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CHARACTERISATION OF THE POLES OF THE ℓ-MODULAR ASAI L-FACTOR

by Robert Kurinczuk & Nadir Matringe

Abstract. — Let $F/F_{\circ}$ be a quadratic extension of non-archimedean local fields of odd residual characteristic, set $G = GL_n(F)$, $G_{\circ} = GL_n(F_{\circ})$ and let ℓ be a prime number different from the residual characteristic of F. For a complex cuspidal representation π of G, the Asai L-factor $L_{As}(X, \pi)$ has a pole at $X = 1$, if and only if π is $G_{\circ}$-distinguished. In this paper, we solve the problem of characterising the occurrence of a pole at $X = 1$ of $L_{As}(X, \pi)$ when π is an ℓ-modular cuspidal representation of G; we show that $L_{As}(X, \pi)$ has a pole at $X = 1$, if and only if π is a *relatively banal* distinguished representation, namely π is $G_{\circ}$-distinguished but not $|\det(\ )|_{F_{\circ}}$-distinguished. This notion turns out to be an exact analogue for the symmetric space $G/G_{\circ}$ of Mínguez and Sécherre's notion of banal cuspidal $\overline{\mathbb{F}_{\ell}}$-representation of $G_{\circ}$. Along the way, we compute the Asai L-factor of all cuspidal ℓ-modular representations of G in terms of type theory and prove new results concerning lifting and reduction modulo ℓ of distinguished cuspidal representations. Finally, we determine when the natural $G_{\circ}$-period on the Whittaker model of a distinguished cuspidal representation of G is non-zero.

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Résumé (*Caractérisation des pôles du facteur d'Asai ℓ-modulaire*). — Soit F/F_o une extension quadratique de corps locaux non archimédiens de caractéristique résiduelle impaire. Posons $G = GL_n(F)$, $G_\mathsf{o} = GL_n(F_\mathsf{o})$ et soit ℓ un nombre premier différent de la caractéristique résiduelle de F. Pour une représentation cuspidale complexe π de G, le facteur L d'Asai $L_{As}(X, \pi)$ admet un pôle en $X = 1$ si et seulement si π est G_o-distinguée. Dans cet article nous résolvons le problème de l'occurence d'un pôle en $X = 1$ de $L_{As}(X, \pi)$ quand π est une représentation cuspidale ℓ-modulaire de G: dans ce cas $L_{As}(X, \pi)$ admet un pôle en $X = 1$ si et seulement si π est *relativement banale* distinguée; autrement dit π est G_o-distinguée mais pas $|\det(\ )|_{F_\mathsf{o}}$-distinguée. Cette notion est l'analogue pour l'espace symétrique G/G_o de la notion de cuspidale banale introduite par Mínguez et Sécherre pour les $\overline{\mathbb{F}_\ell}$-représentations de G_o. En cours de route, on calcule le facteur L d'Asai des représentations cuspidales ℓ-modulaires de G par la théorie des types, et on prouve de nouveaux résultats concernant le relèvement et la réduction modulo ℓ des représentations cuspidales distinguées. Finalement, on détermine quand la G_o-période sur le modèle de Whittaker d'une représentation cuspidale distinguée de G est non nulle.

1. Introduction

Let F/F_o be a quadratic extension of non-archimedean local fields of residual characteristic $p \neq 2$ and set $G = GL_n(F)$ and $G_\mathsf{o} = GL_n(F_\mathsf{o})$. An irreducible representation of G is said to be *distinguished by* G_o, if it possesses a non-zero G_o-invariant linear form. In the case of complex representations, the equality of the Asai L-factor defined by the Rankin–Selberg method and its Galois avatar ([3], [21]) provides a bridge between functorial lifting from the quasi-split unitary group $U_n(F/F_\mathsf{o})$ and G_o-distinction of discrete series representations of G; a discrete series of G is a (stable or unstable depending on the parity of n) lift of a (necessarily discrete series) representation of $U_n(F/F_\mathsf{o})$, if and only if the Asai L-factor of its Galois parameter has a pole at $X = 1$ ([25], [10]), whereas it is G_o-distinguished, if and only if its Asai L-factor obtained by the Rankin–Selberg method has a pole at $X = 1$ ([14], [1]).

Recently, motivated by the study of congruences between automorphic representations, there has been great interest in studying representations of G on vector spaces over fields of positive characteristic ℓ. There are two very different cases: when $\ell = p$ and when $\ell \neq p$. This article focuses on the latter $\ell \neq p$ case, where there is a theory of Haar measure that allows us to define Asai L-factors via the Rankin–Selberg method as in the complex case (Section 7).

The aim of this article is to show that in this case, a connection remains between the poles of Asai L-factor and distinction; however, this connection no longer characterises distinction, but a more subtle notion, which we call a *relatively banal* distinction. The easiest way to state that a cuspidal distinguished ℓ-modular representation is relatively banal is to say that it is not $|\det(\ \cdot\ )|_{F_\mathsf{o}}$-distinguished, where $|\det(\ \cdot\ )|_{F_\mathsf{o}}$ is considered as an $\overline{\mathbb{F}_\ell}$-valued character, but other compact definitions can also be given in terms of type theory, as well as in terms of its supercuspidal lifts:

PROPOSITION 1.1 (Definition 6.2, Theorem 6.11 and Corollary 6.3). — *Let π be an ℓ-modular cuspidal distinguished representation of* G*. Then, the following are equivalent, and when they are satisfied, we say that π is* relatively banal*:*

(i) *π is not $|\det(\,\cdot\,)|_{\mathrm{F}_\circ}$-distinguished.*
(ii) *All supercuspidal lifts of π are distinguished by an unramified character of* $\mathrm{G}_\circ$*.*
(iii) $q_\circ^{n/e_\circ(\pi)} \not\equiv 1[\ell]$*, where $e_\circ(\pi)$ denotes the invariant associated to π in* [4, Lemma 5.10] *(see Section 5.2).*

Relatively banal for $\mathrm{G}_\circ$-distinguished cuspidal representations turns out to be the exact analogue of the definition of *banal cuspidal representations* of $\mathrm{G}_\circ$ (see [24, Remark 8.15] and [23]) after one identifies the cuspidal (irreducible) representations of $\mathrm{G}_\circ$ with the $\Delta(\mathrm{G}_\circ)$-distinguished cuspidal (irreducible) representations of $\mathrm{G}_\circ \times \mathrm{G}_\circ$, where $\Delta : \mathrm{G}_\circ \to \mathrm{G}_\circ \times \mathrm{G}_\circ$ is the diagonal embedding, as we explain in Section 8.3.

The main theorem of this paper characterises the poles of the Asai L-factor:

THEOREM 1.2 (Theorem 8.1). — *Let π be a cuspidal ℓ-modular representation, then $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ has a pole at $\mathrm{X}=1$, if and only if π is relatively banal distinguished.*

Note that the proof of the above theorem is completely different from the proof of characterisation theorem in the complex case (see Remark 8.2 for more on the comparison of the proofs). Here, we show that the Asai L-factor of a cuspidal ℓ-modular representation is equal to 1 whenever π is not the unramified twist of a relatively banal representation using Theorem 6.11, which is the characterisation of relatively banal in terms of supercuspidal lifts. Then, when π is the unramified twist of relatively banal representation, following our paper [18], we get an explicit formula for $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ in Theorem 7.8 from the test vector computation of [4], which due to the relatively banal assumption (more precisely its type theory version) reduces modulo ℓ without vanishing. We then deduce Theorem 1.2 from this computation, together with the computation of the group of unramified characters μ of $\mathrm{G}_\circ$, such that π is μ-distinguished (Corollary 5.17).

Finally, denoting by N the unipotent radical of the group of upper triangular matrices in G, by $\mathrm{Z}_\circ$ the centre of $\mathrm{G}_\circ$ and by $\mathrm{N}_\circ$ the group $\mathrm{N}\cap\mathrm{G}_\circ$, the most natural $\mathrm{G}_\circ$-invariant linear form to consider on the Whittaker model of an ℓ-modular cuspidal representation π with respect to a distinguished nondegenerate character of N trivial on $\mathrm{N}_\circ$ is the local period

$$\mathcal{L}_\pi : \mathrm{W} \mapsto \int_{\mathrm{Z}_\circ\mathrm{N}_\circ\backslash\mathrm{G}_\circ} \mathrm{W}(h)dh.$$

In fact, this period plays an essential role in the proof of Theorem 1.2 over the field of complex numbers (see Remark 8.2). One of the main differences in the

ℓ-modular setting is that $\mathcal{L}_\pi$ can be zero even when π is distinguished. Not only do we show that it can vanish but we say exactly when it vanishes:

THEOREM 1.3 (Theorem 8.3). — *Let π be a cuspidal distinguished ℓ-modular representation of $\mathrm{GL}_n(\mathrm{F})$. Then, the local period $\mathcal{L}_\pi$ is non-zero, if and only if the following two properties are satisfied:*

(i) *π is relatively banal.*
(ii) *ℓ does not divide $e_{\circ}(\pi)$; in other words, if $\widetilde{\pi}$ is a lift of π, the ℓ-adic valuation of n is the same as the ℓ-adic valuation of the number of ℓ-adic unramified characters $\widetilde{\mu}_{\circ}$ of $\mathrm{G}_{\circ}$, such that $\widetilde{\pi}$ is $\widetilde{\mu}_{\circ}$-distinguished.*

This theorem is related to the vanishing modulo ℓ of a rather interesting and subtle scalar related, after fixing an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_\ell}$, to a quotient of the formal degree of a complex cuspidal representation of a unitary group by the formal degree of its base change to $\mathrm{GL}_n(\mathrm{F})$, see Remark 8.4 for a precise statement.

In light of Theorem 1.2, the role of the Asai L-factor in the study of distinguished representations will be less important in the ℓ-modular setting, as some ℓ-modular distinguished representations have Asai L-factors equal to 1 in the cuspidal case already, and new ideas will be required already for non-relatively banal distinguished cuspidal representations. We will focus on the general case of distinguished irreducible ℓ-modular representations, restricting it to small rank in the paper [19].

Finally, we mention that this paper relies heavily on the results from [4] and [27] and can be seen as a natural continuation of the themes developed in these two papers. In particular, our section on lifting distinction for cuspidal representations of finite general linear groups uses the same techniques as [27], and the statements that we obtain here were known to the author of [27].

2. Notation

Let $\mathrm{F}/\mathrm{F}_{\circ}$ be a quadratic extension of non-archimedean local fields of odd residual characteristic p. For any finite extension $\mathrm{E}/\mathrm{F}_{\circ}$, we let $|\ |_{\mathrm{E}}$ be the absolute value, $\mathrm{val}_{\mathrm{E}}$ the additive valuation, $\mathscr{O}_{\mathrm{E}}$ denote the ring of integers of E, with maximal ideal $\mathscr{P}_{\mathrm{E}}$, residue field k_{E}, and put $q_{\mathrm{E}} = \#\mathrm{k}_{\mathrm{E}}$. We put $|\ | = |\ |_{\mathrm{F}}$, $\mathrm{val} = \mathrm{val}_{\mathrm{F}}$, $\mathscr{O} = \mathscr{O}_{\mathrm{F}}$, $\mathscr{P} = \mathscr{P}_{\mathrm{F}}$, $\mathrm{k} = \mathrm{k}_{\mathrm{F}}$, $q = q_{\mathrm{F}}$, $|\ |_{\circ} = |\ |_{\mathrm{F}_{\circ}}$, $\mathrm{val}_{\circ} = \mathrm{val}_{\mathrm{F}_{\circ}}$, $\mathscr{O}_{\circ} = \mathscr{O}_{\mathrm{F}_{\circ}}$, $\mathscr{P}_{\circ} = \mathscr{P}_{\mathrm{F}_{\circ}}$, $\mathrm{k}_{\circ} = \mathrm{k}_{\mathrm{F}_{\circ}}$ and $q_{\circ} = q_{\mathrm{F}_{\circ}}$.

We let ℓ denote a prime not equal to p. Set $\overline{\mathbb{Q}_\ell}$ to be an algebraic closure of the ℓ-adic numbers, $\overline{\mathbb{Z}_\ell}$ its ring of integers, and $\overline{\mathbb{F}_\ell}$ its residue field.

Let G be the F-points of a connected reductive group defined over F and $\mathcal{G}$ be the k-points of a connected reductive group defined over k.

All representations considered are assumed to be smooth. We consider representations of G and $\mathcal{G}$ and their subgroups on $\overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{F}_\ell}$-vector spaces and the relations between them. We let R denote either $\overline{\mathbb{Q}_\ell}$ or $\overline{\mathbb{F}_\ell}$, so that we can make statements valid in both cases more briefly. By an R-representation we mean a representation on an R-vector space.

An R-representation of G or $\mathcal{G}$ is called *cuspidal*, if it is irreducible and does not appear as a quotient of any representation parabolically induced from an irreducible representation of a proper Levi subgroup. It is called *supercuspidal*, if it is irreducible and does not appear as a subquotient of any representation parabolically induced from an irreducible representation of a proper Levi subgroup. Over $\overline{\mathbb{Q}_\ell}$ a representation of G or $\mathcal{G}$ is cuspidal, if and only if it is supercuspidal; however, this is not the case over $\overline{\mathbb{F}_\ell}$, see [28, III] and [16] for examples of cuspidal non-supercuspidal representations.

3. Background on integral representations and distinction

Definition 3.1. — Let G be a locally profinite group and H be a closed subgroup of G. Let π be an R-representation of G and $\chi : \mathrm{H} \to \mathrm{R}^\times$ be a character. We say that π is *χ-distinguished*, if $\mathrm{Hom}_{\mathrm{H}}(\pi, \chi) \neq 0$. We say that π is *distinguished*, if it is 1-distinguished where 1 denotes the trivial character of H.

We will mainly consider cases where H is the group of fixed points $\mathrm{G}^\sigma = \{g \in \mathrm{G} : \sigma(g) = g\}$ of an involution σ. In this case, for any subset $\mathrm{X} \subset \mathrm{G}$, we set $\mathrm{X}^\sigma = \mathrm{X} \cap \mathrm{G}^\sigma$.

Definition 3.2. — We call a triple $(\mathrm{G}, \mathrm{H}, \sigma)$ *an* F*-symmetric pair* when

(i) $\mathrm{G} = \mathbb{G}(\mathrm{F})$ with $\mathbb{G}$ a connected reductive group defined over F, and σ is an involution of $\mathbb{G}$ defined over F.
(ii) H is an open subgroup of the group G^σ.

The main symmetric pair of interest in this note will be $(\mathrm{GL}_n(\mathrm{F}), \mathrm{GL}_n(\mathrm{F}_\circ), \sigma)$, where σ is the involution induced from the non-trivial element of $\mathrm{Gal}(\mathrm{F}/\mathrm{F}_\circ)$. Two important basic results concerning this pair, which we shall use later, are the following ([9], [26] for $\overline{\mathbb{Q}_\ell}$-representations, extended to R-representations in [27, Theorem 3.1]):

Proposition 3.3. — *Let π be an irreducible* R*-representation of* $\mathrm{GL}_n(\mathrm{F})$*, then*

$$\dim(\mathrm{Hom}_{\mathrm{GL}_n(\mathrm{F}_\circ)}(\pi, \mathrm{R})) \leqslant 1.$$

Moreover, if this dimension is equal to 1, then $\pi^\vee \simeq \pi^\sigma$.

Let K be a locally profinite group. An irreducible $\overline{\mathbb{Q}_\ell}$-representation π of K is called *integral*, if it stabilises a $\overline{\mathbb{Z}_\ell}$-lattice in its vector space. An integral irreducible $\overline{\mathbb{Q}_\ell}$-representation π that stabilises a lattice L induces an $\overline{\mathbb{F}_\ell}$-representation on the space $\mathrm{L} \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{F}_\ell}$. When K is either a profinite group or the F-points of a connected reductive F-group, the semi-simplification $r_\ell(\pi) =$ of

$\mathrm{L}\otimes_{\overline{\mathbb{Z}_\ell}}\overline{\mathbb{F}_\ell}$ is independent of the choice of L and is called the *reduction modulo ℓ* of π ([28, 9.6] in the profinite setting, where all representations are automatically defined over a finite extension of $\overline{\mathbb{F}_\ell}$, or [30, Theorem 1] in the context of reductive groups). Given an irreducible $\overline{\mathbb{F}_\ell}$-representation $\overline{\pi}$ of K, we will call any integral irreducible $\overline{\mathbb{Q}_\ell}$-representation π of K that satisfies $r_\ell(\pi)=\overline{\pi}$ a *lift* of $\overline{\pi}$.

We shall see later that the distinction of cuspidal representations of G does not always lift, i.e. that an ℓ-modular cuspidal distinguished representation may have no distinguished lifts. However, we have the following general result, which shows that distinction reduces modulo ℓ:

Theorem 3.4. — *Let $(\mathrm{G},\mathrm{H},\sigma)$ be an* F*-symmetric pair. Let π be an integral ℓ-adic supercuspidal representation of* G *and let χ be an integral character of* H*. Then, if π is χ-distinguished, the representation $r_\ell(\pi)$ is $r_\ell(\chi)$-distinguished.*

Proof. — Note that χ coincides with the central character of π restricted to H, which is also integral on $\mathrm{Z_G}\cap\mathrm{H}$ (where $\mathrm{Z_G}$ is the centre of G), and, hence, we extend it to a character still denoted χ to $\mathrm{Z_GH}$: $\chi(zh)=c_\pi(z)\chi(h)$. Note that $\mathrm{Hom_H}(\pi,\chi)=\mathrm{Hom_{Z_GH}}(\pi,\chi)$. By [15, Proposition 8.1] for $\chi=1$, extended to general χ in [8, Theorem 4.4], for L a non-zero element of $\mathrm{Hom_H}(\pi,\chi)$, the map

$$v\mapsto(g\mapsto L(\pi(g)v))$$

embeds π as a submodule of $\mathrm{ind}_{\mathrm{Z_GH}}^{\mathrm{G}}(\chi)$. Now, by [30, Proposition II.3], $\mathrm{ind}_{\mathrm{Z_GH}}^{\mathrm{G}}(\chi,\overline{\mathbb{Z}_\ell})$ is an integral structure in $\mathrm{ind}_{\mathrm{Z_GH}}^{\mathrm{G}}(\chi)$, and, hence, its intersection π_e with π is an integral structure of π by [28, 9.3] (note that Vignéras works over a finite extension of $\overline{\mathbb{F}_\ell}$, but her results apply here because both π and χ, and, hence, both π and $\mathrm{ind}_{\mathrm{Z_GH}}^{\mathrm{G}}(\chi)$ have E-structures by [28, Section 4]). So, $\pi_e\subset\mathrm{ind}_{\mathrm{Z_GH}}^{\mathrm{G}}(\chi,\overline{\mathbb{Z}_\ell})$, but the map $\Lambda: f\mapsto f(1_{\mathrm{G}})$ is an element of $\mathrm{Hom_H}(\mathrm{ind}_{\mathrm{Z_GH}}^{\mathrm{G}}(\chi,\overline{\mathbb{Z}_\ell}),\chi)$, which is non-zero on any submodule of $\mathrm{ind}_{\mathrm{Z_GH}}^{\mathrm{G}}(\chi,\overline{\mathbb{Z}_\ell})$, in particular on π_e. Up to multiplying Λ by an appropriate non-zero scalar in $\overline{\mathbb{Q}_\ell}$, we can suppose that $\Lambda(\pi_e)=\overline{\mathbb{Z}_\ell}$, and Λ induces a non-zero element of $\mathrm{Hom_H}(\pi_e\otimes\overline{\mathbb{F}_\ell},r_\ell(\chi))$. The result follows. □

Remark 3.5. — If K′ is a closed subgroup of a profinite group K, (smooth) finite dimensional $\overline{\mathbb{Q}_\ell}$-representations of K are always integral and the image of a lattice by a non-zero linear form on such a representation is obviously a lattice of $\overline{\mathbb{Q}_\ell}$, so the reduction modulo ℓ of a (K',χ)-distinguished finite-dimensional $\overline{\mathbb{Q}_\ell}$-representation of K is $(\mathrm{K}',r_\ell(\chi))$-distinguished.

Remark 3.6. — The following observation sheds more light on Theorem 3.4 when $\mathrm{G}=\mathrm{GL}_n(\mathrm{F})$. Let π be an integral supercuspidal $\overline{\mathbb{Q}_\ell}$-representation of $\mathrm{GL}_n(\mathrm{F})$, then its reduction modulo ℓ is (irreducible and) cuspidal, by [28, III 4.25]. This, however, is not true in general, see [16] for an example of an integral supercuspidal $\overline{\mathbb{Q}_\ell}$-representation whose reduction modulo ℓ is reducible.

Let K be a locally profinite group and K′ a closed subgroup. While, in general, it appears to be a subtle question to ascertain when the distinction of $\overline{\mathbb{F}_\ell}$-representations of K lifts, there is, however, one elementary case where it does: when the subgroup for which we want to study distinction K′ is profinite of pro-order prime to ℓ. In this case, an ℓ-modular finite dimensional (smooth) representation of K′ is semi-simple, and reduction modulo ℓ defines a bijection between the set of isomorphism classes of integral irreducible $\overline{\mathbb{Q}_\ell}$-representations of K′ and the set of isomorphism classes irreducible $\overline{\mathbb{F}_\ell}$-representations of K′, and we have:

LEMMA 3.7. — *Let* K *be a locally profinite group and* K′ *be a compact subgroup of* K*. Suppose that the pro-order of* K′ *is prime to* ℓ*. Let* ρ *be an finite-dimensional integral* $\overline{\mathbb{Q}_\ell}$*-representation of* K *and* χ *be a character of* K′*. Then,* ρ *is* χ*-distinguished, if and only if* $r_\ell(\rho)$ *is* $r_\ell(\chi)$*-distinguished.*

REMARK 3.8. — If K is compact modulo centre, an irreducible $\overline{\mathbb{Q}_\ell}$-representation of K is always finite-dimensional and is integral, if and only if its central character is integral.

4. Distinction for finite GL_n

For the rest of this section, we set $\mathcal{G} = \mathrm{GL}_n(\mathrm{k})$, where (as before) k denotes a finite field of odd cardinality q. If $\mathrm{k}/\mathrm{k}_\mathsf{o}$ is a quadratic extension of k_o, we denote by σ the non-trivial element of $\mathrm{Gal}(\mathrm{k}/\mathrm{k}_\mathsf{o})$ and set $\mathcal{G}_\mathsf{o} = \mathrm{GL}_n(\mathrm{k}_\mathsf{o})$.

We recall the definitions of self-dual and σ-self-dual representations of $\mathcal{G}$:

DEFINITION 4.1. —
- (i) Suppose $\mathrm{k}/\mathrm{k}_\mathsf{o}$ is a quadratic extension of finite fields, then a representation ρ of $\mathcal{G}$, over $\overline{\mathbb{Q}_\ell}$ or $\overline{\mathbb{F}_\ell}$, is called *$\sigma$-self-dual*, if $\rho^\sigma \simeq \rho^\vee$.
- (ii) A representation ρ of $\mathcal{G}$, over $\overline{\mathbb{Q}_\ell}$ or $\overline{\mathbb{F}_\ell}$, is called *self-dual* if $\rho \simeq \rho^\vee$.

4.1. Basic results on distinction. — The following multiplicity one results are [27, Remark 3.2 with the adhoc modification in the proof of Theorem 3.1, Proposition 6.10 and Remark 6.11]:

PROPOSITION 4.2. — *Let* ρ *be an irreducible* R*-representation of* $\mathcal{G}$*:*
- (i) *If* $\mathrm{k}/\mathrm{k}_\mathsf{o}$ *is a quadratic extension of finite fields, then or any character* χ *of* $\mathcal{G}_\mathsf{o}$*,* $\dim(\mathrm{Hom}_{\mathcal{G}_\mathsf{o}}(\rho,\chi)) \leq 1$*.*
- (ii) *If* ρ *is cuspidal, and* r *and* s *are two non-negative integers such that* $r + s = n \geq 2$*. Then,* $\dim(\mathrm{Hom}_{(\mathrm{GL}_r\times\mathrm{GL}_s)(\mathrm{k})}(\rho,\chi)) \leq 1$ *for any character* χ *of* $\mathrm{GL}_r \times \mathrm{GL}_s$*, and this dimension is equal to zero if* r *and* s *are positive, and* $r \neq s$*.*

The final goal of this section is to understand when a cuspidal $\overline{\mathbb{F}_\ell}$-representation of a finite general linear group that is distinguished by a maximal Levi subgroup or by a Galois involution has a lift that does not share the same distinction property.

The connection between (σ)-self-dual representations and distinction comes from:

Lemma 4.3. — (i) *A* $\mathrm{GL}_n(\mathrm{k}_\mathsf{o})$*-distinguished irreducible* R*-representation of* $\mathrm{GL}_n(\mathrm{k})$ *is* σ*-self-dual. Moreover, if* ρ *is supercuspidal, we have an equivalence:* ρ *is* σ *self-dual, if and only if it is* $\mathrm{GL}_n(\mathrm{k}_\mathsf{o})$*-distinguished.*

(ii) *A supercuspidal representation of* $\mathrm{GL}_n(\mathrm{k})$ *is self-dual, if and only if either* $n=1$*, and it a quadratic character, or if* n *is even, and it is* $(\mathrm{GL}_{n/2}\times\mathrm{GL}_{n/2})(\mathrm{k})$*-distinguished.*

Proof. — The first assertion of (i) follows from [27, Remark 3.2] and the second from [27, Lemma 8.3]. The second assertion follows from [27, Lemmas 7.1 and 7.3]. □

4.2. Self-dual and σ-self-dual cuspidal representations via the Green–Dipper–James parametrisation. — In this subsection, either k is an arbitrary finite field, and we consider self-dual representations of $\mathrm{GL}_n(\mathrm{k})$, or $\mathrm{k}/\mathrm{k}_\mathsf{o}$ is quadratic, and we consider σ-self-dual representations of $\mathrm{GL}_n(\mathrm{k})$ where $\langle\sigma\rangle=\mathrm{Gal}(\mathrm{k}/\mathrm{k}_\mathsf{o})$.

Let l/k be a degree n extension of k. A character $\theta:\mathrm{l}^\times\to\overline{\mathbb{Q}_\ell}^\times$ is called k-*regular*, if $\#\{\theta^\tau:\tau\in\mathrm{Gal}(\mathrm{l}/\mathrm{k})\}=n$, i.e. the orbit of θ under Gal(l/k) has maximal size. According to [11], there is a surjective map

$$\begin{aligned}&\{\text{k-regular characters of } \mathrm{l}^\times\to\overline{\mathbb{Q}_\ell}^\times\}\\&\quad\to\{\text{supercuspidal } \overline{\mathbb{Q}_\ell}\text{-representations of } \mathcal{G}\}/\simeq\\&\theta\mapsto\rho(\theta).\end{aligned}$$

The character formula given in [11] also implies:

(i) Two such cuspidal representations $\rho(\theta)$ and $\rho(\theta')$ are isomorphic, if and only if there exists $\tau\in\mathrm{Gal}(\mathrm{l}/\mathrm{k})$ such that $\theta'=\theta^\tau$.

(ii) The dual $\rho(\theta)^\vee$ is isomorphic to $\rho(\theta^{-1})$.

(iii) If k/k′ is a finite extension and $\tau\in\mathrm{Gal}(\mathrm{l}/\mathrm{k}')$, we have $\rho(\theta^\tau)\simeq\rho(\theta)^\tau$.

The following is well known, and a similar proof to ours can be found in [27, Lemmas 7.1 & 8.1]. In the greater generality of supercuspidal R-representations, we provide a proof as a warm-up:

Lemma 4.4. — (i) *If there exists a* σ*-self-dual supercuspidal* $\overline{\mathbb{Q}_\ell}$*-representation of* $\mathcal{G}$*, then* n *is odd.*

(ii) *If there exists a self-dual supercuspidal* $\overline{\mathbb{Q}_\ell}$*-representation of* $\mathcal{G}$*, then* n *is either 1 or even.*

Proof. — (i) Suppose that ρ is a σ-self-dual cuspidal $\overline{\mathbb{Q}_\ell}$-representation and write $\rho = \rho(\theta)$ for a k-regular character θ. Choose an extension of σ to $\widetilde{\sigma} \in \mathrm{Gal}(\mathrm{l}/\mathrm{k_o})$. Then, as $\rho(\theta^{-1}) \simeq \rho(\theta)^\vee \simeq \rho(\theta)^\sigma$, necessarily $\theta^{\widetilde{\sigma}} = (\theta^{-1})^\tau$, for some $\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k})$. Hence, $(\tau^{-1} \circ \widetilde{\sigma})^2$ fixes θ, so it is 1, as θ is regular. This implies that $\tau^{-1} \circ \widetilde{\sigma}$ is a $\mathrm{k_o}$-linear involution of l, which extends σ. However, the cyclic group $\mathrm{Gal}(\mathrm{l}/\mathrm{k_o})$ contains a unique element of order 2. If n were even, $\tau^{-1} \circ \widetilde{\sigma}$ would belong to $\mathrm{Gal}(\mathrm{l}/\mathrm{k})$, and this is absurd, as it extends σ, which is not k-linear.

(ii) Suppose that ρ is a self-dual cuspidal $\overline{\mathbb{Q}_\ell}$-representation and write $\rho = \rho(\theta)$ for a k-regular character θ. In this case, reasoning as before, necessarily $\theta = \tau(\theta^{-1})$, for some $\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k})$, and it follows that $\tau^2 = 1$. Either $\tau = 1$, hence, $\theta^2 = 1$, but there is a unique non-trivial quadratic character of $\mathrm{l}^\times$, which is, thus, fixed by all $\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k})$, and the trivial character of $\mathrm{l}^\times$ is also $\mathrm{Gal}(\mathrm{l}/\mathrm{k})$-invariant, so we deduce that $n = 1$, as θ is regular. Or τ has order 2, and, hence, $n = \#\,\mathrm{Gal}(\mathrm{l}/\mathrm{k})$ is even. □

We now recall the classification of cuspidal $\overline{\mathbb{F}_\ell}$-representations of James [13]. We have a surjective map

$$\begin{aligned}\{\text{supercuspidal } \overline{\mathbb{Q}_\ell}\text{-representations of } \mathcal{G}\}/{\simeq} \\ \to \{\text{cuspidal } \overline{\mathbb{F}_\ell}\text{-representations of } \mathcal{G}\}/{\simeq} \\ \rho(\theta) \mapsto \overline{\rho(\theta)}\end{aligned}$$

given by reduction modulo ℓ.

Given a character $\theta : \mathrm{l}^\times \to \overline{\mathbb{Q}_\ell}^\times$ we can uniquely write $\theta = \theta_r\theta_s$ with θ_r of order prime to ℓ and θ_s of order a power of ℓ. James' parametrisation enjoys the following properties:

(i) Two supercuspidal $\overline{\mathbb{Q}_\ell}$-representations $\rho(\theta), \rho(\theta')$ have isomorphic reductions modulo ℓ, if and only if there exists $\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k})$, such that $\theta'_r = \theta_r^\tau$.

(ii) $\overline{\rho(\theta)}$ is supercuspidal, if and only if θ_r is regular.

4.3. σ-self-dual lifts of cuspidal $\overline{\mathbb{F}_\ell}$-representations. — We now specialise to the case $\mathrm{k}/\mathrm{k_o}$ is quadratic. Write $\Gamma = \mathrm{Hom}(\mathrm{l}^\times, \overline{\mathbb{Q}_\ell}^\times)$, then $\Gamma = \Gamma_s \times \Gamma_r$, where Γ_s consists of the characters of ℓ-power order, and Γ_r consists of the characters with order prime to ℓ.

We study σ-self-dual lifts of cuspidal $\overline{\mathbb{F}_\ell}$-representations, and when n is even, there are no σ-self-dual supercuspidal $\overline{\mathbb{Q}_\ell}$-representations by Lemma 4.4. Hence, without loss of generality, we can assume that n is odd. Moreover, as reduction modulo ℓ commutes with taking contragredients and with the action of σ, this implies that when the cuspidal representation $\overline{\rho}$ of $\mathcal{G}$ is not σ-self-dual, it has no σ-self-dual lifts.

For $\gamma \in \Gamma$, we set $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_\gamma = \{\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k}) : \gamma^\tau = \gamma\}$. Letting $\theta \in \Gamma$ we can decompose $\theta = \theta_r\theta_s$ and we have $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_\theta = \mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_r} \cap \mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_s}$. In particular, θ is regular, if and only if $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_r} \cap \mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_s} = \{1\}$.

Let l_o be the unique subextension of $\mathrm{l}/\mathrm{k}_\mathsf{o}$ of degree n as an extension of k_o and put $\Gamma_\mathsf{o} = \mathrm{Hom}(\mathrm{l}_\mathsf{o}^\times, \overline{\mathbb{Q}_\ell}^\times)$. We have an embedding

$$i : \Gamma_\mathsf{o} \hookrightarrow \Gamma, \qquad i : \gamma \mapsto \gamma \circ \mathrm{N}_{\mathrm{l}/\mathrm{l}_\mathsf{o}},$$

by surjectivity of the norm. Hence, $\Gamma^+ = i(\Gamma_\mathsf{o})$ is a unique subgroup of the cyclic group Γ of order $q_\mathsf{o}^n - 1$. Write $\widetilde{\sigma}$ for the unique involution in $\mathrm{Gal}(\mathrm{l}/\mathrm{k}_\mathsf{o})$, which extends σ (as n is odd). By Hilbert's theorem 90, we have

$$\Gamma^+ = \{\gamma \in \Gamma : \gamma^{\widetilde{\sigma}} = \gamma\}.$$

On the other hand, the unique subgroup of the cyclic group Γ of order $q_\mathsf{o}^n + 1$ is

$$\Gamma^- = \{\gamma \in \Gamma : \gamma \circ \mathrm{N}_{\mathrm{l}/\mathrm{l}_\mathsf{o}} = 1\} = \{\gamma \in \Gamma : \gamma^{\widetilde{\sigma}} = \gamma^{-1}\},$$

as the norm map is surjective. Note that $(q_\mathsf{o}^n + 1, q_\mathsf{o}^n - 1) = 2$ because q is odd, so $\Gamma^+ \cap \Gamma^- = \{1, \eta\}$, where η denotes the unique quadratic character in Γ. Moreover, if ℓ is odd:

(i) If $\ell \mid q_\mathsf{o}^n - 1$, then $\Gamma_s \subseteq \Gamma^+$.
(ii) If $\ell \mid q_\mathsf{o}^n + 1$, then $\Gamma_s \subseteq \Gamma^-$.

Before giving the full solution of the lifting σ-self-duality for ℓ-modular cuspidal representations, we characterise ℓ-modular cuspidal σ-self-duality in terms of the Dipper and James parametrisation.

Proposition 4.5. — *Let $\overline{\rho}$ be a cuspidal representation of $\mathcal{G}$ and suppose that n and ℓ are odd, then $\overline{\rho}$ is σ-self-dual, if and only if $\theta_r^{\widetilde{\sigma}} = \theta_r^{-1}$.*

Proof. — Write $\overline{\rho} = \overline{\rho(\theta)}$ for a k-regular character θ and let $\widetilde{\sigma} \in \mathrm{Gal}(\mathrm{l}/\mathrm{k}_\mathsf{o})$ be the unique involution extending σ. One implication is obvious; for the other, we thus suppose that $\overline{\rho(\theta)}$ is σ-self-dual. Then, there exists $\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k})$, such that $\theta_r^{\widetilde{\sigma}\tau} = \theta_r^{-1}$. This implies that $\tau^2 = (\widetilde{\sigma}\tau)^2$ belongs to $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_r}$. On the other hand, the order of τ is odd because n is, and, hence, τ also belongs to $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_r}$, so $\theta_r^{\widetilde{\sigma}} = \theta_r^{-1}$. □

We have the following complete result when ℓ is odd.

Proposition 4.6. — *Assume that n and ℓ are odd. Let $\overline{\rho}$ be a σ-self-dual cuspidal $\overline{\mathbb{F}_\ell}$-representation of $\mathcal{G}$.*

(i) *Suppose that ℓ is prime with $q^n - 1$. Then, the unique supercuspidal lift of $\overline{\rho}$ is σ-self-dual.*

(ii) *Suppose that $\ell \mid q_\mathsf{o}^n - 1$.*

(a) *If $\overline{\rho}$ is supercuspidal, and ℓ^a is the highest power of ℓ dividing $q^n - 1$, then there is a unique σ-self-dual supercuspidal lift amongst the ℓ^a supercuspidal lifts of $\overline{\rho}$. In terms, of Green's parameterisation of supercuspidal $\overline{\mathbb{Q}_\ell}$-representations, if $\rho(\theta)$ is a lift of $\overline{\rho}$, then $\rho(\theta_r)$ is the unique σ-self-dual supercuspidal lift of $\overline{\rho}$.*

(b) *If $\overline{\rho}$ is cuspidal non-supercuspidal, then none of its supercuspidal lifts are σ-self-dual.*

(iii) *Suppose that $\ell \mid q_\mathsf{o}^n + 1$. Then all supercuspidal lifts of $\overline{\rho}$ are σ-self-dual.*

Proof. — Write $\overline{\rho} = \overline{\rho(\theta)}$ for a k-regular character θ of $\mathrm{l}^\times$ and let $\widetilde{\sigma} \in \mathrm{Gal}(\mathrm{l}/\mathrm{k}_\mathsf{o})$ be the unique involution extending σ.

The set of isomorphism classes of supercuspidal $\overline{\mathbb{Q}_\ell}$-representations lifting $\overline{\rho}$ is then

$$\{\rho(\theta_r\mu) : \mu \in \Gamma_s, \theta_r\mu \text{ k-regular}\}/\simeq .$$

Such a representation $\rho(\theta_r\mu)$ is σ-self-dual, if and only if there exists $\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k})$, such that $(\theta_r\mu)^{\widetilde{\sigma}} = (\theta_r^{-1}\mu^{-1})^\tau$. As $\theta_r\mu$ is regular, this condition implies that $\tau^2 = (\widetilde{\sigma}\tau)^2$ is the identity, so that $\tau = \mathrm{Id}$ as n is odd. So, $\rho(\theta_r\mu)$ is σ-self-dual, if and only if $\theta_r^{\widetilde{\sigma}} = \theta_r^{-1}$ and $\mu^{\widetilde{\sigma}} = \mu^{-1}$, and the set of σ-self-dual lifts of $\overline{\rho}$ is equal to

$$\{\rho(\theta_r\mu) : \mu \in \Gamma_s \cap \Gamma^-, \theta_r\mu \text{ k-regular}\}/\simeq$$

because the condition $\theta_r^{\widetilde{\sigma}} = \theta_r^{-1}$ is always satisfied due to Proposition 4.5. In particular, when θ_r is regular, then all $\theta_r\mu$ must be regular as well, and the cardinality of the set of σ-self-dual lifts of $\overline{\rho}$ is that of Γ_s, namely, the highest power of ℓ dividing $q^n - 1$.

In particular, if ℓ is prime to $q^n - 1$, then Γ_s is trivial, and this proves (i).

If $\ell \mid q_\mathsf{o}^n - 1$. Then $\Gamma_s \subseteq \Gamma^+$, and $\Gamma_s \cap \Gamma^- = \Gamma_s \cap \Gamma^- \cap \Gamma^+ = \{1\}$ because $\Gamma^+ \cap \Gamma^- = \{1, \eta\}$ and $\eta \notin \Gamma_s$. Hence, if $\overline{\rho}$ is supercuspidal, i.e. if θ_r is regular, then $\rho(\theta_r)$ is the unique σ-self-dual supercuspidal lift of $\overline{\rho}$, whereas if $\overline{\rho}$ is cuspidal non-supercuspidal, then it has no σ-self-dual supercuspidal lift, and we have shown (ii).

Finally, suppose that $\ell \mid q_\mathsf{o}^n + 1$, then $\Gamma_s \subseteq \Gamma^-$, so all supercuspidal lifts of $\overline{\rho}$ are σ-self-dual, and we have shown (iii). □

In the case $\ell = 2$, we have:

Proposition 4.7. — *Assume that n is odd and $\ell = 2$. Let $\overline{\rho}$ be a σ-self-dual cuspidal $\overline{\mathbb{F}_\ell}$-representation of $\mathcal{G}$, then it has a non-σ-self-dual lift.*

Proof. — Write $\overline{\rho} = \overline{\rho(\theta)}$ for a regular character θ and let $\widetilde{\sigma} \in \mathrm{Gal}(\mathrm{l}/\mathrm{k}_\mathsf{o})$ be the unique involution extending σ.

First note that $q^n - 1 = (q_{\mathsf{o}}^n - 1)(q_{\mathsf{o}}^n + 1)$, so the highest power of 2 dividing $q^n - 1$, which is the order of Γ_s, does not divide $q_{\mathsf{o}}^n + 1$, as 2 also divides $q_{\mathsf{o}}^n - 1$. In particular, Γ_s is not a subgroup of Γ^-, so if μ_0 is a generator of Γ_s, then $\widetilde{\sigma}(\mu_0) \neq \mu_0^{-1}$. Now we claim that $\rho(\theta_r\mu_0)$ is a non-σ-self-dual lift of $\overline{\rho(\theta)}$. First, it is supercuspidal; indeed, $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\mu_0} \subset \mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_s}$ because θ_s is a power of μ_0, and, hence, $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_r} \cap \mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\mu_0}$ is trivial because $\mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_r} \cap \mathrm{Gal}(\mathrm{l}/\mathrm{k})_{\theta_s}$ is trivial. Moreover, suppose that $\rho(\theta_r\mu_0)$ was σ-self-dual, then following the beginning of the proof of Proposition 4.6, this would imply that both θ_r and μ_0 belong to Γ^-, which is absurd. □

4.4. Self-dual lifts of self-dual cuspidal $\overline{\mathbb{F}_\ell}$-representations. — If there exists a self-dual supercuspidal $\overline{\mathbb{Q}_\ell}$-representation of $\mathcal{G}$, then n is 1 or even by Lemma 4.4. The case $n = 1$ is straightforward; a character is self-dual, if and only if it is quadratic, and we treat it separately:

Proposition 4.8. — *Suppose that $n = 1$. Then $1, \eta$ are the unique self-dual supercuspidal $\overline{\mathbb{Q}_\ell}$-representations of $\mathrm{GL}_1(\mathrm{k})$. The reductions $\overline{1}, \overline{\eta}$ of $1, \eta$, respectively, are the unique self-dual cuspidal $\overline{\mathbb{F}_\ell}$-representations of $\mathrm{GL}_1(\mathrm{k})$.*

(i) *Suppose that $\ell \nmid q - 1$. Then $\overline{1}, \overline{\eta}$ have $1, \eta$, respectively, as unique lift.*
(ii) *Suppose that $\ell \mid q-1$ and let ℓ^a be the highest power of ℓ dividing $q-1$. Then $\overline{1}, \overline{\eta}$ each have ℓ^a-supercuspidal lifts of which $1, \eta$ (respectively) is the unique self-dual supercuspidal lift.*

Note that case (ii) contains the case $\ell = 2$, in which case $\overline{1} = \overline{\eta}$. So, in particular in case (ii), non-trivial lifts of the trivial character of $\kappa^\times$ always exist.

Hence, for the rest of this section, we assume that n is even. Let σ' denote the unique involution in $\mathrm{Gal}(\mathrm{l}/\mathrm{k})$ and $\mathrm{l}'_{\mathsf{o}} = \mathrm{l}^{\sigma'}$ denote the σ'-fixed subfield. Then we have an embedding

$$i' : \mathrm{Hom}(\mathrm{Gal}(\mathrm{l}'_{\mathsf{o}}/\mathrm{k}), \overline{\mathbb{Q}_\ell}^\times) \hookrightarrow \Gamma, \qquad i' : \gamma \mapsto \gamma \circ \mathrm{N}_{\mathrm{l}/\mathrm{l}'_{\mathsf{o}}},$$

by surjectivity of the norm, so its image is the unique subgroup of Γ of order $q^{n/2} - 1$:

$$\Gamma_+ = \{\gamma \in \Gamma : \gamma^{\sigma'} = \gamma\}.$$

The unique subgroup of Γ of order $q^{n/2} + 1$ is, thus,

$$\Gamma_- = \{\gamma \in \Gamma : \gamma^{\sigma'} = \gamma^{-1}\},$$

their intersection is given by $\Gamma_+ \cap \Gamma_- = \{1, \eta\}$, as q is odd. As $(q^{n/2}+1, q^{n/2} - 1) = 2$, we deduce that, if ℓ is odd:

(i) If $\ell \mid q^{n/2} - 1$, then $\Gamma_s \subseteq \Gamma_+$.
(ii) If $\ell \mid q^{n/2} + 1$, then $\Gamma_s \subseteq \Gamma_-$.

The results concerning self-duality look very similar to those concerning σ-self-duality; however, in one case, we only consider lifting of distinction.

Proposition 4.9. — *Suppose that $n = 2m \geqslant 2$ is even and that ℓ is odd. Let $\overline{\rho}$ be a self-dual cuspidal $\overline{\mathbb{F}_\ell}$-representation of $\mathcal{G}$.*

(i) *Suppose that ℓ is prime with $q^n - 1$. Then the unique supercuspidal lift of $\overline{\rho}$ is self-dual.*

(ii) *Suppose that $\ell \mid q^{n/2} - 1$.*

  (a) *If $\overline{\rho}$ is supercuspidal, and ℓ^a is the highest power of ℓ dividing $q^n - 1$, then there is a unique self-dual lift of $\overline{\rho}$ amongst its ℓ^a supercuspidal lifts. In terms of Green's parameterisation of supercuspidal $\overline{\mathbb{Q}_\ell}$-representations, if $\rho(\theta)$ is a lift of $\overline{\rho}$, then $\rho(\theta_r)$ is the unique self-dual supercuspidal lift of $\overline{\rho}$.*

  (b) *If $\overline{\rho}$ is cuspidal non-supercuspidal, then none of its supercuspidal lifts are self-dual.*

(iii) *Suppose that $\ell \mid q^{n/2}+1$ and that $\overline{\rho}$ is $\mathrm{GL}_m(\mathrm{k})\times\mathrm{GL}_m(\mathrm{k})$-distinguished. Then all supercuspidal lifts of $\overline{\rho}$ are self-dual.*

Proof. — A lift $\rho(\theta_r\mu)$ with $\mu \in \Gamma_s$ is self-dual, if and only if there exists $\tau \in \mathrm{Gal}(\mathrm{l}/\mathrm{k})$, such that $\theta_r\mu = (\theta_r^{-1}\mu^{-1})^\tau$, and, hence, $\tau^2 = \mathrm{Id}$, i.e. $\tau = \mathrm{Id}$ or σ'. The first case is impossible, as it would imply that $\theta_r\mu$ is the quadratic character in Γ, which would contradict its regularity. Hence, $\rho(\theta_r\mu)$ is self-dual, if and only if $\theta_r^{\sigma'} = \theta_r^{-1}$ and $\mu^{\sigma'} = \mu^{-1}$. Thus, if $\theta_r^{\sigma'} = \theta_r^{-1}$, then the set

$$\{\rho(\theta_r\mu) : \mu \in \Gamma_s \cap \Gamma_-, \theta_r\mu \text{ regular}\}/\simeq$$

is a full set of representatives for the isomorphism classes of self-dual lifts of $\overline{\rho}$ (and there are no self-dual lifts if $\theta_r^{\sigma'} \neq \theta_r^{-1}$).

For parts (i) and (ii)(a), as θ_r is regular, the self-duality of $\overline{\rho}$ implies that $\theta_r^{\sigma'} = \theta_r^{-1}$. Hence, parts (i) and (ii)(a) follow in the same way as their analogues in Proposition 4.6. Part (ii)(b) is obvious, as $\Gamma_s \cap \Gamma_-$ is trivial in this case. Finally, (iii) holds, because in this case, distinction lifts by Lemma 3.7. □

When $\ell = 2$, we have the exact analogue of Proposition 4.7 with the same proof, replacing q_o^n by $q^{n/2}$.

Proposition 4.10. — *Assume that $n \geqslant 2$ is even and $\ell = 2$. Let $\overline{\rho}$ be a self-dual cuspidal $\overline{\mathbb{F}_\ell}$-representation of $\mathcal{G}$, then it has a non-self-dual lift.*

5. Type theory and distinction

From now on we set $\mathrm{G} = \mathrm{GL}_n(\mathrm{F})$, $\mathrm{G}_\mathsf{o} = \mathrm{GL}_n(\mathrm{F}_\mathsf{o})$ and σ, the Galois involution. We use the Bushnell–Kutzko construction of cuspidal representations of G [7], extended by Vignéras to the setting of cuspidal R-representations [28,

§III]. We summarise the properties that we will use and refer the reader to [7] and [28], for more details on this construction.

5.1. Properties of types. — Let π be a cuspidal R-representation of G. Then, associated to it is a family of explicitly constructed pairs $(\mathbf{J}, \boldsymbol{\lambda})$, called *extended maximal simple types*, where $\mathbf{J}$ is an open subgroup of G containing the centre Z_G of G with $\mathbf{J}/Z_G$ compact, and $\boldsymbol{\lambda}$ is an irreducible (hence, finite-dimensional) representation of $\mathbf{J}$, such that

$$\pi \simeq \mathrm{ind}_{\mathbf{J}}^{G}(\boldsymbol{\lambda}).$$

We abbreviate extended maximal simple type to *type* for the rest of the paper and will say R-*type* when we wish to specify the field R considered.

Let $(\mathbf{J}, \boldsymbol{\lambda})$ be an R-type *in* π, i.e. associated to π as described above. Types enjoy the following key properties:

(T-1) Two types in π are conjugate in G, [7, 6.2.4] and [28, III 5.3]

(T-2) The group $\mathbf{J}$ has a unique maximal compact subgroup J, and J has a unique maximal normal pro-p-subgroup J^1, cf. [7, §3.1] for the definitions of these groups.

(T-3) There is a subfield E of $\mathcal{M}_n(F)$ containing F, the multiplicative group of which normalises J, and $\mathbf{J} = E^\times J$. (In fact, we have summarised this construction in reverse; the extension E/F is part of the original data used to construct the type.) The quotient J/J^1 is isomorphic to $\mathrm{GL}_m(k_E)$ with $m = n/[E:F]$. Moreover, $E^\times \cap J = \mathscr{O}_E^\times$, hence $\mathbf{J} = \langle \varpi_E \rangle J$ is the semi-direct product of J with the group generated by ϖ_E.

(T-4) Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a type, let $\boldsymbol{\lambda}'$ be a representation of $\mathbf{J}$, set $\pi = \mathrm{ind}_{\mathbf{J}}^{G}(\boldsymbol{\lambda})$ and $\pi' = \mathrm{ind}_{\mathbf{J}}^{G}(\boldsymbol{\lambda}')$. If $\boldsymbol{\lambda}' |_J \simeq \boldsymbol{\lambda} |_J$, then $(\mathbf{J}, \boldsymbol{\lambda}')$ is a type, and the cuspidal representation π' is an unramified twist of π. Conversely, if $(\mathbf{J}, \boldsymbol{\lambda}')$ is a type, and the cuspidal representation π' is an unramified twist of π, then $\boldsymbol{\lambda}' |_J \simeq \boldsymbol{\lambda} |_J$.

(T-5) The representation $\boldsymbol{\lambda}$ (by construction) decomposes (non-uniquely) as a tensor product $\boldsymbol{\kappa} \otimes \boldsymbol{\tau}$, where:
- $\boldsymbol{\tau}$ is a representation of $\mathbf{J}$ trivial on J^1, which restricts irreducibly to J, and the representation of J/J^1 induced by $\boldsymbol{\tau}$ identifies with a cuspidal representation of $\mathrm{GL}_m(k_E)$.
- $\boldsymbol{\kappa}$ is a representation of $\mathbf{J}$, which restricts irreducibly to J^1.

(T-6) The representation π is supercuspidal, if and only if $\boldsymbol{\tau}$ induces a supercuspidal restriction on J/J^1, [28, III 5.14].

(T-7) The pair $(\mathbf{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\tau}')$ is another type in π, if and only if $\boldsymbol{\tau} \simeq \boldsymbol{\tau}'$.

(T-8) When $R = \overline{\mathbb{Q}_\ell}$, the representation π is integral, if and only if $\boldsymbol{\lambda}$ is integral.

(T-9) The construction is compatible with reduction modulo ℓ in the following sense: given a $\overline{\mathbb{Q}_\ell}$-type $(\mathbf{J}, \boldsymbol{\lambda})$ with $\boldsymbol{\lambda}$ integral, $(\mathbf{J}, r_\ell(\boldsymbol{\lambda}))$ is an $\overline{\mathbb{F}_\ell}$-type, and $r_\ell(\mathrm{ind}_{\mathbf{J}}^{G}(\boldsymbol{\lambda})) = \mathrm{ind}_{\mathbf{J}}^{G}(r_\ell(\boldsymbol{\lambda}))$ is a cuspidal $\overline{\mathbb{F}_\ell}$-representation, [28, III 4.25].

(T-10) The construction lifts [28, III 4.29]: given an $\overline{\mathbb{F}_\ell}$-type $(\mathbf{J}, \boldsymbol{\kappa}\otimes\boldsymbol{\tau})$, there is a unique irreducible $\overline{\mathbb{Q}_\ell}$-representation $\widetilde{\eta}$ of J^1, which lifts $\boldsymbol{\kappa}\,|_{\mathrm{J}^1}$, and we can fix a extension $\widetilde{\boldsymbol{\kappa}}$ of $\widetilde{\eta}$ with $r_\ell(\widetilde{\boldsymbol{\kappa}}) = \boldsymbol{\kappa}$. As all of the extensions of $\widetilde{\eta}$ to $\mathbf{J}$ are related by twisting by a character trivial on J^1, Property (T-7) implies that the set of isomorphism classes of lifts of $\pi = \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\kappa}\otimes\boldsymbol{\tau})$ is in bijection with the set of isomorphism classes of lifts of $\boldsymbol{\tau}$ by $\widetilde{\boldsymbol{\tau}} \mapsto \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\widetilde{\boldsymbol{\kappa}}\otimes\widetilde{\boldsymbol{\tau}})$.

(T-11) We call a field extension E/F associated to a type in π as in (T- 3) a *parameter field* for π. While there are potentially many choices, the ramification index $e(\mathrm{E}/\mathrm{F})$, inertial degree $f(\mathrm{E}/\mathrm{F})$, and (hence) the degree $[\mathrm{E}:\mathrm{F}]$ are invariants of π as follows from [7, 3.5.1]. As such, we write $d(\pi) = [\mathrm{E}:\mathrm{F}]$, $e(\pi) = e(\mathrm{E}/\mathrm{F})$ and $f(\pi) = f(\mathrm{E}/\mathrm{F})$. We write $m(\pi)$ for $n/d(\pi)$. These invariants are compatible with reduction modulo ℓ; for π an integral cuspidal $\overline{\mathbb{Q}_\ell}$-representation we have

$$d(\pi) = d(r_\ell(\pi)),\ e(\pi) = e(r_\ell(\pi)),\ f(\pi) = f(r_\ell(\pi)),\ m(\pi) = m(r_\ell(\pi)).$$

5.2. Galois-self-dual types. — It was recently shown in [4] and [27] that the construction of types also enjoys good compatibility properties with σ-self-duality and distinction. Indeed, according to [4, §4], if π is a cuspidal R-representation of G which is σ-self-dual, then one can chose a type $(\mathbf{J}, \boldsymbol{\lambda})$ in π such that:

(SSDT-I) $\mathbf{J}$ (hence, J and J^1) and E are σ-stable, and $\boldsymbol{\lambda}^\vee \simeq \boldsymbol{\lambda}^\sigma$.

(SSDT-II) Set $\mathrm{E}_\mathsf{o} = \mathrm{E}^\sigma$, then $\mathrm{E}/\mathrm{E}_\mathsf{o}$ is a quadratic extension, and we can choose a uniformiser ϖ_E with $\sigma(\varpi_\mathrm{E}) = \varpi_\mathrm{E}$, if $\mathrm{E}/\mathrm{E}_\mathsf{o}$ is unramified, and $\sigma(\varpi_E) = -\varpi_E$, if $\mathrm{E}/\mathrm{E}_\mathsf{o}$ is ramified as in [27, (5.2)].

(SSDT-III) $\boldsymbol{\kappa}$, hence, $\boldsymbol{\tau}$, are σ-self-dual ([27, Lemma 8.9]).

The ramification index $e(\mathrm{E}/\mathrm{E}_\mathsf{o}) \in \{1, 2\}$ (and is equal to 1 if $\mathrm{F}/\mathrm{F}_\mathsf{o}$ is unramified) is an invariant of π, and we write

$$e_\sigma(\pi) = e(\mathrm{E}/\mathrm{E}_\mathsf{o}).$$

Remark 5.1. — This latter invariant is also equal to the ramification index of the extension $\mathrm{T}/\mathrm{T}^\sigma$, where T is the maximal tamely ramified extension of F contained in E due to [27, Remark 4.15 (2)]. We use this fact when referring to some results of [27].

In [4, §6.2], another invariant, a positive integer $e_\mathsf{o}(\pi)$ dividing n defined only in terms of the σ-stable group $\mathbf{J}$, is associated to π. By [4, Lemma 5.10], we have the following description:

$$e_\mathsf{o}(\pi) = \begin{cases} 2e(\mathrm{E}_\mathsf{o}/\mathrm{F}_\mathsf{o}) & \text{if } e_\sigma(\pi) = 2 \text{ and } m(\pi) \neq 1; \\ e(\mathrm{E}_\mathsf{o}/\mathrm{F}_\mathsf{o}) & \text{otherwise.} \end{cases}$$

Again, these invariants are compatible with reduction modulo ℓ; for π an integral σ-self-dual supercuspidal $\overline{\mathbb{Q}_\ell}$-representation we have

$$e_\sigma(\pi) = e_\sigma(r_\ell(\pi)),\ e_{\mathsf{o}}(\pi) = e_{\mathsf{o}}(r_\ell(\pi)).$$

Definition 5.2. — We call a type $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\tau}$ satisfying Conditions (SSDT-I) to (SSDT-III) a *σ-self-dual type.*

Note that an immediate consequence of the existence of a σ-self-dual cuspidal type in a σ-self-dual cuspidal representation is that in some cases, there are no σ-self-dual cuspidal representations, as follows from [27, Lemma 6.9 and Lemma 8.1]:

Lemma 5.3. — *Let π be a σ-self-dual cuspidal* R*-representation.*

(i) *If $e_\sigma(\pi) = 1$ and π is supercuspidal, then $m(\pi)$ is odd.*
(ii) *If $e_\sigma(\pi) = 2$, then $m(\pi)$ is either equal to 1 or even.*

A crucial property of σ-self-dual types is the following:

Proposition 5.4 ([27, Lemma 5.19]). — *Let $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-self-dual type. Then there exists a unique character $\chi_{\boldsymbol{\kappa}}$ of $\mathbf{J}^\sigma$ trivial on $(\mathrm{J}^1)^\sigma$, such that*

$$\mathrm{Hom}_{(\mathrm{J}^1)^\sigma}(\boldsymbol{\kappa}, \mathrm{R}) = \mathrm{Hom}_{\mathbf{J}^\sigma}(\boldsymbol{\kappa}, \chi_{\boldsymbol{\kappa}}),$$

and the canonical map

$$\mathrm{Hom}_{\mathbf{J}^\sigma}(\boldsymbol{\kappa}, \chi_{\boldsymbol{\kappa}}) \otimes \mathrm{Hom}_{\mathbf{J}^\sigma}(\boldsymbol{\tau}, \chi_{\boldsymbol{\kappa}}^{-1}) \to \mathrm{Hom}_{\mathbf{J}^\sigma}(\boldsymbol{\lambda}, \mathrm{R})$$

is an isomorphism.

In many cases, it is shown in [27] that one can choose $\chi_{\boldsymbol{\kappa}} = 1$ above, including the supercuspidal case:

Proposition 5.5. — *Let π be a σ-self-dual supercuspidal* R*-representation and $(\mathbf{J}, \boldsymbol{\lambda})$ be a σ-self-dual type of π, then one can choose $\boldsymbol{\kappa}$ such that $\chi_{\boldsymbol{\kappa}} = 1$.*

Proof. — The only cases to consider are those that are not ruled out by Lemma 5.3, and the assertion then follows from [27, Propositions 6.15 and 8.10]. □

Remark 5.6. — Note that if $\mathrm{R} = \overline{\mathbb{Q}_\ell}$ above, then π is an integral, and $(\mathbf{J}, r_\ell(\boldsymbol{\tau}))$ is a σ-self-dual type for $r_\ell(\pi)$. Moreover, $\chi_{r_\ell(\boldsymbol{\kappa})} = 1$ if $\chi_{\boldsymbol{\kappa}} = 1$. Indeed, dues to Remark 3.5 applied to $(\mathbf{J}/\mathrm{F}_{\mathsf{o}}^\times, \boldsymbol{\kappa})$, the representation $r_\ell(\boldsymbol{\kappa})$ is distinguished, and, hence, $\chi_{r_\ell(\boldsymbol{\kappa})} = 1$ by the first part of Proposition 5.4.

5.3. Generic types and distinguished types. — There are, in general, more than one G_{o}-conjugacy class of σ-self-dual types in a σ-self-dual cuspidal R-representation (see [27, Section 1.11]). However, there is only one G_o-conjugacy class among those that contain a generic type in the following sense.

We denote by N be the maximal unipotent subgroup of the subgroup of upper triangular matrices in G, and $N_o = N^\sigma$. Let ψ a non-degenerate character of N. Note that such a character is always integral with non-degenerate reduction modulo ℓ, as N is exhausted by its pro-p-subgroups.

Definition 5.7. — Let $(\mathbf{J}, \boldsymbol{\lambda})$ be an R-type; we say that $(\mathbf{J}, \boldsymbol{\lambda})$ is a *ψ-generic type*, if

$$\mathrm{Hom}_{N\cap \mathbf{J}}(\boldsymbol{\lambda}, \psi) \neq \{0\}.$$

We say that it is *generic*, if it is ψ-generic for some non-degenerate character of N.

Remark 5.8. — Note that if μ is a character of G, and $(\mathbf{J}, \boldsymbol{\lambda})$ is a ψ-generic type, then $(\mathbf{J}, \mu\mid_{\mathbf{J}} \otimes \boldsymbol{\lambda})$ is also ψ-generic.

We will also use the following observation later.

Lemma 5.9. — *A $(\mathbf{J}, \boldsymbol{\lambda})$ is an integral $\overline{\mathbb{Q}_\ell}$-type, and ψ be a non-degenerate character of* N*. If $(r_\ell(\mathbf{J}), r_\ell(\boldsymbol{\lambda}))$ is $r_\ell(\psi)$-generic, then $(\mathbf{J}, \boldsymbol{\lambda})$ is ψ-generic.*

Proof. — It is a consequence of Lemma 3.7, once one observes that $\mathbf{J} \cap N$ is a (compact) pro-p group. □

If the type we consider is, moreover, σ-self-dual, we will only consider distinguished non-degenerate characters ψ of N, i.e. those that are trivial on N_o:

Definition 5.10. — A σ-self-dual R-type is called *generic*, if it is ψ-generic with respect to a distinguished non-degenerate character ψ of N.

Our definition of a *generic σ-self-dual type* coincides with the definition given in [27, Definition 9.1] (see the discussion after [4, Definition 5.7]). There are two fundamental facts about these types: first, they always occur in σ-self-dual cuspidal representations.

Proposition 5.11 ([4, Proposition 5.5]). — *Let π be a σ-self-dual cuspidal* R*-representation of* G *and let ψ be a distinguished non-degenerate character of* N*; then π has a ψ-generic σ-self-dual type, which is, moreover, unique up to N_o-conjugacy.*

The second fact concerns distinguished representations, which are σ-self-dual due to Proposition 3.3.

Theorem 5.12 ([4, Corollary 6.6], [27, Theorem 9.3]). — *Let π be a σ-self-dual cuspidal* R*-representation and $(\mathbf{J}, \boldsymbol{\lambda})$ a generic σ-self-dual type of π. Then π is distinguished, if and only if $\boldsymbol{\lambda}$ is $\mathbf{J}^\sigma$-distinguished.*

DEFINITION 5.13. — We call a σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ a *distinguished type* if $\boldsymbol{\lambda}$ is $\mathbf{J}^{\sigma}$-distinguished.

We have the following surprising result, which completes Theorem 5.12 and is evidence of the interplay between genericity and Galois distinction for $\mathrm{GL}_n(\mathrm{F})$:

LEMMA 5.14. — *A distinguished* R*-type is automatically σ-self-dual generic.*

Proof. — Take a σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ such that $\boldsymbol{\lambda}$ is $\mathbf{J}^{\sigma}$-distinguished. Then, $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ is distinguished by Mackey theory and Frobenius reciprocity. However, then by [4, Remark 6.7] (and the equivalence of our definition of generic type with that given in [27, Defintion 9.1]), the type $(\mathbf{J}, \boldsymbol{\lambda})$ must be ψ-generic for some distinguished non-degenerate character ψ of N. □

We end this section with the following important corollary of Proposition 5.4:

COROLLARY 5.15. — *A σ-self-dual type $(\mathbf{J}, \boldsymbol{\lambda})$ is a distinguished type, if and only if $\mathrm{Hom}_{\mathbf{J}^{\sigma}}(\boldsymbol{\kappa}, \chi_{\boldsymbol{\kappa}})$ and $\mathrm{Hom}_{\mathbf{J}^{\sigma}}(\boldsymbol{\tau}, \chi_{\boldsymbol{\kappa}}^{-1})$ are non-zero, in which case both are one-dimensional.*

5.4. The relative torsion group of a distinguished representation. — For the rest of this section, we fix π a distinguished cuspidal R-representation and $(\mathbf{J}, \boldsymbol{\lambda})$ a distinguished type in π. We set $\varpi_{\mathrm{E}_{\mathsf{o}}} = \varpi_{\mathrm{E}}^2$, if $\mathrm{E}/\mathrm{E}_{\mathsf{o}}$ is ramified, and $\varpi_{\mathrm{E}_{\mathsf{o}}} = \varpi_{\mathrm{E}}$, if $\mathrm{E}/\mathrm{E}_{\mathsf{o}}$ is unramified. When $e_{\sigma}(\pi) = 2$, and $m(\pi) = 2r$ is even, we denote by w the element of $\mathbf{J}$ corresponding to $\left(\begin{smallmatrix} 0 & 1_r \\ 1_r & 0 \end{smallmatrix}\right)$ as in [27, Lemma 6.19]. We define the *relative torsion group of* π to be the following group:

$$\mathrm{X}_{\mathsf{o}}(\pi) = \{\mu_{\mathsf{o}} \in \mathrm{Hom}(\mathrm{G}_{\mathsf{o}}, \mathrm{R}^{\times}) : \mu_{\mathsf{o}} \text{ is unramified, } \mathrm{Hom}_{\mathrm{G}_{\mathsf{o}}}(\pi, \mu_{\mathsf{o}}) \neq \{0\}\}.$$

THEOREM 5.16. — *Let π be a cuspidal distinguished* R*-representation of* G *and set $\varpi' = \varpi_E w$, if $e_{\sigma}(\pi) = 2$, and $m(\pi)$ is even and $\varpi' = \varpi_{\mathrm{E}_{\mathsf{o}}}$ otherwise. Then we have:*

(i) *Let μ_{o} be an unramified character of G_{o}, then $\mu_{\mathsf{o}} \in \mathrm{X}_{\mathsf{o}}(\pi)$, if and only if $\mu_{\mathsf{o}}(\varpi') = 1$.*

(ii) *Let χ_{o} be an unramified character of $\mathrm{F}_{\mathsf{o}}^{\times}$, then $\chi_{\mathsf{o}} \circ \det \in \mathrm{X}_{\mathsf{o}}(\pi)$, if and only if $\chi_{\mathsf{o}}(\varpi_{\mathsf{o}})^{n/e_{\mathsf{o}}(\pi)} = 1$.*

Proof. — Note that if $e_{\sigma}(\pi) = 2$, then $m(\pi)$ is even or equal to 1 due to Lemma 5.3. Then, $\mathbf{J}^{\sigma}$ is generated by ϖ' and J^{σ}, due to [27, Lemmas 6.18, 6.19 and 8.7]. Let μ_{o} be an unramified character of G_{o} and denote by μ an unramified extension of it to G; hence, $\mu_{\mathsf{o}} \in \mathrm{X}_{\mathsf{o}}(\pi)$, if and only if $\mu \otimes \pi$ is distinguished. Suppose that $\mu \otimes \pi$ is distinguished, then $(\mathbf{J}, \boldsymbol{\kappa} \otimes (\mu \otimes \boldsymbol{\tau}))$ is a σ-self-dual type, which is, in fact, a distinguished type due to Remark 5.8 and Theorem 5.12, and conversely, if $(\mathbf{J}, \boldsymbol{\kappa} \otimes (\mu \otimes \boldsymbol{\tau}))$ is a distinguished type, then $\mu \otimes \pi$ is distinguished.

So, $\mu_\circ \in \mathrm{X}_\circ(\pi)$, if and only if $(\mathbf{J}, \boldsymbol{\kappa} \otimes (\mu \otimes \boldsymbol{\tau}))$ is a distinguished type, which is, if and only if $\mathrm{Hom}_{\mathbf{J}^\sigma}(\mu \otimes \boldsymbol{\tau}, \chi_{\boldsymbol{\kappa}}^{-1})$ has dimension 1 according to Corollary 5.15.

However, $\mathrm{Hom}_{\mathbf{J}^\sigma}(\boldsymbol{\tau}, \chi_{\boldsymbol{\kappa}}^{-1})$ has dimension 1 by the same corollary, but $\mathrm{Hom}_{\mathrm{J}^\sigma}(\boldsymbol{\tau}, \chi_{\boldsymbol{\kappa}}^{-1})$ is already at most one-dimensional due to Proposition 4.2, so from these multiplicity one statements, we deduce that $\mu_\circ \in \mathrm{X}_\circ(\pi)$, if and only if $\mathrm{Hom}_{\mathbf{J}^\sigma}(\mu \otimes \boldsymbol{\tau}, \chi_{\boldsymbol{\kappa}}^{-1}) = \mathrm{Hom}_{\mathbf{J}^\sigma}(\boldsymbol{\tau}, \chi_{\boldsymbol{\kappa}}^{-1})$. Finally, this translates as: $\mu_\circ \in \mathrm{X}_\circ(\pi)$, if and only if $\mu(\varpi') = 1$ and proves (i).

Now, let $\chi_\circ$ be an unramified character of $\mathrm{F}_\circ^\times$ and let χ be an unramified character of $\mathrm{F}^\times$ extending it. If $e_\sigma(\pi) = 2$, then $\mathrm{F}/\mathrm{F}_\circ$ is ramified according to [27, Lemma 4.14].

Suppose that $e_\sigma(\pi) = 2$ and $m(\pi)$ is even; we have:

$$\begin{aligned}\chi_\circ(\det(w')) &= \chi(\det(\varpi_E)) = \chi(\mathrm{N}_{\mathrm{E}/\mathrm{F}}(\varpi_{\mathrm{E}}))^{m(\pi)} = \chi(\varpi)^{f(\mathrm{E}/F)m(\pi)} \\ &= \chi(\varpi)^{n/e(\mathrm{E}/\mathrm{F})} = \chi(\varpi)^{ne(\mathrm{F}/\mathrm{F}_\circ)/e_\sigma(\pi)e(\mathrm{E}_\circ/\mathrm{F}_\circ)} \\ &= \chi(\varpi)^{n/e(\mathrm{E}_\circ/\mathrm{F}_\circ)} = \chi(\varpi_\circ)^{n/2e(\mathrm{E}_\circ/\mathrm{F}_\circ)} = \chi(\varpi_\circ)^{n/e_\circ(\pi)}\end{aligned}$$

due to [4, Lemma 5.10]. Otherwise, we have:

$$\begin{aligned}\chi_\circ(\det(w')) &= \chi_\circ(\mathrm{N}_{\mathrm{E}/\mathrm{F}}(\varpi_{E_\circ}))^{m(\pi)} = \chi_\circ(\mathrm{N}_{\mathrm{E}_\circ/\mathrm{F}_\circ}(\varpi_{E_\circ}))^{m(\pi)} \\ &= \chi_\circ(\varpi_\circ)^{f(\mathrm{E}_\circ/\mathrm{F}_\circ)m(\pi)} = \chi_\circ(\varpi_\circ)^{n/e(\mathrm{E}_\circ/\mathrm{F}_\circ)} = \chi(\varpi_\circ)^{n/e_\circ(\pi)}\end{aligned}$$

due to [4, Lemma 5.10] again. □

It has the following two corollaries.

Corollary 5.17. — *Let π be a distinguished cuspidal* R*-representation of* G*. Write $n/e_\circ(\pi) = a(\pi)\ell^r$ with $a(\pi)$ prime to ℓ. Then, $\mathrm{X}_\circ(\pi)$ is a cyclic group, of order $n/e_\circ(\pi)$, if $\mathrm{R} = \overline{\mathbb{Q}_\ell}$, and of order $a(\pi)$, if $\mathrm{R} = \overline{\mathbb{F}_\ell}$.*

Corollary 5.18. — *Let π be a distinguished cuspidal (hence, integral) $\overline{\mathbb{Q}_\ell}$-representation of* G*. Then, the homomorphism*

$$r_\ell : \mu_\circ \mapsto r_\ell(\mu_\circ)$$

is surjective from $\mathrm{X}_\circ(\pi)$ to $\mathrm{X}_\circ(r_\ell(\pi))$, and its kernel is the ℓ-singular part of $\mathrm{X}_\circ(\pi)$.

Proof. — It suffices to verify the assertion on the kernel. It is clear that the ℓ-singular part of $\mathrm{X}_\circ(\pi)$ belongs to the kernel of r_ℓ. Conversely, if $r_\ell(\mu_\circ) = 1$, write $\mu_\circ = (\mu_\circ)_r(\mu_\circ)_s$ with $(\mu_\circ)_r$ of order prime to ℓ and $(\mu_\circ)_s$ of order a power of ℓ, then $r_\ell((\mu_\circ)_r) = r_\ell(\mu_\circ) = 1$ so $(\mu_\circ)_r = 1$ because r_ℓ induces a bijection between the group of roots of unity of order prime to ℓ in $\overline{\mathbb{Q}_\ell}^\times$ and the group of roots of unity in $\overline{\mathbb{F}_\ell}^\times$. □

6. Relatively banal cuspidal representations of p-adic GL_n

In [24] and [23], Mínguez and Sécherre single out a class of irreducible representations called *banal*, for which the Zelevinski classification works particularly nicely. For cuspidal representations, the following definition can be given ([24, Remarque 8.15] and [24, Lemme 5.3]).

Definition 6.1. — A cuspidal $\overline{\mathbb{F}_\ell}$-representation π is called *banal*, if $q^{n/e(\pi)} \not\equiv 1[\ell]$.

The following definition is new and is motivated by our cuspidal L-factor computation later and an analogy with banal cuspidal representations and the Rankin–Selberg computation of [18]. We show in Section 8.3 how it is a natural analogue of banal for the symmetric pair $(\mathrm{G}, \mathrm{G}_\circ, \sigma)$.

Definition 6.2. — Let π be a distinguished cuspidal $\overline{\mathbb{F}_\ell}$-representation. We say that it is *relatively banal*, if $q_\circ^{n/e_\circ(\pi)} \not\equiv 1[\ell]$.

Theorem 5.16 (ii) has the following third consequence:

Corollary 6.3 (of Theorem 5.16). — *Let π be a distinguished cuspidal $\overline{\mathbb{F}_\ell}$-representation, it is relatively banal, if and only if it is not $|\det(\cdot)|_\circ$-distinguished.*

Before stating the next lemma, we make the following observation, which shows that the statement of the lemma in question (Lemma 6.6) is, indeed, complete.

Remark 6.4. — For any character χ of $\mathrm{G}_\circ$, there are no χ-distinguished cuspidal R-representations π of G with $e_\sigma(\pi) = 2$ when $m(\pi) \geqslant 3$ is odd, by Proposition 5.4 and [27, Lemma 6.9].

While we have defined relatively banal distinguished representations in terms of the invariant $e_\circ(\ )$, we will use the following equivalent formulation:

Lemma 6.5. — *Let π be a σ-self-dual cuspidal* R*-representation of* G*. Let* E *be a σ-self-dual parameter field for π. Then,*

(i) *If $e_\sigma(\pi) = 1$, then $q_\circ^{n/e_\circ(\pi)} = q_{\mathrm{E}_\circ}^{m(\pi)}$ (and is also equal to $q_\circ^{n/e(\pi)}$, if $\mathrm{F}/\mathrm{F}_\circ$ is unramified, and $q^{n/2e(\pi)}$, if $\mathrm{F}/\mathrm{F}_\circ$ is ramified).*

(ii) *If $e_\sigma(\pi) = 2$ and $m(\pi) = 1$, then $q_\circ^{n/e_\circ(\pi)} = q_{\mathrm{E}_\circ}$ (and is also equal to $q^{n/e(\pi)} = q_{\mathrm{E}}$).*

(iii) *If $e_\sigma(\pi) = 2$ and $m(\pi) \geqslant 2$ is even, then $q_\circ^{n/e_\circ(\pi)} = q_{\mathrm{E}_\circ}^{m(\pi)/2}$ (and is also equal to $q^{n/2e(\pi)} = q_{\mathrm{E}}^{m(\pi)/2}$).*

Proof. — In all cases, we have $q^{n/e(\pi)} = q_{\mathrm{E}}^{m(\pi)}$. In case (i), $q_{\mathrm{E}_\circ}^{m}$ is the positive square root of q_{E}^{m}. However, by [4, Lemma 5.10], we have

$$\begin{aligned} e_\circ(\pi) &= e(\mathrm{E}_\circ/\mathrm{F}_\circ) = e(\mathrm{E}/\mathrm{F}_\circ)/e_\sigma(\pi) = e(\mathrm{E}/\mathrm{F}_\circ) \\ &= e(\mathrm{E}/\mathrm{F})e(\mathrm{F}/\mathrm{F}_\circ) = e(\pi)e(\mathrm{F}/\mathrm{F}_\circ). \end{aligned}$$

If $\mathrm{F}/\mathrm{F}_\circ$ is unramified, then $q_\circ^{n/e_\circ(\pi)} = q_\circ^{n/e(\pi)}$ is the positive square root of $q^{n/e(\pi)}$. Now, if $\mathrm{F}/\mathrm{F}_\circ$ is ramified, then $q_\circ^{n/e_\circ(\pi)} = q_\circ^{n/2e(\pi)}$ and (i) is proved.

In case (ii) by [4, Lemma 5.10], again, we have

$$\begin{aligned} e_\circ(\pi) &= e(\mathrm{E}_\circ/\mathrm{F}_\circ) = e(\mathrm{E}/\mathrm{F}_\circ)/e_\sigma(\pi) = e(\mathrm{E}/\mathrm{F}_\circ)/2 \\ &= e(\mathrm{E}/\mathrm{F})e(\mathrm{F}/\mathrm{F}_\circ)/2 = e(\pi)e(\mathrm{F}/\mathrm{F}_\circ)/2. \end{aligned}$$

However, $e(\mathrm{F}/\mathrm{F}_\circ) = 2$ by [27, Lemma 4.14], so $e_\circ(\pi) = e(\pi)$ and $q = q_\circ$, which proves case (ii).

Finally, in case (iii) by [4, Lemma 5.10], again, we have

$$\begin{aligned} e_\circ(\pi) &= 2e(\mathrm{E}_\circ/\mathrm{F}_\circ) = 2e(\mathrm{E}/\mathrm{F}_\circ)/e_\sigma(\pi) = e(\mathrm{E}/\mathrm{F}_\circ) \\ &= e(\mathrm{E}/\mathrm{F})e(\mathrm{F}/\mathrm{F}_\circ) = e(\pi)e(\mathrm{F}/\mathrm{F}_\circ), \end{aligned}$$

and $e(\mathrm{F}/\mathrm{F}_\circ) = 2$ by [27, Lemma 4.14], so $e_\circ(\pi) = 2e(\pi)$ and $q = q_\circ$, which proves case (iii). □

Immediately, from Remark 6.4 and Lemma 6.5, we have:

Corollary 6.6. — *A banal distinguished cuspidal $\overline{\mathbb{F}_\ell}$-representation of* G *is relatively banal.*

Remark 6.7. — A banal cuspidal $\overline{\mathbb{F}_\ell}$-representation is supercuspidal. However, there are relatively banal distinguished cuspidal non-supercuspidal $\overline{\mathbb{F}_\ell}$-representations. For example, when $n = 3$ and $\ell \neq 2$, the non-normalised parabolic induction of the trivial representation of the Borel subgroup has a cuspidal subquotient St(3) when $q_\circ^3 \equiv -1[\ell]$, and when $\mathrm{F}/\mathrm{F}_\circ$ is unramified, it is relatively banal distinguished (see [19]).

Before proving the main result of this section, it will be useful to know that there are no relatively banal distinguished cuspidal representations π when $e_\sigma(\pi) = 1$ and $m(\pi)$ is even:

Lemma 6.8. — *Let π be a cuspidal $\overline{\mathbb{F}_\ell}$-representation of* G*, which is σ-self-dual. Suppose that $e_\sigma(\pi) = 1$, that $m(\pi)$ is even, and $q_\circ^{n/e_\circ(\pi)} \not\equiv 1[\ell]$, then π is not distinguished.*

Proof. — Let $(\mathbf{J}, \boldsymbol{\kappa} \otimes \boldsymbol{\tau})$ be a σ-self-dual generic $\overline{\mathbb{F}_\ell}$-type for π (Proposition 5.11) with σ-stable parameter field E. Suppose π is distinguished. Then, by Theorem 5.12, we can suppose that $\boldsymbol{\kappa} \otimes \boldsymbol{\tau}$ is distinguished as well. By Proposition 5.4, $\boldsymbol{\tau}$ is $\chi_{\boldsymbol{\kappa}}^{-1}$-distinguished, and, hence, $\rho = \boldsymbol{\tau} \mid_{\mathrm{J}}$ is seen as a representation of $\mathrm{GL}_{m(\pi)}(\mathrm{k_E})$ is $\chi_{\boldsymbol{\kappa}}^{-1}$-distinguished by the group $\mathrm{GL}_{m(\pi)}(\mathrm{k}_{\mathrm{E}_\circ})$, i.e. that $\rho' = \chi \otimes \rho$ is distinguished for an extension χ of $\chi_{\boldsymbol{\kappa}}$ to $\mathrm{k}_{\mathrm{E}}^{\times}$. Now, by Lemma 6.5 and Lemma 3.7, ρ' has a distinguished lift, which contradicts Lemma 4.4. □

Remark 6.9. — Note that the statement of Lemma 6.8 is not empty, as σ-self-dual representations π exist under the hypothesis $e_\sigma(\pi) = 1$ and $q_{\mathrm{o}}^{n/e_{\mathrm{o}}(\pi)} \not\equiv 1[\ell]$; for example, when $n = 2$ and $\mathrm{F}/\mathrm{F}_{\mathrm{o}}$ is unramified, the non-normalised parabolic induction of the trivial representation of the Borel subgroup has a cuspidal subquotient $\mathrm{St}(2)$ when $q \equiv -1[\ell]$, which is σ-self-dual and $e_\sigma(\mathrm{St}(2)) = 1$, as $\mathrm{F}/\mathrm{F}_{\mathrm{o}}$ is unramified.

Lemma 6.10. — *Let $(\mathbf{J}, \boldsymbol{\lambda})$ be an* R*-type, such that* $\mathbf{J}$ *is σ-stable and put $\pi = \mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$. If $\boldsymbol{\lambda}\mid_{\mathrm{J}}$ is distinguished, then π is the unramified twist of a distinguished representation. Conversely, suppose that, moreover, $\boldsymbol{\lambda} = \boldsymbol{\kappa} \otimes \boldsymbol{\tau}$ is generic and that $\boldsymbol{\kappa}$ is distinguished and σ-self-dual, if π is the unramified twist of a distinguished representation, then $\boldsymbol{\tau}\mid_{\mathrm{J}}$ is distinguished.*

Proof. — If $\boldsymbol{\lambda}\mid_{\mathrm{J}}$ is distinguished, then we can extend $\boldsymbol{\lambda}$ to a distinguished representation $\boldsymbol{\lambda}_{\mathrm{F}}$ of $\mathrm{F}^\times\mathrm{J}$ by setting $\boldsymbol{\lambda}_{\mathrm{F}}(\varpi_{\mathrm{F}}) = 1$. The induced representation $\mathrm{ind}_{\mathrm{F}^\times \mathrm{J}}^{\mathrm{E}^\times \mathrm{J}}(\boldsymbol{\lambda}_{\mathrm{F}})$ is distinguished, and because $\mathbf{J}/\mathrm{F}^\times\mathrm{J} \simeq \langle\varpi_{\mathrm{E}}\rangle/\langle\varpi_{\mathrm{F}}\rangle$ is cyclic, all of its irreducible subquotients extend $\boldsymbol{\lambda}_{\mathrm{F}}$ by Clifford theory, so one extension $\boldsymbol{\lambda}_{\mathrm{E}}$ of $\boldsymbol{\lambda}_{\mathrm{F}}$ to $\mathbf{J}$ is distinguished. Hence, $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda}_{\mathrm{E}})$ is distinguished and an unramified twist of $\mathrm{ind}_{\mathbf{J}}^{\mathrm{G}}(\boldsymbol{\lambda})$ by Property (T-4).

For the partial converse, by twisting by an unramified character without loss of generality we can suppose that π is distinguished (and $\boldsymbol{\kappa}$ is the same). Then, $\boldsymbol{\tau}$ is σ-self-dual due to Property (T-7), and, hence, $\boldsymbol{\lambda}$ as well, and it is distinguished because of Theorem 5.12. Then Proposition 5.4 implies that $\boldsymbol{\tau}$, and, hence, $\boldsymbol{\tau}\mid_{\mathrm{J}}$ is distinguished. □

Relatively banal distinguished cuspidal $\overline{\mathbb{F}_\ell}$-representations enjoy very nice lifting properties:

Theorem 6.11. — *Let π be a cuspidal and distinguished $\overline{\mathbb{F}_\ell}$-representation of* G.

(i) *Then π is relatively banal, if and only if all of its lifts are unramified twists of distinguished representations.*
(ii) *If it is relatively banal, then it has a distinguished lift.*

Proof. — Suppose that π is relatively banal distinguished. Choose a distinguished type $(\mathbf{J}, \boldsymbol{\lambda})$ in π and let $\widetilde{\pi}$ be a lift of π. We can choose a type in $\widetilde{\pi}$ of the form $(\mathbf{J}, \widetilde{\boldsymbol{\lambda}})$ with $r_\ell(\widetilde{\boldsymbol{\lambda}}) = \boldsymbol{\lambda}$ by property (T-10). As ℓ is coprime to J^σ, we can apply Lemma 3.7, and $\widetilde{\boldsymbol{\lambda}}\mid_{\mathrm{J}}$ is distinguished because so is $\boldsymbol{\lambda}\mid_{\mathrm{J}}$. Hence, $\widetilde{\pi}$ is a unramified twist of a distinguished representation by Lemma 6.10, and this proves one implication in (i).

We now prove (ii). Suppose that π is relatively banal. By the implication already proved in (i) we know that π has a lift $\widetilde{\pi}$, which is $\widetilde{\mu_\circ}$-distinguished for $\widetilde{\mu_\circ}$ an unramified character of $G_\circ$. Let $\widetilde{\mu}$ be an unramified character of G extending $\widetilde{\mu_\circ}$; then $\widetilde{\mu}^{-1}\otimes\widetilde{\pi}$ is distinguished. However, because π is distinguished, setting $\mu = r_\ell(\widetilde{\mu})$, the representation $\mu^{-1}\otimes\pi$ is $\mu_\circ^{-1}$-distinguished for $\mu_\circ = r_\ell(\widetilde{\mu_\circ}) = \mu\mid_{G_\circ}$. Due to Corollary 5.18, $\mu_\circ$ has a lift $\widetilde{\mu_\circ}' \in X_\circ(\widetilde{\mu}^{-1}\otimes\widetilde{\pi})$. Writing $\mu = \chi\circ\det$, it is possible to extend $\widetilde{\mu_\circ'}$ to an unramified character $\widetilde{\mu}'$ of G such that, if $\mu' = r_\ell(\mu) = \chi'\circ\det$, then $\chi'(\varpi) = \chi(\varpi)$; indeed, as μ and μ' both extend $\mu_\circ$, this is automatic, if $F/F_\circ$ is unramified, whereas, if $F/F_\circ$ is ramified $\chi'(\varpi) = \pm\chi(\varpi)$ is automatic, and we can always change $\widetilde{\mu}'$ so that this sign is $+$. With such choices, the representation $\widetilde{\mu}'\widetilde{\mu}^{-1}\otimes\widetilde{\pi}$ is a distinguished lift of π.

It remains to prove the second implication of (i). Suppose that π is not relatively banal, i.e. $q_\circ^{n/e_\circ(\pi)} \equiv 1[\ell]$. Suppose, for the sake of contradiction, that all lifts of π are distinguished up to an unramified twist and let $\widetilde{\pi}$ be a lift of π. Under this assumption the argument used to prove (ii) shows that π has a distinguished supercuspidal lift $\widetilde{\pi}$. This lift has a distinguished type $(\mathbf{J}, \widetilde{\boldsymbol{\kappa}}\otimes\widetilde{\boldsymbol{\tau}})$ with $\chi_{\widetilde{\boldsymbol{\kappa}}} = 1$ due to Theorem 5.12 and Proposition 5.5, and we set $\boldsymbol{\kappa} = r_\ell(\widetilde{\boldsymbol{\kappa}})$ and $\boldsymbol{\tau} = r_\ell(\widetilde{\boldsymbol{\tau}})$, so in particular, $(\mathbf{J}, \boldsymbol{\kappa}\otimes\boldsymbol{\tau})$ is a distinguished type due to Remark 3.5 (and also Lemma 5.14). Proposition 5.4 together with Lemma 4.4 imply that if $e_\sigma(\pi) = 2$, then either $m(\pi) = 1$ or it is even, and if $e_\sigma(\pi) = 1$, then $m(\pi)$ is odd. Then, $\boldsymbol{\tau}\mid_{J}$ is distinguished according to Remark 3.5, but the assumption $q_\circ^{n/e_\circ(\pi)} \equiv 1[\ell]$ translated in terms of $GL_{m(\pi)}(k_E)$ due to Lemma 6.5 together with Propositions 4.6 (ii), 4.7, 4.8 (ii), 4.6 (ii) and 4.10 imply that $\boldsymbol{\tau}\mid_J$ has a non-distinguished lift $\widetilde{\boldsymbol{\tau}}'$. This lift extends to $\langle\varpi_\circ\rangle J$ to a lift of $\boldsymbol{\tau}\mid_{\langle\varpi_\circ\rangle J}$ by setting $\widetilde{\boldsymbol{\tau}}'(\varpi_\circ) = 1$. Then, by Clifford theory, because the quotient $\mathbf{J}/\langle\varpi_\circ\rangle J \simeq \langle\varpi_E\rangle/\langle\varpi_\circ\rangle$ is cyclic, the representation $\mathrm{ind}_{\langle\varpi_\circ\rangle J}^{\mathbf{J}}(\widetilde{\boldsymbol{\tau}}')$ contains a lift of $\boldsymbol{\tau}$ that extends $\widetilde{\boldsymbol{\tau}}'$, and we again denote it $\widetilde{\boldsymbol{\tau}}'$. Then, the representation $\pi' = \mathrm{ind}_{\mathbf{J}}^{G}(\widetilde{\boldsymbol{\kappa}}\otimes\widetilde{\boldsymbol{\tau}}')$ is a supercuspidal lift of π. As $\widetilde{\boldsymbol{\kappa}}\otimes\widetilde{\boldsymbol{\tau}}'$ reduces to the generic type $\boldsymbol{\kappa}\otimes\boldsymbol{\tau}$, it is generic due to Lemma 5.9, and, hence, it cannot be an unramified twist of a distinguished representation according to the second part of Lemma 6.10. □

Remark 6.12. — Note that as an unramified character of $G_\circ$ always has an unramified extension to G, Part (i) of Theorem 6.11 can also be stated as π is relatively banal, if and only if all its lifts are distinguished by an unramified character.

7. Asai L-factors of cuspidal representations

7.1. Asai L-factors. — Let N be the maximal unipotent subgroup of the subgroup of upper triangular matrices in G, and $N_\circ = N^\sigma$. Let ψ be a non-degenerate R-valued character of N trivial on $N_\circ$. Let π be an R-representation of G of *Whittaker type* (i.e. of finite length with a one-dimensional space of Whittaker functionals) with Whittaker model $\mathcal{W}(\pi,\psi)$. We refer to [17, Section 2] for more details as well as basic facts about Whittaker functions and their analytic behaviour. For $W \in \mathcal{W}(\pi,\psi)$ and $\Phi \in \mathscr{C}_c^\infty(F_o^n)$ and $l \in \mathbb{Z}$, we define the local *Asai coefficient* to be

$$\mathrm{I}_{\mathrm{As}}^l(\mathrm{X},\Phi,\mathrm{W}) = \int_{\substack{\mathrm{N_o}\backslash \mathrm{G_o}\\ \mathrm{val}(\det(g))=l}} \mathrm{W}(g)\Phi(\eta_n g)\, dg, \tag{1}$$

where η_n denotes the row vector $(0 \cdots 0\ 1)$, and dg denotes a right invariant measure on $\mathrm{N_o} \setminus \mathrm{G_o}$ with values in R. We refer the reader to [17, Section 2.2] for details on R-valued equivariant measures on homogeneous spaces and their properties. The integrand in the Asai coefficient has compact support, so it is well defined and it, moreover, vanishes for $l << 0$. We define the *Asai integral* of $W \in \mathcal{W}(\pi,\psi)$ and $\Phi \in \mathscr{C}_c^\infty(F_o^n)$ to be the formal Laurent series

$$\mathrm{I}_{\mathrm{As}}(\mathrm{X},\Phi,\mathrm{W}) = \sum_{l\in\mathbb{Z}} \mathrm{I}_{\mathrm{As}}^l(\mathrm{X},\Phi,\mathrm{W})\mathrm{X}^l. \tag{2}$$

In exactly the same way as in [17, Theorem 3.5], we deduce the following lemma:

Lemma 7.1. — *For* $W \in \mathcal{W}(\pi,\psi)$ *and* $\Phi \in \mathscr{C}_c^\infty(F_o^n)$, $\mathrm{I}_{\mathrm{As}}(\mathrm{X},\Phi,\mathrm{W}) \in \mathrm{R}(\mathrm{X})$ *is a rational function. Moreover, as* W *varies in* $\mathcal{W}(\pi,\psi)$, *and* Φ *varies in* $\mathscr{C}_c^\infty(F_o^n)$, *these functions generate a* $\mathrm{R}[\mathrm{X}^{\pm 1}]$*-fractional ideal of* $\mathrm{R}(\mathrm{X})$ *independent of the choice of* ψ.

In the setting of the lemma, it follows that there is a unique generator $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$, which is a Euler factor and is independent of the character ψ. We call $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ the *Asai* L*-factor* of π.

For $s(X)$ of the form $1/P(X)$ for $P(X) \in\in \overline{\mathbb{Z}_\ell}[X]$ with non-zero reduction modulo ℓ, we write $r_\ell(s(X)) = 1/r_\ell(P(X))$. If P and Q are two non-zero elements of $k[X]$ for any field k, we write

$$1/P(X) \mid 1/Q(X) \quad \text{if } P(X) \mid Q(X).$$

Lemma 7.2. — *Let* π *be an integral cuspidal* $\overline{\mathbb{Q}_\ell}$*-representations of* G *and* $\overline{\pi}$ *its reduction modulo* ℓ.

(i) *Then,* $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ *is the inverse of a polynomial in* $\overline{\mathbb{Z}_\ell}[\mathrm{X}]$.
(ii) *Moreover,*

$$\mathrm{L}_{\mathrm{As}}(\mathrm{X},\overline{\pi}) \mid r_\ell(\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi))$$

with constant term equal to 1.

Proof. — The first part (i) follows from the asymptotic expansion of Whittaker functions as in [17, Corollary 3.6]. The second part (ii) follows by imitating the proof of [17, Theorem 3.13]. We recall the argument here: by definition, we can write the L-factor $\mathrm{L_{As}}(\mathrm{X},\overline{\pi})$ as a finite sum of Asai integrals; for $i\in\{1,\dots,r\}$, there are $\Phi_i\in\mathscr{C}_c^\infty(\mathrm{F}_\mathsf{o}^n)$ and $\mathrm{W}_i\in\mathcal{W}(\overline{\pi},\overline{\psi})$, such that

$$\mathrm{L_{As}}(\mathrm{X},\overline{\pi})=\sum_{i=1}^r \mathrm{I_{As}}(\mathrm{X},\Phi_i,\mathrm{W}_i).$$

By [17, Lemma 2.23] there are Whittaker functions $\mathrm{W}_{i,e}\in\mathcal{W}(\pi,\psi)$ that take values in $\overline{\mathbb{Z}_\ell}$, such that $\mathrm{W}_i=r_\ell(\mathrm{W}_{i,e})$, and clearly there are Schwartz functions $\Phi_{i,e}\in\mathscr{C}_c^\infty(\mathrm{F}_\mathsf{o}^n)$ that take values in $\overline{\mathbb{Z}_\ell}$, such that $\Phi_i=r_\ell(\Phi_{i,e})$. Moreover,

$$\sum_{i=1}^r \mathrm{I_{As}}(\mathrm{X},\Phi_{i,e},\mathrm{W}_{i,e})\in \mathrm{L_{As}}(\mathrm{X},\pi)\overline{\mathbb{Q}_\ell}[\mathrm{X}^{\pm1}]\cap\overline{\mathbb{Z}_\ell}((\mathrm{X}))=\mathrm{L_{As}}(\mathrm{X},\pi)\overline{\mathbb{Z}_\ell}[\mathrm{X}^{\pm1}],$$

and, hence, $\mathrm{L_{As}}(\mathrm{X},\overline{\pi})=\sum_{i=1}^r \mathrm{I_{As}}(\mathrm{X},\Phi_i,\mathrm{W}_i)\in r_\ell(\mathrm{L_{As}}(\mathrm{X},\pi))\overline{\mathbb{F}_\ell}[\mathrm{X}^{\pm1}]$. □

As we shall see later, strict divisions occur.

7.2. Test vectors. — In [4], test vectors for the Asai integral of a distinguished supercuspidal $\overline{\mathbb{Q}_\ell}$-representation were given with the Asai integral computed explicitly. The pro-order of a compact open subgroup of G_o may be zero in $\overline{\mathbb{F}_\ell}$, and so one cannot normalise a right Haar measure with values in $\overline{\mathbb{F}_\ell}$ arbitrarily. So, for compatibility with reduction modulo ℓ, we need to be more careful with normalisation of measures over $\overline{\mathbb{Q}_\ell}$. We set $\mathrm{K}=\mathrm{GL}_n(\mathscr{O})$ and $\mathrm{K}_\mathsf{o}=\mathrm{K}^\sigma$, $\mathrm{K}_\mathsf{o}^1=\mathrm{I}_n+\mathcal{M}_n(\mathscr{O}_\mathsf{o})$, and P the σ-stable mirabolic subgroup of G of all elements with final row $(\,0\cdots 0\;1\,)$.

Definition 7.3. — A triple $(\mathbf{J},\boldsymbol{\lambda},\psi)$ with $(\mathbf{J},\boldsymbol{\lambda})$ a σ-self-dual R-type, and ψ a distinguished non-degenerate character of N satisfying conditions (i) and (ii) of [4, Lemma 6.8] will be called an *adapted type.*

Remark 7.4. —
(i) In particular, if $(\mathbf{J},\boldsymbol{\lambda},\psi)$ is an adapted type, the type $(\mathbf{J},\boldsymbol{\lambda})$ is a σ-self-dual ψ-generic type (in particular, ψ is distinguished).
(ii) By the proof of [4, Lemma 6.8], if π is a σ-self-dual cuspidal R-representation, it contains an adapted type, the point of the remark being that the N of [4, Lemma 6.8] can be chosen to be our N: the group of unipotent upper triangular matrices in G.

Now let π be a σ-self-dual cuspidal R-representation and $(\mathbf{J},\boldsymbol{\lambda},\psi)$ be an adapted type of π. We associate to $(\mathbf{J},\boldsymbol{\lambda},\psi)$ the *Paskunas–Stevens Whittaker function* $\mathrm{W}_{\boldsymbol{\lambda}}\in\mathcal{W}(\pi,\psi)$ defined in [4, (6.3)]. Note that $\mathrm{W}_{\boldsymbol{\lambda}}$ takes values in $\overline{\mathbb{Z}_\ell}$ as soon as π is integral (see, for example, [18, Lemma 10.2]). One of the main results of [4] is that this Whittaker function is a test vector for the Asai L-factor:

Proposition 7.5 ([4, Theorem 7.14]). — *Let π be a distinguished supercuspidal $\overline{\mathbb{Q}_\ell}$-representation of* G *and* $\mathrm{W}_{\boldsymbol{\lambda}}$ *be the explicit Whittaker function defined* *above. There is a unique normalisation of the invariant measure on* $\mathrm{N_o} \backslash \mathrm{G_o}$*, such that*

$$\mathrm{I_{As}}(\mathrm{X}, \mathbf{1}_{\mathfrak{o}_{\mathrm{F_o}}^n}, \mathrm{W}_{\boldsymbol{\lambda}}) = (q_{\mathrm{o}} - 1)(q_{\mathrm{o}}^{n/e_{\mathrm{o}}(\pi)} - 1)\mathrm{L_{As}}(\mathrm{X}, \pi).$$

The volume of $\mathrm{N_o} \cap \mathrm{K_o^1} \backslash \mathrm{K_o^1}$ *is of the form* p^l *for* $l \in \mathbb{Z}$ *with this normalisation.*

Proof. — We start with Haar measures dg and dn on $\mathrm{G_o}$ with values in $\overline{\mathbb{Q}_\ell}$ normalised by $dg(\mathrm{K_o^1}) = 1$ and $dg(\mathrm{N_o} \cap \mathrm{K_o^1}) = 1$, which, in turn, normalises the measure (still denoted dg) on the quotient $\mathrm{N_o} \backslash \mathrm{G_o}$.

With this normalisation, which is the exact parallel of the normalisation used in [18] for the analogue Rankin–Selberg computation, first of all we get an extra factor of $(q_{\mathrm{o}} - 1)$ on the top of the Tate factor defined before [4, Lemma 7.11]. Then there is a factor $dk((\mathrm{P}^\sigma \cap \mathrm{K}^\sigma) \backslash \mathrm{J}^\sigma)$, which appears in [4, Lemma 6.11], and we have

$$dk((\mathrm{P}^\sigma \cap \mathrm{J}^\sigma) \backslash \mathrm{J}^\sigma) = dk((\mathrm{P}^\sigma \cap (\mathrm{J}^1)^\sigma) \backslash (\mathrm{J}^1)^\sigma)|\mathrm{J}^\sigma/(\mathrm{P}^\sigma \cap \mathrm{J}^\sigma)(\mathrm{J}^1)^\sigma)|.$$

As $dk((\mathrm{P}^\sigma \cap (\mathrm{J}^1)^\sigma) \backslash (\mathrm{J}^1)^\sigma)$ is a (possibly negative) power of p, we can renormalise our measure to remove it. The image of $\mathrm{P} \cap \mathrm{J}$ modulo J^1 is a σ-stable mirabolic $\overline{\mathrm{P}}_m(\mathrm{k_E})$ of J/J^1, and we thus have

$$|\mathrm{J}^\sigma/(\mathrm{P}^\sigma \cap \mathrm{J}^\sigma)(\mathrm{J}^1)^\sigma)| = |\,\mathrm{GL}_m(\mathrm{k_E})^\sigma/\overline{\mathrm{P}}_m(\mathrm{k_E})^\sigma| = q_0^{n/e_{\mathrm{o}}(\pi)} - 1$$

due to Lemma 6.5. □

Corollary 7.6. — *Suppose that* π *is an unramified twist of a relatively banal distinguished cuspidal* $\overline{\mathbb{F}_\ell}$*-representation, and let* $\widetilde{\pi}$ *be a supercuspidal lift of* π*. Then,*

$$\mathrm{L_{As}}(\mathrm{X}, \pi) = r_\ell(\mathrm{L_{As}}(\mathrm{X}, \widetilde{\pi})).$$

Proof. — Let $\widetilde{\pi}$ be such a lift. Due to Theorem 6.11 there is an unramified character $\widetilde{\chi}$ of $\mathrm{F}^\times$, such that $\widetilde{\pi}_0 = (\widetilde{\chi} \circ \det)^{-1} \otimes \widetilde{\pi}$ is distinguished. Let $(\mathbf{J}, \widetilde{\boldsymbol{\lambda}}, \widetilde{\psi})$ be an adapted type of $\widetilde{\pi}_{\mathrm{o}}$. Proposition 7.5 then implies that

$$\mathrm{I_{As}}(\mathrm{X}, \mathbf{1}_{\mathfrak{o}_{\mathrm{F_o}}^n}, \mathrm{W}_{\boldsymbol{\lambda}}) = (q_{\mathrm{o}} - 1)(q_{\mathrm{o}}^{n/e_{\mathrm{o}}(\pi)} - 1)\mathrm{L_{As}}(\mathrm{X}, \pi_{\mathrm{o}}).$$

Then, setting $\pi_{\mathrm{o}} = r_\ell(\widetilde{\pi}_{\mathrm{o}})$, we deduce that

$$\mathrm{L_{As}}(\mathrm{X}, \pi_{\mathrm{o}}) = r_\ell(\mathrm{L_{As}}(\mathrm{X}, \widetilde{\pi}_{\mathrm{o}}))$$

in the exact same way that [18, Corollary 10.1] follows from [18, Proposition 9.3]. We obtain the statement of the corollary by twisting $\widetilde{\pi}_{\mathrm{o}}$ by $\widetilde{\chi} \circ \det$ in this equality, as it sends X to $\chi(\varpi_{\mathrm{o}})\mathrm{X}$ on the left-hand side and to $\widetilde{\chi}(\varpi_{\mathrm{o}})\mathrm{X}$ on the right-hand side. □

7.3. Asai L-factors of cuspidal representations. — We first recall the computation of the Asai L-function of a cuspidal $\overline{\mathbb{Q}_\ell}$-representation:

PROPOSITION 7.7 ([4, Corollary 7.6] and Remark 7.7). — *Let π be a cuspidal $\overline{\mathbb{Q}_\ell}$-representation. If no unramified twist of π is distinguished, then* $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi) = 1$. *If π is distinguished, then*

$$\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi) = \frac{1}{1-\mathrm{X}^{n/e_{\mathsf{o}}(\pi)}}.$$

This gives a complete description in the cuspidal case; as for an unramified character $\chi : \mathrm{F}^\times \to \mathrm{K}^\times$, we have

$$\mathrm{L}_{\mathrm{As}}(\mathrm{X},(\chi\circ\det)\otimes\pi) = \mathrm{L}_{\mathrm{As}}(\chi(\varpi_{\mathsf{o}})\mathrm{X},\pi).$$

THEOREM 7.8. — *Let π be a cuspidal $\overline{\mathbb{F}_\ell}$-representation of* $\mathrm{GL}_n(\mathrm{F})$.

(i) *If π is an unramified twist $(\chi\circ\det)\otimes\pi_0$ of a relatively banal distinguished representation π_0, then*

$$\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi) = \frac{1}{1-(\chi(\varpi_{\mathsf{o}})\mathrm{X})^{n/e_{\mathsf{o}}(\pi)}}.$$

(ii) *If π is not an unramified twist of a relatively banal distinguished representation, then*

$$\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi) = 1.$$

Proof. — If π is an unramified twist of a relatively banal distinguished representation, the statement follows, for example, from Corollary 7.6 and Proposition 7.7.

If π is not an unramified twist of a relatively banal distinguished representation, it has a supercuspidal lift $\widetilde{\pi}$, which is not an unramified twist of a distinguished representation due to Theorem 6.11. By Proposition 7.7 we have $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\widetilde{\pi}) = 1$, and, hence, $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi) = 1$ as $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi) \mid r_\ell(\mathrm{L}_{\mathrm{As}}(\mathrm{X},\widetilde{\pi}))$ by Lemma 7.2. □

REMARK 7.9. — Note that when $\ell = 2$, we are in case (ii) of Theorem 7.8, and $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi) = 1$. This can also be seen directly from the asymptotics of Whittaker functions. Without entering the details as we do not need to, the asymptotic expansion of Whittaker functions on the diagonal torus allow one to express the Asai integrals in terms of Tate integrals for $\mathrm{F}_{\mathsf{o}}^\times$, and these Tate integrals are all 1 because $q = 1[2]$, as shown in [22].

8. Distinction and poles of the Asai L-factor

8.1. Characterisation of the poles of the Asai L-factor. — We are now in position to prove the main results of this paper:

Theorem 8.1. — *Let π be a cuspidal $\overline{\mathbb{F}_\ell}$-representation of* G*. Then,* $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ *has a pole at* $\mathrm{X}=1$*, if and only if π is relatively banal distinguished. In this case, the pole is of order ℓ^r, where $n/e_{\mathsf{o}}(\pi)=a\ell^r$ with a prime to ℓ.*

Proof. — If $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ has a pole at $\mathrm{X}=1$, in particular $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ is not equal to 1, and, hence, the representation π is an unramified twist of a relatively banal distinguished cuspidal $\overline{\mathbb{F}_\ell}$-representation π_0, say $\pi=(\chi\circ\det)\otimes\pi_0$, with χ an unramified character of $\mathrm{F}^\times$. Denote by χ_{o} the restriction of χ to $\mathrm{F}_{\mathsf{o}}^\times$, then by Theorem 7.8,

$$\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)=\frac{1}{1-(\chi_{\mathsf{o}}(\varpi_{\mathsf{o}})\mathrm{X})^{n/e_{\mathsf{o}}(\pi)}}=\frac{1}{(1-(\chi_{\mathsf{o}}(\varpi_{\mathsf{o}})\mathrm{X})^{a})^{\ell^r}}$$

which has a pole at $\mathrm{X}=1$, if and only if $\chi_{\mathsf{o}}(\varpi_{\mathsf{o}})^{n/e_{\mathsf{o}}(\pi)}=1$. By Theorem 5.16 this implies that $\chi_{\mathsf{o}}\circ\det$ belongs to $\mathrm{X}_{\mathsf{o}}(\pi)$, i.e. that $\pi=(\chi\circ\det)\otimes\pi_0$ is distinguished. The converse is just Theorem 7.8. □

Remark 8.2. — Note that our proof of Theorem 8.1 is very different from the proof over the field of complex numbers. In the proof above, the direction π relatively banal distinguished implies $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ having a pole at 1 is an immediate consequence of Theorem 7.8 and works for complex representations as well (in which case, we consider all cuspidal distinguished $\mathbb{C}$-representations as "relatively banal") due to [4, Corollary 7.6]. Saying this is not enough to claim a proof in the case of complex cuspidal representations different from the original one given in [1, Theorem 1.4], as the way the equality of [4, Corollary 7.6] is obtained is a consequence of [20, Proposition 6.3], which itself follows either from [20, Theorem 3.1] or from [1, Theorem 1.4] and [14, Theorem 4], together with the fact that the poles of the Asai L-factor are simple in the cuspidal case. However, the first equality in [4, Theorem 7.14] is independent of the results cited above, and it in particular implies that, if π is a cuspidal distinguished $\mathbb{C}$-representation, its Asai L-factor has a pole at $\mathrm{X}=1$. The proof of the other implication that we give also works in the complex case, and is again different from the original proof given in [14, Theorem 4]. Kable shows that, if $\mathrm{L}_{\mathrm{As}}(\mathrm{X},\pi)$ has a pole at $\mathrm{X}=1$, the rational function $(1-\mathrm{X})\mathrm{I}_{\mathrm{As}}(X,\mathrm{W},\Phi)$ is regular at $\mathrm{X}=1$ and that up to a non-zero constant independent of W and Φ its value at $\mathrm{X}=1$ is given by

$$\Phi(0)\int_{\mathrm{Z}^\sigma\mathrm{N}^\sigma\backslash\mathrm{G}^\sigma}\mathrm{W}(h)dh.$$

As by assumption the Asai L-factor has a pole at $\mathrm{X}=1$, the G^σ-invariant linear form

$$\mathcal{L}_\pi:\mathrm{W}\mapsto\int_{\mathrm{Z}^\sigma\mathrm{N}^\sigma\backslash\mathrm{G}^\sigma}\mathrm{W}(h)dh$$

is non-zero. Note that to adapt this proof to the modular setting with $R = \overline{\mathbb{F}_\ell}$ we would need to take $1-X^{n/e_{\mathsf{o}}(\pi)}$, where Kable takes $1-X$ (this does not matter over $\mathbb{C}$, as both polynomials have a simple zero at $X = 1$) to get the correct order of the pole, although from Kable's proof, one sees that the natural choice is, in fact, $1-X^n$. However, we claim that this cannot be done in general, as we shall now see that the local period $\mathcal{L}_\pi$, although well defined for cuspidal $\overline{\mathbb{F}_\ell}$-representations, might vanish even for relatively banal distinguished cuspidal $\overline{\mathbb{F}_\ell}$-representations.

8.2. The G_{o}-period of cuspidal distinguished representations. — Let π be a cuspidal distinguished R-representation; we still denote by ψ a distinguished non-degenerate character of N. There are two natural G^σ-invariant linear forms on $\mathcal{W}(\pi,\psi)$. The first is

$$\mathcal{P}_\pi : W \mapsto \int_{N^\sigma\backslash P^\sigma} W(p)dp$$

which is well-defined and non-zero due to [28, Chapter III, Theorem 1.1]. Although it does not look G^σ-invariant, it is so by [4, Proposition B.23]. Let $(\mathbf{J},\boldsymbol{\lambda},\psi)$ be an adapted type in π. A natural test vector for this linear form is the Paskunas–Stevens Whittaker function $W_{\boldsymbol{\lambda}}$; we have $\mathcal{P}_\pi(W_{\boldsymbol{\lambda}}) \neq 0$ according to the proof of [4, Proposition 6.5].

The second is

$$\mathcal{L}_\pi : W \mapsto \int_{Z^\sigma N^\sigma\backslash G^\sigma} W(h)dh.$$

It is G^σ-invariant by definition, and well defined, as all $W \in \mathcal{W}(\pi,\psi)$ have compact support on $N \setminus P$; they have compact support of $ZN \setminus G$ due to the Iwasawa decomposition $G = PZK$. By cuspidal multiplicity one for the pair (P, P^σ) ([4, Proposition B.23]), $\mathcal{L}_\pi$ is a multiple of $\mathcal{P}_\pi$, and the proportionality constant between them turns out to be a very interesting quantity; this scalar is related to the formal degrees of complex discrete series representations of unitary groups (see [2] and Remark 8.4). When $R = \overline{\mathbb{Q}_\ell}$ the linear form, $\mathcal{L}_\pi$ is non-zero (Remark 8.2); here, we solve the problem of understanding when $\mathcal{L}_\pi$ is non-zero when $R = \overline{\mathbb{F}_\ell}$:

Theorem 8.3. — *Let π be a cuspidal distinguished $\overline{\mathbb{F}_\ell}$-representation of* G, *then $\mathcal{L}_\pi$ is non-zero, if and only if:*

(i) *π is relatively banal.*
(ii) *ℓ does not divide $e_{\mathsf{o}}(\pi)$.*

Proof. — Due to the Iwasawa decomposition $G = PZK$, we have the equality:

$$\int_{Z^\sigma N^\sigma\backslash G^\sigma} W(h)dh = \int_{K^\sigma\cap P^\sigma\backslash K^\sigma}\int_{N^\sigma\backslash P^\sigma} W(pk)|\det(pk)|_{\mathsf{o}}^{-1}dpdk.$$

We introduce the power series

$$\mathrm{I}_{\mathrm{As},(0)}(\mathrm{X},\mathrm{W}) = \sum_{l\in\mathbb{Z}} \left(\int_{\mathrm{N}^\sigma\backslash\mathrm{P}^{\sigma(l)}} \mathrm{W}(p)|\det(p)|_\circ^{-1}dp\right)\mathrm{X}^l$$

where $\mathrm{P}^{\sigma(l)} = \{p\in\mathrm{P}^\sigma,\ \mathrm{val}_{\mathrm{F}_\circ}(\det(p)) = l\}$ which is, in fact, a Laurent polynomial, as π is cuspidal, so that

$$\mathcal{P}_\pi(\mathrm{W}) = \mathrm{I}_{\mathrm{As},(0)}(1,\mathrm{W}).$$

Now suppose that π is not relatively banal; then, π is $|\det(\,\cdot\,)|_\circ$-distinguished, and appealing to [4, Proposition B.23], it means that the linear form

$$\mathcal{P}_{\pi,|\det(\,\cdot\,)|_\circ} : \mathrm{W}\mapsto \int_{\mathrm{N}^\sigma\backslash\mathrm{P}^\sigma} \mathrm{W}(p)|\det(p)|_\circ^{-1}dp$$

is $|\det(\,\cdot\,)|_\circ$-equivariant under the action of $\mathrm{G}_\circ$. So, in particular, up to possible renormalisation of the invariant measure,

$$\begin{aligned}\int_{\mathrm{Z}^\sigma\mathrm{N}^\sigma\backslash\mathrm{G}^\sigma} \mathrm{W}(h)dh &= \mathrm{vol}(\mathrm{K}^\sigma\cap\mathrm{P}^\sigma\setminus\mathrm{K}^\sigma)\mathcal{P}_{\pi,|\det(\,\cdot\,)|_\circ}(\mathrm{W})\\ &= (q_\circ^n-1)\mathcal{P}_{\pi,|\det(\,\cdot\,)|_\circ}(\mathrm{W}) = 0\end{aligned}$$

as $q_\circ^{n/e_\circ(\pi)} = 1$.

So it remains to understand what happens when π is relatively banal. As we said, by multiplicity one we know that $\mathcal{L}_\pi = \lambda\mathcal{P}_\pi$ and we noticed that $\mathcal{P}_\pi(\mathrm{W}_{\boldsymbol\lambda}) \neq 0$. Hence, $\mathcal{L}_\pi = 0$, if and only if $\mathcal{L}_\pi(\mathrm{W}_{\boldsymbol\lambda}) = 0$. However, following the proof of [18, Theorem 9.1] at the end of [18, p. 19] or the proof of [4, Theorem 7.14], one gets up to a possible renormalisation of invariant measures:

$$\begin{aligned}\mathcal{L}_{\pi,\mathrm{X}}(\mathrm{W}_{\boldsymbol\lambda}) &:= \int_{\mathrm{K}^\sigma\cap\mathrm{P}^\sigma\backslash\mathrm{K}^\sigma} \mathrm{I}_{\mathrm{As},(0)}(\mathrm{X},\rho(k)\mathrm{W}_{\boldsymbol\lambda})dk\\ &= (q_\circ-1)(q_\circ^{n/e_\circ(\pi)}-1)\frac{1-\mathrm{X}^n}{1-\mathrm{X}^{n/e_\circ(\pi)}}.\end{aligned}$$

Now, the value at $\mathrm{X}=1$ of $\mathcal{L}_{\pi,\mathrm{X}}$ is $\mathcal{L}_\pi$, so $\mathcal{L}_\pi$ vanishes, if and only if $\frac{1-\mathrm{X}^n}{1-\mathrm{X}^{n/e_\circ(\pi)}}$ vanishes at $\mathrm{X}=1$. However, the order of the zero of $1-\mathrm{X}^n$ is the $\ell^{\mathrm{val}_\ell(n)}$, whereas that of the zero of $1-\mathrm{X}^{n/e_\circ(\pi)}$ is $\ell^{\mathrm{val}_\ell(n/e_\circ(\pi))}$. This means that, if π is relatively banal, $\mathcal{L}_\pi$ is non-zero, if and only if $\mathrm{val}_\ell(n) = \mathrm{val}_\ell(n/e_\circ(\pi))$, i.e., if and only if ℓ does not divide $e_\circ(\pi)$. □

Remark 8.4. — Here, we explain how this vanishing result modulo ℓ is related to the vanishing of the ℓ-adic proportionality constant between $\mathcal{L}$ and $\mathcal{P}$. For an algebraically closed field $\mathbf{C}$, write $\mathrm{Cusp}_{\mathbf{C},\mathrm{dist}}(\mathrm{G})$ for the set of isomorphism classes of distinguished cuspidal $\mathbf{C}$-representations. Fix an isomorphism $\mathbb{C}\simeq\overline{\mathbb{Q}_\ell}$; this induces a bijection $\mathrm{Cusp}_{\mathbb{C},\mathrm{dist}}(\mathrm{G})\to\mathrm{Cusp}_{\overline{\mathbb{Q}_\ell},\mathrm{dist}}(\mathrm{G})$, depending on the choice of isomorphism.

Let $\pi \in \mathrm{Cusp}_{\overline{\mathbb{Q}_\ell},\mathrm{dist}}(\mathrm{G})$ and $\psi : \mathrm{N} \to \overline{\mathbb{Z}_\ell}^\times$ be an N^σ-distinguished non-degenerate character of N. Then, by [1, Corollary 1.2], there exists $\mu \in \overline{\mathbb{Q}_\ell}$, such that

$$\mathcal{L}_\pi = \mu \mathcal{P}_\pi, \tag{3}$$

Let c_π denote the central character of π and Res_P denote the restriction of Whittaker functions to P. Then,

$$\mathcal{W}(\pi,\psi) \subset \mathrm{ind}_{\mathrm{ZN}}^{\mathrm{G}}(c_\pi \otimes \psi) \text{ and } \mathrm{Res}_\mathrm{P}(\mathcal{W}(\pi,\psi)) \subset \mathrm{ind}_{\mathrm{N}}^{\mathrm{P}}(\psi),$$

the first fact being a consequence of the second, which has been known since [5]. Now let $\mathcal{W}(\pi,\psi)_e$ denote the $\overline{\mathbb{Z}_\ell}$-submodule of $\mathcal{W}(\pi,\psi)$ consisting of Whittaker functions with values in $\overline{\mathbb{Z}_\ell}$. It follows from [30, Theorem 2] and [29, Theorem 2] that $\mathcal{W}(\pi,\psi)_e$ is a lattice in $\mathcal{W}(\pi,\psi)$, and $\mathrm{Res}_\mathrm{P}(\mathcal{W}(\pi,\psi)_e) = \mathrm{ind}_{\mathrm{N}}^{\mathrm{P}}(\psi,\overline{\mathbb{Z}_\ell})$ is a lattice in $\mathrm{Res}_\mathrm{P}(\mathcal{W}(\pi,\psi))$, reducing to $\mathcal{W}(\pi,\psi)$ and $\mathrm{Res}_\mathrm{P}(\mathcal{W}(\pi,\psi))$, respectively.

Finally, from [17, Section 2.2], there are appropriate ℓ-adic and ℓ-modular invariant measures on $\mathrm{Z}^\sigma\mathrm{N}^\sigma \setminus \mathrm{G}^\sigma$ and $\mathrm{N}^\sigma \setminus \mathrm{P}^\sigma$, such that

$$r_\ell(\mathcal{L}_\pi(\mathrm{W}_e)) = \mathcal{L}_{r_\ell(\pi)}(r_\ell(\mathrm{W}_e)), \tag{4}$$

$$r_\ell(\mathcal{P}_\pi(\mathrm{W}_e)) = \mathcal{P}_{r_\ell(\pi)}(r_\ell(\mathrm{W}_e)) \tag{5}$$

for all $\mathrm{W}_e \in \mathrm{W}(\pi,\psi)_e$ and $\mathcal{P}_\pi(\mathrm{Res}_\mathrm{P}(\mathcal{W}(\pi,\psi)_e)) = \overline{\mathbb{Z}_\ell}$.

Evaluating Equation 3 on an element $\mathrm{W}_e \in \mathcal{W}(\pi,\psi)_e$, such that $\mathcal{P}_\pi(\mathrm{W}_e) = 1$, we deduce that $\mu \in \overline{\mathbb{Z}_\ell}$. Now, Theorem 8.3 and Equations 4 and 5 imply that ℓ divides μ, if and only if either $r_\ell(\pi)$ is not relatively banal or $\ell \mid e_\mathsf{o}(\pi)$. Running over all $\ell \neq p$, we recover the radical of the p-regular part of μ (explicitly using the type theoretic definition of relatively banal, Definition 6.2).

As was mentioned already, this scalar μ is a very interesting and subtle quantity. By [2, Theorem 7.1], we have

$$\mathcal{L}_\pi = \lambda \frac{d(\rho)}{d(\pi)} \mathcal{P}_\pi,$$

where λ is a constant independent of π, ρ is the cuspidal $\overline{\mathbb{Q}_\ell}$-representation of the quasi-split unitary group in n-variables defined over F_o which base changes to π (stably or unstably depending on the parity of n), and $d(\rho)$ and $d(\pi)$ denote the formal degrees of ρ and π, respectively, under the normalisation of invariant measures of [12]. One could check that the formal degrees are rational for our well-chosen measures (and λ as well), and preserved under the bijection $\mathrm{Cusp}_{\mathbb{C},\mathrm{dist}}(\mathrm{G}) \to \mathrm{Cusp}_{\overline{\mathbb{Q}_\ell},\mathrm{dist}}(\mathrm{G})$.

While we have explained how Theorem 8.3 tells us exactly when μ vanishes modulo ℓ, we could also go in the other direction. By [12] and [6], the constant μ could be computed explicitly, and its explicit description would give a different proof of Theorem 8.3. It should be clear to the reader that the amount of work required for such a proof is much more considerable than that of the proof given above.

8.3. Comparison of banal and relatively banal. — Finally, we compare our notion of relatively banal distinguished with the notion of banal representation introduced in [24] for cuspidal representations.

By [24, Remarque 8.15] a cuspidal $\overline{\mathbb{F}_\ell}$-representation π of $\mathrm{G}_\circ$ is banal, if and only if

$$|\det(\ )|_\circ \otimes \pi \not\simeq \pi.$$

However, the map

$$b : \pi \mapsto \pi \otimes \pi^\vee$$

is a bijection between the set of (isomorphism classes of) irreducible representations of $\mathrm{G}_\circ$ and the set of $\Delta(\mathrm{G}_\circ)$-distinguished irreducible representations of $\mathrm{G}' = \mathrm{G}_\circ \times \mathrm{G}_\circ$, where Δ is the diagonal embedding of $\mathrm{G}_\circ$ into G'. In particular, π (seen as the distinguished representation $b(\pi)$ of G') is banal, if and only if $|\det(\ )|_\circ \otimes b(\pi) = (|\det(\ )|_\circ \otimes \pi) \otimes \pi^\vee$ is not distinguished. Note that $|\ |_\circ$ plays the same role for the split quadratic algebra $(\mathrm{F}_\circ \times \mathrm{F}_\circ)/\mathrm{F}_\circ$ that $|\ |_\circ$ plays for $\mathrm{F}/\mathrm{F}_\circ$, i.e. it is a square root of the absolute value on the bigger algebra. So, this proves the exact analogy of banal cuspidal representations of $\mathrm{G}_\circ$ and relatively banal distinguished cuspidal representations of G according to Corollary 6.3.

The analogy can also be seen at the L-factor level; it follows from [18, Theorem 4.9] that, if $\pi \otimes \pi'$ is a cuspidal representation of G', then the Rankin–Selberg L-factor $\mathrm{L}(\mathrm{X}, \pi, \pi')$ (which can be thought of as the Asai L-factor of $\pi \otimes \pi'$) has a pole at $\mathrm{X} = 1$, if and only if $\pi \otimes \pi'$ is $\Delta(\mathrm{G}_\circ)$-distinguished, and π is banal, which is the exact analogue of Theorem 8.1 replacing banal with relatively banal.

Finally, in terms of the type theory definition, a cuspidal representation π of $\mathrm{G}_\circ$ is banal, if and only if $q_\circ^{n/e(\pi)} \not\equiv 1[\ell]$, but tracking down how $e(\pi)$ is defined with respect to π in terms of type theory (more precisely lattice periods) shows that it plays the same role for the $\Delta(\mathrm{G}_\circ)$-distinguished representation $b(\pi)$ of G' that $e_\circ(\tau)$ plays for a distinguished cuspidal representation τ of G.

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