Remark that the equivalence in the above definition follows directly from the admissibility of the Galois structure Γ.

By a *monadic extension* we mean a morphism $p\colon E \to B$ in $\mathscr{E}$ that is also an *effective descent morphism* (see [11, 12], for instance).

Definition 1.5. — A morphism $f\colon A \to B$ in $\mathscr{E}$ is a *(Γ-)central extension* when there exists a monadic extension $p\colon E \to B$ such that $p^*(f)\colon E \times_B A \to E$ is a (Γ-)trivial extension, that is, the following diagram

$$\begin{tikzcd} E \times_B A \arrow[r, "\eta_{E\times_B A}"] \arrow[d, "p^*(f)=\pi_1"'] & UF(E \times_B A) \arrow[d, "UF(\pi_1)"] \\ E \arrow[r, "\eta_E"'] & UF(E) \end{tikzcd}$$

is a pullback, where π_1 is the first projection in the pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "p^*(f)=\pi_1"'] & A \arrow[d, "f"] \\ E \arrow[r, "p"'] & B. \end{tikzcd}$$

Definition 1.6. — An arrow $f\colon A \to B$ in $\mathscr{E}$ is a *(Γ-)normal extension* when f is a monadic extension, and $f^*(f)$ is a (Γ-)trivial extension.

Clearly, any trivial extension is central and any normal extension is central. The admissibility of the Galois structure Γ also guarantees that any trivial extension is normal.

2. The reflector $C\colon \mathsf{PreOrdGrp} \to \mathsf{PreOrdAb}$ and its induced admissible Galois structure

The thorough study of the properties of the category $\mathsf{PreOrdGrp}$ carried out in [4] provides in particular a description of the limits and the colimits of this category that will be needed for our work. We briefly recall these constructions for the reader's convenience.

First, the product of two preordered groups (G, P_G) and (H, P_H) is given by the preordered group $(G \times H, P_G \times P_H)$. Then the equalizer of two parallel arrows

$$(G, P_G) \underset{(g,\bar{g})}{\overset{(f,\bar{f})}{\rightrightarrows}} (H, P_H) \tag{5}$$

is given by $((E, P_E), (e, \bar{e}))$, where (E, e) is the equalizer of f and g in the category Grp of groups and $P_E = E \cap P_G$, that is, the following square is a pullback in Mon:

$$\begin{tikzcd} P_E \arrow[r, tail, "\bar{e}"] \arrow[d, tail] & P_G \arrow[d, tail] \\ E \arrow[r, tail, "e"'] & G. \end{tikzcd}$$

It is not difficult to see that this is equivalent to saying that (E, e) is the equalizer of f and g in Grp and $(P_E, \bar{e})$ the equalizer of $\bar{f}$ and $\bar{g}$ in Mon. As a consequence, the pullback of two morphisms $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ and $(g, \bar{g})\colon (C, P_C) \to (H, P_H)$ with the same codomain (H, P_H) is given by $((P, P_P), (p_1, \bar{p}_1), (p_2, \bar{p}_2))$, where (P, p_1, p_2) is the pullback of f and g in Grp and $(P_P, \bar{p}_1, \bar{p}_2)$ the pullback of $\bar{f}$ and $\bar{g}$ in Mon. More generally, all limits of $\mathsf{PreOrdGrp}$ are computed "componentwise". Note that in particular the kernel (K, P_K) of a morphism $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ can also be computed by taking K to be the kernel $\mathsf{Ker}(f)$ of f in Grp and P_K the intersection $K \cap P_G$.

Colimits in $\mathsf{PreOrdGrp}$ are a bit more difficult to describe, in general, since they are not simply computed "componentwise". We shall be mainly interested in coequalizers; given two parallel arrows as in (5) their coequalizer is given by $((Q, P_Q), (q, \bar{q}))$ with (Q, q) the coequalizer of f and g in Grp and $P_Q = q(P_H)$ (i.e. $\bar{q}$ is surjective):

$$\begin{tikzcd} P_H \arrow[r, two heads, "\bar{q}"] \arrow[d, tail] & P_Q \arrow[d, tail] \\ H \arrow[r, two heads, "q"'] & Q. \end{tikzcd}$$

In particular the cokernel of a morphism $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ in $\mathsf{PreOrdGrp}$ is then given by a pair $(q, \bar{q})$ making the diagram above commute, with q the cokernel of f in Grp and $\bar{q}$ surjective.

As recalled in the Introduction, an important result of [4] is that the category $\mathsf{PreOrdGrp}$ of preordered groups is *normal*. Although the category $\mathsf{PreOrdGrp}$ is not *Barr-exact* [1] (see [4, Remark 2.6]) its *effective descent morphisms* (see [11], for instance) are easy to characterize. In this context, effective descent morphisms coincide with normal epimorphisms, so that an effective descent morphism in the category $\mathsf{PreOrdGrp}$ of preordered groups has a fairly simple description; it is a morphism $(f, \bar{f})\colon (G, P_G) \to (H, P_H)$ as in (1) where both f and $\bar{f}$ are surjective. Epimorphisms in $\mathsf{PreOrdGrp}$ are given by morphisms $(f, \bar{f})$ where only f is required to be surjective, and monomorphisms are morphisms $(f, \bar{f})$ where f is injective (which, in turn, implies that also $\bar{f}$ is injective).

Let us then denote by $\mathsf{PreOrdAb}$ the full subcategory of $\mathsf{PreOrdGrp}$ whose objects (G, P_G) are preordered abelian groups, i.e. such that G is abelian.

Proposition 2.1. — *There is an adjunction*

$$
\begin{tikzcd}
\mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{PreOrdAb} \arrow[l, bend left, hook, "V"]
\end{tikzcd}
\tag{6}
$$

between the category $\mathsf{PreOrdGrp}$ *of preordered groups and its full subcategory* $\mathsf{PreOrdAb}$ *of preordered abelian groups, where the right adjoint* V *is the inclusion functor and the left adjoint* C *is defined, for any* $(G, P_G) \in \mathsf{PreOrdGrp}$*, by* $C(G, P_G) = (G/[G,G], \eta_G(P_G))$

$$
\begin{tikzcd}
P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail] & \eta_G(P_G) \arrow[d, tail] \\
G \arrow[r, two heads, "\eta_G"'] & G/[G,G] = ab(G),
\end{tikzcd}
$$

where $\eta_G(P_G)$ *is the direct image of* P_G *along the quotient* η_G *of* G *by its derived subgroup* $[G,G]$*.*

Proof. — Let (G, P_G) be any preordered group. Then the (G, P_G)-component of the unit of the adjunction (6) is given by the above morphism $(\eta_G, \bar{\eta}_G)$. Indeed, consider any preordered abelian group (A, P_A) and any morphism $(f, \bar{f})\colon (G, P_G) \to (A, P_A)$ in $\mathsf{PreOrdGrp}$:

$$
\begin{tikzcd}
P_G \arrow[rr, two heads, "\bar{\eta}_G"] \arrow[dd, tail] \arrow[dr, "\bar{f}"'] & & \eta_G(P_G) \arrow[dl, dotted, "\bar{g}"] \arrow[dd, tail, "i_{ab(G)}"] \\
& P_A \arrow[dd, tail, "i_A"] & \\
G \arrow[rr, two heads, "\eta_G"] \arrow[dr, "f"'] & & ab(G). \arrow[dl, dotted, "g"] \\
& A &
\end{tikzcd}
$$

Then, the universal property of the abelianization of G yields a unique arrow $g\colon ab(G) \to A$ such that $g \cdot \eta_G = f$ in the category Grp. Knowing that $\bar{\eta}_G$ is a strong epimorphism in the category Mon of monoids, there is, moreover, a unique monoid morphism $\bar{g}\colon \eta_G(P_G) \to P_A$ making the following diagram commute:

$$
\begin{tikzcd}
P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, "\bar{f}"'] & \eta_G(P_G) \arrow[d, "g \cdot i_{ab(G)}"] \arrow[dl, dotted, "\bar{g}"] \\
P_A \arrow[r, tail, "i_A"'] & A.
\end{tikzcd}
$$

Accordingly, there exists a unique morphism $(g, \bar{g})\colon (ab(G), \eta_G(P_G)) \to (A, P_A)$ in $\mathsf{PreOrdGrp}$ such that $(g, \bar{g}) \cdot (\eta_G, \bar{\eta}_G) = (f, \bar{f})$. □

Corollary 2.2. — *If $\mathscr{E}_C$ denotes the class of normal epimorphisms in* $\mathsf{PreOrdGrp}$ *and $\mathscr{Z}_C$ the class of normal epimorphisms in* $\mathsf{PreOrdAb}$*, then*

$$\Gamma_C = (\mathsf{PreOrdGrp}, \mathsf{PreOrdAb}, C, V, \mathscr{E}_C, \mathscr{Z}_C) \tag{7}$$

is a Galois structure.

We will now prove that this Galois structure is actually admissible. In order to do this, we will use the following result, which is a reformulation of a result due to Janelidze and Kelly [10].

Proposition 2.3. — *Let $\mathscr{C}$ be a normal category. Let $\Gamma = (\mathscr{C}, \mathscr{F}, F, U, \mathscr{E}, \mathscr{Z})$ be a Galois structure, where $\mathscr{E}$ is the class of normal epimorphisms in $\mathscr{C}$ and $\mathscr{F}$ a full reflective subcategory of $\mathscr{C}$ closed under subobjects and quotients. If the lattice $\mathsf{Norm}_X(\mathscr{C})$ of normal subobjects of any X in $\mathscr{C}$ is modular, then the Galois structure Γ is admissible.*

Proof. — Consider any pullback in $\mathscr{C}$ of the following form

$$\begin{tikzcd} P \arrow[r, two heads, "\pi_2"] \arrow[d, two heads, "\pi_1"'] & A \arrow[d, two heads, "\phi"] \\ B \arrow[r, two heads, "\eta_B"'] & F(B), \end{tikzcd} \tag{8}$$

where η_B is the B-component of the unit of the adjunction $F \dashv U$, A an object of the subcategory $\mathscr{F}$ and ϕ any morphism in the class $\mathscr{E}$. We write $s\colon S \rightarrowtail P$ and $t\colon T \rightarrowtail P$ for the kernels of π_1 and π_2, respectively. We need to prove that this pullback is preserved by the reflector F. In order to do this, we consider the next commutative diagram

$$\begin{tikzcd} R \vee S \arrow[dr, tail] & S \arrow[l, tail] \arrow[d, tail, "s"] & & \\ R \arrow[u, tail] \arrow[r, tail, "r"] \arrow[d, dotted, tail, "m"] & P \arrow[r, two heads, "\eta_P"] \arrow[d, two heads, "\pi_1"'] \arrow[dr, two heads, "q"] \arrow[rr, bend left, two heads, "\pi_2"] & F(P) \arrow[r, two heads, "\psi"] \arrow[d, two heads, "F(\pi_1)"] & A \arrow[d, two heads, "\phi"] \\ T \arrow[ur, tail, "t"'] \arrow[uu, dotted, tail, bend left, "n"] & B \arrow[r, two heads, "\eta_B"'] & F(B) \arrow[r, equal] & F(B) \end{tikzcd} \tag{9}$$

where $r\colon R \rightarrowtail P$ is the kernel of the P-component η_P of the unit of the adjunction $F \dashv U$, $R \vee S$ is the supremum of R and S in $\mathsf{Norm}_P(\mathscr{C})$, q is the composite $\eta_B \cdot \pi_1 = F(\pi_1) \cdot \eta_P$, and ψ is the unique morphism induced by the universal property of η_P such that $\psi \cdot \eta_P = \pi_2$. We compute that $\pi_2 \cdot r = \psi \cdot \eta_P \cdot r = 0$ so that, by the universal property of kernels, there exists a unique arrow $m\colon R \to T$ such that $t \cdot m = r$, hence $R \leq T$. The assumption that $\mathscr{F}$ is a full reflective subcategory of $\mathscr{C}$ closed under subobjects and quotients implies that the middle square of the above diagram (9) is a pushout (see

Proposition 3.1 in [10]). As a consequence, the supremum $R \vee S$ exists, and it is the kernel of the morphism q. Accordingly, since $q \cdot t = F(\pi_1) \cdot \eta_P \cdot t = \phi \cdot \psi \cdot \eta_P \cdot t = \phi \cdot \pi_2 \cdot t = 0$, there exists a unique morphism $n \colon T \to R \vee S$ such that $\ker q \cdot n = t$, so that $T \leq R \vee S$ in $\mathsf{Norm}_P(\mathscr{C})$. Observe then that

$$R = R \vee \{0\} = R \vee (S \wedge T) = (R \vee S) \wedge T = T,$$

where the second equality follows from the fact that π_1 and π_2 are jointly monomorphic, the third equality by modularity of $\mathsf{Norm}_P(\mathscr{C})$ (since $R \leq T$), and the last one from the fact that $T \leq R \vee S$. This means that $\ker(\eta_P) = \ker(\pi_2)$, and the induced morphism ψ is then an isomorphism. If we apply the reflector F to the pullback (8), we then get a pullback in $\mathscr{F}$, and the Galois structure Γ is therefore admissible, as desired. □

We will now apply this proposition to the adjunction we are interested in. It just remains to prove that the lattice of normal subobjects in $\mathsf{PreOrdGrp}$ is modular. In order to do this, we first need to describe the supremum of two normal subobjects in $\mathsf{PreOrdGrp}$.

Lemma 2.4. — *Let (A, P_A) and (B, P_B) be two normal subobjects in* $\mathsf{PreOrdGrp}$ *of a preordered group (G, P_G). The supremum of (A, P_A) and (B, P_B) in* $\mathsf{Norm}_{(G,P_G)}(\mathsf{PreOrdGrp})$ *is given by*

$$(A, P_A) \vee (B, P_B) = (A \cdot B, (A \cdot B) \cap P_G),$$

where $A \cdot B = \{a + b \mid a \in A \text{ and } b \in B\}$ is the supremum of the normal subgroups A and B in the category Grp *of groups.*

Proof. — We first note that $(A \cdot B, (A \cdot B) \cap P_G)$ is a normal subobject of (G, P_G). Indeed, $A \cdot B$ is a normal subgroup of G and the square

$$\begin{tikzcd} (A \cdot B) \cap P_G \arrow[r, tail] \arrow[d, tail] & P_G \arrow[d, tail] \\ A \cdot B \arrow[r, tail] & G \end{tikzcd}$$

is a pullback in the category Mon of monoids. It is also clear that $(A, P_A) \leq (A \cdot B, (A \cdot B) \cap P_G)$ and that $(B, P_B) \leq (A \cdot B, (A \cdot B) \cap P_G)$ (since $P_A = A \cap P_G$ and $P_B = B \cap P_G$). Consider next that we have another normal subobject (N, P_N) of (G, P_G) such that $(A, P_A) \leq (N, P_N)$ and $(B, P_B) \leq (N, P_N)$. Then, of course, $A \cdot B \leq N$ since $A \cdot B = A \vee B$ in Grp. As a consequence, $(A \cdot B, (A \cdot B) \cap P_G) \leq (N, P_N)$. □

Proposition 2.5. — *The lattice of normal subobjects in the category* $\mathsf{PreOrdGrp}$ *of preordered groups is modular: for any triple (A, P_A), (B, P_B) and (C, P_C) of* *normal subobjects in* PreOrdGrp *of a given preordered group* (G, P_G)*, such that* $(C, P_C) \leq (A, P_A)$*, we have that*

$$(A, P_A) \wedge ((B, P_B) \vee (C, P_C)) = ((A, P_A) \wedge (B, P_B)) \vee (C, P_C).$$

Proof. — We already know that $A \wedge (B \vee C) = (A \wedge B) \vee C$ since the lattice $\mathsf{Norm}_G(\mathsf{Grp})$ of normal subgroups of G is modular. We, therefore, compute that

$$\begin{aligned}(A, P_A) \wedge ((B, P_B) \vee (C, P_C)) &= (A, P_A) \wedge (B \cdot C, (B \cdot C) \cap P_G) \\ &= (A \cap (B \cdot C), P_A \cap (B \cdot C) \cap P_G) \\ &= (A \cap (B \cdot C), A \cap P_G \cap (B \cdot C) \cap P_G) \\ &= (A \cap (B \cdot C), (A \cap (B \cdot C)) \cap P_G) \\ &= (A \wedge (B \vee C), (A \wedge (B \vee C)) \cap P_G) \\ &= ((A \wedge B) \vee C, ((A \wedge B) \vee C) \cap P_G) \\ &= (A \cap B) \cdot C, ((A \cap B) \cdot C) \cap P_G) \\ &= (A \cap B, P_A \cap P_B) \vee (C, P_C) \\ &= ((A, P_A) \wedge (B, P_B)) \vee (C, P_C),\end{aligned}$$

where we have used the fact that the infimum of two normal subobjects in the lattice of normal subobjects of (G, P_G) is given by their pullback. □

Corollary 2.6. — *The Galois structure* (7) *is admissible.*

Proof. — This is a direct consequence of Propositions 2.3 and 2.5. □

We will prove in Theorem 2.9 that the Γ_C-normal extensions are given by the normal epimorphisms $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ such that

(i) $f\colon G \twoheadrightarrow H$ is an *algebraically central extension*, i.e. $\mathsf{Ker}(f) \subseteq Z(G)$;
(ii) the following Condition $(\star)$ holds for $(f, \bar{f})$:

$(\star)$ for any $(a, b, c) \in Eq(\bar{\eta}_G) \times_{P_G} Eq(\bar{f})$, $a - b + c \in P_G$,

where $\bar{\eta}_G\colon P_G \twoheadrightarrow \eta_G(P_G)$ is the restriction of $\eta_G\colon G \twoheadrightarrow G/[G, G]$ to the positive cones.

The following two lemmas will be helpful to establish this characterization.

Lemma 2.7. — *Consider the following pullback*

$$\begin{tikzcd} (P, P_P) \arrow[r, two heads, "(p_2, \bar{p}_2)"] \arrow[d, two heads, "(p_1, \bar{p}_1)"'] & (G, P_G) \arrow[d, two heads, "(f, \bar{f})"] \\ (E, P_E) \arrow[r, two heads, "(p, \bar{p})"'] & (H, P_H) \end{tikzcd}$$

in PreOrdGrp*, where all the arrows are regular epimorphisms.*

(1) *If Condition* $(\star)$ *holds for* $(f, \bar{f})$*, then it holds for* $(p_1, \bar{p}_1)$.
(2) *If* $(p, \bar{p}) = (f, \bar{f})$*, then Condition* $(\star)$ *holds for* $(p_1, \bar{p}_1)$ *if and only if it holds for* $(f, \bar{f})$.

Proof. — 1. Let $((e_1, x_1), (e_2, x_2), (e_3, x_3)) \in Eq(\bar{\eta}_P) \times_{P_P} Eq(\bar{p}_1)$. Then, this means that

- $e_i \in P_E$ and $x_i \in P_G$ for any $i = 1, 2, 3$;
- $p(e_i) = f(x_i)$ for any $i = 1, 2, 3$;
- $\eta_P(e_1, x_1) = \eta_P(e_2, x_2)$, which implies that
$$(e_1 - e_2, x_1 - x_2) \in \mathsf{Ker}(\eta_P) = [P, P],$$
hence, in particular, $x_1 - x_2 \in [G, G]$;
- $p_1(e_2, x_2) = p_1(e_3, x_3)$, i.e. $e_2 = e_3$.

As a consequence,

- $\eta_G(x_1) = \eta_G(x_2)$, i.e. $(x_1, x_2) \in Eq(\bar{\eta}_G)$;
- $f(x_2) = p(e_2) = p(e_3) = f(x_3)$, i.e. $(x_2, x_3) \in Eq(\bar{f})$.

In other words, $(x_1, x_2, x_3) \in Eq(\bar{\eta}_G) \times_{P_G} Eq(\bar{f})$. By assumption the element $x_1 - x_2 + x_3$ then belongs to P_G, and one has that
$$\begin{aligned}(e_1, x_1) - (e_2, x_2) + (e_3, x_3) &= (e_1 - e_2 + e_3, x_1 - x_2 + x_3)\\ &= (e_1, x_1 - x_2 + x_3) \in P_P,\end{aligned}$$
as desired.

2. Thanks to (1), it suffices to check that, if Condition $(\star)$ holds for $(p_1, \bar{p}_1)$ (with $(p_1, \bar{p}_1)$ the first projection of the kernel pair of $(f, \bar{f})$), then it also holds for $(f, \bar{f})$.

Let $(a, b, c) \in Eq(\bar{\eta}_G) \times_{P_G} Eq(\bar{f})$. In particular, this means that $a, b, c \in P_G$, $a - b \in \mathsf{Ker}(\eta_G) = [G, G]$, and $f(b) = f(c)$. The fact that $a - b \in [G, G]$ implies that
$$\begin{aligned}(a, a) - (b, b) &= (a - b, a - b)\\ &= \big([x_1, x_2] + [x_3, x_4] + \cdots + [x_{n-1}, x_n],\\ &\qquad [x_1, x_2] + [x_3, x_4] + \cdots + [x_{n-1}, x_n]\big),\end{aligned}$$
for some suitable elements $x_i \in G$ (with $i \in \{1, \ldots, n\}$). It follows that
$$\begin{aligned}(a, a) - (b, b) = {}&(x_1, x_1) + (x_2, x_2) - (x_1, x_1) - (x_2, x_2)\\ &+ \cdots + (x_{n-1}, x_{n-1}) + (x_n, x_n) - (x_{n-1}, x_{n-1}) - (x_n, x_n),\end{aligned}$$
so that $(a, a) - (b, b) \in [Eq(f), Eq(f)] = \mathsf{Ker}(\eta_{Eq(f)})$. This means that $((a, a), (b, b)) \in Eq(\bar{\eta}_{Eq(f)})$. On the other hand, $p_1(b, b) = p_1(b, c)$, and this implies that $((b, b), (b, c)) \in Eq(\bar{p}_1)$, and then
$$((a, a), (b, b), (b, c)) \in Eq(\bar{\eta}_{Eq(f)}) \times_{Eq(\bar{f})} Eq(\bar{p}_1).$$

So, by assumption,

$$(a, a-b+c) = (a,a) - (b,b) + (b,c) \in Eq(\bar{f}),$$

hence, in particular, $a-b+c \in P_G$. $\square$

LEMMA 2.8. — *Condition* $(\star)$ *holds for any morphism* $(f,\bar{f})\colon (G,P_G) \to (H,P_H)$ *in* $\mathsf{PreOrdGrp}$ *where* $(G,P_G) \in \mathsf{PreOrdAb}$.

Proof. — Let $(a,b,c) \in Eq(\bar{\eta}_G) \times_{P_G} Eq(\bar{f})$. Since $G \in \mathsf{Ab}$, we have that $\eta_G = 1_G$, and then that $a = \eta_G(a) = \eta_G(b) = b$. It follows that $a-b+c = c \in P_G$. $\square$

THEOREM 2.9. — *Let* $(f,\bar{f})\colon (G,P_G) \twoheadrightarrow (H,P_H)$ *be a regular epimorphism in* $\mathsf{PreOrdGrp}$. *The following conditions are equivalent:*

(1) (i) $\mathsf{Ker}(f) \subseteq Z(G)$;

(ii) *for any* $(a,b,c) \in Eq(\bar{\eta}_G) \times_{P_G} Eq(\bar{f})$, *we have* $a-b+c \in P_G$.

(2) $(f,\bar{f})$ *is a* $(\Gamma_C$-$)$*normal extension.*

Proof. — (1) $\Rightarrow$ (2): we need to prove that the first projection $(\pi_1,\bar{\pi}_1)$ of the kernel pair of $(f,\bar{f})$ is a $(\Gamma_C$-$)$trivial extension, i.e. that the square below

$$\begin{tikzcd}
Eq(f,\bar{f}) \arrow[r, two heads, "(\eta_{Eq(f)},\bar{\eta}_{Eq(f)})"] \arrow[d, two heads, "(\pi_1,\bar{\pi}_1)"'] & C(Eq(f,\bar{f})) \arrow[d, two heads, "C(\pi_1,\bar{\pi}_1)=(C(\pi_1),C(\bar{\pi}_1))"] \\
(G,P_G) \arrow[r, two heads, "(\eta_G,\bar{\eta}_G)"'] & C(G,P_G)
\end{tikzcd} \tag{10}$$

is a pullback in the category $\mathsf{PreOrdGrp}$ of preordered groups. First of all it is well known that (i) implies that its restriction to the category Grp of groups is a pullback (in Grp) [7]. It remains to show that the external square in the diagram

$$\begin{tikzcd}
Eq(\bar{f}) \arrow[rr, two heads, "\bar{\eta}_{Eq(f)}"] \arrow[dd, two heads, "\bar{\pi}_1"'] \arrow[dr, dotted, tail, "\bar{\phi}"'] & & \eta_{Eq(f)}(Eq(\bar{f})) \arrow[dd, two heads, "C(\bar{\pi}_1)"] \\
& P_P \arrow[ur, two heads, "\bar{q}_2"'] \arrow[dl, two heads, "\bar{q}_1"] & \\
P_G \arrow[rr, two heads, "\bar{\eta}_G"'] & & \eta_G(P_G)
\end{tikzcd}$$

is a pullback in Mon. Consider then the pullback (P,P_P) of $(\eta_G,\bar{\eta}_G)$ and $C(\pi_1,\bar{\pi}_1)$ (denoted with a slight abuse of notation by $(C(\pi_1),C(\bar{\pi}_1))$) in $\mathsf{PreOrdGrp}$ (with the two projections $(q_1,\bar{q}_1)$ and $(q_2,\bar{q}_2)$), as well as the induced morphism

$(\phi,\bar{\phi})\colon (Eq(f),Eq(\bar{f}))\to(P,P_P)$ to the pullback (with ϕ an isomorphism, as already observed). Since ϕ is an isomorphism, obviously $\bar{\phi}$ is a monomorphism by commutativity of the following square:

$$\begin{tikzcd} Eq(\bar{f}) \arrow[r, tail, "\bar{\phi}"] \arrow[d, tail] & P_P \arrow[d, tail] \\ Eq(f) \arrow[r, "\cong", "\phi"'] & P. \end{tikzcd}$$

So it suffices to show that $\bar{\phi}$ is surjective. Let $(a,\eta_{Eq(f)}(b,c))\in P_P$. In other words, $a,b,c\in P_G$ are such that $f(b)=f(c)$ and $\eta_G(a)=C(\bar{\pi}_1)(\eta_{Eq(f)}(b,c))=(\eta_G\cdot\pi_1)(b,c)=\eta_G(b)$, that is, $(a,b,c)\in Eq(\bar{\eta}_G)\times_{P_G}Eq(\bar{f})$. By (ii) we then have that $(a,a-b+c)\in Eq(\bar{f})$. We are going to show that this element is sent by ϕ to $(a,\eta_{Eq(f)}(b,c))$. We first observe that, since $(a,b)\in Eq(\bar{\eta}_G)$, there exist $x_i\in G$ (for $i\in\{1,\dots,n\}$) such that

$$a-b=x_1+x_2-x_1-x_2+x_3+x_4-x_3-x_4+\cdots+x_{n-1}+x_n-x_{n-1}-x_n.$$

This implies that

$$\begin{aligned}(a-b,a-b)&=(x_1,x_1)+(x_2,x_2)-(x_1,x_1)-(x_2,x_2)\\&\quad+\cdots+(x_{n-1},x_{n-1})+(x_n,x_n)-(x_{n-1},x_{n-1})-(x_n,x_n),\end{aligned}$$

i.e. $(a-b,a-b)\in[Eq(f),Eq(f)]=\mathsf{Ker}(\eta_{Eq(f)})$. As a consequence, we can compute

$$\begin{aligned}\phi(a,a-b+c)&=(a,\eta_{Eq(f)}(a,a-b+c))\\&=(a,\eta_{Eq(f)}(a-b,a-b)+\eta_{Eq(f)}(b,c))\\&=(a,\eta_{Eq(f)}(b,c)),\end{aligned}$$

and deduce that the homomorphism $\bar{\phi}$ is then surjective, hence an isomorphism. This proves that the square (10) is a pullback in $\mathsf{PreOrdGrp}$, i.e. that $(f,\bar{f})$ is a (Γ_C-)normal extension.

(2) $\Rightarrow$ (1): since $(f,\bar{f})$ is a (Γ_C-)normal extension, the two squares below are then pullbacks in $\mathsf{PreOrdGrp}$:

$$\begin{tikzcd} (Eq(f),Eq(\bar{f})) \arrow[r, two heads, "(\pi_2,\bar{\pi}_2)"] \arrow[d, two heads, "(\pi_1,\bar{\pi}_1)"'] & (G,P_G) \arrow[d, two heads, "(f,\bar{f})"] \\ (G,P_G) \arrow[r, two heads, "(f,\bar{f})"'] & (H,P_H) \end{tikzcd}$$

$$\begin{tikzcd}
(Eq(f), Eq(\bar{f})) \arrow[r, two heads, "(\eta_{Eq(f)}, \bar{\eta}_{Eq(f)})"] \arrow[d, two heads, "(\pi_1, \bar{\pi}_1)"'] & (ab(Eq(f)), \eta_{Eq(f)}(Eq(\bar{f}))) \arrow[d, two heads, "C(\pi_1, \bar{\pi}_1)"] \\
(G, P_G) \arrow[r, two heads, "(\eta_G, \bar{\eta}_G)"'] & (ab(G), \eta_G(P_G)).
\end{tikzcd}$$

In this second diagram, $C(\pi_1, \bar{\pi}_1) \in \mathsf{PreOrdAb}$. So, thanks to Lemma 2.8, we know that $C(\pi_1, \bar{\pi}_1)$ satisfies Condition ($\star$). Now, using Lemma 2.7(1) with the second pullback, we get that Condition ($\star$) holds for $(\pi_1, \bar{\pi}_1)$. The application of Lemma 2.7(2) to the first pullback gives the validity of Condition ($\star$) for $(f, \bar{f})$, which corresponds to (ii). Condition (i) follows directly from the fact that $f\colon G \twoheadrightarrow H$ is a normal extension with respect to the admissible Galois structure induced by the abelianization functor $\mathsf{ab}\colon \mathsf{Grp} \to \mathsf{Ab}$ (see [7, 10]). □

Remark 2.10. — *It would be interesting to know whether Lemma 2.7(2) holds when $(p, \bar{p}) \neq (f, \bar{f})$. This problem is related to the possibility of finding a characterization of the (Γ_C)-central extensions.*

3. Commutative and abelian objects in the category of preordered groups

This section is devoted to the characterization of the commutative and the abelian objects in the category of preordered groups. For the reader's convenience, we first recall some definitions and results (we refer to [2] for more details).

A pointed category $\mathscr{C}$ with finite limits is said to be *unital* when, for any objects $X, Y \in \mathscr{C}$, the pair (l_X, r_Y) in the diagram

$$X \xrightarrow{l_X} X \times Y \xleftarrow{r_Y} Y$$

is strongly epimorphic, where $l_X = \langle 1_X, 0\rangle$ and $r_Y = \langle 0, 1_Y\rangle$.

Definition 3.1. — Let $\mathscr{C}$ be a unital category. An object $X \in \mathscr{C}$ is said to be *commutative* when there exists a morphism $\phi\colon X \times X \to X$ making the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "l_X"] \arrow[dr, equal] & X \times X \arrow[d, dotted, "\phi"] & X \arrow[l, "r_X"'] \arrow[dl, equal] \\
& X. &
\end{tikzcd}$$

Note that the morphism ϕ above is necessarily unique by unitality of the category $\mathscr{C}$. It turns out that any commutative object in a unital category is an (internal) commutative monoid.

Proposition 3.2. — [2] *Let* $\mathscr{C}$ *be a unital category. The category* $\mathsf{ComMon}(\mathscr{C})$ *of* internal commutative monoids *in* $\mathscr{C}$ *is the full subcategory of commutative objects in* $\mathscr{C}$*.*

Definition 3.3. — [2] Let $\mathscr{C}$ be a unital category. An object $X \in \mathscr{C}$ is said to be *abelian* when it is commutative and the corresponding internal commutative monoid is an internal abelian group.

The following proposition gives a useful characterization of abelian objects in a unital category:

Proposition 3.4. — [2] *Let* $\mathscr{C}$ *be a unital category. Then an object* X *of* $\mathscr{C}$ *is abelian if and only if there exists a morphism* $\phi\colon X \times X \to X$ *making the diagram*

$$\begin{tikzcd} X \arrow[r, "r_X"] \arrow[dr, equal] & X \times X \arrow[d, dotted, "\phi"] & X \arrow[l, "\Delta_X"'] \arrow[dl, "0"] \\ & X & \end{tikzcd}$$

commute, where $\Delta_X = \langle 1_X, 1_X \rangle$ *is the diagonal of* X*.*

A pointed category $\mathscr{C}$ with finite limits is said to be *strongly unital* when, for any object $X \in \mathscr{C}$, the pair (r_X, Δ_X) in the diagram

$$X \xrightarrow{r_X} X \times X \xleftarrow{\Delta_X} X$$

is strongly epimorphic, where $r_X = \langle 0, 1_X \rangle$ and $\Delta_X = \langle 1_X, 1_X \rangle$.

Any abelian object is commutative (by definition), but the converse is not true in general. In the strongly unital context though, the converse holds:

Proposition 3.5. — [2] *In a strongly unital category, any commutative object is an abelian object.*

As proved in [4], the category $\mathsf{PreOrdGrp}$ of preordered groups is a unital category, so that the notions of commutative and abelian objects both make sense in this setting. It turns out that the commutative objects in $\mathsf{PreOrdGrp}$ are precisely the preordered abelian groups:

Proposition 3.6. — *The category* $\mathsf{PreOrdAb}$ *of preordered abelian groups coincides with the category* $\mathsf{ComMon}(\mathsf{PreOrdGrp})$ *of internal commutative monoids in* $\mathsf{PreOrdGrp}$*:*

$$\mathsf{PreOrdAb} = \mathsf{ComMon}(\mathsf{PreOrdGrp}).$$

Proof. — Let (G, P_G) be a commutative object in $\mathsf{PreOrdGrp}$. Then, this means that there exists a morphism $(\phi, \bar{\phi})\colon (G, P_G) \times (G, P_G) \to (G, P_G)$ in $\mathsf{PreOrdGrp}$ such that the diagram

$$\begin{tikzcd} (G,P_G) \arrow[r, "(l_G,\bar{l}_G)"] \arrow[dr, equal] & (G\times G, P_G\times P_G) \arrow[d, dotted, "(\phi,\bar{\phi})"] & (G,P_G) \arrow[l, "(r_G,\bar{r}_G)"'] \arrow[dl, equal] \\ & (G,P_G) & \end{tikzcd}$$

commutes. In particular, for any $x \in G$ and any $y \in G$,

$$\begin{aligned} x + y &= (\phi \cdot l_G)(x) + (\phi \cdot r_G)(y) \\ &= \phi(x,0) + \phi(0,y) \\ &= \phi((x,0) + (0,y)) \\ &= \phi(x,y) \\ &= \phi((0,y) + (x,0)) \\ &= \phi(0,y) + \phi(x,0) \\ &= (\phi \cdot r_G)(y) + (\phi \cdot l_G)(x) \\ &= y + x, \end{aligned}$$

which means that G is an abelian group and then that $(G, P_G) \in \mathsf{PreOrdAb}$.

Conversely, consider $(G, P_G) \in \mathsf{PreOrdAb}$ and define, for any $(x, y) \in G \times G$, the morphism $\phi\colon G\times G \to G$ as follows: $\phi(x, y) = x+y$. Then it is easy to check that ϕ is a group morphism and that $\phi \cdot l_G = 1_G$ and $\phi \cdot r_G = 1_G$. Moreover, it is clear that the restriction $\bar{\phi}$ of ϕ to $P_G \times P_G$ takes its values in P_G, since P_G is a submonoid of G. As a consequence, $(\phi, \bar{\phi})\colon (G, P_G) \times (G, P_G) \to (G, P_G)$ is a morphism in $\mathsf{PreOrdGrp}$ making the above diagram commute in $\mathsf{PreOrdGrp}$. This means that (G, P_G) is a commutative object in $\mathsf{PreOrdGrp}$. The result then follows from Proposition 3.2. □

PROPOSITION 3.7. — *The abelian objects in* $\mathsf{PreOrdGrp}$ *are the preordered groups* (G, P_G) *where* G *is an abelian group and* P_G *a (normal) subgroup of* G.

Proof. — Let (G, P_G) be an abelian object in $\mathsf{PreOrdGrp}$. In particular, it is a commutative object, so that $G \in \mathsf{Ab}$ (by Proposition 3.6). Thanks to Proposition 3.4, we also have that there exists a morphism $(\phi, \bar{\phi})\colon (G, P_G) \times (G, P_G) \to (G, P_G)$ making the following diagram commute:

$$\begin{tikzcd} (G,P_G) \arrow[r, "(r_G,\bar{r}_G)"] \arrow[dr, equal] & (G\times G, P_G\times P_G) \arrow[d, dotted, "(\phi,\bar{\phi})"] & (G,P_G) \arrow[l, "(\Delta_G,\bar{\Delta}_G)"'] \arrow[dl, "{(0,0)}"] \\ & (G,P_G). & \end{tikzcd}$$

The restriction $\bar{\phi}$ of ϕ to $P_G \times P_G$ takes its values in P_G. Accordingly, for any $x \in P_G$, $\phi(x,0) \in P_G$. Now we compute that

$$\begin{aligned}\phi(x,0) &= \phi((0,-x)+(x,x)) = \phi(0,-x)+\phi(x,x)\\ &= (\phi\cdot r_G)(-x) + (\phi\cdot\Delta_G)(x) = -x+0 = -x.\end{aligned}$$

As a consequence, $-x \in P_G$ for any $x \in P_G$, which proves that P_G is a subgroup of G, as desired.

Conversely, consider any preordered abelian group (G, P_G) where the positive cone P_G is a subgroup of G. Define the morphism $\phi\colon G \times G \to G$ by $\phi(x,y) = -x+y$, for any $(x,y) \in G \times G$. It is easily seen that ϕ is a group morphism since the group G is abelian. Moreover, for any $x \in G$,

- $(\phi\cdot r_G)(x) = \phi(0,x) = -0+x = x$, so that $\phi \cdot r_G = 1_G$;
- $(\phi\cdot\Delta_G)(x) = \phi(x,x) = -x+x = 0$, so that $\phi\cdot\Delta_G = 0$.

Let us now prove that the restriction $\bar{\phi}$ of ϕ to $P_G \times P_G$ takes its values in P_G. Let $(x,y) \in P_G \times P_G$. Since P_G is a subgroup of G, then $-x \in P_G$ and then $-x+y = \phi(x,y) \in P_G$. Accordingly, the pair $(\phi,\bar{\phi})\colon (G,P_G)\times(G,P_G) \to (G,P_G)$ is a morphism in $\mathsf{PreOrdGrp}$ making the diagram above commute. By Proposition 3.4 we conclude that (G,P_G) is an abelian object in $\mathsf{PreOrdGrp}$. □

Corollary 3.8. — *The category* $\mathsf{Ab}(\mathsf{PreOrdGrp})$ *of internal abelian groups in* $\mathsf{PreOrdGrp}$ *is isomorphic to the category* $\mathsf{Mono}(\mathsf{Ab})$ *of monomorphisms in the category* Ab *of abelian groups:*

$$\mathsf{Ab}(\mathsf{PreOrdGrp}) = \mathsf{Mono}(\mathsf{Ab}).$$

Proof. — This is a direct consequence of Proposition 3.7 and Definition 3.3. □

Another consequence of Proposition 3.7 is the following remark, which was first observed in [4]:

Remark 3.9. — The category $\mathsf{PreOrdGrp}$ of preordered groups is not strongly unital.

Proof. — This follows from Propositions 3.6, 3.7 and 3.5. For instance, the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ of the monoid $\mathbb{N}$ of non-negative integers in the abelian group $\mathbb{Z}$ of integers is an example of a commutative object in $\mathsf{PreOrdGrp}$ that is not an abelian object. □

Corollary 3.10. — *The category* $\mathsf{PreOrdGrp}$ *is not* subtractive *(in the sense of* [13]*).*

Proof. — This follows from the Remark 3.9 and the well-known fact that

$$\text{strongly unital} = \text{unital} + \text{subtractive}$$

(see [6], for instance). □

We will now consider the functor F from the category $\mathsf{PreOrdGrp}$ of preordered groups to its subcategory $\mathsf{Mono(Ab)}$ of abelian objects. We will see it as the composite of the following two functors

$$\mathsf{PreOrdGrp} \xrightarrow{C} \mathsf{PreOrdAb} \xrightarrow{A} \mathsf{Mono(Ab)},$$

where C is defined as in Section 2 and A is defined, for any preordered abelian group (G, P_G), by

$$A(G, P_G) = (G, grp(P_G))$$

with $grp(P_G)$ the *group completion* of the monoid P_G. As a consequence, for any $(G, P_G) \in \mathsf{PreOrdGrp}$,

$$F(G, P_G) = (ab(G), grp(\eta_G(P_G))).$$

$$\begin{tikzcd}
P_G \arrow[r, two heads, "\bar{\eta}_G"] \arrow[d, tail] \arrow[rr, bend left, "\hat{\eta}_G"] & \eta_G(P_G) \arrow[r, dotted, "j_G"] \arrow[d, tail] & grp(\eta_G(P_G)) \arrow[d, tail, dotted, "i_{ab(G)}"] \\
G \arrow[r, two heads, "\eta_G"'] \arrow[rr, bend right, two heads, "\eta_G"'] & ab(G) \arrow[r, equal] & ab(G)
\end{tikzcd} \tag{11}$$

Let us recall the general construction of the *group completion* (also called *Grothendieck group*) of an additive commutative monoid M. On the Cartesian product $M \times M$ one defines an equivalence relation $\sim$ in the following way: $(m_1, m_2) \sim (n_1, n_2)$ if and only if there exists an element k in M such that $m_1 + n_2 + k = m_2 + n_1 + k$. The Grothendieck group $grp(M)$ of M is then given by the quotient $(M \times M)/\sim$, which turns out to be an abelian group. Note that there is a monoid homomorphism $j \colon M \to grp(M)$ sending any element m of M to the equivalence class $[(m, 0)]$ (with respect to $\sim$). This homomorphism satisfies a universal property: for any monoid homomorphism $\phi \colon M \to X$ from M to an abelian group X, there is a unique group homomorphism $\psi \colon grp(M) \to X$ such that $\phi = \psi \cdot j$.

This universal property yields the existence of the unique morphism $i_{ab(G)}$ making the diagram (11) commute. The following lemma implies that $grp(\eta_G(P_G))$ is in addition a submonoid of $ab(G)$, so that $(ab(G), grp(\eta_G(P_G)))$ is then, indeed, a preordered group.

Lemma 3.11. — *Let M be a submonoid of an abelian group X and*

$$Y = \{x \in X \mid x = a - b, \text{ for } a, b \in M\},$$

which is a (normal) subgroup of X. Then there is a group isomorphism

$$grp(M) \cong Y,$$

which implies that $grp(M)$ is a (normal) subgroup of X.

Proof. — By definition, $grp(M) = (M \times M)/\sim$. Now, we observe that, for $(m_1, m_2), (n_1, n_2) \in M \times M$, $(m_1, m_2) \sim (n_1, n_2)$ if and only if there exists a k in M such that $m_1 + n_2 + k = m_2 + n_1 + k$. We can see this equality in the group X so that, by the cancellation property, $(m_1, m_2) \sim (n_1, n_2)$ if and only if $m_1 + n_2 = m_2 + n_1$, which is equivalent to $m_1 - m_2 = n_1 - n_2$. Accordingly, the group homomorphism $\Phi\colon grp(M) \to Y$ defined, for any $[(m_1, m_2)]$ in $grp(M)$, by $\Phi([(m_1, m_2)]) = m_1 - m_2$, is an isomorphism. □

4. The Galois theory corresponding to the group completion functor

If we take a look at the restriction of the functor A to the positive cones (i.e. to the category ComMon of commutative monoids), we then get the *group completion functor grp* studied in [15]. In this section, we recall the results of this article that will be useful for our work.

In their paper [15], Montoli, Rodelo and Van der Linden study the adjunction

$$\begin{tikzcd} \mathsf{Mon} \arrow[r, bend left, "grp"] \arrow[r, phantom, "\perp"] & \mathsf{Grp} \arrow[l, bend left, "mon"] \end{tikzcd} \tag{12}$$

between the categories Mon of monoids and Grp of groups, where the right adjoint *mon* is the forgetful functor, while the left adjoint *grp* is the *group completion functor*.

A significant part of the article [15] is devoted to the proof that the Galois structure

$$\Gamma_{grp} = (\mathsf{Mon}, \mathsf{Grp}, grp, mon, \mathscr{E}_{grp}, \mathscr{Z}_{grp})$$

is admissible when $\mathscr{E}_{grp}$ and $\mathscr{Z}_{grp}$ are the classes of surjective homomorphisms in Mon and Grp, respectively. At the end of the paper, a characterization of (Γ_{grp}-)normal and (Γ_{grp}-)central extensions is given; both notions coincide with the one of *special homogeneous surjection* that we are now going to recall.

Definition 4.1. — [3] Let $f\colon X \to Y$ be a split epimorphism in Mon, with section s and kernel k:

$$\begin{tikzcd} K \arrow[r, tail, "k"] & X \arrow[r, shift left, "f"] & Y. \arrow[l, shift left, "s"] \end{tikzcd} \tag{13}$$

The split epimorphism (f, s) is *homogeneous* when, for any $y \in Y$, the functions $\mu_y\colon K \to f^{-1}(y)$ and $\nu_y\colon K \to f^{-1}(y)$, defined, for any $x \in K$, by $\mu_y(x) = x + s(y)$ and $\nu_y(x) = s(y) + x$, are bijective.

Definition 4.2. — [3] Let $f\colon X \twoheadrightarrow Y$ be a surjective homomorphism in the category Mon of monoids. Consider its kernel pair $(Eq(f), \pi_1, \pi_2)$ with the diagonal Δ:

$$\begin{tikzcd} Eq(f) \arrow[r, shift left=2, "\pi_1"] \arrow[r, shift right=2, "\pi_2"'] & X \arrow[l, "\Delta" description] \arrow[r, two heads, "f"] & Y. \end{tikzcd}$$

The morphism f is said to be a *special homogeneous surjection* when (π_1, Δ) is a homogeneous split epimorphism.

Special homogeneous surjections are pullback-stable and, moreover, one has the following property:

Proposition 4.3. — [3] *Consider in* Mon *the following pullback where both f and f' are surjective homomorphisms:*

$$\begin{tikzcd} P \arrow[r, two heads, "f'"] \arrow[d, two heads, "g'"'] & Z \arrow[d, two heads, "g"] \\ X \arrow[r, two heads, "f"'] & Y. \end{tikzcd}$$

If g' is a special homogeneous surjection, then so is g.

We now recall in Theorem 4.5 the characterization of (Γ_{grp}-)normal and (Γ_{grp}-)central extensions considered in [15].

Proposition 4.4. — [15] *Consider an arbitrary split epimorphism (f, s) as in* (13)*. Then the following conditions are equivalent:*

1. *f is a (Γ_{grp}-)trivial extension.*
2. *f is a special homogeneous surjection.*

Theorem 4.5. — [15] *Let $f\colon X \twoheadrightarrow Y$ be a surjective homomorphism of monoids. Then the following conditions are equivalent:*

1. *f is a special homogeneous surjection.*
2. *f is a (Γ_{grp}-)normal extension.*
3. *f is a (Γ_{grp}-)central extension.*

5. The functor $F\colon \mathsf{PreOrdGrp} \to \mathsf{Mono(Ab)}$ and its induced admissible Galois structure

Proposition 5.1. — *There is an adjunction*

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{Mono(Ab)} \arrow[l, hook, bend left, "U"] \end{tikzcd} \tag{14}$$

between the category $\mathsf{PreOrdGrp}$ *of preordered groups and its full subcategory* $\mathsf{Mono}(\mathsf{Ab})$ *of abelian objects, where the right adjoint* U *is the inclusion functor, and the left adjoint* F *is defined as in Section 3: for any* $(G, P_G) \in \mathsf{PreOrdGrp}$*,*

$$F(G, P_G) = (ab(G) = G/[G,G], grp(\eta_G(P_G))).$$

Proof. — The adjunction (14) can be seen as the composite of the two adjunctions

$$\begin{tikzcd} \mathsf{PreOrdGrp} \arrow[r, bend left, "C"] \arrow[r, phantom, "\perp"] & \mathsf{PreOrdAb} \arrow[l, bend left, "V"] \arrow[r, bend left, "A"] \arrow[r, phantom, "\perp"] & \mathsf{Mono}(\mathsf{Ab}) \arrow[l, bend left, "W"] \end{tikzcd}$$

where the left-hand one has been studied in Section 2 and (the restriction to the positive cones of) the right-hand one has been considered in Section 4. Note that the (G, P_G)-component of the unit of the composite adjunction is given by the morphism $(\eta_G, \hat{\eta}_G)$ (in the notations of diagram (11)). $\square$

Remark 5.2. — *For a given preordered group* (G, P_G)*, the* (G, P_G)*-component* $(\eta_G, \hat{\eta}_G)$ *of the unit of the adjunction* (14) *is not a regular epimorphism, in general. The morphism* $(\eta_G, \hat{\eta}_G)$ *is just an epimorphism, since* η_G *is surjective, while* $\hat{\eta}_G$ *is not necessarily surjective. Note also that the morphism* $j_G \colon \eta_G(P_G) \to grp(\eta_G(P_G))$ *is a monomorphism because* $\eta_G(P_G)$ *is a commutative monoid with cancellation.*

Remark 5.3. — *The restriction to the positive cones of the adjunction* $A \dashv W$ *is not exactly the same as the adjunction* (12) *studied in* [15]*. As a matter of fact, we are considering the adjunction*

$$\begin{tikzcd} \mathsf{ComMon}_{\mathsf{can}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{Ab} \arrow[l, shift left] \end{tikzcd}$$

which is the restriction of (12) *to the subcategory* $\mathsf{ComMon}_{\mathsf{can}}$ *of commutative monoids with cancellation. However, the results of* [15] *that we use in this article can be easily adapted to this situation (see Remark 2.5 in* [15]*).*

Proposition 5.4. — *If* $\mathscr{E}$ *denotes the class of regular epimorphisms in* $\mathsf{PreOrdGrp}$ *and* $\mathscr{Z}$ *the class of regular epimorphisms in* $\mathsf{Mono}(\mathsf{Ab})$*, then*

$$\Gamma = (\mathsf{PreOrdGrp}, \mathsf{Mono}(\mathsf{Ab}), F, U, \mathscr{E}, \mathscr{Z}) \tag{15}$$

is a Galois structure.

Since the Galois structure (15) is a composite of two "compatible" Galois structures that are admissible, we may use the following known result in order to prove its admissibility (the argument given to prove Lemma 6.2 in [5], for instance, still holds in our situation).

Proposition 5.5. — *Consider the following chain of adjunctions*

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "J"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "L"] \arrow[r, bend left, "I"] \arrow[r, phantom, "\perp"] & \mathscr{A} \arrow[l, bend left, "H"] \end{tikzcd} \tag{16}$$

where $\mathscr{A}$ is a full subcategory of $\mathscr{B}$ and $\mathscr{B}$ a full subcategory of $\mathscr{C}$. Assume, moreover, that

- $\Gamma_J = (\mathscr{C}, \mathscr{B}, J, L, \mathscr{E}_J, \mathscr{Z}_J)$*;*
- $\Gamma_I = (\mathscr{B}, \mathscr{A}, I, H, \mathscr{E}_I, \mathscr{Z}_I)$

are admissible Galois structures that are "compatible" in the sense that $J(\mathscr{E}_J) \subset \mathscr{E}_I$ and $H(\mathscr{Z}_I) \subset \mathscr{Z}_J$. Then the composite of these two admissible Galois structures

$$\Gamma = (\mathscr{C}, \mathscr{A}, I \cdot J, L \cdot H, \mathscr{E}, \mathscr{Z}),$$

where $\mathscr{E} = \mathscr{E}_J$ and $\mathscr{Z} = \mathscr{Z}_I$, is an admissible Galois structure.

Corollary 5.6. — *The Galois structure* (15) *is admissible.*

6. Characterization of Γ-normal and Γ-central extensions

This section is devoted to Γ-normal and Γ-central extensions, which will be characterized in Theorem 6.3. The following two lemmas will be needed for the proof of this result.

Lemma 6.1. — *Consider a chain of adjunctions as in* (16) *and assume that we have admissible Galois structures Γ_J and Γ_I as in Proposition 5.5. Assume, moreover, that any component of the unit of the adjunction $J \dashv L$ is a descent morphism. Then, for an extension $f \colon X \to Y$ in $\mathscr{C}$, the following conditions are equivalent:*

1. *f is a Γ-trivial extension for the admissible Galois structure Γ described in Proposition 5.5.*
2. *f is a Γ_J-trivial extension and $J(f)$ is a Γ_I-trivial extension.*

Proof. — (1) ⇒ (2): by assumption, the square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & (I \cdot J)(X) \arrow[d, "(I \cdot J)(f)"] \\ Y \arrow[r, "\eta_Y"'] & (I \cdot J)(Y) \end{tikzcd}$$

is a pullback in $\mathscr{C}$. Note that this pullback decomposes into the following two squares:

$$\begin{tikzcd} X \arrow[r, "\eta^J_X"] \arrow[d, "f"'] & J(X) \arrow[r, "\eta^I_{J(X)}"] \arrow[d, "J(f)"] & (I\cdot J)(X) \arrow[d, "(I\cdot J)(f)"] \\ Y \arrow[r, "\eta^J_Y"'] & J(Y) \arrow[r, "\eta^I_{J(Y)}"'] & (I\cdot J)(Y) \end{tikzcd} \tag{17}$$

(where η^J and η^I are the units of the adjunctions $J \dashv L$ and $I \dashv H$, respectively). Equivalently, the fact that f is a Γ-trivial extension can also be formulated by saying that

$$f = (L\cdot H)^Y(g)$$

for some extension $g\colon A \to (I\cdot J)(Y)$ in $\mathscr{A}$. Now,

$$(L\cdot H)^Y(g) = L^Y(H^{J(Y)}(g)),$$

so that $f = L^Y(\bar{g})$, for an extension $\bar{g} = H^{J(Y)}(g)$ in $\mathscr{B}$ with codomain $J(Y)$. This means precisely that f is a Γ_J-trivial extension; hence in the diagram (17) both the left-hand square and the external rectangle are pullbacks. Since η^J_Y is a descent morphism, it then follows that also the right-hand square is a pullback, meaning that $J(f)$ is a Γ_I-trivial extension.

(2) $\Rightarrow$ (1): statement (2) means that both squares in diagram (17) are pullbacks, so that the external rectangle is also a pullback, and f is a Γ-trivial extension. □

Lemma 6.2. — *Let* $(f,\bar{f})\colon (G,P_G) \twoheadrightarrow (H,P_H)$ *be a regular epimorphism in* PreOrdGrp*. Then the following conditions are equivalent:*

1. *for any* $(x,y) \in Eq(\bar{f})$*,* $y-x \in P_G$ *and* $-x+y \in P_G$*;*
2. $\bar{f}$ *is a special homogeneous surjection in* Mon*;*
3. *for any* $(x,y) \in Eq(\bar{f})$*,* $y-x \in P_G$*;*
4. $\bar{f}$ *is a special Schreier surjection in* Mon*.*

Proof. — The surjective homomorphism $\bar{f}$ is special homogeneous precisely when $Eq(\bar{f}) \underset{\bar{\Delta}}{\overset{\bar{\pi}_1}{\rightleftarrows}} P_G$ is a homogeneous split epimorphism (see Definition 4.2). This happens if and only if, for any $x \in P_G$, the functions

$$\mu_x\colon \mathsf{Ker}(\bar{\pi}_1) \cong \mathsf{Ker}(\bar{f}) \to \bar{\pi}_1^{-1}(x); (0,a) \mapsto (0,a) + \bar{\Delta}(x) = (x, a+x)$$

and

$$\nu_x\colon \mathsf{Ker}(\bar{\pi}_1) \cong \mathsf{Ker}(\bar{f}) \to \bar{\pi}_1^{-1}(x); (0,b) \mapsto \bar{\Delta}(x) + (0,b) = (x, x+b)$$

are bijective. This is equivalent to the fact that, for any $(x,y) \in Eq(\bar{f})$, there exist a unique $a \in \mathsf{Ker}(\bar{f})$ and a unique $b \in \mathsf{Ker}(\bar{f})$ such that $(x,y) = (x, a+x)$ and $(x,y) = (x, x+b)$. This condition also amounts to asking for the existence of a unique $a \in \mathsf{Ker}(f) \cap P_G$ and a unique $b \in \mathsf{Ker}(f) \cap P_G$ such that $y = a + x$ and $y = x + b$, so that a and b are elements of G of the form $a = y - x$ and $b = -x + y$. As a consequence, the surjective homomorphism $\bar{f}$ is special homogeneous if and only if, for any $(x,y) \in Eq(\bar{f})$, $y - x \in P_G$ and $-x + y \in P_G$. This proves the equivalence between (1) and (2).

The equivalence between (3) and (4) is proved similarly. Indeed, the fact that $\bar{f}$ is a special Schreier surjection is equivalent to the fact that the function μ_x defined above is bijective for any $x \in P_G$.

It remains to check that condition (3) implies condition (1). Let $(x,y) \in Eq(\bar{f})$. Then, by assumption, we already have that $y - x \in P_G$. Now, we compute that $-x + y = -x + y + (-x + x) = -x + (y - x) + x$. This allows one to conclude that $-x + y \in P_G$ since P_G is closed in G under conjugation. $\square$

Theorem 6.3. — *Let* $(f, \bar{f})\colon (G, P_G) \twoheadrightarrow (H, P_H)$ *be a regular epimorphism in* $\mathsf{PreOrdGrp}$. *Then the following conditions are equivalent:*

(1) (a) $\mathsf{Ker}(f) \subseteq Z(G)$;
  (b) $\bar{f}$ *is a special homogeneous surjection in* Mon.
(2) (a) $\mathsf{Ker}(f) \subseteq Z(G)$;
  (b) $\bar{f}$ *is a special Schreier surjection in* Mon.
(3) $(f, \bar{f})$ *is a* $(\Gamma\text{-})$*normal extension.*
(4) $(f, \bar{f})$ *is a* $(\Gamma\text{-})$*central extension.*

Proof. — (1) $\Leftrightarrow$ (2): this follows directly from Lemma 6.2.

(1) $\Rightarrow$ (3): first remark that condition (b) implies that $a - b + c \in P_G$ for any $(a,b,c) \in Eq(\bar{\eta}_G) \times_{P_G} Eq(\bar{f})$. Indeed, thanks to Lemma 6.2, we know that $-b + c \in P_G$ since $(b,c) \in Eq(\bar{f})$, and then $a - b + c \in P_G$ since $a \in P_G$, and P_G is a submonoid of G. So condition (b) implies condition (ii) of Theorem 2.9, and it then follows, thanks to the validity of condition (a), that $(f, \bar{f})$ is a Γ_C-normal extension. In other words, the first projection $(\pi_1, \bar{\pi}_1)\colon (Eq(f), Eq(\bar{f})) \twoheadrightarrow (G, P_G)$ of the kernel pair of $(f, \bar{f})$ is a Γ_C-trivial extension. According to Lemma 6.1, it now remains to prove that $C(\pi_1, \bar{\pi}_1)$ is a Γ_A-trivial extension (where Γ_A denotes the admissible Galois structure associated with the reflection $A \dashv W$). By condition (b), $\bar{f}$ is a special homogeneous surjection, which implies that also $\bar{\pi}_1$ is a special homogeneous surjection by pullback-stability. This, in turn, implies that the restriction $C(\bar{\pi}_1)$ of $C(\pi_1, \bar{\pi}_1)$ to the positive cones is a special homogeneous

surjection thanks to Proposition 4.3. Indeed, the square

$$\begin{tikzcd}
Eq(\bar{f}) \arrow[r, two heads, "\bar{\eta}_{Eq(f)}"] \arrow[d, two heads, "\bar{\pi}_1"'] & \eta_{Eq(f)}(Eq(\bar{f})) \arrow[d, two heads, "C(\bar{\pi}_1)"] \\
P_G \arrow[r, two heads, "\bar{\eta}_G"'] & \eta_G(P_G)
\end{tikzcd}$$

is a pullback of regular epimorphisms in Mon, since $(\pi_1, \bar{\pi}_1)$ is a Γ_C-trivial extension. Accordingly, $C(\bar{\pi}_1)$ is a split epimorphism, which is a special homogeneous surjection, and is then a Γ_{grp}-trivial extension by Proposition 4.4. This means that the square

$$\begin{tikzcd}
\eta_{Eq(f)}(Eq(\bar{f})) \arrow[r, tail, "j_{Eq(f)}"] \arrow[d, two heads, "C(\bar{\pi}_1)"'] & grp(\eta_{Eq(f)}(Eq(\bar{f}))) \arrow[d, two heads, "grp(C(\bar{\pi}_1))"] \\
\eta_G(P_G) \arrow[r, tail, "j_G"'] & grp(\eta_G(P_G))
\end{tikzcd}$$

is a pullback in the category ComMon of commutative monoids. As a consequence, the square

$$\begin{tikzcd}
(ab(Eq(f)), \eta_{Eq(f)}(Eq(\bar{f}))) \arrow[r, tail, "(1, j_{Eq(f)})"] \arrow[d, two heads, "C(\pi_1, \bar{\pi}_1)"'] & (ab(Eq(f)), grp(\eta_{Eq(f)}(Eq(\bar{f})))) \arrow[d, two heads, "(A\cdot C)(\pi_1, \bar{\pi}_1) = F(\pi_1, \bar{\pi}_1)"] \\
(ab(G), \eta_G(P_G)) \arrow[r, tail, "(1, j_G)"'] & (ab(G), grp(\eta_G(P_G)))
\end{tikzcd}$$

is a pullback in the category $\mathsf{PreOrdAb}$ of preordered abelian groups, and this proves that $C(\pi_1, \bar{\pi}_1)$ is a Γ_A-trivial extension. By Lemma 6.1, we conclude that $(\pi_1, \bar{\pi}_1)$ is a Γ-trivial extension, which is equivalent to saying that $(f, \bar{f})$ is a Γ-normal extension.

(3) $\Rightarrow$ (4): any normal extension is central by definition.

(4) $\Rightarrow$ (1): by definition of a (Γ-)central extension, there exists an effective descent morphism (i.e. a regular epimorphism) $(p, \bar{p}) \colon (E, P_E) \twoheadrightarrow (H, P_H)$ in $\mathsf{PreOrdGrp}$ such that the pullback $(p, \bar{p})^*(f, \bar{f})$ (which we will denote by $(\pi_1, \bar{\pi}_1)$) of $(f, \bar{f})$ along $(p, \bar{p})$ is a Γ-trivial extension. Using Lemma 6.1, this means that

(α) $(\pi_1, \bar{\pi}_1)$ is a Γ_C-trivial extension, and

(β) $C(\pi_1, \bar{\pi}_1)$ is a Γ_A-trivial extension.

The first statement (α) means, of course, that $(f, \bar{f})$ is a Γ_C-central extension. In particular, f is algebraically central, that is, $f \colon G \twoheadrightarrow H$ satisfies condition (a). The second statement (β) now implies that the square

$$\begin{tikzcd}
(ab(P), \eta_P(P_P)) \arrow[r, tail, "{(1,j_P)}"] \arrow[d, two heads, "{(C(\pi_1),C(\bar{\pi}_1))=C(\pi_1,\bar{\pi}_1)}"'] & (ab(P), grp(\eta_P(P_P))) \arrow[d, two heads, "{(A\cdot C)(\pi_1,\bar{\pi}_1)=F(\pi_1,\bar{\pi}_1)}"] \\
(ab(E), \eta_E(P_E)) \arrow[r, tail, "{(1,j_E)}"'] & (ab(E), grp(\eta_E(P_E)))
\end{tikzcd}$$

(where $P = E \times_H G$ and $P_P = P_E \times_{P_H} P_G$) is a pullback in $\mathsf{PreOrdAb}$. In particular, its restriction to ComMon

$$\begin{tikzcd}
\eta_P(P_P) \arrow[r, tail, "j_P"] \arrow[d, two heads, "C(\bar{\pi}_1)"'] & grp(\eta_P(P_P)) \arrow[d, two heads, "grp(C(\bar{\pi}_1))"] \\
\eta_E(P_E) \arrow[r, tail, "j_E"'] & grp(\eta_E(P_E))
\end{tikzcd}$$

is a pullback. This means that $C(\bar{\pi}_1)$ is a Γ_{grp}-trivial extension and then a Γ_{grp}-normal extension. Indeed, since Γ_{grp} is admissible, any Γ_{grp}-trivial extension is Γ_{grp}-normal. Thanks to Theorem 4.5, we can conclude that $C(\bar{\pi}_1)$ is a special homogeneous surjection. Since special homogeneous surjections are pullback-stable, it then follows that $\bar{\pi}_1$ has this same property. Indeed, the statement (α) implies (among other things) that the square

$$\begin{tikzcd}
P_P \arrow[r, two heads, "\bar{\eta}_P"] \arrow[d, two heads, "\bar{\pi}_1"'] & \eta_P(P_P) \arrow[d, two heads, "C(\bar{\pi}_1)"] \\
P_E \arrow[r, two heads, "\bar{\eta}_E"'] & \eta_E(P_E)
\end{tikzcd}$$

is a pullback in Mon. Applying now Proposition 4.3 to the following pullback of regular epimorphisms

$$\begin{tikzcd}
P_P \arrow[r, two heads, "\bar{\pi}_2"] \arrow[d, two heads, "\bar{\pi}_1"'] & P_G \arrow[d, two heads, "\bar{f}"] \\
P_E \arrow[r, two heads, "\bar{p}"'] & P_H,
\end{tikzcd}$$

we obtain that $\bar{f}$ is a special homogeneous surjection, i.e. condition (b) is satisfied. This concludes the proof. $\square$

BIBLIOGRAPHY

[1] M. Barr – "Exact categories", in *Exact categories and categories of sheaves*, Lecture notes in mathematics, no. 236, Springer-Verlag, 1971.

[2] F. Borceux & D. Bourn – *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and Its Applications, no. 566, Kluwer, 2004.

[3] D. Bourn, N. Martins-Ferreira, A. Montoli & M. Sobral – *Schreier split epimorphisms in monoids and in semirings*, Textos de Matemática (Série B), no. 45, Departamento de Matemática da Universidade de Coimbra, 2014.

[4] M. Clementino, N. Martins-Ferreira & A. Montoli – "On the categorical behaviour of preordered groups", *J. Pure Appl. Algebra* **223** (2019), p. 4226–4245.

[5] M. Duckerts, T. Everaert & M. Gran – "A description of the fundamental group in terms of commutators and closure operators", *J. Pure Appl. Algebra* **216** (2012), no. 8–9, p. 1837–1851.

[6] M. Gran & D. Rodelo – "On the characterization of Jonsson-Tarski and of subtractive varieties", *Diagrammes* **S67-68** (2012), p. 101–115.

[7] G. Janelidze – "Pure Galois theory in categories", *J. Algebra* **132** (1990), no. 2, p. 270–286.

[8] ______, *Categorical Galois theory: revision and some recent developments*, Math. Appl., vol. 565, Kluwer Acad. Publ., 2004.

[9] ______, "Galois groups, abstract commutators, and Hopf formula", *Appl. Categor. Struct.* **16** (2008), p. 653–668.

[10] G. Janelidze & G. Kelly – "Galois theory and a general notion of central extension", *J. Pure Appl. Algebra* **97** (1994), p. 135–161.

[11] G. Janelidze, M. Sobral & W. Tholen – "Beyond Barr exactness: effective descent morphisms", in *Categorical Foundations* (M. Pedicchio & W. Tholen, eds.), Encyclopedia of Mathematics and Its Applications, 2004, p. 359–405.

[12] G. Janelidze & W. Tholen – "Facets of descent, II", *Appl. Categ. Struct.* **5** (1997), p. 229–248.

[13] Z. Janelidze – "Subtractive categories", *Appl. Categor. Struct.* **13** (2005), p. 343–350.

[14] ______, "The pointed subobject functor, 3×3 lemmas, and subtractivity of spans", *Theory Appl. Categ.* **23** (2010), no. 11, p. 221–242.

[15] A. Montoli, D. Rodelo & T. Van der Linden – "A Galois theory for monoids", *Theory Appl. Categ.* **29** (2014), no. 7, p. 198–214.