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MAURICE AUSLANDER
RAGNAR-OLAF BUCHWEITZ

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THE HOMOLOGICAL THEORY OF MAXIMAL COHEN-MACAULAY APPROXIMATIONS

by

Maurice Auslander (Brandeis) and Ragnar-Olaf Buchweitz (Toronto)

Summary: Let R be a commutative noetherian Cohen-Macaulay ring which admits a dualizing module. We show that for any finitely generated R-module N there exists a maximal Cohen-Macaulay R-module M which surjects onto N and such that any other surjection from a maximal Cohen-Macaulay module onto N factors over it. Dually, there is a finitely generated R-module I of finite injective dimension into which N embeds, universal for such embeddings. We prove and investigate these results in the broader context of abelian categories with a suitable subcategory of "maximal Cohen-Macaulay objects" extracting for this purpose those ingredients of Grothendieck-Serre duality theory which are needed.

Résumé: Soit R un anneau commutatif, noethérien et de Cohen-Macaulay, tel que un module dualisant existe pour R. On démontre que pour chaque R-module N de type fini il existe un R-module M de profondeur maximale et un homomorphisme surjectif de M sur N, tel que toute autre surjection d'un tel module sur N s'en factorise. De manière duale, il existe aussi un plongement de N dans un R-module I de type fini et de dimension injective finie, universelle pour telles plongements. Nous démontrons et examinons ces résultats dans le cadre des catégories abéliennes avec une sous-catégorie convenable des "objets de Cohen-Macaulay maximaux", à cet effet mettant en évidence les propriétés de la théorie de dualité de Grothendieck-Serre dont on a besoin.

§0. *A Commutative Introduction*

The aim of this work is to analyze the framework in which the theory of *maximal Cohen-Macaulay approximations* can be developed. Instead of outlining right away the abstract results, we want to start by describing the situation in the classical case of a commutative local noetherian ring R with maximal ideal $\mathbf{m}$ and residue class field $k = R/\mathbf{m}$.

Assume that R admits a *dualizing module* ω. Then R is Cohen-Macaulay, and the finitely generated R-modules M which are maximal Cohen-Macaulay in the sense that $\mathrm{depth}_{\mathbf{m}} M = \dim R$ can be characterized homologically as those modules for which $\mathrm{Ext}^i_R(M, \omega) = 0$ for $i \neq 0$.

Our main result can then be paraphrased as saying that **R-mod**, the category of finitely generated R-modules, is obtained by glueing together the orthogonal subcategories of *modules of finite injective dimension* over R and the category of *maximal Cohen-Macaulay modules* along their common intersection which is spanned by ω.

More precisely, let us recall that ω is *dualizing* for the local ring $(R,\mathfrak{m},k)$ if and only if it satisfies the following three conditions:

(i) ω is finitely generated and of finite injective dimension over R.

(ii) The natural ring homomorphism which is given by multiplication with scalars from R on ω, $R \to \mathrm{Hom}_R(\omega,\omega)$ is an isomorphism.

(iii) For any integer $i \neq 0$, one has $\mathrm{Ext}^i_R(\omega,\omega) = 0$.

Now our main results in this context are

Theorem A: (Existence of the decomposition). Let $(R,\mathfrak{m},k)$ be a commutative, local noetherian ring with dualizing module ω. For any finitely generated R-module N there exist finitely generated R-modules M_N and I^N together with an R-linear map

$$d_N : M_N \to I^N$$

such that

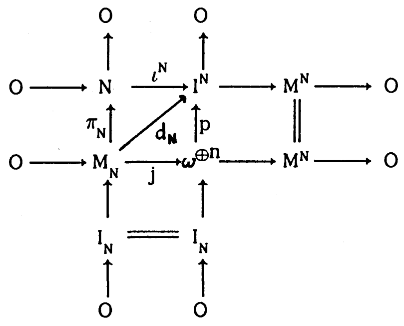
(a) The image of d_N is isomorphic to N.

(b) M_N is maximal Cohen-Macaulay and $I_N = \mathrm{Ker}\, d_N$ is an R-module of finite injective dimension.

(c) I^N is of finite injective dimension and $M^N = \mathrm{Cok}\, d_N$ is maximal Cohen-Macaulay.

(d) There exists an integer $n \geq 0$ such that d_N can be factored into an injection $j : M_N \to \omega^{\oplus n}$ and a surjection $p : \omega^{\oplus n} \to I^N$. ■

If $d_N = \iota^N \circ \pi_N$ denotes the factorization of d_N over its image N, we can arrange the data given in the theorem into the following exact commutative diagram of R-modules:



Theorem B: (Essential Uniqueness)

(a) Assume given a second homomorphism $d'_N: M'_N \to {}'I^N$ satisfying Theorem A for the same module N. If the image factorization of d'_N is given as

$$M'_N \xrightarrow{\pi'_N} N \xrightarrow{\iota'^N} {}'I^N,$$

there exist modules P, P' and Q, 'Q which are each finite direct sums of copies of ω, and R-module isomorphisms μ, κ so that the following diagram commutes:

$$\begin{array}{ccccc} M'_N \oplus P & \xrightarrow{\pi'_N \oplus O} & N & \xrightarrow{\iota'^N \oplus O} & {}'I^N \oplus Q \\ \mu \downarrow & & \| & & \downarrow \kappa \\ M_N \oplus P' & \xrightarrow{\pi_N \oplus O} & N & \xrightarrow{\iota^N \oplus O} & I^N \oplus {}'Q \end{array}$$

(b) If $f: M \to N$ is any homomorphism from a maximal Cohen-Macaulay R-module M into N, it factors over π_N. If $g: N \to J$ is any homomorphism from N into an R-module J of finite injective dimension, it factors over ι^N. ■

These results suggest to call $0 \to I_N \to M_N \xrightarrow{\pi_N} N \to 0$ a *maximal Cohen-Macaulay approximation of* N and $0 \to N \xrightarrow{\iota^N} I^N \to M^N \to 0$ a *hull of finite injective dimension* for N.

To give a simple illustration, consider the case where N itself is a Cohen-Macaulay R-module, hence satisfying $\operatorname{depth}_{\mathfrak{m}} N = \dim N$.
Set $n = \operatorname{codepth}_{\mathfrak{m}} N = \dim R - \dim N$. Then local duality theory implies:

(i) $\operatorname{Ext}^i_R(N, \omega) = 0$ for $i \neq n$.

(ii) $N^\vee = \operatorname{Ext}^n_R(N, \omega)$ is again Cohen-Macaulay of codepth n.

(iii) $N = \operatorname{Ext}^n_R(N^\vee, \omega) = N^{\vee\vee}$.

Using this information, let

$$0 \to \Omega_n(N) \to R^{\oplus b_{n-1}} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_0} R^{\oplus b_0} \longrightarrow N^\vee \to 0$$

be an exact sequence obtained by truncating a free resolution of $N^\vee$. It follows that $\Omega_n(N)$ is maximal Cohen-Macaulay and that dualizing with respect to ω results in an exact sequence

$$0 \longrightarrow \omega^{\oplus b_0} \longrightarrow \cdots \xrightarrow{d^\vee_{n-2}} \omega^{\oplus b_{n-1}} \longrightarrow \operatorname{Hom}_R(\Omega_n(N), \omega) \xrightarrow{\pi} N^{\vee\vee} = N \longrightarrow 0.$$

Then $M_n = \operatorname{Hom}_R(\Omega_n(N), \omega) \xrightarrow{\pi} N$ is a desired maximal Cohen-Macaulay approximation of N, and $I_N = \operatorname{Cok} d^\vee_{n-2}$ admits a finite resolution "by ω", which shows that I_N is of finite injective dimension. The hull of finite injective dimension I^N is then simply the cokernel of the ω-dual of the next differential in the resolution of $N^\vee$, namely
$I^N = \operatorname{Cok} \operatorname{Hom}_R(d_{n-1}, \omega)$.

If R is a domain, for example, we get even more precise information:

(i) The rank of M_N equals the alternating sum

$$\sum_{i=1}^{n}(-1)^{i+1}b_{n-i} + (-1)^n \text{ rk } N,$$

(ii) $M_N^{\vee} = \mathrm{Hom}_R(M_N, \omega) = \Omega_n(N)$ embeds into $R^{\oplus b_{n-1}}$,

(iii) M_N contains no copy of ω as a direct summand if and only if $\Omega_n(N)$, the n-th syzygy module of N, contains no free summand.

It follows that one can attach new numerical invariants to an R-module N in this way. The minimum number of copies of ω necessarily contained in M_N or I^N, the rank of the ω-free summand of either M_N or I^N, their minimum number of generators and so forth.

Here, we are not concerned with these more detailed consequences of the theory but rather with its general framework.

The first author first proved an essentially equivalent version of Theorem A but for the category of additive functors on **R-mod**, see [Ausl], where the result was phrased by saying that the category of maximal Cohen-Macaulay modules is "coherently (co-)finite". The essential step then was to establish the representability of the functors involved.

This background illuminates our approach here. Although the primary applications of the theory might be within the classical theory of rings and algebras, to a large extent it can be developed in any abelian category **C** which admits a suitable subcategory **X** of "maximal Cohen-Macaulay objects".

Here we establish sufficient conditions on **X** to guarantee the categorical analogues of Theorems A and B. Section 1 deals with the decomposition theorem and section 2 addresses the uniqueness question. Sections 3 and 4 investigate the circumstances under which - in the terminology of the above example - the category of modules with "finite ω-resolution" are *all* the modules of finite injective dimension. Section 5 assembles a few remarks on finiteness conditions and section 6 contains more examples, among other purposes highlighting the differences in the theory when applied to either commutative or non-commutative rings.

§1. *The Basic Decomposition Theorem*

In this section we prove the basic decomposition theorem on which this paper rests. Before stating the result, we give some definitions and notations.

Throughout, **C** will be an *abelian* category. By a *subcategory* **A** of **C** we will always mean a *full, additive* and *essential* subcategory of **C**, so that **A** is closed under finite direct sums in **C** and such that any object C in **C** which is isomorphic to an object in **A** is already an object in **A**.

A subcategory of **C** is said to be *additively closed* (or *karoubian* in the terminology of [SGA IV] or [Qu]), if it is closed under direct summands in **C**, or, equivalently, if any projector ($p = p^2$) in the subcategory admits an image in that subcategory. Any subcategory **A** of **C** admits an *additive closure* **add A** in **C**, consisting of all those objects C in **C** which are isomorphic to a direct summand (in **C**) of an object in **A**. Clearly **A** is additively closed in **C** if and only if **A** = **add A**.

More generally, given any collection $\{C_\iota\}_{\iota\in I}$ of objects in **C**, there is a unique smallest additively closed subcategory **add** $\{C_\iota\}_{\iota\in I}$ containing each object C_ι, $\iota\in I$. It can be described by the following "universal mapping property": If $F:\mathbf{C} \to \mathbf{D}$ is any additive functor from **C** into another additive category **D** such that $F(C_\iota)$ is a zero-object in **D** for each $\iota\in I$, then F(**add** $\{C_\iota\}_{\iota\in I}$) consists entirely of zero-objects.

In particular, (cf. also [He]), there exists the *additive quotient category* $\pi:\mathbf{C} \to$ **C/add** $\{C_\iota\}_{\iota\in I}$, where **C/add** $\{C_\iota\}_{\iota\in I}$ has the same objects as **C** and π is a full, additive functor which is the identity on objects.

The projection functor π is characterized by the property that any additive functor F as before factors uniquely over π. Of course, even if **C** is assumed to be abelian, as here, **C/add** $\{C_\iota\}_{\iota\in I}$ need not to be so.

If **A** is an additively closed subcategory of **C**, the morphism groups in **C/A** are given by

$$\mathrm{Hom}_{\mathbf{C/A}}(C_1,C_2) = \frac{\mathrm{Hom}_{\mathbf{C}}(C_1,C_2)}{\{\phi:C_1\to C_2 \mid \phi \text{ factors over an object in } \mathbf{A}\}} .$$

Now suppose again that **A** is any subcategory of **C** in the sense fixed above. We say that a sequence of morphisms $\ldots \to A_{i+1} \to A_i \to A_{i-1} \to \ldots$ in **A** is *exact*, if when viewed as a sequence in **C** it is exact.

Suppose C is an object in **C**. We define **A**–resol.dimC, the **A**-*resolution dimension* of C, to be the smallest nonnegative integer n such that there exists an exact sequence $0 \to A_n \to A_{n-1} \to \ldots \to A_0 \to C \to 0$, with each A_i in **A**, if such an integer exists. We say that **A**–resol.dim$C < \infty$ if **A**–resol.dim$C = n$ for some non-negative integer n. The subcategory of **C** consisting of all C in **C** such that **A**–resol.dim$C < \infty$ will be denoted $\hat{\mathbf{A}}$.

Finally, we say a subcategory **B** of **A** is a *cogenerator* for **A** if for each object A in **A** there is an exact sequence $0 \to A \to B \to A' \to 0$ in **A** with B in **B**.

With these notations, we fix throughout the rest of this paper an additively closed subcategory **X** of **C** which is furthermore *closed under extensions*, i.e. if $0 \to C_1 \to C_2 \to C_3 \to 0$ is exact in **C** with C_1 and C_3 in **X**, then also C_2 is in **X**. (In the terminology of [Qu], for example, **X** is a *karoubian exact* subcategory of **C**.) Also we assume given an additively closed subcategory $\boldsymbol{\omega}$ of **X** which is a cogenerator of **X**.

The paper is now devoted to studying how the categories **X**, $\boldsymbol{\omega}$, $\hat{\mathbf{X}}$ and $\hat{\boldsymbol{\omega}}$ are related.

All of our results depend on the following

Theorem 1.1. For each C in $\hat{\mathbf{X}}$ there are exact sequences

$$0 \to Y_C \to X_C \to C \to 0 \quad \text{and}$$

$$0 \to C \to Y^C \to X^C \to 0$$

with Y_C and Y^C in $\hat{\omega}$ and X_C and X^C in $\mathbf{X}$.

Proof. The proof proceeds by induction on $\mathbf{X}$–resol.dimC and is based on the following two easily proven observations.

Lemma 1.2. Suppose given exact sequences $0 \to K \to X \to C \to 0$ and $0 \to K \to Y^K \to X^K \to 0$ with X and X^K in $\mathbf{X}$ and Y^K in $\hat{\omega}$. Then in the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y^K & \longrightarrow & U & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & X^K & = & X^K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

the exact sequence

$$0 \to Y^K \to U \to C \to 0$$

has the property that Y^K is in $\hat{\omega}$ and U is in $\mathbf{X}$.

Proof. As Y^K is in $\hat{\omega}$ by assumption, it remains to be seen that U is in $\mathbf{X}$. This follows from the fact that both X and X^K are in $\mathbf{X}$ and $\mathbf{X}$ is closed under extensions. ∎

The other observation we need is the following.

Lemma 1.3. Suppose that we have an exact sequence $0 \to Y_C \to X_C \to C \to 0$ with Y_C in $\hat{\omega}$ and X_C in $\mathbf{X}$. Let $0 \to X_C \to W \to X \to 0$ be exact with X in $\mathbf{X}$ and W in ω. Then in the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_C & = & Y_C & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_C & \longrightarrow & W & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & C & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

the exact sequence

$$0 \to C \to Z \to X \to 0$$

has the property that Z is in $\hat{\omega}$ and X is in $\mathbf{X}$.

Proof. As again X is in $\mathbf{X}$ by assumption, it is only required to prove that Z is in $\hat{\omega}$. But in the exact sequence $0 \to Y_C \to W \to Z \to 0$, we have Y_C in $\hat{\omega}$ and W in ω, so that Z is in $\hat{\omega}$ by definition of that category. ∎

The proof of theorem 1.1. follows now easily from these lemmas.

Suppose $\mathbf{X}$–resol.dimC = n and let $0 \to X_n \to \dots \to X_1 \xrightarrow{d_0} X_0 \to C \to 0$ be exact with each X_i in $\mathbf{X}$. If n = 0, we have that C is already in $\mathbf{X}$. Since ω is a cogenerator for $\mathbf{X}$, there is an exact sequence $0 \to C \to W \to X \to 0$ in $\mathbf{X}$ with W in ω which is one of our desired exact sequences. The other one is $0 \to 0 \to C \xrightarrow{=} C \to 0$. Now suppose that n > 0 and set $K = \operatorname{Im} d_0$, so that we have exact sequences $0 \to K \to X_0 \to C \to 0$ and $0 \to X_n \to \dots \to X_1 \to K \to 0$ with each X_i in $\mathbf{X}$. By the inductive hypothesis we know there is an exact sequence $0 \to K \to Y^K \to X^K \to 0$ with Y^K in $\hat{\omega}$ and X^K in $\mathbf{X}$. Therefore, by Lemma 1.2, the pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & X_0 & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y^K & \longrightarrow & U & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

has the property that U is in $\mathbf{X}$. Hence we may choose $0 \to Y^K \to U \to C \to 0$ as one of our desired sequences for C. From the existence of this exact sequence, it follows by Lemma 1.3 that we also have an exact sequence $0 \to C \to Y^C \to X^C \to 0$ with Y^C in $\hat{\omega}$ and X^C in $\mathbf{X}$. This finishes the proof of theorem 1.1. ∎

For ease of reference, we call an exact sequence $0 \to Y_C \to X_C \xrightarrow{\pi_C} C \to 0$ with X_C in $\mathbf{X}$ and Y_C in $\hat{\omega}$ an *$\mathbf{X}$-approximation* of C. Dually, we call an exact sequence $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$ with Y^C in $\hat{\omega}$ and X^C in $\mathbf{X}$ an *$\hat{\omega}$-hull* of C.

From now on, we assume that $\mathbf{X}$ has the property that if $0 \to X_0 \to X_1 \to X_2 \to 0$ is an exact sequence with X_1 and X_2 in $\mathbf{X}$, then X_0 is also in

X, in addition to **X** being an additively closed subcategory of **C** which is closed under extensions. (In D.Quillen's terminology, (loc. cit.), all epimorphisms from **C** in **X** are *admissible.*)

It should be noted that in all our examples the categories **X** satisfy this additional condition. As a consequence of this further hypothesis on **X**, we get the following

<u>*Lemma 1.4.*</u> Suppose C in **C** has an $\hat{\omega}$-hull $0 \to C \to Y^C \to X^C \to 0$. Then it also admits an **X**-approximation $0 \to Y_C \to X_C \to C \to 0$. Furthermore, Y_C can be chosen such that ω–resol.dimY_C < ω–resol.dim Y^C if Y^C is not already in ω.

<u>*Proof.*</u> Let $0 \to W_n \to W_{n-1} \to \ldots \xrightarrow{d_0} W_0 \to Y^C \to 0$ be exact with the W_i in ω. Then we obtain the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & L & \to & W_0 & \to & X^C & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & C & \to & Y^C & \to & X^C & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $K = \operatorname{Im} d_0$. Since X^C and W_0 are in **X**, the additional assumption yields that L is in **X** too. By definition, K is in $\hat{\omega}$ and so $0 \to K \to L \to C \to 0$ is an **X**-approximation of C. Now set $Y_C = K$ and $X_C = L$. ∎

As a consequence of this lemma, we obtain the following characterization of the objects in $\hat{\mathbf{X}}$.

<u>*Proposition 1.5.*</u> Let **X** be an additively closed and exact subcategory of **C** in which every epimorphism is admissible. If ω is a cogenerator of **X**, the following are equivalent for an object C in **C**:

(a) C is in $\hat{\mathbf{X}}$.

(b) There exists an **X**-approximation $0 \to Y_C \to X_C \to C \to 0$ of C.

(c) There is an $\hat{\omega}$-hull $0 \to C \to Y^C \to X^C \to 0$ of C.

<u>*Proof.*</u> Since (a) implies (b) and (c) by theorem 1.1, it is only required to show that (b) implies (a) and (c) implies (b).

(b) ⇒ (a): Since Y_C is in $\hat{\omega}$ by assumption, there is an exact sequence $0 \to W_n \to \ldots \to W_0 \to Y_C \to 0$ with each W_i in ω. Since ω is a subcategory of **X**, it follows from the exact sequence $0 \to W_n \to \ldots \to W_0 \to X_C \to C \to 0$ that C is in $\hat{\mathbf{X}}$.

(c) $\Rightarrow$ (b): This is just a restatement of Lemma 1.4. ∎

We end this section with three examples, illustrating the theory developed so far.

Example 1. Let $X \to \operatorname{Spec} k$ be a scheme of finite type over a field k. Assume that X is equidimensional of dimension d and locally Cohen-Macaulay in the sense that $\mathcal{O}_{X,x}$ is a local Cohen-Macaulay ring for each x in X. Let $\mathbf{C}$ be the category of coherent sheaves of $\mathcal{O}_X$*-modules* and define $\mathbf{X}$ to be the subcategory of maximal Cohen-Macaulay coherent sheaves, where a coherent $\mathcal{O}_X$-module $\mathcal{M}$ is said to be maximal Cohen-Macaulay if for every $x \in X$ one has $\operatorname{depth}_{\mathbf{m}_x} \mathcal{M}_x = \dim \mathcal{O}_{X,x}$; $\mathbf{m}_x$ the unique maximal ideal of $\mathcal{O}_{X,x}$.

It is then clear that if $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ is an exact sequence in $\mathbf{C}$, then

(a) $\mathcal{M}_2$ is in $\mathbf{X}$ if $\mathcal{M}_1$ and $\mathcal{M}_3$ are in $\mathbf{X}$, and

(b) $\mathcal{M}_1$ is in $\mathbf{X}$ if $\mathcal{M}_2$ and $\mathcal{M}_3$ are in $\mathbf{X}$.

Remark also that, by hypothesis, the structure sheaf $\mathcal{O}_X$ is in $\mathbf{X}$ and that consequently $\mathbf{X}$ contains all locally free sheaves of $\mathcal{O}_X$-modules. Conversely, a maximal Cohen-Macaulay $\mathcal{O}_X$-module is locally free on the regular locus $X_{reg} \subseteq X$. Moreover, $\mathbf{C} = \hat{\mathbf{X}}$, and if $C \neq 0$ is in $\mathbf{C}$, then $\mathbf{X}\text{-resol.dim} C = n$ if and only if n is the largest integer such that $\mathcal{E}xt^n_{\mathcal{O}_X}(C, \omega_X) \neq 0$, where ω_X is a dualizing sheaf for X.

Now assume that either X admits a very ample invertible sheaf $\mathcal{L}$ or that X is affine (in which case $\mathcal{L} = \mathcal{O}_X$ in the following). Then X can be embedded into a projective space over k, say $i: X \to \mathbf{P}^N_k$, with $\mathcal{L} = i^*\mathcal{O}_{\mathbf{P}^N}(1)$.

Denoting by $\omega_{\mathcal{L}}$ the smallest additively closed subcategory which contains the family of objects $\{\omega_X \otimes \mathcal{L}^{\otimes n}\}_{n \in \mathbf{Z}}$, it follows easily from Grothendieck-Serre duality theory that $\omega_{\mathcal{L}}$ is a cogenerator for $\mathbf{X}$.

Proposition 1.6. For each coherent sheaf C of $\mathcal{O}_X$-modules we have both an $\mathbf{X}$-approximation with respect to $\omega_{\mathcal{L}}$ of the form $0 \to Y_C \to X_C \to C \to 0$ and an $\hat{\omega}_{\mathcal{L}}$-hull $0 \to C \to Y^C \to X^C \to 0$. ∎

Remark that in this example the category $\mathbf{X}$ depends only on the scheme X whereas its cogenerator depends on the choice of both a dualizing module ω_X and a very ample sheaf $\mathcal{L}$. Also the $\mathbf{X}$-approximations and $\hat{\omega}_{\mathcal{L}}$-hulls will vary with these choices.

Next consider the following modified version of Example 1.

Example 2. As in Example 1, we let $X \to \operatorname{Spec} k$ be an equidimensional Cohen-Macaulay scheme over a field k. Let $X' \subset X$ be the *Gorenstein locus* of X, which is the set of all points x in X for which $\mathcal{O}_{X,x}$ is a Gorenstein local ring. Let $\mathbf{X}'$ be the subcategory of $\mathbf{C}$, the category of coherent sheaves of $\mathcal{O}_X$-modules, consisting of those Cohen-Macaulay sheaves $\mathcal{M}$ such that $\mathcal{M}_x$ is $\mathcal{O}_{X,x}$-free for all x in X'. It is clear again that $\mathbf{X}'$ is an exact subcategory of $\mathbf{C}$ in which every epimorphism is admissible. Also $\hat{\mathbf{X}}'$ consists of all those

$\mathcal{M}$ in **C** for which $\mathcal{M}_x$ is of finite projective dimension over $\mathcal{O}_{X,x}$ for each x in X'. This implies that an exact sequence $0 \to A \to B \to C \to 0$ in **C** is in $\hat{\mathbf{X}}'$ if any two of A,B or C are in $\hat{\mathbf{X}}'$.

Again, any invertible sheaf of $\mathcal{O}_X$–modules is in **X'**, and in particular for any dualizing sheaf ω_X and each very ample invertible sheaf $\mathcal{L}$ on X, the category $\omega_{\mathcal{L}}$ defined above is a cogenerator of **X'**. We leave it to the reader to give in this case the analogue of Proposition 1.6.

Our final example in this section treats a not necessarily commutative version of Gorenstein rings of finite Krull dimension.

Example 3. Let R be a ring with unit which is noetherian on both sides and such that the *injective dimension of* R *as a right module over itself is finite,* say equal to d.

Take **C** = **R-mod**, the category of finitely generated left R-modules, and let **X** be the subcategory consisting of all modules M in **R-mod** which satisfy $\mathrm{Ext}^i_R(M,R) = 0$ for $i \neq 0$.

Then **X** is certainly additively closed and has the property that an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is in **X** as soon as either M_1 and M_3 or M_2 and M_3 are in **X**. Hence **X** satisfies our general assumptions.

For ω, take the subcategory of all finitely generated *projective* left R-modules. Then ω is by definition a subcategory of **X** which is additively closed.

For our theory to apply, we have hence to show that ω constitutes a cogenerator for **X**. To obtain this result we need our assumption on R. Namely, let $\dots \xrightarrow{d_j} P_j \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_0} P_0 \longrightarrow M \longrightarrow 0$ be a projective resolution of a module M in **X**. By definition of **X**, the dualized complex

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{d_0^*} P_1^* \longrightarrow \dots \longrightarrow P_j^* \xrightarrow{d_j^*} \dots$$

is acyclic. But then our hypothesis furnishes the following more precise information.

Lemma 1.7. With notations and assumptions as above, for every module M in **X** one has

(a) For all integers $j \geq 0$, the right R-modules $K_j = \mathrm{Ker}\, d_j^*$ satisfy $\mathrm{Ext}^i_R(K_j,R) = 0$ for $i \neq 0$.

(b) M is *reflexive,* that is, the natural morphism of left R-modules $M \to M^{**}$ is an isomorphism.

(c) If $0 \longrightarrow L \longrightarrow Q \xrightarrow{p} M^* \longrightarrow 0$ is an exact sequence of right R-modules with Q finitely generated projective, then L^* satisfies $\mathrm{Ext}^i_R(L^*,R) = 0$ for $i \neq 0$.

Proof. (a) As all the modules P_j^* are finitely generated projective right R-modules, they satisfy necessarily $\mathrm{Ext}^i_R(P_j^*,R) = 0$ for $i \neq 0$. But this implies that for any integer $n \geq 0$ one has natural isomorphisms $\mathrm{Ext}^i_R(K_{j-n},R) \xrightarrow{\sim} \mathrm{Ext}^{i+n}_R(K_j,R)$ for any $i > 0$. Since by assumption $\operatorname{Ext}^k_R(-,R) = 0$ as soon as $k > d$, it suffices to take $n \geq d$ above to conclude $\operatorname{Ext}^i_R(K_j,R) = 0$ for all $i > 0$ and $j \geq 0$.

(b) This is a consequence of (a). By [A-B; 2.1.], for any left R-module M the natural morphism $M \to M^{**}$ fits into an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_R(D(M),R) \longrightarrow M \longrightarrow M^{**} \longrightarrow \operatorname{Ext}^2_R(D(M),R) \longrightarrow 0,$$

where $D(M) = \operatorname{Cok} d_0^*$. But if M is in **X**, we have $\operatorname{Cok} d_0^* = \operatorname{Ker} d_2^*$ and (a) shows that the extreme terms of this exact sequence vanish, establishing (b).

(c) As $M^* = K_0$, part (a) implies that the sequence

$$(*)\ 0 \longrightarrow M^{**} \xrightarrow{p^*} Q^* \longrightarrow L^* \longrightarrow 0$$

is exact. From (b) we have $M \cong M^{**}$ and as M is in **X**, it follows already that $\operatorname{Ext}^i_R(L^*,R) = 0$ for $i > 1$. It hence only remains to be seen that $\operatorname{Ext}^1_R(L^*,R) = 0$, or, equivalently, that the dual sequence of (*):

$$0 \longrightarrow L^{**} \longrightarrow Q^{**} \xrightarrow{p^{**}} M^{***} \longrightarrow 0$$

is again exact. But this is obvious as both Q and M^* are reflexive right R-modules and $p^{**} = p$. ■

Combining (b) and (c) of this lemma, we have now that any module M in **X** embeds into the finitely generated projective module $\operatorname{Hom}_R(Q,R)$ and that the cokernel, isomorphic to L^*, is again in **X**. This shows that ω is indeed a cogenerator for **X**.

Finally observe that $\hat{\mathbf{X}}$ consists of all left R-modules N in **C** satisfying $\operatorname{Ext}^i_R(N,R) = 0$ for all sufficiently large i, and that $\hat{\omega}$ is the category of all finitely generated left R-modules *of finite projective dimension.*

Now Theorem 1.1 yields in this context the following.

Theorem 1.8. Let R be a ring which is noetherian on both sides and of finite injective dimension as a right module over itself. Then for any finitely generated left R-module N satisfying $\operatorname{Ext}^i_R(N,R) = 0$ for all sufficiently large i, there are modules Y_N and Y^N in **R-mod** of finite projective dimension and modules X_N and X^N in **R-mod** with $\operatorname{Ext}^i_R(X_N,R) = \operatorname{Ext}^i_R(X^N,R) = 0$ for $i \neq 0$ which fit into exact sequences

$$0 \longrightarrow Y_N \longrightarrow X_N \longrightarrow N \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow N \longrightarrow Y^N \longrightarrow X^N \longrightarrow 0.$$ ■

§2. *Injective Cogenerators*

Having established the *existence* of **X**-approximations and $\hat{\omega}$–hulls for a pair $(\mathbf{X},\omega)$ of subcategories as in the preceding section, the important question which remains is their *uniqueness.*

To see which conditions ought to be imposed, assume given two $\mathbf{X}$-approximations for the same object C in $\hat{\mathbf{X}}$, say $0 \to Y_C \to X_C \to C \to 0$ and $0 \to Y'_C \to X'_C \to C \to 0$. Then the least one should ask for is that these $\mathbf{X}$-approximations can be *compared* in the sense that there exists a morphism $\phi : X_C \to X'_C$ making the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_C & \longrightarrow & X_C & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow & & \\ 0 & \longrightarrow & Y'_C & \longrightarrow & X'_C & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Apparently, the existence of such a comparison morphism is guaranteed as soon as $\mathrm{Ext}^1_{\mathbf{C}}(X_C, Y'_C) = 0$.

Hence, for comparisons to exist and to yield an equivalence relation, it certainly suffices to have that $\mathrm{Ext}^1_{\mathbf{C}}(X,Y) = 0$ for *all* X in $\mathbf{X}$ and Y in $\hat{\omega}$. This section is devoted to a study of this condition and its consequences. First, once again, some general remarks and notations.

Let A and C be objects in $\mathbf{C}$. As $\mathbf{C}$ is supposed to be abelian, the groups $\mathrm{Ext}^i_{\mathbf{C}}(A,C)$ are defined for all $i \geq 0$. If there is an integer n such that $\mathrm{Ext}^i_{\mathbf{C}}(A,C) = 0$ for all $i > n$, then the smallest nonnegative such integer n is called the A-*injective dimension of* C, (notation: A−inj.dimC), or the C-*projective dimension of* A, (notation: C−proj.dimA).

Otherwise we set A−inj.dimC $= \infty =$ C−proj.dimA . If $\mathbf{B}$ is a subcategory of $\mathbf{C}$, for each A in $\mathbf{C}$ we define A−inj.dim$\mathbf{B}$ to be the maximum (in $\mathbf{Z} \cup \{\infty\}$) of A−inj.dimB for all B in $\mathbf{B}$. Dually, for each C in $\mathbf{C}$, we define C−proj.dim$\mathbf{B}$ to be the maximum of C−proj.dimB for all B in $\mathbf{B}$.

Clearly A−inj.dim$\mathbf{B}$ = $\mathbf{B}$−proj.dimA.

Suppose now that $\mathbf{A}$ and $\mathbf{B}$ are subcategories of $\mathbf{C}$. Then define $\mathbf{A}$−proj.dim$\mathbf{B}$ to be the maximum of A−proj.dimB for all A in $\mathbf{A}$ and B in $\mathbf{B}$. We define dually $\mathbf{A}$−inj.dim$\mathbf{B}$ to be the maximum of A−inj.dimB for all A in $\mathbf{A}$ and B in $\mathbf{B}$. Again, one has clearly $\mathbf{A}$−inj.dim$\mathbf{B}$ = $\mathbf{B}$−proj.dim$\mathbf{A}$.

If for two such subcategories $\mathbf{A}$−inj.dim$\mathbf{B}$ = 0 = $\mathbf{B}$−proj.dim$\mathbf{A}$, we follow J.L. Verdier, [SGA 4½, C.D.; I.2.6.1.], and say that $\mathbf{A}$ is *left orthogonal* to $\mathbf{B}$ and $\mathbf{B}$ is *right orthogonal* to $\mathbf{A}$ - with respect to the "augmented" bilinear $\mathbf{Z}$-graded pairing induced by $(\mathrm{Ext}^i_{\mathbf{C}}(-,-))_{i>0}$ on the monoid of isomorphism classes of objects of $\mathbf{C}$.

Consequently, if $\mathbf{A}$ consists precisely of those objects A in $\mathbf{C}$ for which A−inj.dim$\mathbf{B}$ = 0, we call $\mathbf{A}$ the *left orthogonal complement* of $\mathbf{B}$ in $\mathbf{C}$, denoted $\mathbf{A} = {}^{\perp}\mathbf{B}$. Dually again, $\mathbf{A}^{\perp}$, the *right orthogonal complement* of $\mathbf{A}$ in $\mathbf{C}$, is the subcategory $\mathbf{B}$ consisting of all objects B in $\mathbf{C}$ for which $\mathbf{A}$−inj.dimB = 0.

One has obviously $\mathbf{A} \subseteq {}^{\perp}(\mathbf{A}^{\perp})$ and $\mathbf{A} \subseteq ({}^{\perp}\mathbf{A})^{\perp}$, but not necessarily ${}^{\perp}(\mathbf{A}^{\perp}) = ({}^{\perp}\mathbf{A})^{\perp}$. If $\mathbf{B}'$ is a subcategory of $\mathbf{B}$ in $\mathbf{C}$, then ${}^{\perp}\mathbf{B}$ is contained in ${}^{\perp}\mathbf{B}'$ and similary for right orthogonal complements. Remark also that by definition ${}^{\perp}\mathbf{C}$, the *left radical* of $\mathbf{C}$ with respect to the pairing $(\mathrm{Ext}^i_{\mathbf{C}}(-,-))_{i>0}$, consists precisely of all *projective* objects of $\mathbf{C}$, whereas $\mathbf{C}^{\perp}$, the *right radical* of **C**, is given by all *injective* objects of **C**.

Furthermore, it is obvious that orthogonal complements are additively closed and exact subcategories of **C** and that in a left orthogonal complement ${}^{\perp}\mathbf{B}$ all epimorphisms are admissible, whereas in a right orthogonal complement $\mathbf{A}^{\perp}$ all monomorphisms are admissible.

Returning to our subcategories **X** and ω of **C** from the previous section, we say that ω is an *injective cogenerator* for **X** if $\mathbf{X}$–inj.dim$\omega = 0$, that is, $\omega \subseteq \mathbf{X}^{\perp}$. If there is a cogenerator for **X** in $\mathbf{X} \cap \mathbf{X}^{\perp}$, we say also that the exact category **X** has *enough relatively injective objects.*

Unless stated to the contrary, we assume from now on that ω is an injective cogenerator for **X**. Our next aim is to explore some important properties of **X**-approximations and $\hat{\omega}$–hulls implied by this additional assumption.

We begin with the following relations between some of the dimensions we have just introduced for an object C in $\hat{\mathbf{X}}$. These relations do not require that any epimorphism in **X** is admissible.

Proposition 2.1. Given an object C in $\hat{\mathbf{X}}$, where **X** is an additively closed exact subcategory of **C** and ω is an injective cogenerator for **X**, the following are equivalent for any integer $n \geq 0$.

(a) $\mathbf{X}$–resol.dim$C = n$,

(b) C–inj.dim$\omega = n$,

(c) C–inj.dim$\hat{\omega} = n$,

(d) $\mathrm{Ext}_{\mathbf{C}}^{n+1}(C, Y) = 0$ for all Y in $\hat{\omega}$.

Proof: Proceed by induction on $n = \mathbf{X}$–resol.dim$C$, the case $n = 0$ being settled as follows.

(a) $\Rightarrow$ (b) is true because ω is contained in $\mathbf{X}^{\perp}$ by assumption.

(b) $\Rightarrow$ (c) follows from the usual dimension shift argument.

(c) $\Rightarrow$ (d) is the definition of C–inj.dim$\hat{\omega}$.

(d) $\Rightarrow$ (a): Since C is in $\hat{\mathbf{X}}$ by the general hypothesis, there is an **X**-approximation $0 \to Y_C \to X_C \to C \to 0$ which splits by (d). Hence C is a direct summand of X_C in **X** and so C is in **X**.

The proof of the inductive step follows easily from what we have just shown and is left to the reader. ∎

As an obvious consequence of this proposition we have

Corollary 2.2. $\mathbf{X}$–inj.dim$\hat{\omega} = 0$. ∎

This corollary yields the following important properties of **X**-approximations and $\hat{\omega}$-hulls.

Theorem 2.3. Let $0 \to Y_C \to X_C \xrightarrow{\pi_C} C \to 0$ be an $\mathbf{X}$-approximation for C in $\hat{\mathbf{X}}$. Then for each X in $\mathbf{X}$ we have

(a) $0 \to \mathrm{Hom}_C(X, Y_C) \to \mathrm{Hom}_C(X, X_C) \to \mathrm{Hom}_C(X, C) \to 0$ is exact,

(b) π_C induces isomorphisms $\mathrm{Ext}^i_C(X, X_C) \to \mathrm{Ext}^i_C(X, C)$ for all $i > 0$.

Proof: As $\mathbf{X}$–inj.dim$\hat{\omega} = 0$, one has $\mathrm{Ext}^i_C(X, Y_C) = 0$ for all $i > 0$. ■

The exact sequence $0 \to Y_C \to X_C \to C \to 0$ is called an $\mathbf{X}$-approximation precisely because $\mathrm{Hom}_C(X, X_C) \to \mathrm{Hom}_C(X, C) \to 0$ is exact for all X in $\mathbf{X}$. This property of $\mathbf{X}$-approximations of C gives rise to a weak sort of uniqueness for such approximations as we now explain.

Let us call two morphisms $f: B \to C$ and $f': B' \to C$ in $\mathbf{C}$ *equivalent* if there are morphisms $g: B \to B'$ and $h: B' \to B$ such that $f = f'g$ and $f' = fh$. Also, we say that two exact sequences $0 \to A \to B \xrightarrow{f} C$ and $0 \to A' \to B' \xrightarrow{f'} C$ are (*right*) *equivalent*, if f and f' are equivalent, which amounts to the same as saying that there is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{f'} & C \\
 & & \downarrow & & \downarrow{\scriptstyle h} & & \| \\
0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{f} & C \\
 & & \downarrow & & \downarrow{\scriptstyle g} & & \| \\
0 & \longrightarrow & A' & \longrightarrow & B' & \xrightarrow{f'} & C
\end{array}$$

In particular, $\mathrm{id}_B - hg$ factors over A and $\mathrm{id}_{B'} - gh$ factors over A', so that h,g become inverse isomorphisms in $\mathbf{C}/_{\mathbf{add}\{A, A'\}}$.

As an immediate consequence of theorem 2.3 we obtain the following uniqueness result.

Corollary 2.4. $\mathbf{X}$-approximations for an object C in $\hat{\mathbf{X}}$ are unique up to equivalence, that is, any two $\mathbf{X}$-approximations for C are (right) equivalent exact sequences. ■

There are also similar results for $\hat{\omega}$-hulls of an object C in $\hat{\mathbf{X}}$ as we now point out.

Theorem 2.5. Let $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$ be an $\hat{\omega}$-hull for C in $\hat{\mathbf{X}}$. Then for each Y in $\hat{\omega}$ we have the following

(a) $0 \to \mathrm{Hom}_C(X^C, Y) \to \mathrm{Hom}_C(Y^C, Y) \to \mathrm{Hom}_C(C, Y) \to 0$ is exact,

(b) ι^C induces isomorphisms $\mathrm{Ext}^i_C(Y^C, Y) \to \mathrm{Ext}^i_C(C, Y)$ for all $i > 0$.

Proof: This follows again from the fact that $\mathbf{X}$–inj.dim$\hat{\omega} = 0$. ■

The exact sequence $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$ is called an $\hat{\omega}$-hull precisely because $\mathrm{Hom}_C(Y^C, Y) \to \mathrm{Hom}_C(C, Y) \to 0$ is exact for all Y in $\hat{\omega}$. Again, this property gives rise to a weak sort of uniqueness for $\hat{\omega}$-hulls, similar to that already discussed for **X**-approximations, as we now explain.

Dually to the above, we say two morphisms $f:C \to D$ and $f':C \to D'$ are equivalent if there are morphisms $g:D \to D'$ and $h:D' \to D$ such that $f' = gf$ and $f = hf'$. Also, we say that two exact sequences $C \xrightarrow{f} D \to E \to 0$ and $C' \xrightarrow{f'} D' \to E' \to 0$ are (*left*) ***equivalent*** if f and f' are equivalent, which is the same thing as saying that there is a commutative diagram

$$\begin{array}{ccccccc} C & \xrightarrow{f} & D & \longrightarrow & E & \longrightarrow & 0 \\ \| & & \downarrow g & & \downarrow & & \\ C & \xrightarrow{f'} & D' & \longrightarrow & E' & \longrightarrow & 0 \\ \| & & \downarrow h & & \downarrow & & \\ C & \xrightarrow{f} & D & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

In particular, $id_D - hg$ factors over E and $id_{D'} - gh$ factors over E', so that h and g become inverse isomorphisms in $\mathbf{C}/_{\mathbf{add}\{E,E'\}}$.

As an immediate consequence of theorem 2.5 we have the following uniqueness theorem.

Corollary 2.6. $\hat{\omega}$-hulls for an object C in $\hat{\mathbf{X}}$ are unique up to equivalence, that is, any two $\hat{\omega}$-hulls are (left) equivalent exact sequences. ■

We may reformulate and sharpen these uniqueness results slightly by considering the situation "modulo ω". This depends on the following simple observation.

Lemma 2.7. Let $f:X \to C$ be a morphism in **C** with X in **X** and C in $\hat{\mathbf{X}}$. Then the following conditions on f are equivalent.

(a) f factors through an object in $\hat{\omega}$.

(b) f factors through an object in ω.

Proof: As (b) is a priori a special case of (a), we need only to show that in fact (a) implies (b). Hence assume that $f = gh$ where $h:X \to Y$ and $g:Y \to C$ are morphisms in **C** and Y is in $\hat{\omega}$. By definition of $\hat{\omega}$, there is an exact sequence $0 \to K \to W \to Y \to 0$ with W in ω and K again in $\hat{\omega}$. By corollary 2.2, $\mathbf{X}$–inj.dim $\hat{\omega} = 0$ and so $\mathrm{Ext}^1_{\mathbf{C}}(X,K) = 0$. This shows that h, and then also f, factor over W in ω. ■

Now choose for any object C in $\hat{\mathbf{X}}$ an **X**-approximation $0 \to Y_C \to X_C \xrightarrow{\pi_C} C \to 0$ and an $\hat{\omega}$-hull $0 \to C \xrightarrow{\iota^C} Y^C \to X^C \to 0$, as well as for any morphism $f:C \to D$ in $\hat{\mathbf{X}}$ liftings $f_*:X_C \to X_D$ and $f^*:Y^C \to Y^D$ which exist by the above.

By the uniqueness results just established, it follows that given a second morphism $g:D \to E$ in $\hat{\mathbf{X}}$, the differences $g^*f^* - (gf)^*$ and $g_*f_* - (gf)_*$ factor over objects in ω, hence become zero-morphisms in $\hat{\mathbf{X}}/\omega$, the full subcategory spanned by $\hat{\mathbf{X}}$ in $\mathbf{C}/\omega$.

From this we obtain immediately the following

Theorem 2.8. Denote $i: \hat{\omega} \to \hat{\mathbf{X}}$ and $j:\mathbf{X} \to \hat{\mathbf{X}}$ the natural inclusion functors. Then

(a) The induced functor $j_!:\mathbf{X}/\omega \to \hat{\mathbf{X}}/\omega$ is fully faithful and admits a ***right adjoint*** $j^!:\hat{\mathbf{X}}/\omega \to \mathbf{X}/\omega$ which associates to an object C in $\hat{\mathbf{X}}$ the chosen **X**-approximation X_C. The adjunction morphism $j_!j^!C \to C$ is given by the class of $\pi_C:X_C \to C$ in $\mathrm{Hom}_{\hat{\mathbf{X}}/\omega}(X_C, C)$.

(b) The induced functor $i_*: \hat{\omega}/\omega \to \hat{\mathbf{X}}/\omega$ is fully faithful and admits a *left adjoint* $i^*: \hat{\mathbf{X}}/\omega \to \hat{\omega}/\omega$ which associates to an object C in $\hat{\mathbf{X}}$ the chosen $\hat{\omega}$-hull Y^C. The adjunction morphism $C \to i_*i^*C$ is given by the class of $\iota^C:C \to Y^C$ in $\mathrm{Hom}_{\hat{\mathbf{X}}/\omega}(C, Y^C)$.

(c) One has $j^!i_* = 0$ and $i^*j_! = 0$.

(d) The composition of the adjunction morphisms

$$j_!j^! \xrightarrow{\pi} \mathrm{id}_{\hat{\mathbf{X}}/\omega} \xrightarrow{\iota} i_*i^*$$

is zero in $\hat{\mathbf{X}}/\omega$.

Proof: The remarks preceding the theorem show that X_- and Y^- define functors from $\hat{\mathbf{X}}$ into $\mathbf{X}/\omega$ and $\hat{\omega}/\omega$ respectively. By the universal property of quotient categories these functors factor over $\hat{\mathbf{X}}/\omega$, yielding $j_!$ and i^*. To prove that $j_!$ is indeed right adjoint to the inclusion functor $j^!:\mathbf{X}/\omega \to \hat{\mathbf{X}}/\omega$, it suffices to give the natural isomorphisms $\phi_{X,C}:\mathrm{Hom}_{\mathbf{X}/\omega}(X, j_!C) \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbf{X}}/\omega}(j^!X, C)$. Now composition with $\pi_C:X_C = j_!C \to C$ defines the natural map $\mathrm{Hom}_{\mathbf{X}}(X, X_C) \to \mathrm{Hom}_{\hat{\mathbf{X}}}(X, C)$ which is surjective by theorem 2.3.(a). Let $\phi_{X,C}$ be the induced map on the quotient groups, which is hence still surjective. To prove that it is injective, let f in $\mathrm{Hom}_{\mathbf{X}}(X, X_C)$ be a morphism such that $\pi_C f:X \to C$ factors over some object W in ω. This means that there is a commutative diagram in **C**

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_C & \longrightarrow & X_C & \xrightarrow{\pi_C} & C & \longrightarrow & 0 \\ & & & & f\uparrow & & \uparrow g & & \\ & & & & X & \xrightarrow[h]{} & W & & \end{array}$$

As W is a priori in **X** and Y_C is in $\hat{\omega}$, corollary 2.2 applies once again to yield $\mathrm{Ext}^1_{\mathbf{C}}(W, Y_C) = 0$ and hence to establish the existence of a morphism $g':W \to X_C$ such that $\pi_C g' = g$. But then $f-g'h$ satisfies $\pi_C(f-g'h) = \pi_C f-(\pi_C g')h = gh-gh = 0$, so that $f-g'h$ factors over Y_C. Then lemma 2.7 shows that $f-g'h$ factors already over some object W' in ω and hence the class of $f-g'h$ in $\mathrm{Hom}_{\mathbf{X}/\omega}(X, X_C)$ is the zero-morphism. As $g'h$ factors over W in ω, its class is zero as well, which shows that f and $f-g'h$ define the same morphism in $\mathrm{Hom}_{\mathbf{X}/\omega}(X, X_C) = \mathrm{Hom}_{\mathbf{X}/\omega}(X, j_!C)$. Hence already the class of f is the zero-morphism and $\phi_{X,C}$ is injective as claimed.

The definition of $\phi_{X,C}$ is natural in both arguments, so that the adjointness of $j_!$ and $j^!$ is established. Furthermore, the construction of $\phi_{X,C}$ shows that π_C induces the adjunction morphism $j^!j_!C \to C$.

This proves (a).
As the proof of (b) is completely analogous, it is left to the reader.
For (c), just remark again that by definition of $\hat{\omega}$, any object Y in $\hat{\omega}$ appears in an exact sequence $0 \to K \to W \to Y \to 0$ with W in ω and K in $\hat{\omega}$. But this sequence serves as an **X**-approximation for Y whence $j^! i_* Y$, the chosen **X**-approximation of Y, is isomorphic to W in $\mathbf{X}/\omega$, i.e. it is a zero object. This shows $j^! i_* = 0$ and $i^* j_! = 0$ follows then by adjunction.
(d) follows now from (c), as one has by naturality the commutative diagram of morphisms of functors

$$\begin{array}{ccc} j_! j^! & \xrightarrow{\pi} & id_{\hat{\mathbf{X}}/\omega} \\ {\scriptstyle \iota*(j_! j^!)} \downarrow & & \downarrow {\scriptstyle \iota} \\ i_* i^* j_! j^! & \xrightarrow[(i_* i^*)*\pi]{} & i_* i^* \end{array}$$

in which the lower left corner is zero by (c). (In more concrete terms, (d) says that for any object C in $\hat{\mathbf{X}}$ there is a commutative diagram

$$\begin{array}{ccc} X_C & \xrightarrow{\pi_C} & C \\ {\scriptstyle j} \downarrow & & \downarrow {\scriptstyle \iota^C} \\ W & \longrightarrow & Y^C \end{array}$$

with W in ω, and we have seen indeed in lemma 1.3 and the proof of theorem 1.1 that ι^C can be obtained as the push-out of such a morphism j along π_C.) This finishes the proof of theorem 2.8. ∎

The reader puzzled by the notations used in the preceding theorem should compare it with the treatment of the "glueing of categories" in [BBD; 1.4]. It shows that in our situation one should think of $\hat{\mathbf{X}}$ as being obtained by "glueing together the open subcategory **X** and the closed subcategory $\hat{\omega}$ along ω ". What is missing for a complete glueing in the sense of (loc. cit.) is the existence of the other adjoints j_* and $i^!$.

The statements (c) and (d) in theorem 2.8 also explain why we think of theorem 1.1 as a "decomposition theorem": an object C in $\hat{\mathbf{X}}$ is *decomposed* - at least in $\hat{\mathbf{X}}/\omega$ - into its **X**-approximation X_C and its $\hat{\omega}$-hull Y^C, which belong to "orthogonal" subcategories of $\hat{\mathbf{X}}/\omega$.

The property which is desirable but missing yet is that **X** and $\hat{\omega}$ should have ω as their common intersection. This will be addressed later on in Proposition 3.6.

For now, we return to the examples 1 and 2 of the previous section. As soon as $X \to \operatorname{Spec} k$ is *projective*, $\omega_{\mathcal{L}}$ is not an injective generator in either **X** or **X'**, as $\operatorname{Ext}^d_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X \otimes \mathcal{L}^{\otimes n}) = H^d(X, \omega_X \otimes \mathcal{L}^{\otimes n})$ does not vanish for all integers n. None the less, the following analogues of the results for injective cogenerators are valid for these examples if one substitutes $\mathcal{E}xt^i_{\mathcal{O}_X}(A, B)$ for $\operatorname{Ext}^i_{\mathcal{O}_X}(A, B)$.

Lemma 2.9. With notations as in examples 1 and 2, the following are equivalent for a sheaf $\mathcal{M}$ in **C**.

(a) $\mathcal{M}$ is in **X**.

(b) $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, \omega_X) = 0$ for all $i > 0$.

(c) $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, \omega_X \otimes \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ and all n in **Z**.

(d) $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, Y) = 0$ for all $i > 0$ and Y in $\omega_{\mathcal{L}}$.

Proof: Easy consequence of the fact that the corresponding statements hold for Cohen-Macaulay local rings with a dualizing module. ∎

Proposition 2.10. With the same assumptions and notations as above, let $0 \to Y_C \to X_C \to C \to 0$ be an **X**-approximation for C in **C**.
Then we have for any $\mathcal{M}$ in **X**:

(a) $0 \to Hom_{\mathcal{O}X}(\mathcal{M}, Y_C) \to Hom_{\mathcal{O}X}(\mathcal{M}, X_C) \to Hom_{\mathcal{O}X}(\mathcal{M}, C) \to 0$ is exact.

(b) The induced morphisms $\mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, X_C) \to \mathcal{E}xt^i_{\mathcal{O}X}(\mathcal{M}, C)$ are isomorphisms for $i > 0$.

Proof: Immediate consequence of lemma 2.7. ∎

Remark that in Example 3 the category ω is in fact an injective cogenerator as by definition there $\mathbf{X} = {}^{\perp}\omega$. Furthermore, in that example $\mathbf{X}/\omega$ is the category of left *maximal Cohen-Macaulay R-modules* - in the sense that $\mathrm{Ext}^i_R(M,R) = 0$ for $i \neq 0$ - *modulo stable equivalence*: two modules M and M' from **X** become isomorphic in $\mathbf{X}/\omega$ if and only if there are finitely generated projective left R-modules P and P' such that $M \oplus P'$ is isomorphic to $M' \oplus P$ in **R-mod**.

We end this section with two more illustrations of situations where ω is an injective cogenerator for **X**.

Example 4. Suppose R is a commutative noetherian Cohen-Macaulay ring in the sense that all its localizations $R_{\mathbf{p}}$ at primes **p** are local Cohen-Macaulay rings. We say that a finitely generated R-module M is *maximal Cohen-Macaulay* (MCM for short), if $M_{\mathbf{p}}$ satisfies depth $M_{\mathbf{p}} = \dim R_{\mathbf{p}}$ for all primes **p**.

Now suppose that R is a *Gorenstein* ring and that S is a commutative R-algebra which is MCM as an R-module. Let **C** = **S-mod** be the category of finitely generated S-modules and let **X** be the subcategory of **C** consisting of those S-modules M which are maximal Cohen-Macaulay as R-modules. Then **X** satisfies the usual properties. Set $\omega_{S/R} = \mathrm{Hom}_R(S,R)$, which is a relative dualizing module for the algebra $R \to S$. Then $\omega = \mathbf{add}\{\omega_{S/R}\}$ consists of all S-modules of the form $\mathrm{Hom}_R(P,R)$ with P finitely generated projective over S. It is easily seen - and well-known - that ω is an injective cogenerator for **X**. Also, if the Krull dimension of R is finite, then $\hat{\mathbf{X}} = \mathbf{C}$.

To acknowledge the scope of this example and to emphasize its relevance for Grothendieck duality theory, we quote the following from [FGR; Cor. 5.9.].

Theorem. Suppose S is a commutative ring with finite Krull dimension and with connected prime spectrum. Then S *admits a canonical module* if and only if S is a homomorphic image of a Gorenstein ring R such that S is maximal Cohen-Macaulay as an R-module. ∎

Our final illustration of this section is the following variant of Examples 4 and 2.

Example 5. Maintain the hypotheses on S and R from Example 4. Let $X \subset \operatorname{Spec} R$ have the property that if $\mathbf{p}$ is in X, then $S_{\mathbf{p}}$ is a Gorenstein ring, or equivalently, $(\omega_{S/R})_{\mathbf{p}}$ is $S_{\mathbf{p}}$-free.

Set again $\mathbf{C} = \mathbf{S\text{-}mod}$ and let $\mathbf{X'}$ consist of those S-modules M which are MCM over R and satisfy furthermore that $M_{\mathbf{p}}$ is $S_{\mathbf{p}}$-projective for all $\mathbf{p}$ in X. Then $\mathbf{X'}$ satisfies the usual properties and contains $\omega = \mathbf{add}\{\omega_{S/R}\}$. Again, ω is an injective cogenerator for $\mathbf{X'}$ and $\hat{\mathbf{X}}'$ consists of all S-modules C such that $\text{proj.dim}_{S_{\mathbf{p}}} C_{\mathbf{p}} < \infty$ for all $\mathbf{p}$ in X.

§3. *Exactness properties of* $\hat{\mathbf{X}}$ *and* $\hat{\omega}$.

We maintain our general assumption that $\mathbf{X}$ is an additively closed and exact subcategory of $\mathbf{C}$ in which every epimorphism is admissible, and that ω is an injective cogenerator for $\mathbf{X}$.

In this situation, we show that $\hat{\mathbf{X}}$ is an additively closed subcategory of $\mathbf{C}$ which has the property that an exact sequence $0 \to A \to B \to C \to 0$ is in $\hat{\mathbf{X}}$ whenever two of A, B and C are in $\hat{\mathbf{X}}$. This result is then used to prove that $\hat{\omega}$ is an additively closed subcategory of $\mathbf{C}$ having the property that an exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{C}$ is already in $\hat{\omega}$ if either A and C are in $\hat{\omega}$ or A and B are in $\hat{\omega}$. Hence $\hat{\omega}$ is seen to be an *additively closed exact subcategory* of $\mathbf{C}$ in which *every monomorphism is admissible.*

We begin with the following

Lemma 3.1. The category $\hat{\mathbf{X}}$ is closed under extensions.

Proof: Suppose $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathbf{C}$ with A and C in $\hat{\mathbf{X}}$. Proceed by induction on $n = \mathbf{X}\text{-resol.dim}C$. Suppose $n = 0$, which means that C is in $\mathbf{X}$. As A is in $\hat{\mathbf{X}}$, there is an $\mathbf{X}$-approximation $0 \to Y_A \to X_A \to A \to 0$ of A. Since C is in $\mathbf{X}$, we know by theorem 2.3, that the induced map $\operatorname{Ext}^1_{\mathbf{C}}(C, X_A) \to \operatorname{Ext}^1_{\mathbf{C}}(C, A)$ is an isomorphism. Hence there exists an exact commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_A & = & Y_A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & X_A & \to & Z & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Since X_A and C are in **X**, the object Z is also in **X**, as that category is closed under extensions. Now Y_A is in $\hat{\omega}$, hence in $\hat{\mathbf{X}}$, and it follows that B is in $\hat{\mathbf{X}}$ as required.

Suppose now that $n > 0$ and let $0 \to L \to X_0 \to C \to 0$ be exact with **X**–resol.dim$L = n-1$. Since X_0 is in **X**, we have that $\mathrm{Ext}^1_{\mathbf{C}}(X_0,X_A) \to \mathrm{Ext}^1_{\mathbf{C}}(X_0,A)$ is an isomorphism by theorem 2.3, and so there exists an exact commutative diagram in **C**

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Y_A & \to & U & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X_A & \to & V & \to & X_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

This shows that B is in $\hat{\mathbf{X}}$ since V is necessarily in **X** and U is in $\hat{\mathbf{X}}$ by the inductive hypothesis. ∎

We now use the fact that $\hat{\mathbf{X}}$ is closed under extensions to prove the following

Lemma 3.2. Let $0 \to K \to X \to C \to 0$ be an exact sequence in **C** with X in **X**. Then C is in $\hat{\mathbf{X}}$ if and only if K is in $\hat{\mathbf{X}}$.

Proof: By definition, if K is in $\hat{\mathbf{X}}$ then also C is in $\hat{\mathbf{X}}$. Hence assume that C is in $\hat{\mathbf{X}}$ and let $0 \to Y_C \to X_C \to C \to 0$ be an **X**-approximation of C. Since X is in **X**, we have by theorem 2.3 that $\mathrm{Hom}_{\mathbf{C}}(X,X_C) \to \mathrm{Hom}_{\mathbf{C}}(X,C)$ is surjective. Therefore we obtain a commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_C & \longrightarrow & X_C & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Since Y_C is in $\hat{\omega}$, we know there is an epimorphism $W \to Y_C$ with W in ω. Adding this epimorphism to the foregoing diagram, we obtain the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & V & = & V & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & K\oplus W & \longrightarrow & X\oplus W & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y_C & \longrightarrow & X_C & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $K\oplus W \to X\oplus W$ is the sum of $K \to X$ and the identity on W. Since W and X are both in $\mathbf{X}$, we have that V is in $\mathbf{X}$, as any epimorphism of $\mathbf{C}$ in $\mathbf{X}$ is admissible by assumption. Therefore $K\oplus W$ is in $\hat{\mathbf{X}}$, since Y_C and V are in $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}$ is closed under extensions. We now show that this implies that K is in $\hat{\mathbf{X}}$. Since $K\oplus W$ is in $\hat{\mathbf{X}}$, we obtain the following exact commutative diagram from an $\mathbf{X}$-approximation of $K\oplus W$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & Y_{K\oplus W} & \longrightarrow & Z & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_{K\oplus W} & = & X_{K\oplus W} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W & \longrightarrow & K\oplus W & \longrightarrow & K & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Hence we have the exact sequence $0 \to Y_{K\oplus W} \to Z \to W \to 0$. Since W and $Y_{K\oplus W}$ are in $\hat{\mathbf{X}}$, (in fact already in $\hat{\omega}$), and $\hat{\mathbf{X}}$ is closed under extensions, we have that Z is in $\hat{\mathbf{X}}$ as well. This implies that K is in $\hat{\mathbf{X}}$ since $X_{K\oplus W}$ is in $\mathbf{X}$. This completes the proof of the lemma. ■

We now apply the foregoing lemma to prove

Proposition 3.3. Suppose C is an object in $\hat{\mathbf{X}}$. Then the following are equivalent for any integer $n \geq 0$:

(a) $\mathbf{X}$–resol.dimC $\leq n$,

(b) If $0 \to U \to X_{n-1} \to \dots \to X_0 \to C \to 0$ is exact with X_i in $\mathbf{X}$ for $i = 0,\dots,n\text{-}1$, then U is in $\mathbf{X}$.

Proof: For $n = 0$ there is nothing to prove. So suppose $n > 0$. Assuming (a), repeated application of lemma 3.2 shows that U is in $\hat{\mathbf{X}}$. Also we have $\mathrm{Ext}^1_{\mathbf{C}}(U,W) = \mathrm{Ext}^{n+1}_{\mathbf{C}}(C,W) = 0$ for all W in $\boldsymbol{\omega}$ since $\mathbf{X}$–inj.dim$\boldsymbol{\omega} = 0$ and $\mathbf{X}$–resol.dimC $\leq n$. Therefore by proposition 2.1, it follows that U is in $\mathbf{X}$ proving that (a) implies (b). As (a) follows from (b) by definition of $\mathbf{X}$–resol.dimC, we are done. ∎

As a first application of this proposition we prove the following

Proposition 3.4. $\hat{\mathbf{X}}$ is an additively closed subcategory of $\mathbf{C}$, that is $\hat{\mathbf{X}} = \mathbf{add}\,\hat{\mathbf{X}}$.

Proof: Suppose $C_1 \oplus C_2$ is in $\hat{\mathbf{X}}$ for two objects C_1 and C_2 in $\mathbf{C}$. Proceed by induction on $n = \mathbf{X}$–resol.dim $(C_1 \oplus C_2)$. If $n = 0$, the summands C_1 and C_2 are in $\mathbf{X}$ as $\mathbf{X}$ is an additively closed subcategory of $\mathbf{C}$. Suppose $n > 0$. Since $C_1 \oplus C_2$ is in $\hat{\mathbf{X}}$, there is an epimorphism $X \to C_1 \oplus C_2 \to 0$ with X in $\mathbf{X}$. Therefore we obtain exact sequences $0 \to L_i \to X \to C_i \to 0$ for $i = 1,2$ which yield the exact sequence $0 \to L_1 \oplus L_2 \to X \oplus X \to C_1 \oplus C_2 \to 0$. Now by Lemma 3.2, we know that $L_1 \oplus L_2$ is in $\hat{\mathbf{X}}$ and proposition 3.3 shows that $\mathbf{X}$–resol.dim$(L_1 \oplus L_2) \leq n-1$. By the inductive hypothesis L_1 and L_2 are in $\hat{\mathbf{X}}$ and another application of lemma 3.2 shows that then also C_1 and C_2 are in $\hat{\mathbf{X}}$. ∎

We are now in position to establish one of the results promised in the beginning of this section.

Proposition 3.5. An exact sequence $0 \to A \to B \to C \to 0$ from $\mathbf{C}$ is in $\hat{\mathbf{X}}$ if any two of A, B and C are in $\hat{\mathbf{X}}$.

Proof: Since we already know that $\hat{\mathbf{X}}$ is closed under extensions by lemma 3.1, it suffices to show that if B is in $\hat{\mathbf{X}}$ then A is in $\hat{\mathbf{X}}$ if and only if C is in $\hat{\mathbf{X}}$. We first show that if A and B are in $\hat{\mathbf{X}}$ then C is in $\hat{\mathbf{X}}$. Choose an $\mathbf{X}$-approximation $0 \to Y_B \to X_B \to B \to 0$ for B. It gives rise to an exact commutative diagram

(*)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & Y_B & \longrightarrow & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & X_B & = & X_B & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

from which we get an exact sequence $0 \to Y_B \to L \to A \to 0$. It follows that L is in $\hat{\mathbf{X}}$ since Y_B and A are in $\hat{\mathbf{X}}$ and $\hat{\mathbf{X}}$ is closed under extensions. Therefore C is in $\hat{\mathbf{X}}$ since X_B is in $\mathbf{X}$.

Suppose now that B and C are in $\hat{\mathbf{X}}$. Using lemma 3.2, the exact sequence $0 \to L \to X_B \to C \to 0$ from (*) shows that L is in $\hat{\mathbf{X}}$. Applying the just established result to the exact sequence $0 \to Y_B \to L \to A \to 0$, it follows that A is in $\hat{\mathbf{X}}$. This completes the proof of the proposition. ■

We now turn our attention to $\hat{\omega}$. We begin with the characterization of $\hat{\omega}$ as a subcategory of $\hat{\mathbf{X}}$, proving that $\hat{\omega} = \mathbf{X}^{\perp} \cap \hat{\mathbf{X}}$ in $\mathbf{C}$.

Proposition 3.6. The following statements are equivalent for an object C in $\hat{\mathbf{X}}$.

(a) C is in $\hat{\omega}$.

(b) $\mathbf{X}$-inj.dim C = 0, that is: C is in $\mathbf{X}^{\perp} \cap \hat{\mathbf{X}}$.

(c) If $0 \to Y_C \to X_C \to C \to 0$ is any $\mathbf{X}$-approximation of C, then X_C is in ω.

Proof: That (a) implies (b) was seen in corollary 2.2, and it is obvious that (c) implies (a). Hence we only need to show that (b) implies (c).
Since $\mathbf{X}$-inj.dim C = 0 = $\mathbf{X}$-inj.dim Y_C, it follows that $\mathbf{X}$-inj.dim X_C = 0. Our desired result is therefore a trivial consequence of the following, which proves $\omega = \mathbf{X} \cap \hat{\omega}$.

Lemma 3.7. The following are equivalent for an object X in $\mathbf{X}$.

(a) X is in ω.

(b) X is in $\hat{\omega}$.

(c) $\mathbf{X}$-inj.dim X = 0.

Proof: Again it is obvious that (a) implies (b), and Corollary 2.2 shows that (b) implies (c). It remains to prove
(c) ⇒ (a): Let $0 \to X \to W \to X' \to 0$ be an exact sequence in $\mathbf{X}$ with W in ω which exists as ω is a cogenerator for $\mathbf{X}$. Then by (c) this sequence splits. Therefore X is a direct summand of W which implies that X is in ω as that category is assumed to be additively closed.
This establishes lemma 3.7 and finishes the proof of proposition 3.6. ■

These results prove the following fact, already announced in the introduction to this section.

Proposition 3.8. $\hat{\omega}$ is an additively closed and exact subcategory of $\mathbf{C}$ in which any monomorphism is admissible. In more detail, let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathbf{C}$. Then

(a) B is in $\hat{\omega}$ if A and C are in $\hat{\omega}$, and

(b) C is in $\hat{\omega}$ if A and B are in $\hat{\omega}$.

Proof: We know by now that $\hat{\omega} = \mathbf{X}^{\perp} \cap \hat{\mathbf{X}}$ in $\mathbf{C}$. But the statements hold for $\hat{\mathbf{X}}$ by propositions 3.4 and 3.5, and as $\mathbf{X}^{\perp}$ is a right orthogonal complement, it also is an additively closed and exact subcategory of $\mathbf{C}$ in which every monomorphism is admissible. As all these properties are stable under intersection in $\mathbf{C}$, the result for $\hat{\omega}$ follows. ■

We sum up the foregoing results as

Theorem 3.9. Let $\mathbf{X}$ be an additively closed and exact subcategory of an abelian category $\mathbf{C}$. Assume that

(i) all epimorphisms from $\mathbf{C}$ in $\mathbf{X}$ are admissible, and

(ii) $\mathbf{X}$ has enough relatively injective objects.

Let ω be an injective cogenerator for $\mathbf{X}$. Then there results a diagram of additively closed and exact subcategories of $\mathbf{C}$

$$\begin{array}{ccccc} \mathbf{X} & \longrightarrow & \hat{\mathbf{X}} & \longrightarrow & \mathbf{C} \\ \uparrow & & \uparrow & & \uparrow \\ \omega & \longrightarrow & \hat{\omega} & \longrightarrow & \mathbf{X}^{\perp} \end{array}$$

such that

(a) each square is *cartesian*, i.e.: $\hat{\omega} = \hat{\mathbf{X}} \cap \mathbf{X}^{\perp}$ and $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$,

(b) in $\hat{\mathbf{X}}$ all *mono- or epimorphisms* from $\mathbf{C}$ *are admissible*,

(c) in $\mathbf{X}^{\perp}$ and $\hat{\omega}$ *all monomorphisms* from $\mathbf{C}$ *are admissible*.

In particular, there is a *unique* injective cogenerator ω for $\mathbf{X}$ in $\mathbf{C}$, namely $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$. ■

To reformulate it once again modulo $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$, let us say that a sequence $0 \to \mathbf{A} \to \mathbf{B} \xrightarrow{p} \mathbf{C} \to 0$ of additive functors between additive categories is *exact* if and only if $\mathbf{A}$ is a full, essential and additively closed subcategory of $\mathbf{B}$ and p is equivalent to the projection functor $\pi : \mathbf{B} \to \mathbf{B}/\mathbf{A}$.

With the notations of theorem 2.8 we have then the following

Corollary 3.10. The adjoint pairs of functors (i^*, i_*) and $(j_!, j^!)$ fit into the commutative diagram of exact sequences of additive categories

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{\omega}/\omega & \xrightarrow{i_*} & \hat{X}/\omega & \xrightarrow{j^!} & X/\omega & \longrightarrow & 0 \\
 & & \Downarrow & & \| & & \Uparrow & & \\
0 & \longleftarrow & \hat{\omega}/\omega & \xleftarrow{i^*} & \hat{X}/\omega & \xleftarrow{j_!} & X/\omega & \longleftarrow & 0
\end{array}$$

§4. *The category $\hat{\omega}$.*

Our aim in this section is to describe under which assumptions $\hat{\omega}$ has the further property that an exact sequence $0 \to A \to B \to C \to 0$ is in $\hat{\omega}$ if B and C are in $\hat{\omega}$.

To investigate this problem, we first define $\tilde{\omega}$ to be the subcategory of **C** consisting of all objects C in **C** which appear in an exact sequence $0 \to C \to Y_0 \to \dots \to Y_n \to 0$ with each Y_i in $\hat{\omega}$. Clearly such an object C is in $\hat{\mathbf{X}}$ since the Y_i are in $\hat{\omega} \subseteq \hat{\mathbf{X}}$ and the kernel of an epimorphism in $\hat{\mathbf{X}}$ is again in $\hat{\mathbf{X}}$ by proposition 3.5. Also it is obvious that $\hat{\omega}$ is a subcategory of $\tilde{\omega}$.

Lemma 4.1. The following statements are equivalent:

(a) An exact sequence $0 \to A \to B \to C \to 0$ is in $\hat{\omega}$ if B and C are in $\hat{\omega}$.

(b) $\hat{\omega} = \tilde{\omega}$.

The proof is trivial. ∎

This simple observation explains the relevance of the category $\tilde{\omega}$ to our problem about $\hat{\omega}$. The following description of $\tilde{\omega}$ is basic to the results in this paragraph.

Proposition 4.2. For an object C in $\hat{\mathbf{X}}$ the following are equivalent:

(a) C is in $\tilde{\omega}$,

(b) **X**-inj.dim $C < \infty$.

Proof: (a) $\Rightarrow$ (b). Let $0 \to C \to Y_0 \to \dots \to Y_n \to 0$ be exact with each Y_i in $\hat{\omega}$. Since **X**-inj.dim $\hat{\omega} = 0$, it follows by induction on n that **X**-inj.dim $C \leq n < \infty$.
(b) $\Rightarrow$ (a): Since C is in $\hat{\mathbf{X}}$, it admits an $\hat{\omega}$-hull $0 \to C \to Y^C \to X^C \to 0$. The assumption that **X**-inj.dim $C < \infty$ and the fact that **X**-inj.dim $Y^C = 0$ imply that **X**-inj.dim $X^C < \infty$. Therefore if we show that an object X from **X** which satisfies **X**-inj.dim $X < \infty$ is necessarily in $\tilde{\omega}$, we will be done. Indeed we have the following more specific result.

Lemma 4.3. Let X be an object in **X**, n a nonnegative integer. Then **X**-inj.dim $X \leq n$ if and only if there is an exact sequence $0 \to X \to W_0 \to W_1 \to \dots \to W_n \to 0$ with W_i in ω for $i = 0, \dots, n$.

Proof: The if-part follows as before from **X**-inj.dim $\omega = 0$. Hence suppose that **X**-inj.dim $X = n$. Since ω is a cogenerator for **X**, we can construct an exact sequence $0 \to X \to W_0 \to \dots \to W_{n-1} \to X' \to 0$ in **X** such that each W_i is in ω for $i = 0,\dots,n-1$. As **X**-inj.dim $\omega = 0$ and **X**-inj.dim $X \leq n$ by assumption, it follows that for any integer $i > 0$ and all objects Z in **X** one has $\mathrm{Ext}^i_{\mathbf{C}}(Z,X') \cong \mathrm{Ext}^{n+i}_{\mathbf{C}}(Z,X) = 0$. But by lemma 3.7 this shows that X' is already in ω as desired.
This concludes the proof of lemma 4.3 and proposition 4.2. ∎

As a first application we get the following.

Corollary 4.4. $\check{\omega}$ is an additively closed subcategory of **C** with the property that an exact sequence $0 \to A \to B \to C \to 0$ is in $\check{\omega}$ if any two of A, B and C are in $\check{\omega}$.

Proof: Since $\hat{\mathbf{X}}$ is additively closed, it contains with $\check{\omega}$ also **add** $\check{\omega}$. It then follows from proposition 4.2 that $\check{\omega} =$ **add** $\check{\omega}$. Also if $0 \to A \to B \to C \to 0$ is an exact sequence in **C** with two of A, B and C in $\check{\omega}$, then all of A, B and C are in $\hat{\mathbf{X}}$ by Proposition 3.5. It then follows from proposition 4.2 that they are all in $\check{\omega}$. ∎

As another immediate consequence of proposition 4.2 we get the following.

Corollary 4.5. The following are equivalent:

(a) $\hat{\omega} = \check{\omega}$,

(b) If C is an object in $\hat{\mathbf{X}}$ with **X**-inj.dim $C < \infty$, then **X**-inj.dim $C = 0$.

Proof: Let C be in $\hat{\mathbf{X}}$. By proposition 4.2 we have that C is in $\check{\omega}$ if and only if **X**-inj.dim $C < \infty$. By Proposition 3.6 we have that C is in $\hat{\omega}$ if and only if **X**-inj.dim $C = 0$.
Hence the equivalence of (a) and (b). ∎

We now give criteria for the property $\hat{\omega} = \check{\omega}$ in terms of the categories ω and **X** themselves.

Proposition 4.6. The following are equivalent:

(a) $\hat{\omega} = \check{\omega}$.

(b) If $0 \to C \to W_0 \to W_1 \to 0$ is exact in **C** with W_0 and W_1 in ω, then C is in ω.

(c) If $0 \to C \to W_0 \to W_1 \to \dots \to W_n \to 0$ is exact with each W_i in ω for $i = 0,\dots,n$, then C is in ω.

(d) If X is in **X** and **X**-inj.dim $X < \infty$, then X is in ω.

Proof: (a) ⇒ (b). Since W_0 and W_1 are objects in ω, they are in **X**, so C is in **X**. Clearly C is in $\check{\omega}$ which means by the assumption that it is in $\hat{\omega}$. Therefore C is in $\mathbf{X} \cap \hat{\omega}$ which category equals ω by Lemma 3.7.
(b) ⇒ (c) by induction on n.
(c) ⇒ (d). Suppose X is in **X** with **X**-inj.dim $X < \infty$. Then by Lemma 4.3, we know there

is an exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ with W_i in ω for $i = 0,\dots,n$. Therefore by (c) we have that X is in ω.
(d) $\Rightarrow$ (a). Let C be an object in $\check{\omega}$. Then we can choose an $\mathbf{X}$-approximation $0 \to Y_C \to X_C \to C \to 0$ for C and proposition 4.2 shows that $\mathbf{X}$-inj.dim $C < \infty$. But $\mathbf{X}$-inj.dim $Y_C = 0$ so that $\mathbf{X}$-inj.dim $X_C < \infty$. Therefore X_C is in ω by the hypothesis (d), which shows that C is in $\hat{\omega}$. ∎

Now we establish the following.

Proposition 4.7. Let C be an object in $\check{\omega}$. Then ω-inj.dim C = $\mathbf{X}$-inj.dim C.

Proof: Since ω is a subcategory of $\mathbf{X}$, we have that always ω-inj.dim C $\leq$ $\mathbf{X}$-inj.dim C . So it suffices to show that ω-inj.dim C $\geq$ $\mathbf{X}$-inj.dim C. As C is in $\check{\omega}$ by assumption, we also know from proposition 4.2 that $\mathbf{X}$-inj.dim C is finite.

To begin with, we prove the proposition when C = X is an object in $\mathbf{X} \cap \check{\omega}$. By lemma 4.3, we have that then there is an exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ with each W_i in ω for $i = 0,\dots,n$. Assume that ω-inj.dim X = 0. Since ω-inj.dim $\omega = 0$, it follows by induction on n that the exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ splits. Hence $\mathbf{X}$ is already in ω which implies $\mathbf{X}$-inj.dim X = 0. This result shows furthermore that ω-inj.dim X $\leq$ n if and only if there is an exact sequence $0 \to X \to W_0 \to \dots \to W_n \to 0$ with each W_i in ω. But we have seen in lemma 4.3 that the existence of such an exact sequence is equivalent to $\mathbf{X}$-inj.dim X $\leq$ n. Hence we have shown that ω-inj.dim C = $\mathbf{X}$-inj.dim C when C is in $\mathbf{X} \cap \check{\omega}$.

Assume now that C is an arbitrary object in $\check{\omega}$. Let $0 \to C \to Y^C \to X^C \to 0$ be an $\hat{\omega}$-hull of C. Since C and Y^C are in $\check{\omega}$ by assumption, we get that X^C is in $\check{\omega}$ by corollary 4.4. Suppose now ω-inj.dim C = 0. Then also ω-inj.dim $X^C = 0$ which implies that X^C is in ω by our previous result. But our current hypothesis then implies that $\mathrm{Ext}^1_{\mathbf{C}}(X^C, C) = 0$, which means that the chosen $\hat{\omega}$-hull of C splits. So C is a direct summand of Y^C in $\hat{\omega}$ and is hence itself in $\hat{\omega}$, as $\hat{\omega}$ is additively closed by proposition 3.8. Therefore $\mathbf{X}$-inj.dim C = 0 by corollary 2.2 and we are done in this case.

Finally suppose ω-inj.dim C = n > 0. Since ω-inj.dim $Y^C = 0$, it follows that ω-inj.dim $X^C = n-1$. Therefore $\mathbf{X}$-inj.dim $X^C = n-1$ by our first result, which implies that $\mathbf{X}$-inj.dim C $\leq$ n. Hence ω-inj.dim C $\geq$ $\mathbf{X}$-inj.dim C for all C in $\check{\omega}$, which completes the proof of the proposition. ∎

The following is an immediate consequence of our earlier results and summarizes sufficient conditions for $\hat{\omega} = \check{\omega}$ to hold.

Corollary 4.8. Consider the following conditions:
(a) ω-inj.dim $\mathbf{X} = 0$,
(b) ω-inj.dim $\hat{\mathbf{X}} = 0$,
(c) ω-inj.dim $\check{\omega} = 0$,
(d) Every epimorphism $W' \to W \to 0$ in $\mathbf{C}$ with W and W' in ω admits a section.
(e) $\hat{\omega} = \check{\omega}$.

Then one has (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e).

Proof. As $\mathbf{X}$ and $\check{\omega}$ are subcategories of $\hat{\mathbf{X}}$, the implications (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are trivial. That (a) $\Rightarrow$ (b) follows from the existence of an $\mathbf{X}$-approximation $0 \to Y_C \to X_C \to C \to 0$ for any object C in $\hat{\mathbf{X}}$ and the fact that ω-inj.dim $\hat{\omega}$ = $\mathbf{X}$-inj.dim $\hat{\omega} = 0$. That (c) $\Rightarrow$ (d) follows from the fact that the kernel K of any epimorphism $W' \to W \to 0$ between objects from ω is by definition an object in $\check{\omega}$. But (c) implies $\mathrm{Ext}^1_C(W,K) = 0$, whence the exact sequence $0 \to K \to W' \to W \to 0$ splits. The remaining implication (d) $\Rightarrow$ (e) is a special case of proposition 4.2. ■

Example 6. A special case in which $\hat{\omega} = \check{\omega}$ holds, has been investigated already by A.Heller [He]. Following him let us say that $\mathbf{X}$ in $\mathbf{C}$ is a *Frobenius category* if it satisfies our usual assumptions of being additively closed and exact in $\mathbf{C}$ with every epimorphism from $\mathbf{C}$ in $\mathbf{X}$ being admissible and if furthermore $\omega = \mathbf{X} \cap \mathbf{X}^{\perp}$ is also a *projective generator* for $\mathbf{X}$, which is equivalent to ω^{op} being an injective cogenerator of $\mathbf{X}^{op}$.
This means hence that

(i) $\mathbf{X}$-inj.dim ω = $\mathbf{X}$-proj.dim ω = 0 and

(ii) for every object X in $\mathbf{X}$ there exists both a monomorphism $i : X \to W$ as well as an epimorphism $p : W' \to X$ with W, W' in ω such that the objects Ker p and Cok i, calculated in $\mathbf{C}$, are again objects in $\mathbf{X}$.

A.Heller himself gave already some examples of such categories and further such categories are discussed in [Ha]. Also, it is clear from the definitions that in Example 3 the category $\mathbf{X}$ of maximal Cohen-Macaulay R-modules is Frobenius.

§5. *Some remarks on the* $\mathbf{X}$*-resolution dimension of* $\hat{\mathbf{X}}$.

We define $\mathbf{X}$-resol.dim $\hat{\mathbf{X}}$ to be the maximum (including ∞) of $\mathbf{X}$-resol.dim C for all objects C in $\hat{\mathbf{X}}$. This paragraph is devoted to interpreting some of our previous results when $\mathbf{X}$-resol.dim $\hat{\mathbf{X}}$ is *finite.* So for the remainder of this section we assume $\mathbf{X}$-resol.dim $\hat{\mathbf{X}} = d < \infty$.

Our remarks are based on the following observation.

Lemma 5.1. The following statements are equivalent for an object C in $\mathbf{C}$.
(a) $\mathbf{X}$-inj.dim C $< \infty$,
(b) $\hat{\mathbf{X}}$-inj.dim C $< \infty$.
Moreover, if $\mathbf{X}$-inj.dim C $= m < \infty$, then $\hat{\mathbf{X}}$–inj.dim C $\leq d+m$.

Proof: Usual dimension shift argument. ■

This lemma implies immediately the following.

Proposition 5.2. Suppose C is an object in $\hat{\mathbf{X}}$.
(a) $\hat{\mathbf{X}}$-inj.dim$C < \infty$ if and only if C is in $\tilde{\omega}$.
(b) If C is an object in $\hat{\omega}$, then $\hat{\mathbf{X}}$-inj.dim$C \leq d$.

Proof: (a): By lemma 5.1, we know that $\hat{\mathbf{X}}$-inj.dim$C < \infty$ if and only if $\mathbf{X}$-inj.dim$C < \infty$. But by proposition 4.2, we know that $\mathbf{X}$-inj.dim$C < \infty$ if and only if C is in $\tilde{\omega}$.
(b): Since $\mathbf{X}$-inj.dim $\hat{\omega} = 0$ by corollary 2.2, the result follows from lemma 5.1. ∎

As a special case we obtain the following consequence.

Corollary 5.3. If $\hat{\mathbf{X}} = \mathbf{C}$, then we have:
(a) C is in $\tilde{\omega}$ if and only if inj.dim$C < \infty$.
(b) If C is in $\hat{\omega}$, then inj.dim$C \leq d$. ∎

Remark that the injective dimension of an object C in $\mathbf{C}$ is defined here in terms of vanishing of the functors $\mathrm{Ext}^{*}_{\mathbf{C}}(-,C)$. As soon as $\mathbf{C}$ itself has enough injective objects it coincides with the notion obtained from the length of a shortest injective resolution.

Applying the foregoing result to our decomposition into $\mathbf{X}$-approximations and $\hat{\omega}$-hulls we have the following.

Corollary 5.4. Suppose again $\hat{\mathbf{X}} = \mathbf{C}$ and let $0 \to Y_C \to X_C \to C \to 0$ and $0 \to C \to Y^C \to X^C \to 0$ be an $\mathbf{X}$-approximation and an $\hat{\omega}$-hull of an object C in $\mathbf{C}$ respectively. Then inj.dim $Y_C <$ inj.dim $Y^C \leq d$ or both Y_C and Y^C are already injective. ∎

Finally, consider the case where $\hat{\omega} = \tilde{\omega}$. Then we have first the following consequence of lemma 5.1.

Proposition 5.5. Suppose $\hat{\omega} = \tilde{\omega}$. Then the following statements are equivalent for an object C in $\hat{X}$.
(a) C is in $\hat{\omega}$.
(b) $\hat{\mathbf{X}}$-inj.dim $C \leq d$.
(c) $\hat{\mathbf{X}}$-inj.dim $C < \infty$.

Proof: (a) ⇒ (b) by proposition 5.2.
(b) ⇒ (c) is trivial.
(c) ⇒ (a): Since $\hat{\mathbf{X}}$-inj.dim $C < \infty$, we have that $\mathbf{X}$-inj.dim $C < \infty$.
Therefore C is in $\tilde{\omega}$ by proposition 4.2. Hence C is in $\hat{\omega}$ since $\tilde{\omega} = \hat{\omega}$ by assumption. ∎

As an obvious consequence of this proposition we have the following.

Corollary 5.6. Suppose $\hat{\mathbf{X}} = \mathbf{C}$ and $\check{\omega} = \hat{\omega}$. Then the following conditions are equivalent for an object C in **C**.

(a) C is in $\hat{\omega}$,

(b) inj.dim $C \leq d$,

(c) inj.dim $C < \infty$. ∎

§6. *More Examples*

In this section we describe various situations where the theory we have developed is applicable.

First we consider a generalization of Example 4.

Example 7. Let R be a commutative noetherian Gorenstein ring of finite dimension d. Let Λ be an R-algebra, not necessarily commutative, which is a *maximal Cohen-Macaulay R-module.* Set $\mathbf{C} = \mathbf{mod}\text{-}\Lambda$, the category of finitely generated right Λ-modules, and let **X** be the full subcategory of **C** whose objects are the Λ-modules which are MCM if considered as R-modules. Then **X** is again additively closed, exact and has all its epimorphisms admissible. Also we have that $\hat{\mathbf{X}} = \mathbf{mod}\text{-}\Lambda$ and that **X**-resol.dim $\hat{\mathbf{X}} = d < \infty$.

As in Example 4, we let ω consist of all Λ-modules isomorphic to $\mathrm{Hom}_R(P,R)$ for some finitely generated projective Λ^{op}–module P. Again, ω is just the additive closure of $\omega_{\Lambda/R} = \mathrm{Hom}_R(\Lambda, R)$, and it is an injective cogenerator for **X**.

Applying the results in section 5, we have the following.

Proposition 6.1. Let C be in $\mathbf{mod}\text{-}\Lambda$.

(a) inj.dim $C < \infty$ if and only if C is in $\check{\omega}$.

(b) If C is in $\hat{\omega}$ then inj.dim $C \leq d$.

Proof: See Corollary 5.3. ∎

As a consequence of this we obtain hence the following.

Corollary 6.2. Let C be in $\mathbf{mod}\text{-}\Lambda$. Then $0 \to Y_C \to X_C \to C \to 0$, the **X**–approximation of C, and $0 \to C \to Y^C \to X^C \to 0$, the $\hat{\omega}$-hull of C, have the property that X_C and X^C are maximal Cohen-Macaulay R-modules and that inj.dim $Y_C \leq d-1$ and inj.dim $Y^C \leq d$. ∎

We now turn our attention to the question of when $\hat{\omega} = \check{\omega}$ in this context.

Proposition 6.3. The following statements are equivalent for Λ.

(a) $\hat{\omega} = \check{\omega}$.

(b) If X is a Λ^{op}–module which is MCM as an R-module and such that $\mathrm{proj.dim}_{\Lambda^{op}} X < \infty$, then X is a projective Λ^{op}-module.

Proof: We know by Proposition 4.6 that $\check{\omega} = \hat{\omega}$ if and only if an exact sequence $0 \to C \to W_0 \to \dots \to W_n \to 0$ is in ω as soon as each W_i is in ω for $i = 0,\dots n$.
(a) $\Rightarrow$ (b): Suppose $0 \to P_m \to \dots \to P_0 \to X \to 0$ is a projective resolution for a given Λ^{op}-module X which is MCM over R. Then

$$0 \to \mathrm{Hom}_R(X,R) \to \mathrm{Hom}_R(P_0,R) \to \dots \to \mathrm{Hom}_R(P_m,R) \to 0$$

is exact in **mod**-Λ with each $\mathrm{Hom}_R(P_i,R)$ in ω for $i = 0,\dots,m$. By (a) we have that $\mathrm{Hom}_R(X,R)$ is necessarily in ω. Therefore $\mathrm{Hom}_R(X,R) \cong \mathrm{Hom}_R(P,R)$ for some projective Λ^{op}-module P, which then yields $X \cong P$.
(b) $\Rightarrow$ (a): Suppose that X in **mod**-Λ is MCM over R and that $0 \to X \to W_0 \to \dots \to W_m \to 0$ is an exact sequence with W_i in ω for all $i = 0,\dots,m$. Then $0 \to \mathrm{Hom}_R(W_m,R) \to \dots \to \mathrm{Hom}_R(W_0,R) \to \mathrm{Hom}_R(X,R) \to 0$ is exact and $\mathrm{Hom}_R(W_i,R)$ is a projective Λ^{op}-module for each i. The Λ^{op}-module $\mathrm{Hom}_R(X,R)$ is still MCM as an R-module and hence $\mathrm{Hom}_R(X,R) \cong P$ for some projective Λ^{op}-module by our assumption. As MCM's are reflexive, $X \cong \mathrm{Hom}_R(P,R)$ and X is therefore in ω. ∎

This proposition gives the following generalization of a result of R. Sharp [Sh].

Corollary 6.4. Suppose Λ is a commutative ring. Then the following are equivalent for a finitely generated Λ-module M.

(a) $\mathrm{inj.dim}_\Lambda\, M < \infty$.
(b) There is an exact sequence $0 \to W_m \to \dots \to W_0 \to M \to 0$ with W_i in ω for all $i = 0,\dots,m$.

Proof: The equivalence of (a) and (b) is nothing more than the statement that $\check{\omega} = \hat{\omega}$. But this follows from Proposition 6.3 since it is well-known for commutative rings, that a maximal Cohen-Macaulay module of finite projective dimension is projective. ∎

However, if Λ is not commutative, it is not necessarily true that a Λ^{op}-module which is MCM over R and of finite projective dimension over Λ^{op} is necessarily projective. For example, let R be a regular local ring of dimension $d > 0$ and let Λ be the algebra of lower triangular $n \times n$ matrices over R with $n \geq 2$. Then Λ is a free and finitely generated R-module and $\mathrm{gl.dim}\, \Lambda^{op} = d+1$. Let

$$0 \to P_{d+1} \to P_d \to \dots \to P_0 \to M \to 0$$

be a projective Λ^{op}-resolution of a Λ^{op}-module M with $\mathrm{proj.dim}_{\Lambda^{op}} M = d+1$. Then $\mathrm{Im}(P_d \to P_{d-1})$ is an MCM over R which is of projective dimension one over Λ^{op} and is hence not Λ^{op}-projective.

Example 8. Let k be a field and $P = k[x_0,\dots,x_n]$ a polynomial ring over k in n+1 variables which we grade by assigning arbitrary positive integral weights to the variables. Let I be a homogeneous ideal in P and set $S = P/I$ which is hence a positively graded k-algebra.

We assume that S is a Cohen-Macaulay ring. It is known then that there exists a sequence $y_1, \ldots, y_m$ of homogeneous elements of strictly positive degrees in I which is a regular $k[x_0, \ldots, x_n]$-sequence, and such that $R = P/(y_1, \ldots, y_m)$ has the same dimension as S. As R is a complete intersection, it is a Gorenstein ring and by construction the natural surjection $R \to S \to 0$ is a degree-preserving homomorphism of rings. Let **C** = **S-grmod** be the category of finitely generated graded S-modules with degree zero graded maps as morphisms. Also let **X** be the subcategory of **C** consisting of all maximal Cohen-Macaulay modules. In addition to the usual properties, **X** also satisfies $\hat{\mathbf{X}} = \mathbf{C}$ and **X**-resol.dim $\hat{\mathbf{X}} = n+1-m = d$, the dimension of S.

Set $\omega_{S/R} = \mathrm{Hom}_R(S, R)$, which is a dualizing module of S over R, and define ω to be the subcategory of **C** consisting of all $\omega_{S/R}(n)$ for n in **Z**. Then ω is an injective cogenerator for **X**. Moreover we know that X in **X** is of finite projective dimension if and only if it is isomorphic to a direct sum $\oplus_i S(a_i)$.

As in the previous example, this implies $\hat{\omega} = \check{\omega}$. We leave it to the reader to write down in detail what this means for **X**-approximations, $\hat{\omega}$-hulls and modules of finite injective dimension.

We now give our last example.

Example 9. Let $\Lambda \to \Gamma$ be a ring homomorphism with Λ both left and right noetherian and Γ a finitely generated projective Λ-module on both left and right. Let **C** = Γ-**mod** be the category of all finitely generated left Γ-modules and let **X** consist of all M in **C** such that M is a projective Λ-module. In addition to the usual properties, we have that $\hat{\mathbf{X}}$ consists of all N in **C** such that $\mathrm{proj.dim}_\Lambda\, N < \infty$.

Define ω to be the category of all modules isomorphic to $\mathrm{Hom}_\Lambda(P, \Lambda)$ for some finitely generated projective Γ^{op}-module P. Then ω is an injective cogenerator for **X**. In general $\hat{\omega} \neq \check{\omega}$, but if all the modules in ω are projective Γ-modules, then we do have $\hat{\omega} = \check{\omega}$ by Corollary 4.8 and in fact **X** becomes a Frobenius category, see Example 6.

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CONJECTURE JACOBIENNE ET OPÉRATEURS DIFFÉRENTIELS*

Hyman BASS**

à Pierre SAMUEL

1.– Introduction et énoncés

Soient

$$A = \mathbb{C}[x] \subset B = \mathbb{C}[t]$$

deux algèbres de polynômes en n variables $x = (x_1, \ldots, x_n)$ et $t = (t_1, \ldots, t_n)$. Nous supposons que

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(t_1,\ldots,t_n)} = 1 . \tag{1}$$

Alors la Conjecture Jacobienn affirme que $A = B$. La condition (1) entraîne que B est étale (= plat et non ramifié) sur A , donc que toute dérivation de A se prolonge à B . Par exemple $\partial_i = \partial/\partial x_i$ opère sur $b = b(t) \in B$ par

$$\partial_i(b) = \frac{\partial(x_1,\ldots,x_{i-1}, b, x_{i+1}, \ldots, x_n)}{\partial(t_1, \ldots \qquad \ldots, t_n)} .$$

On peut donc considérer A et B comme des modules sur l'algèbre de Weyl

$$W = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n] .$$

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* Une exposition plus élaborée des résultats présentés ici se trouve dans [B].

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Dans W est contenue l'algèbre de Lie $\underline{gl}_n(\mathbb{C})$, de base $\epsilon_{ij} = x_i\partial_j$ $(i,j = 1,\dots,n)$. Soit $\underline{g} \subset \underline{gl}_n(\mathbb{C})$ une sous-algèbre de Lie.

1.1.– THEOREME. *Si* $\dim_{\mathbb{C}}(\underline{g}) > n$, ***alors*** B ***est un module de torsion sur l'algèbre enveloppante*** $U = U\underline{g}$.

En effet supposons que $b \in B$ engendre un U–module libre, $Ub \simeq U$. D'après le théorème de Poincaré–Birkhoff–Witt, U est de croissance polynomiale de degré $\dim_{\mathbb{C}}(\underline{g})$. D'autre part $Ub \subset B$ et B est de croissance polynomial de degré Krull $\dim(B) = n$, d'où $\dim_{\mathbb{C}}(\underline{g}) \leq n$ (cf. [B], Prop. (3.2)).

Ce résultat fournit la stratégie suivante pour approcher la Conjecture Jacobienne. Choisissons une algèbre de Lie $\underline{g} \subset \underline{gl}_n(\mathbb{C})$ de dimension $> n$ et d'algèbre enveloppante $U = U\underline{g}$. Alors la condition :

(2) B/A **est un** U**-module sans torsion**

entraîne, d'après (1.1), que $A = B$. Plus explicitement, (2) signifie que :

(2′) **Si** $\phi \in U$, $\phi \neq 0$, $f \in B$, **et** $\phi f \in A$, **alors** $f \in A$.

Un élément non constant ϕ du U est dit irréductible si dans toute factorisation $\phi = \phi_1 \cdot \phi_2$ dans U on a $\phi_1 \in \mathbb{C}$ ou $\phi_2 \in \mathbb{C}$. La filtration de Poincaré–Birkhoff–Witt montre que tout élément de U est produit d'éléments irréductibles. Si on vérifie (2′) pour ϕ irréductible alors il est vrai en général (par récurrence sur le nombre de facteurs irréductibles de ϕ). **Il suffit donc de considérer (2′) pour** ϕ **irréductible.**

Quitte à remplacer $x(t)$ par $x(t) - x(0)$, on peut supposer que $x(0) = 0$. Alors la condition (1) entraîne que les complétés $\hat{A}$ et $\hat{B}$ à l'origine coïncident. D'où les inclusions :

$$A = \mathbb{C}[x] \subset B \subset \hat{A} = \mathbb{C}[[x]] .$$

Notons A_d le $\mathbb{C}$–module des polynômes en $x = (x_1,\dots, x_n)$ homogènes de degré d. On a

(3) $$A = \bigoplus_{d \geq 0} A_d \subset \hat{A} = \prod_{d \geq 0} A_d .$$

Les opérateurs $\epsilon_{ij} = x_i\partial_j$ sont homogènes, donc les A_d dans (3) sont des U–modules.

Soient $\phi \in U$ et $f \in B$ comme dans (2′). Ecrivons $f = \sum_{d\geq 0} f_d$ où $f_d \in A_d$. On a $\phi f = \sum_{d\geq 0} \phi f_d$ avec $\phi f_d \in A_d$. La condition $\phi f \in A$ signifie que $\phi f_d = 0$ pour tout $d > N$, pour N suffisamment grand. Si $f_{(N)} = \sum_{d\leq N} f_d$ on a $f_{(N)} \in A$ et $\phi(f - f_{(N)}) = 0$; d'ailleurs $f \in A \Leftrightarrow f - f_{(N)} = 0$; d'ailleurs $f \in A \Leftrightarrow f - f_{(N)} \in A$. Ainsi dans (2′) on peut remplacer l'hypothèse "$\phi f \in A$" par "$\phi f = 0$".

Notation. Pour tout opérateur linéaire D sur un module V notons

$$V_0^D = \operatorname{Ker}(V \xrightarrow{D} V) .$$

Avec cette notation, on voit maintenant que les conditions (2) et (2′) sont équivalents à la condition :

(2″) $\qquad B_0^\phi \subset A$ **pour tout élément irréductible** ϕ **de** $U = U_{\mathfrak{g}}$.

Pour le cas $n = 2$ nous avons obtenu les résultats suivants. Posons $x = x_1$, $y = x_2$, $\epsilon_x = x\frac{\partial}{\partial x}$, $\epsilon_y = y\frac{\partial}{\partial y}$, $\delta = x\frac{\partial}{\partial y}$, $\epsilon = \epsilon_x + \epsilon_y$. On a

$$A = \mathbb{C}[x,y] \subset B \subset \hat{A} = \mathbb{C}[[x,y]] .$$

1.2.– **Proposition.** *B/A est un module sans torsion sur $\mathbb{C}[\epsilon,\delta]$. Plus précisément :*

(a) $\quad B_0^\delta = \mathbb{C}[x] \subset A$ *et* $\hat{A}_0^\delta = \mathbb{C}[[x]]$.

(b) $\quad$ *Si* $\phi = \phi(\epsilon,\delta)$ *et* $\phi(\epsilon,0) \neq 0$ *alors* $B_0^\phi = \hat{A}_0^\phi \subset A$ *et* $\dim_{\mathbb{C}}(\hat{A}_0^\phi) < \infty$.

1.3.– **Theoreme.** *Soit* $\phi = a\epsilon_x - b\epsilon_y - c$ *où* $a,b,c \in \mathbb{Z}$, $a,b \geq 0$, $\operatorname{pgdc}(a,b) = 1$, $c \leq 0$ *si* $a = 0$, *et* $c \geq 0$ *si* $b = 0$. *Posons* $u = x^b y^a$. *Alors*

$$B_0^\phi = x^r y^s \mathbb{C}[u] \subset A \quad et \quad \hat{A}_0^\phi = x^r y^s \mathbb{C}[[u]] ,$$

où (r,s) *est le point de* $\mathbb{N}^2$ *le plus près de l'origine sur la droite* $ax - by = c$.

Appelons **spécial linéaire** tout élément ϕ de la forme $\phi = a\epsilon_x - b\epsilon_y - c$ comme dans (1.3).

1.4.– THEOREME. *B/A est un module sans torsion sur $\mathbb{C}[\epsilon_x, \epsilon_y]$. Plus précisément soit ϕ un élément irréductible de $\mathbb{C}[\epsilon_x, \epsilon_y]$. Il y a trois cas possibles :*

(a) $B_0^\phi = \hat{A}_0^\phi \subset A$ *et* $\dim_{\mathbb{C}}(\hat{A}_0^\phi) < \infty$.

(b) *Il existe un $k \neq 0$ dans $\mathbb{C}$ tel que $k\phi$ soit spécial linéaire* (cf. (1.3))

(c) $\dim_{\mathbb{C}}(A_0^\phi) = \infty$ *mais tout $f \in \hat{A}_0^\phi$ qui est algébrique sur $\mathbb{C}(x,y)$ appartient à A. En particulier $B_0^\phi \subset A$.*

Pour démontrer la Conjecture Jacobienne (pour $n = 2$) il suffit, d'après (1.1), de montrer que B/A est un module sans torsion sur

$$U = \mathbb{C}[\epsilon_x, \epsilon_y, \delta] = U\underline{g},$$

où $\underline{g}$ désigne l'algèbre triangulaire supérieure de $\underline{gl}_2(\mathbb{C})$.

1.5.– THEOREME. *Soit $\phi = \phi(\epsilon_x, \epsilon_y, \delta) \in U$ et posons $\phi_o = \phi(\epsilon_x, \epsilon_y, 0) \in \mathbb{C}[\epsilon_x, \epsilon_y]$. Supposons que ϕ_o ne soit pas multiple d'un élément spécial linéaire. Alors tout $f \in \hat{A}_0^\phi$ qui est algébrique sur $\mathbb{C}(x,y)$ appartient à A. En particulier $B_0^\phi \subset A$.*

Les démonstrations de (1.4) et (1.5) font intervenir le théorème de Siegel sur les courbes algébriques ayant un nombre infini de points entiers, ainsi que le théorème de Fabry sur les séries lacunaires.

2.– Deux lemmes préliminaires.

Soient $A \subset B$ des anneaux commutatifs intègres de corps de fractions $F \subset E$. Notons $A^\times$ et $B^\times$ leurs groupes des éléments inversibles

$$\begin{array}{ccc} F & \subset & E \\ \cup & & \cup \\ A & \subset & B \end{array}.$$

2.1.– LEMME. *Supposons que A soit factoriel, que B soit plat sur A, et que $A^\times = B^\times$. Alors $F \cap B = A$ (autrement dit, B/A est un A-module sans torsion).*

En effet soit $p/q \in F \cap B$ où $p,q \in A$ sont étrangers. Alors la suite $0 \longrightarrow A/qA \xrightarrow{p} A/qA$ est exacte. Tensorisant avec le A–module plat B on trouve que la suite $0 \longrightarrow B/qB \xrightarrow{p} B/qB$ est exacte. Or $p/q \in B$, c'est–à–dire $p \in qB$, donc la dernière application injective est nulle ; d'où $q \in B^\times = A^\times$ et $p/q \in A$.

Supposons maintenant que

$$A = \bigoplus_{d \geq 0} A_d \,, \quad A_o = \mathbb{C} \,,$$

soit une $\mathbb{C}$–algèbre factorielle graduée, que B soit étale sur A, que $B^\times = A^\times \ (= \mathbb{C}^\times)$, et que E soit unirationel sur $\mathbb{C}$.

2.2.– Lemme. *Soit* $u \in A_d$, $d > 0$, ***tel que*** u ***ne soit pas une puissance supérieure à*** 1 ***d'un élément de*** A. ***Alors*** $\mathbb{C}(u)$ ***est algébriquement clos dans*** E, ***et*** $\mathbb{C}(u) \cap B = \mathbb{C}[u]$.

En effet soit L la clôture algébrique de $\mathbb{C}(u)$ dans E. Puisque E est unirationnel sur $\mathbb{C}$ le théorème de Lüroth ([N], p. 137) entraîne que L est une extension rationnelle de $\mathbb{C}$, $L = \mathbb{C}(v)$. Soit $R = L \cap B \supset \mathbb{C}[u]$. Alors R est intégralement clos dans L (car B est normal), donc L est le corps de fractions de R. De plus $R^\times \subset B^\times = \mathbb{C}^\times$, donc R est une algèbre de polynômes, et on peut choisir le générateur v de L tel que $R = \mathbb{C}[v]$.

Notons D la dérivation "d'Euler" de A ; $D(a) = ma$ pour $a \in A_m$. Le prolongement de D à E laisse B invariant, car B est étale sur A. On a $D(u) = du$, donc D laisse invariant $\mathbb{C}(u)$, aussi bien que son extension algébrique L. Il s'ensuit que $R = L \cap B$ est D–invariant. Alors $u \in R = \mathbb{C}[v]$, donc $du = D(u) = \frac{du}{dv} \cdot D(v)$. On voit ainsi que $\deg_v(D(v)) = 1$, $D(v) = av + b$ $(a \in \mathbb{C}^\times, b \in \mathbb{C})$. Quitte à remplacer v par $v - a^{-1}b$, on peut supposer que $D(v) = av$. Par suite $D(v^m) = amv$ pour tout $m \geq 0$. Puisque $u \in \mathbb{C}[v]$ est un vecteur propre de D on conclut que $u = cv^m$ ou $c \in \mathbb{C}^\times$ et $am = d$. On a donc $uB = (vB)^m$. D'autre part, par hypothèse, l'idéal uA n'est pas une puissance supérieure à 1 dans A. Puisque B est étale, donc non ramifié, sur A l'idéal uB ne peut pas être non plus une puissance supérieure à 1. Donc $m = 1$, $u = cv$, et $\mathbb{C}[u] = \mathbb{C}[v]$; d'où le lemme.

2.3.– COROLLAIRE. *Supposons ci-dessus que* $A = \mathbb{C}[x_1 ,..., x_n]$ *soit une algèbre de polynômes, et soit* ∂_i *le prolongement de* $\frac{\partial}{\partial x_i}$ *à* E. *Alors*

$$\bigcap_{i=2}^{n} B_0^{\partial_i} = \mathbb{C}[x_1] ,$$

et $\mathbb{C}(x_1)$ *est algébriquement clos dans* E.

En effet on voit facilement que $L = \bigcap_{i=2}^{n} E_0^{\partial_i}$ est la clôture algébrique de $\mathbb{C}(x_1)$ dans E. Il suffit donc d'appliquer (2.2) avec $u = x_1$.

3.– Les calculs en dimension 2 .

Revenons au cadre des résultats (1.2),...,(1.5) . On a

$$\begin{array}{ccccc} A & = & \mathbb{C}[x,y] \subset B \subset & \hat{A} & = \mathbb{C}[[x,y]] \\ \| & & & \| & \\ \bigoplus_{d \geq 0} A_d & & & \prod_{d \geq 0} A_d & \end{array}$$

où A_d désigne le $\mathbb{C}$–module des polynômes en x,y homogènes de degré d. Il nous suffit de supposer que B soit étale sur A, que $B^{\times} = \mathbb{C}^{\times}$, et que le corps de fractions de B soit (uni)rationnel sur $\mathbb{C}$.

Rappelons que $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$, $\epsilon_x = x\partial_x$, $\epsilon_y = y\partial_y$, $\delta = x\partial_y$, et $\epsilon = \epsilon_x + \epsilon_y$. Ces dérivations opèrent sur les monômes par :

$$(1) \qquad \begin{cases} \epsilon_x(x^p y^q) = px^p y^q \\ \epsilon_y(x^p y^q) = px^p y^q \\ \delta(x^p y^q) \ = qx^{p+1} y^{q-1} \end{cases}$$

Remarquons d'abord que

$$\hat{A}_0^{\delta} = \hat{A}_0^{\partial_y} = \mathbb{C}[[x]]$$

et que

$$B_0^\delta = \mathbb{C}[x] \subset A ,$$

d'après (2.3), d'où (1.2)(a).

Prenons pour base de A_d les monômes :

$$x^d , x^{d-1}y ,..., xy^{d-1} , y^d .$$

Alors les matrices de ϵ_x et ϵ_y sont diagonales, et celle de δ est

(3) $$\delta|_{A_d} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 2 & & 0 \\ & & 0 & \ddots & \\ & 0 & & \ddots & d \\ & & & & 0 \end{bmatrix}$$

Soit $\phi = \phi(\epsilon_x ,\epsilon_y , \delta) \in \mathbb{C}[\epsilon_x ,\epsilon_y ,\delta]$. On peut écrire ϕ sous la forme :

$$\phi = \sum_{i=0}^{N} \phi_i(\epsilon_x ,\epsilon_y)\delta^i .$$

Alors la matrice de ϕ sur A_d est triangulaire supérieure,

(4) $$\phi|_{A_d} = \begin{bmatrix} \phi_o(d,0) & x\ x & & \\ & \phi_o(d-1,1) & & * \\ & & \ddots & \\ & 0 & & \phi_o(0,d) \end{bmatrix}$$

Posons

$$C = \{(p,q) \in \mathbb{C}^2 \mid \phi_o(p,q) = 0\} .$$

Supposons que $\phi_o \neq 0$, de sorte que C est une courbe plane. Pour $d \in \mathbb{N}$ posons

$$N_d = \{(p,q) \in \mathbb{N}^2 \mid p + q = d\}$$
$$\{(d,0), (d-1,1), \ldots, (0,d)\} .$$

Alors on voit que :

$$(5)\ \begin{cases} \mathrm{Card}(C\cap N_d) \text{ est la multiplicité de } 0 \text{ comme valeur propre de particulier} \\ \phi|_{A_d}\ .\ \text{En} \quad \dim_{\mathbb{C}}((A_d)_0^{\phi}) \geq \mathrm{Card}(C\cap N_d) \\ \text{et} \quad \phi|_{A_d} \text{ est inversible} \Leftrightarrow C\cap N_d = \emptyset\ . \end{cases}$$

Remarquons que :

$$\hat{A}_0^{\phi} = \prod_{d\geq 0} (A_d)_0^{\phi}\ .$$

Par conséquent :

Les conditions suivantes sont équivalentes

$$(6)\qquad \begin{cases} \text{(a)}\ \hat{A}_0^{\phi} \subset A \\ \text{(b)}\ \dim_{\mathbb{C}}(\hat{A}_0^{\phi}) < \infty \\ \text{(c)}\ \mathrm{Card}(C\cap \mathbb{N}^2) \text{ est fini.} \end{cases}$$

Exemple : Supposons que $\phi_o = \phi_o(\epsilon) \in \mathbb{C}[\epsilon]$ (rappelons que $\epsilon = \epsilon_x + \epsilon_y$), par exemple que $\phi \in \mathbb{C}[\epsilon,\delta]$. Alors les éléments de la diagonale de $\phi|_{A_d}$ sont tous égaux à $\phi_o(d)$. Puisque $\phi_o \neq 0$, $\phi_o(d)$ ne s'annule que pour un nombre fini de valeurs de d, d'où les conditions de (6). Cecic donne (1.2) (b).

Démonstration de (1.3). Considérons un élément ϕ non constant de la forme $\phi = \phi_o = a\epsilon_x - b\epsilon_y - c$ $(a,b,c\in \mathbb{C})$, et soit C la droite

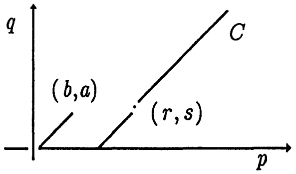
$$C = \{(p,q) \mid ap - bq = c\}\ .$$

Pour montrer que $B_0^{\phi} \subset A$ il suffit, d'après (6), de traiter le cas où $C\cap \mathbb{N}^2$ est infini, donc C est une droite rationnelle. Quitte à multiplier ϕ par une constante on peut supposer que a et b sont des sentiers étrangers (donc $c\in \mathbb{Z}$ aussi) et que $a, b \geq 0$ (donc $c \leq 0$ si $a = 0$, et $c \geq 0$ si $b = 0$). Ainsi ϕ est "special lineaire" au sens de (1.3). Soit (r,s) le point de $C\cap \mathbb{N}^2$ le plus près de l'origine. Alors

$$C \cap \mathbb{N}^2 = \{(r,s) + n(b,a) \mid n \in \mathbb{N}\} .$$

Donc

(7) $$\hat{A}_0^{\phi} = x^r y^s \mathbb{C}[[u]] , \text{ où } u = x^b y^a .$$



Posons $\partial = a\epsilon_x - b\epsilon_y$, de sorte que $\hat{A}_0^{\partial} = \mathbb{C}[[u]]$. Soit E le corps de fractions de B. Alors E_0^{ϕ} est une extension algébrique de $\mathbb{C}(u)$. D'autre part $u = x^b y^a$ n'est pas une puissance supérieure à 1 d'un élément de A. Du lemme (2.2) on conclut que $E_0^{\phi} = \mathbb{C}(u)$. Soit alors $f \in B_0^{\phi}$. D'après (7) on a $f = x^r y^s g(u)$ où $g(u) \in \mathbb{C}[[u]] \cap E \subset E_0^{\phi} = \mathbb{C}(u) \subset \mathbb{C}(x,y)$. Donc $f \in \mathbb{C}(x,y) \cap B = A$, d'après le lemme (2.1) ; d'où (1.3).

Démonstration de (1.4) : Soit $\phi = \phi_0(\epsilon_x, \epsilon_y)$ un élément **irréductible** de $\mathbb{C}[\epsilon_x, \epsilon_y]$, et posons $C = \{(p,q) \mid \phi(p,q) = 0\}$. Puisque $\phi|_{A_d}$ est diagonal on a

(8) $$\hat{A}_0^{\phi} = \prod_{(p,q) \in C \cap \mathbb{N}^2} \mathbb{C} x^p y^q .$$

Si $C \cap \mathbb{N}^2$ est fini (cf. (6)) on a le cas (1.4) (a). Supposons désormais que

(9) $C \cap \mathbb{N}^2$ **soit infini**.

Alors d'après un théorème classique de Siegel (cf. [L], Ch. 8) la courbe irréductible C est rationnelle. De plus les résultats de Siegel nous permettent de donner une paramétrisation rationnelle de C (cf. [B], Appendix E), à savoir :

$$(10)\begin{cases}\text{Il existe des fonctions rationnelles } p = p(t),\ q = q(t) \text{ dans } \mathbb{Q}(t) \\ \text{telles que } \mathbb{Q}(t) = \mathbb{Q}(p,q) \text{ et } C \cap \mathbb{Q}^2 = \{(p(\tau), q(\tau)) \mid \tau \in \mathbb{Q}\}. \\ \text{De plus on a l'un des cas suivants :} \\ \text{(a) } p(t),\ q(t) \in Q[t] \text{, ils ont des coefficients dominants } \geq 0, \\ \text{et } (p(\tau), q(\tau)) \in \mathbb{Z}^2 \Rightarrow \tau \in \mathbb{Z} \\ \text{(b) } p(t) = P(t)/(t^2-d)^e \text{ et } q(t) = Q(t)/(t^2-d)^e, \text{ où } P, Q \in \mathbb{Q}[t] \\ \text{sont de degrés} > 2e, \text{ et } d \text{ est un entier} > 1 \text{ sans facteur carré.}\end{cases}$$

Si ϕ est linéaire c'est–à–dire si C est une droite, alors la démonstration de (1.3) nous donne le cas (1.4) (b). Supposons donc que C ne soit pas une droite. Ceci entraîne que :

(11) **Dans le cas (10)(a), l'un au moins de $p(t)$ et $q(t)$ est de degré ≥ 2.**

Sous ces conditions on peut montrer que :

(12) **Les points de $C \cap \mathbb{N}^2$ "tendent rapidement vers l'infini".**

(voir [B], Appendix E pour un énoncé précis).
Soit $f \in \hat{A}_0^{\phi}$. D'après (8) et (10) on peut écrire

$$f(x,y) = \sum_{\tau} a_{\tau} x^{p(\tau)} y^{q(\tau)} ,$$

où $a_{\tau} \in \mathbb{C}$ et τ parcourt les éléments de $\mathbb{Q}$ tels que $(p(\tau), q(\tau)) \in \mathbb{N}^2$. Pour $b \in \mathbb{C}$ posons

$$f_b(x) = f(x,bx) = \sum_{\tau} a_{\tau} b^{q(\tau)} x^{r(\tau)} \in \mathbb{C}[[x]] ,$$

où $r(t) = p(t) + q(t)$. Si $f \notin A = \mathbb{C}[x,y]$ on a $f_b \notin \mathbb{C}[x]$ pour tout $b \in \mathbb{C}$ en dehors d'un ensemble dénombrable. De plus si f est algébrique sur $\mathbb{C}(x,y)$ alors f_b sera algébrique sur $\mathbb{C}(x)$ pour tout sauf un nombre fini de valeurs de b.

Pour achever la démonstration de (1.4) il nous reste à montrer que tout $f \in \hat{A}_0^{\phi}$ qui est algébrique sur $C(x,y)$ est un polynôme $(f \in A)$. Supposons au contraire que $f \in \hat{A}_0^{\phi}$, $f \notin A$, et f soit algébrique sur $C(x,y)$. Choisissons b tel que $f_b \notin \mathbb{C}[x]$ et f_b est algébrique sur $\mathbb{C}(x)$. Ecrivons

$$f_b = \sum_{m=0}^{\infty} c_m x^{n_m} ,$$

où $c_m \in \mathbb{C}^\times$ et $n_0 < n_1 < n_2 < \dots$. Alors il résulte de (12) que

(13) $$n_m/m \longrightarrow \infty \quad \text{lorsque} \quad m \longrightarrow \infty ,$$

Autrement dit f_b est une série "lacunaire". Il résulte alors du théorème de Fabry (1896) (cf. [D], Ch. XI, 93.II) que tout point du cercle de convergence de f_b est singulier. Mais une fonction algébrique ne peut avoir qu'un nombre fini de points singuliers. Par suite f_b converge partout dans $\mathbb{C}$. Mais une série partout convergente qui est algébrique sur $\mathbb{C}(x)$ est un polynôme (cf. [B], Prop. (D.1)). Cette contradiction conclut la démonstration.

Remarque. C'est R. Narasimhan qui m'a signalé le théorème de Fabry et son application dans ce cadre. Je tiens à l'en remercier.

Démonstration de (1.5) : Revenons au cas général, $\phi = \sum_{i=0}^{N} \phi_i(\epsilon_x, \epsilon_y)\delta^i$, et $C = \{(p,q) \mid \phi_o(p,q) = 0\}$. Supposons que ϕ_o n'est pas multiple d'un polynôme spécial linéaire (au sens de (1.3)). Soit $C = C_1 \cup \dots \cup C_h$ la décomposition de C comme réunion de courbes irréductibles. Notre hypothèse entraîne, vu la démonstration de (1.4), que pour tout $i = 1,\dots,h$ soit $C_i \cap \mathbb{N}^2$ est fini, soit les points de $C_i \cap \mathbb{N}^2$ tendent rapidement vers l'infini. Ainsi si $f \in \hat{A}_0^\phi$, $f \notin A$, on a $f = \sum_{m=0}^{\infty} f_{d_m}$, où $f_{d_m} \in A_{d_m}$ et $d_m/m \longrightarrow \infty$ lorsque $m \longrightarrow \infty$. Comme dans la démonstration de (1.4), on voit qu'une telle fonction ne peut pas être algébrique sur $\mathbb{C}(x,y)$.

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Hyman BASS
Mathematics Department
Columbia University
Morningside Heights
New York, N.Y. 10027 (U.S.A)

SPENCER BLOCH
M. PAVAMAN MURTHY
LUCIEN SZPIRO

Zero cycles and the number of generators of an ideal

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ZERO CYCLES AND THE NUMBER OF GENERATORS OF AN IDEAL

Spencer BLOCH, M. Pavaman MURTHY and Lucien SZPIRO

(to Pierre Samuel)

RESUME

Soit X une surface dans l'espace affine $\mathbb{A}^4$ sur un corps algébriquement clos k. On montre que X est ensemblistement intersection complète si $k = \overline{\mathbb{F}}_p$ ou si X n'est pas birationnelle à une surface projective de type général.

On donne aussi des exemples de variétés affines lisses de dimension n qui ne sont pas des sous–variétés fermées dans $\mathbb{A}^{2n}$. La plupart des résultats s'appuie sur les théorèmes de Mumford et de Roitman concernant le groupe de Chow $CH_0(X)$.

ABSTRACT

Let X be a local complete intersection surface in $\mathbb{A}^4$ over an algebraically closed field k. We show that X is set–theoretic complete intersection if $k = \overline{\mathbb{F}}_p$ or if X is smooth and not birational to a surface of general type.

We also give examples of smooth affine varieties of dimension n, not admitting a closed immersion in $\mathbb{A}^{2n}$. Most of the results here depend crucially on the results of Mumford and Roitman on the Chow group $CH_0(X)$.

Introduction.

Let k be a field and X a closed codimension two local complete intersection sub–scheme of the affine n–space $\mathbb{A}^n_k$. Let I be the defining ideal of X in $k[X_1,...,X_n]$ and suppose that there is a surjection $I/I^2 \twoheadrightarrow \omega_X$, where ω_X is the dualizing module of X. Then, the Ferrand–Szpiro Theorem ([Sz], see Cor. 0.2 below) asserts that X is a set–theoretic complete intersection. When X is a curve of dimension one, the surjection $I/I^2 \twoheadrightarrow \omega_X$ always exists and thus Ferrand–Szpiro showed that a local complete intersection curve in $\mathbb{A}^3_k$ is a set–theoretic complete intersection. The question whether any local complete intersection sub–scheme of $\mathbb{A}^n_k$ is a set–theoretic complete intersection is open.

In sections 1 and 2, we examine this question for surfaces in $\mathbb{A}^4$. It is shown that local complete intersection surfaces in $\mathbb{A}^4_{\mathbb{F}_p}$ are set–theoretic complete intersections.

For a smooth surface X in $\mathbb{A}^4_k$ (k algebraically closed), the existence of a surjection $I/I^2 \twoheadrightarrow \omega_X$ turns out to be equivalent to the vanishing of c_1^2 ($c_1 = c_1(\Omega^1_X)$) in the Chow group of zero–cycles. In view of this, it follows by looking at the classification of surfaces, that if X is not birationally equivalent to a surface of general type, then X is a set–theoretic complete intersection (Th. 2.9). We also show that for a smooth affine variety X in $\mathbb{A}^n_k$, the ideal I_X of X in $k[X_1,...,X_n]$ is generated by $n-1$ element if and only if Ω^1_X has a free direct summand of rank one (Th. 1.11).

In section 3, we give a partial converse to the Ferrand–Szpiro theorem. More precisely, we show that if $X \subset \mathbb{A}^4$ is a smooth surface which is an intersection of two surfaces $F_1 = F_2 = 0$ such that at each point of X either F_1 or F_2 is smooth, then $c_1^2 = 0$ (Cor. 3.7). In section 4, we prove a result about zero–cycles on the product of two curves, which enables us to produce examples of surfaces $X = C_1 \times C_2$, with C_i smooth affine curves such that X does not admit a closed immersion in $\mathbb{A}^4$. Further for this example Ω^1_X is not generated by three elements and hence X cannot be immersed in $\mathbb{A}^3$. In section 3, for all n,d with $1 \leq d \leq n \leq 2d+1$ we make examples of smooth d–dimensional affine varieties X such that X admits a closed immersion in $\mathbb{A}^n$, but not in $\mathbb{A}^{n-1}$. Further for any embedding of X in $\mathbb{A}^m$, the prime ideal $I(X)$ of X is not generated by $m-1$ element. When $d=2$ this also provides an example of a smooth surface in $\mathbb{A}^4$ with $c_1^2 \neq 0$. The example in sections 4 and 5 are constructed by showing that the appropriate obstructions in zero–dimensional Chow groups do not vanish.

In this paper we use extensively the results of Roitman ([Ro 1], [Ro 2], [Ro 3]) and Mumford ([Mum]) on the Chow group of zero–cycles. In section 5, we need a result about embedding of affine varieties (Th. 5.7). The simple and elegant proof of this theorem we have included here is due to M.V. Nori. Our thanks are due to him for this proof which replaces our earlier lengthly proof of Theorem 5.7. Thanks are also due to V. Srinivas for asking us a question about embedding of affine varieties. Results in section 5 were rewritten and refined recently in response to his question.

The work in this paper began in 1977. A part of this work was outlined in the survey article [Mu 3]. A major portion of this work was done in 1978 when the first and second named authors were visiting IHES and Ecole Normale Supérieure at Paris, respectively, and the third named author was at Ecole Normale Supérieure. We are grateful to these institutions for hospitality and support. The first two authors were also supported by NSF grants.

We have mentioned some of the recent work relevant to this paper in the form of "remarks".

§0. Notations and preliminaries.

We consider only commutative noetherian rings. Let A be such a ring and $I \subset A$ an ideal. We recall that I is a *complete intersection of height* r *if* I is generated by an A–regular sequence of length r. The ideal I is a *local complete intersection of height* r if for all maximal ideals M containing I, the ideal $I_M \subset A_M$ is a complete intersection of height r. The ideal I is a *set-theoretic complete intersection of height* r if there is an ideal J such that $\sqrt{J} = \sqrt{I}$ and J is a complete intersection of height r. If $I \subset A$ is a local complete intersection of height r, we write $\omega_1 = \mathrm{Ext}^r_A(A/I,A)$. It is well known that $\omega_1 \simeq \mathrm{Hom}(\Lambda^r I/I^2, A/I)$. Note that if X is a smooth affine variety and $V \subset X$ is a local complete intersection sub–scheme of codimension r and I the defining ideal of V in the coordinate ring A of V, then ω_I is the module of sections of $\omega_V \otimes \omega_X^{-1}$, where ω_V and ω_X are the canonical sheaves of V and X respectively.

We recall the following result of Ferrand–Szpiro [Sz], which is crucial for this paper.

Theorem 0 (*Ferrand-Szpiro*). *Let A be a commutative noetherian ring and $I \subset A$ local complete intersection ideal of height* 2. *Suppose there is a surjection $I \twoheadrightarrow \omega_I$. Then there is an exact sequence $0 \to A \to P \to J \to 0$, with P a projective A-module of rank* 2 (*for proof see* [Sz] *or* [Mu 2]).

For a projective R–module L of rank 1, we write $L^n = L^{\otimes n}$, $L^{-n} = \mathrm{Hom}(L^n,R)$, $n \geq 0$.

Remark 0.1. The existence of surjection $I \twoheadrightarrow \omega_I$ is easily seen to be equivalent to the isomorphism $I/I^2 \approx \omega_I^{-2} \oplus \omega_I$, where $\omega_I^{-2} = \mathrm{Hom}(\omega_I^{\otimes 2}, A/I)$. If every projective A/I–module splits as a direct sum of a free module and a module of rank one (e.g. dim $A/I = 1$), then every projective A/I–module P of rank r is completely determined by $\Lambda^r P$ and hence in this case the surjection $I \twoheadrightarrow \omega_I$ is immediate. This remark and the fact that projective modules over polynomial rings over fields are free (Quillen–Suslin Theorem) led Ferrand–Szpiro to deduce that local complete intersection curves in $\mathbb{A}^3$ are set–theoretic complete intersections. Later, Mohan Kumar (MK1] generalized the Ferrand–Szpiro argument to show that any local complete intersection curve in $\mathbb{A}^n$ is a set–theoretic complete intersection.

We do not know the answer even when $n = 4$ and $V = V(I)$ is a smooth surface and k is algebraically closed.

Lemma 1.2. *Let* $R = k[X_1,X_2,X_3,X_4]$, *where* k *is an algebraically closed field and* $I \subset R$ *a local complete intersection ideal of height* 2. *Let* $A = R/I$ *and let* $\omega_I = \omega = \mathrm{Ext}^2_R(A,R)$. *Then*

1) $I/I^2 \approx A \oplus \omega^{-1}$
2) *Consider the following conditions*
    a) ω *is generated by two elements.*
    b) $I/I^2 \approx \omega \oplus \overline{\omega}^2$
    c) ω^{-2} *is generated by two elements. We have* a) $\Rightarrow$ b) $\Rightarrow$ c).

Proof : 1) Since projective R–modules are free and I has projective dimension one, we have an exact sequence $0 \to R^{\ell-1} \to R^{\ell} \to I \to 0$. Tensoring this sequence with $A = R/I$, we get an exact sequence $0 \to L \to A^{\ell-1} \to A^{\ell} \to I/I^2 \to 0$ with L a projective A–module of rank one. Thus in $K_0(A)$, we have $[I/I^2] = [A]+[L]$, and hence $L \approx \Lambda^2 I/I^2 = \omega^{-1}$. Since cancellation holds for projectives over A [Su], we have $I/I^2 \simeq A \oplus \omega^{-1}$.

2) a) $\Rightarrow$ b). By (1), $I/I^2 \oplus A \approx A^2 \oplus \omega^{-1}$. Since ω is generated by two elements, whe have $\omega \oplus \omega^{-1} \approx A^2$. So

$$I/I^2 \oplus A \approx \omega \oplus \omega^{-1} \oplus \omega^{-1} \approx \omega \oplus (\omega^{-1} \otimes A^2)$$
$$\approx \omega \oplus \omega^{-1} \otimes (\omega \oplus \omega^{-1}) \approx A \oplus \omega \oplus \omega^{-2}.$$

Now b) follows from [Su].

3) b) $\Rightarrow$ c). 1) and b) imply that $\omega^2 \oplus \omega^{-1} \approx A \oplus \omega$. Hence $\omega^2 \oplus \omega^{-2} \oplus \omega \approx \omega^2 \oplus A \oplus \omega^{-1} \approx A^2 \oplus \omega$.

Hence by cancelling ω, we have $\omega^2 \oplus \omega^{-2} \approx A^2$, i.e., ω^2 is generated by two elements.

Remarks 1.3. $SK_0(A) = \ker(\tilde{K}_0(A) \xrightarrow{\det} \operatorname{Pic} A)$ has no 2–torsion if char $k \neq 2$ [Le] or $(A/I)_{\text{red}}$ is smooth (Prop. 2.1). Using this fact, it is not hard to show that, in these cases, the implication c) $\Rightarrow$ a) holds and hence a), b), c) are equivalent.

Corollary 1.4. *Let $V \subset \mathbb{A}^4_k$ $(k = \bar{k})$ be a smooth irreducible affine surface. Then V lies on a smooth hypersurface.*

Proof : If $I \subset R = k[X_1,X_2,X_3,X_4]$ is the ideal of V, then we have $I/I^2 \approx A\bar{f} \oplus \omega^{-1}$ $(A = R/I)$. Let $f \in I$ be a lift of $\bar{f}$ and let $I^2 = \Sigma_{i=1}^{\ell} Rf_i$. Then $f,f_1,\ldots,f_\ell$ have no common zeros on $\mathbb{A}^4 - V$. Hence by Bertini's theorem (as given in [Sw] applied to $\mathbb{A}^4 - V$), there exist linear polynomials $h_1,\ldots,h_\ell$ such that Spec $R/f'R$ is smooth and integral on $\mathbb{A}^4 - V$, where $f' = f + \Sigma_{i=1}^{\ell} h_i f_i$. Since V is smooth and f' is a lift of $\bar{f}$ it follows that Spec $R/f'R$ is smooth at points of V. Hence the hypersurface $f' = 0$ is smooth and integral.

Remark 1.5. Recently it has been shown that (1.4) is true for smooth n–dimensional affine varieties V in $\mathbb{A}^{2n}$ ([Mu 5]).

Proposition 1.6. *Let R, I, A and ω be as in Lemma 1.2. Suppose $\omega^{\otimes r}$ is generated by two elements for some $r \neq 0$. Then I is a set-theoretic complete intersection.*

Proof : We may assume $r > 0$. Let $f \in I$ such that $I/I^2 \approx A\bar{f} \oplus \omega_I^{-1}$. Set $J = I^r + Rf$. It is easy to see that J is a local complete intersection of height two. Further $J/J^2 \approx R/J.\tilde{f} \oplus \omega_J^{-1}$ (use i) of Lemma 1.2), where $\tilde{f}$ is the image of f in J/J^2. Hence $\omega_J^{-1} = J/(J^2 + Rf)$. By Corollary 0.2 and Lemma 1.2, 2), it suffices to show that ω_J^{-1} is generated by two elements. Since I/J is a nilpotent ideal in R/J it suffices to show that $\omega_J^{-1} \otimes R/I$ is generated by two elements. But

$$\omega_J^{-1} \otimes R/I = \frac{J}{IJ + Rf} = \frac{I^r + Rf}{I^{r+1} + Rf} \approx \omega_I^{-r} .$$

Hence ω_J^{-1} is generated by two elements and the proof of the proposition is complete.

Theorem 1.7. *Let $k = \bar{\mathbb{F}}_p$ and $I \subset k[X_1,X_2,X_3,X_4]$ a local complete intersection of height two. Then I is a set-theoretic complete intersection.*

Proof : Immediate from Proposition 1.6 and the following lemma.

Lemma 1.8. *Let $K/\mathbb{F}_p$ be an algebraic extension and A a d-dimensional affine ring over K $(d > 0)$. Let L be a projective A-module of rank one. Then $L^{\otimes r}$ is generated by d elements for some $r > 0$ (depending on L).*

Proof : We prove the lemma by induction on d. Suppose the lemma is proved for $d = 1$ and assume $d > 1$. Without loss, we may assume that A is reduced, Spec A is connected and $L = I \subset A$ is an invertible ideal. Then I/I^2 is a projective A/I–module of rank one and $\dim A/I = d-1$. Hence by induction hypothesis $(I/I^2)^{\otimes r} = I^r/I^{r+1}$ is generated by $d-1$ elements and hence J is generated by d elements (e.g. see [MK2]).

Thus we may assume $d = 1$. Since Pic A commutes with direct limits, we may assume K is finite. Let A' be the integral closure of A (in its total quotient ring) and F the conductor ideal from A' to A. Then A'/F is finite. Furthermore, Pic A' is finite [We ; p. 207, Th. 5–3–11]). Now the standard exact sequence [Ba, 5.6] $(A'/F)^* \longrightarrow \text{Pic } A \longrightarrow \text{Pic } A'((A'/F)^* = \text{units in } A'/F)$ show that Pic A is finite. This proves the lemma when $d = 1$ and the proof of the lemma is complete.

Remark 1.9. The proof of Lemma 1.8 works verbatim when $k = \mathbb{Z}$. Also recently it has been shown [MKMR] that cancellation theorem similar to [Su] holds for finitely generated rings over $\mathbb{Z}$. Hence (1.2), (1.6) and hence (1.7) hold when R is replaced by $\mathbb{Z}[X_1,X_2,X_3]$ or $\mathbb{F}_q[X_1,X_2,X_3,X_4]$.

Theorem 1.10. *Let k be an algebraically closed field and $I \subset R = k[X_1,\dots,X_n]$ a local complete intersection of height two and $\omega = \omega_1 = \text{Ext}^2_R(R/I,R)$. Then the following conditions are equivalent.*

a) *I is generated by $n-1$ elements.*

b) *I/I^2 is generated by $n-1$ elements.*

c) *ω is generated by $n-2$ elements.*

Proof : a) $\Longrightarrow$ b). Obvious.

b) $\Longrightarrow$ c). Put $A = R/I$. As in the proof of 1) of Lemma 1.2, we have $[I/I^2] = [A \oplus \omega^{-1}]$ in $K_0(A)$. Hence $I/I^2 \oplus A^\ell \approx A^{\ell+1} \oplus \omega^{-1}$ for some $\ell \geq 0$. Now b) implies that $A^{\ell+1} \oplus \omega^{-1}$ is generated by $n-1+\ell$ elements. Let $\varphi : A^{n-1+\ell} \twoheadrightarrow A^{\ell+1} \oplus \omega^{-1}$ be a surjection with kernel M. Then

$$A^{n-2} \oplus A^{\ell+1} = A^{n-1+\ell} \approx M \oplus A^{\ell+1} \oplus \omega^{-1}.$$

Since $\dim A = n-2$, by Suslin's cancellation theorem [Su], we get $A^{n-2} \approx M \oplus \omega^{-1}$. This shows that ω^{-1} and therefore ω is generated by $n-2$ elements.

c) $\Longrightarrow$ a). Since $\mathrm{Ext}^1(I,R) \approx \omega$, by [Mu2, p. 180], there exists an exact sequence $0 \longrightarrow R^{n-2} \longrightarrow P \longrightarrow I \longrightarrow 0$ with P a projective R–module. Now a) is immediate by Quillen–Suslin Theorem.

Theorem 1.11. *Let $X \subset \mathbb{A}^n_k$ be a smooth affine variety of dimension d and $I = I(X)$, the defining ideal of X in $k[X_1,\dots,X_n]$ $(k = \bar{k})$. Let A be the coordinate ring of X. Then the following conditions are equivalent.*

1) *I is generated by $n-1$ elements.*
2) *I/I^2 is generated by $n-1$ elements.*
3) *$\Omega_{A/k}$ has a free direct summand of rank one.*

Proof : 1) $\Longrightarrow$ 2). Trivial.

2) $\Longrightarrow$ 3). We have $I/I^2 \oplus \Omega^1_A \approx A^n$. 2) implies that $I/I^2 \oplus Q \approx A^{n-1}$ for some Q. Hence in $K_0(A)$, $[\Omega_A] = [A] + [Q]$. By [Su], $\Omega_A \approx A \oplus Q$.

3) $\Longrightarrow$ 2). Let $\Omega_A \approx A \oplus Q$. Then $I/I^2 \oplus A \oplus Q \approx A^n$. Therefore by [Su], $I/I^2 \oplus Q \approx A^{n-1}$ and hence I/I^2 is generated by $n-1$ elements.

2) $\Longrightarrow$ 1). Suppose $n-1 \geq d+2$ i.e., $n \geq d+3$. Then by [MK1], I is generated by $n-1$ elements. If $n = d+1$, I is principal and there is nothing to prove. Hence we may assume $n = d+2$. In this case, the result is immediate from Theorem 1.10.

Remark 1.12. By [Mu5], it follows that Condition 2 in Theorem 1.11 is equivalent to $c_d(\Omega^1_X) = 0$, where $c_d(\Omega_X)$ is the dth Chern class of Ω^1_X with values in the Chow group of zero cycles (see also [MKM, Cor. 2.6]). In particular, for example if X is rational, $I(X)$ is generated by $n-1$ elements.

Proposition 1.13. *Let X be a smooth affine variety of dimension d over a field k (not necessarily algebraically closed) with coordinate ring A. Suppose X admits a closed immersion in $\mathbb{A}^{d+2}_k$. Then $\Omega^1_{A/k}$ is generated by $d+1$ elements.*

PROOF : Let $I \subset k[X_1,...,X_{d+2}]$ be the prime ideal of X. Then, as in the proof of 1) of Lemma 1.2, I/I^2 is stably isomorphic to $A \oplus \omega_I^{-1}$. Now $I/I^2 \oplus \Omega_A^1 \approx A^{d+2}$. So $A \oplus \omega_I^{-1} \oplus \Omega_A^1$ is stably isomorphic to A^{d+2}. So by Bass' cancellation theorem, we have $\Omega_A^1 \oplus \omega_I^{-1} \approx A^{d+1}$, i.e., Ω_A^1 is generated by $d+1$ elements.

§2. Smooth surfaces in $\mathbb{A}^4$.

For an algebraic scheme X over an algebraically closed field k, we denote by $A_p(X)$ the group of p–dimensional cycles modulo rational equivalence. If X is an irreducible scheme of dimension n, we write $A^p(X) = A_{n-p}(X)$. If X is complete, we denote by $A_{00}(X)$, the group of zero cycles of degree zero modulo rational equivalence. The following proposition is an easy consequence of Roitman's theorem (see [RO3] and [Mi]) on torsion in $A_0(X)$.

PROPOSITION 2.1. *Let X be a smooth affine variety of dimension $d \geq 2$ over an algebraically closed field k. Suppose* $\dim X = 2$ *or* $\operatorname{char} k = 0$. *Then $A_0(X)$ is torsion-free.*

PROOF : Because of resolution of singularities, we may choose a smooth projective completion V of X. Let $V - X = \bigcup_{i=1}^{r} C_i = C$, where C_i are irreducible sub–varieties of codimension one. We may also assume that C_i are all smooth. Since C is connected, we have an exact sequence

$$\oplus A_{00}(C_i) \xrightarrow{\varphi} A_{00}(V) \longrightarrow A_0(X) \longrightarrow 0 .$$

Since $A_{00}(C_i)$ are divisible, $A_{00}(V) \approx \operatorname{Im} \varphi \oplus A_0(X)$. Further C generates $\operatorname{Alb}(V)$. (To see this, cutting V by hyperplane sections, we may assume V is a smooth surface. Yhen by Goodman's theorem [Go] C supports an ample divisor and hence generates $\operatorname{Alb}(V)$). Thus, we have a surjective map of abelian varieties, $\operatorname{Alb}(C_i) \twoheadrightarrow \operatorname{Alb}(V)$. This induces a surjective map

$$\psi : \bigoplus_i \operatorname{Alb}(C_i)_{torsion} \twoheadrightarrow \operatorname{Alb}(V)_{torsion} .$$

Thus we have the commutative diagram

$$\begin{array}{ccc} \oplus A_{00}(C_i)_{torsion} & \xrightarrow{\varphi} & A_{00}(V)_{torsion} \\ \approx\downarrow & & \downarrow\approx \\ \oplus \mathrm{Alb}(C_i)_{torsion} & \xrightarrow{\psi} & \mathrm{Alb}(V)_{torsion} . \end{array}$$

By Roitman's theorem ([Ro3] and [Mi]), the vertical maps are isomorphisms. Since ψ is surjective, it follows that $(\mathrm{Im}\ \varphi)_{torsion} = A_{00}(V)_{torsion}$ and hence $A_0(X)$ is torsion–free.

Remark 2.2. Proposition 2.1 is valid for any smooth affine variety X over k $(k=\bar{k})$, in all characteristics (see [Sr] and [Mu5]).

Let X be a smooth affine variety of dimension d with coordinate ring A. For a projective module P, we denote by $c_p(P) \in A^p(X)$, the pth chern class of P [Fu]. Suppose now that $\dim X = 2$. It is weell known [MPS] that

$$A_0(X) = A^2(X) = SK_0(A) \overset{def}{=} \ker(\tilde{K}_0(A) \xrightarrow{det} \mathrm{Pic}(A).$$

If P is a projective A–module of rank r, it is not hard to see that $c_2(P) =$ class of $[A^{r-1}] + [\Lambda^r P] - [P]$ in $SK_0(A)$. In view of this and the cancellation theorem for projectives, we have

Remark 2.3. Let X be a smooth affine surface over an algebraically closed field k and let A be the coordinate ring of X. Let P be a projective A–module of rank r. Then

1) $c_2(P) = 0 \Leftrightarrow P \approx A^{r-1} \oplus \Lambda^r P$.

2) If P and Q are projective A–modules, then $P \approx Q \Leftrightarrow \mathrm{rank}\, P = \mathrm{rank}\, Q$; $c_i(P) = c_i(Q)$, $i = 1,2$.

3) $L \in \mathrm{Pic}\, A$ is generated by two elements $\Leftrightarrow L \oplus L^{-1} \approx A^2 \Leftrightarrow c_1(L)^2 = 0$ in $A^2(X)$.

The following corollary is immediate from Proposition 2.1 and Remark 2.3.

Corollary 2.4. *With the notation as in Remark* 2.3, *let* $L \in \mathrm{Pic}\, A$. *Then the following conditions are equivalent.*

1) L *is generated by two elements.*

2) $c_1(L)^2 = 0$ *in* $A^2(X)$.

3) $L^{\otimes r}$ *is generated by two elements for some* $r \neq 0$.

Corollary 2.5. *Let $X \subset \mathbb{A}^4$ be a smooth affine surface. Let $I \subset R = k[X_1,X_2,X_3,X_4]$ be its prime ideal. Then the following conditions are equivalent.*

1) $I/I^2 \approx \omega_I \oplus \omega_I^{-2}$.

2) $K_X^2 = 0$ *in* $A^2(X)$ ($K_X = c_1(\omega_I)$ *is the canonical divisor of* X).

3) $rK_X^2 = 0$ *for some* $r \neq 0$.

4) ω_I *is generated by two elements.*

5) I *is generated by three elements.*

If further any of these conditions is satisfied then V *is a set-theoretic complete intersection in* $\mathbb{A}^4$.

Proof : 2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5) is immediate from Corollary 2.4 and Theorem 1.10. Further, $I/I^2 \approx A \oplus \omega_I^{-1}$ by Lemma 1.3, 1). Hence 1) holds if and only if

$$c_2(\omega_I \oplus \omega_I^{-2}) = -2K_X^2 = 0 \Leftrightarrow K_X^2 = 0 .$$

The last assertion follows from Proposition 1.6.

Remark 2.6. a) When $\dim X = n \geq 3$, and $L \in \operatorname{Pic} X$, it has been proved that L is generated by n elements if and only if $c_1(L)^n = 0$. For $n = 3$ see [MKM] and for arbitrary n see [Mu5].

b) For further results about set–theoretic complete intersections see [Ly], [Bo] and [MK3].

For a smooth variety X, we write $c_i(X) = c_i(\Omega^1_X) \in A^i(X)$ and $c(X) = 1 + c_1(X) + c_2(X) + \ldots$, the total chern class of X. Following [F], let $s(X) = c(X)^{-1} = \Sigma_{p \geq 0}\, s_p(X)$, $s_p(X) \in A_p(X)$ be the total Segre class of Ω^1_X. If $X \hookrightarrow \mathbb{A}^n$ is a closed immersion with normal bundle N_X, then $s(X) = c(\check{N})_X$, where $\check{N}_X =$ dual of N_X.

Lemma 2.7. *Let* $X \xrightarrow{i} \mathbb{A}^n$ *be a smooth d-dimensional variety. Then* $s_{2d-n}(X) = 0$.

Proof : By the self–intersection formula (cf. [Fu ; Cor. 6.3]), $0 = (i^*i_*[X] = c_{n-d}(N_X))$. Hence $s_{2d-n}(X) = c_{n-d}(\check{N}_X) = 0$.

Lemma 2.8. *Let V be a smooth projective minimal surface. Suppose there exist integers r, s such that $rc_1(U)^2 + sc_2(U) = 0$ (in $A^2(U)$) for all affine open sets U of V. Then for any smooth affine surface X birationally equivalent to V, $rc_1(X)^2 + sc_2(X) = 0$.*

Proof: If V is ruled, then $A^2(X) = 0$ and there is nothing to prove. Otherwise, let $\tilde{V}$ be a smooth projective completion of X. Then $\tilde{V}$ dominates V birationally and therefore there exists $E \subset X$, $E = \cup_{i=1}^{\ell} E_i$, E_i rational curves such that the affine surface $U = X - E$ is an open set of V. Let $j : U \hookrightarrow X$ be the inclusion. Then we have the surjective ring homomorphism $j^* A(X) \twoheadrightarrow A(U)$. Now

$$j^*(rc_1(X)^2 + sc_2(X)) = rc_1(U)^2 + sc_2(U) = 0 .$$

Since $A_0(E) = 0$, we have $j^* : A^2(X) \xrightarrow{\sim} A^2(U)$. Hence $rc_1(X)^2 + sc_2(X) = 0$.

Theorem 2.9. *Let $X \subset \mathbb{A}^4_k$ $(k = \bar{k})$ be a smooth affine surface. Then X is a set-theoretic complete intersection in the following cases.*

1) *X is not birationally equivalent to a surface of general type.*
2) *X is not birationally equivalent to a projective surface in $\mathbb{P}^3$.*
3) *X is not birationally equivalent to a product of two curves.*

Proof: In view of Corollary 2.5, it suffices to check that $rc_1(X)^2 = 0$ for some $r > 0$. Let V be a smooth projective completion of X. If X is birationally equivalent to a ruled surface, then $A^2(X) = 0$, so $c_1(X)^2 = 0$ and we are done. So assume that V is not birationally equivalent to a ruled surface. First assume that V is a minimal surface. Then V is one of the following types:

a) $\kappa(V) = 0$, $12c_1(V) = 0$. Thus $12c_1(X)^2 = 0$ and $c_1(X)^2 = 0$ by Proposition 2.1.

b) $\kappa(V) = 1$; there exists r such that $r^2 c_1(V)^2 = 0$. Hence again $c_1(X)^2 = 0$, by Proposition 2.1.

According to our hypothesis, if V is i) of general type, then V is a smooth surface in $\mathbb{P}^3$ or degree ≥ 5 or ii) $V = C_1 \times C_2$ where the C_i are smooth non-rational curves.

In case i), let $r = \deg C$, $C = V - X$. Let $i : X \hookrightarrow \mathbb{P}^3 - C$ be the closed immersion. If h is the restriction of a hyperplane to $\mathbb{P}^3 - C$, then $rh^2 = 0$. Hence $ri^*(h)^2 = 0$. Since $c_1(X)$ is a multiple of $i^*(h)$, it follows by Proposition 2.1, that $c_1(X)^2 = 0$.

In case ii), let K_i, $i = 1,2$ be the pullback to $V = C_1 \times C_2$ of the canonical divisors on C_i. Then $\Omega^1_V = \mathcal{O}(K_1) \oplus \mathcal{O}(K_2)$. Then $K_i^2 = 0$, $i = 1,2$ and $c_2(V) = K_1K_2$, and $c_1(V) = (K_1+K_2)^2 = 2K_1K_2 = 2c_2(V)$.

Now let X be any smooth affine surface in $\mathbb{A}^4$ satisfying the hypothesis of the theorem. Then by Lemma 2.8 and the discussion above, either $c_1(X)^2 = 0$ or $c_1(X)^2 = 2c_2(X)$ (the latter holds when X is birational to product of two curves). But by Lemma 2.7, $s_0(X) = c_1(X)^2 - c_2(X) = 0$. Hence in any case $c_1(X)^2 = 0$ and the proof the theorem is complete.

Remark 2.10. If X is birational to product of two curves and is embedded in $\mathbb{A}^4$, then $2c_1(X) = c_1(X)^2 = c_2(X)$. Hence $c_1(X) = c_2(X) = 0$.

Remark 2.11. Mohan Kumer [MK3] has recently shown that if $X \subset \mathbb{A}^n$ $(n \geq 5)$ is a smooth affine surface birational to a product of curves, then X is a set–theoretic complete intersection.

§3. A criterion for vanishing of c_1^2.

In this section we give a partial converse to Corollary 2.5. We begin with the following well known lemma.

Lemma 3.1. *Let A be a noetherian ring and M, N finite A-modules. Let $x_1,\ldots,x_r$ be a N-regular sequence which annihilates M. Then* $\mathrm{Ext}^r_A(M,N) \approx \mathrm{Hom}_A(M,N/(x_1,\ldots,x_r)N)$.

Proof: We use induction on r, the case $r = 0$ being trivial. Assume $r > 0$ and put $\overline{N} = N/x_1N$. By induction hypothesis,

$$\mathrm{Ext}^{r-1}(M,\overline{N}) \approx \mathrm{Hom}_A(M,N/(x_1,\ldots,x_r)N).$$

The exact sequence

$$0 \longrightarrow N \xrightarrow{x_1} N \longrightarrow \overline{N} \longrightarrow 0$$

given

$$\operatorname{Ext}_A^{r-1}(M,N) \longrightarrow \operatorname{Ext}_A^{r-1}(M,\overline{N}) \longrightarrow \operatorname{Ext}_A^{r-1}(M,N) \xrightarrow{\;0\;} \operatorname{Ext}_A^{r}(M,N).$$

Since $x_1,\ldots,x_r$ is a regular N–sequence annihilating M, we have $\operatorname{Ext}_A^{r-1}(M,N) = 0$. Hence

$$\operatorname{Ext}_A^{r}(M,N) \approx \operatorname{Ext}_A^{r-1}(M,\overline{N}) \approx \operatorname{Hom}(M,N/(x_1,\ldots,x_r)N).$$

This completes the proof of Lemma 3.1.

Corollary 3.2. *Let A be a noetherian ring and $I \subset A$ a local complete intersection of height r. Let J be a complete intersection of height r contained in I. Then*

$$\omega_I \approx \operatorname{Ext}^r(A/I,A) \approx \operatorname{Hom}(A/I,A/J) = \frac{J:I}{J}.$$

Lemma 3.3. *Let P be a projective A-module of rank $r-1$ generated by r elements. Then* $\det P = \Lambda^{r-1}P$ *is generated by r elements.*

Proof : We have the surjection $A^r \twoheadrightarrow P$. This induces the surjection $A^r \approx \Lambda^{r-1}A^r \twoheadrightarrow \Lambda^{r-1}P$.

Lemma 3.4. (Swan) *Let P be a projective A-module of rank* 1. *Suppose P is generated by r elements. Then $\otimes^n P$ is generated by r elements for all n.*

Proof : We have a surjection $A^r \overset{\varphi}{\twoheadrightarrow} P$, so that $P \oplus \ker \varphi \simeq A^r$. Taking duals, we see that $P^* = P^{-1}$ is generated by r elements. Hence we may assume $n > 0$. Let $x_1,\ldots,x_r$ generated P. Set $\otimes^n x = \underbrace{x \otimes \ldots \otimes x}_{n \text{ times}}$. Then $\otimes^n x_1, \ldots \otimes^n x_r$ generate $\otimes^n P$: (check locally).

Theorem 3.5. *Let A be a noetherian ring and I a prime ideal which is a local complete intersection of height r. Let $J = (f_1,\ldots,f_r)$ be a complete intersection of height r with $\sqrt{J} = I$. Assume that for every maximal ideal $M \supset I$, the ideal J_M contains $r-1$ elements of a minimal set of generators of I/M* (i.e., $\dim_{A/M}\operatorname{Coker}(J \longrightarrow I/MI) \leq 1$). *Let $k(I)$ denote the quotient field of A/I and $n = \operatorname{length}_{k(I)}(\frac{A_I}{JA_I})$. Then*

1) $\omega_I = \operatorname{Ext}^r(A/I,A)$ *is divisible by $n-1$ in* $\operatorname{Pic} A/I$.

2) $\omega_I^{\otimes n}$ *is generated by r elements, where $n = \operatorname{length}_{k(I)}(\frac{A_I}{JA_I})$.*

Proof : By hypothesis for every maximal ideal $M \supset I$ *there existe* $g_1,\dots,g_r \in A_M$ such that $IA_M = (g_1,\dots,g_r)A_M$ with $g_1,\dots,g_{r-1} \in J$. So in $A_M/(g_1,\dots,g_{r-1})$, $\bar{I}_M = I_M/(g_1,\dots,g_{r-1})$ is a principal prime ideal generated by the non-zero divisor $\bar{g}_r$. Since $J_M/(g_1,\dots,g_{r-1})$ is $\bar{I}$-primary, $J_M = (g_1,\dots,g_{r-1},g_r^k)$. Further

$$k = \operatorname{length}_{k(I)} \frac{A_I}{(J_M)_I} = \operatorname{length}_{k(I)} \frac{A_I}{JA_I} = n\,.$$

Hence for every maximal ideal M, there exist $g_1,\dots,g_r \in A_M$ such that $I_M = (g_1,\dots,g_r)A_M$ and $J_M = (g_1,\dots,g_{r-1},g_r^n)$. So for $\ell < n$, $(I^\ell + J)_M$ is generated by $g_1,\dots,g_{r-1},g_r^\ell$. In particular, $I^\ell + J$ is a local complete intersection of height r. Now by Corollary 3.2, $\omega_I = \operatorname{Hom}(A/I, A/J) = (J{:}I)/J$. We claim that $J{:}I = I^{n-1}+J$. Since $\sqrt{I{:}J} = \sqrt{I^{n-1}+J} = I$, we have to check this locally at maximal ideals $M \supset I$. In A_M, the equality reduces to

$$(g_1,\dots,g_{r-1},g_r^n) : (g_1,\dots,g_r) = (g_1,\dots,g_{r-1},g_r^{n-1}).$$

This is obvious since $g_1,\dots,g_r$ is a regular A_M-sequence. Hence $\omega_I = I^{n-1}+J/J$. By the local description one also easily sees that I/I^2+J (in fact, all $I^k+J/I^{k+1}+J$, $1 \le k \le n-1$) are projective A/I-modules of rank 1. Set $L = I/I^2+J$. Then we have a natural surjection of

$$L^{\otimes n-1} \xrightarrow{\varphi} \frac{I^{n-1}+J}{J} = \omega_I\,.$$

φ is in fact an isomorphism, since L and ω_I are projective modules of rank 1. This establishes 1). We have the split exact sequence

$$0 \longrightarrow \frac{I^2+J}{I^2} \longrightarrow \frac{I}{I^2} \longrightarrow \frac{I}{I^2+J} = L \longrightarrow 0\,.$$

Now $\frac{I^2+J}{I^2}$ is a projective A/I-module of rank $r-1$ and is generated by r elements since J is generated by r elements. So by Lemma 3.3, $Q = \det(\frac{I^2+J}{I^2})$ is generated by r elements. Taking determinants, we see that $\Lambda^r I/I^2 = \omega_I^{-1} \approx L \otimes Q$. Hence $Q \approx L^{-1} \otimes \omega_I^{-1}$ and therefore

$$Q^{\otimes n-1} \approx (L^{\otimes n-1})^{-1} \otimes (\omega_I^{-1})^{\otimes n-1} = \omega_I^{-1} \otimes \omega_I^{1-n} = \omega_I^{\otimes -n}\,.$$

Since Q is generated by r elements, it follows by Lemma 3.4 that $\omega_I^{\otimes -n}$ and hence $\omega_I^{\otimes n}$ is generated by r elements.

Corollary 3.6. *Let $V \subset \mathbb{A}^n$ be a closed smooth variety of codimension r. Suppose V is a set-theoretic complete intersection of r hypersurfaces $H_i = (f_i = 0)$, $f_i \in k[X_1,\ldots,X_n]$, $1 \le i \le r$ with $(f_1,\ldots,f_r)$ containing $r-1$ minimal set of generators of $I(V)_P$ for all $P \in V$. ($I(V)$ = prime ideal of V in $k[X_1,\ldots,X_n]$). If $H_1 \ldots H_r = mV$ then $m^r c_1(V)^r = 0$ in $A^r(V)$.*

Proof: Let A be the coordinate ring of V and L be a projective A–module of rank 1. If L is generated by r elements, we have $L \oplus P \approx A^r$, for projective A–modules P of rank $r-1$. Now $c(P) = (1+c_1(L))^{-1}$. Since rank $P = r-1$, $c_r(P) = (-1)^r c_1(L)^r = 0$. By Theorem 3.5, $\omega_V^{\otimes m}$ is generated by r elements. Hence $(mc_1(V))^r = 0$ in $A^r(V)$.

Corollary 3.7. *Let $X \subset \mathbb{A}_k^{2r}$ be a smooth affine variety of dimension r satisfying the hypothesis of Corollary 3.6. Then $c_1(V)^r = 0$.*

Proof: Immediate from Corollary 3.6, Proposition 1.2 and Remark 2.2.

§4. Zero cycles on product of two curves.

If X is a smooth affine surface in $\mathbb{A}^4$ which is birationally equivalent to a product of curves, then we have seen that $c_1(X)^2 = c_2(X) = 0$. Here we prove a result about zero cycles on product of two curves which shows that there exists smooth affine curves C_i, $i = 1,2$ such that for $X = C_1 \times C_2$, $c_1(X)^2 \neq 0$ and $c_2(X) \neq 0$. This gives in particular an example of a surface not embeddable in $\mathbb{A}^4$.

Theorem 4.1. *Let $X = C_1 \times C_2$, where C_i are smooth projective curves. Let Δ be a zero cycle of positive degree on X. Suppose for all $(P_1,P_2) \in X$, there is a positive integer m (depending on (P_1,P_2) such that $m\Delta$ is rationally equivalent to a zero cycle supported on $P_1 \times C_2 \cup C_1 \times P_2$. Let $V = C_1' \times C_2'$, where $C_i' = C_i - \operatorname{Supp} p_{i*}(\Delta)$, and p_i is the projection of X onto C_i, $i = 1,2$. Then $A_0(V) = 0$.*

Proof : We write $\sim$ for rational equivalence $\text{Fix}(P_1,P_2) \in X$. Suppose $m\,\Delta \sim D$, with D supported on $P_1 \times C_2 \cup C_1 \times P_2$. Write $D = D_1 \times P_2 + P_1 \times D_2$, where the D_i are zero cycles on C_i, $i = 1,2$. We have

$$mp_{1*}(\Delta) \sim D_1 + (\deg D_2)P_1 \text{ and } mp_{2*}(\Delta) \sim D_2 + (\deg D_1)P_1 .$$

Reading these equivalences on $C_1 \times P_2$ and $P_1 \times C_2$ respectively, we get

$$mp_{1*}(\Delta) \times P_2) \sim D_1 \times P_2 + (\deg D_2).(P_1,P_2)$$

and

$$m(P_1 \times p_{2*}(\Delta)) \sim P_1 \times D_2 + (\deg D_1).(P_1,P_2).$$

Adding these two rational equivalences and restricting to V, we get

$$m \deg \Delta.j^*(P_1,P_2) = -j^*(D) = -j^*(m\,\Delta) = 0 ,$$

where $j : V \hookrightarrow X$ is the inclusion. Since this holds for all $(P_1,P_2) \in X$, we get that $A_0(V)$ is torsion. Hence $A_0(V) = 0$ by Proposition 2.1.

Corollary 4.2. *Let $X = C_1 \times C_2$, where C_i are smooth projective curves of positive genus over $\mathbb{C}$. Let Δ be a zero cycle of positive degree. Then there exists a $(P_1,P_2) \in X$ such that $mi^*(\Delta) \neq 0$ in $A_0(V)$, for any $m > 0$, where $V = C_1' \times C_2'$, $C_i' = C_i - \{P_i\}$, $i = 1,2$, and $i : V \hookrightarrow X$ is the inclusion.*

Proof : Since $p_g(X) > 0$, by [Mum], $A_0(V) \neq 0$ for any open set V of X. Now the corollary is immediate from Theorem 4.1.

Lemma 4.3. *Let $X \subset \mathbb{A}^n$ be a smooth affine variety of dimension d. Let $I \subset k[X_1,\ldots,X_n]$ be the prime ideal of X and A its coordinate ring.*

1) *If I is generated by r elements, then $\Omega^1_{A/k}$ has a free direct summand of rank $n-r$. Consequently, $c_i(X) = 0$ for $i > d+r-n$.*

2) *If* $\Omega_{A/k}$ *is generated by* s *elements, then* $s_i(X) = 0$ *for* $i < 2d-s$, (s_i = ith Segre class).

Proof : 1) If I is generated by r elements, then $I/I^2 \oplus L \approx A^r$ for some L. Hence $I/I^2 \oplus L \approx A^{n-r} \approx A^n \approx I/I^2 \oplus \Omega_A$. By [Su], $\Omega_A \approx L \oplus A^{n-r}$.

2) If Ω_A is generated by s elements, then $\Omega_A \oplus L \approx A^s$, for some L. Then I/I^2 and L are stably isomorphic.

Since rank $L = s-d$, we have $s_{d-i}(X) = c_i(I/I^2) = 0$, for $i > s-d$, i.e. $s_i(X) = 0$ for $i < 2d-s$.

Corollary 4.4. *Let* $X = C_1 \times C_2$, *where* C_i *are smooth projective curves of genus* $g_i \geq 2$, $i = 1,2$ *over* $k = \mathbb{C}$. *There exists a* $(P_1,P_2) \in X$ *such that the affine surface* $V = C_1' \times C_2'$, $C_i' = C_i - \{P_i\}$, $i = 1,2$ *has the following properties.*

1) $c_1(V)^2 \neq 0$, $c_2(V) \neq 0$, $c_1(V)^2 \neq c_2(V)$.

2) *For any closed immersion* $V \hookrightarrow \mathbb{A}^n$ *the prime ideal* $I(V) \subset \mathbb{C}[X_1,\dots,X_n]$ *of* V *is not generated by* $n-1$ *elements.*

3) Ω^1_V *is not generated by three elements. In particular, there does not exist any un ramified morphism* $V \longrightarrow \mathbb{A}^3$.

4) V *does not admit a closed immersion in* $\mathbb{A}^4$.

5) $\Lambda^2\Omega^1_V$ *is not genered by two elements.*

Proof : Fix canonical divisors K_i on C_i. Let $\overline{K}_i = p_i^*(K_i)$, $p_i : X \longrightarrow C_i$, being the projection. Then $\overline{K}_1+\overline{K}_2$ is the canonical divisor and $\Omega^1_X = \mathcal{O}(\overline{K}_1) \oplus \mathcal{O}(\overline{K}_2)$. Hence

$$c_1(X)^2 = 2\overline{K}_1.\overline{K}_2 \ , \ c_2(X) = \overline{K}_1.\overline{K}_2 = c_1(X)^2 - c_2(X) \ .$$

Further, $\deg(\overline{K}_1.\overline{K}_2 = 4(g_1-1)(g_2-1) > 0$. Now 1) follows from Corollary 4.2, with $\Delta = \overline{K}_1.\overline{K}_2$. The assertions 2) and 3) are immediate from 1) and Lemma 4.3 since, $c_2(V) \neq 0$, and $s_0(V) = c_1(V)^2 - c_2(V) \neq 0$. Again 4) is immediate from Lemma 2.7, since $s_0(V) \neq 0$. 5) follows from Corollary 2.4, since $c_1(V)^2 \neq 0$.

§5. Examples of surfaces in $\mathbb{A}^4$ with $c_1^2 \neq 0$.

Let X be a smooth affine variety of dimension d. It is well known that X can be embedded in $\mathbb{A}^{2d+1}$. In this section in the range, $d+1 \leq n \leq 2d+1$, we give examples of affine varieties X admitting a closed immersion in $\mathbb{A}^n$ but not admitting a closed immersion in $\mathbb{A}^{n-1}$. These examples will also have $c_d(X) \neq 0$ so that its ideals $I(X)$ aree not generated by $n-1$ elements. When $d=2$, $n \geq 4$, this also provides an example of a smooth surface in $\mathbb{A}^4$ with $c_1^2 \neq 0$ (cf. Corollary 3.7).

We first collect some facts which follow easily from Roitman's methods [Ro 1], [Ro 2]. As before, for a variety X, $A_0(X)$ = group of zero cycles modulo rational equivalence.

LEMMA 5.1. [B1] *Let X be a smooth projective variety of dimension d over $k = \mathbb{C}$. Let $N > 0$ be an integer, and let $\gamma : X^N \times X^N \longrightarrow A_0(X)$ denote the map $\gamma(x_1,\ldots,x_N;y_1,\ldots,y_N) = \Sigma\, x_i - \Sigma\, y_i$. Let Z be a non-singular variety and suppose given a morphism $f = (f_1,f_2) : Z \longrightarrow X^N \times X^N$ such that the composition $\gamma \circ f : Z \longrightarrow A_0(X)$ is the zero map. Let $\omega \in \Gamma(X,\Omega_X^q)$ be a q-form on X for some $q \geq 1$. Define a differential $\tilde{\omega} \in \Gamma(X^N,\Omega^q)$ by $\tilde{\omega} = \Sigma_{i=1}^N p_i^*(\omega)$, where $p_i : X^N \longrightarrow X$ is the projection on the ith factor. Then $f_1^*(\tilde{\omega}) = f_1^*(\tilde{\omega})$ on Z.*

LEMMA 5.2. *Let X be a smooth projective variety over $\mathbb{C}$. Let Y be any complete variety (possibly reducible) of dimension q. Let $\varphi : Y \longrightarrow X$ be a morphism such that the induced map $\varphi_* : A_0(Y) \longrightarrow A_0(X)$ is surjective. Then $H^0(X,\Omega_X^\ell) = 0$ for $\ell > q$.*

PROOF : By using Chow lemma first and then resolution of singularities, we may assume Y is smooth and projective. Let $Y_1,\ldots,Y_r$ be irreducible components of Y. For r–tuple of non–negative integers $(\alpha_1,\ldots,\alpha_r)$ we put $|\alpha| = \Sigma\, \alpha_i$ and $Y_\alpha = \Pi_{i=1}^r Y_i^{\alpha_i}$. (Here $\alpha_i = 0$ means that Y_i is omitted). For $|\alpha| = n$, the restriction to Y_α of $\varphi_n : Y^n \longrightarrow A_0(X)$, given by $\varphi_n(y_1,\ldots,y_n) = \varphi(y_1) + \ldots + \varphi(y_n)$, induces a morphism of $\varphi_\alpha : Y_\alpha \longrightarrow A(X)$ in the sense of [Ro2]. Similarly $Y_\alpha \times X \longrightarrow A(X)$ induced by $Y^n \times X \longrightarrow X$, $((y_1,\ldots,y_n,x) \longmapsto \varphi(y_1) + \ldots + \varphi(y_n) + x)$ is a morphism. Hence

$$Z_{\alpha,\beta,n} \subset Y_\alpha \times Y_\beta \times X,\ |\alpha| = n,\ |\beta| = n-1$$
$$Z_{\alpha,\beta,n} = \{(z_\alpha,z_\beta,x) \mid \varphi_\alpha(z_\alpha) \underset{\text{rat}}{\sim} \varphi_\beta(z_\beta)+x\}$$

is c–closed (i.e., countable union of irreducible closed sets) [Ro2 ; Lemma 3]. Let $p_{\alpha,\beta,n}$ denote the projection of $Z_{\alpha,\beta,n}$ on X. Since every $x \in X$ is in $\operatorname{Im} \varphi_*$

$$X = \bigcup_{\substack{\alpha,\beta,n \\ |\alpha|=n, |\beta|=n-1 \\ n\geq 1}} \operatorname{Im} p_{\alpha,\beta,n} .$$

Hence there exists an irreducible variety $Z \subset Y_\alpha \times Y_\beta \times X$ for some α,β,n, such that Z dominates X (under projection). Let $f : \overline{Z} \to Z$ be a desiingularization. Let p_α, q_β denote projection of Z onto Y_α and $Y_\beta \times X$ respectively. $p_\alpha \circ f$ and $q_\beta \circ f$ composed with natural product maps $Y_\alpha \to X^n$ and $Y_\beta \times X \to X^n$ give the morphisms $f_i : \overline{Z} \to X^n$, $i = 1,2$, such that the composite

$$\overline{Z} \xrightarrow{(f_1,f_2)} X^n \times X^n \xrightarrow{\gamma} A_0(X)$$

is zero, wheere γ as in Lemma 5.1 is the natural difference map. Let $\omega \in H^0(X,\Omega^\ell_X)$ with $\ell > q$ and $\tilde{\omega} = \Sigma\, p_i^*(\omega)$, $p_i : X^n \times X$ is the ith projection. Since $\dim Y_i < \ell$, $(\varphi| Y_i)^*(\omega) = 0$. Hence $f_1^*(\tilde{\omega}) = 0$. On the other hand $f_2^*(\tilde{\omega}) = g^*(\omega)$, where g is the composite map $\overline{Z} \xrightarrow{f} Z \xrightarrow{\text{proj.}} X$. Hence by Lemma 5.1, $f_2^*(\tilde{\omega}) = g^*(\omega) = 0$. Since we are in characteristic zero and $\overline{Z}$ dominates X, we have $\omega = 0$.

Corollary 5.3. *Let X be a smooth affine variety of dimension d over $\mathbb{C}$. Let $\tilde{X}$ be a smooth projective completion. Suppose $H^0(\tilde{X},\Omega^\ell_{\tilde{X}}) \neq 0$. Then $A_0(X)$ is a non-zero torsion-free divisible group.*

Proof : By Lemma 5.2, $A_0(\tilde{X}-X) \to A_0(\tilde{X})$ is not surjective. Hence $A(X) \neq 0$. The rest follows from Proposition 2.1.

Remark 5.4. As in [MS], using Roitman's methods one can show that there is a homomorphism of an abelian variety $J \to A(X)$ with countable kernel so that $\operatorname{rank}_{\mathbb{Q}} A(X) = \operatorname{card} \mathbb{C}$. But we do not need this here.

Lemma 5.5. *Let K be a field of characteristic zero. Let H_r denote the vector space of homogeneous polynomials of degree r in $X_1,\dots,X_n$. Then H_r is spanned over K by $\{L^r \mid L$ linear polynomials in $X_1,\dots,X_n\}$.*

Proof : Exercise.

Corollary 5.6. *Let X be a smooth affine variety of dimension d over $\mathbb{C}$. Suppose $A(X)$ is generated by the intersection products $c_1(L_1)\dots c_1(L_d)$, with $L_i \in \operatorname{Pic} X = A^1(X)$. If $A_0(X) \neq 0$, then there exists an $L \in \operatorname{Pic} X$ such that $c_1(L_1)^d \neq 0$ in $A_0(X)$.*

Proof : By Lemma 5.5 (with $K = \mathbb{Q}$), some integral multiple of $c_1(L_1)\dots c_1(L_d)$ is an integral linea combination of $(c_1(L_1)^d \mid L \in \operatorname{Pic}(X))$. Since $A_0(X)$ is torsion–free, the corollary is immediate.

Next, we need the following result about embeddings of affine varieties. The proof we have given here is due to M.V. Nori. This proof replaces our lengthy proof.

Theorem 5.7. *Let X be an integral variety of dimension d over an algebraically closed field. There exists a smooth affine open set V of such that V admits a closed immersion in $\mathbb{A}^{d+1}$*

Proof : (M.V. Nori). By taking a generic projection to $\mathbb{A}^{d+1}$, we get a finite birational map $\pi : X \to X'$, such that X' is a hypersurface in $\mathbb{A}^{d+1}$ and π induces an isomorphism $\pi^{-1}(X'_{\text{reg}}) \xrightarrow{\sim} X'_{\text{reg}}$ on regular points. Hence we may assume that X is an integral hypersurface (possibly singular) in $\mathbb{A}^{d+1}$. Let $A = k[x_1,\dots,x_d,x_{d+1}]$ be the coordinate ring of X. Let $F = \sum_{i=0}^m f_i x_{d+1}^i = 0$ be the equation of X, with $f_i \in k[x_1,\dots,x_d]$ and $f_0 \neq 0$. For any $h \in k[x_1,\dots,x_d]$, put $x'_{d+1} = x_{d+1}/(hf_0^2)$. Then we have $\sum_{i=0}^m f_i (hf_0^2)^i x'^{\,i}_{d+1} = 0$. Dividing this equation by hf_0^2, we see that $\frac{1}{hf_0} \in k[x_1,\dots,x_d,x'_{d+1}]$. It is easily seen that $A_{hf_0} = k[x_1,\dots,x_d,x'_{d+1}]$. Hence for any $h \in k[x_1,\dots,x_d]$, $h \neq 0$, Spec A_{hf_0} admits a closed immersion $\mathbb{A}^{d+1}$. Let h be any nonzero element in $J \cap k[x_1,\dots,x_d]$, when $J \subset A$ is the ideal defining the singular locus. Then Spec A_{hf_0} is a smooth affine hypersurface in $\mathbb{A}^{d+1}$.

THEOREM 5.8. *Let d,n be positive integers such that $d+1 \leq n \leq 2d$. Then there exists a smooth affine variety X of dimension D over $\mathbb{C}$ such that*

1) *X admits a closed immersion in $\mathbb{A}^{n+1}$, but X does not admit a closed immersion in $\mathbb{A}^n$.*

2) *$c_d(\Omega^1_X) \neq 0$ and the prime ideal $I(X)$ of X in $\mathbb{C}[X_1,\dots,X_m]$ for any closed immersion $X \hookrightarrow \mathbb{A}^m$ is not generated by $m-1$ elements.*

3) *Ω^1_X is not generated by $n-1$ elements.*

4) *$c_1(X)^d \neq 0$ and $\Lambda^d\Omega^1_X$ is not generated by d elements.*

PROOF : Let Y be a product of n elliptic curves. Clearly for any open set V of Y, $A_0(V)$ is generated by the products $c_1(L_1)\dots c_1(L_n)$, with $L_i \in \operatorname{Pic} V$. Further, by Lemma 5.2, since $H^0(Y,\Lambda^n\Omega_Y) \neq 0$, we get that $A_0(V) \neq 0$, for any open set V of Y; By Theorem 5.7, choose an affine open set V of Y such that V admits a closed immersion in $\mathbb{A}^{n+1}$. In view of Corollary 5.6, there exists an $L \in \operatorname{Pic} V$ such that $c_1(L)^n \neq 0$. Since V is affine, by Bertini's theorem, we can choose "generic" D_i, $1 \leq i \leq n-d$ such that the D_i are smooth integral divisors with $\mathcal{O}_V(D_i) \approx L$ and $X = \cap_{i=1}^{n-d} D_i$ is a smooth integral variety of dimension d. We claim that X has all the properties listed in the theorem. Let $I = I_X \subset \mathcal{O}_V$ be the defining ideal of X. Then $I = \mathcal{O}(-D_1) + \dots + \mathcal{O}(-D_{n-d})$. Hence I/I^2 is a direct sum of $n-d$ line bundles each isomorphic to $i^*(L)$, where $i : X \hookrightarrow V$ is the inclusion. Hence the total chern class $c(I/I^2) = (1-i^*c_1(L))^{n-d}$. Since Ω_Y and hence Ω_V is trivial, we have $c(\Omega_X) = (1-i^*c_1(L))^{-n-d}$. Hence

$$c_{n-d}(I/I^2) = (-1)^{n-d} i^*c_1(L)^{n-d}$$
$$c_d(X) = c_d(\Omega^1_X) = (-1)^d \binom{d-n}{d}.i^*c_1(L)^d = \binom{n-1}{d}.i^*c_1(L)^d$$

and

$$c_1(X) = c_1(\Omega^1_X) = (n-d)i^*c_1(L).$$

Since

$$i_*i^*c_1(L)^d = c_1(L)^d i_*[X] = c_1(L)^d c_1(L)^{n-d} = c_1(L)^n \neq 0\ ,$$

it follows that $i^*c_1(L)^d \neq 0$. Since $n-d \leq d$ and $A_0(X)$ is torsion–free, it follows that $c_d(\Omega^1_X)$, $s_{2d-n}(X) = c_{n-d}(I/I^2)$ and $c_1(X)^d$ are all non–zero in the Chow ring of X . Now 1) follows from Lemma 2.7 and 2) and 3) are immediate from Lemma 4.2.

Since $c_1(X)^d \neq 0$ and for any $H \in \operatorname{Pic} X$, H is generated by d elements implies $c_1(H)^d = 0$, it follows that $\Lambda^d \Omega_X$ is not generated by d elements.

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S. BLOCH
Chicago University
Dept. of Mathematics
5734 University Avenue
CHICAGO, Ill. 60637 (U.S.A.)

M.P. MURTHY
Chicago University
Dept. of Mathematics
5734 University Avenue
CHICAGO, Ill. 60637 (U.S.A.)

L. SZPIRO
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 PARIS CEDEX 05

MÉMOIRES DE LA S. M. F.

ALBERTO COLLINO
WILLIAM FULTON

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INTERSECTION RINGS OF SPACES OF TRIANGLES

Alberto COLLINO and William FULTON

In 1880 Schubert [12] described a space which compactifies the set of (ordered) plane triangles, and described its intersection ring – giving a basis for the cycles in each dimension, and giving algorithms for computing products. In 1954 Semple [13] gave a modern construction of this space, which we denote X, as an algebraic submanifold of a product of projective and Grassmann manifolds. Tyrrell [15] verified Schubert's prescription of the cycles and their relations in codimension one, and calculated a few other intersection products. The aim of this note is to complete this analysis. We give a formula for the Chow ring (or cohomology ring) of this space : it is generated by seven classes in codimension one, with an ideal of relations generated by twelve classes. In particular we verify that Schubert's basis is correct in all dimensions, and the intersections are as he specified. It is interesting, however, that one of the defining relations for the intersection ring is independent of those given by Schubert before he lists the basis.

The proof is remarkably easy. Since the torus of diagonal matrices in $SL(3)$ acts on X with finitely many (72) fixed points, it follows from the work of Bialynicki–Birula [1], [2] that the total Chow group $A^{\cdot}(X)$ of X is free on 72 generators. We define, purely algebraically, a graded ring $A^{\cdot}$ with seven generators and certain relations, and verify that $A^{\cdot}$ has 72 generators – the same basis as given by Schubert. It is easy to verify that there is a homomorphism from the ring $A^{\cdot}$ to the Chow ring $A^{\cdot}(X)$. Since the generator of A^6 maps to the generator of $A^6(X)$, Poincaré duality implies that this homomorphism is an isomorphism.

Because the algorithms for writing any classes in terms of the basic classes are given explicitly, it becomes a simple algebraic exercise to compute any intersection products, and in particular any enumerative formula, involving the basic 72 generators.

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Although modern machinery has often been used to give rigorous proofs of classical formulas in enumerative geometry, this appears to us to be one of the rare instances where a modern framework actually simplifies the classical calculations. Only part of the first few pages of Schubert's calculations appear in this approach. Perhaps the most obscure part of Schubert's paper (pp. 167–181), which may be regarded as a calculation of the Kunneth components of the class of the diagonal on $X \times X$, can be dispensed with, since this is equivalent to knowing the intersection products of all pairs of generators in complementary dimensions.

In this paper we also compute the Chow ring of the space of triangles in a projective bundle over a given variety. This includes the space of triangles in $\mathbb{P}^n$; for $n=3$ a few equations were included at the end of Schubert's paper [Sch]. As he implies, there are few new ideas needed for this generalization ; the present framework makes it quite automatic.

Another approach to the computation of intersections on the space X of plane triangles has been developed by Roberts and Speiser [9], [10]. They show how X can be constructed by starting with $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, and forming two blowups, followed by one blowdown. This allows one to work out, although with some difficulty, any intersection products one may wish. That approach requires delving considerably deeper into the geometry of the space X, which is of independent interest. Our approach, on the other hand, gives the whole intersection theory on X all at once, with minimal knowledge needed about its geometry, and no need to verify intersection multiplicities of any but the simplest intersection products.

We were led to this idea by reading the preprint of Ellingsrud and Strømme [5], who used the Bialynicki–Birula theorem to compute the Chow groups of the Hilbert schemes of points in the plane. The simple observation of the present note is that the same theorem will yield the Chow ring of a variety, provided one can guess (say with the help of Schubert !) what the ring should be, and one can produce a suitable homomorphism from this abstract ring to the actual ring.

Le Barz [8] has used Hilbert scheme methods to construct a space of triangles in any non–singular variety. We comment on this in §5.

Schubert gives many applications, of which we discuss only one : to calculate the number of triangles which are simultaneously inscribed in a given plane curve C, and circumscribed about a given plane curve D, assuming C and D are suitably general. Here Schubert makes an error and gives an incorrect formula. This is remarkable not only because of the rarity of any errors in Schubert's formulas, but also because the correct formula had been given a decade earlier by

Caylay [3] ! Schubert's error was not in his discussion of the intersection theory of the space of triangles. Rather, he ignored the fact that the dual of a smooth curve of degree greater than two has singularities. When this is taken into account, the correct formula comes out.

The first section discusses the space X of complete triangles, reviewing that part of the work of Schubert and Semple that we need. The second section is pure algebra, describing the ring $A^{\cdot}$ and giving algorithms for writing any element of $A^{\cdot}$ as a linear combination of 72 basic classes. The proof that $A^{\cdot}$ is the intersection ring of X is given in §3, and the application to inscribed and circumscribed triangles in §4. The extension to higher dimensions, with a few complementary remarks occupies §5. Appendix A contains some algebraic manipulations needed for §2 (and for [12], but Schubert assumed the reader could supply them). Appendix B contains the tables of intersection products of classes of complementary dimensions. In Appendix C we prove a simple "Leray Hirsh" theorem for Chow groups of fibre bundles whose fibre is a variety such as the variety of plane triangles, or any smooth projective variety with $\mathbb{C}^*$ action with finitely many fixed points.

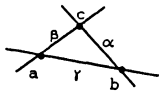
We thank Joe Harris for useful advice about the influence of plane curve singularities on enumerative formulas, and Steven Kleiman for pointing us to Cayley's paper.

Section 1. The compactified space of triangles.

We follow Schubert's notation for ordered triangles in the plane. We sketch a typical member of each type, according to dimension of the loci of such triangles.

A general triangle has vertices a, b, c, with the opposite sides being lines α, β, γ :

Dimension 6



Five–dimensional families :

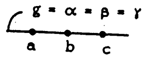
ϵ : the three lines coincide in one line denoted g , on which there are three vertices a, b, c .

τ : dually, the three vertices coincide in a point s , through which pass three lines α, β, γ .

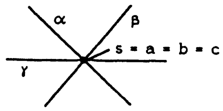
θ_a : the two lines β and γ coincide in a line g , the two points b and c coincide in a point s on g ; a is another point on g , while α is another line through s .

θ_b and θ_c are defined similarly, by permuting the vertices and edges.

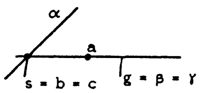
Dimension 5



Type ϵ



Type τ



Type θ_a

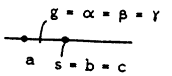
Four–dimensional families :

ω_a : one line g, with the two vertices b and c coinciding in one point s on g, with a another point on g. Similarly for ω_b and ω_c.

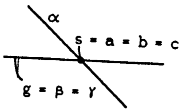
ω_α : the dual specialization of type τ, with $\beta = \gamma = g$; similarly for ω_β and ω_γ.

ψ : the three sides coincide in one line g, and the three vertices coincide in one point s on g. In addition, a net of conics is specified, which contains the pencil of conics consisting of g and an arbitrary line through s, and is contained in the web of conics consisting of all conics which are tangent to g at s. (This net is therefore a plane in the $\mathbb{P}^5$ of conics, containing a certain $\mathbb{P}^1$ and contained in a certain $\mathbb{P}^3$).

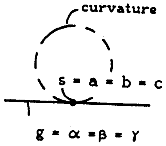
Dimension 4



Type ω_a



Type ω_α



Type ψ

Three–dimensional families :

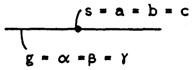
The two special nets described in the following types η and ζ should be regarded as exceptions from type ψ.

η : one line g, one point s on g; the net of conics consists of those conics which contain g as a component, i.e. consist of g and an arbitrary line.

ζ : one line g, one point s, and the net of conics consists of those which are singular at s, i.e. consist of two lines through s.

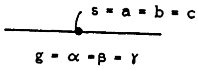
Dimension 3

curvature = 0



Type η

curvature $= \infty$



Type ζ

The set X of complete triangles is the union of the set of general triangles and the special triangles described in the above list. Schubert also described the topology of X, in the sense that he specified which triangles are to be regarded as specializations of which other types :

ω_a is a specialization of ϵ and θ_a ;
ω_α is a specialization of τ and θ_a ;
ψ is a specialization of θ_a , θ_b and θ_c , but not a specialization of ϵ or τ ;
η is a specialization of type ω_a , ω_b and ω_c and ψ, but not of ω_α , ω_β or ω_γ ;
ζ is a specialization of type ω_α , ω_β and ω_γ and ψ, but not of ω_a , ω_b or ω_c .

Each complete triangle has an associated net of conics ; except for types ψ , η and ζ it is determined by the vertices and edges.

For a general triangle, the net is the net of conics passing through the three vertices a, b, c of the triangle.

For a triangle of type ϵ or ω_a , the net consists of all conics which contain the triple line g.

For type τ or ω_a , the net consists of conics which are singular at the point s.

For a triangle of type θ_a , the net consists of all conics which are tangent to α at the point s , and which pass through the point a.

Over the real numbers a net of conics contains a unique circle – the conic which passes through the two circular points $(1: \pm i: 0)$ at infinity. With this interpretation the net of conics corresponds to a radius of curvature ; if the three vertices of a triangle lie on a curve, and approach a non–singular point of the curve, the limiting circle will be the osculating circle to the curve.

Semple [13] defined the space X to be the closure in the space

$$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2 \times G,$$

where G is the Grassmannian of planes in the $\mathbb{P}^5$ of conics, of the locus consisting of all $(a, b, c, \alpha, \beta, \gamma, \Lambda)$ for which a, b, c are the vertices, and α, β, γ the sides of a general triangle as above, and Λ is the net of conics passing through the three vertices a, b, c. He showed, by straight–forward calculations in local coordinates, that X is a non–singular six–dimensional subvariety of this product variety, and that the points of X are precisely of the types described above, with the prescribed nets of conics and specialization relations. Each of the types makes up a locally closed algebraic subvariety of X, of the dimension specified with its description.

The main goal of this note is to describe the intersection ring $A^{\cdot}(X)$ of X. Following Schubert and Semple, we use the notations ϵ, τ, θ_b, θ_c, ψ, etc. to denote the classes in $A^{\cdot}(X)$ determined by the closures in X of the corresponding loci of special triangles.

There are also classes in $A^1(X)$ determined by subvarieties of X consisting of triangles in special positions :

a : the vertex "a" is required to lie on a given line. This condition defines a hypersurface in X, whose class is independent of the choice of line. In fact, a is the pull–back of the generator of $A^1(\mathbb{P}^2)$ via the projection to the first factor in the above product.

b and c are defined similarly, and are pull–backs from the second and third factors.

α : the side "α" is required to pass through a given point. This is the pull–back of the generator of $A^1(\check{\mathbb{P}}^2)$ via the projection to the fourth factor in the product. Similarly for β and γ.

d : the net of conics is required to meet a given net of conics. This is the pull–back of the standard generator of $A^1(G)$ via the projection to the last factor.

Tyrrell [15] proved that the relations among these divisor classes were as stated by Schubert. To do this he has to compute some intersection products ; we give direct proofs of these relations in §5. Among these relations are :

(1)
$$b+c-\alpha = \tau+\theta_a$$
$$c+a-\beta = \tau+\theta_b$$
$$a+b-\gamma = \tau+\theta_c$$

(2)
$$\beta+\gamma-a = \epsilon+\theta_a$$
$$\gamma+\alpha-b = \epsilon+\theta_b$$
$$\alpha+\beta-c = \epsilon+\theta_c$$

(3)
$$d = \alpha+\beta+\gamma+\tau = a+b+c+\epsilon .$$

Some other relations are obvious from the definition, or the fact that the classes are pull–backs from divisors on surfaces :

(4)
$$a^3 = b^3 = c^3 = \alpha^3 = \beta^3 = \gamma^3 = 0 .$$

From the fact that the vertex a is always contained in the side β, i.e. that the projection of X to the product of the first and fourth factors $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ lies in the incidence variety gives the first of the following relations :

(5)
$$a\beta = a^2+\beta^2 , \ a\gamma = a^2+\gamma^2 , \ b\gamma = b^2+\gamma^2 ,$$
$$b\alpha = b^2+\alpha^2 , \ c\alpha = c^2+\alpha^2 , \ c\gamma = c^2+\gamma^2 .$$

Since the points b and c (and sides β and γ) coincide on a triangle of type θ_a, we have equations

(6)
$$\theta_a b = \theta_a c , \ \theta_b c = \theta_b a , \ \theta_c a = \theta_c b ,$$
$$\theta_a \beta = \theta_a \gamma , \ \theta_b \gamma = \theta_b \alpha , \ \theta_c \alpha = \theta_c \beta .$$

All the above, with the exception of the trivial equations (4), are among those given by Schubert. A final equation which we shall need, however,

(7)
$$\epsilon\tau = 0$$

is not among those in Schubert*. It follows immediately from the definitions that the geometric loci describing the types ϵ and τ have disjoint closures, because the nets of conics can never coincide : those of type ϵ have curvature 0, while those of type τ have curvature ∞ .

* We cannot help commenting on the fact that Schubert omits such useful equations. It is now universally agreed that what Schubert was doing is exactly equivalent to the modern calculation of intersection products of cycles on manifolds, and we do not pretend to deny this. But to anyone now calculating intersection products, the first relations written down would be that products of classes determined by disjoint subvarieties are zero. In fact, Schubert only explicitly writes down products of classes where at most **one** of the factors describes figures of a special type ; all the other factors describe figures in special position with regard to given but variable objects. Of course several classes involving special type are more likely to meet improperly, and perhaps, in the absence of foundations, he wanted to avoid such dangers.

It should also be pointed out that Schubert's equations given in the beginning of his paper do **not** generate all equations in codimension ≥ 2 ; the equation $\epsilon\tau = 0$ is independent of the equations he lists.

Section 2. The algebraic intersection ring.

The ring $A^{\cdot}$ is defined to be the polynomial ring in seven variables, subject to certain relations. That is

$$A^{\cdot} = \mathbb{Z}[a,b,c,\alpha,\beta,\gamma,d)/I$$

where I is the ideal generated by the polynomials listed in (1) – (4) :

(1) a^3 , b^3 , c^3 , α^3 , β^3 , γ^3 ;

(2) $a\beta-a^2-\beta^2$, $a\gamma-a^2-\gamma^2$, $b\gamma-b^2-\gamma^2$,
$b\alpha-b^2-\alpha^2$, $c\alpha-c^2-\alpha^2$, $c\beta-c^2-\beta^2$;

(3) $(b+c+\beta+\gamma-d)(b-c)$, $(a+c+\alpha+\gamma-d)(c-a)$, $(a+b+\alpha+\beta-d)(a-b)$;
$(b+c+\beta+\gamma-d)(\beta-\gamma)$, $(a+c+\alpha+\gamma-d)(\gamma-\alpha)$, $(a+b+\alpha+\beta-d)(\alpha-\beta)$;

(4) $(d-a-b-c)(d-\alpha-\beta-\gamma)$.

Remark. This list of generators for I is not minimal. In fact, modulo relations (2), the six equations in (3) are equivalent (see equation (A.8) of the appendix) to the four equations

$$2a^2+a\alpha-\alpha^2-ad = 2b^2+b\beta-\beta^2-bd = 2c^2+c\gamma-\gamma^2-cd\ ,$$
$$2\alpha^2+a\alpha-a^2-\alpha d = 2\beta^2+b\beta-b^2-\beta d = 2\gamma^2+c\gamma-\gamma^2-\gamma d\ ;$$

so two generators, say the first and fourth of (3), could be omitted. In addition, the six generators in (1) can be replaced by any one of them (e.g., to see $a^3 \equiv \beta^3$, combine $a^2\beta \equiv a^3+a\beta^2$ with $a\beta^2 \equiv \beta^3+a^2\beta$). When this is done one has 12 generators for I which are a minimal set of generators. In fact, relations (2), (3), and (4) put 11 relations on the 28 monomials of degree 2, so all of these equations are needed to get $\dim A^2 = 17$. One may check that a relation (1) must be added to cut the dimension of A^3 from 23 to 22, or even to cut $A^{\cdot}$ down to a 0–dimensional ring.

Note that the group $G = \mathfrak{S}_3 \times \mathfrak{S}_2$ acts on $A^{\cdot}$; the symmetric group $\mathfrak{S}_3$ on three letters permutes a, b and c and simultaneously α , β and γ ; the group $\mathfrak{S}_2$ acts via the "duality" operation which interchanges a and α , b and β , c and γ ; all elements of G fix d . The generators of I are chosen so that I is clearly taken into itself by thic action of G .

For simplicity as well as to clarify the relations with geometry we define polynomials ϵ , τ , θ_a , θ_b and θ_c and express some of the generators of I in terms of them ; we set

(i)
$$\begin{aligned}
\epsilon &= d-a-b-c\\
\tau &= d-\alpha-\beta-\gamma\\
\theta_a &= b+c+\beta+\gamma-d\\
\theta_b &= a+c+\alpha+\gamma-d\\
\theta_c &= a+b+\alpha+\beta-d\,.
\end{aligned}$$

The generators (3) and (4) for I can be written :

(3′)
$$\begin{aligned}
&\theta_a(b-c), \quad \theta_b(c-a), \quad \theta_c(a-b) \quad ;\\
&\theta_a(\beta-\gamma), \quad \theta_b(\gamma-\alpha), \quad \theta_c(\alpha-\beta) \quad ;
\end{aligned}$$

(4′)
$$\epsilon\tau\,.$$

The same notation will be used for the corresponding elements a, b, c, α, β, γ, d, ϵ, τ, θ_a, θ_b, θ_c, in $A^\cdot$. It follows immediately from the definitions that any of the elements ϵ, τ, θ_a, θ_b or θ_c could have been used instead of d as the seventh generator of $A^\cdot$, and that we have the equations :

(ii)
$$\begin{aligned}
\theta_a &= b+c-\alpha-\tau = \beta+\gamma-a-\epsilon\\
\theta_b &= c+a-\beta-\tau = \gamma+\alpha-b-\epsilon\\
\theta_c &= a+b-\gamma-\tau = \alpha+\beta-c-\epsilon\,.
\end{aligned}$$

In addition, since $\theta_a b = \theta_a c$ in $A^\cdot$ by (3′), we denote this common element of $A^\cdot$ by $\theta_a s$; similarly $\theta_a g$ denotes $\theta_a\beta = \theta_a\gamma$, and the same is done for θ_b and θ_c. That is, we define :

(iii)
$$\begin{aligned}
\theta_a s &= \theta_a b = \theta_a c \;,\quad \theta_a g = \theta_a\beta = \theta_a\gamma\\
\theta_b s &= \theta_b c = \theta_b a \;,\quad \theta_b g = \theta_b\gamma = \theta_b\alpha\\
\theta_c s &= \theta_c a = \theta_c b \;,\quad \theta_c g = \theta_c\alpha = \theta_c\beta\,.
\end{aligned}$$

From equation (A.3) of Appendix A follow the equations $\epsilon\alpha = \epsilon\beta = \epsilon\gamma$, which we denote by ϵg, and dually for τ; that is, we define :

(iv)
$$\begin{aligned}
&\epsilon g = \epsilon\alpha = \epsilon\beta = \epsilon\gamma\\
&\tau s = \tau a = \tau b = \tau c\,.
\end{aligned}$$

Similarly (see (A.10)) $\theta_a b^2 = \theta_a c^2 = \theta_a bc$ is denoted $\theta_a s^2$, with analogous formulae for θ_b and θ_c. Likewise $\tau a^2 = \tau b^2 = \tau ab = \ldots$, denoted τs^2, and dually for ϵg^2 :

(v)
$$\begin{aligned}
\theta_a s^2 &= \theta_a b^2 = \theta_a c^2 = \theta_a bc\\
\theta_b s^2 &= \theta_b c^2 = \theta_b a^2 = \theta_b ca\\
\theta_c s^2 &= \theta_c a^2 = \theta_c b^2 = \theta_c ab\\
\tau s^2 &= \tau a^2 = \tau b^2 = \tau c^2 = \tau ab = \tau bc = \tau ca\\
\epsilon g^2 &= \epsilon \alpha^2 = \epsilon \beta^2 = \epsilon \gamma^2 = \epsilon \alpha\beta = \epsilon \beta\gamma = \epsilon \gamma\alpha .
\end{aligned}$$

Another simple calculation (A.5) shows that $\theta_a \theta_b = \theta_b \theta_c = \theta_c \theta_a$, and this element is denoted ψ:

(vi)
$$\psi = \theta_a \theta_b = \theta_b \theta_c = \theta_c \theta_a ,$$

and we have (A.19) the formula $\psi a^2 = \psi b^2 = \psi ab = \ldots$ which is denoted ψs^2, and similarly for ψg^2:

(vii)
$$\begin{aligned}
&\psi s^2 = \psi a^2 = \psi b^2 = \psi c^2 = \psi ab = \psi bc = \psi ca\\
&\psi g^2 = \psi \alpha^2 = \psi \beta^2 = \psi \gamma^2 = \psi \alpha\beta = \psi \beta\gamma = \psi \gamma\alpha .
\end{aligned}$$

Finally note (A.27) that $(\psi s^2)\alpha = (\psi s^2)\beta = (\psi s^2)\gamma$, which is denoted $\psi s^2 g$, and similarly (A.19) for $(\theta_a s^2)\beta = (\theta_a s^2)\gamma$:

(vii)
$$\begin{aligned}
\psi s^2 g &= (\psi s^2)\alpha = (\psi s^2)\beta = (\psi s^2)\gamma\\
\theta_a s^2 g &= (\theta_a s^2)\beta = (\theta_a s^2)\gamma\\
\theta_b s^2 g &= (\theta_b s^2)\alpha = (\theta_b s^2)\gamma\\
\theta_c s^2 g &= (\theta_c s^2)\alpha = (\theta_c s^2)\beta .
\end{aligned}$$

For convenience we set

(viii)
$$[\star] = a^2 b^2 c^2$$

in A^6. We will see shortly that $[\star]$ is also equal to $\alpha^2\beta^2\gamma^2$.

PROPOSITION. *The ring* $A^{\cdot}$ *is generated as an additive group over* $\mathbb{Z}$ *by the* 72 *elements*:

1 *in* A^0

$a, b, c, \alpha, \beta, \gamma, d$ in A^1

$a^2, b^2, c^2, \alpha^2, \beta^2, \gamma^2, a\alpha, b\beta, c\gamma, \tau s, \epsilon g, \theta_a s, \theta_b s,$
$\theta_c s, \theta_a g, \theta_b g, \theta_c g$ in A^2

$a^2\beta,\ a^2\gamma,\ b^2\gamma,\ b^2\alpha,\ c^2\alpha,\ c^2\beta,\ abc,\ \alpha\beta\gamma,\ \epsilon a^2,\ \epsilon b^2,\ \epsilon c^2,$
$\tau\alpha^2,\ \tau\beta^2,\ \tau\gamma^2,\ \epsilon g^2,\ \tau s^2,\ \theta_a s^2,\ \theta_b s^2,\ \theta_c s^2, \theta_a g^2,\ \theta_b g^2,\ \theta_c g^2$ in A^3

$b^2c^2,\ c^2a^2,\ a^2b^2,\ \beta^2\gamma^2,\ \gamma^2\alpha^2,\ \alpha^2\beta^2,\ \tau s^2\alpha,\ \tau s^2\beta,\ \tau s^2\gamma,$
$\epsilon g^2 a,\ \epsilon g^2 b,\ \epsilon g^2 c,\ \theta_a s^2 g,\ \theta_b s^2 g,\ \theta_c s^2 g,\ \psi s^2,\ \psi g^2$ in A^4

$\epsilon b^2c^2,\ \epsilon c^2a^2,\ \epsilon a^2b^2,\ \tau\beta^2\gamma^2,\ \tau\gamma^2\alpha^2,\ \tau\alpha^2\beta^2,\ \psi s^2 g$ in A^5

$[\star]$ in A^6.

In fact, we give recipes to write any monomials in $a,b,c,\alpha,\beta,\gamma,d$ in $A^\cdot$ as integral linear combinations of these 72 basic classes. Most of these rules are formulas of Schubert ; the point is simply to verify that they all follow algebraically from the basic relations (1) – (4). We list and verify those of Schubert's formulas which we need in Appendix A.

Because of the action of $G = \mathfrak{S}_3 \times \mathfrak{S}_2$ on $A^\cdot$, each relation that is proved to hold in $A^\cdot$ may give rise to up to 12 relations by applying the symmetries in G to it.

In this regard note that ϵ and τ are dual and fixed under $\mathfrak{S}_3$, that θ_a, θ_b, and θ_c are self–dual, and are permuted as the subscripts indicate by $\mathfrak{S}_3$; ψ is self–dual ; $\theta_a s$ and $\theta_a g$ are dual, as are τs and ϵg, $\theta_a s^2$ and $\theta_a g^2$, τs^2 and ϵg^2, and ψs^2 and ψg^2 ; we will see that $\theta_a s^2 g$ and $\psi s^2 g$ are self–dual.

We note also that the set of proposed generators of each A^k is closed under the action of G. Except in degrees 3 and 6, this follows immediately from the previous paragraph. For degree 3 one needs to add the equation

(5) $$a\beta^2 = (a^2+\beta^2)\beta = a^2\beta\ ,$$

which follows from (2) and (1). For degree 6, to show that $\alpha^2\beta^2\gamma^2 = a^2b^2c^2$, note first that

(6) $$a^2\beta^2 = a(a\beta^2) = a(a^2\beta) = 0\ ,$$
(7) $$ab\gamma^2 = (a^2+\gamma^2)(b^2+\gamma^2) = a^2b^2\ ,$$
(8) $$ab\alpha\beta = (a^2+\beta^2)(b^2+\alpha^2) = a^2b^2+\alpha^2\beta^2+a^2\alpha^2+b^2\beta^2\ .$$

Multiplying (8) by γ^2 and applying (6) and (7) it follows that

(9) $$\alpha^2\beta^2\gamma^2 = ab\alpha\beta\gamma^2 = a^2b^2\alpha\beta\ .$$

Now $a^2b^2\alpha\beta = ab\alpha^2\beta^2$ by (5), and by the duals of the preceding steps, this is $a^2b^2c^2$, as required.

To show how the indicated elements generate in a given degree k, it suffices to show first how to write monomials of degree k of the form ST, where S is a monomial in a, b, c, and T is a monomial in α, β, γ, as a linear combination of the given elements. Using the action of G, one need only check one monomial in each G–orbit ; for example, by duality, it suffices to consider $\deg(S) \geq \deg(T)$. Next, for each ST as above, but of degree k–1 , one must show how to write the product of ST with any one of the elements d, ϵ, τ, θ_a, θ_b, or θ_c, as a linear combination of the given elements. To see this one uses the equations (i). Because of (4) we have

$$(10) \qquad d^2 = (a+b+c+\alpha+\beta+\gamma)d + (a+b+c)(\alpha+\beta+\gamma),$$

so we never have to consider products of any of these last elements. The details of these computations are included in Appendix A. □

This makes the calculation of any product in $A^\cdot$ a simple algebra exercise. In particular one computes easily that the 7×7 , 17×17 , and 22×22 matrices obtained by multiplying the basic classes in A^1 and A^5, A^2 and A^4, A^3 and A^3, respectively and picking off coefficients of $[\star]$, are all unimodular (see Appendix B).

In the next section we will construct a homomorphism from the ring $A^\cdot$ to the Chow ring $A^\cdot(X)$. We will apply the following simple lemma to this homomorphism, to deduce that this homomorphism is an isomorphism, and that the above classes form a basis for $A^\cdot$ and $A^\cdot(X)$.

DEFINITION. *A graded ring* $A^\cdot = A^0 \oplus A^1 \oplus .. \oplus A^n$ *will be called an* n*-dimensional Poincaré duality ring if* A^n *has one generator* $[\star]_A$ *over* $\mathbb{Z}$, *and each* A^i *has a finite number of generators* $\alpha_p^{(i)}$; *in addition, for each* i *there should be integers* $a_{pq}^{(i)}$ *such that*

$$\alpha_p^{(i)} . \alpha_q^{(n-i)} = a_{pq}^{(i)} [\star]_A$$

and the matrices $(a_{pq}^{(i)})$ *are unimodular. We will call such a ring a strong Poincaré duality ring if, in addition, the generator* $[\star]_A$ *is not a torsion element (or zero) ; it follows that the elements* $\alpha_p^{(i)}$ *form a basis for* A^i *over* $\mathbb{Z}$, *and the product* $A^i \otimes A^{n-i} \longrightarrow A^n \cong \mathbb{Z}$ *is a perfect pairing over* $\mathbb{Z}$.

LEMMA. *Let* $A^\cdot$ *and* $B^\cdot$ *be* n*-dimensional Poincaré duality rings, with* $B^\cdot$ *assumed strong. Suppose* $\varphi^\cdot : A^\cdot \longrightarrow B^\cdot$ *is a homomorphism of graded rings, and that* φ^n *maps* $[\star]_A$ *onto* $[\star]_B$. *Then* $A^\cdot$ *is also strong. Suppose that the total number of generators of* $A^\cdot$ *is the same as the number of generators of* $B^\cdot$. *Then* $\varphi^\cdot$ *is an isomorphism.*

Proof: The first assertion is obvious, since a torsion element of A^n could not map to a torsion–free element of B^n. For the second, to see that $\varphi^{\cdot}$ is injective, suppose $x \in A^i$ and $\varphi^i(x) = 0$; choose $y \in A^{n-i}$ with $x.y = [\star]_A$. Then

$$[\star]_B = \varphi^n([\star]_A) = \varphi^i(x).\varphi^{n-i}(y) = 0 ,$$

a contradiction. Since $A^{\cdot}$ and $B^{\cdot}$ have the same ranks, each φ^i must map A^i onto a lattice in B^i. Consider the commutative diagram

$$\begin{array}{ccccc} A^i & \otimes & A^{n-i} & \longrightarrow & A^n \\ \varphi^i\downarrow & & \downarrow\varphi^{n-i} & & \downarrow\varphi^n \\ B^i & \otimes & B^{n-i} & \longrightarrow & B^n . \end{array}$$

Since the bottom pairing is perfect over $\mathbb{Z}$, the index of $\varphi^i(A^i)$ in B^i must divide the determinant of the matrix which describes the upper pairing. But this determinant is assumed to be 1, so $\varphi^i(A^i) = B^i$, as required. □

Section 3. The Chow ring for plane triangles.

Let $A^{\cdot}$ be the graded ring constructed in Section 2, and let $A^{\cdot}(X)$ be the Chow ring of the space X of complete triangles.

PROPOSITION. *The Chow ring $A^{\cdot}(X)$ is a free abelian group on 72 generators, and the canonical map from $A^{\cdot}(X)$ to the homology ring $H^{\cdot}(X)$ is an isomorphism.*

Proof : This follows from the theorem of Bialynicki–Birula [1], [2], once we prove that a torus T acts on X, with 72 fixed points ; this uses the fact that X is a non–singular projective variety. The torus T is the group of diagonal matrices in $SL(3)$, which acts on the projective plane by linear transformations, and hence acts on X. The fixed points of this action are easy to list, since the only fixed points of this action are the three points (1:0:0), (0:1:0), and (0:0:1), and the only fixed lines are the axes joining them. There are 6 honest (ordered) triangles, obtained by ordering these three points as vertices. There are 18 of type θ_a , 18 of type ω_a , 18 of type ω_α , 6 of type η , and 6 of type ζ . □

THEOREM. *There is an isomorphism from $A^{\cdot}$ to $A^{\cdot}(X)$ which takes the elements a, b, c, α, β, γ, and d to the classes described by Schubert (which are the pullbacks of the positive generators of divisor classes via the projections to the six factors). In addition, the elements ϵ, τ, θ_a, θ_b, θ_c in $A^{\cdot}$ map to the classes in $A^{\cdot}(X)$ of the closures of the corresponding loci in X. The classes listed in the proposition of §2 map to a basis for $A^{\cdot}(X)$.*

Proof : Map $\mathbb{Z}[a,b,c,\alpha,\beta,\gamma,d]$ to $A^{\cdot}(X)$, with generators going to the pullbacks of the designated hyperplane classes. To obtain a homomorphism from $A^{\cdot}$ to $A^{\cdot}(X)$ it must be verified that the generators of the ideal I map to zero, which was already proved in Section 1, the essential point being the formulae relating the divisors ϵ, τ, θ_a, θ_b, θ_c to the divisors a, b, c, α, β, γ, d proved in [15] or §5 below. We proved in Section 2 that $A^{\cdot}$ is a Poincaré duality ring with 72 generators. Since X is a smooth projective variety whose Chow ring is isomorphic to its cohomology ring, $A^{\cdot}(X)$ is a strong Poincaré duality ring, and we know it has 72 generators. The class $[\star] = a^2b^2c^2$ maps to the class of a point in X, namely that representing the unique triangle with three given general vertices. By the lemma of §2, it follows that the map from $A^{\cdot}$ to $A^{\cdot}(X)$ is an isomorphism. □

Remark. The classes of the closures of the loci described in Section 1 by the notations ψ, ω_a, ω_b, ω_c, ω_α, ω_β, ω_γ, η, and ζ also correspond to the elements in $A^{\cdot}$ specified by Schubert. These can be deduced from the formulae

$$\psi = \theta_{\mathrm{a}}\theta_{\mathrm{b}}\ ,\ \omega_{\mathrm{a}} = \epsilon\theta_{\mathrm{a}}\ ,\ \omega_{\alpha} = \tau\theta_{\alpha}\ ,\ \eta = \epsilon\psi\ ,\ \zeta = \tau\psi\ ,$$

by the algorithms of Appendix A. To verify these formulae, it suffices to show that, at a generic point of a locus on the left side of the equation, the two loci on the right meet transversally ; this can be done in local coordinates, as in [13], [15], or [8].

Section 4. Inscribed and circumscribed triangles

Among Schubert's applications is a calculation of the number of triangles which are simultaneously inscribed in one curve C and circumscribed about another curve D, i.e., the vertices lie on C and the sides are tangent to D. In this section we carry out this application, while correcting an error of Schubert's.

Let V_C be the subvariety of the space of complete triangles consisting of those which are inscribed in C. More precisely, V_C is the closure in X of the set of triangles with non–collinear vertices which lie on C. As the image of a rational map from $C\times C\times C$ to X, V_C is an irreducible three–dimensional subvariety. Let $v = v_C$ denote the class of $[V_C]$ in $A^3(X)$. To compute the coefficients for v of the 22 basic generators, it will suffice to compute the intersection of v with 22 independent elements of $A^3(X)$.

LEMMA 4.1. *If C is an irreducible curve of degree n with δ ordinary nodes, and κ ordinary cusps as its only singularities, then*

(i) *the intersection numbers of v with the following classes are* 0 : a^2b, $a^2\beta$, $a^2\alpha$, $\theta_a s^2$, $\theta_a a^2$, ϵa^2, τs^2, $\theta_a \alpha^2$, $\tau\alpha^2$.

(ii) $v.\alpha^2\beta = n(n-1)^2$, $v.a\alpha^2 = n^2(n-1)$, $v.\epsilon g^2 = n(n-1)(n-2)$, $v.abc = n^3$, $v.\theta_a sa = n^2$, $v.\theta_a g^2 = n(n-1)$.

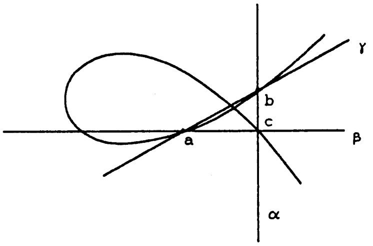
(iii) $v.\tau\alpha\beta = 2\delta+3\kappa$.

Proof : (i) all conditions but the last two require one of the vertices to be a fixed general point, which would not be on C. For the intersection with $\theta_a\alpha^2$, a degenerate triangle of the form θ_a is in V_C only if the line α is tangent to C at the point $b=c$. Since in this condition α is fixed and general, so transversal to C, the intersection is empty. A similar argument works for $\tau\alpha^2$.

(ii) For the first, the general line α meets C transversally, giving $n(n-1)$ choices for the points b and c; for each of these, the line β is determined, and there are $n-1$ choices for the point a on this line. The other cases in (ii) are similar.

(iii) For a triangle of type τ to be in V_C it is necessary that the point s of τ is the limit of three non–collinear points of C. If s is a smooth point of C, the three lines must come together in the tangent line to C at s. Two general points are fixed for the lines α and β to pass through. The only possibility for intersection therefore comes from the singular points of C. We analyze these locally. Assume $(0,0) = (0:0:1)$ is a node of C, that C has affine equation of the form $y^2 = x^2 +$ higher terms, and that α must pass through $(0:1:0)$ and β

must pass through (1:0:0). To get a triangle of type τ as a limit of honest triangles with vertices on C, two of the three vertices must move on one branch, one on the other ; the limiting triangle has sides $\alpha : x = 0$; $\beta : y = 0$, and γ: either $y = x$ or $y = -x$. We must show that each of these counts for 1 in the intersection product. It suffices to consider the first case. Let $y = x+g(x)$, $y = -x+h(x)$ define the two branches, with g and h power series vanishing to order at least two at the origin.



The four–dimensional variety Y of triangles with α passing through (0:1:0) and β passing through (1:0:0) can be parametrized by coordinates s, t, u, v, where $x = s$, $y = t$ are equations for α and β respectively, $c = (s,t)$, $a = (s-v,t)$, $b = (s,t+(1+u)v)$, and γ has equation $y = (1+u)(x-s+v) + t$. The intersection of Y with V_C is described by equations

$t = s-v+g(s-v)$	(the point b is on the first branch)
$t+(1+u)v = s+g(s)$	(the point a is on the first branch)
$t = -s+h(s)$	(the point c is on the second branch).

This curve is parametrized by s; $t = -s$ + higher terms, $v = 2s$ + higher terms, and $u = (g(s)-g(s-v))/v = \ldots$. The hypersurface of triangles of type τ is defined in Y by the equation $v=0$. Since the order of v as a function of s is 1, the intersection multiplicity is 1, as required.

Similarly for a cusp of C at the origin, say defined by $y^2 = x^3$ + higher terms. Let Z be the locus in X of triangles such that α passes through $(1{:}p{:}\dot{a})$, and β passes through $(1{:}q{:}0)$, for p and q general constants. The point P in $Z \cap \tau$ will have sides $\alpha : Y = pX$,

$\beta: Y = qX$, $\gamma: Y = 0$. We will parametrize the curve $Z \cap V_C$, and intersect with the hypersurface τ.

Parametrize C near the origin by $t \longmapsto c_t = (t^2, t^3+\ldots)$. The point b_t is the other point on $C \cap \alpha$, which has the parametrization

$$t \longmapsto b_t = (t^2 - \frac{2}{p}t^3+\ldots, -t^3+\ldots).$$

Similarly for a_t, replacing p by q. One verifies easily[1] that the hypersurface $\tau \cap Z$ is defined near P by the equality of the x–coordinates of the points a and b. Pulling this hypersurface back to the t–disk, one has the equation

$$(t^2 - \frac{2}{p}t^3) - (t^2 - \frac{2}{q}t^3) + \ldots = (\frac{2}{q} - \frac{2}{p})t^3 + \ldots .$$

Since the order of vanishing in t is three, the intersection number is three[2], as required. □

PROPOSITION. *If C is an irreducible plane curve of degree n with only δ ordinary nodes and κ ordinary cusps as singularities, then*

$$\begin{aligned}[V_C] = {} & n(n-1)(\tau\alpha^2+\tau\beta^2+\tau\gamma^2) + 2n(n-1)(\theta_a g^2 + \theta_b g^2 + \theta_c g^2) \\ & + \check{n}\tau s^2 + (3n^2-2n)\epsilon g^2 + n(n-1)(n-2)abc,\end{aligned}$$

where $\check{n} = (n-1) - 2\delta - 3\kappa$ is the class of C.

Proof: It suffices to check that both sides have the same intersection numbers with the 22 basis elements of $A^3(X)$. For $[V_C]$, all but the intersection with $\alpha\beta\gamma$ are listed in the lemma, up to permutations. From Appendix A one can write $\tau\alpha\beta$ in terms of the basic elements :

$$\tau\alpha\beta = c^2\alpha + c^2\beta + 2\epsilon g^2 + 2\tau s^2 + \epsilon a^2 + \epsilon b^2 + \theta_a s^2 + \theta_b s^2 + 2\theta_c s^2 + \theta_c g^2 - \alpha\beta\gamma .$$

---

[1] Z has local coordinates (u,v,g,h), where the sides have equations $\alpha: Y = pX+u$, $\beta: Y = qX+v$, $\gamma: Y = gX+h$. One solves for the points a, b and c in terms of these coordinates and checks that an equation for τ is $(v-h)(g-p) = (u-h)(g-q)$, which is equivalent to equating the x–coordinates of a and b.

[2] This follows from the projection formula for the parametrization from the disk to the curve.

From the lemma, we derive

$$2\delta+3\kappa = 2n(n-1)(n-2) + n(n-1) - [V_C].\alpha\beta\gamma .$$

One then checks the table (Appendix B) to see that the coefficients agree. □

By duality we have :

PROPOSTION. *If D is an irreducible plane curve of degree m with only τ ordinary bitangents and ι ordinary inflections (as singularities on the dual curve), and W_D is the locus of triangles circumscribed about D, then*

$$[W_D] = \check{m}(\check{m}-1)(\epsilon a^2+\epsilon b^2+\epsilon c^2) + 2\check{m}(\check{m}-1)(\theta_a s^2+\theta_b s^2+\theta_c s^2)$$
$$+ m\epsilon g^2 + (3\check{m}^2-2\check{m})\tau s^2 + \check{m}(\check{m}-1)(\check{m}-2)\alpha\beta\gamma ,$$

where $\check{m}$ is the class of D. Note that $m = \check{m}(\check{m}-1)-2\tau-3\iota$.

COROLLARY. *With C and D as in the propositions, and in general position in the plane, the number of triangles simultaneously inscribed in C and circumscribed about D is one-sixth of*

$$2n(n-1)(n-2)\check{m}(\check{m}-1)(\check{m}-2) + n(n-1)(n-2)m + \check{m}(\check{m}-1)(\check{m}-2)\check{n} .$$

Proof : The fact that $[V_C].[W_D]$ is equal to the displayed number follows from the two propositions and the tables for intersecting basic 3–cycles with each other. One must also verify that V_C meets W_D transversally at points which correspond to honest triangles ; this follows as usual from the transitive action of the projective linear group. One must use the actual description of X to see that there are no others. For example, a triangle of type θ_a is in V_C if the line α is tangent to the curve at the point $b=c$, and the point a is another point on C. Dually, this triangle will belong to W_D if a is on D and the line $\beta=\gamma$ is tangent do D at a, and α is another tangent to D. Thus, if θ_a is in $V_C \cap W_D$, a is one of the mn points on $C \cap D$, α is one of the $\check{m}\check{n}$ common tangents to C and D, and the tangent to D at a meets α at its point of tangency to C; this does not happen if C and D are in general position. Similar arguments apply to other types of degenerate triangles. □

Remarks. (1) Schubert's formula $n(n-1)\check{m}(\check{m}-1)(2n\check{m}-3n-3\check{m}+4)$ differs from the correct answer by the quantity

$$(2\delta+3\kappa)\check{m}(\check{m}-1)(\check{m}-2) + (2\tau+3\iota)n(n-1)(n-2) .$$

If C is smooth – which Schubert presumably assumed – the first term can be ignored, but the second term is non–zero when D is a smooth curve of degree ≥ 3 (if degree $C \geq 3$). Schubert gives the intersection of $[V_C]$ with $\tau\alpha\beta$ as zero, which is only correct if C is smooth ; the dual formula for $[W_D].\epsilon ab = 0$ is false when D has flexes and bitangents, even if D is smooth.

(2) The formula of the corollary depends only on the degrees and classes of the two curves. From this one might expect that the same formula is valid for curves with arbitrary singularities, as is the case in the contact formula [7]. However, this is not the case. A singularity of the form $y^p = x^q$, with $p < q$ coprime, contributes $q(p-1)^2$ to the intersection product $v.\tau\alpha\beta$, while its contribution to the class number formula – intersecting the curve with a polar curve – is only $q(p-1)$. For a discussion of variations of numerical invariants of singularities in families see the article of Diaz and Harris [4].

Section 5. Triangles in a projective bundle.

Let E be a vector bundle on a smooth quasiprojective variety S. Let $Y = G_3(E) = G_2(\mathbb{P}(E))$ be the Grassmann bundle of 2–planes in the projective bundle $\mathbb{P}(E)$ of lines in E. Let U be the universal 3–plane bundle on Y, and $\mathbb{P}(U)$ the bundle of projective planes. The space of triangles of $\mathbb{P}(E)$ is defined to be the fiber bundle X over Y whose fiber over a plane is the space of (complete) triangles in that plane. In this section we determine the Chow ring $A^{\cdot}(X)$ as an algebra over $A^{\cdot}(Y)$, and hence as an algebra over $A^{\cdot}(S)$.

Take three copies of $\mathbb{P}(U)$, with tautological sub–line bundles of U denoted A, B, and C. Take three copies of $G_2(U)$, with tautological sub–plane bundles of U denoted $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$. We can construct X globally as the closure in

$$\mathbb{P}(U)\times_Y\mathbb{P}(U)\times_Y\mathbb{P}(U)\times_Y G_2(U)\times_Y G_2(U)\times_Y G_2(U)\times_Y G_3(\mathrm{Sym}^2(U)$$

of the set of honest triangles. Over any open set Y^o of Y where U is trivial, X is the product of Y^o by the triangle space discussed in §1. In particular we have the loci of triangles of special type θ_a, ϵ, τ, etc., and we denote the classes in $A^{\cdot}(X)$ of such subvarieties by the same Greek letters.

On X we have inclusions of vector bundles (denotes by the same letters)

$$A \subset \mathscr{B} \subset U,\ A \subset \mathscr{C} \subset U,\ B \subset \mathscr{A} \subset U,\ B \subset \mathscr{C} \subset U,\ C \subset \mathscr{A} \subset U,\ C \subset \mathscr{B} \subset U,$$

corresponding to the inclusions of points in lines. We define classes in $A^1(X)$ by :

$$\begin{aligned} \mu_1 &= c_1(U^{\vee}) \\ a &= c_1(A^{\vee}), \quad b = c_1(B^{\vee}), \quad c = c_1(C^{\vee}), \\ \alpha &= c_1(\mathscr{A}^{\vee}), \quad \beta = c_1(\mathscr{B}^{\vee}), \quad \gamma = c_1(\mathscr{C}^{\vee}), \end{aligned}$$

and define $\mu_2 = c_2(U^{\vee}) \in A^2(X)$, $\mu_3 = c_3(U^{\vee}) \in A^3(X)$.

When S is a point, so Y is a Grassmann variety, μ_1 is represented by the condition that the plane of the triangle meet a given linear space of codimension three, μ_2 the condition that the plane meets a given codimension two space in at least a line, and μ_3 the condition for the plane to be contained in a given hyperplane. For $Y = \check{\mathbb{P}}^3$, we have $\mu_i = \mu^i$, with $\cdot\mu$ the condition for the plane to pass through a point ; this is the notation used by Schubert [12], §11.

The class a is represented by the condition for the vertex "a" to lie on a given hyperplane, while α by the condition that the side "a" meet a given space of codimension two. The notations for special triangles θ_a, ϵ, τ, ψ, have the same meaning as before.

LEMMA. *The following equations are valid in* $A^{\cdot}(X)$:

(i) $\theta_c = a+b-\gamma-\tau$

(ii) $\theta_c = \alpha+\beta-c-\epsilon-\mu_1$

(iii) $a^3 = \mu_1 a^2-\mu_2 a+\mu_3$

(iv) $\alpha^3 = 2\mu_1\alpha^2-\mu_1^2\alpha-\mu_2\alpha+\mu_1\mu_2-\mu_3$

(v) $a\beta = a^2+\beta^2-\mu_1\beta+\mu_2$.

Proof : For (i), we look at the locus where the points "a" and "b" coincide. This is given by the vanishing of the composite map of line bundles

$$A \longrightarrow \mathscr{C} \longrightarrow \mathscr{C}/\mathrm{B} .$$

Thus this locus represents the class $c_1(\mathscr{C}/\mathrm{B}) - c_1(A) = -\gamma+b+a$. On the other hand, the locus where these two points coincide consists of all triangles of type θ_c or type τ (or the closure of these two types) ; one checks easily, say in local coordinates (cf. [13]), that the map of line bundles vanishes to order 1 along each of these divisors, which proves (i).

The proof of (ii) is similar, the locus where "α" and "β" coincide being the zero scheme of the composite

$$\mathscr{A}/C \longrightarrow U/C \longrightarrow U/\mathscr{B} .$$

This locus, which is $\theta_c+\epsilon$, therefore represents the class

$$c_1(U/\mathscr{B}) - c_1(\mathscr{A}/C) = (-\mu_1+\beta) - (-\alpha+c) = \alpha+\beta-c-\mu_1 .$$

Equation (iii) is just the universal equation for the tautological bundle A on the first copy of $\mathbb{P}(U)$; i.e., $U \otimes A^{\vee}$ has a nowhere vanishing section. Likewise (iv) is from the universal line bundle $U/\mathscr{A}$ on the first copy of $G_2(U) = \mathbb{P}(U^{\vee})$; i.e., $U^{\vee} \otimes (U/\mathscr{A})$ has a nowhere vanishing section. Finally $A \subset \mathscr{B}$ gives a nowhere vanishing section of $\mathscr{B} \otimes A^{\vee}$, which gives

$$c_2(\mathscr{B}) - \beta a + a^2 = 0 .$$

Since $0 = c_2(U/\mathscr{B}) = \mu_2 - \mu_1\beta + \beta^2 - c_2(\mathscr{B})$, (v) follows. □

From (i) and (ii) we may set

$$d = \alpha+\beta+\gamma+\tau = a+b+c+\epsilon+\mu_1 .$$

Hence

(vi) $\quad \tau = d-\alpha-\beta-\gamma$, and $\epsilon = d-a-b-c-\mu_1$,

and from the lemma, $\theta_c = a+b-\gamma-(d-\alpha-\beta-\gamma)$, or

(vii) $\quad \theta_c = a+b+\alpha+\beta-d$.

Although we do not need this, in fact one has $d = c_1(D^\vee)$, where D is the universal subbundle on the Grassmann bundle $G_3(\mathrm{Sym}^2(U))$. To see this, note that the canonical map of rank three bundles

$$A^{\otimes 2} \oplus B^{\otimes 2} \oplus C^{\otimes 2} \longrightarrow D$$

(which determines the map to the Grassmann bundle over the locus of honest triangles) vanishes on the loci θ_a, θ_b, θ_c, and τ , the latter to order two. Hence

$$c_1(D) - (2c_1(A)+2c_1(B)+2c_1(C)) = \theta_a+\theta_b+\theta_c+2\tau ,$$

so $c_1(D^\vee) = \theta_a+\theta_b+\theta_c+2\tau-2a-2b-2c$, which, by (i) and (vi), is d .

There is a **duality** map from X to X'= space of triangles in $\mathbb{P}(U^\vee)$. A triangle Δ in X determines a triangle Δ^2 in X^2. In terms of bundles, given $A, B, C, \mathscr{A}, \mathscr{B}, \mathscr{C} \subset U$, the dual is determined by $A^6, B^6, C^6, \mathscr{A}^6, \mathscr{B}^6, \mathscr{C}^6 \subset U^\vee$, where

$$A^2 = \mathrm{Ker}(U^\vee \longrightarrow \mathscr{A}^\vee) , \mathscr{A}^2 = \mathrm{Ker}(U^\vee \longrightarrow \mathscr{A}^\vee) , \text{etc.}$$

It follows that the duality map acts as follows on the classes :

$$a^5 = \alpha-\mu , \alpha^5 = a-\mu , \epsilon^5 = \tau , \tau^5 = \epsilon , \theta_a{}^5 = \theta_a ,$$
$$\mu^4 = -\mu , \mu_2^4 = \mu_2 , \mu_3^4 = -\mu_3 , \text{ and } d^4 = d-4\mu .$$

THEOREM. *We have*

$$A^{\cdot}(X) = A^{\cdot}(Y)[a,b,c,\alpha,\beta,\gamma,d]/I\,,$$

where I *is the ideal generated by the polynomials listed in* (1)–(4) *below ; for each polynomial listed, it is to be understood that the polynomials obtained by the action of the symmetric group* $\mathfrak{S}_3$ *on* a, b, c *and simultaneously* α, β, γ *are included* :

(1) $a^3-\mu_1 a^2+\mu_2 a-\mu_3$,
$\alpha^3-2\mu_1\alpha^2+\mu_1^2\alpha+\mu_2\alpha-\mu_1\mu_2+\mu_3$;

(2) $a\beta-a^2-\beta^2+\mu_1\beta-\mu_2$

(3) $(b+c+\beta+\gamma-d)(b-c)$,
$(b+c+\beta+\gamma-d)(\beta-\gamma)$;

(4) $(d-a-b-c-\mu_1)(d-\alpha-\beta-\gamma)$.

Proof : Let $B^{\cdot} = A^{\cdot}(Y)$, and define $A^{\cdot}$ to be the graded algebra over $B^{\cdot}$ with generators $a,b,c,\alpha,\beta,\gamma,d$, and relations specified in the theorem. We have a canonical homomorphism of graded $B^{\cdot}$–algebras from $A^{\cdot}$ to $A^{\cdot}(X)$. Indeed, the lemma shows that relations (1) and (2) map to zero ; by equation (vii), (3) follows from the fact that $a=b$ and $\alpha=\beta$ on the locus θ_c . Relation (4) follows from (vi), as before, and the fact that the loci ϵ and τ are disjoint.

Define elements τ, ϵ, θ_a, θ_b, θ_c in $A^{\cdot}$ by formulas (vi) and (vii). The formulas (iii)–(viii) of §2 can be used to construct classes $\theta_a s,\ldots,\psi,\ldots,\theta_c s^2 g$, [$\star$] in $A^{\cdot}$ contains the 72 elements named in the Proposition of §2. We call these 72 elements the **basic classes.** While completing the proof of the theorem, we prove :

PROPOSITION. $A^{\cdot}(X)$ *is a free module over* $A^{\cdot}(Y)$ *on the 72 basic classes.*

Proof : If $\zeta_1,\ldots,\zeta_{72}$ are the basic classes in $A^{\cdot}(X)$, the proposition follows from the "Leray–Hirsh" theorem proved in Appendix C : the map

$$\bigoplus_1^{72} A_{\cdot}(Y) \longrightarrow A_{\cdot}(X)\ ,\ \oplus\ \alpha_i \longmapsto \Sigma\ \zeta_i \cap f^*(\alpha_i),$$

is an isomorphism, where $f : X \longrightarrow Y$ is the projection. □

The theorem now follows. Indeed, from Appendix A it follows that the 72 basic classes generate the algebra $A^{\cdot}$ as a module over the ring $B^{\cdot}$. From the proposition we have a surjection $A^{\cdot} \longrightarrow A^{\cdot}(X)$ of $B^{\cdot}$–modules, and the second is free over $B^{\cdot}$ on the images of these 72 generators. Hence the map is an isomorphism. □

Remarks. (1) Note that if E has rank e over S, and $Y = G_3(E)$, then $A^{\cdot}(Y) = A^{\cdot}(S)[\mu_1,\mu_2,\mu_3]/J$ where J is the ideal generated by three universal homogeneous polynomials $P_i(\mu_1,\mu_2,\mu_3,c_1(E),\ldots,c_e(E))$ of degrees $i = e-2, e-1$, and e, which express the vanishing of $c_i(E/U)$ for these indices, as Grothendieck showed (cf. [6], Ex. 14.6.6). Therefore $A^{\cdot}(X)$ is a polynomial ring in ten variables $a,b,c,\alpha,\beta,\gamma,d,\mu_1,\mu_2,\mu_3$ over $A^{\cdot}(S)$, modulo the ideal generated by these three polynomials P_i together with the nineteen polynomials specified in the theorem.

If S is allowed to be a singular variety, similar arguments show that the Chow group $A.(X)$ is a direct sum of $12e(e-1)(e-2)$ copies of $A.(S)$.

The same results hold when $A^{\cdot}$ and $A.$ are replaced by cohomology $H^{\cdot}$ and (Borel–Moore) homology $H.$; this version of the theorem follows from the standard Leray–Hirsch theorem for fibre bundles.

(2) The full working out of intersection products in higher dimensions can be tedious, but Appendix A contains a complete recipe for computing all such products. For a simple application to triangles in three space, one can verify that *the number of triangles each of whose sides meets three given space curves is* 8 *times the product of the degrees of the nine given curves.* To see this ; note that the condition for the side α to meet a curve of degree n is the class $n\alpha$. We are reduced to showing that $\alpha^3\beta^3\gamma^3 = 8$. But $\alpha^3 = 2\mu\alpha^2 - 2\mu^2\alpha$, so

$$\alpha^3\beta^3\gamma^3 = 8\mu^3\alpha\beta\gamma(\alpha-\mu)(\beta-\mu)(\gamma-\mu) = 8\mu^3\alpha^2\beta^2\gamma^2 = 8 .$$

(3) The method of this paper can be used in other situations where the intersection rings are rather simple. For example, it can be applied to the space Ψ of "infinitely small triangles", which is the four–dimensional locus in the triangle space X whose class is denoted ψ above. The pull–backs of divisors on X give the basic divisors on Ψ : if i is the inclusion of Ψ in X, and we define

$$s = i^*(a) = i^*(b) = i^*(c),$$
$$g = i^*(\alpha) = i^*(\beta) = i^*(\gamma),$$
$$\eta = i^*(\epsilon),$$
$$\zeta = i^*(\tau),$$

while the pull–back of d is still denoted d. Equation (3) of §1 pulls back to the equation

$$3s+\eta = 3g+\zeta = d\,.$$

PROPOSITION. (1) *$A^{\cdot}(\Psi) = \mathbb{Z}[s,g,d]/K$, where K is the ideal generated by $sg\text{-}s^2\text{-}g^2$, $(d-3s)(d-3g)$, s^3, and g^3.*

(2) *The ranks of its cycles are* 1, 3, 4, 3, 1 ; *a basis for $A^{\cdot}(\Psi)$ is*

$$1\ ;\ s\ ,\ g\ ;\ s^2\ ,\ g^2\ ,\ \eta g\ ,\ \zeta s\ ;\ \eta g^2\ ,\ \zeta s^2\ ,\ s^2 g\ ;\ \textit{and}\ \ ds^2 g\,.$$

The intersection tables are

Products of A^1 and A^3

	s	g	d
ηg^2	1		
ζs^2		1	
$s^2 g$			1

Products of A^2 and A^2

	s	g	ηg	ζs
s^2			1	
g^2				1
ηg	1			
ζs		1		

Proof : The relations in K pull back from basic equations on X, so one has a homomorphism from $\mathbb{Z}[s,g,d]/K$ to $A^{\cdot}(\Psi)$. Since Ψ has twelve fixed points by the torus action, and the stated classes clearly generate $\mathbb{Z}[s,g,d]/K$, and ds^2g maps to the class of a point, the map is an isomorphism. □

More generally, for the space Ψ of infinitely small triangles in a varying plane $\mathbb{P}(U)$, with U as at the beginning of this section,

$$A^{\cdot}(\Psi) = A^{\cdot}(Y)[s,g,d]/K\,,$$

where K is generated by $sg-s^2-g^2+\mu_1 g-\mu_2$, $(d-3s-\mu)(d-3g)$, $s^3-\mu_1 s^2+\mu_2 s-\mu_3$, and $g^3-2\mu_1 g^2+\mu_1^2 g+\mu_2 g-\mu_1\mu_2+\mu_3$.

(4) The results of this paper extend to arbitrary charateristic, avoiding only characteristics 2 and 3 for the discussion of inscribed and circumscribed triangles (§4). The theorem of Bialynicki–Birula is still valid for the Chow groups, as discussed in [14].

(5) Le Barz [8] has given another construction of the space of plane triangles which generalizes to give a space of triples of points in any smooth variety V. This space is the closure of the space of honest triangles in the product of three copies of V, three copies of the Hilbert scheme $\mathrm{Hilb}_2 V$ of length two subschemes of V, and one copy of $\mathrm{Hilb}_3 V$. Le Barz shows by calculations in local coordinates that this closure is smooth.

In fact the variety constructed by Le Barz represents a natural functor. From this fact the smoothness of the variety follows from a simpler calculation of its deformations. To use notation which agrees with Schubert, the data given by a family of triangles in V parametrized by a scheme S is a collection of subschemes $a,b,c,\alpha,\beta,\gamma,d$ of $V\times S$, finite and flat over S of degrees 1, 1, 1, 2, 2, 2, 3, with inclusions $a\subset\beta\subset d$, $a\subset\gamma\subset d$, and similarly for the permutations by $\mathfrak{S}_3$. The key condition of Le Barz is that the corresponding ideal sheaves satisfy :

$$\mathcal{I}(a).\mathcal{I}(\alpha)\subset I(d)$$

for each "vertex" a and its opposite "side" α.

This construction can be used for plane triangles in place of that of Semple, and can be generalized to smooth families. It would be interesting to describe the cohomology of this space in terms of the cohomology of V.

Appendix A. Algebra

In this appendix we use the relations (1)–(4) of the main theorem of §5 to deduce formulas that are valid in the ring $A^{\cdot}$, and use these to give recipes to write any element of $A^{\cdot}$ as a $B^{\cdot}$–linear combination of the 72 basic elements. When $B^{\cdot} = \mathbb{Z}$ this specializes to the assertions needed in §2. We put the terms involving the classes μ_i in braces $\{\ \}$; these terms are to be ignored for the case of plane triangles. For formulae in low degree that are used frequently, we have written out the term in braces : the recipes we give determine them in all degrees, but the expressions become rather long to write out.

The equations labelled with a star $\star$ have all terms (outside braces) on the right appearing in the proposed list of 72 generators ; they are also, modulo the terms in braces, equations which appear in Schubert [12]. They give an effective algorithm for computing all products in these intersection rings ; note that the terms in braces involved only elements of $A^{\cdot}$ of lower degree.

Recall that the ring $A^{\cdot}$ is defined as a polynomial ring in variables a, b, c, α, β, γ, d over $B^{\cdot}$ modulo relations (1)–(4). We sometimes denote the element μ_1 of $B^{\cdot}$ by μ for brevity.

We use freely the symmetry under the group $\mathfrak{S}_3$, and usually write only one equation to represent the 1, 2, 3, or 6 equations resulting from the action of $\mathfrak{S}_3$. In fact $B^{\cdot}$ has an involution which takes

$$\mu_1 \longmapsto -\mu_1\ ,\ \mu_2 \longmapsto \mu_2\ ,\ \text{and}\ \ \mu_3 \longmapsto -\mu_3\ .$$

Then $G = \mathfrak{S}_3 \times \mathfrak{S}_2$ acts on $A^{\cdot}$, compatibly with this involution ; the dual of an equation is obtained by the substitutions :

$$a \longmapsto \alpha - \mu\ ,\ \alpha \longmapsto a - \mu\ ,\ d \longmapsto d - 4\mu\ .$$

It is easy to verify that the defining equations I are preserved by this duality operation, so G acts as automorphisms of $A^{\cdot}$. We include a few of the most useful dual equations, labelled with a prime $'$; since dual statements follow formally, proofs will be omitted.

In the ring $A^{\cdot}$ we have defined

$$\begin{aligned} \tau &= d-\alpha-\beta-\gamma \\ \epsilon &= d-a-b-c-\mu \\ \theta_c &= a+b+\alpha+\beta-d\,. \end{aligned}$$

These are preserved by the duality map : $\epsilon \longmapsto \tau$, $\tau \longmapsto \epsilon$, $\theta_a \longmapsto \theta_a$.

By equation (3) we have $\theta_c a = \theta_c b$, which we call $\theta_c s$, and similarly $\theta_c \alpha = \theta_c \beta$ is called $\theta_c g$. All the equations (i)–(vii) in and after the Lemma in §5 are valid in the ring $A^{\cdot}$.

We shall need one more equation in degree one :

(A.1)* $\qquad \epsilon+\tau+\theta_b+\theta_c = a+\alpha+\{-\mu\}$.

Proof : By (i) and (ii), $(\theta_c+\tau) + (\theta_b+\epsilon) = (\alpha+\gamma-b) + (a+b-\gamma-\mu) = \alpha+a-\mu$.

We turn to degree two :

(A.2)* $\qquad bc = \alpha^2+\tau s+\theta_a s+\{-\mu\alpha+\mu_2\}$.

(A.2′) $\qquad \beta\gamma = a^2+\epsilon g+\theta_a g+\{\mu_2\}$.

Proof : By (i) and (v), $\theta_a s+\tau s = (b+c-\alpha)b = b^2+bc-(\alpha^2+b^2-\mu\alpha+\mu_2) = bc-\alpha^2+\mu\alpha-\mu_2$.

(A.3) $\qquad \epsilon\alpha = \epsilon\beta$.

(A.3′) $\qquad \tau a = \tau b$.

Proof : By (ii), $\epsilon = \alpha+\beta-c-\mu-\theta_c$, so $\epsilon\alpha = \alpha^2+\alpha\beta-c\alpha-\mu\alpha-\theta_c\alpha = \alpha^2+\alpha\beta-(c^2+\alpha^2-\mu\alpha+\mu_2)-\mu\alpha-\theta_c\alpha = \alpha\beta-c^2-\mu_2-\theta_c\alpha$. Interchanging the roles of α and β , $\epsilon\beta = \beta\alpha-c^2-\mu_2-\theta_c\beta$. The equality of $\epsilon\alpha$ and $\epsilon\beta$ then follows from the equation $\theta_c\alpha = \theta_c\beta$.

By symmetry we therefore have $\epsilon\alpha = \epsilon\beta = \epsilon\gamma$ and $\tau a = \tau b = \tau c$; these elements are denoted ϵg and τs respectively.

(A.4)* $\qquad \epsilon a = a^2+a\alpha-\tau s-\theta_b s-\theta_c s+\{-\mu a\}$.

(A.4′) $\qquad \tau\alpha = \alpha^2+a\alpha-\epsilon g-\theta_b g-\theta_c g+\{-\mu\alpha\}$.

Proof : $\epsilon a+\tau s+\theta_b s+\theta_c s = (\epsilon+\tau+\theta_b+\theta_c)a = a^2+a\alpha-\mu a$ by (A6).

We now have more that enough equations to write any element of degree 2 in $A^{\cdot}$ as a linear combination of the 17 basic elements of A^2 of degree two, following the discussion of Section 2. For monomials S in a, b and c, use (A.2). For products ST of linear monomials use the defining equation (2). For products $S\epsilon$ use (A.4). As shown in Section 2, this finishes the proof for degree 2.

It will be useful to make a few more calculations in degree two :

(A.5) $$\theta_a \theta_b = \theta_a \theta_c \,.$$

Proof : $\theta_b - \theta_c = (a+c-\beta-\tau) - (a+b-\gamma-\tau) = c-b+\gamma-\beta$. The result follows by multiplying by θ_a and using the equations $\theta_a b = \theta_a c$ and $\theta_a \beta = \theta_a \gamma$.

The element $\theta_a \theta_b = \theta_a \theta_c = \theta_b \theta_c$ is denoted ψ. We list the equation for ψ which follows from the preceding rules, although this is not needed here ; the equation (4), that $\epsilon\tau = 0$, is needed :

(A.6)*
$$\psi = \theta_a g + \theta_b g + \theta_c g + \theta_a s + \theta_b s + \theta_c s - a\alpha - b\beta - c\gamma + 2\tau s + 2\epsilon g + \{-\mu\alpha - \mu\beta - \mu\gamma + \mu d + 2\mu_2\}$$

(A.7)* $$\theta_a a = \beta^2 + \gamma^2 - a\alpha + \theta_b s + \theta_c s + \tau s + \{-\mu\beta - \mu\gamma + 2\mu_2\} \,.$$
(A.7′) $$\theta_a \alpha = b^2 + c^2 - a\alpha + \theta_b g + \theta_c g + \epsilon g + \{-\mu\alpha + 2\mu_2\} \,.$$

Proof : $\theta_a a = (\beta+\gamma-a-\epsilon-\mu)a = (a^2+\beta^2-\mu\beta+\mu_2) + (a^2+\gamma^2-\mu\gamma+\mu_2) - a^2 - \epsilon a - \mu a =$ $\beta^2+\gamma^2+a^2-\epsilon a+\{-\mu\beta-\mu\gamma-\mu a+2\mu_2\}$ by (ii) and (2). Substituting for ϵa from (A.4) finishes the proof.

(A.8) $$2b^2 + b\beta - \beta^2 - bd + \{\mu\beta\} = 2c^2 + c\gamma - \gamma^2 - cd + \{\mu\gamma\} \,.$$

Proof : $\theta_a b = (b+c+\beta+\gamma-d)b = b^2+bc+b\beta+(b^2+\gamma^2-\mu\gamma+\mu_2)-bd$, so (A.8) is equivalent to the equation $\theta_a b = \theta_a c$.

Next we deduce some equations of degree three :

(A.9) $$a\beta^2 = a^2\beta + \{\mu\beta^2 - \mu^2\beta + \mu\mu_2 - \mu_3\} \,.$$

Proof : $a\beta^2 = \beta(a^2+\beta^2-\mu\beta+\mu_2) = a^2\beta + (2\mu\beta^2-\mu^2\beta-\mu_2\beta+\mu\mu_2-\mu_3)-\mu\beta^2+\mu_2\beta$.

(A.10) $\theta_a b^2 = \theta_a c^2 = \theta_a bc\ ,\ \tau a^2 = \tau b^2 = \tau ab\ .$

(A.10′) $\theta_a \beta^2 = \theta_a \gamma^2 = \theta_a \beta\gamma\ ,\ \epsilon\alpha^2 = \epsilon\beta^2 = \epsilon\alpha\beta\ .$

Proof : $\theta_a bb = \theta_a cb$ by (3), from which the first formula follows. The second is similar, using (A.3) in place of (3).

These elements are denoted $\theta_a s^2$, τs^2 , $\theta_a g^2$ and ϵg^2 respectively.

(A.11)* $b^2 c = b^2\alpha+\tau s^2+\theta_a s^2+\{-\mu b^2+\mu_2 b-\mu_3\}\ .$

(A.11′) $\beta^2\gamma = a^2\beta+\epsilon g^2+\theta_a g^2+\{\mu_2\beta\}\ .$

Proof. Multiply (A.2) by b , getting $b^2 c = b\alpha^2+\tau sb+\theta_a sb-\mu\alpha b+\mu_2 b$, and by (A.9) this is $(b^2\alpha+\{\mu\alpha^2-\mu^2\alpha+\mu\mu_2-\mu_3\})+\tau s^2+\theta_a s^2-\mu(b^2+\alpha^2-\mu\alpha+\mu_2)+\mu_2 b\ .$

(A.12)* $a^2\alpha = \tau s^2+\epsilon a^2+\theta_b s^2+\theta_c s^2+\{\mu_2 a-\mu_3\})\ .$

(A.12′) $a\alpha^2 = \epsilon g^2+\tau\alpha^2+\theta_b g^2+\theta_c g^2+\{-\mu\alpha^2+\mu^2\alpha+\mu_2\alpha-\mu\mu_2+\mu_3\}\ .$

Proof : Multiply (A.1) by a^2 .

(A.13)* $\theta_a sa = abc-\epsilon g^2-\tau s^2-\tau\alpha^2-\theta_b g^2-\theta_c g^2+\{\mu\alpha^2+\mu a\alpha-\mu^2\alpha-\mu_2 a-\mu_2\alpha+\mu\mu_2-\mu_3\}\ .$

Proof : Multiply the equation $\theta_a = b+c-\alpha-\tau$ by ab , getting
$\theta_a sa = ab^2+abc-ab\alpha-\tau ab = ab^2+abc-a(b^2+\alpha^2-\mu\alpha+\mu_2)-\tau s^2 = abc-a\alpha^2-\tau s^2+\{\mu a\alpha-\mu_2 a\}\ .$
Applying (A.12′) to replace $a\alpha^2$ in the right side of this equation gives (A.13).

(A.14)* $\theta_a a^2 = a^2\beta+a^2\gamma-\epsilon a^2+\{-2\mu a^2+\mu_2 a-\mu_3\}\ .$

Proof. Multiply $\theta_a = \beta+\gamma-a-\epsilon-\mu$ by a^2, and use (iii) of the lemma.

(A.15)* $\theta_a a\alpha = b^2\gamma+c^2\beta+\tau s^2+\epsilon g^2+\theta_b s^2+\theta_c s^2+\theta_b g^2+\theta_c g^2$
$+ \{-\mu(b^2+c^2+a\alpha+\epsilon g+\theta_b g+\theta_c g) + \mu_2(2a+2\alpha+b+c+\beta+\gamma-d)-\mu\mu_2-\mu_3\}\ .$

Proof : Multiply (A.7) by α , yielding
$\theta_a a\alpha = \alpha\beta^2+\alpha\gamma^2-a\alpha^2+\theta_b s\alpha+\theta_c s\alpha+\tau s\alpha+\{-\mu\alpha\beta-\mu\alpha\gamma+2\mu_2\alpha\}$. The first two terms on the right of this equation are known by (A.11′), the third by (A.12′). For the fourth and fifth terms we have by (v), $\theta_b s\alpha = \theta_b c\alpha = \theta_b c^2+\theta_b\alpha^2-\mu\theta_b\alpha+\mu_2\theta_b = \theta_b s^2+\theta_b g^2-\mu\theta_b g+\mu_2\theta_b$, and similarly for the sixth, $\tau s\alpha = \tau c\alpha = \tau s^2+\tau\alpha^2-\mu\tau\alpha+\mu_2\tau$. (A15) follows by substituting these six expressions.

(A.16) $$ab\gamma = a^2\gamma+b^2\gamma+\tau s^2+\theta_c s^2+\{-\mu a^2-\mu b^2+\mu_2 a+\mu_2 b-2\mu_3\}\ .$$

Proof: From (v) we have $ab\gamma = a(b^2+\gamma^2-\mu\gamma+\mu_2)$. Using (A.11), (A.9), and (v) this becomes $(b^2\gamma+\tau s^2+\theta_c s^2-\mu b^2+\mu_2 b-\mu_3)+(a^2\gamma+\mu\gamma^2-\mu^2\gamma+\mu\mu_2-\mu_3) - \mu(a^2+\gamma^2-\mu\gamma+\mu_2) + \mu_2 a$, which simplifies as required.

These formulae suffice to write any element of A^3 in terms of the 22 basic elements of degree three. Indeed, from (A.11) and (1) we obtain any monomial S. From (A.16) and (A.12) come all products ST. To obtain the products $S\theta_a$, one has $\theta_a b^2 = \theta_a bc = \theta_a s^2$, $\theta_a a^2$ by (A.14), $\theta_a ab = \theta_a sa$ by (A.13). Finally, to obtain $ST\theta_a$, one has $\theta_a a\alpha$ by (A.15), $\theta_a b\beta = \theta_a c\beta = \theta_a\beta^2-\mu\theta_a\beta+\mu_2\theta_a$ from (v), and the remaining follow similarly using (v).

We also have

(A.17) $$\psi a = \psi b\ .$$
(A.17′) $$\psi\alpha = \psi\beta\ .$$

Proof: $\psi a = \theta_b\theta_c a = \theta_b\theta_c b = \psi b$ by (3).

These elements are denoted ψs and ψg.

For later use we record another equation of degree 3, which follows from the preceding prescription. The notation {...} indicates an expression involving μ_i's and lower degree terms in the basic classes. In the proofs we write ≡ to denote that two expressions differ by a class of the form {...}.

(A.18)* $$\psi s = abc-\tau\alpha^2-\tau\beta^2-\tau\gamma^2-2\tau s^2-\epsilon g^2-\theta_a g^2-\theta_b g^2-\theta_c g^2+\{...\}\ .$$

Now for equations of degree four:

(A.19) $$\theta_a s^2\beta = \theta_a s^2\gamma\ ,\ \psi a^2 = \psi ab = \psi b^2\ ,\text{ and }\ \psi\alpha^2 = \psi\alpha\beta = \psi\beta^2\ .$$

Proof: The first equation follows immediately from the equation $\theta_a\beta = \theta_a\gamma$. The others follow from the first, and the definition of ψ.

These elements are denoted $\theta_a s^2 g$, ψs^2 and ψg^2. Note that from (v) we have $\theta_a g^2 b = \theta_a g^2 c = \theta_a s^2 g + \{...\}$. Similarly $\epsilon g a^2 = \epsilon g^2 a + \{...\}$, and $\tau s\alpha^2 = \tau s^2\alpha + \{...\}$.

(A.20)* $a^2\alpha^2 = \epsilon g^2 a + \tau s^2\alpha + \theta_b s^2 g + \theta_c s^2 g + \{...\}$.

Proof : Multiply (A.1) by $a^2\alpha$, getting $a^2\alpha^2 \equiv (a+\alpha)a^2\alpha \equiv \epsilon a^2\alpha + \tau s^2\alpha + \theta_b s^2 g + \theta_c s^2 g$, which suffices, since $\epsilon a^2\alpha \equiv \epsilon g^2 a$.

(A.21)* $\theta_a g a^2 = \beta^2\gamma^2 - \epsilon g^2 a + \{...\}$.

Proof : Multiply (A.14) by β, getting $\theta_a g a^2 \equiv a^2\beta^2 + a^2\beta\gamma - \epsilon g a^2$. But $a^2\beta^2 \equiv 0$ and $a^2\beta\gamma \equiv \beta^2\gamma^2$, as in equations (6) and (7) of §2, and the result follows.

(A.22)* $a^2bc = \beta^2\gamma^2 + \theta_b s^2 g + \theta_c s^2 g + \tau s^2\alpha + \tau s^2\beta + \tau s^2\gamma + \psi s^2 + \{...\}$.

Proof : By (A.18),
$\psi s^2 \equiv \psi s a = a^2bc - \tau s\alpha^2 - \tau s\beta^2 - \tau s\gamma^2 - 0 - \epsilon g^2 a - \theta_a g^2 a - \theta_b g^2 a - \theta_c g^2 a \equiv$
$a^2bc - \tau s^2\alpha - \tau s^2\beta - \tau s^2\gamma - \epsilon g^2 a - \theta_a g a^2 - \theta_b s^2 g - \theta_c s^2 g$. Subtituting $\beta^2\gamma^2$ for $\theta_a g a^2 + \epsilon g^2 a$ by (A.21), one obtains (A.22).

(A.23)* $\theta_a s^2\alpha = \theta_a s^2 g + \tau s^2\beta + \tau s^2\gamma + \psi s^2 + \{...\}$.

Proof : Multiplying (A.13) by b, one has
$\theta_a s^2\alpha = (\theta_a s a)b \equiv ab^2c - \epsilon g^2 b - 0 - \tau s\alpha^2 - \theta_b g^2 b - \theta_c g^2 b \equiv ab^2c - \epsilon g^2 b - \tau s^2\alpha - \theta_b g b^2 - \theta_c s^2 g$, and one concludes by substituting from (A.22) and (A.21).

The next three sets of equations are essentially equations (6), (7) and (8) of §2 ; the equalities there become congruences here :

(A.24) $a^2\beta^2 = \{...\}$.

(A.25) $ab\gamma^2 = a^2b^2 + \{...\}$, $c^2\alpha\beta = \alpha^2\beta^2 + \{...\}$.

(A.26) $ab\alpha\beta = a^2b^2 + \alpha^2\beta^2 + a^2\alpha^2 + b^2\beta^2$.

To verify that the 17 basic elements generate in degree 4, one has all monomials S by (A.22) and (1). For monomials ST we have $a^2\alpha^2$ by (A.20), $a^2b\alpha \equiv a^2b^2 + a^2\alpha^2$ by (v), $abc\alpha \equiv abc^2 + ab\alpha^2$, which one has by duals of preceding equations ; using (A.12) one has

$$a^2b\beta \equiv ab(a^2+\beta^2) \equiv a(\epsilon g^2 + \tau\beta^2 + \theta_a g^2 + \theta_c g^2) \equiv \epsilon g^2 a + \tau s^2\beta + \theta_a g a^2 + \theta_a s^2 g \, ,$$

and therefore also $a^2\alpha\beta \equiv a\alpha(a^2+\beta^2) \equiv a\alpha\beta^2$; likewise, using (A.24), $a^2b\gamma \equiv a^2b^2+a^2\gamma^2 \equiv a^2b^2$, and similarly $a^2\beta\gamma \equiv \beta^2\gamma^2$, $ab\alpha\beta \equiv a^2\alpha^2+b^2\beta^2$, and $ab\alpha\gamma \equiv a^2b^2+a^2\gamma^2$. It remains to show how to obtain the product of each ST times one of the elements θ_a , θ_b or θ_c . For a^2b or abc use (A.23) ; for $a^2\beta$ use (A.21) ; one then has $ab\gamma \equiv ab^2+a\gamma^2$; for $a^2\alpha$ one has $\theta_b a^2\alpha = \theta_b s^2 g$, which completes the proof.

Among the equations in degree 5 we need

(A.27) $$\psi s^2\beta = \psi s^2\gamma .$$

Proof : This follows from (A.17).

We denote this class by $\psi s^2 g$. Note that $\psi g^2 a = \psi g^2 b = \psi s^2 g + \{...\}$.

(A.28)* $$\theta_b s^2 b^2 = \tau\alpha^2\gamma^2+\psi s^2 g+\{...\} .$$

Proof : By (A.23), $(\theta_b s^2 b)\alpha \equiv \theta_b s^2 g\alpha+\tau s^2\alpha^2+\tau s^2\alpha\gamma+\psi s^2\alpha \equiv \tau\alpha^2\gamma^2+\psi s^2 g$ by (A.24) and (A.25). And $(\theta_b s^2 b)\alpha \equiv \theta_b cb\alpha^2 \equiv \theta_b c^2 b^2 = \theta_b s^2 b^2$.

(A.29)* $$b^2c^2\beta = \epsilon b^2c^2+\tau\alpha^2\beta^2+\psi s^2 g+\{...\} .$$

Proof : By (A.9) and (A.20), $b^2c^2\beta \equiv (b^2\beta^2)c \equiv \epsilon g^2 bc+\theta_c s^2 cg$. Now $\epsilon g^2 bc \equiv \epsilon b^2c^2$ by (A.25), and, by (A.23), $\theta_c s^2 cg \equiv (\theta_c s^2 g+\tau s^2\alpha+\tau s^2\beta+\psi s^2)\alpha \equiv \tau\alpha^2\beta^2+\psi s^2 g$, using (A.24) and (A.25) again.

(A.30)* $$ab^2c^2 = \epsilon b^2c^2+\tau\alpha^2\gamma^2+\tau\alpha^2\beta^2+2\psi s^2 g+\{...\} .$$

Proof : By (vii), $ab^2c^2 \equiv b^2c^2(\epsilon+\tau+\theta_b+\theta_c-\alpha) \equiv \epsilon b^2c^2+\theta_b s^2b^2+\theta_c s^2c^2$, and one concludes by (A.28).

To verify that A^5 is generated by the required 7 classes, consider first products ST . Note that we have a^2b^2c and $a^2b^2\alpha$ by (A.30) and (A.29), and $a^2b^2\gamma \equiv a^2b\gamma^2 \equiv 0$, $a^2bc\alpha \equiv a^2b^2c+a^2c\alpha^2 \equiv a^2b^2c+a^2c^2\alpha$, and similarly $a^2bc\beta \equiv a^2bc^2+a^2b^2\beta$. This gives any product where $\deg(S) = 5$ or 4. When S is a cubic, the use of equation (2) and the resulting $a\beta^2 \equiv a^2\beta$ and $a^2\beta^2 \equiv 0$ reduce to the previous cases : $a^2b\gamma^2 \equiv a^2b^2\gamma$, $a^2b\beta^2 \equiv 0$, $a^2b\alpha^2 \equiv a^2b^2\alpha$, $a^2b\beta\gamma \equiv a^2b^2\beta+a^2\beta\gamma^2 \equiv a^2b^2\beta$, $a^2b\alpha\gamma \equiv a^2b^2\alpha+a^2\alpha\gamma^2 \equiv a^2b^2\alpha$, $a^2b\alpha\beta \equiv a^2b^2\beta+a^2\alpha^2\beta \equiv a^2b^2\beta+a\alpha^2\beta^2$, $abc\alpha^2 \equiv abc^2\alpha$, and $abc\alpha\beta \equiv abc^2\beta+ab\alpha^2\beta$. Finally consider $ST\tau$ for S and T as above. This is zero if $\deg(S) \geq 3$, and if $\deg(S) = 2$, it is $\tau s^2 T$; for $T = \alpha\beta$, one gets $\tau c^2\alpha\beta \equiv \tau\alpha^2\beta^2$ by

(A.25) ; for $T = \alpha\beta$, one gets $\tau c^2\alpha\beta \equiv \tau\alpha^2\beta^2$ by (A.25) ; for $T = \alpha^2$ one similarly gets zero ; this completes the proof in degree 5.

To see that $[\star] = a^2b^2c^2$ generates A^6, we have $a^2b^2c\alpha \equiv a^2b^2c^2 + a^2b^2\alpha^2 \equiv [\star]$, and similarly $a^2b^2c\gamma \equiv 0$. For ST with $\deg(S) = 4$, for $S = a^2b^2$, the products are zero if $T = \alpha^2$, β^2 or γ^2. Equation (A.26) yields $a^2b^2\alpha\beta \equiv [\star]$, and $a^2b^2\alpha\gamma \equiv a^2b\alpha\gamma^2 \equiv 0$. For $S = a^2bc$, the product : with α^2 is $a^2b^2c\alpha \equiv [\star]$; with β^2 is $\equiv 0$; with $\alpha\beta$ is $a^2b^2c\beta + a^2c\alpha^2\beta \equiv [\star]$; with $\beta\gamma$ is $\equiv [\star]$ by (A.24), (A.25) and (A.26). For products ST of cubic monomials, consider first $S = a^2b$. The product with T is $\equiv 0$ if T contains β^2 or γ^2 , the product with $\alpha^2\gamma$ is $a^2b^2\alpha^2 + a^2\alpha^2\gamma^2 \equiv 0$, with $\alpha^2\beta$ is $a^2b^2\alpha\beta \equiv [\star]$, and with $\alpha\beta\gamma$ is $a^2b^2\alpha\beta + a^2\alpha\beta\gamma^2 \equiv [\star]$. Finally, the product of abc with $\alpha\beta\gamma$ is $(a^2+\beta^2)(b^2+\gamma^2)(c^2+\alpha^2) \equiv a^2b^2c^2 + \alpha^2\beta^2\gamma^2 \equiv 2[\star]$. Lastly, we must consider products of ST with either ϵ or τ . If $\deg(S) \geq 3$, the product with τ is $\equiv 0$, while if $\deg(T) \geq 3$, the product with ϵ is $\equiv 0$.

Remark. This includes proofs of many of the formulas in Schubert [12], pp. 153–164. Most of those not listed above are obtained by symmetry, i.e. the action of G . The two remaining equations involving the above classes.

(A.31)* $$\theta_a s a^2 = \beta^2\gamma^2 - \epsilon g^2 a + \tau s^2\beta + \tau s^2\gamma + \psi s^2 + \{\ldots\} ,$$

(A.32)* $$\tau s\alpha\gamma = \gamma^2\alpha^2 - \epsilon g^2 b + \tau s^2\alpha + \theta_b s^2\gamma - \psi g^2 + \{\ldots\} ,$$

are obtained from the others by using the above prescription. In addition, these pages of Schubert contain formulae for the classes ω_a, ω_b, ω_c, ω_α,ω_β, ω_γ, η and ζ . These follow similarly, starting from

$$\omega_a = \epsilon\theta_a \ , \ \omega_\alpha = \tau\theta_\alpha \ , \ \eta = \epsilon\psi \ , \ \zeta = \tau\psi \ .$$

For example, one deduces easily the formula

(A.33)* $$3\psi s + \eta = 3\psi g + \zeta + \{-\mu\psi\} ,$$

used for infinitely small triangles. In fact $3\psi s + \eta = \psi(a+b+c+\epsilon) = \psi(d-\mu) = \psi(\alpha+\beta+\gamma+\tau) - \mu\psi = 3\psi g + \zeta - \mu\psi$.

Appendix B. Intersection tables of basic classes

By the preceding appendix, all products in the ring $A^{\cdot}$ of §2 can be derived. In particular, given two elements of complementary dimension we may compute an integer such that the product of these elements is that integer times $[\star]$. (The uniqueness of this coefficient then follows from the theorem in §2).

Each entry in a table denotes the coefficient of $[\star]$ in the product of the entry labelling the row and column.

Table for products of A^1 and A^5

	a	b	c	α	β	γ	d
εb^2c^2	1						
εc^2a^2		1					
εa^2b^2			1				
$\tau\beta^2\gamma^2$				1			
$\tau\gamma^2\alpha^2$					1		
$\tau\alpha^2\beta^2$						1	
ψs^2g							1

Table for products of A^2 and A^4

	a^2	b^2	c^2	α^2	β^2	γ^2	$a\alpha$	$b\beta$	$c\gamma$	τs	εg	$\theta_a s$	$\theta_b s$	$\theta_c s$	$\theta_a g$	$\theta_b g$	$\theta_c g$
b^2c^2	1																
c^2a^2		1															
a^2b^2			1														
$\beta^2\gamma^2$				1													
$\gamma^2\alpha^2$					1												
$\alpha^2\beta^2$						1											
$\tau s^2\alpha$															1		
$\tau s^2\beta$																1	
$\tau s^2\gamma$																	1
$\varepsilon g^2 a$												1					
$\varepsilon g^2 b$													1				
$\varepsilon g^2 c$														1			
$\theta_a s^2 g$							1										
$\theta_b s^2 g$								1									
$\theta_c s^2 g$									1								
ψs^2											1				-1	-1	-1
ψg^2										1		-1	-1	-1			

Table for products of A^3 and A^3

	$a^2\beta$	$a^2\gamma$	$b^2\gamma$	$b^2\alpha$	$c^2\alpha$	$c^2\beta$	abc	$\alpha\beta\gamma$	εa^2	εb^2	εc^2	$\tau\alpha^2$	$\tau\beta^2$	$\tau\gamma^2$	εg^2	τs^2	$\theta_a s^2$	$\theta_b s^2$	$\theta_c s^2$	$\theta_a g^2$	$\theta_b g^2$	$\theta_c g^2$
$a^2\beta$				1																		
$c^2\gamma$					1																	
$b^2\gamma$						1																
$b^2\alpha$	1																					
$c^2\alpha$		1																				
$c^2\beta$			1																			
abc							1	2							1							
$\alpha\beta\gamma$							2	1								1						
εa^2										-1	-1						1					
εb^2									-1		-1							1				
εc^2									-1	-1									1			
$\tau\alpha^2$													-1	-1						1		
$\tau\beta^2$												-1		-1							1	
$\tau\gamma^2$												-1	-1									1
εg^2							1															
τs^2								1														
$\theta_a s^2$									1													
$\theta_b s^2$										1												
$\theta_c s^2$											1											
$\theta_a g^2$												1										
$\theta_b g^2$													1									
$\theta_c g^2$														1								

Appendix C. A Leray–Hirsh theorem for Chow groups

We consider a smooth proper morphism $f : X \longrightarrow Y$ which is locally trivial in the Zariski topology, with fibre F. As in topology, we will assume that there are elements in the Chow ring of the total space X which restrict to a basis for the Chow ring of the fibre, and we want to conclude that these elements give a basis for $A^{\cdot}X$ over $A^{\cdot}Y$, at least when Y is non–singular. Unlike the situation in topology, however, there is no Künneth theorem for Chow groups, so we need to make rather strong assumptions on the fibre. We will assume that F has a filtration by closed subschemes

$$F = F_0 \supset F_1 \supset \ldots \supset F_r = 0$$

such that each $F_i \setminus F_{i+1}$ is a disjoint union of affine spaces. This assumption guarantees that $A^{\cdot}F$ is a free abelian group generated by the closures of these affine spaces (cf. [6] and [11]). We will also assume that *satisfies Poincaré duality*, i.e., the degree map from A_0F to $\mathbb{Z}$ is an isomorphism, and, if $d = \dim(F)$, the intersection pairings

$$A^iF \otimes A^{d-i}F \longrightarrow A^dF \cong A_0F \cong \mathbb{Z}$$

are perfect pairings for all i. Any smooth projective variety with a action by the multiplicative group $\mathbb{G}_m$ with a finite number of fixed points satisfies these conditions ([1], [2], [14]). If the ground field k has characteristic zero, Poincaré duality is automatic from the existence of a filtration, since, when $k = \mathbb{C}$, the map from $A^{\cdot}F$ to cohomology $H^{\cdot}F$ is an isomorphism. In positive characteristic one may use ℓ–adic homology and cohomology to prove analogous statements, although one may need to take coefficients in a field. At any rate, these assumptions are verified for many varieties which occur in enumerative geometry.

It follows easily from the definitions that if F is a variety over k which satisfies the above conditions, then $F_K = F \otimes_k K$ is a variety over K which also satisfies the conditions.

Let $f : X \longrightarrow Y$ be a proper smooth morphism of relative dimension d, locally trivial in the Zariski topology, with fibre F satisfying the above conditions. We assume that for all fibres X_y, the restriction map from $A^{\cdot}(X)$ to $A^{\cdot}(X_y)$ is surjective ; this will be true for all fibres if it holds for one point y in each component of Y, for example the generic point of each irreducible component.

PROPOSITION. *Let $\zeta_1 \ldots \zeta_m$ be homogeneous elements of $A^{\cdot}X$ whose restrictions to fibres form a basis over $\mathbb{Z}$. Then every element in $A.X$ has a unique expression of the form*

$$\sum_{i=1}^{m} \zeta_i \cap f^*\alpha_i \ , \ \alpha_i \in A.Y .$$

Equivalently, the homomorphism

$$\varphi : \bigoplus_{i=1}^{m} A.Y \longrightarrow A.X \ , \ \varphi(\oplus\, \alpha_i) = \Sigma\, \zeta_i \cap f^*\alpha_i \ ,$$

is an isomorphism.

When Y is non-singular, this says that the ζ_i form a free basis for $A^{\cdot}X$ as a module over $A^{\cdot}Y$.

Proof : The proof of surjectivity of φ is the standard argument by Noetherian induction on the dimension of Y. One can assume Y is a variety, with function field K; regard the generic fibre $X_K \cong F \otimes_k K$ as a variety over K. Since $A.(X_K)$ is generated by the images of the ζ_i, one need only consider classes in $A.X$ whose restriction to the generic fibre are zero. Such classes will restrict to zero in $A.(f^{-1}U)$ for some open U in Y, hence will be in the image of $A.(f^{-1}Z)$, where Z is the complement of U in Y. By induction one knows the result for $f^{-1}Z \longrightarrow Z$, and the proof concludes as usual (cf. [6], §1.9).

For the injectivity of φ, let $[\star] \in A^dF$ be the generator corresponding to $1 \in \mathbb{Z}$ by the degree isomorphism. We may assume Y is connected. We first verify that if η is any element of A^dX, and the restriction of η to a fibre is $n[\star]$, for some integer n, then $f_*(\eta \cap f^*\alpha) = n\alpha$ for all $\alpha \in A.Y$. This too is standard. To prove it one may assume $\alpha = [V]$, with V a variety, then replace Y by V, in which case $f_*(\eta \cap f^*\alpha)$ must be $n'[Y]$ for some integer n'; one sees that n' equals n by restricting to a fibre. Similarly one sees that $f_*(\eta \cap f^*\alpha) = 0$ if $\eta \in A^pX$ with $p < d$.

We relabel the elements ζ_i with double subscripts, so that ζ_{pj} are the elements which are in A^pX. Since the restriction from $A^{\cdot}X$ to fibres is assumed to be surjective, we may choose elements ω_{pj} in $A^{d-p}X$ whose restrictions to fibres give the dual basis of the restrictions of ζ_{pj}, i.e., the restriction of $\omega_{pj}\zeta_{pk}$ to a fibre is $\delta_{jk}[\star]$. Now if $\Sigma_{ij}\zeta_{ij} \cap f^*\alpha_{ij} = 0$, consider the maximum p for which some $\alpha_{pk} \neq 0$. By the previous assertions

$$0 = f_*(\omega_{pk}(\Sigma\, \zeta_{ij} \cap f^*\alpha_{ij})) = f_*(\omega_{pk}\zeta_{pk} \cap f^*\alpha_{pk}) + 0 = \alpha_{pk} \ ,$$

which concludes the proof. □

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W. Fulton
The University of Chicago
Department of Mathematics
5734 University Avenue
Chicago, Illinois 60637 (USA)

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MELVIN HOCHSTER
CRAIG HUNEKE

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TIGHT CLOSURE AND STRONG F–REGULARITY

by Melvin HOCHSTER[1] and Craig HUNEKE[1]

This paper is written in celebration of the contributions of Pierre Samuel to commutative algebra.

1. Introduction.

Throughout this paper all rings are commutative, with identity, and Noetherian, unless otherwise specified. In [HH1] and [HH2] the authors introduced the notion of the tight closure of an ideal and the tight closure of a submodule of a finitely generated module for Noetherian rings which are either of positive prime characteristic p or else are algebras essentially of finite type over a field of characteristic 0. This notion enabled us to give new proofs, which are especially simple in characteristic p, of a number of results (not all of which were perceived to be particularly related) : that rings of invariants of linearly reductive groups acting on regular rings are Cohen–Macaulay, that the integral closure of the n^{th} power of an n generator ideal of a regular ring is contained in the ideal (the Briançon–Skoda theorem), of the monomial conjecture, and of the syzygy theorem. The new proofs yield much more general theorems. For example, we can show by these methods that if S is any Noetherian regular ring containing a field and R is a direct summand of S as an R–module (we shall sometimes say, briefly, that R is a *summand* of S to describe this situation : we always mean $R \longrightarrow S$ is R–split) then R is Cohen–Macaulay. This result was not previously known in this generality. Moreover, this illustrates the general principle that results proved using tight closure techniques but which do not refer specifically to tight closure can be extended to the general equicharacteristic case by using Artin approximation to reduce to a situation in which tight closure is defined.

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One of the most important characteristics of tight closure is that in a regular ring every ideal is tightly closed. We call the Noetherian rings all of whose localizations have this property "F–regular". (The "F" in "F–regular stands for "Frobenius" : the reason for this usage will become clear later). This is an important class of rings which includes the rings of invariants of linearly reductive groups acting on regular rings. A key point is that if S is F–regular and R is a direct summand of S as an R–module then R is F–regular. It turns out that, under mild conditions (like being a homomorphic image of a Cohen–Macaulay ring or a weakening of the requirements for excellence), F–regular rings, which are always normal, are Cohen–Macaulay as well. This is the basis for our new proof that direct summands of regular rings are Cohen–Macaulay in the equicharacteristic case.

Our objectives in this paper are, first, to recap briefly some of the features of tight closure, and then to focus on the notion of a "strongly F–regular" ring. It turns out that rings of invariants of reductive groups have, in fact, this stronger property, and that the stronger property has numerous apparent advantages over F–regularity. We should point out right away that we do not know whether the notions of F–regularity and strong F–regularity are really different in good cases. It would be very worthwhile if it could be proved that the two notions coincide.

2. A survey of tight closure.

Unless otherwise specified A , R , and S denote Noetherian commutative rings with 1. By a "local ring" we always mean a Noetherian ring with a unique maximal ideal. R^0 denotes the complement of the union of the minimal primes of R . I and J always denote ideals. Unless otherwise specified given modules M and N are assumed to be finitely generated.

We make the following notational conventions for discussing "characteristic p". We shall always use p to denote a positive prime integer. We shall use e for a variable element of $\mathbb{N}$, the set of nonnegative integers, and q for a variable element of the set $\{p^e : e \in \mathbb{N}\}$.

If R is reduced of characteristic p we write $R^{1/q}$ for the ring obtained by adjoining all q^{th} roots of elements of R : the inclusion map $R \subseteq R^{1/q}$ is isomorphic with the map $F^e : R \longrightarrow R$, where $q = p^e$, F is the Frobenius endomorphism of R and F^e is the e^{th} iteration of F , i.e. $F^e(r) = r^q$. When R is reduced we write R^∞ for the R–algebra $\cup_q R^{1/q}$. Note that R^∞ is an exception to the rule that the rings we consider be Noetherian.

If $I \subseteq R$ and $q = p^e$ then $I^{[q]}$ denotes $(i^q : i \in I) = F^e(I)R$. If S generates I then $\{i^q : i \in S\}$ generates $I^{[q]}$.

We are now ready to define tight closure for ideals in the characteristic p case.

DEFINITION. *Let* $I \subseteq R$ *of characteristic* p *be given. We say that* $x \in I^*$, *the* *__tight closure__* *of* I, *if there exists* $c \in R^o$ *such that* $cx^q \in I^{[q]}$ *for all* $q \gg 0$, i.e. *for all sufficiently large* q *of the form* p^e. *If* $I = I^*$ *we say that* I *is* *__tightly closed__*.

Remarks. Note that if R is a domain, which is the most important case, the condition that $c \in R^0$ is simply the condition that $c \neq 0$. Note also that if R is reduced then $cx^q \in I^{[q]}$ iff $c^{1/q}x \in IR^{1/q}$. Thus, if $x \in I^*$ then for some $c \in R^0$ we have that $c^{1/q}x \in IR^\infty$ for all q (this condition gets stronger as q gets larger). This gives a heuristic argument for regarding x as being "nearly" in I or, at least, IR^∞ : it is multiplied into IR^∞ by elements which, in a formal sense, are getting "closer and closer" to 1 (since $1/q \longrightarrow 0$ as $q \longrightarrow \infty$).

We also note that if R is reduced or if I has positive height it is not hard to show that $x \in I^*$ iff there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for all q.

We extend this notion to finitely generated algebras over a field of characteristic 0 as follows :

DEFINITION. *Let* R *be a finitely generated algebra over a field* K *of characteristic* 0, $I \subseteq R$, *and* $x \in R$. *We say that* x *is in the* *__tight closure__* I^* *of* I *if there exist an element* $c \in R^0$, *a finitely generated* $\mathbb{Z}$*–subalgebra* D *of* K, *a finitely generated* D*–subalgebra* R_D *of* R *containing* x *and* c, *and an ideal* I_D *of* R_D *such that* I_D *and* R_D/I_D *are* D*–free, the canonical map* $K \otimes_D R_D \longrightarrow R$ *induced by the inclusions of* K *and* R_D *in* R *is a* K*–algebra isomorphism,* $I = I_D R$, *and for every maximal ideal* m *of* D, *if* $\kappa = D/m$ *and* p *denotes the characteristic of* κ, *then* $c_\kappa x_\kappa^q \in I_\kappa^{[q]}$ *in* $R_\kappa \cong R_D/mR_D$ *for every* $q = p^e \gg 0$, *where the subscript* κ *denotes images after applying* $\kappa \otimes_D$. *If* $I = I^*$ *we say that* I *is* *__tightly closed__*.

It is not even completely clear from this definition that I^* is an ideal, although it is not difficult to establish. Our attitude in this survey is as follows : we give a number of proofs in characteristic p to illustrate how easy many arguments are while in characteristic 0 we state results but omit discussion of the proofs (generally speaking, the arguments are rather technical but hold few surprises).

We also note that if R is an algebra essentially of finite type over a field K of characteristic 0, and $I \subseteq R$, we can define the *__tight closure__*, I^*, of I as $\cup_B (I \cap B)^*$, where the union is extended over all finitely generated K–subalgebras B of R such that R is a localization of B. However, we shall not discuss the situation for algebras essentially of finite type over a field in any detail in this paper.

The next result shows, among other things, how one uses tight closure to prove that direct summands of regular rings are Cohen–Macaulay (C–M).

2.1. THEOREM. *Let R, S denote Noetherian rings which are either of characteristic p or else essentially of finite type over a field.*

a) *If R is regular, every ideal of R is tightly closed.*

b) *If $R \subseteq S$ are domains and J is tightly closed in S then $J \cap R$ is tightly closed in R. (When R and S are not necessarily domains we may assume instead that $R^0 \subseteq S^0$).*

c) *Let $R \subseteq S$ be domains such that every ideal of R is contracted from S (this holds, in particular, if R is a direct summand of S as an R–module). If every ideal of S is tightly closed then every ideal of R is tightly closed.*

d) *The tight closure of an ideal I of R is contained in the integral closure I^- of I.*

e) *If R is a locally unmixed homomorphic image of a C–M ring and $x_1,\dots,x_n \in R$ have the property that any t of the x's generate an ideal of height $\geq t$, then $(x_1,\dots,x_{n-1}) :_R x_nR \subseteq (x_1,\dots,x_{n-1})R^*$, where $I :_R J = \{r \in R : rJ \subset I\}$.*

Sketch of the proof in characteristic p. a) Suppose that $I \subseteq R$, that R is regular, and that $x \in I^*-I$. By localizing at a prime containing $I :_R xR$ we may assume that (R,m) is local as well. If $cx^q \in I^{[q]}$ for all $q > q'$ then $c \in \cap_{q>q'}(I^{[q]}:x^q) = \cap_{q>q'}(I:x)^{[q]}$ (the flatness of the Frobenius endomorphism for regular rings implies that $(I^{[q]}:x^q) = (I:x)^{[q]}) \subseteq \cap_{q>q'} m^q = (0)$, so that $c = 0$.

b) is immediate from the definition of tight closure and c) is immediate from b).

d) (The reader may want to look at the discussion of integral closure given in (2.8) below before going through this argument). We may use a). Suppose $x \in I^*$ and $c \in R^0$ is such that $cx^q \in I^{[q]}$ for all $q \gg 0$. Let $h : R \longrightarrow V$ with $\ker h$ a minimal prime of R, where V is a DVR. Then $h(c)h(x)^q \in (IV)^{[q]}$ for all $q \gg 0$ and $h(c) \neq 0$, and so $h(x) \in (IV)^* = IV$ (since V is regular), and we are done. On the other hand, we may argue directly as follows: Let $I = (x_1,\dots,x_h)$. Applying the discrete valuation v to the equation $cx^q = \Sigma_{t=1}^h r_{qt}\, x_t^q$ yields $v(c) + qv(x) \geq q \min\{v(x_t):t\}$. Dividing by q and taking the limite as $q \longrightarrow \infty$ yields the result.

e) We shall not prove the result stated in full generality here : we refer the reader to [HH2]. However, we shall give the argument in the special case where the x_i are contained in a regular ring $A \subseteq R$ and R is module–finite over A. In many good cases it is possible to reduce to this case by localizing and completing R and then choosing A properly. In the interesting case (where the x_i do not generate the unit ideal) we may reduce to the situation where A is local, the x_i are part of a system of parameters for A, and R is module–finite over A. The unmixedness hypothesis then translates into the condition that R be torsion–free as an A–module. The result stated then follows from Lemma 2.2 below. QED.

2.2. LEMMA. *Let R be a Noetherian ring of characteristic p module-finite and torsion-free over regular domain A. Let I, J be ideals of A. Then $IR :_R JR \subseteq ((I:_A J)R)^*$ and $IR \cap JR \subseteq ((I \cap J)R)^*$.*

Proof: Let $F \cong A^t$ be an A–free submodule of R whose rank t is equal to the torsion–free rank of R is an A–module. Then R/F is a torsion A–module, and we can choose a nonzero element $c \in A$ such that $cR \subseteq F$. Let $x \in IR:_R JR$ (resp. $IR \cap JR$). Then, for all q, $x^q \in I^{[q]}R :_R J^{[q]}R$ (resp. $I^{[q]}R \cap J^{[q]}R$), whence $cx^q \in I^{[q]}F :_A J^{[q]}$ (resp. $I^{[q]}F \cap J^{[q]}F$). Since F is A–free, we see $cx^q \in (I^{[q]}:J^{[q]})F$ (resp. $I^{[q]} \cap J^{[q]})F$), and by the flatness of the Frobenius endomorphism of A we then have $cx^q \in (I:_A J)^{[q]}F \subseteq ((I:_A J)R)^{[q]}$ (resp. $(I \cap J)^{[q]}F \subseteq ((I \cap J)R)^{[q]}$) for all q, which yields the desired result. QED

This also completes the proof of (2.1e).

We next give a number of corollaries of (2.1) as well as some remarks about how it is used.

We first recall that a ring for which tight closure is defined is called *weakly F-regular* if every ideal is tightly closed and *F-regular* if this is true in all localizations as well. The authors do not know at present whether every weakly F–regular ring is F–regular. With this terminology we have the following corollaries of Theorem 2.1.

2.3. COROLLARY. *Every regular ring is F-regular.*

2.4. COROLLARY. *If $R \subseteq S$ is a direct summand as an R-module and S is F-regular then R is F-regular.*

2.5. COROLLARY. *A weakly F-regular ring which is a homomorphic image of a C–M ring is C–M.*

2.6. THEOREM. *In the equicharacteristic case, a direct summand of a regular ring (as in 2.3) is C–M.*

2.3 is immediate from 2.1a and 2.4 from 2.1c. 2.5 is then clear from 2.1e. 2.6 is obvious from 2.4 and 2.5 in the case where tight closure is defined. In the general case one reduces to the case of complete local rings and then uses Artin approximation to prove a subtle generalization of 2.1e which yields the result. We refer to [HH2] for details.

2.6 includes the result that rings of invariants of linearly reductive groups G over a field K acting K–rationally on a regular K–algebra R have rings of invariants R^G which are C–M. See [HR1], [K], and [B], as well as the discussion of rational singularities following the statement of the Briançon–Skoda theorem below. It is worth noting that in many cases in characteristic p where the group is reductive but not linearly reductive and where, in fact, R^G is not a direct summand of R, it is nonetheless true that R^G is F–regular : for example, the rings defined by the vanishing of the minors of a given size of a matrix of indeterminates are F–regular. See [HH2].

Remark. The theory of tight closure permits a very substantial generalization of 2.1e. Under the same hypothesis one may perform a sequence of operations including sum, product, colon, and intersection on ideals generated by monomials in the x_i. If the ring is C–M (or, more generally, if the x_i form a permutable regular sequence) then it is easy to compute the result of these iterated operations : the x_i behave as though they were indeterminates. It turns out to be extremely useful to be able to restrict the possibilitities for the answer in the general case, when the x_i are parameters but not necessarily a regular sequence. The key point is that the actual ideal resulting from the iterated operations is in the tight closure of what one gets when the x_i form a permutable regular sequence. We should note that there are some restrictions on the use of colon in doing iterated operations : we are not giving the precise statement here. One very special case of this is that $(x_1^t,\dots,x_n^t):(x_1\dots x_n)^{t+1} \subseteq (x_1,\dots,x_n)^*$, which implies the monomial conjecture (and, hence, the direct summand, canonical element, and new improved intersection conjectures, all of which are equivalent : see [H3]).

Remark. If $I \subseteq R$, tight closure is defined in R, and $R \subseteq S$, where S is regular and, for simplicity, a domain, then $I^*S = IS$. This is a remarkable consequence of the theory of tight closure. For example, it implies in the situation of 2.1e that $(x_1,\dots,x_{n-1})R:_R x_nR \subseteq (x_1,\dots,x_{n-1})S$, and similar remarks apply to the iterated operations discussed in the preceding remark. Moreover, one can prove extremely powerful theorems parallel to this one in a somewhat different direction as follows :

2.7. THEOREM (Vanishing theorem). *Let $A \subseteq R \xrightarrow{f} S$ be excellent equicharacteristic rings such that A, S are regular domains, f is injective, and R is module-finite over A. Let M be a finitely generated A–module. Then the map*

$$\mathrm{Tor}_i^A(M,R) \longrightarrow \mathrm{Tor}_i^A(M,S)$$

is 0 *for all* $i \geq 1$.

More general statements are given in [HH2], but (2.7) is already quite strong : the case where $M = A/(x_1,...,x_i)$, where the x_t are s.o.p. in A , and R is a direct summand of S implies that summands of regular rings are Cohen–Macaulay, while the case where $M = A/J$ and S is a DVR dominating the local ring R implies the canonical element conjecture [H3]. The proof of 2.7 uses the notion of the tight closure of a submodule of a module : one shows that certain cycles are in the tight closure of the boundaries and hence are boundaries once one passes to the regular ring S, where every submodule is tightly closed. (In char. p, if $N \subseteq M$ we say $y \in M$ is in N^* if there exists $c \in R^0$ such that for all $e \in \mathbb{N}$, $c(1 \otimes y)$ maps to 0 in $F^e(M/N)$, where F is the Peskine–Szpiro functor [PS, p. 330, Def. 1.2].)

To prove 2.7 in characteristic 0 a statement is needed which can be preserved while applying Artin approximation: in consequence, we prove a more general result in which the condition that f be injective is weakened to the condition that the image of Spec(S) in Spec(R) meet the Cohen–Macaulay locus in Spec(R).

2.8. Discussion. We recall that an element x of a ring R is *integral* over an ideal I provided that there is an integer $k > 0$ and an equation $x^k + i_1x^{k-1} +...+ i_jx^j +...+ i_{k-1}x + i_k = 0$ where $i_j \in I^j$ for $1 \leq j \leq k$. This is easily seen to be equivalent to the assertion that there is an integer $k \geq 1$ such that $x^k \in I(I+Rx)^{k-1}$, and this holds iff $(I+Rx)^k = I(I+Rx)^{k-1}$. From this it is trivial to prove by induction on m that

$$(\#)\ (I+Rx)^{k+m} = I^{m+1}(I+Rx)^{k-1}$$

for every integer $m \in \mathbb{N}$. Thus, x is integral over I iff there exists a positive integer k such that (#) holds for all $m \in \mathbb{N}$.

The integral closure I^- of I is simply the set of elements integral over I, and is an ideal.

Another characterization of integral closure for ideals is given by valuations : let R be a ring with finitely many minimal prime ideals (this is, of course, automatic when R is Noetherian) and $I \subseteq R$. Then x is integral over I iff for every homomorphism h of R into a valuation domain V such that Ker h is a minimal prime of R, $hx \in IV$. If R is Noetherian the same result holds with V restricted to be a discrete valuation ring (by which we always mean a rank one discrete valuation ring).

It is instructive to compare integral closure of ideals with tight closure. If x and y are any two elements of a ring R then $(x^n,y^n)^- \supseteq (x^n,x^{n-1}y,...,x^{n-i}y^i,...,xy^{n-1},y^n) = (x,y)^n$ since the monomial $x^{n-i}y^i$ satisfies $z^n - (x^n)^{n-i}(y^n)^i = 0$. On the other hand, if R is regular or

F–regular, e.g. if $R = K[x,y]$ where K is a field, then $(x^n,y^n)^* = (x^n,y^n)$, since every ideal is tightly closed. Thus, the tight closure is, in general, much smaller than the integral closure. The tight closure is a "tight fit" for the original ideal, which is the reason for the choice of the term.

Suppose we define the regular closure I^{reg} of an ideal I in a Noetherian ring R as follows : $x \in I^{\text{reg}}$ precisely if for every homomorphism $h : R \longrightarrow S$ with S regular such that Ker h is a minimal prime of R, $h(x) \in h(I)S$. Roughly speaking, I^{reg} is the largest ideal which cannot be distinguished from I by maps to regular rings whose kernel is a minimal prime, just as I^- plays this role for maps to valuation rings.

A crucial observation is that $I^* \subseteq I^{\text{reg}}$ whenever tight closure is defined : this is immediate from the definition of tight closure and the fact that every ideal in a regular ring is tightly closed. This explains much of the usefulness of I^*. The trouble with working with I^{reg} itself is that it appears to be very difficult to prove anything interesting about its behavior directly. We have many useful results about I^{reg} all of which are proved by studying I^*. While I^{reg} is defined in mixed characteristic (where I^* is not), we cannot prove anything really useful about it. So far as we know it is possible that $I^* = I^{\text{reg}}$ when I^* is defined : this appears to be a difficult question.

We next note that the theory of tight closure provides an easy proof of the Briançon–Skoda theorem, and, in fact, generalizes it. The theorem was first proved by analytic methods ([BrS]) and later by algebraic techniques ([LT], [LS]), but the argument below is simpler, and, in a certain direction, improves the result.

2.9. THEOREM (generalized Briançon–Skoda theorem). *Let R be a Noetherian ring for which tight closure is defined and let I be an ideal of positive height generated by n elements. Then* $(I^n)^- \subseteq I^*$.

Hence, if R is weakly F–regular and, in particular, if R is regular, then $(I^n)^- \subseteq I$.

Sketch of the proof in characteristic p. Let $a = I^n$. If a is contained in the union of I^* and the minimal primes of R it must be contained in I^*. If not choose y in a not in any minimal prime of R. From (#) in Discussion 2.8 concerning definitions of integral closure we have $y^{k+m} \in a^{m+1}(a+yR)^{k+1} \subseteq a^m$ for a certain integer $k \geq 1$ and all $m \in \mathbb{N}$. Let $c = y^k$, $m = q = p^e$ and note that $a^q = I^{nq} \subseteq I^{[q]}$ (since I has n generators), i.e. $cy^q \in I^{[q]}$ for all q. QED

It is easy to deduce the result for arbitrary equicharacteristic regular rings from the characteristic p case using Artin approximation.

We have already observed that direct summands of regular rings are F–regular and so C–M. This was the basis for our new proof that rings of invariants of linearly reductive groups acting rationally on regular K–algebras are C–M: see [HR1], [K], and [B]. The result of [B] is actually that in the case of algebras finitely generated over a field of characteristic 0 and in the analytic case, direct summands of rings with rational singularities have rational singularities. This suggests a connection between F–regularity and rational singularity. We observe :

2.10. THEOREM. *Suppose that R is of finite type over a field K of char.* 0 *and that $I = I^*$ for all $I \subseteq R$. If either*: a) *R has isolated singularities or* b) *R is $\mathbb{N}$–graded with $R_0 = K$, $m = \bigoplus_{i=1}^{\infty} R_i$, and R has rational singularities except possibly at m, then R has rational singularities.*

In particular, F–regular surfaces have rational singularities. The authors conjecture that all F–regular rings of finite type over a field of char. 0 have rational singularities. The converse is not true : an example of [W] shows that a surface in char. 0 may have rational singularities without even being of F–pure type in the sense of [HR2]. The proof of part a) depends on the fact 2.9 that the conclusion of the Briançon–Skoda theorem is valid in a weakly F–regular ring. A key point in the proof of 2.10a is that an n–dimensional isolated singularity of a local ring (R,m) of an algebra of finite type over a field of char. 0 is rational iff for every m–primary ideal I, $\overline{I^n} \subseteq I$. (One may replace I by an ideal generated by a system of parameters on which it is integrally dependent). An alternative equivalent condition for the isolated singularity to be rational is that for some s.o.p. $x_1,\ldots,x_n$ whose normalized blow–up is regular, $\overline{(x_1^{tn},\ldots,x_n^{tn})} \subseteq (x_1^t,\ldots,x_n^t)$ for all t.

Remark. As we shall see in the next section, a Gorenstein local ring has the property that $I = I^*$ for all I iff the ideal generated by a single s.o.p. is tightly closed. If a Gorenstein local ring has an isolated nonrational singularity it follows that the ideal generated by any s.o.p. is not tightly closed. E.g. in $K[[X,Y,Z]]/(X^2+Y^3+Z^7) = K[[x,y,z]]$ the ideal generated by any two of the elements x, y, z fails to be tightly closed : $x \in (y,z)^* - (y,z)$, $y^2 \in (x,z)^* - (x,z)$, and $z^6 \in (x,y)^* - (x,y)$.

3. Strongly F–regular rings.

The notion of weak F–regularity for a ring R, that every ideal be tightly closed, is clearly a valuable one, but has annoying technical drawbacks. One of the worst of these is that we have not been able to prove that it passes, in general, to localizations. The situation is as follows : the property of weak F–regularity passes to localizations at maximal ideals, and, in the case of algebras of finite type over a field, to localizations at an element. In the Gorenstein case it passes to all localizations. However we have not, in general, been able to obtain the result for localizations at an arbitrary prime, even for algebras of finite type over a field. For this reason, we made the property of passing to localizations part of the definition of F–regularity.

In this section we shall study an *a priori* stronger property in characteristic p that may be equivalent to F–regularity or even to weak F–regularity : we simply do not know. It is only defined for reduced rings R such that $R^{1/p}$ is module–finite over R : however, this class contains finitely generated algebras over a perfect field K and complete local rings R with perfect residue class field K (one only needs that $K^{1/p}$ be finite over K), and so is not too restrictive.

Throughout the rest of this section R denotes a reduced Noetherian ring of positive prime characteristic p such that $R^{1/p}$ is module–finite over R (although we often reiterate this hypothesis in stating theorems). Of course, $R^{1/q}$ is then module–finite over R for all $q = p^e$.

In this note we shall restrict attention to the domain case : very little is lost in doing so, since, in general, a strongly F–regular ring is a finite product of strongly F–regular domains.

DEFINITION. ***We say that a domain*** R ***as above is*** ***<u>strongly</u>*** F-***<u>regular</u>*** ***if for every*** $c \in R^0$ ***there exists*** q ***such that the*** R***-linear map*** $R \longrightarrow R^{1/q}$ ***which sends*** 1 ***to*** $c^{1/q}$ ***splits as a map of*** R***-modules,*** i.e. ***iff*** $Rc^{1/q} \subseteq R^{1/q}$ ***splits over*** R .

Remarks. a) The issue of whether a homomorphism of finitely generated modules over a Noetherian ring splits is local and is unaffected by a faithfully flat extension of the base ring (since the question can be translated into whether a certain map of Hom′s is onto : see [H1]).

b) If $R \subseteq S$ and $f : R \longrightarrow M$ is split by g, where M is an S–module, then $R \subseteq S$ splits : send s to $g(sf(1))$.

c) In the definition above, if a splitting exists for one choice of $c \in R^0$ and q then $R \subseteq R^{1/q'}$ splits for every q'. (It suffices to split $R \subseteq R^{1/p}$ and hence $R \subseteq R^{1/q}$: now use b)).

d) Note also that if $R \longrightarrow R^{1/q}$ sending 1 to $c^{1/q}$ splits for one choice of q, the map $R \subseteq R^{1/q'}$ sending 1 to $c^{1/q'}$ splits for every $q' \geq q$: the map $R \longrightarrow R^{1/q}$ described is isomorphic to the map $R^{q/q'} \longrightarrow R^{1/q'}$ sending 1 to $c^{1/q'}$ and so that map splits over $R^{q/q'}$, and this splitting may be composed with the R–splitting $R \subseteq R^{q/q'}$ whose existence we showed in c).

The following result exhibits a number of the good properties of strong F–regularity.

3.1. THEOREM. *Let R be a Noetherian domain of positive prime characteristic p such that $R^{1/p}$ is module-finite over R.*

a) *R is strongly F-regular iff R_P is strongly F-regular for every prime (respectively, for every maximal) ideal P of R. Hence, if R is strongly F-regular, so is $S^{-1}R$ for every multiplicative system S.*

b) *If S is faithfully flat over R and strongly F-regular then so is R.*

c) *If R is regular, then R is strongly F-regular.*

d) *If R is strongly F-regular, then R is F-regular.*

e) *If R' is strongly F-regular and R is a direct summand of R' as an R-module, then R is strongly F-regular. In particular, a direct summand (as a module over itself) of a regular ring is strongly F-regular.*

f) *If R is weakly F-regular and Gorenstein, then R is strongly F-regular.*

Proof: a) First suppose that R is strongly F–regular. We show that $S^{-1}R$ is strongly F–regular for every S. Let a nonzero element $c = c'/s$ in $S^{-1}R$ be given, where $c' \in R^0$ and $s \in S$. Choose q such that $h : R \to R^{1/q}$ sending 1 to $c'^{1/q}$ has a splitting g. Then $(1/s^{1/q})(S^{-1}h)$ has as a splitting the map which sends x to $(S^{-1}g)(s^{1/q}x)$. (Note that $(S^{-1}R)^{1/q} \cong S^{-1}(R^{1/q})$ canonically).

On the other hand, suppose that R_m is strongly F–regular for every maximal ideal m and let $c \in R^0$ be given. For each maximal ideal m choose $q(m)$ such that the map $R_m \to R_m^{1/q(m)} \cong (R^{1/q(m)})_m$ which sends 1 to $c^{1/q(m)}$ splits. One then gets a splitting for the same $q(m)$ on a Zariski open neighborhood. Taking a finite subcover and the supremum q of the finite set of values of $q(m)$ used in constructing it, we obtain a splitting of h_m, where h is the map $R \to R^{1/q}$ sending 1 to $c^{1/q}$, for every maximal ideal m of R, and this implies that h has a splitting.

b) Suppose $c \in R^0$. Choose q such that $Sc^{1/q} \subseteq S^{1/q}$ splits over S. Since S is faithfully flat over R, $Rc^{1/q} \subseteq R^{1/q}$ splits iff it splits after applying $S \otimes_R$. But we get a splitting by composing $S \otimes_R R^{1/q} \to S^{1/q}$ with the map $S^{1/q} \to Sc^{1/q}$ which splits the inclusion $Sc^{1/q} \subseteq S^{1/q}$.

c) Suppose that R is regular. To prove F–regularity, it suffices to do so locally, by a). We may assume without loss of generality that (R,m) is local. Since R is regular, $R^{1/q}$ is free over R. Let $c \in R^0$ be given and choose q so large that $c \notin m^q$. Then $c^{1/q} \notin m(R^{1/q})$ and so is part of a free basis for $R^{1/q}$ over R. The existence of the required map is then obvious.

d) Suppose $x \in I^*$ and that $c \in R^0$ is such that $cx^q \in I^{[q]}$ for all $q \geq q'$. Then $c^{1/q}x \in IR^{1/q}$ and we can choose q so large that there is an R–linear map $R^{1/q} \to R$ sending $c^{1/q}$ to 1. Applying this map yields that $x \in I$. Thus, $I^* = I$ for every ideal I.

Let $c \in R^0$ and an R'–splitting of the map $R' \to R'^{1/q}$ which sends $c^{1/q}$ to 1 be given. Simply compose with an R–linear map splitting $R \subseteq R'$ and restrict the composition to $R^{1/q}$ to obtain the desired splitting.

f) The issue is local on the maximal ideals of R. Thus, it suffices to prove that if (R,m) is a local Gorenstein domain which is weakly F–regular and $R^{1/q}$ is module–finite over R then for every $c \in R^0$, $Rc^{1/q} \subseteq R^{1/q}$ splits for sufficiently large q. Let $x_1,\ldots,x_d$ be a system of parameters for R and let the image of $u \in R$ generate the socle in $R/(x_1,\ldots,x_d)$. By [H1, Remark 2, pp. 30 and 31], since R is Gorenstein the map $R \to M$ sending 1 to m splits iff $(x_1^t \ldots x_d^t)um \notin (x_1^{t+1},\ldots,x_d^{t+1})M$ for all nonnegative integers t, and when M has depth d this simply is the condition that $um \notin (x_1,\ldots,x_d)M$. We see that it will suffice to choose q such that $uc^{1/q} \notin (x_1,\ldots,x_d)R^{1/q}$, i.e. such that $cu^q \notin ((x_1,\ldots,x_d)R)^{[q]}$. It is possible to do this, since $u \notin (x_1,\ldots,x_d)R = I$ and I is tightly closed. QED

Remark. The argument for part f) actually shows that a Gorenstein local ring is strongly F–regular provided the ideal generated by a single system of parameters is tightly closed. We remark that, quite generally, a Gorenstein local ring is F–regular (not just weakly F–regular) if the ideal generated by one system of parameters is tightly closed : it is not necessary that $R^{1/p}$ be module–finite over R. See [HH2].

Remark. When $R^{1/p}$ is module–finite over the domain R we can always choose $c \in R^0$ such that $(R^{1/p})_c \cong (R_c)^{1/p}$ is free over R_c : for such a c, R_c is regular and, hence, strongly F–regular.

Remark 3.2. Suppose that $R^{1/p}$ is module–finite over the domain R and that $c \in R^0$ is such that R_c is strongly F–regular. Then for every $d \in R^0$ there is an integer $q = p^e$, an integer $t \geq 0$ and an R–linear map $R^{1/q} \to R$ which sends $d^{1/q}$ to c^t. To see this, choose q sufficiently large that there is an R_c–linear map $g : (R^{1/q})_c \to R_c$ such that $g(d^{1/q}) = 1$. Since $R^{1/q}$ is module–finite over R, $c^t g(R^{1/q}) \subseteq R$ for sufficiently large t, and then $c^t g$ restricted to $R^{1/q}$, has the required property. Notice that we may replace t by any larger integer : in particular, we may assume that it is a power of p.

3.3. THEOREM. *Let R be a Noetherian domain of positive prime characteristic p such that $R^{1/p}$ is module-finite over R.*

a) *Let c be any element of R^0 such that R_c is strongly F-regular (such elements always exist). Then R is strongly F-regular if and only if there exists $q = p^e$ such that $R[c^{1/q}] \subseteq R^{1/q}$ splits over R.*

b) *The set $\{P \in \operatorname{Spec} R : R_P$ is strongly F-regular$\}$ is Zariski open in* Spec R.

Proof: a) The existence of such a c is proved in the first of the two remarks just preceding the Theorem. Let $d \in R^0$ be given. Since R_c is strongly F–regular, by Remark 3.2 we can choose an R–linear map $R^{1/q'}$ to R taking $d^{1/q'}$ to $c^{q''}$. The inverse of the iterated Frobenius endomorphism gives an isomorphism of the map $R \subseteq R^{1/q'}$ with the map $R^{1/qq''} \subseteq R^{1/qq'q''}$ and it follows that there is an $R^{1/qq''}$–linear map of $R^{1/qq'q''}$ to $R^{1/qq''}$ which sends $d^{1/qq'q''}$ to $c^{q''/qq''} = c^{1/q}$, taking $(qq'')^{th}$ roots. We may compose this with an $R^{1/q}$–linear map of $R^{1/qq''}$ to $R^{1/q}$ which sends 1 to 1 (and, hence, $c^{1/q}$ to $c^{1/q}$), and then with the R–linear map from $R^{1/q}$ to R which sends $c^{1/q}$ to 1 guaranteed by the hypothesis. This establishes part a).

b) Choose $c \in R^0$ such that R_c is strongly F–regular. Suppose R_P is strongly F–regular for a certain prime P. Then we can choose q and a splitting of $R_P c^{1/q} \subseteq (R_P)^{1/q}$ or, equivalently, of $(Rc^{1/q} \subseteq R^{1/q})_P$, and this splitting extends to a Zariski neighborhood, so that for a certain $d \notin P$, we have a splitting of $(Rc^{1/q} \subseteq R^{1/q})_d$, and it then follows from part a) that R_Q is strongly F–regular for all primes Q with $d \notin Q$. QED

One of the apparently mysterious aspects of tight closure is the nature of the element c such that $cx^q \in I^{[q]}$ for all sufficiently large q. In the definition c is allowed to vary with both x and I. As mentioned earlier, if R is reduced or if I has positive height one can replace the condition "for all sufficiently large q" by the condition "for all q".

It is natural to ask whether, for "good" choices of R, there is an element $c \in R^0$ such that for every ideal $I \subset R$ and every element $x \in R$, $x \in I^*$ iff $cx^q \in I^{[q]}$ for all q. We refer to such an element as a *test element.* It is proved in [HH2, §6] that if R is module–finite, torsion–free and generically smooth over a regular domain A then R has a test element, as does every localization of R. We note that, in particular, every algebra essentially of finite type over a field has a test element.

The constructions of test elements given in [HH2] provide only a very limited class. One of the pleasant consequences of the theory of strong F–regularity is that one can use it to show that every R such that $R^{1/p}$ is module–finite over R has a test element and, in fact, an abundance of test elements: every element in the ideal which defines the locus of primes P where R_P is not strongly F–regular has a power which is a test element. In particular, in the case of an isolated singularity, there is a power of the maximal ideal defining the singular point all of whose elements not in R^0 are test elements. This follows from:

3.4. THEOREM. *Let R be a Noetherian domain of positive prime characteristic p such that $R^{1/p}$ is module-finite over R. Then every element c' of R^0 such that $R_{c'}$ is strongly F-regular has a power which is a test element.*

More precisely, c' has a power c such that there is an R-linear map h of $R^{1/p}$ to R which sends 1 to *c, and for $c \in R^0$ with this property such that R_c is strongly F-regular, c^3 is a test element.*

Proof : The existence of h follows from Remark 3.2 applied with $d = 1$. We next observe that the existence of h implies that for all q, there is an R–linear map $R^{1/q} \to R$ which sends 1 to c^2: ch works if $q = p$, while given such a map $g: R^{1/q} \to R$ we get a map $g': R^{1/pq} \to R^{1/p}$ by taking p^{th} roots which is $R^{1/p}$–linear and sends 1 to $c^{2/p}$. Multiplying by $c^{(p-2)/p}$ yields a map which sends 1 to c and we may then apply h, for $h(c) = c^2$.

Now suppose $x \in I^*$. Then there is a $d \in R^0$ such that $dx^q \in I^{[q]}$ for all q. We must show that c^3 has the same property. As in Remark 3.2 we may choose q' and q'' such that there is an R–linear map of $R^{1/q'}$ to R sending $d^{1/q'}$ to $c^{q''}$. Let q be a varying power of p. Taking $qq''{}^{th}$ roots we obtain an $R^{1/qq''}$–linear map f of $R^{1/qq'q''}$ to $R^{1/qq''}$ sending $d^{1/qq'q''}$ to $c^{1/q}$. Since $dx^{qq'q''} \in I^{[qq'q'']}$ taking $qq'q''{}^{th}$ roots yields that $d^{1/qq'q''}x \in IR^{1/qq'q''}$. Applying the map f we find that $c^{1/q}x \in IR^{1/qq''}$. From the first paragraph we know that there is an R–linear map $g: R^{1/q''} \to R$ sending 1 to c^2 and hence c to c^3. It follows that there is an $R^{1/q}$–linear map $R^{1/qq''} \to R^{1/q}$ sending $c^{1/q}$ to $(c^3)^{1/q}$. Applying this map we see that $(c^3)^{1/q}x \in IR^{1/q}$, and taking q^{th} powers yields exactly the fact we need. QED

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M. Hochster
Department of Mathematics
University of Michigan
ANN ARBOR, MI 48109-1003
(USA)

C. Huneke
Department of Mathematics
Purdue University
WEST LAFAYETTE, IN 47907
(USA)

MÉMOIRES DE LA S. M. F.

N. MOHAN KUMAR

Set-theoretic generation of ideals

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SET–THEORETIC GENERATION OF IDEALS

N. MOHAN KUMAR

(Dedicated to Professor P. Samuel)

SUMMARY

We study the problem of whether a given surface in affine space is a set–theoretic complete intersection. We show, in particular, that surfaces which are birational to a product of curves are set–theoretic complete intersections.

RESUME

On étudie le problème de savoir si une surface donnée dans un espace affine est une intersection complète ensembliste. On démontre en particulier qu'une surface birationellement équivalente à un produit de courbes est une telle intersection.

§0. Introduction.

In this paper, we study set–theoretic generators of ideals in affine algebras. We will be working over an algebraically closed field k. We will prove a sufficient condition for a smooth surface X to be a set–theoretic complete intersection in $\mathbb{A}^n$ $n \geq 5$). This condition is trivially satisfied by a birationally ruled surface. We will show that this condition is satisfied by surfaces birational to product of curves. Spencer Bloch has recently shown to me that this condition is also satisfied by surfaces birational to abelian surfaces.

Another problem we attempt in this article is whether a codimension one subvariety of a smooth affine variety X of dimension n is set–theoretically defined by n–1 equations. The main interest in this problem, at least for the author, is that if this were not so, then one can find stably trivial non–trivial bundles of rank n–1 on such varieties. To see why this case is interesting, the reader may see [3]. Of course, the problem is easy when n=1 or 2. The real difficulty is from n=3. We will show that when $n \geq 3$, a subvariety as above is set–theoretically the zeroes of a section of a stably free, rank n–1 module. For a precise statement, see Theorem 2.

I thank Professors M. Raynaud and L. Szpiro for including me in the Samuel Colloquium. I thank Professor M.P. Murthy for many discussions on the subject matter of this article and Professor Spencer Bloch for showing me how my results apply to the case of surfaces birational to abelian surfaces as well.

§1. Surfaces.

Let $X \subset \mathbb{A}^n$ be a smooth affine surface. Let A denote the coordinate ring of X. Let P = the conormal module of X in $\mathbb{A}^n$.

THEOREM (Boratynski [1]) $X \subset \mathbb{A}^n$ ***is a set-theoretic complete intersection if and only if the ideal*** $S_+(P)$ = ***positively graded elements in*** $R = S(P)$, ***the symmetric algebra of*** P ***over*** A, ***is a set-theoretic complete intersection in*** R.

We say that A satisfies $(\star)$ if for any $z \in A_0(A)$ = zero–cycles modulo rational equivalence, there exists $L_1,\ldots,L_n \in \text{Pic } A$ such that $z = \Sigma_{i=1}^n (L_i.L_i)$, where $(L.L)$ denotes the intersection product in the Chow–ring.

THEOREM 1. *Let A be the co-ordinate ring of a smooth surface. Let P be any A-projective module with rang $P \geq 3$. Let $R = S(P) =$ symmetric algebra of P over A and $I = S_+(P)$, the ideal of positively graded elements. If A satisfies $(\star)$, then I is a set-theoretic complete intersection in R.*

To prove this theorem, we introduce the notion of modifications. Let the notation be as in the theorem. A projective module Q over A is said to be a modification of P, written $Q[P]$, if

i) rank $Q =$ rank P,

ii) there exists an A–algebra homomorphism $f : S(Q) \longrightarrow S(P)$, such that $\text{rad}(f(S_+Q)) = S_+(P)$.

REMARKS :

i) If $Q_2[Q_1]$ and $Q_3[Q_2]$ then $Q_3[Q_1]$.

ii) If $P \approx Q \oplus L$ where $L \in \text{Pic } A$ then $(Q \oplus L^m)[P]$ for any $m \geq 1$.

The first remark is obvious and the second remark follows, once we use the natural map $S(L^m) \longrightarrow S(L)$ for any $m \geq 1$.

PROOF OF THE THEOREM : We need only to show that P can be modified to a free module. Let $L = \det P$. Since $\dim A = 2$ and rank $P \geq 3$, by Serre's theorem [9], there exists a projective module Q such that $P \approx Q \oplus L^{-1}$. Then $\det Q = L^{\otimes 2}$. By remark ii), $Q \oplus L^{-\otimes 2}$ is a modification of P. Also $\det(Q \oplus L^{-\otimes 2}) = A$. Thus we may assume that $\det P = A$. Let $c_2(P) \in A_0(A)$ be the second chern class of P. $A_0(A)$ is divisible [see e.g. [6], Lemma 2.3]. So we may write $c_2(P) = 3z$. Since A satisfies $(\star)$, we may write $z = \Sigma_{i=1}^{n}(L_i.L_i)$ with $L_i \in \text{Pic } A$. Now, the proof is by induction on n. If $n = 0$, then $z = 0$ and by [5], P is free.

We will show that P can be modified to a projective module P' with $\det P' = A$ and $c_2(P) = 3z'$, where $z' = \Sigma_{i=1}^{n-1}(L_i.L_i)$. This will complete the proof.

For notational simplicity let $M = L_n$. As before we may write $P = P_1 \oplus M$. Let c denote the total chern class. Then we have

a) $c(p) = c(P_1).(1+c_1(M))$.

By Remark ii), $P_1 \oplus M^{\otimes 2}$ is a modification of P. Again we may write $P_1 \oplus M^{\otimes 2} = P_2 \oplus M^{\otimes 1}$. Then we have

b) $c(P_1).(1+2c_1(M)) = c(P_2).(1-c_1(M))$.

Again by Remark ii), $P_2 \oplus M^{\otimes 2}$ is a modification of $P_1 \oplus M^{\otimes 2}$ and hence by Remark i), a modification of P. Using a) and b) we may compute $c(P_2 \oplus M^{\otimes 2})$ and then we will get

$$c(P_2 \oplus M^{\otimes 2}) = 2+3z-3(M.M).$$

Thus $P' = P_2 \oplus M^{\otimes 2}$ has all the properties we wanted to achieve. This finishes the proof of the theorem.

COROLLARY 1. (Murthy) *If $X \subset \mathbb{A}^n$, X a smooth surface which is birationally ruled, then X is a set-theoretic complete intersection.*

PROOF : For $n \leq 4$ see [4].

PROPOSITION. *If A is birational to a product of curves then A satisfies* $(\star)$.

PROOF : Let A be birational to $C_1 \times C_2$ where C_i are smooth projective curves. We may also assume that C_i's have positive genus ; if not A is birationally ruled and so A satisfies $(\star)$ trivially. Let Y be a smooth projective completion of $X = \operatorname{Spec} A$. Then we have a birational morphism $\pi : Y \longrightarrow C_1 \times C_2$, by uniqueness of minimal models. Let Z denote the union of exceptional curves of Y. Then Z is the union of rational curves. So the natural map $A_0(X) \longrightarrow A_0(X-Z)$ is an isomorphism. Also $\operatorname{Pic} X \longrightarrow \operatorname{Pic}(X-Z)$ is a surjection. Thus we need only prove $(\star)$ for X an affine open subset of $C_1 \times C_2$.

Now, since $A_0(X)$ is divisible, we may write any zero cycle $z = 2t$. Also, since X is affine, we may write t as a sum of points of X. So it suffices to prove that for any point $p \in X$, $2p = (L.L)$ in $A_0(X)$ where $L \in \operatorname{Pic} X$. Write $p = (p_1,p_2) \in C_1 \times C_2$. Then $M_1 = p_1 \times C_2$ and $M_2 = C_1 \times p_2$ are divisors on $C_1 \times C_2$. $(M_1.M_2) = p$ and $(M_i.M_i) = 0$ for $i = 1,2$ in $A_0(C_1 \times C_2)$. Then $(M_1 \otimes M_2.M_1 \otimes M_2) = 2p$ in $A_0(C_1 \times C_2)$. Restricting $M_1 \otimes M_2$ to X, we get the desired result.

COROLLARY 2. *If $X \subset \mathbb{A}^n$, is a smooth surface birational to a product of curves then X is a set-theoretic complete intersection.*

PROOF : When $n \leq 4$, this was proved by M.P. Murthy [4].

REMARK. Spencer Bloch has shown me that if X is a smooth affine surface birational to an abelian surface, then it satisfies $(\star)$. So our theorem applies and it is also a set–theoretic complete intersection.

§2. Divisors.

This section grew out of an attempt to decide whether stably trivial modules over a 3–fold are trivial or not. Unfortunately, the following theorem that I prove is inconclusive.

For a module M , $\mu(M)$ will denote the minimum number of generators of M .

THEOREM 2. *Let* $Y \subset X = \operatorname{Spec} A$ *be a divisor on a smooth variety* X *of dimension* n *over an algebraically closed field. Assume* $n \geq 3$. *Let* I *be the defining ideal of* Y *in* X . *Then there exists an ideal* $I' \subset I$ *such that*

i) $\operatorname{rad} I' = \operatorname{rad} I$;

ii) $\mu(I'/I'^2) \leq n-1$;

iii) *if* $n = 3$, *there exists a stably trivial module of rank* 2 *mapping onto* I' ;

iv) *if all stably trivial* (rank 2) *modules on all affine* 3–*folds over an algebraically closed field are trivial then we have an* I' *satisfying* i) *above with* $\mu(I') = n-1$, *for any* $n \geq 3$.

PROOF : We will first prove the theorem in the crucial case of $n = 3$. The proof is a judicious application of Ferrand construction [7].

To avoid confusion, let L denote the element in $\operatorname{Pic} A$ corresponding to the divisor Y. That is, L is a module isomorphic to I. Choose a general homomorphism $f : L \longrightarrow A$ so that, $J' = f(L)+I$ is a local complete intersection ideal of height 2. Thus, we have the following Koszul resolution for J' :

$$(\star) \qquad 0 \longrightarrow L^2 \longrightarrow L \oplus I \longrightarrow J' \longrightarrow 0$$

[L^n denotes $L \otimes ... \otimes L$, n times].

Since J' is a local complete intersection ideal of height 2, J'/J'^2 is a projective module of rank 2 over the one–dimensional ring A/J'. So by Serre's theorem [9], we can find a surjective homomorphism, $J'/J'^2 \longrightarrow L^{-6} \otimes A/J'$. Thus we have an exact sequence,

$$(a) \qquad 0 \longrightarrow K/J'^2 \longrightarrow J'/J'^2 \longrightarrow L^{-6} \otimes A/J' \longrightarrow 0$$

where $J'^2 \subset K \subset J'$, K an ideal of A. It is easy to check that K is also a local complete intersection ideal of height 2. So by the above reasoning, we can get another exact sequence

$$\text{(b)} \qquad 0 \longrightarrow J/K^2 \longrightarrow K/K^2 \longrightarrow A/K \longrightarrow 0 .$$

Again J is a local complete intersection ideal of height 2 with $K^2 \subset J \subset K$. So $\operatorname{rad} J = \operatorname{rad} K = \operatorname{rad} J' \supset I$.

Claim : $\operatorname{Ext}^1_A(J,L^{-4}) \simeq A/J$.

Since J is a local complete intersection ideal of height 2, by local checking, one can see that $\operatorname{Ext}^1_A(J,L^{-4})$ is a projective module of rank one over A/J. So to prove the claim it suffices to prove that $\operatorname{Ext}^1_A(J,L^{-4}) \otimes A/J' \simeq A/J'$ since $\operatorname{rad} J = \operatorname{rad} J'$. One has

$$\operatorname{Ext}^1(J,L^{-4}) \simeq \overset{2}{\Lambda}(\operatorname{Hom}(J/J^2,A/J)) \otimes L^{-4} .$$

[See e.g. [10]]. Since one has a natural filtration

$$0 \longrightarrow K^2/KJ \longrightarrow J/KJ \longrightarrow J/K^2 \longrightarrow 0 ,$$

and J/JK is a projective module of rank 2 over A/K, we see that,

$$\overset{2}{\Lambda}(J/J^2) \otimes A/K \simeq J/K^2 \otimes K^2/KJ .$$

But

$$K^2/KJ \simeq K/J \otimes K/J \approx A/K \otimes A/K \approx A/K$$

from (b). Thus

$$\overset{2}{\Lambda}(J/J^2) \otimes A/K \approx J/K^2 \otimes A/K \approx \overset{2}{\Lambda}(K/K^2)$$

from (b). A similar computation done with (a) will yield,

$$\overset{2}{\Lambda}(K/K^2) \otimes A/J' \simeq \overset{2}{\Lambda}(J'/J^2) \otimes L^{-6} .$$

Putting these together, one will get

$$\begin{aligned}\mathrm{Ext}^1(J,L^{-4})\otimes A/J' &\simeq \overset{2}{\Lambda}(K/K^2)\otimes L^{-4}\otimes A/J'\\ &\simeq \overset{2}{\Lambda}(J'/J'^2)^*\otimes L^6\otimes L^{-4}\\ &\simeq \overset{2}{\Lambda}(J'/J'^2)^*\otimes L^2\\ &\simeq \mathrm{Ext}^1(J',L^2).\end{aligned}$$

But $(\star)$ implies $\mathrm{Ext}^1(J,L^2)\simeq A/J'$, proving the claim. Thus, by Serre's construction [8] we get an exact sequence,

$$0\longrightarrow L^{-4}\longrightarrow P\longrightarrow J\longrightarrow 0$$

where P is an A–projective module of rank 2. Computing the chern classes, one has

$$c_1(P)=L^{-4}\ \text{ and }\ c_2(P)=[A/J]=4[A/J']=4(c_1(L).c_1(L)).$$

Thus $c(P)=c(L^{-2}\oplus L^{-2})$. By [2], this implies that P is stably isomorphic to $L^{-2}\oplus L^{-2}$. Tensoring the above exact sequence by L^2 and noting that $L\simeq I$, we get an exact sequence

$$0\longrightarrow L^{-2}\longrightarrow P\otimes L^2\longrightarrow I^2J\longrightarrow 0\,.$$

If we take $I'=I^2J$, then $\mathrm{rad}\,I'=\mathrm{rad}\,I$, since $\mathrm{rad}\,J\subset I$. Thus we have part iii) of the theorem, as well as part i) for $n=3$. By [5], $P\otimes L^2\otimes A/I'$ is free and thus we havee ii) for $n=3$. iv) is now obvious for $n=3$.

Now, to do the general case, let $\dim A=n>3$. Chosse a sufficiently general map,

$$\varphi:\overset{n-3}{\underset{1}{\oplus}}L^{-2}\xrightarrow{\varphi}A\,,$$

L as before, so that $B=A/\mathrm{Im}\,\varphi$ is a smooth 3–dimensional affine ring and $I_1=$ image of I in B is a locally principal ideal of B. From the earlier part, we can find an ideal J_1 of B such that there exists an exact sequence of B–modules

(c) $$0\longrightarrow L^{-4}\otimes B\longrightarrow Q\longrightarrow J\longrightarrow 0$$

with J a local complete intersection ideal of B containing I_1 up to radical and Q a B–projective module of rank 2, stably isomorphic to $(L^{-2}\oplus L^{-2})\otimes B$. Let $J=$ inverse image of J_1 in A and let $I'=I^2.J$. We will show that I' has all the properties asserted in the theorem. Since rad $J_1 \supset I_1$, it is clear that rad $I' =$ rad I . By [5],

$$Q \otimes B/I_1 \simeq (L^{-2} \oplus L^{-2}) \otimes B/I_1 .$$

So we may find an element $f \in A$, $f \equiv 1 \pmod{I}$ such that

$$Q \otimes B_f \simeq (L^{-2} \oplus L^{-2}) \otimes B_f .$$

Notice that by our choice of f ,

$$I'/I'^2 \simeq I'_f/I'^2_f .$$

The map from $Q \otimes B_f \longrightarrow J_{1f}$ can be lifted to a map $(L^{-2} \oplus L^{-2}) \otimes A_f \xrightarrow{\psi} J_f$. Also im $\varphi \subset J_f$ and im $\varphi \otimes A_f +$ im $\psi = J_f$. So we get a surjective map, $\oplus_1^{n-1} L_f^{-2} \longrightarrow J_f$; thus a surjective map

(d) $$\overset{n-1}{\underset{1}{\oplus}} A_f \longrightarrow I_f^2 . J_f = I'_f .$$

So $\mu(I'/I'^2) = \mu(I'_f/I'^2_f) \leq n-1$. This proves ii).

If the hypothesis in iv) were satisfied then we could have chosen $f = 1$. Then (d) implies $I'_f = I'$ is $n-1$ generated. This completes the proof of the theorem.

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N. MOHAN KUMAR
Department of Mathematics
Tata Institute of fundamental research
Homi Bhabba road
BOMBAY 400005 (Inde)

Mémoires de la S. M. F.

Paul Roberts

Local Chern classes, multiplicities, and perfect complexes

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LOCAL CHERN CLASSES, MULTIPLICITIES, AND PERFECT COMPLEXES

Paul ROBERTS

ABSTRACT : We define an invariant associated to a homomorphism of free modules and show, first, that this generalizes the multiplicity in the sense of Samuel and, second, that in the situation we are considering, the local Chern character of a perfect complex can be defined in terms of this invariant. Some questions are raised as to the positivity of these numbers and connections with mixed multiplicities are described.

One of the common methods in studying ideals and modules over a commutative ring has been to define numerical invariants which reflect their properties. In this paper we look at a few of these invariants, which have been defined in various contexts, and describe some relations between them. Let A be a local ring with maximal ideal $\mathfrak{m}$, and let I be an ideal of A primary to the maximal ideal, so that A/I is a module of finite length. This length is the simplest invariant associated to the ideal, and it could be considered to be the most important one, but Samuel [7] defined a somewhat more complicated one, called the multiplicity of I, and showed that it was often more fundamental in studying both algebraic and geometric questions; since then, of course, this has become a standard part of Commutative Algebra.

The comparison of invariants we discuss in this paper is analogous to the comparison of length and multiplicity of an $\mathfrak{m}$–primary ideal. Take now a bounded complex of free A–modules, which we denote F_*. In place of the assumption that I be primary, we assume that the homology of F_* is of finite length. Again, there are two invariants one can associate to F_*. The first is the Euler characteristic, denoted $\chi(F_*)$, which is the alternating sum of lengths of the homology modules. The second was defined by Baum, Fulton and MacPherson and is defined in terms of the local Chern character. This theory has been extended by Fulton [2], and certain applications have made it appear that here also this more complicated invariant may be more fundamental in studying homological questions in Commutative Algebra (see Roberts [5] [6]).

We give here an alternative construction of this invariant. More precisely, we define an invariant of a map of free modules (or of a matrix, if one chooses to look at it that way) with certain properties (corresponding to finite length, specified below). On the one hand, if this map goes to a rank one free module, the image is a primary ideal, and this is the multiplicity of Samuel. On the other hand, the alternating sum of these is the local Chern character in the second example. We define this, which we call the *multiplicity* of the homomorphism, in section 1, and, in the process, we show that the connection with multiplicities is more that simply an analogy, since the definition itself is in terms of the so-called *mixed multiplicities* of ideals of minors of the matrix. In section 2 we show that it does agree with the other invariants mentioned above. In the third section we consider homomorphisms which can be put into a complex of length equal to the dimension of the ring with homology of finite length and ask some questions concerning the properties of this invariant in that case. Finally, in the last section, we work out a couple of special cases to explain how one step of the construction works in practice.

We remark that one motivation behind this work was to investigate the contributions of the individual boundary homomorphisms of a perfect complex (i.e. a bounded complex of free modules) to the local Chern character. The fact that a complex can be divided up in this way was proven in a construction of Fulton ([2], ex. 18.3.12) to prove his local Riemann-Roch theorem. The construction we give here carries this out explicitly, specifies which locally free sheaves occur in the decomposition in terms of determinants, and gives a formula for each contribution in terms of mixed multiplicities. In addition, it is applied to an independent map of free modules, so that, in particular, it is defined whether the map fits into a perfect complex or not. What this number means when the map does not fit into a perfect complex is not clear, but it is interesting that an invariant like this can be defined in this generality.

1. The multiplicity of a homomorphism of free modules.

Let A be a local ring of dimension d and maximal ideal $\mathfrak{m}$, and let $\phi : E \longrightarrow F$ be a homomorphism of free A-modules. We wish to assume that ϕ is generically of constant rank, and, to simplify the situation here, we assume that A is an integral domain. Let r be the generic rank of ϕ. We define the support of ϕ to be the set of prime ideals of A for which the localization at P is not split of rank r, by which we mean that it is not of the form

$$A^s \otimes A^r \longrightarrow A^r \otimes A^t$$

where the map is $\begin{pmatrix} 0 & 1_{A^r} \\ 0 & 0 \end{pmatrix}$. Let e denote the rank of E and f the rank of F. We assume that the support of ϕ is the maximal ideal of A. We wish to define a number associated to ϕ which satisfy the properties outlined in the introduction.

Let M denote the matrix which defines ϕ. We assume that the bases are chosen so that both the first r rows and the first r columns of M have rank r.

We first define two sequences of ideals associated to the matrix M. We note that these are not canonically defined by the map itself, but depend on the bases chosen for E and F (or, more precisely, on filtrations by free direct summands defined by them). First, for $k = 0,1,...,r$ we let $\mathfrak{e}_k$ denote the ideal generated by the k by k minors of the first k rows of M (for $k = 0$ this is defined to be the unit ideal, i.e. A itself; we include this to avoid special cases in later notation). Next, for $k = 0,1,...,r$ we let $\mathfrak{f}_k$ denote the ideal generated by the r by r minors of the first r columns of M which include the first k rows. Note that these ideals are not necessarily $\mathfrak{m}$–primary. We also note that $\mathfrak{e}_r$ and $\mathfrak{f}_0$ are, respectively, the ideals generated by the r by r minors of the first r rows and the first r columns of M.

The invariant we define is in terms of mixed multiplicities, so we next recall some facts on mixed multiplicities of sets of ideals. These were introduced for two ideals by Bhattacharya [1] and later also by Teissier [8], and more recently the definition was extended to a set of d ideals, where d is the dimension of the ring by Rees (see [3]). We briefly recall the situation we need for our construction. This appears to by slightly different than that considered by Rees; he considered d ideals (not necessarily distinct) such that it is possible to choose one element from each of the ideals to form a system of parameters for the ring A. We require instead that at least one of the ideals be $\mathfrak{m}$–primary. So let $\mathfrak{a}_1,...,\mathfrak{a}_n$ be n ideals of A such that $\mathfrak{a}_l$ is $\mathfrak{m}$–primary. If all of the ideals were $\mathfrak{m}$–primary, there would be a polynomial P in n variables of degree d such that we would have

$$P(s_1,...,s_n) = \text{length}(A/\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n})$$

for large values of $s_1,...,s_n$. In our case these lengths are not finite, so this does not make sense. However, since $\mathfrak{a}_l$ is $\mathfrak{m}$–primary, there is still a polynomial P' in n variables of degree $d-1$ such that we have

$$P'(s_1,...,s_n) = \text{length}(\mathfrak{a}_1^{s_1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n}/\mathfrak{a}_1^{s_1+1}\mathfrak{a}_2^{s_2}...\mathfrak{a}_2^{s_2}...\mathfrak{a}_n^{s_n})$$

for large values of $s_1,...,s_n$. In the case in which all ideals are $\mathfrak{m}$–primary, this is the difference $P(s_1 + 1,...,s_n) - P(s_1,...,s_n)$ and one can recover those coefficients of P which invove at least one factor of $\mathfrak{a}_1$. In our case, this gives a well–defined coefficient for each term of the polynomial for which at least one $\mathfrak{m}$–primary factor occurs. We summarize this in the following definition :

DEFINITION. *Let* $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$ *be* n *ideals of* A *such that* $\mathfrak{a}_1,\ldots,\mathfrak{a}_k$ *are* $\mathfrak{m}$*–primary. We call the mixed multiplicity polynomial of* $\mathfrak{a}_1,\ldots,\mathfrak{a}_k;\ \mathfrak{a}_{k+1},\ldots,\mathfrak{a}_n$ *the homogeneous polynomial* P *in* n *variables of degree* d *such that*

(1) *for* $i = 1,\ldots,k$ *we have* $P(s_1,\ldots,s_i + 1,\ldots,s_n) - P(s_1,\ldots,s_i,\ldots,s_n) =$ *the part of degree* $d-1$ *of the polynomial which gives the length of* $\mathfrak{a}_1^{s_1}\ldots\mathfrak{a}_i^{s_i}\ldots\mathfrak{a}_n^{s_n}/\mathfrak{a}_1^{s_1}\ldots\mathfrak{a}_i^{s_i+1}\ldots\mathfrak{a}_n^{s_n}$. *For large* $s_1,\ldots,s_n$,

(2) *all coefficients involving only the last* n - k *variables are zero.*

We make two remarks on this definition. First, it might seem reasonable to call it the Hilbert–Samuel polynomial in analogy with the case of one ideal; the terminology we have chosen is because we have taken only the part of degree d, and these coefficients are (up to certain multinomial coefficients) the mixed multiplicities of the ideals. The second is that the last condition, letting those coefficients which are not well–defined be zero, may seem arbitrary, but it turns out to be exactly what is needed in our formula.

We give an alternative description of the coefficients of the polynomial which will be useful later. We begin by taking the multigraded ring whose $s_1,\ldots,s_n$ component is $\mathfrak{a}_1^{s_1}\,\mathfrak{a}_2^{s_2}\ldots\mathfrak{a}_n^{s_n}$. In conformity with the usual terminology for one ideal, we call this the Rees ring associated to $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$. By taking the projective scheme associated to this, one gets a scheme X proper over Spec A with an imbedding into the product of projective space over Spec(A); this imbedding is defined by choosing a set of generators for each of the ideals. Finally, on X there are invertible sheaves of ideals $O(-A_1),\ldots,O(-A_n)$ associated to divisors $A_1,\ldots,A_n$ defined by the ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_n$. The coefficients of the mixed multiplicity polynomial can then be defined as the degrees of the intersections of these divisors. More precisely, one has coefficient of

$$s_1^{k_1}s_2^{k_2}\ldots s_n^{k_n} = (-1)^{d-1}\Big(\frac{1}{k_1!\ldots k_n!}\Big)A_1^{k_1}A_2^{k_2},\ldots,A_n^{k_n}.$$

In this intersection product one must first take the exceptional divisor corresponding to an ideal which is $\mathfrak{m}$–primary, which reduces the situation to a subscheme which lies over the closed point of Spec(A), and then intersect with the other divisors. In ring–theoretic terms, this can be done by first dividing the Rees ring by the image of one of the ideals which is $\mathfrak{m}$–primary, which reduces the situation to a multigraded ring over an Artinian ring, and then dividing by generic enough elements in appropriate graded pieces of the Rees ring (this works at least if the residue field of A is infinite). The sign occurs because every intersection after the first is with one of the hyperplanes coming from the embedding into a product of projective spaces, and this is the negative of the corresponding exceptional divisor. The mixed multiplicity polynomial can thus be expressed more simply as

$$(-1)^{d-1}(\frac{1}{d!})((A_1s_1 + A_2s_2 + \ldots + A_ns_n)^d - (A_{k+1}s_{k+1} + \ldots + A_ns_n)^d).$$

We remark that this expression is simpler, but that to actually compute the polynomial it is necessary to compute the individual mixed multiplicities. On the other hand, sometimes some of the divisors can be combined and this can be used to simplify the computations.

We now define the invariant of the homomorphism ϕ in terms of mixed multiplicities of the ideals $\mathfrak{e}_k$ and $\mathfrak{f}_k$ defined above plus some other ones defined in terms of these. Let k be an integer between 1 and r. We consider the four ideals $\mathfrak{e}_{k-1}, \mathfrak{e}_k, \mathfrak{f}_{k-1}$, and $\mathfrak{f}_k$. As described above, there is a Rees ring associated to these ideals, as well as an associated projective scheme X proper over Spec(A) with four divisors which we denote E_{k-1}, E_k, F_{k-1}, and F_k. Take the map :

$$O(E_{k-1} - E_k) \longrightarrow O(F_{k-1} - F_k)$$

defined locally as follows : the scheme X is covered by affine pieces corresponding to choices of one generator of each of the ideals $\mathfrak{e}_{k-1}, \mathfrak{e}_k, \mathfrak{f}_{k-1}$, and $\mathfrak{f}_k$. Choose four such generators to be the determinants $\Delta^E_{k-1}, \Delta^E_k, \Delta^F_{k-1}$, and Δ^F_k. The local expression for the map above is then multiplication by the element $\dfrac{\Delta^E_k\Delta^F_{k-1}}{\Delta^E_{k-1}\Delta^F_k}$.

Lemma. *The element* $\dfrac{\Delta^E_k\Delta^F_{k-1}}{\Delta^E_{k-1}\Delta^F_k}$ *is in the coordinate ring defined by the generators* $\Delta^E_{k-1}, \Delta^E_k, \Delta^F_{k-1}$, *and* Δ^F_k *of* $\mathfrak{e}_{k-1}, \mathfrak{e}_k, \mathfrak{f}_{k-1}$, and $\mathfrak{f}_k$.

Proof : What must be shown is that the element in question can be written as a sum of quotients with denominators $\Delta^E_{k-1}\Delta^F_k$ and with numerators products of elements in the original ring times elements in $\mathfrak{e}_{k-1}$ and $\mathfrak{f}_k$. If the minor of M defining Δ^F_{k-1} happens to included the k^{th} row, this is easy to show by expanding Δ^E_k along the k^{th} row. If not, one must first expand Δ^E_k along the k^{th} row and then, for each element α of the k^{th} row of the minor defining Δ^E_k, add the corresponding row and column of this entry to the minor defining Δ^F_{k-1}, and, using the fact that this $r+1$ by $r+1$ determinant must be zero, expand it along the column of α to express it as a sum of other entries in that column multiplied by the corresponding cofactors. When this is all worked out, all terms drop out except those for minors including the first k rows, which are of the desired form.

We assume next that there are $\mathfrak{m}$–primary ideals $\mathfrak{g}_k$ and $\mathfrak{g}_{k-1}$ such that, with notation as above, we have

$$O(E_{k-1} - E_k) \longrightarrow O(F_{k-1} - F_k) \cong O(E_{k-1} - E_k) \otimes (O \longrightarrow O(G_k - G_{k-1})).$$

In many cases these ideals can be calculated explicitly – we give some examples below to show how this works out in practice. We now let P_k be the mixed multiplicity polynomial associated to $\mathfrak{g}_k, \mathfrak{g}_{k-1}; \mathfrak{e}_k, \mathfrak{e}_{k-1}$ (we note that the first two of these are $\mathfrak{m}$–primary but the last two might not be). Then our formula is :

$$m(\phi) = (d!) \sum_{k=1}^{r} P_k(1,-1,1,-1).$$

Actual computation of this number is fairly complicated, but we give some examples later to show that it can be done. We note also that using the last form of the mixed multiplicity polynomial this becomes

$$((G_k - G_{k-1} + E_{k-1} - E_k)^d - (E_{k-1} - E_k)^d).$$

2. Relationships with other invariants.

First, to justify the term multiplicity given to this number, we must show that it agrees with the definition of multiplicity of an ideal. We first do a more general case where the formula simplifies considerably; this is the case of a homomorphism of maximal rank. Recall that $\phi : E \longrightarrow F$ is a homomorphism of free A–modules of ranks e and f respectively. We now assume that r, the rank of ϕ, is equal to f, the rank of F. In this case the matrix defining ϕ has r rows, and the ideals $\mathfrak{f}_k$ defined in the last section are all principal (generated by the same element, the first r by r minor) and this term cancels out in the formulas. Hence we can omit this in the discussion and we left with $O(E_{k-1} - E_k) \longrightarrow O$. In this case we can clearly let $\mathfrak{g}_k = \mathfrak{e}_k$; these ideals are all $\mathfrak{m}$–primary in this case. We note that the formulas give $(G_k - G_{k-1} + E_{k-1} - E_k)^d - (E_{k-1} - E_k)^d$, and in this case the first term is zero so that we are left with $-(E_{k-1} - E_k)^d$, thus if Q_k represents the mixed multiplicity polynomial of $\mathfrak{e}_k, \mathfrak{e}_{k-1}$ then $m(\phi) = Q_1(1,-1) + \ldots + Q_r(1,-1)$.

Now we return to the multiplicity of an $\mathfrak{m}$–primary ideal in the sense of Samuel. In this case we are in the above situation with $r = f = 1$; that is, we have a map from A^e to A defined by a 1 by e matrix whose entries are a set of generators for the ideal. Thus the only determinantal ideal which occurs is $\mathfrak{e}_1$, which is simply the ideal we started with. Hence there we are left with $Q_1(1,-1)$, and since the first ideal is trivial (this is just $\mathfrak{e}_0$) this is just the usual multiplicity of the ideal.

The other connection is with the local Chern character as defined by Fulton. We let now F_* denote a bounded complex of free modules with support the maximal ideal of A (i.e. for every prime ideal other than $\mathfrak{m}$, the localization is (split) exact). In this case one has a number associated to the complex, and if we let $[A]$ denote the fundamental class of Spec(A) in the part of the Chow group of Spec(A) of dimension d, this is $ch(F_*)([A])$. We refer to Fulton ([2]) Chapter 18) for a description of what this is well as the properties this invariant satisfies. It is mostly these properties which we need in the proof we give below.

We first note that the condition on the support of F_* implies that the individual homomorphisms of the complex, which we denote δ_i, (δ_i will be the map from F_i to F_{i-1}) satisfy the hypotheses to make $m(\delta_i)$ defined. The formula we wish to prove is :

$$ch(F_*)([A]) = \Sigma(-1)^i(m(\delta_i)).$$

There are three main steps in this proof. Let r_i be the generic rank of δ_i. The first step is to blow up the ideals of r_i by r_i minors of the matrices defining δ_i to split the complex up into maps of rank r_i locally free sheaves on the blown up scheme X. Next, we show that, by blowing up further, each of these pieces can be filtered with quotients maps of invertible sheaves defined locally by determinants in the ideals $\mathfrak{e}_k$ and $\mathfrak{f}_k$. Finally, we put this together and derive the formula given in the first section. This is similar to the process used by Fulton ([2], Example 18.3.12) to prove his local Riemann–Roch theorem; he shows there that this can be done, at least in theory, and we show here how to carry it out.

We first introduce some notation. We wish to construct a rank r_i locally free quotient Q_i of F_i and a rank r_i locally free subsheaf R_i of F_{i-1} (such that the inclusion of R_i into F_{i-1} is locally split) such that the map δ_i factors through a map ρ_i from Q_i to R_i. The first step, as mentioned above, is to blow up the ideal of r_i by r_i minors of the matrices defining each of the maps δ_i. Call the resulting scheme X and denote the proper map from X to Spec(A) by π. If the quotients Q_i and the subsheaves R_i as above exist, we have a short exact sequence for each i:

$$0 \longrightarrow R_{i+1} \longrightarrow F_i \longrightarrow Q_i \longrightarrow 0.$$

Thus the complex can be broken up over X into the maps $Q_i \xrightarrow{\rho_i} R_i$ and it follows from the additivity of local Chern characters on short exact sequences and the compatibility with proper maps that we have

$$ch(F_*)([A]) = \pi_*(ch(\pi^*(F_*))([X])) = \Sigma\ \pi_*(ch(Q_i \xrightarrow{\rho_i} R_i)([X])).$$

To show that this decomposition does exist it suffices to do it for each i separately, and we now return to our previous notation, replacing F_i, F_{i-1}, Q_i, R_i, ρ_i and δ_i by E, F, Q, R, ρ and ϕ. Let M, as above, be the matrix defining ϕ and let I denote a set of r rows and J a set of r columns of M. For any choice of a set of rows K and a set of columns L we let $M_{K,L}$ denote the submatrix with entries from those rows and columns. We recall that we have blown up the ideal of r by r minors of M. For each set of columns J we take the map $A^e \longrightarrow A^r$ defined by $M_{I,J}^{-1}M_{I,e}$ (where e denotes all e columns; similarly for f below).

Lemma. *This matrix does not depend on the row I chosen.*

Proof : We note that the matrix $M_{I,J}^{-1}M_{I,e}$ has an identity matrix in the J columns, no matter which I is chosen. Since the entire matrix M had rank r, if I' is another set of rows, there is a matrix N at least with entries in the quotient field of A such that

$$M_{I,J}^{-1}M_{I,e} = N(M_{I',J}^{-1}M_{I',e}).$$

But these are the same in the J columns, so N is the identity matrix and these two matrices are the same.

It follows from this lemma that we can take I to be the first r rows of M. Recall that $\mathfrak{e}_r$ is the ideal of r by r minors of the first r rows of M. It then follows from Cramer's rule that the matrix $M_{I,J}^{-1}M_{I,e}$ has entries in the part of the blow up of $\mathfrak{e}_r$ corresponding to the determinant in the J columns. Also, since the matrix $M_{I,J}^{-1}M_{I,e}$ contains an identity matrix in the J columns this map is surjective. Thus we have a quotient onto a rank r locally free sheaf over the blow up of $\mathfrak{e}_r$; this locally free sheaf has transition matrices from J to J' given by $M_{I,J'}^{-1}M_{I,J}$ (as above, this does not depend on I).

We remark here that for this part it was only necessary to blow up $\mathfrak{e}_r$, and not the entire ideal of r by r minors of M. On the other hand, the ideal of all r by r minors is isomorphic to the product $\mathfrak{e}_r\mathfrak{f}_0$, so it would have amounted to the same thing to blow up $\mathfrak{e}_r$ and $\mathfrak{f}_0$ (which we need to do in the next step) instead.

We next define a rank r vector bundle over the blow up of $\mathfrak{f}_0$ and a map which is locally split into $A^f = F$. The maps are indexed by sets I of r rows and the maps are given by $M_{f,J}M_{I,J}^{-1}$. The transition matrices are $M_{I',J}M_{I,J}^{-1}$. As before, it does not depend on which column J is chosen. Furthermore, for each I and J, we can take these maps and put them into a commutative diagram

$$\begin{array}{ccc} A^e & \xrightarrow{M} & A^f \\ M_{I,J}^{-1}M_{I,e} \downarrow & & \uparrow M_{f,J}M_{I,J}^{-1} \\ A^r & \xrightarrow{M_{I,J}} & A^r. \end{array}$$

Denote the rank r quotient of E by Q and the rank r locally free subsheaf of F by R. This diagram says that we have a map from Q to R defined locally by $M_{I,J}$. The support of this map is a closed subscheme of X lying over the maximal ideal of A.

This construction shows that we can split up the complex F_* as claimed above, so we have the formula

$$ch(F_*)([A]) = \pi_*(ch(\pi^*(F_*))([X])) = \Sigma\, \pi_*(ch(Q_i \xrightarrow{\rho_i} R_i)([X]).$$

We note here that this would also give a definition of the multiplicity of a homomorphism of free modules in terms of MacPherson's graph construction for morphisms of locally free sheaves on a blown up scheme; we refer to Fulton ([2], Example 18.1.6) for this construction. In addition, it follows from this part of the proof that the number we define does not depend on choice of basis, since up to now we have blown up only the ideal of all r by r minors of M, and this does not depend on the bases chosen.

We now come to the main part of this section, the fact that the formula we gave in section 1 is the right one. To accomplish this we examine in detail a filtration of the map $\rho : Q \longrightarrow R$ with quotients maps of invertible sheaves.

We define a sequence of quotients Q_k and R_k of Q and R respectively of rank k for each $k = 1,\dots,r-1$ together with compatible maps from Q_k to R_k induced by the map ρ. There will also be maps from Q_k to Q_{k-1} and from R_k to R_{k-1}; their kernels will be invertible sheaves which we denote $\mathscr{L}_k$ and $\mathscr{N}_k$. We then express $\pi_*(ch(Q_i \xrightarrow{\rho_i} R_i)([X]))$ in terms of the induced maps from $\mathscr{L}_k$ to $\mathscr{N}_k$ and this will give the formula.

We first define the Q_k's and the maps between them. This will be done by specifying the transition maps between the local pieces of each locally free sheaf and the local expressions for the maps between the different ones. First, these are defined on the scheme obtained by blowing up certain determinantal ideals, and a local affine piece is defined by choosing one of these, say Δ, and taking the ring generated by all Δ'/Δ, where Δ' is also one of the generators of the ideal. The matrices we define below will have entries which are quotients of determinants of this form (this usually follows directly from Cramer's rule) and we will not go over this point again at each point in the construction.

We first give the local expression for the map from Q to Q_k. Choose a set L of k columns of the matrix M. We denote the k by k matrix with entries the first k rows and the columns in L by $M_{k,L}$. Choose also an r by r submatrix $M_{I,J}$ of M. The local expression for the projection of Q onto Q_k is then $M_{k,L}^{-1}M_{k,J}$. The transition matrix on Q_k from L to L' is $M_{k,L'}^{-1}M_{k,L}$. The transition matrix from I,J to I',J' is the identity map. We verify this last statement : it must be shown that for I,J and I',J' as above, the diagram

$$
\begin{array}{ccccc}
 & A^r & \xrightarrow{M^{-1}_{I,J'}M_{I,J}} & A^r & \\
M^{-1}_{k,L}M_{k,J} & \downarrow & & \downarrow & M^{-1}_{k,L}M_{k,J'} \\
 & A^k & \xrightarrow{\quad I \quad} & A^k &
\end{array}
$$

commutes. Since the map in the top row does not depend on which set of rows I is chosen, we can choose the first r rows, where the commutativity is clear.

It must also be verified that the projections are (locally) surjective; if J contains L, the projection matrix contains a k by k identity matrix, so this is obvious, and the general case can be deduced by using the compatibility in the above diagram to change J.

To define the map from Q_k to Q_{k-1}, we choose sets L_k of k columns and L_{k-1} of $k-1$ columns of M respectively and define the map locally to be given by $(M_{k-1,L_{k-1}})^{-1}M_{k,L_k}$. The fact that the required diagrams commute and the maps are locally surjective follows as above.

We next define the rank k quotients R_k of R and the corresponding maps in this case. It is more convenient here to construct the rank $r-k$ locally free subsheaves which are the kernels of the projections from R to R_k instead, so we do this. We denote this kernel T_{r-k}.

Blow up the ideal $\mathfrak{f}_k$. The r by r determinants generating this ideal have their entries in the first r columns and a set of rows containing the first k; we index this by the set N of r rows. Choose one of these, and an r by r submatrix $M_{I,J}$, and define the imbedding of T_{r-k} into R locally by letting it be given by the matrix $M_{I,J}M^{-1}_{N,J}\begin{pmatrix}0\\ I\end{pmatrix}$, where the last factor is an r by $r-k$ matrix with an identity matrix in the last $r-k$ rows. This is, of course, the same as taking the last $r-k$ columns of $M_{I,J}M^{-1}_{N,J}$.

We next define the transition matrices for T_{r-k}. Take N and N' choices of r rows containing the first k rows, and choose r by r submatrices $M_{I,J}$ and $M_{I',J'}$. We must then find a matrix P such that the following diagram commutes :

$$
\begin{array}{ccccc}
 & A^{r-k} & \xrightarrow{\qquad P \qquad} & A^{r-k} & \\
M_{I',J}M^{-1}_{N',J}\begin{pmatrix}0\\ I\end{pmatrix} & \downarrow & & \downarrow & M_{I,J}M^{-1}_{N,J}\begin{pmatrix}0\\ I\end{pmatrix} \\
 & A^r & \xrightarrow{M_{I,J}M^{-1}_{I',J}} & A^r. &
\end{array}
$$

We define P to be the $r-k$ by $r-k$ submatrix of $M_{N,J}M^{-1}_{N',J}$ defined by choosing the last $r-k$ rows and the last $r-k$ columns. Since the first k rows of $M_{N,J}$ and $M_{N',J}$ are the same $M_{N,J}M^{-1}_{N',J}$ is of the form

$$\begin{pmatrix} I & 0 \\ * & P \end{pmatrix}.$$

Hence one has

$$M_{\mathrm{I,J}}M_{\mathrm{N,J}}^{-1}\begin{pmatrix} 0 \\ I \end{pmatrix} P = M_{\mathrm{I,J}}M_{\mathrm{N,J}}^{-1}\begin{pmatrix} 0 \\ P \end{pmatrix} = M_{\mathrm{I,J}}M_{\mathrm{N,J}}^{-1}(M_{\mathrm{N,J}}M_{\mathrm{N',J}}^{-1})\begin{pmatrix} 0 \\ I \end{pmatrix} = M_{\mathrm{I,J}}M_{\mathrm{N',J}}^{-1}\begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Hence the diagram commutes.

We note also from this that the determinant of P is $\Delta_{\mathrm{N}}/\Delta_{\mathrm{N'}}$.

The maps from T_{r-k} to are (locally) split injections – this can be seen by comparing with the case in which the rows of N other than the first k are contained in I using the above compatibility.

The maps from T_{r-k} to T_{r-k+1} are defined locally by matrices defined analogously to the transition matrices just described : fixing N_k and N_{k-1}, the map from T_{r-k} to T_{r-k+1} is given by the lower right $r-k+1$ by $r-k$ submatrix of the matrix $M_{\mathrm{N}_{k-1},\mathrm{J}}M_{\mathrm{N}_k,\mathrm{J}}^{-1}$. The commutativity of the required diagrams is proven as above. Thus we have locally free sheaves R_k together with compatible maps from R_k to R_{k-1} for $k = 1,\ldots,r$, and we denote the invertible kernels of these maps by $\mathcal{N}_k$. We note that $\mathcal{N}_k$ can also be described as the cokernel of the map from T_{r-k} to T_{r-k+1}.

We must next show that the original map defined by M defines compatible maps from Q_k to R_k, and hence also from $\mathcal{L}_k$ to $\mathcal{N}_k$. We use the following lemma, which simplifies the situation :

Lemma. *Consider the Rees ring of* $\mathfrak{f}_0$, $\mathfrak{f}_k$, *and let* X *denote the corresponding blow up. Then* X *is covered by distinguished open sets corresponding to* (I,N) *where the rows of* N *other than* $1,\ldots,k$ *are in* I.

Proof : Fix I. This part of the blow up is covered by all (I,N) if we put no condition on N other than that it contain the first k rows. Thus it suffices to show that if a bigraded prime ideal of the Rees ring which does not contain Δ_{I} (the Δ_{I} in degree (1,0)) contains Δ_{N} for those N satisfying the condition of the hypothesis it contains all of them. If N has at least one row which is neither one of the first k nor in I, we can use the Plücker identities to write $\Delta_{\mathrm{N}}\Delta_{\mathrm{I}}$ as a sum of products $\Delta_{\mathrm{N'}}\Delta_{\mathrm{I'}}$ where each N' has one more row in common with I than N does. Thus, using induction, one has that $\Delta_{\mathrm{N}}\Delta_{\mathrm{I}}$ is in the prime ideal, and since Δ_{I} is not, Δ_{N} must be in the ideal. This proves the lemma.

We now fix I, N, and L with I and N as in the lemma. Let $\tilde{N}$ denote the set of rows of N other that $1,\ldots,k$; our assumption then states that $\tilde{N}$ is contained in I. We define two matrices U and V as follows :

$U =$ the k by k submatrix of $M^{-1}_{I,J}$ obtained by taking rows in $I-\tilde{N}$ and columns $1,\ldots,k$.

$V =$ the k by $r-k$ submatrix of $M_{I,J}M^{-1}_{N,J}$ obtained by taking rows in $I-\tilde{N}$ and columns $\tilde{N}$.

We claim that the map from Q_k to R_k is locally defined by $UM_{k,L}$. The factor $M_{k,L}$ cancels $M^{-1}_{k,L}$ in the projection from Q to Q_k, and what must be proven is the commutativity of the following diagram :

$$\begin{array}{ccc} A^r & \xrightarrow{M_{I,J}} & A^r \\ M_{k,J}\downarrow & & \downarrow (-I\ V) \\ A^k & \xrightarrow{U} & A^k. \end{array}$$

In representing the local projection of R to R_k by $(-I\ V)$ we have grouped the columns in $\tilde{N}$ at the end. Doing the same for $M_{I,J}M^{-1}_{N,J}$ we represent the $I-\tilde{N}$ rows of $M_{I,J}M^{-1}_{N,J}$ as $(U\ V)$. We then have :

the $I-\tilde{N}$ rows of $M_{I,J} = (U\ V)\begin{pmatrix} M_{k,J} \\ M_{\tilde{N},J} \end{pmatrix} = UM_{k,J} + VM_{\tilde{N},J}$, or $UM_{k,J} =$ (the $I-\tilde{N}$ rows of $M_{I,J}) - VM_{\tilde{N},J}$. This says that the above diagram commutes.

Since the rows and columns omitted from to get U correspond to rows of $\tilde{N}$, which are common to both of them, the determinant of U is Δ_I/Δ_N. Hence the determinant of the map from Q_k to R_k is given locally by $\Delta_L\Delta_{I,J}/\Delta_{N,J}$.

We are now in a position to verify the formula we have for the mixed multiplicities. To do this we list first the determinants of the transition maps for Q_k and R_k and of the local expression for the map from Q_k to R_k. We give the determinants of maps which go from local coordinates corresponding to L and N to those corresponding to L' and N'. From the above discussion, these are, respectively :

For $Q_k : \Delta_L/\Delta_{L'}$.

For $T_{r-k} : \Delta_{N'}/\Delta_N$.

For $Q_k \longrightarrow R_k : \Delta_L\Delta_{I,J}/\Delta_{N,J}$.

We have defined the invertible sheaves $\mathscr{L}_k$ and $\mathscr{N}_k$ as the kernel of the map $Q_k \longrightarrow Q_{k-1}$ and the cokernel of the map $T_{r-k} \longrightarrow T_{r-k+1}$ respectively. Using the multiplicativity of the determinant on short exact sequences gives us the transition maps on $\mathscr{L}_k$ and $\mathscr{N}_k$ and the map from $\mathscr{L}_k$ to $\mathscr{N}_k$; these are :

For $\mathscr{L}_k : \Delta_{L_k}\Delta_{L'_{k-1}}/\Delta_{L'_k}\Delta_{L_{k-1}}$.

For $\mathscr{N}_k : \Delta_{N_k}\Delta_{N'_{k-1}}/\Delta_{N'_k}\Delta_{N_{k-1}}$.

For $\mathscr{L}_k \longrightarrow \mathscr{N}_k : \Delta_{L_k}\Delta_{N_{k-1}}/\Delta_{N_k}\Delta_{L_{k-1}}$.

Now these determinants also define the transition maps for the invertible sheaves E_k and F_k defined in the previous section; more precisely, the transition matrix for coordinates on $O(-E_k)$ are given by $\Delta_L\Delta_{L'}$ (since the local generator at L is Δ_L and at L' is $\Delta_{L'}$ and we have $r\Delta_L = ((\Delta_L/\Delta_{L'})r)\Delta_{L'})$, and similarly for F_k. Putting this together, we have that the map of invertible sheaves from $\mathscr{L}_k$ to $\mathscr{N}_k$ is :

$$O(-E_k + E_{k-1}) \longrightarrow O(-F_k + F_{k-1}).$$

Under the assumptions of section 1 this can also be represented :

$$O(-E_k + E_{k-1}) \otimes (O \longrightarrow O(G_k - G_{k-1})).$$

Using the formula for the local Chern character of a map of invertible sheaves in terms of the exponential map : this is, the local Chern character of a map of invertible sheaves $O(D_1) \otimes (O \longrightarrow O(D_2)$ is $(\sum_{n\geq 0} \frac{D_1^n}{n!})(\sum_{n\geq 1} \frac{D_2^n}{n!})$ (see Fulton [2], Ch. 18), the additivity of local Chern characters and the fact we have proven, that the original map from Q to R has a filtration with given subquotients gives the required formula.

3. Homomorphisms which can be extended to a perfect complex of minimal length.

As mentioned in the introduction, one of the motivations behind this work was to study the contributions to the local Chern character of a perfect complex from the individual boundary maps of the complex. In particular, this was of interest for a perfect complex of length d, where d is the dimension of the ring. It was shown in Roberts [6] that the number one obtains from the local Chern character is positive when the local ring has positive characteristic (and some cases which can be deduced from this one). The question which arises is whether the contributions of the individual boundary maps are positive. We first show that this set of maps of free modules can be described explicitly.

Proposition. *Let ϕ be a homomorphism of free modules with support* $\mathfrak{m}$. *Let i be the smallest integer such that there exists a complex $0 \to E_i \to \ldots \to E_0 \to E \to F$ with homology supported at* $\mathfrak{m}$, *and let j be the integer defined in the same way for the dual of ϕ. Then*

1. $i + j \geq d - 1$.

2. *ϕ is a boundary map of a complex with homology of finite length and of length d if and only if $i + j = d - 1$.*

Proof : If both ϕ and its dual have resolutions as in the hypothesis, the resolution of ϕ and the dual of the resolution of the dual of ϕ can be put together to give a complex with homology of finite length and of length $i + j + 1$. Thus one direction of statement 2 is clear, and the other direction and the inequality of statement 1 are easy consequences of the Peskine–Szpiro Intersection Theorem.

If A is Cohen–Macaulay, there is only one possibility for the complexes of the hypothesis of this Proposition, and that is to take free resolutions of the cokernels. It is also easy to see that in these cases the complex is unique. Is this true in general ? In any case, if there are two resolutions, there cannot exist a map from one to the other lifting the identity, since the mapping cone would again violate the intersection theorem. Another question along the same line is whether, as in the Cohen–Macaulay case, there is a unique best choice for the resolution which can be determined at each stage; that is, for example, whether one can give a criterion for what E_0 must be in terms of ϕ whithout extending the resolution further.

The other questions we raise here concern the positivity of the multiplicity in this case. This is a question even for the case of a map of maximal rank considered in the second section; this should be positive even though the formula involves negative terms. We have seen that there the expression in terms of mixed multiplicities is particularly simple; it is $\Sigma\, Q(1,-1)$ for certain polynomials Q. One could ask if even these components are positive. For dimension 1 this is easy since it is the difference of multiplicities and one ideal is contained in the other. For $d = 2$ it is deeper : in this case it follows from an inequality of Teissier [9] (proven in the general case by Rees and Sharp [4]) which implies that for any two $\mathfrak{m}$–primary ideals this number must be positive. It could be asked whether these numbers are always positive for any two $\mathfrak{m}$–primary ideals where one is contained in the other, but Rees has given some examples (not determinantal ideals of the kind which arise here, however) where they are negative. One could also ask, if ϕ can be extended to a perfect complex of length d, say

$$0 \to F_d \to \ldots \to F_0 \to 0,$$

and if it occurs as the map from F_{i+1} to F_i, whether $(-1)^i m(\phi)$ must be positive. We note that it follows from the above proposition that the integer i is uniquely determined by ϕ.

4. The ideals $\mathfrak{g}_k$ in some special cases.

We work out here two special cases. The first is the opposite extreme from the first one we discussed in section 2; we here look at a homomorphism of rank one. In this case there is only one map of invertible sheaves to consider, which we denote $\mathscr{L} \longrightarrow \mathscr{N}$. The map is defined after blowing up the ideals generated by the first row and the first column of the matrix defining ϕ, and denoting the matrix as (m_{ij}), the map $\mathscr{L} \longrightarrow \mathscr{N}$ is locally defined (see the formula above) by $m_{1j}m_{i1}m_{11}$, and, since the matrix has rank one, $m_{1j}m_{i1}m_{11} = m_{ij}$. Thus if we let $\mathfrak{g}$ be the ideal generated by all entries of M and $\mathfrak{e} = \mathfrak{e}_1$ the ideal generated by the entries in the first row, we have, using notation as above,

$$\mathscr{L} \longrightarrow \mathscr{N} \cong O(-E) \otimes (O \longrightarrow O(G)).$$

Hence the multiplicity is defined in terms of the mixed multiplicities of $\mathfrak{e}$ and $\mathfrak{g}$. If one works out the formula, for example, for the matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ over $k[[X,Y,Z,W]]/(XW-YZ)$, one finds

$$\begin{aligned} m(\phi) &= (G^3 - 3G^2E + 3GE^2) \\ &= (2 - 3(2) - 3(0)) \quad \text{(this is shown by looking at the Rees ring)} \\ &= -4. \end{aligned}$$

We next show a simple example of a Koszul complex; we do the case of the middle morphism in of the Koszul complex on three elements, and we take these elements to be a regular system of parameters, denoted X, Y, Z, for regular local ring. In this case, since each end gives the multiplicity of the maximal ideal of a regular local ring, which is 1, the total alternating sum must be 6, and this term occurs in odd degree, the answer must come out to be -4. This map has rank two and there are two terms in the formula. The matrix is :

$$\begin{pmatrix} -Y & -Z & 0 \\ X & 0 & -Z \\ 0 & X & Y \end{pmatrix}.$$

The ideals are as follows :

$\mathfrak{e}_0$: the unit ideal.
$\mathfrak{e}_1$: (Y,Z)
$\mathfrak{e}_2$: (XY, Y^2, YZ)

$\mathfrak{f}_0$: (X^2, XY, XZ)
$\mathfrak{f}_1$: (XY, XZ)
$\mathfrak{f}_2$: (XZ).

In this case the ideals $\mathfrak{g}_k$ can be found easily; they are :

$\mathfrak{g}_0$: the unit ideal
$\mathfrak{g}_1$: (X, Y, Z)
$\mathfrak{g}_2$: $(X, Y, Z)^2$.

The formula can be simplified a little since $O(G_2 - G_1) \cong O(G_1) \cong O(E_2)$, and we get :

$$
\begin{aligned}
m(\phi) &= ch(O(-E_1) \otimes (O \longrightarrow O(G_1)) + ch(O(-E_2+E_1) \otimes (O \longrightarrow O(G_1)) \\
&= (G_1^3 - 3G_1^2E_1 + 3G_1E_1^2) + (G_1^3 - 3(G_1^2E_1 - G_1^2E_2) + 3(G_1E_1^2 - 2G_1E_1E_2 + G_1E_2^2)) \\
&= (1 - 3(1) + 3(0)) + (1 - 3(1-1) + 3(0-2+1)) \\
&= -4.
\end{aligned}
$$

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Paul ROBERTS
Department of Mathematics
University of Utah
Salt Lake City,
Utah, 84112
U.S.A.