MÉMOIRES DE LA S. M. F.

ANTONIO AMBROSETTI

Critical points and nonlinear variational problems

Mémoires de la S. M. F. 2e série, tome 49 (1992)

<http://www.numdam.org/item?id=MSMF_1992_2_49__1_0>

La chaire Lagrange

La chaire Lagrange est née d'une idée de Monsieur Jean-Claude Arditti, Conseiller Scientifique près l'Ambassade de France à Rome: il s'agissait d'établir en France et en Italie un fonds permettant à un éminent mathématicien de chaque pays de donner un cours dans une Université de l'autre pays, commémorant ainsi la carrière de Joseph Louis Lagrange (1736-1813), né à Turin et professeur dans cette ville, puis à Paris. En outre, ce fonds devait permettre l'accueil d'un jeune chercheur de chaque pays dans une équipe de l'autre pays.

Cette initiative a recueilli le patronage des ministres français et italien chargés de la Recherche, Messieurs Curien et Roberti; le Ministère Français de la Recherche et de la Technologie a assuré le financement français de la première année de ce programme, et en a confié la gestion scientifique et administrative au Comité National Français de Mathématiciens.

C'est le professeur A. Ambrosetti, de l'Ecole Normale Supérieure de Pise, qui a été choisi pour inaugurer la chaire Lagrange: le présent volume est issu du cours qu'il a donné à l'Université Paris-Dauphine en mai et octobre 1991. La subvention du M.R.T. a permis en outre de financer une "Bourse Lagrange" d'un an : cette bourse a été attribuée à Monsieur G. Pareschi.

Au nom du C. N. F. M. , je tiens à exprimer ma vive reconnaissance au Ministère de la Recherche et de la Technologie pour avoir permis le démarrage de ce programme, ainsi qu'à la Société Mathématique de France pour avoir ménagé une place pour les cours de la Chaire Lagrange dans la série des "Mémoires".

Je forme le vœu que la qualité de l'ouvrage du Professeur Ambrosetti contribue à inciter des organismes publics ou privés à nous aider à faire vivre la Chaire et la Bourse Lagrange, renforçant ainsi une coopération mathématique séculaire.

Jean-Michel Lemaire
Président du C.N.F.M.

Société Mathématique de France
Mémoire n° 49
Supplément au Bulletin de la S.M.F.
Tome 120, 1992, fascicule 2

Critical points and nonlinear variational problems

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Abstract. This monograph deals with critical point theory and its applications to some classes of nonlinear variational problems. The abstract setting includes the Lusternik-Schnirelman theory and minimax methods for unbounded functionals. Applications to elliptic boundary value problems, Vortex theory, homoclinic orbits and conservative systems with singular potentials are discussed.

Résumé. Cette monographie traite de la théorie des points critiques et de ses applications à quelques classes de problèmes variationels non linéaires. Le cadre abstrait comprend la théorie de Lusternik-Schnirelman et les méthodes de minimax pour des fonctionelles non bornées. Nous examinons des applications à la théorie des problèmes aux limites elliptiques, à celle du vortex, aux orbites homocliniques, et aux systèmes conservatifs avec potentiels singuliers.

(*) Texte reçu le 13 décembre 1991
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Introduction

This monograph is based on a series of 12 Lectures given, partly in May and partly in October 1991, at CEREMADE (University of Paris IX) in the frame of the 'Chaire Lagrange'.

In these Lectures we intended to discuss some classes of nonlinear problems, variational in nature, with the common feature that their solutions arise as saddle points of suitable functionals.

In the first part, from section 1 to section 5, we deal with the Theory of Critical Points which provides the underlying abstract setting for the applications. Our review covers both the classical Lusternik- Schnirelman theory, as well as the more recent min-max results, such as the Mountain-Pass and the Linking theorems, that permit to handle unbounded functionals. In this part, many proofs are omitted or simply outlined. In some more details we have reported the proofs that do not require many technicalities or that are slightly unusual.

The second part is concerned with applications to nonlinear variational problems.

Semilinear elliptic boundary problems, which motivated much work in critical point theory, are studied in sections 6 and 7. Existence and multiplicity results are discussed, depending on the behaviour of the nonlinearity at zero and at infinity.

In section 8 we report on some recent papers dealing with elliptic equations with discontinuous nonlinearities which model several problems in Plasma Physics. The approach we propose is rather simple and allows us to obtain several precise results.

Another classical problem where discontinuous nonlinearities arise in a natural way is the existence of vortex rings in an ideal axisymmetric fluid, discussed in section 9. A new feature of this problem with respect to the previous ones is that it gives rise to a free boundary problem on an unbounded domain and the solutions are found using a limiting procedure. Our discussion covers both the case of a superlinear 'vorticity function', as well as the case, perhaps more interesting, of a bounded 'vorticity function'. A remarkable feature in the latter is that the solution is found as limit of 'Mountain-Pass' critical points of the functionals related to the approximating problems. The limiting process converges just because of the specific topological features of those Mountain-Pass critical points.

A similar approach is used in section 10 to prove a general result concerning the existence of homoclinic orbits for a class of conservative systems with n degrees of freedom.

In sections 11, 12 and 13 we report on some recent works dealing with Conservative Systems with Singular Potentials. The systems we deal with include, as particular cases, the classical problems of Celestial Mechanics, like perturbation of Kepler's problem or the N-body problem. These classical mechanical systems are studied here from the perspective of Nonlinear Functional Analysis (or, more precisely, of the Calculus of Variations in the Large) rather than that of Celestial Mechanics. Our main interest is not so much in the stability or in other precise properties, perturbative in nature, of specific orbits. Our goal is rather to show that the Critical Point Theory can be adapted to obtain solutions in the large of these classes of problems and to understand which properties of the potentials play the role, and where.

In the second part, many proofs are given in detail, some others are outlined, only a few are omitted for the sake of conciseness. For these latters, however, precise references are given.

The Bibliography does not escape the usual rule of being incomplete. In general, we have listed those papers which are more close to the topics discussed here. But, even for those papers, the list is far from being exhaustive and we apologize for omissions. Further references can be found in [73], [98], [120], or in the monographs [102] and [110].

This paper would never have appeared without the collaboration with several collegues and friends. It is a pleasure to warmly thank all of them, especially Vittorio Coti Zelati, Ivar Ekeland, Mario Girardi, Gianni Mancini, Michele Matzeu, Giovanni Prodi, Paul Rabinowitz and Bob Turner.

I am greatly indebted to CEREMADE for the very kind hospitality.

Last, but not least, I would like to express my gratitude to Marino Badiale, Maria Letizia Bertotti, Ugo Bessi, Anna Maria Candela, Monica Lazzo, Pietro Majer, Lorenzo Pisani and Enrico Serra, for all the very stimulating and fruitfull discussions.

1 Preliminaries

Let E be a (real) Banach space with norm $\|\cdot\|$ and $f : E \to \mathbf{R}$ a C^1 *functional*.

A *critical* (or *stationary*) point of f is a $u \in E$ such that $df(u) = 0$. We say that $c \in \mathbf{R}$ is a *critical level* of f if there exists a critical point u of f such that $f(u) = c$.

If E is a Hilbert space with scalar product $(\cdot|\cdot)$, the gradient f' of f is defined by setting

$$(f'(u)|v) = df(u) \cdot v \quad \forall\, v \in E. \tag{1.1}$$

Hence, in this case, a critical point of f is nothing but a solution of the equation

$$f'(u) = 0 \ .$$

An operator $A : E \to E$ is called *variational* if there exists a functional $f \in C^1(E, \mathbf{R})$ such that $A = f'$.

A problem that is translated into a functional equation $A(u) = 0$ is called a *variational problem* whenever the operator A is a variational operator.

The following examples illustrate the typical kind of nonlinear variational problems we will deal with in the sequel.

Example 1.1 Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ and $E = H_0^1(\Omega)$.

Let $p : \Omega \times \mathbf{R} \to \mathbf{R}$ be a continuous function such that

$$|p(x,s)| \leq a_1 + a_2|s|^{\ell} \tag{1.2}$$

where

$$\ell \le \frac{N+2}{N-2} \text{ if } N>2 \text{ , if } N=1,2 \quad \ell \text{ is unrestricted } .$$

From now on in the rest of the paper we will always consider the case $N > 2$; if $N = 1,2$ the arguments can be carried over with minor changes.

Let $P(x,s) = \int_0^s p(x,\tau)d\tau$. Then

$$|P(x,s)| \le a_3|s| + a_4|s|^{\ell+1}.$$

Since $\ell+1 \le \frac{2N}{N-2} = 2^*$, then $E = H_0^1(\Omega) \subset L^{\ell+1}(\Omega)$ (Sobolev embedding theorem) and it makes sense to define $\phi : E \to \mathbf{R}$,

$$\phi(u) = \int_\Omega P(x,u(x))dx.$$

Moreover, using also (1.2), it is easy to verify that $\phi \in C^1(E,\mathbf{R})$ and

$$d\phi(u)v = \int_\Omega p(x,u)v\ dx.$$

See [25, Ch I, Section 2].

Let us remark, for future reference, that if $\ell < \frac{N+2}{N-2}$ then E is compactly embedded into $L^{\ell+1}(\Omega)$ and this readily implies that the gradient ϕ' of ϕ is compact.

Let $f \in C^1(E,\mathbf{R})$ be defined by setting

$$f(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 dx - \phi(u).$$

According to (1.1), if $u \in E$ is critical point of f then there results

$$(f'(u)|v) = \int_\Omega [\nabla u \cdot \nabla v - p(x,u)v]\, dx = 0, \quad \forall\, v \in H_0^1(\Omega).$$

Hence u is a weak solution of the semilinear Dirichlet boundary value problem

$$\begin{cases} -\Delta u &= p(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

If p is locally Hölder continuous, a standard boot-strap argument shows that u is, in fact, a classical solution. ■

Example 1.2 Let $V : \mathbf{R}^n \to \mathbf{R}$ be a C^1 potential and L a Lagrangian function of the form

$$L(q,\dot q) = \sum_{1\le i,j\le n} a_{ij}(q)\dot q_i \dot q_j - V(q)$$

where dots denote time-derivatives and $a_{ij} \in C^1(\mathbf{R}^n, \mathbf{R})$ satisfy

$$\sum a_{ij}(q)\xi_i\xi_j \geq \alpha|\xi|^2 \ , \ \alpha > 0, \forall\, \xi,\ q \in \mathbf{R}^n.$$

Setting $S_T = [0,T]/\{0,T\}, T > 0$, and $E = H^{1,2}(S_T, \mathbf{R}^n)$, let us define the functional $f : E \to \mathbf{R}$ by:

$$f(u) = \int_0^T L(u, \dot{u})dt.$$

A critical point of f turns out to be a (weak and, by regularity, classical) T-periodic solution of the Lagrangian system

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

We anticipate that in sections 11, 12 and 13 we will deal with gravitational-like potentials. They are not defined on all of $\mathbf{R}^n$ but have rather singularities: an example is the Newtonian potential $-|x|^{-1}$. The functional setting appropriate to these classes of problems will be discussed in the sections above.■

Let us remark explicitely that in the sequel the symbol L will be also used with different meaning. For example, in Sections 6, 7, 8 and 9, it will denote an elliptic partial differential operator.

Let M be a C^1 Riemannian manifold modeled on a Hilbert space E, and let $f \in C^1(M, \mathbf{R})$. A *critical point* of f (*constrained*) on M is a $u \in M$ such that $f'_M(u) = 0$. Here f'_M stands for the gradient of f on M.

In the sequel we will deal with the specific case in which M is a *manifold* of *codimension* 1 *in* E. By this we mean that there is a functional $g \in C^1(E, \mathbf{R})$ such that

$$M = \{u \in E : g(u) = 0\}$$

and $g'(u) \neq 0 \quad \forall\ u \in M$. Here the tangent space to M at u is given by $T_uM = \{v \in E : (g'(u)|v) = 0\}$ and a critical point of f on M is a $u \in M$ such that $(f'(u)|v) = 0$ for all $v \in T_uM$. Hence u satisfies

$$f'(u) = \lambda g'(u)$$

for some $\lambda \in \mathbf{R}$ (*Lagrange multiplier rule*).

In applications, constrained critical points correspond to solutions of variational eigenvalue problems

$$A(u) = \lambda B(u)$$

where A, B are variational operators.

Example 1.3 Let Ω, E and P be as in Example 1.1, and set

$$f(u) = \int_\Omega P(x,u)dx$$

$$g(u) = \int_\Omega |\nabla u|^2 dx - 1.$$

Then M is the Hilbert unit sphere $\{u \in E : \int_\Omega |\nabla u|^2 dx = 1\}$ and the critical points of f on M are the solutions of the semilinear eigenvalue problem

$$\begin{cases} -\lambda \Delta u &= p(x,u) & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

In some cases it can be convenient to look for stationary points of f (i.e. $u \in E$ such that $f'(u) = 0$) as critical points of f constrained on a suitable manifold. Since in the sequel we will be interested to find nontrivial (i.e. $u \neq 0$) critical points of f, we will consider this specific situation only.

Let E be a Hilbert space, $f \in C^2(E, \mathbf{R})$ and set

$$M_f = \{u \in E : u \neq 0, (f'(u)|u) = 0\}.$$

Proposition 1.4 *Let $g(u) = (f'(u)|u)$ and suppose*

$$(g'(u)|u) \neq 0 \quad \forall\, u \in M_f\,. \tag{1.3}$$

Then $f'(u) = 0$, $u \neq 0$, if and only if u is a critical point of f constrained on M_f.

Proof If (1.3) holds, then M_f is a (C^1) manifold of codimension 1 in E. If $u \in M_f$ is such that

$$f'(u) = \lambda g'(u)$$

for some $\lambda \in \mathbf{R}$, it follows

$$(f'(u)|u) = \lambda (g'(u)|u).$$

Since $(f'(u)|u) = g(u) = 0$ and $(g'(u)|u) \neq 0$, then $\lambda = 0$ and $f'(u) = 0$.

Conversely, it suffices to remark that any $u \in E$, $u \neq 0$, satisfying $f'(u) = 0$ belongs to M_f because $g(u) = (f'(u)|u) = 0$. ∎

Let E be, say, a Hilbert space, and let $f \in C^1(E, \mathbf{R})$be a functional which is weakly lower semi-continuous, namely $f(u) \leq \liminf f(u_n)$ for each sequence $u_n \rightharpoonup u$, and coercive, namely

$$f(u_n) \to +\infty \quad \text{as} \quad \|u_n\| \to +\infty \, .$$

Then it is well known that f is bounded from below and attains the (global) minimum: there exists $u^* \in E$ such that

$$f(u^*) = \min\{f(u) : u \in E\}$$

Such a u^* is obviously a critical point of f.

Besides variational problems whose solutions correspond to minima (or maxima), there is a broad variety of cases where one looks for critical points different from minima. This can happen either because the functional f is not bounded from below (nor from above), or because f is not coercive, or because the minimum (exists, but) is not relevant for the problem (for example, it corresponds to the "trivial" solution), or else for some of the preceding reasons together. Another case arises when the problem inherits a symmetry and one expects "many" critical points.

The main goal of the next 4 sections is to discuss some topological tools that will allow us to handle these situations.

Notation.

In addition to those introduced in the preceding section, we list here the notation we will more ordinarily use in the rest of the paper.
If E is a Banach space and $u \in E$, we set $B_r(u) = \{v \in E : \|v - u\| < r\}$ and $\partial B_r(u) = \{v \in E : \|v - u\| = r\}$. For brevity, when $u = 0$ we will simply write B_r (resp. ∂B_r) for $B_r(0)$ (resp $\partial B_r(0)$).

Function Spaces. Let Ω be an open subset of $\mathbf{R}^n$. We denote:

$L^p(\Omega)$, $1 \le p \le +\infty$: Lebesgue spaces;
$|\cdot|_p$: norm in $L^p(\Omega)$;
$H^{k,p}(\Omega)$: Sobolev spaces;
$\|\cdot\|_{k,p}$: norm in $H^{k,p}(\Omega)$;

If M is a manifold on E (see section 1) and $f \in C^1(\Omega, \mathbf{R})$, we set:

$f^c = \{u \in M : f(u) \le c\}$;
$f_a^b = \{u \in M : a \le f(u) \le b\}$;
$K = \{u \in M : f'_M(u) = 0\}$;
$K_c = \{u \in K : f(u) = c\}$.

2 Lusternik-Schnirelman theory.

The Lusternik-Schnirelman Theory is, jointly with the Morse Theory (that will not be discussed in this paper), one of the most classical and powerful tool in Critical Point Theory. In this section we will discuss those results we will need in the sequel. For conciseness reasons we will not carry out the details of many proofs, which would require several technicalities. We will rather attempt to highlight the main ideas of the theory.

Let X be a topological space and $A \subset X \;,\; A \neq \emptyset$. A map $\varphi \in C(A, X)$ is a *deformation* if there is a homotopy $h \in C([0,1] \times A, X)$ such that

$$h(0,\cdot) = \varphi \;,\; h(1,\cdot) = \text{ identity }.$$

A is contractible (to a point u_0) in X if there is a deformation $\varphi \in C(A, X)$ such that $\varphi(u) = u_0$.

The *category* of A relative to X, $cat(A; X)$, is the smallest integer k such that

$$A \subset A_1 \cup \ldots \cup A_k$$

with A_i closed and contractible in X, for each $i = 1, \ldots, k$. If there are no such integers, we set $cat(A; X) = +\infty$. We set also $cat(\emptyset; X) = 0$, and abbreviate $cat(X)$ for $cat(X; X)$.

The main properties of the category are collected in the following

Lemma 2.1 *Let $A, B \subset X$.*
(i) if $A \subseteq B$ then $cat(A; X) \leq cat(B; X)$;
(ii) $cat(A \cup B; X) \leq cat(A; X) + cat(B; X)$;
(iii) if $\varphi \in C(A, X)$ is a deformation and A is closed, then

$$cat(\varphi(A); X) \geq cat(A; X);$$

(iv) let X be an ANR (Absolute Neighbourhood Retract) and $\mathcal{K} \subset X$ be compact. Then $cat(\mathcal{K}; X) < +\infty$ and there exists a neighbourbood U of $\mathcal{K}$ such that $cat(\bar{U}; X) = cat(\mathcal{K}; X)$.

Examples 2.2 (i) Let $S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$. One has: $cat(S^{n-1}) = 2$.
(ii) Let $T^k = S^1 \times S^1 \times \ldots \times S^1$ (k times) denote the k-dimensional torus. There results $cat(T^k) = k + 1$.
(iii) Let us consider the representation of $\mathbf{Z}_2$ over $\mathbf{R}^n$

$$T(0) = \text{ id}, \ T(1) = -\text{ id},$$

and let $\mathbf{P}^n = S^{n-1}/\mathbf{Z}_2$ denote the corresponding projective space. As a consequence of the *Borsuk Antipodensatz* [98, chapter 5] one proves that $cat(\mathbf{P}^n) = n$. In addition, if E is a separable, infinite dimensional Hilbert space and

$$S^\infty = \{u \in E : \|u\| = 1\} \ ,$$

letting

$$\mathbf{P}^\infty = S^\infty / \mathbf{Z}_2 \ ,$$

there results $cat(\mathbf{P}^\infty) = +\infty$.
(iv) Let Λ denote the loop space of those $u \in H^1(S_T, \mathbf{R}^N)$ (see notation introduced in Example 1.2) such that $|u(t)| \equiv 1$. It has been shown [76] that $cat(\Lambda) = +\infty$. ■

The category can be employed to find critical levels of min-max type.

For all $k \le cat(X)$, we consider the class $\mathcal{A}_k$ of all subsets $A \subset X$ such that $cat(A; X) \ge k$ and define

$$c_k = \inf_{A \in \mathcal{A}_k} [\sup\{f(u) : u \in A\}]. \tag{2.1}$$

Note that since $\mathcal{A}_k \supset \mathcal{A}_{k+1}$ then

$$c_1 \le c_2 \le \ldots \le c_k \le c_{k+1} \le \ldots .$$

In order to show that c_k's are critical levels, we suppose that

(A0) $X = M$ is a complete, C^1 Hilbert manifold and $f \in C^1(M, \mathbf{R})$.

In addition, the following compactness condition introduced by Palais and Smale [105] is in order. We say that (M, f) satisfies (PS), or simply that (PS) holds, if *any sequence* $\{u_n\} \subset M$ *such that*

$$|f(u_n)| \leq \text{ const.} \tag{2.2}$$

$$f'_M(u_n) \to 0, \tag{2.3}$$

has a converging subsequence.

Any sequence satisfying (2.2) and (2.3) will be called a *PS-sequence.*

Theorem 2.3 *Let* M, f *satisfy (A0), (PS) and let* f *be bounded from below on* M*:*

$$f(u) \geq a_0, \quad \forall\, u \in M.$$

Then:
(i) *each* $c_k < +\infty$ *is a critical level for* f *on* M*;*
(ii) *if* $c := c_k = c_{k+1} = \ldots = c_{k+m}$ *then* $cat(K_c; M) \geq m+1$*;*
(iii) *if* $c_k = +\infty$ *for some* k*, then* $\sup\{f(u) : u \in K\} = +\infty$*.*
In particular, f *has at least* $cat(M)$ *critical points on* M*.*

If E is finite dimensional and M is compact, theorem 2.3 goes back to Lusternik-Schnirelman [96]. For the extension to infinite dimension (under the assumption that both f and M are C^2) see, for example, [91] and [113]. Palais [104] handled the case of Finsler manifolds M modeled on a Banach space (see also Browder [53]) and C^1 functionals. Finally, Szulkin [123] has weakened the regularity assumption on M, showing that C^1 suffices.

In order to highlight the role of (PS), let us outline the proof of (ii). First of all, one uses (PS) to deduce

Lemma 2.4 *Let* $c \in \mathbf{R}$ *and suppose* (M, f) *verifies (PS). Then*
(i) K_c *is compact;*
(ii) for all $\varepsilon > 0$ *and any neighbourhood* U *of* K_c *there exists* $\alpha > 0$ *such that* $\|f'_M(u)\| \geq \alpha > 0$
for all $u \in f_{c-\varepsilon}^{c+\varepsilon}$ - U*.*

Using Lemma 2.4 (ii) one proves the following

Lemma 2.5 (Deformation Lemma) *Let* f *be as in Theorem 2.3.*
(i) *If* $K_c = \emptyset$, $\forall\, c \in [a, b]$*, then* f^b *can be deformed in* f^a*.*
(ii) Given $c \in \mathbf{R}$, $\varepsilon \in (0, \frac{1}{2}]$ *and any neighbourhood* U *of* K_c *there exist* $\delta \in (0, \varepsilon[$ *and* $\sigma \in C([0,1] \times M, M)$

such that, for all $\delta < d < \frac{1}{2}$, there results

$$\sigma(0,u) = u, \ \ \forall\, u \in M \tag{2.4}$$

$$\sigma(t,u) = u, \ \ \forall\, 0 \le t \le 1, \ \ \forall\, u \notin f_{c-d}^{c+d} \tag{2.5}$$

$$f(\sigma(t,u)) \le f(u), \ \ \forall\, 0 \le t \le 1, \ \ \forall\ u \in M \tag{2.6}$$

$$f(\sigma(1,u)) < c - \delta, \ \ \forall\ u \in f^{c+\delta} - U \tag{2.7}$$

Roughly, if f is C^2, σ is found by using the flow generated by a Cauchy problem like

$$\begin{cases} \sigma' &= X(\sigma) \\ \sigma(0) &= u \end{cases}$$

where $\sigma' = d\sigma/dt$ and X is the locally Lipschitzian vector field such that $X = -f'_M$ in the strip $f_{c-\delta}^{c+\delta}$ and $X = 0$ in the complement of the strip f_{c-d}^{c+d}. If f is merely C^1 one uses the so called *Pseudo-gradient Vector Fields* introduced in [104].

It is worth pointing out that (2.7) follows from Lemma 2.4 (ii).

Finally, the Deformation Lemma is used to define a deformation $\varphi := \sigma(1,\cdot)$ with the property

$$\varphi(A - U) \subset f^{c-\delta}$$

for all $A \subset f^{c+\delta}$ ($\delta > 0$ small enough, U neighbourhood of K_c).

We are now in position to prove the claim (ii) of Theorem 2.3. Suppose, by contradiction, that

$$cat(K_c; M) \le m.$$

Since K_c is compact (Lemma 2.4 (i)) we can use Lemma 2.1 (iv) to find a neighbourhood U of K_c such that

$$cat(\bar{U}; M) = \ cat(K_c; M) \le m \ \ .$$

Using the definition of $c = c_{k+m}$, there is $A \in \mathcal{A}_{k+m}$ such that $A \subset f^{c+\delta}$. Let $A' = A - U$. From Lemma 2.1 (ii) it follows that $\text{cat}(A'; M) \ge cat(A; M) - cat(\bar{U}; M) \ge k + m - m = k$, namely $A' \in \mathcal{A}_k$. Then, for $\varphi(A')$ one has:

$$\varphi(A') \in \mathcal{A}_k \ \ \text{(Lemma 2.1 (iii))}$$

$$\varphi(A') \subset f^{c-\delta} \ \ \text{(from (2.4))}$$

These two relationships are in contradiction with the definition of $c = c_k$. ■

Remarks 2.6 (i) In order to show that a certain value $c \in \mathbf{R}$ is a critical level, one can assume, instead of (PS), the following

$(PS)_c$ any sequence $\{u_n\} \in M$ such that $f(u_n) \to c$ and $f'_M(u_n) \to 0$, has a converging
subsequence.

Actually, $(\text{PS})_c$ suffices to prove the Deformation Lemma.

(ii) Let us state, for future reference, a result which can be deduced from the Deformation lemma. *Let $a < b$ be such that $(PS)_c$ holds for all $c \in [a, b]$, $K_b = \emptyset$ and $cat(f^a; M) < +\infty$. Then $cat(f^b; M) < +\infty$.*

To see this, let $K_a^b = K \cap f_a^b$. Since $(PS)_c$ holds for all $c \in [a, b]$, it follows that K_a^b is compact. Let U be a neighbourhood of K_a^b such that $cat(U; M) = cat(K_a^b; M) < +\infty$ and let u be any point in K_a^b. Applying the Deformation Lemma with $c = f(u)$, and U as above, we find a $\delta = \delta(u)$ such that $f^{c+\delta} - U$ can be deformed in $f^{c-\delta}$. Let $[c_i - \delta_i, c_i + \delta_i]$, $1 \le i \le m$, be a finite covering of $[a, b]$, with $c_m + \delta_m \le b$. Using the properties of the category, it follows that

$$cat(f^{c_i-\delta_i}; M) \ge cat(f^{c_i+\delta_i} - U; M) \ge cat(f^{c_i-\delta_i}; M) - cat(U; M), \qquad (1 \le i \le m).$$

This, together with statement (i) of the Deformation Lemma, yields:

$$\begin{aligned} cat(f^b; M) = cat(f^{c_m+\delta_m}; M) &\le cat(f^{c_1-\delta_1}; M) + m \cdot cat(U; M) \\ &\le cat(f^a; M) + m \cdot cat(U; M) \end{aligned}$$

and the claim follows. ■

Among the possible applications of Theorem 2.3 let us recall the case when M is homeomorphic to the unit sphere S^∞ of an infinite dimensional, separable Hilbert space E, through an *even* homeomorphism. If $0 \notin M$ and $\tilde{M} = M/\mathbf{Z}_2$, then $\tilde{M} \cong \mathbf{P}^\infty$ and $\text{cat}(\tilde{M}) = +\infty$. Let $f \in C^1(E, \mathbf{R})$ be even. Then f induces a C^1 functional $\tilde{f}$ whose critical points on $\tilde{M}$ correspond to pairs of critical points $(u, -u)$ of f on M. Then, if such an f is bounded from below on M one finds

Theorem 2.7 *Suppose (M, f) satisfy (A0) and (PS) and let f be bounded from below on M. Moreover let $0 \notin M$, let M be homeomorphic to S^∞ through an even homeomorphism and let f be even. Then f has infinitely many (pairs of) critical points on M.*

Proof. It suffices to take into account the preceding discussion and note that (M, f) satisfies (PS) whenever $(\tilde{M}, \tilde{f})$ does. Then Theorem 2.3 applies to $(\tilde{M}, \tilde{f})$ and $\tilde{f}$ has $\mathrm{cat}(\tilde{M}) = \mathrm{cat}(\mathbf{P}^\infty) = +\infty$ critical points on $\tilde{M}$. ■

Remark 2.8 A counterexample [91, Chapter VI, section 4] shows that if an even f is perturbed through a non-even functional f_1, then $f + \varepsilon f_1$ can have only a finite number of critical points on, say S^∞. ■

As another application of Theorem 2.3, let us consider a functional $f : E \to \mathbf{R}$ of the form

$$f(u) = \frac{1}{2}\|u\|^2 - \phi(u)$$

where E is a separable, infinite dimensional Hilbert space and $\phi \in C^2(E, \mathbf{R})$.

Let $\psi(u) \equiv (\phi'(u)|u)$. We suppose ϕ satisfies $(A1)$:

(A1.1) $\exists\, 0 < \theta < \frac{1}{2} : \phi(u) \leq \theta(\phi'(u)|u) \forall\, u \in E$;

(A1.2) $\phi(0) = 0$ and $\forall\, u \neq 0$, $\psi(su) = o(s^2)$ as $s \to 0$;

(A1.3) $\forall\, u \neq 0$, $s^{-2}\psi(su) \to +\infty$ as $s \to +\infty$;

(A1.4) $(\phi'(u)|u) < (\phi''(u)u|u)\ \ \forall\, u \neq 0$;

(A1.5) ϕ is weakly continuous; ϕ' and ψ' are compact.

Theorem 2.9 *Suppose $\phi \in C^2(E, \mathbf{R})$ satisfies (A1) and is even. Then $f(u) = \frac{1}{2}\|u\|^2 - \phi(u)$ has infinitely many (pairs of) critical points.*

Proof. We use Proposition 1.4. Here one has

$$g(u) = \|u\|^2 - (\phi'(u)|u)$$

and hence

$$M_f = \{u \in E : u \neq 0,\ \ \|u\|^2 = (\phi'(u)|u)\}.$$

Using (A1.4), for all $u \in M_f$ there results

$$\begin{aligned}(g'(u)|u) &= 2\|u\|^2 - (\phi'(u)|u) - (\phi''(u)u|u) \\ &= (\phi'(u)|u) - (\phi''(u)u|u) < 0. \end{aligned} \tag{2.8}$$

Moreover (A1.2) plainly implies

$$\exists\, \rho > 0 : \|u\| \geq \rho\ \ \forall\, u \in M_f\ . \tag{2.9}$$

Using (A1.1) we find for all $u \in M_f$

$$f(u) = \frac{1}{2}\|u\|^2 - \phi(u) \geq \frac{1}{2}\|u\|^2 - \theta(\phi'(u)|u)$$

$$= \left(\frac{1}{2} - \theta\right) \|u\|^2. \tag{2.10}$$

In particular, since $\theta < \frac{1}{2}$, f is bounded below on M_f. Indeed, from (2.9) $f(u) \geq \left(\frac{1}{2} - \theta\right) \rho > 0 \ \forall\, u \in M_f$.

Next, we claim that (M_f, f) satisfies (PS). Indeed, let $u_n \in M_f$ be such that

$$f(u_n) \leq \text{ const.} \tag{2.11}$$

$$\sigma_n \equiv f'_M(u_n) \to 0. \tag{2.12}$$

Since $\theta < \frac{1}{2}$, (2.10) and (2.11) imply that $\|u_n\| \leq$ const., and $u_n \rightharpoonup \bar{u}$, up to a subsequence. There results

$$f(u_n) = \frac{1}{2}\|u_n\|^2 - \phi(u_n) = \frac{1}{2}(\phi'(u_n)|u_n) - \phi(u_n).$$

Since ϕ is weakly continuous and ϕ' is compact, it follows that $f(u_n) \to f(\bar{u})$. This, jointly with (2.9) and (2.10), yields $f(\bar{u}) > 0$ and hence $\bar{u} \neq 0$. Next, one has

$$\sigma_n = f'(u_n) - \lambda_n g'(u_n)$$

where $\lambda_n = (f'(u_n)|g'(u_n)) \cdot \|g'(u_n)\|^{-2}$. Taking the inner product with u_n and recalling that $(f'(u_n)|u_n) = g(u_n) = 0$, we find

$$\lambda_n(g'(u_n)|u_n) = -(\sigma_n|u_n). \tag{2.13}$$

As $u_n \rightharpoonup \bar{u}$, using the compactness of ϕ' and ψ', one has

$$(g'(u_n)|u_n) = (\phi'(u_n)|u_n) - (\phi''(u_n)u_n|u_n) \to$$

$$\to (\phi'(\bar{u})|\bar{u}) - (\phi''(\bar{u})\bar{u}|\bar{u}) < 0 \tag{2.14}$$

because $\bar{u} \neq 0$. Since $\sigma_n \to 0$, (2.13) and (2.14) imply $\lambda_n \to 0$. Taking into account that $f'(u_n) = u_n - \phi'(u_n)$ and $g'(u_n) = 2u_n - \psi'(u_n)$, it follows

$$(1 - 2\lambda_n)u_n = \phi'(u_n) - \lambda_n \psi'(u_n) + \sigma_n\ .$$

Since $\lambda_n \to 0$, $\sigma_n \to 0$ and ϕ' , ψ' are compact, it follows that $u_n \to \bar{u}$ (up to a subsequence). This proves (PS).

Finally, let us show that M_f is radially diffeomorphic to $S^\infty = \{u \in E : \|u\| = 1\}$. Indeed, $\forall\, u \in S^\infty$ and $s > 0$, one has

$$\gamma(s) \equiv \frac{1}{s^2} g(su) = 1 - \frac{1}{s^2}(\phi'(su)|su).$$

Hence, by $(A1.2-3)$, it follows that the equation $\gamma(s)=0$ has a solution $s>0$. This solution is unique, because

$$\gamma'(s)=\frac{1}{s^3}\left[(\phi'(su)|su)-(\phi''(su)su|su)\right]<0,$$

by $(A1.4)$. Moreover, since ϕ is even, then M_f is symmetric with respect to 0. Let $\tilde{M}_f=M_f/\mathbf{Z}_2$. It follows that $\text{cat}(\tilde{M}_f)=\text{cat}(\mathbf{P}^\infty)=+\infty$, and Theorem 2.7 applies. ■

Example 2.10 Keeping the notation introduced in Example 1.1, we let

$$\|u\|^2=\int_\Omega|\nabla u|^2dx$$

and suppose $p\in C^2(\mathbf{R})$[1] satisfies: (i) $\tilde{p}(s)=s^{-1}p(s)$ is convex; (ii) $\tilde{p}'(s)>0$; (iii) $\tilde{p}\to 0$ (resp. $\to+\infty$) as $s\to 0$ (resp. $+\infty$); (iv) p is odd; and (v) p, $sp'(s)$ and $s^2p''(s)$ satisfy the growth restriction (1.2). It is easy to check that (A1) holds true and Theorem 2.9 applies, yielding the existence of infinitely many solutions of

$$\begin{cases}-\Delta u &= p(u) & \text{in } \Omega\\ u &= 0 & \text{on } \partial\Omega\end{cases}$$

Semilinear elliptic problems will be discussed in greater generality in Sections 6 and 7. ■

When dealing with even functionals on a symmetric (here symmetry means $\mathbf{Z}_2$-symmetry) manifold, an alternative way to proceed is to define on the class

$$\Sigma=\{A\subset E-\{0\}: A \text{ is closed and symmetric}\}$$

a map (called genus) $\gamma:\Sigma\to\mathbf{N}\cup\{+\infty\}$ by setting $\gamma(\emptyset)=0$ and, if $A\neq\emptyset$, by letting $\gamma(A)$ be the smallest integer k such that there exists $\psi\in C(A,\mathbf{R}^k)$, ψ odd and $\psi(u)\neq 0\ \ \forall\, u\in A$. One sets also $\gamma(A)=+\infty$ if there are no integers with the above property.

The genus verifies properties similar to those listed in Lemma 2.1, namely

Lemma 2.11 *Let $A,B\in\Sigma$.*
(i) if A is finite (and non-empty), then $\gamma(A)=1$;
(ii) $\gamma(A)\leq\gamma(B)$ if $A\subset B$;

[1] for simplicity, p is taken to be independent of x

(iii) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;
(iv) $\gamma(\varphi(A)) \geq \gamma(A)$ *if* φ *is continuous and odd;*
(v) *if* $\mathcal{K} \in \Sigma$ *is compact then* $\gamma(\mathcal{K}) < +\infty$ *and there exists a neighbourhood* U *of* $\mathcal{K}$,
$\bar{U} \in \Sigma$, *such that* $\gamma(\bar{U}) = \gamma(\mathcal{K})$;
(vi) *if* $\mathcal{N} \subset \mathbf{R}^n$ *is a bounded, symmetric neighbourhood of* 0 *then* $\gamma(\partial\mathcal{N}) = n$.

Let us recall that Property (vi) above follows from the *Borsuk Antipodensatz.* As a consequence one has

$$\gamma(S^{n-1}) = n$$

as well as

$$\gamma(S^{\infty}) = +\infty.$$

When f is even and $M \in \Sigma$ is a C^1 submanifold of E (namely $M = \{u \in E : u \neq 0, g(u) = 0\}$ with $g \in C^1(E, \mathbf{R})$, $g'(u) \neq 0$, for all $u \in M$ and g even), one can use the genus to define critical levels of min-max type. Let us outline the procedure, starting with the counterpart of (i) and (ii) of Theorem 2.3.

Lemma 2.12 *Let* $M \in \Sigma$ *be a* C^1 *submanifold of* E *and let* $f \in C^1(E, \mathbf{R})$ *be even. Suppose that* M, f *satisfy (PS) and let*

$$\tilde{c}_k = \inf_{\substack{A \in \Sigma \cap M \\ \gamma(A) \geq k}} \left[\max\{f(u) : u \in A\}\right].$$

If $\tilde{c}_k \in \mathbf{R}$ *then* $\tilde{c}_k$ *is a critical value for* f. *Moreover if* $\tilde{c} = \tilde{c}_k = \ldots = \tilde{c}_{k+m}$ *then* $\gamma(K_{\tilde{c}}) \geq m + 1$. *In particular, if* $m > 1$ *then* $K_{\tilde{c}}$ *contains infinitely many critical points.*

The proof can be carried out as in Theorem 2.3, taking into account that: (a) since f is even and $M = g^{-1}(0)$, with g even, then f'_M is odd and the deformation φ found in the Deformation Lemma can be chosen to be odd; (b) this allows us to use Lemma 2.11-(iv); (c) the property $\gamma(K_{\tilde{c}}) > 1$ implies that $K_{\tilde{c}}$ contains infinitely many critical points, in view of Lemma 2.11-(i).

Among others, let us remark that Lemma 2.12 permits to re-obtain Theorem 2.3.

As an application of the preceding arguments, let us consider a functional $f \in C^1(E, \mathbf{R})$ of the form

$$f(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}(Au|u) + g(u) \tag{2.15}$$

where

$$A \in L(E,E) \text{ is positive, selfadjoint and compact} \tag{2.16}$$

$$g(u) = o(\|u\|^2) \text{ at } u = 0. \tag{2.17}$$

Let $0 < \mu_1 \leq \mu_2 \leq \cdots$ denote the characteristic values of A with corresponding orthonormal set of eigenfunctions v_i:

$$\mu_i A v_i = v_i \ .$$

Theorem 2.13 *Let $f \in C^1(E, \mathbf{R})$ be of the form (2.15) with A and g satisfying (2.16) and (2.17), respectively. Moreover suppose f is even, bounded below on E and that (PS) holds.*
Then f has at least k (pairs of) nontrivial critical points $(u_i, -u_i)$, $i = 1, \cdots, k$, whenever $\mu_k < 1 \leq \mu_{k+1}$. Moreover $f(u_i) < 0$.

Proof. Consider the min-max levels $\tilde{c}_j$ defined in Lemma 2.12 (here it is understood that $M = E - \{0\}$) and consider the sets

$$B_{k,\varepsilon} = \left\{ u \in M : u = \sum_{i=1}^{k} \alpha_i v_i, \quad \sum_{i=1}^{k} \alpha_i^2 = \varepsilon^2 \right\}.$$

Plainly, $B_{k,\varepsilon} \simeq S^{k-1}$ and thus $\gamma(B_{k,\varepsilon}) = k$. For $u \in B_{k,\varepsilon}$ one readily finds

$$f(u) = \frac{1}{2}\sum_{i=1}^{k}(1 - \frac{1}{\mu_i})\alpha_i^2 + o(\varepsilon^2).$$

Since $\mu_1 \leq \cdots \leq \mu_k < 1$, it follows

$$f(u) \leq \frac{1}{2}(1 - \frac{1}{\mu_k})\varepsilon^2 + o(\varepsilon^2) < 0 \tag{2.18}$$

whenever $\varepsilon > 0$ is small enough. Since $\gamma(B_{k,\varepsilon}) = k$, (2.18) implies that $\tilde{c}_1 \leq \cdots \leq \tilde{c}_k < 0$. Since f is bounded from below , then $-\infty < \tilde{c}_1$. Lastly, since (PS) holds, then by Lemma 2.12 f has at least k pairs of critical points; they are nontrivial because $f(u_i) = \tilde{c}_i < 0$. This completes the proof. ∎

Let us point out that the procedure discussed above can be carried over in other problems which inherit a symmetry. For example, this is the case dealing with autonomous Hamiltonian Systems, when the corresponding functional f turns out to be S^1 invariant. We will not carry over this kind of problems. The reader is referred, for ex., to [41].

3 The Mountain-Pass Theorem

The purpose of this section is to discuss min-max procedures to find critical points for a class of functionals which are possibly unbounded, both from above and from below.

The first case we will deal with concerns roughly a functional f, defined on a Hilbert E, that has a strict local minimum, say at 0, is negative somewhere else, and satisfies (PS) (by this we obviously mean that condition (PS) introduced in the preceding section holds, with $M = E$). A functional of this kind has been considered in Theorem 2.10, and has the form

$$f(u) = \frac{1}{2}\|u\|^2 - \phi(u)$$

where ϕ is "superquadratic" (see condition A1-i).

The main existence result, the so called Mountain-Pass Theorem [26], has a large variety of applications to concrete problems arising in Mathematical Physics.

A second case is concerned, roughly, with functionals of the form

$$f(u) = \frac{1}{2}(Au|u) - \phi(u)$$

where ϕ is still "superquadratic", but A is possibly not positive definite. This is also a case which arises frequently in applications and will be discussed in Section 4 below.

Let E be a Hilbert [2] space and let $f \in C^1(E, \mathbf{R})$. We suppose that there exist two points u_0 and $u_1 \in E$ and numbers $\rho > 0$ and a such that the following conditions $(A2)$ hold true:

[2]most of the results can be extended to Banach spaces with minor changes.

(A2.1) $f(u) \geq a$ for all $u \in \partial B_\rho(u_0) = \{u \in E : \|u - u_0\| = \rho\}$;
(A2.2) $\|u_0 - u_1\| > \rho$;
(A2.3) $f(u_0)$ and $f(u_1) < a$.

In correspondence to u_0 and u_1 we define

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = u_0 \text{ and } \gamma(1) = u_1\}$$

and

$$c = c(\Gamma, f) = \inf_{\gamma \in \Gamma} [\max\{f(\gamma(t)) : 0 \leq t \leq 1\}]. \tag{3.1}$$

Note that $c < +\infty$.

Theorem 3.1 *Let $f \in C^1(E, \mathbf{R})$ satisfies (A2) and $(PS)_c$. Then f has a critical point $\bar{u} \neq 0$, $\bar{u} \neq u_0, u_1$ such that $f(\bar{u}) = c \geq a$.*

Proof. Let $a' = max\{f(u_0), f(u_1)\}$. Since $\|u_0 - u_1\| > \rho$ then each $\gamma \in \Gamma$ intersects the sphere $\|u - u_0\| = \rho$. Then (A2.1) and (A2.3) imply

$$c \geq \inf\{f(u) : \|u - u_0\| = \rho\} \geq a > a' .$$

To prove that c is a critical level, let us suppose, by contradiction, that $K_c = \emptyset$. Letting $\varepsilon = \min\{c - a', \frac{1}{2}\}$, an application of the Deformation Lemma yields a $0 < \delta < c - a'$ and a $\varphi = \sigma(1, \cdot) \in C(E, E)$ such that

$$\varphi(u) = u, \ \ \forall\, u \in f^{c-d}, \ \ \forall\, \delta < d < c - a' \tag{3.2}$$

$$f(\varphi(u)) < c - \delta \quad \forall\, u \in f^{c+\delta}. \tag{3.3}$$

Since both $f(u_0)$ and $f(u_1)$ are $\leq a' < c - d$, then (3.2) implies $\varphi(u_0) = u_0$ and $\varphi(u_1) = u_1$. Thus $\varphi \circ \gamma \in \Gamma \ \forall\, \gamma \in \Gamma$. By the definition of c, there exists $\gamma \in \Gamma$ such that

$$\max\{f(\gamma(t)) : 0 \leq t \leq 1\} \leq c + \delta.$$

Using (3.3) it follows that $\max\{f(\varphi(\gamma(t))) : 0 \leq t \leq 1\} < c - \delta$, a contradiction because $\varphi \circ \gamma \in \Gamma$. ■

Remark 3.2 It has been shown [51] that $(PS)_c$ can be substituted by the following weaker condition:

$(PS)^*_c$ whenever $\{u_n\} \in M$ is a sequence such that $f(u_n) \to c$ and $f'_M(u_n) \to 0$ then c is a critical value of f. ■

Theorem 3.1 has been improved in [83]. Let $u_0, u_1 \in E$ and C be a closed subset of E. We say that C separates u_0 and u_1 if they belong to disjoint connected components of $E - C$. The following result holds:

Theorem 3.3 *Let $f \in C^1(E, \mathbf{R})$*[3] *and suppose there exist $u_0, u_1 \in E$ and $C \subset E$, closed, such that C separates u_0 and u_1. Let Γ and c be defined as in Theorem 3.1; we assume f satisfies $(PS)_c$ and*

$$c = \max\{f(u_0), f(u_1)\}$$

$$f(u) \geq c \quad \forall\, u \in C \ .$$

Then there is $\bar{u} \in C$, such that $f'(\bar{u}) = 0$ and $f(\bar{u}) = c$.

In some applications it will be useful to sharpen Theorem 3.1 by saying something more on the nature of the critical point found through the min-max procedure (3.1). If $f \in C^2(E, \mathbf{R})$ and u is a critical point of f, we set $E^0 = \ker f''(u)$ and let E^- (resp. E^+) denote the subspaces where $f''(u)$ is negative (positive) definite. The Morse Index, $m(u)$, of u is the dimension of $E^- \oplus E^0$; u is said non-degenerate if $E^0 = \{0\}$.

Theorem 3.4 *Let $f \in C^2(E, \mathbf{R})$ satisfy (A2) and $(PS)_c$ and suppose K_c is discrete. Then there exists $u^* \in K_c$ such that $m(u^*) \leq 1$. Moreover, if $K_c = \{u^*\}$ and u^* is nondegenerate, then $m(u^*) = 1$.*

Theorem 3.4 has been found independently in [85] and [6], in the case when u^* is nondegenerate.

Let us sketch the arguments of the last statement. Without loss of generality we can take $u^* = 0$. By contradiction, let us suppose that $m(u^*) \geq 2$. Then $E = E^- \oplus E^+$, with $\dim E^- \geq 2$ and each u can be written in a canonical form $u = u^- + u^+$, where $u^\mp \in E^\mp$.

If $u^* = 0$ is nondegenerate, by the Morse Lemma one has, up to a regular change of co-ordinates

$$f(u) = c - \|u^-\|^2 + \|u^+\|^2 + R(u)$$

where $R(0) = R'(0) = 0$.
Let U denote the neighbourhood of $u^* = 0$

$$U = \{u = u^- + u^+ : \|u^-\| < \alpha \ , \ \|u^+\| < \beta\} \tag{3.4}$$

[3] Actually the regularity assumption $f \in C^1(E, \mathbf{R})$ can be weakened using the Ekeland *ε-Principle* [72].

where $\beta > \alpha > 0$.
For all $u \in \bar{U}$, such that $\|u^+\| = \beta$, one has

$$f(u) \geq c - \alpha^2 + \beta^2 + o(\alpha^2 + \beta^2)$$

and hence, taking $\beta > \alpha > 0$ small enough, one infers that

$$\inf\{f(u) : u \in \bar{U} \ , \ \|u^+\| = \beta\} \geq d > c \ . \tag{3.5}$$

Let $\delta > 0$ be such that $\delta < d - c$. By the definition of c there exists $\gamma \in \Gamma$ such that $f(\gamma(t)) \leq c + \delta \ \forall\, 0 \leq t \leq 1$. In correspondence to the same $\delta > 0$ and to U given by (3.4) we find $\varphi \in C(E, E)$ such that (cf. the Deformation Lemma)

$$\varphi(f^{c+\delta} - U) \subset f^{c-\delta} \ . \tag{3.6}$$

Plainly, $\varphi \circ \gamma \in \Gamma$. If γ does not intersect U, (3.6) yields immediately a contradiction. Then let $t_0, t_1 \in (0,1)$ be such that $\gamma(t_0) = z_0$ and $\gamma(t_1) = z_1 \in \partial U$ while $\gamma(t) \notin \bar{U} \ \ \forall\, t < t_0$ and $t > t_1$. Since $f(z_i) \leq c + \delta < d$, (3.5) implies $\|z_i^-\| = \alpha \ , \ \|z_i^+\| < \beta$ for $i = 0, 1$. Let σ_i denote the segment joining z_i and z_i^-. It is easy to see that $f_{|\sigma_i} \leq c + \delta$. Lastly, if $\dim E^- \geq 2$ we can connect z_0^- and z_1^- by an arc τ contained in $\partial U \cap E^-$. In particular, $f_{|\tau} < c$. Let $\tilde{\gamma}$ be the path which coincides with γ for $t \in [0, t_0] \cup [t_1, 1]$ and with $\{\sigma_0\} \cup \{\sigma_1\} \cup \{\tau\}$ elsewhere. One has that $\tilde{\gamma} \in \Gamma$, $f_{|\tilde{\gamma}} \leq c + \delta$ and $\{\tilde{\gamma}\} \cap U = \emptyset$. Then, by (3.6), $f_{|\varphi \circ \tilde{\gamma}} \leq c - \delta$, a contradiction. This shows that $m(u^*) \leq 1$. If $m(u^*) = 0$, we could take a neighbourhood U of u^* in such a way that $f_{|\partial U} \geq d > c$ and the conclusion follows as before. ∎

4 Linking Theorems

Let E be a Hilbert space, $E = V \oplus W$ with $\dim V < +\infty$ and let $\bar{w} \in W$ be given, with $\|\bar{w}\| = R$. We set

$$D_R = (\bar{B}_R \cap V) \oplus [0, \bar{w}]$$

and consider a functional $f \in C^1(E, \mathbf{R})$ satisfy $(A3)$:
(A3.1) $f(0) = 0$ and $\exists\ \alpha, r > 0$ such that $f(u) \geq \alpha\ \ \forall\, u \in \partial B_r \cap W$;
(A3.2) $\exists\ R > r$ such that $f(u) \leq 0\ \ \forall\, u \in \partial D_R$[4].

Let

$$\tilde{\Gamma} = \{h \in C(D_R, E) : h(u) = u \ \text{ for all } \ u \in \partial D_R\}$$

The following Lemma can be proved by means of topological degree arguments (taking advantage that $\dim V < +\infty$).
Lemma 4.1 *For all $r < R$ and all $h \in \tilde{\Gamma}$, there results*

$$h(D_R) \cap (\partial B_r \cap W) \neq \emptyset \ .$$

From Lemma 4.1 one infers that

$$\max\{f(h(u)) : h \in \tilde{\Gamma}\} \geq \inf\{f(u) : u \in \partial B_r \cap W\} \ . \tag{4.1}$$

Define

$$\tilde{c} = \inf_{h \in \tilde{\Gamma}} [\max\{f(h(u)) : u \in D_R\}] \ .$$

From (4.1) and using (A3.1) it follows that $\tilde{c} \geq \alpha > 0$. Suppose f satisfies $(PS)_{\tilde{c}}$ and let $K_{\tilde{c}} = \emptyset$. We apply the Deformation Lemma with $c = \tilde{c}$ and

[4] ∂D_R stands for the boundary of D_R relative to $V \oplus \mathbf{R}\bar{w}$.

$\varepsilon > 0$ such that $\tilde{c} - \varepsilon > 0$ yielding a deformation $\varphi = \sigma(1, \cdot)$. Since $f_{|\partial D_R} \le 0$ (see $(A3.2)$), property (2.5) implies that $\varphi \circ h \in \tilde{\Gamma}$ for all $h \in \tilde{\Gamma}$. On the other hand there exists $h \in \tilde{\Gamma}$ such that $\max\{f(h(u)) : u \in D_R\} \le c + \delta$ ($\delta > 0$ small enough). Using (2.7) one infers that

$$\max\{f(\varphi \circ h(u)) : u \in D_R\} \le c - \delta \ ,$$

a contradiction because $\varphi \circ h \in \tilde{\Gamma}$.

This shows:

Theorem 4.2 *Suppose* $E = V \oplus W$, $\dim V < +\infty$ *and let* $f \in C^1(E, \mathbf{R})$ *satisfy* $(PS)_{\tilde{c}}$ *and (A3). Then* f *has a critical point* $\tilde{u}$ *such that* $f(\tilde{u}) = \tilde{c}\ (> 0)$.

Complete proofs of the above statements can be found, for example, in [98].

Remarks 4.3 (i) If $V = \{0\}$ Theorem 4.2 is nothing but the Mountain Pass Theorem, with $u_0 = 0$ and $u_1 = \bar{w}$.
(ii) As in Remark 3.2, also here $(PS)_c$ can be substituted by the weaker $(PS)_c^*$. ■

A remarkable improvement of Theorem 4.2 is due to Benci and Rabinowitz [45] who eliminated the condition $\dim V < +\infty$. To describe their result, some preliminaries are in order. Let E be a Hilbert space, $E = V \oplus W$, with $V = W^{\perp}$ and let P, Q denote the canonical projections onto V and W, respectively. Let

$$\hat{\Gamma} = \ \{h \in C([0,1] \times E, E) : h(0,u) = 0 \ \text{ and } \ Qh(t,u) = Qu - K(t,u),$$

$$\text{where } \ K \in C([0,1] \times E, W) \ \text{ is compact } \ \}.$$

Given S and $D \subset E$, with $D \subset \tilde{E}$, $\tilde{E}$ subspace of E, we say that S and ∂D link if for all $h \in \hat{\Gamma}, h(t, D) \cap S \ne \emptyset$ provided $h(t, \partial D) \cap S = \emptyset$, $\forall\, t \in [0,1]$.

Theorem 4.4 *Let* $f \in C^1(E, \mathbf{R})$ *satisfy (PS) and:*
(i) $f(u) = \frac{1}{2}(Au|u) + \phi(u)$, *where* $A = A_1 P + A_2 Q, A_i$ *are selfadjoint and* $A_1 \in L(W, W)$;
(ii) ϕ' *is compact;*
(iii) there exist $\alpha > 0$ *and* $S \subset E, D \subset \tilde{E}$, $\tilde{E}$ *subspace of* E *such that* $S \subset V, f_{|S} \ge \alpha; D$ *is bounded,* $f_{|\partial D} \le 0; S$ *and* ∂D *link.*
Then f *has a critical point* u *such that* $f(u) \ge \alpha > 0$.

For the proof we refer to [45]. Here we limit ourselves to note that the assumption of the theorem allows us to show, by using the Leray-Schander topological degree, that

$$h(t, D) \cap S \neq \emptyset \ \ \forall\, t \in [0,1]$$

provided $h \in \hat{\Gamma}$.

5 Lusternik-Schnirelman Theory for Unbounded Functionals

In this section we will deal with the existence of multiple critical points for *even* functionals which are not bounded (from above nor from below). The discussion follows [26].

Let $f \in C^1(E, \mathbf{R})$ and set $E_+ = \{u \in E : f(u) \geq 0\}$. Our first result deals with a class of functionals satisfying $(A4)$:
(A4.1) $f(0) = 0$ and $\exists\ \rho, a > 0 : f(u) > 0, \forall\ u \in B_\rho - 0, f(u) \geq a, \forall\ u \in \partial B_\rho$;
(A4.2) for any finite dimensional subspace $E^n \subset E$, $E^n \cap E_+$ is bounded;
(A4.3) $f(-u) = f(u)$.

Remark 5.1 Assumption (A4.1) is nothing but (A2.1), while (A4.2) is the natural generalization of (A2.2-3). ■

Let $\mathcal{H}$ denote the class of maps $h \in C(E, E)$ which are odd homeomorphism and such that $h(B_1) \subset E_+$. Let us remark that $\mathcal{H}$ is not empty because the map $h_\rho : u \to \rho u$ belongs to $\mathcal{H}$.

We set

$$\Gamma_n = \{A \subset \Sigma : A \text{ is compact, and } \gamma(A \cap h(\partial B_1)) \geq n, \ \forall\ h \in \mathcal{H}\}$$

where $\Sigma = \{A \subset E - \{0\} : A \text{ is closed and symmetric}\}$ and γ denotes the *genus* (see section 2). The following lemma describes the properties of Γ_n.

Lemma 5.2 *Let f satisfy $(A4)$. Then*
(i) $\Gamma_n \neq \emptyset$ *for all n;*
(ii) $\Gamma_{n+1} \subset \Gamma_n$*;*

(iii) if $A \in \Gamma_n$ and $U \in \Sigma$, with $\gamma(U) \le r < n$, then $\overline{A-U} \in \Gamma_{n-r}$;
(iv) if φ is an odd homeomorphism in E such that $\varphi^{-1}(E_+) \subset E_+$, then $\varphi(A) \in \Gamma_n$ provided
$A \in \Gamma_n$.

Proof. (i) By (A4.2) there exists $R > 0$ such that

$$A := \overline{B_R} \cap E^n \supset E^n \cap E_+ .$$

Let $h \in \mathcal{H}$. Since $h(B_1) \subset E_+$, it follows that $A \supset E^n \cap h(B_1)$, and therefore

$$A \cap h(\partial B_1) = E^n \cap h(\partial B_1).$$

But h is an odd homeomorphism in E and hence $E^n \cap h(B_1)$ is a symmetric neighbourhood $\mathcal{N}$ of 0, with boundary contained in $E^n \cap h(\partial B_1)$. Then Lemma 2.11-(vi) implies that

$$\gamma(A \cap h(\partial B_1)) = \gamma(E^n \cap h(\partial B_1)) \ge \gamma(\partial \mathcal{N}) = n.$$

Hence $A \in \Gamma_n$ proving (i).
(ii) is trivial.
(iii) $\overline{A-U}$ is compact, and for every $h \in \mathcal{H}$ there results

$$\gamma([\overline{A-U}] \cap h(\partial B_1)) = \gamma(\overline{[A - h(\partial B_1)] - U}).$$

By Lemma 2.1-(iii) it follows

$$\gamma(\overline{[A - h(\partial B_1)] - U}) \ge \gamma(A - h(\partial B_1)) - \gamma(\overline{U}) \ge n - r$$

and hence $\overline{A-U} \in \Gamma_{n-r}$.
(iv) First of all, note that $\varphi(A)$ is compact. Moreover, if $h \in \mathcal{H}$ and $\varphi^{-1}(E_+) \subset E_+$, then $\varphi^{-1} \circ h \in \mathcal{H}$. Therefore, for all $A \in \Gamma_n$ there results

$$\gamma(A \cap \varphi^{-1} \circ h(\partial B_1)) \ge n.$$

Since φ is odd, Lemma 2.11-(iv) implies

$$\gamma(\varphi(A) \cap h(\partial B_1)) = \gamma(\varphi(A \cap \varphi^{-1} \circ h(\partial B_1))) \ge \gamma(A \cap \varphi^{-1} \circ h(\partial B_1)) \ge n,$$

and hence $\varphi(A) \in \Gamma_n$. This completes the proof of the Lemma. ∎

Remark 5.3 Each deformation $\sigma_t = \sigma(t, \cdot)$ such that $f(\sigma_t(u)) \le f(u)$, for all $t \ge 0$ and all $u \in E$ satisfies $\sigma_t^{-1}(E_+) \subset E_+$. ∎

Theorem 5.4 *Let $f \in C^1(E, \mathbf{R})$ satisfy (A4) and (PS). Then f possesses infinitely many critical points.*
In particular, for each positive integer n, setting

$$b_n = \inf_{A \in \Gamma_n} \max\{f(u) : u \in A\},$$

there results:
(i) $b_{n+1} \geq b_n \geq a > 0$ *for all n;*
(ii) *each b_n is a critical level for f;*
(iii) *if $b = b_n = b_{n+1} = \cdots = b_{n+r}$, then $\gamma(K_b) \geq r + 1$.*

Proof. Since $h_\rho \in \mathcal{H}$, then $A \cap \partial B_\rho \neq \emptyset$ for any $A \in \Gamma_n$. Therefore

$$b_n \geq \inf\{f(u) : u \in \partial B_\rho\} \geq a > 0$$

which, jointly with Lemma 5.2-(ii), proves (i).

Let us prove the stronger statement (iii). Let $\gamma(K_b) \leq r$ and let U be a symmetric neighbourhood of K_b such that ($\overline{U} \in \Sigma$ and)

$$\gamma(\overline{U}) = \gamma(K_b) \leq r.$$

Applying the Deformation Lemma, we find a homeomorphism $\varphi = \sigma(1, \cdot)$ and a $\delta < a$ such that (see Remark 5.3)

$$\varphi^{-1}(E_+) \subset E_+ \tag{5.1}$$

and

$$f(\varphi(u)) < b - \delta, \quad \forall\, u \in f^{b+\delta} - U. \tag{5.2}$$

Moreover φ is odd because f is even. By the definition of $b = b_{n+r}$, there exists $A \in \Gamma_{n+r}$ such that $A \subset f^{b+\delta}$. By Lemma 5.2-(iii) it follows that $\tilde{A} := \overline{A - U} \in \Gamma_n$. In view of (5.1), Lemma 5.2-(iv) applies yielding $\varphi(\tilde{A}) \in \Gamma_n$. Finally (5.2) implies

$$\varphi(\tilde{A}) \subset f^{b-\delta}$$

a contradiction with the definition of $b = b_n$. This completes the proof of the Theorem. ■

In the remainder of this section we will discuss some weakening of (A4.1) and (A4.2). The first case is to be related to the Linking Theorem 4.2. Let us assume that $E = V \oplus W$, with $d = dimV < +\infty$, $W = V^\perp$ and let $f \in C^1(E, \mathbf{R})$ satisfy $f(0) = 0$ and

(A4.1') $\exists\ \rho, a > 0 : f(u) > 0$, $\forall u \in (B_\rho - \{0\}) \cap W$ and $f(u) \geq a$, $\forall u \in \partial B_\rho \cap W$.

Let us define the counterpart of $\mathcal{H}$ and Γ_n by setting

$$\tilde{\mathcal{H}} = \{h \in C(E,E) : h \text{ is an odd homeomorphism, and } h(B_1) \subset E_+ \cup \overline{B_\rho}\}$$

and

$$\tilde{\Gamma}_n = \{A \subset E : A \text{ is compact, and } \gamma(A \cap h(\partial B_1)) \geq n \quad \forall\, h \in \tilde{\mathcal{H}}\}.$$

As before, $\tilde{\mathcal{H}} \neq \emptyset$ because $h_\rho \in \tilde{\mathcal{H}}$.

Lemma 5.5 *If f satisfies (A4.1'-2-3) then*
(i) $\tilde{\Gamma}_n \neq \emptyset$ *for all n;*
(ii) $\tilde{\Gamma}_{n+1} \subset \tilde{\Gamma}_n$*;*
(iii) if $A \in \tilde{\Gamma}_n$ and $U \in \Sigma$, with $\gamma(U) \leq r < n$, then $\overline{A - U} \in \tilde{\Gamma}_{n-r}$;
(iv) if φ is an odd homeomorphism in E such that $\varphi(u) = u$ for all u with $f(u) < 0$ and
$\varphi^{-1}(E_+) \subset E_+$, then $\varphi(A) \in \tilde{\Gamma}_n$ provided $A \in \tilde{\Gamma}_n$.

Proof. The proof of (iii) is exactly as that of Lemma 5.2-(iii) and (ii) is trivial.To prove (i), let us first take $n \geq d$, and $E^n \supset V$. Let $A := \overline{B}_\rho \cap E^n$. Assumption (A4.1') implies

$$A \supset (E_+ \cap B_\rho) \cap E^n \supset h(B_1) \cap E^n$$

for any $h \in \tilde{\mathcal{H}}$. Then the same arguments used in Lemma 5.2-(i) show that $A \in \tilde{\Gamma}_n$ for $n \geq d$. For $n < d$, (i) follows from (ii).

(iv) It suffices to show that $\varphi^{-1} \circ h \in \tilde{\mathcal{H}}$, whenever $h \in \tilde{\mathcal{H}}$. Actually, since $\varphi^{-1} \circ h$ is plainly an odd homeomorphism, it remains to prove that $\varphi^{-1} \circ h(B_1) \subset E_+ \cup \overline{B_\rho}$. Indeed, by definition one has that $h(B_1) \subset E_+ \cup \overline{B_\rho}$; if $h(B_1) \subset E_+$ then $\varphi^{-1}(E_+) \subset E_+$ implies immediately $\varphi^{-1} \circ h(B_1) \subset E_+$. If $h(B_1) \subset \overline{B_\rho}$ but $h(B_1)$ is not contained in E_+, then $\varphi(u) = u$ on the set $\{u : f(u) < 0\}$ implies that $\varphi^{-1} \circ h(B_1) = h(B_1) \subset \overline{B_\rho}$. This proves (iv) and completes the proof of the Lemma. ■

We are now in position to state a result which improves Theorem 5.4. For $n > d$, we define

$$\tilde{b}_n = \inf_{A \in \tilde{\Gamma}_n} \{f(u) : u \in A\}$$

Theorem 5.6 *Let $f \in C^1(E, \mathbf{R})$ satisfy (A4.1'-2-3) and (PS). Then f has infinitely many critical points. In particular, for all $n > d$ there results:*
(i) $\tilde{b}_{n+1} \geq \tilde{b}_n \geq a > 0$
(ii) *each $\tilde{b}_n$ is a critical level for f;*
(iii) *if $\tilde{b} = \tilde{b}_n = \cdots = \tilde{b}_{n+r}$, then $\gamma(K_{\tilde{b}}) \geq r + 1$.*

Proof. Let $A \in \tilde{\Gamma}_n$ with $n > d$. Then, taking $h = h_\rho$ there results

$$\gamma(A \cap h_\rho(\partial B_1)) = \gamma(A \cap \partial B_\rho) \geq n > d. \tag{5.3}$$

This implies that

$$(A \cap \partial B_\rho) \cap W \neq \emptyset. \tag{5.4}$$

To see this, we can argue by contradiction: if $(A \cap \partial B_\rho) \cap W = \emptyset$, then, denoted by P the canonical projection onto V, there results

$$P(A \cap \partial B_\rho) \subset V - \{0\}.$$

Since $dimV = d$, then the definition of the genus would imply $\gamma(A \cap \partial B_\rho) \leq d$, a contradiction with (5.3).
From (5.4) and (A4.1') it follows

$$\tilde{b}_n \geq \max\{f(u) : u \in A \cap \partial B_\rho\} \geq a > 0.$$

The rest of the Theorem is proved as Theorem 5.4, remarking that Lemma 5.6-(iii) applies because the map φ given by the Deformation Lemma can be obviously taken to satisfy $\varphi(u) = u$ for all u such that $f(u) < 0$. ∎

The next result deals, roughly, with a functional f which has a strict local minimum at 0, is bounded from below and has a negative global minimum. According to the Mountain-Pass Theorem, such an f possesses a second, nontrivial critical point at a positive level. We will show that if f is even and $\{u : f(u) < 0\}$ has genus $\tilde{d}$, then f has $2\tilde{d}$ pairs of nontrivial critical points.

Precisely, let us substitute (A4.2) with

(A4.2') there exist a subspace $\tilde{V}$ of E with $dim(\tilde{V}) = \tilde{d}$ and a compact, symmetric set
$\mathcal{K} \subset \tilde{V}$ such that $f < 0$ on $\mathcal{K}$ and 0 lies in a bounded component in $\tilde{V}$of $\tilde{V} - \mathcal{K}$.

Theorem 5.7 *Let $f \in C^1(E,\mathbf{R})$ satisfy (A4.1-2'-3) and (PS). Then each $\tilde{b}_n$, $1 \le n \le \tilde{d}$, is a positive critical level for f, and f possesses at least $\tilde{d}$ pairs of non trivial critical points $\pm u_n$, with $f(\pm u_n) > 0$.*

If, in addition, f is bounded below on E, then f possesses at least other $\tilde{d}$ pairs of nontrivial critical points $\pm v_n$, $1 \le n \le \tilde{d}$, with $f(\pm v_n) < 0$.

Proof. To prove the first statement, let us remark that the only role played by (A4-ii) was to show that $\Gamma_n \neq \emptyset$. We shall prove that this is still the case for $1 \le n \le \tilde{d}$, whenever (A4.2') holds. Let again $A = \overline{B}_R \cap E^n$. For R large and $1 \le n \le \tilde{d}$, (A4.2') implies that $A \supset \mathcal{K} \cap E^n$. Therefore the component $\mathcal{Q}$ of $E_+ \cap E^n$ containing 0 lies in A. Thus, for all $h \in \mathcal{H}$ there results $A \cap h(\partial B_1) \supset \mathcal{Q} \cap h(\partial B_1)$ and hence

$$\gamma(A \cap h(\partial B_1)) \ge \gamma(\mathcal{Q} \cap h(\partial B_1)) \ge n.$$

The last inequality is due to the fact that $\mathcal{Q} \cap h(\partial B_1)$ contains the boundary of a symmetric, bounded, neighbourhood of 0 in E. Then, repeating the arguments of Theorem 5.4, the result follows.

Let f be, in addition, bounded from below on E and consider the min-max level (see Lemma 2.12)

$$\tilde{c}_k = \inf_{\gamma(A) \ge k} \max\,[f(u) : u \in A]\,.$$

Since $\mathcal{K}$ contains the boundary of symmetric, bounded neighbourhood of 0 in $\tilde{V}$, then $\gamma(\mathcal{K}) = \tilde{d}$ and there results

$$\tilde{c}_{\tilde{d}} \le \max\,[f(u) : u \in \mathcal{K}] < 0.$$

As a consequence, for all $1 \le n \le \tilde{d}$ one has $\tilde{c}_n \le \tilde{c}_{\tilde{d}} < 0$ and each $\tilde{c}_n$ carries a pair of nontrivial critical points. This completes the proof of the Theorem.
■

6 Semilinear Elliptic Dirichlet Problems (I)

As a first application of Critical point theory we will discuss here and in Sections 7 and 8 some existence and multiplicity results for semilinear elliptic problems.

Let us introduce the notation we will use throughout this and the following sections. We will deal with Dirichlet boundary value problems like

$$\begin{cases} Lu = p(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{D}$$

where, hereafter, it is understood that Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$,

$$Lu = -\sum (a_{ij} u_{x_i})_{x_j}$$

$a_{ij} = a_{ji}$ are smooth on Ω and $\exists\ \mu_0 > 0$ such that

$$\sum a_{ij}(x)\xi_i\xi_j \geq \mu_0 |\xi|^2 \quad \forall\, x \in \Omega \ , \ \forall\, \xi \in \mathbf{R}^N.$$

We will work on the Sobolev space $E = H_0^1(\Omega)$. Equipped with the norm

$$||u||^2 = \int_\Omega \sum a_{ij} u_{x_i} u_{x_j}\, dx$$

and with corresponding scalar product

$$((u|v)) = \int_\Omega \sum a_{ij} u_{x_i} v_{x_j}\, dx$$

E is a Hilbert space.

Recall that the Poincaré inequality implies that $\|\cdot\|$ is equivalent to usual H_0^1 norm $\|\cdot\|_{1,2}$.

In the sequel $(\cdot|\cdot)$ will denote the scalar product in L^2.

Let $\lambda_i, 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ denote the eigenvalues of

$$\begin{cases} Lu = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

(repeated according to their multiplicity) and let φ_i denote a corresponding orthonormal system of eigenfunctions. We take φ_1 to be positive in Ω.

For $\rho \in L^\infty(\Omega)$, let $\lambda_j[\rho]$ denote the j-th eigenvalue of $Lu = \lambda \rho u$ with zero Dirichlet boundary conditions.

As for the nonlinearity, we will consider, for simplicity, functions p independent of x, with the only exception of problems handled in subsection 6.B below. In Sections 6 and 7 we will always assume that p satisfies

(p0) *$p \in C(\mathbf{R})$ and is locally Hölder continuous.*

A. Coercive Problems. We will start with a class of nonlinearities (satisfying $(p0)$ and) such that

(p1) $|p(s)| \leq a|s| + b$, $a, b > 0 \ \forall\, s \in \mathbf{R}$.

Let

$$P(u) = \int_0^u p(s)ds.$$

Since (p0-1) hold then (cf. also Example 1.1)

$$\phi(u) = \int_\Omega P(u(x))dx$$

defines a C^1 functional on E and the critical points of

$$f(u) = \frac{1}{2}\|u\|^2 - \phi(u)$$

on E are (weak and by regularity strong) solutions of (D) .

If $a < \lambda_1$ then it is plain that f is coercive on E and weakly lower semicontinuous, and has a (global) minimum, which gives rise to a solution of (D).

On more precise information about the behaviour of p at $u = 0$ and at infinity, it is possible to prove a multiplicity result.

Theorem 6.1 *Suppose p satisfies (p0) and (p2):*

(p2.1) $\limsup_{|s|\to+\infty} s^{-1}p(s) \le a < \lambda_1$;
(p2.2) $p(s) = \lambda s - sh(s)$ *with* $h(0) = 0$.

Then there results
(i) *if* $\lambda > \lambda_1$, *(D) has at least a positive (negative) solution* u^+ *(resp.* u^-*);*
(ii) *if* $\lambda > \lambda_2$, *(D) has at least a third solution* $\tilde{u} \ne u^\pm$, $\tilde{u} \ne 0$*;*
(iii) *if* $p(-u) = -p(u)$ *then (D) has at least* k *(pairs of) nontrivial solutions whenever* $\lambda > \lambda_k$.

Proof. First of all we note that we can suppose, without loss of generality, that p satisfies (p1) with $a < \lambda_1$. In fact, if $\exists\ s^+ > 0$ (resp. $s^- < 0$) such that $p(s^+) \le 0$ (resp. $p(s^-) \ge 0$), we can substitute p with a locally Hölder continuous $\tilde{p}$ such that $\tilde{p}(s) < 0 \quad \forall\ s > s^+$ (resp. $\tilde{p}(s) > 0 \quad \forall\ s < s^-$), and $|\tilde{p}| \le$ const. By the maximum principle, any solution of the modified problem

$$\begin{cases} Lu = \tilde{p}(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

satisfies $s^- \le u(x) \le s^+$ and hence solves (D).

To find a positive solution of (D) we can make another truncation, taking a smooth p^+, $p^+(s) = p(s)$ for $s \ge 0$ and $p^+(s) < 0$ (and bounded) for $s < 0$. Once again, by the maximum principle, any solution u of

$$\begin{cases} Lu = p^+(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $u(x) > 0$ in Ω and hence solves (D). One has (f^+ stands for f with p substituted by p^+)

$$\begin{aligned} f^+(\varepsilon\varphi_1) &= \frac{1}{2}\varepsilon^2\|\varphi_1\|^2 - \frac{1}{2}\varepsilon^2\lambda|\varphi_1|_2^2 + o(\varepsilon^2) \\ &= \frac{1}{2}\varepsilon^2(\lambda_1 - \lambda)|\varphi_1|_2^2 + o(\varepsilon^2). \end{aligned}$$

Then, if $\lambda > \lambda_1$, $f(\varepsilon\varphi_1)$ is negative for $|\varepsilon| > 0$ small enough and thus $\min_E f^+ < 0$. It follows that the minimum is achieved at some $u^+ \ne 0$. By the preceding remark $u^+ > 0$.

A similar argument yields a solution $u^- < 0$.

Next, let us prove (ii) under the additional assumption that p is differentiable and

$$\lambda - h(u) > p'(u) \quad \forall\, u \ne 0. \tag{6.1}$$

From $Lu^+ = p(u^+)$, namely

$$\begin{cases} Lu^+ &= (\lambda - h(u^+))u^+ & x \in \Omega \\ u^+ &= 0 & x \in \partial\Omega \end{cases}$$

one infers that 1 is an eigenvalue of $Lu = \lambda\rho u$, with $\rho = (\lambda - h(u^+))$, and corresponding eigenfunction u^+. Since $u^+ > 0$, then it follows that

$$\lambda_1[\lambda - h(u^+)] = 1.$$

From (6.1) and the comparison property of the eigenvalues, we infer

$$\lambda_1[p'(u^+)] > 1.$$

This means that (u^+ is non-degenerate and has) Morse index $m(u^+) = 0$, namely that u^+ is a minimum of f. Similar argument for u^-. We now apply the Mountain-Pass Theorem to f with $u_0 = u^+$ and $u_1 = u^-$, yielding a critical point $\tilde{u} \neq u^{\pm}$. We claim that, whenever $\lambda > \lambda_2$ then $\tilde{u} \neq 0$. To see this, it suffices to note that the Morse index of 0 is ≥ 2 provided $\lambda > \lambda_2$. If the critical points of f are u^+, u^- and 0, only, then theorem 3.4 would apply yielding $m(0) \leq 1$, a contradiction.

The general case can be handled by a similar argument, up to a Lyapunov-Schmidt reduction (see [19]).

Lastly, to prove (iii), we use Theorem 2.13, with the operator A_λ defined by the formula

$$((A_\lambda u|v)) = \lambda(u|v).$$

Plainly μ is a characteristic value of A_λ whenever

$$Lu = \mu \cdot \lambda u, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega.$$

Thus the characteristic values of A_λ are nothing but $\mu_k = \frac{\lambda_k}{\lambda}$ and the condition $\mu_k < 1$ follows from the assumption $\lambda > \lambda_k$. ∎

Remarks 6.2. (i) When $N = 1$, much stronger results can be proved. Indeed, from each $\lambda_k = \frac{T^2k^2}{4\pi^2}$ bifurcates a global branch of solutions of

$$\begin{cases} -(a(x)u')' &= \lambda u - uh(u) \text{ in } [0, T] \\ u(0) &= u(T) = 0 \end{cases}$$

and such a Sturm-Liouville problem has, for $\lambda > \lambda_k$, at least k nontrivial solutions.

(ii) It is an open question to see if the number of solutions of (D) increases as $\lambda \to +\infty$ (of course, without any oddness assumption on p).
(iii) When $p(u) = \lambda u - uh(u)$ and (6.1) holds it has been shown [20] that (D) has precisely 2 nontrivial solutions for all $\lambda_1 < \lambda \leq \lambda_2$. Moreover, if λ_2 is simple then for $\lambda_2 < \lambda < \lambda_2 + \varepsilon$, ε small enough, (D) has precisely 4 nontrivial solutions. ■

B. Asymptotically Homogeneous Problems. Our next application deals with the case when p is asymptotically homogeneous. Let

$$p(x,u) = \beta u^+ - \alpha u^- + b(u) + h(x) \tag{6.2}$$

where $\alpha, \beta > 0$ and

$$\lim_{|s| \to \infty} \frac{b(s)}{s} = 0. \tag{6.3}$$

When $\max\{\alpha, \beta\} < \lambda_1$, f is coercive and (D) has always a solution.

If $\alpha = \beta = \lambda_j$, (D) becomes the problem at resonance

$$\begin{cases} Lu &= \lambda_j u + b(u) + h(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

which has been extensively studied, beginning with the paper by Landesman and Lazer [92].

When $\alpha \neq \beta$, say $\alpha < \beta$, and there is at least an eigenvalue λ_j in $]\alpha, \beta[$, (D) is called "problem with a jumping nonlinearity" [78].

A first result is concerned with the case

$$\alpha < \lambda_1 < \beta < \lambda_2$$

and goes back to [24]. Under the assumption that $p(x,s)$ is of class C^2 with respect to s and $p_{ss}(x,s) > 0 \ \forall \ (x,s) \in \Omega \times \mathbf{R}$, it is proved that the Hölder space $Y = C^{0,\nu}(\Omega)$ can be split into two open components Y_0 and Y_2, with common boundary Y_1, such that (D) has precisely 1,2 or no solutions whenever $h \in Y_1$, $h \in Y_2$ or $h \in Y_0$. Such kind of result is obtained by means of a suitable "Global Inversion Theorem" for maps $\Phi \in C^2(X,Y)$ which possess singularities $\mathcal{S}$, (X, Y Banach spaces); such $\mathcal{S}$ is a manifold of codimension one in X and Y_1 turns out to be $\Phi(\mathcal{S})$.

Let us take, for simplicity, $h(x) = t\varphi_1(x)$, where t plays the role of a parameter. In such a case, the preceding result can be expressed by saying that $\exists \ t_0$ such that (D) has 2 solutions $\forall \ t < t_0$, 1 solution for $t = t_0$ and no solution for all $t > t_0$.

In [2] the convexity assumption on p has been eliminated at the expenses of the sharpness of the result. See also [89].

The case $\beta > \lambda_2$ has been investigated by Lazer and McKenna [93] by using topological degree arguments in the case when λ_2 is simple.

Hereafter we expose a result of [6], which improves that of [93].

Theorem 6.3 *Suppose p has the form (6.2) with b satisfying (6.3). If*

$$\alpha < \lambda_1 < \lambda_2 < \beta, \quad \beta \neq \lambda_j,$$

then $\exists\, t_1$ such that (D) has at least 3 solutions for all $t < t_1$.

Proof. First of all, let us consider the boundary value problem

$$Lu = \beta u + b(u) + t\varphi_1 \quad x \in \Omega; u = 0 \quad x \in \partial\Omega \tag{6.4}$$

Since $\beta \neq \lambda_j$, then (6.4) has a solution $\bar{u}_t$, for all $t \in \mathbf{R}$. This fact can be easily proved by topological degree arguments.

Let us set

$$\bar{v}_t = \bar{u}_t - \frac{t}{\lambda_1 - \beta}\varphi_1.$$

A direct calculation shows that

$$L\bar{v}_t = \beta\bar{v}_t + b(\bar{u}_t).$$

Since $\beta \neq \lambda_j$ and $\frac{b(u)}{u} \to 0$ as $|u| \to \infty$ one infers

$$\|\bar{v}_t\|_{C^1} \leq c.$$

Then from $\bar{u}_t = \bar{v}_t + \frac{t}{\lambda_1 - \beta}\varphi_1$ and $\beta > \lambda_1$, it follows that $\exists\, \bar{t}_0 < 0$ such that

$$\bar{u}_t \geq 0 \quad \forall\, t \leq \bar{t}_0. \tag{6.5}$$

From (6.5) one deduces that $\bar{u}_t$ actually solves

$$L\bar{u}_t = \beta\bar{u}_t^+ - \alpha\bar{u}_t^- + b(\bar{u}_t) + t\varphi_1$$

and hence is a solution of (D). A similar argument shows that $\exists\, \hat{t}_0 < 0$ such that (D) has a negative solution $\hat{u}_t$, for all $t \leq \hat{t}_0$, and

$$w_t = \hat{u}_t - \frac{t}{\lambda_1 - \alpha}\varphi_1$$

is bounded in C^1 norm.

To complete the proof, the following Lemma is in order.

Lemma 6.4 *There exists $t_1 < 0$ such that for all $t < t_1$ there results:*
(i) $m(\hat{u}_t) = 0$;
(ii) $\bar{u}_t$ *is nondegenerate and* $m(\bar{u}_t) = k$, *where k is such that $\lambda_k < \beta$.*

Proof. As $t \to -\infty$

$$\bar{u}_t = \bar{v}_t + \frac{t}{\lambda_1 - \beta}\varphi_1 \to +\infty$$

pointwise in Ω. Then $p_s(x, \bar{u}_t) \to \beta$ as $t \to -\infty$, uniformly in x, and the continuity of eigenvalues yields

$$\lambda_j\left[p_s(x, \bar{u}_t)\right] \to \lambda_j(\beta) = \frac{\lambda_j}{\beta} \quad (t \to -\infty).$$

Similarly one has $p_s(x, \hat{u}_t) \to \alpha$ as $t \to -\infty$ and

$$\lambda_j\left[p_s(x, \hat{u}_t)\right] \to \lambda_j(\alpha) = \frac{\lambda_j}{\alpha} \quad (t \to -\infty).$$

Since $\alpha < \lambda_1$ and $\beta > \lambda_k$, then $\exists\, t_1 < \min(\bar{t}_0, \hat{t}_0)$, such that

$$\lambda_1[p_s(\hat{u}_t)] > 1$$

$$\lambda_k[p_s(x, \bar{u}_t)] < 1 < \lambda_{k+1}[p_s(x, \bar{u}_t)].$$

This proves the Lemma. ■

Proof of Theorem 6.3 completed. Let us consider now the functional f whose critical points give rise to solutions of (D). It is easy to see that (PS) holds. For $t < t_1$ (given by Lemma 6.4) $\hat{u}_t$ is a local minimum of f because $m(\hat{u}_t) = 0$. Moreover for $s > 0$ one has

$$f(s\varphi_1) = \frac{1}{2}s^2\|\varphi_1\|^2 - \int P(s\varphi_1)dx.$$

Since p has the form (6.2), b satisfies (6.3), $\varphi_1 > 0$ and $h = t\varphi_1$, it follows:

$$f(s\varphi_1) \leq \frac{1}{2}s^2\lambda_1 - \frac{1}{2}s^2\beta + c_0 s - st$$

for some constant c_0. Since $\beta > \lambda_1$ it follows that $f(s\varphi_1) \to -\infty$ as $s \to +\infty$. Therefore the Mountain Pass Theorem applies and yields a critical point $\tilde{u}_t \neq \hat{u}_t$ (for all $t < t_1$). Such a $\tilde{u}_t$ cannot coincide with $\bar{u}_t$; indeed, otherwise,

$\tilde{u}_t = \bar{u}_t$ would be, by Lemma 6.4, nondegenerate and $m(\tilde{u}_t) = m(\bar{u}_t) \geq 2$, because $\beta > \lambda_2$. This would be in contradiction with Theorem 3.4. ∎

Remarks 6.5. (i) A counterexample [69] shows that if $\beta > \lambda_k$ (D) can have only 4 solutions. This is obtained by perturbing a problem posed on an Ω such that $\lambda_2 = \lambda_3 = \ldots = \lambda_k$. Also here an interesting question is to study the number of solutions of (D) when $\beta \to +\infty$.
(ii) The case in which $]\alpha, \beta[$ contains an eigenvalue $\lambda_k \neq \lambda_1$ has been studied, for example, in [79]. ∎

7 Semilinear Elliptic Problems (II)

Here we will discuss Elliptic Dirichlet boundary value problems with nonlinearities of the type $\lambda u + |u|^{\ell-1} u$ with $\ell > 1$. The section is divided in 3 parts; the first two deal, respectively, with existence and multiplicity results and mainly follow Section 3 of [26]. The latter is concerned with the so called *Critical Sobolev exponent*, namely with the case $\ell + 1 = 2^*$. We will keep the notation of Section 6.

A. EXISTENCE RESULTS. The first problem we consider is the existence of positive solutions for Dirichlet problems (D) when the nonlinearity p satisfies $(p3)$:

(p3.1) $p \in C(\mathbf{R}^+)$ is locally Hölder continuous and differentiable at 0;
(p3.2) $\exists\, r > 0$ and $\theta \in (0, \frac{1}{2})$ such that

$$P(u) \leq \theta u p(u) \quad \forall\, u \geq r \tag{7.1}$$

(p3.3) $p(u) \leq a_1 + a_2 u^\ell$, for all $u > 0$, $1 < \ell < \frac{N+2}{N-2}$, if $N > 2$, ℓ is unrestricted if $N = 1, 2$.

In the sequel $a_1, a_2, \ldots$ denote positive constants.

As anticipated in Section 1, we will consider the case $N > 2$ and, for simplicity, we will deal with nonlinearities independent of x.

According to the discussion in Example 1.1, assumption (p3.3) allows us to define $\phi, f : E \to \mathbf{R}$ ($E = H_0^1(\Omega)$) by setting

$$\phi(u) = \int_\Omega P(u(x))dx$$

$$f(u) = \frac{1}{2}\|u\|^2 - \phi(u).$$

Moreover (ϕ and hence) $f \in C^1(E, \mathbf{R})$ and critical points of f give rise to solutions of (D).

Let us remark explicitely that from (p3.2) it follows

$$P(u) \geq a_3 u^{\frac{1}{\theta}}, \quad \text{for } u \geq r \tag{7.2}$$

with $\frac{1}{\theta} > 2$. In this sense we say that such a p is "superlinear" as $u \to \infty$.

We want to prove

Theorem 7.1 *Suppose p satisfies (p3) and let $p(0) = 0$ and $\lambda := p'(0+) < \lambda_1$. Then (D) has a positive solution.*

First of all, dealing with positive solutions, we can assume, without loss of generality, that $p(u) \equiv 0$ for all $u < 0$. See the discussion in the proof of Theorem 6.1.

We begin showing

Lemma 7.2 *f satisfies (PS) on E.*

Proof. First, let us remark that (7.1) yields

$$\begin{aligned} \phi(u) &= \int_{u \leq r} P(u(x))dx + \int_{u \geq r} P(u(x))dx \\ &\leq b_1 + \theta \int_{u \geq r} p(u(x))u(x)dx \\ &\leq b_2 + \theta \int_\Omega p(u(x))u(x)dx = b_2 + \theta(\phi'(u)|u). \end{aligned} \tag{7.3}$$

Let $u_n \in E$ be such that $f(u_n) \leq b$ and $z_n = f'(u_n) \to 0$. Using (7.3) it follows

$$\begin{aligned} b &\geq f(u_n) = \frac{1}{2}\|u_n\|^2 - \phi(u_n) \\ &\geq \frac{1}{2}\|u_n\|^2 - b_2 - \theta(\phi'(u_n)|u_n) \end{aligned}$$

and hence

$$\frac{1}{2}\|u_n\|^2 \leq b_3 + \theta(\phi'(u_n)|u_n). \tag{7.4}$$

Since $z_n \to 0$, then for n large

$$|(z_n|u_n)| = \left| \|u_n\|^2 - (\phi'(u_n)|u_n) \right| \le \varepsilon \|u_n\|. \tag{7.5}$$

Combining (7.4) and (7.5) we find

$$\|u_n\|^2 \le 2\theta(\phi'(u_n)|u_n) + 2b_3 \ \le 2\theta\|u_n\|^2 + 2\theta\varepsilon\|u_n\| + 2b_3$$

Since $2\theta < 1$ one deduces that $\|u_n\|^2 \le b_4$. Hence, up to a subsequence, $u_n \rightharpoonup \bar{u}$ in E. Recall that, since $\ell < \frac{N+2}{N-2}$, namely $\ell + 1 < 2^*$, then E is compactly embedded in $L^{\ell+1}(\Omega)$ and this, in turn, implies that ϕ' is compact. This and $z_n = u_n - \phi'(u_n) \to 0$ imply $u_n = \phi'(u_n) + z_n \to \phi'(\bar{u})$. This proves (PS). ■

Proof of Theorem 7.1. By Lemma 7.2, (PS) holds. From (p3) one deduces

$$|\phi(u)| \le \frac{\lambda}{2}|u|_2^2 + b_4|u|_{\ell+1}^{\ell+1} . \tag{7.6}$$

Using the Poincaré and Sobolev inequalities it follows

$$|\phi(u)| \le \frac{\lambda}{2\lambda_1}\|u\|^2 + b_5\|u\|^{\ell+1} \tag{7.7}$$

and hence

$$f(u) \ge \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^2 + o(\|u\|^2).$$

Since $\lambda < \lambda_1$ then $u_0 = 0$ is a strict, local minimum for f. Moreover, for any $z > 0$, using (7.2), one deduces

$$\begin{aligned} f(tz) &= \frac{1}{2}t^2\|z\|^2 - \int_\Omega P(tz(x))dx \\ &\le \frac{1}{2}t^2\|z\|^2 - b_6 t^{1/\theta}\int_\Omega |z|^{1/\theta} - b_7 . \end{aligned}$$

Since $\frac{1}{\theta} > 2$ then $f(tw) \to -\infty$ as $t \to +\infty$ and there exists u_1, $\|u_1\|$ large enough, such that $f(u_1) < 0$. This suffices to apply the Mountain-Pass Theorem to f, yielding a critical point $u \ne 0$. Such a critical point gives rise, by the maximum principle, to a positive solution of (D). ■

Condition λ $(= p'(0+)) < \lambda_1$ can be eliminated by using the Linking Theorem instead of the Mountain-Pass Theorem.

Theorem 7.3 *Suppose p satisfies (p3.1) and*

(p3.2') $\exists\, r > 0$ *and* $\theta \in (0, \frac{1}{2})$ *such that* $P(u) \le \theta u p(u) \quad \forall\, |u| \ge r$;

(p3.3') $|p(u)| \le a_1 + a_2|u|^\ell$, $1 < \ell < \frac{N+2}{N-2}$, *(* $N > 2$*)*
Let $p(0) = 0$. *Then (D) has a nontrivial solution.*

Proof. Let $\lambda_k \le \lambda < \lambda_{k+1}$ and set

$$V = span\,\{\varphi_1, \cdots, \varphi_k\},$$

$$W = span\,\{\varphi_j : j > k\}.$$

Starting again from (7.3) and recalling that for $w \in W$ there results

$$\|w\|^2 \ge \lambda_k |w|_2^2$$

(7.7) becomes

$$|\phi(w)| \le \frac{1}{2}\frac{\lambda}{\lambda_k}\|w\|^2 + b_5\|w\|^{\ell+1}. \tag{7.8}$$

Plainly (7.8) $w = 0$ is a strict, local mimimum for f_W, namely that (A3.1) holds. ∎

Remark 7.4 If $\lambda \ge \lambda_1$, (D) might not have positive solutions at all. To see this, let us consider the boundary value problem

$$-\Delta u = \lambda u + u^\ell, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega \tag{7.9}$$

with $1 < \ell < 2^* - 1$ and $\lambda \ge \lambda_1$. Let $u_0 > 0$ be a solution of (7.9). Then

$$-\Delta u_0 = \left(\lambda + u_0^{\ell-1}\right) u_0$$

shows that the *first* eigenvalue of $-\Delta v = \mu\left(\lambda + u_0^{\ell-1}\right) v$, with zero Dirichlet boundary conditions, is 1 :

$$\lambda_1 \left[\lambda + u_0^{\ell-1}\right] = 1.$$

Since $\lambda + u_o^{\ell-1} > \lambda \ge \lambda_1$, the comparison property of the eigenvalues gives immediately rise to a contradiction. ∎

Remark 7.5 In [27] , instead of using the Linking Theorem, the Mountain-Pass Theorem has been emploied, jointly with the *Dual Variational Principle.* A somewhat similar argument will be discussed in some more details in section 8.

B. MULTIPLICITY RESULTS. When p is odd Theorems 7.1 and 7.3 can be greatly improved. Indeed, if this is the case, then

$$f(u) = \frac{1}{2}\|u\|^2 - \phi(u)$$

is even. Repeating the same arguments as before, one readily shows that f satisfies (A4) if $\lambda < \lambda_1$ and (A4.1'-2-3) otherwise (with $V = \text{span}\{\varphi_1, \cdots, \varphi_k\}$, whenever $\lambda_k < \lambda \leq \lambda_{k+1}$). Then Theorems 5.4 and 5.6 apply, yielding

Theorem 7.6 *Suppose p satisfies (p3) and is odd. Then (D) has infinitely many (pairs of) solutions.*

Remark 7.7 The existence of infinitely many solutions was first proved in [4] for a class of convex, superlinear nonlinearities, using Theorem 2.9, instead of Theorem 5.4. ■

A natural question is whether (D) possesses infinitely many solutions when p is not odd. For perturbed problems like

$$Lu = |u|^{\ell-1}u + \varepsilon h(x,u), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega \tag{7.10}$$

a partial answer has been given in [5] proving, by means of perturbation techniques, that (7.10) possesses an *arbitrarly large* number of solutions provided ε is small enough. Here $\ell + 1 < 2^*$ and h is assumed to satisfy the same growth restriction as p in (p3.3).

Such a result has been improved in [35] and [119]. It is proved that

$$Lu = |u|^{\ell-1}u + h(x,u), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega \tag{7.11}$$

has infinitely many solutions provided $\ell + 1 \leq \ell^*$ where ℓ^* is strictly smaller than 2^*.

A further improvement has been obtained in [37] by means of Morse theory.

Theorem 7.8 *Let $h \in C^{0,\nu}(\Omega \times \mathbf{R})$ satisfy*

$$|h(x,s)| \leq a_1 + a_2|s|^{\frac{N+2}{N-2}}$$

$$\exists\, \alpha \in (0,2) \;:\; \left|\int_0^s h(x,t)dt\right| \leq a_3 + a_4|s|^\alpha.$$

If $1 < \ell < \frac{N+2(1-\alpha)}{N-2}$, then (7.11) has infinitely many solutions.

Remark 7.9. When $h = h(x)$ is independent of u, then the range of permissable ℓ is

$$1 < \ell < \frac{N}{N-2}.$$

On the other side, a result of [34] shows that

$$Lu = |u|^{\ell-1}u + h(x), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega$$

has infinitely many solutions for any $\ell + 1 < 2^*$ and h is a residual set in $L^2(\Omega)$. ■

Another strategy to find multiple solutions can be to use the specific structure of the boundary value problem (D), rather than through a comparison with the model nonlinearity $|u|^{\ell-1}u$.

Let p satisfy (p3.1), (p3.2'-3') and let $p(0) = p'(0) = 0$. Then one can apply Theorem 7.1, with p substituted by its positive (respectively negative) part p_+ (resp. p_-) to find a positive (resp. negative) solution u^+ (resp. u^-) of (D). These $u^\pm$ are critical points of the functionals $f_\pm$ corresponding to $p_\pm$. Suppose, in addition that one can use the procedure indicated in Proposition 1.4 and Theorem 2.9, and that $u^\pm$ are non-degenerate minima of $f_\pm$ constrained on $M_{f_\pm}$ (see notation introduced in Section 1). Here one has

$$g_\pm(u) = \int_\Omega \left[\frac{1}{2}p_\pm'(u)u - p_\pm(u)\right] dx.$$

Since $u^\pm$ do not change sign, it follows that $T_{u^\pm}M_{f_\pm} = T_{u^\pm}M_f$. As a consequence, if $f_\pm''(u^\pm)[v,v] > 0$ for all $v \in T_{u^\pm}M_{f_\pm}$ then one also has

$$f_\pm''(u^\pm)[v,v] > 0 \ \forall \, v \in T_{u^\pm}M_f.$$

Since, plainly, $f_\pm''(u^\pm)[v,v] = f''(u^\pm)[v,v]$, one deduces that $u^\pm$ are also non-degenerate local minima for f constrained on M_f. Using a Mountain-Pass argument with base points at $u^\pm$ (or else using the Morse relationships) one finds a third critical point of f on M_f, hence a third solution of (D).

In the general case, a third critical point of f can be found by means of an appropriate linking argument, still starting from $u^\pm$. See [128]. This leads to

Theorem 7.10 *Suppose p satisfies (p3.1-2'-3') and let $p(0) = p'(0) = 0$. Then (D) has at least 3 nontrivial solutions.*

We end this subsection with some remarks concerning

$$-\Delta u = \lambda u + |u|^{\ell-1}u, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega \tag{7.12}$$

It has been proved in [42], see also [54] and [94] for extensions, that (7.12) has multiple positive solutions provided $1 < \ell < 2^* - 1$ and $\lambda < 0$ and $|\lambda|$ is sufficiently large. Precisely, the number of positive solutions of (7.12) is bounded below from the Lusternik-Schnirelman category of Ω. On the other side, it has been shown in [117] that if Ω is a ball in $\mathbf{R}^N$ then (7.12) possesses a unique positive solution, whenever $1 < \ell < 2^* - 1$ and $\lambda \in (0, \lambda_1)$.

C. CRITICAL EXPONENT. Let us begin with a celebrated *Identity* due to Pohozaev [106]. Let ν_x denote the unit outward normal at $x \in \partial\Omega$ and u_ν the normal derivative of u along ν.

Lemma 7.11 *Let u be a smooth solution of (D). Then there results*

$$N \int_\Omega P(u)dx + \frac{2-N}{2} \int_\Omega up(u)dx = \frac{1}{2} \int_{\partial\Omega} u_\nu^2 (x \cdot \nu_x) d\sigma.$$

As a consequence one immediately has

Corollary 7.12 *If $x \cdot \nu_x \geq o$ on $\partial\Omega$, namely if Ω is star-shaped with respect to 0, then the boundary value problem*

$$-\Delta u = |u|^{q-1}u, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega$$

has only the trivial solution, whenever $q \geq \frac{N+2}{N-2}$ $(N > 2)$.

On the light of Corollary 7.12, few more words are in order concerning (p3.3'). First of all, a growth restriction like (1.2), with $\ell \leq 2^* - 1$, is needed in order to define the functional f on E. Corollary 7.12 shows that, in general, one has to take $\ell < 2^* - 1$ in order (D) possesses nontrivial solutions. On the other side, it does not exclude that (D) has non-trivial solutions when (p3.3') is violated but p is not homogeneous, or Ω is not star-shaped. The latter question will not be discussed here: the interested reader is referred to [36], [62].

As for the former, it suffices to consider problem (7.12) and remark that it has nontrivial solutions bifurcating from the eigenvalues λ_k of $-\Delta$ with zero Dirichlet boundary conditions. In view of the specific feature of the nonlinearity, such a bifurcation is backword and hence solutions of (7.12) exist in any left neighbourhood of λ_k.

A global existence result is a much more subtle question. For the equation

$$- \Delta u = \lambda u + |u|^{2^*-2}u, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega \tag{7.13}$$

the problem has been first faced by variational tools in [52]. Roughly, the functional f defined above still makes sense, because $u^{2^*} \in L^1(\Omega)$ whenever $u \in H_0^1(\Omega)$. But now Lemma 7.2 concerning the (PS) condition cannot be carried out in the same way, because the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$ is not compact. Overcoming this difficulty, it is possible to show that

Lemma 7.13 *Let*

$$S = inf\{\|u\|^2 \ : u \in H_0^1(\Omega), |u|_{2^*}^{2^*} = 1\}.$$

Then for any

$$c < \frac{1}{N} S^{\frac{N}{2}}$$

f satisfies $(PS)_c$.

Lemma 7.13 leads to

Theorem 7.14 *Let $N > 3$. Then (7.13) has a nontrivial solution whenever $\lambda > 0$.*
If $N = 3$, there exists $\lambda_0 > 0$ such that (7.13) has a solution for all $\lambda > \lambda_0$.

Remarks 7.15 (i) If $\lambda < \lambda_1$ one finds positive solutions and the result goes back to [52]; the case $\lambda \geq \lambda_1$ has been studied in [55] and [28].

(ii) The Srikanth uniqueness result cited above is valid for (7.13), too. ∎

8 Elliptic Problems with Discontinuous Nonlinearities

In this section we deal with a class of elliptic equations with the specific feature that the nonlinearity is discontinuous. They serve as model in several concrete problems in Mathematical Physics. Postponing the case of vortex rings in an ideal fluid, which will be discussed in Section 9, let us begin with a Free Boundary Problem arising in Plasma Physics.

Example 8.1 Consider a cylinder with bounded cross-section $\Omega \subset \mathbf{R}^{\mathbf{N}}$, containing a ionized gas. Let τ_0 denote the temperature of the lateral surface of the cylinder; $\delta \geq \tau_0$ that of discharge in the gas; ϑ the termal conductivity and e, the electric field. Let us assume all these quantities are *positive constants.*

Let v denote the variable temperature in the gas and $\sigma = \sigma(v)$ denote the electrical conductivity. If we assume that

$$\sigma(v) = \begin{cases} 0 & \text{if } v \leq \delta \\ v & \text{if } v > \delta \end{cases}$$

then v satisfies

$$\begin{aligned} -\Delta v &= 0 \text{ if } v \leq \delta \\ -\Delta v &= \frac{|e|^2}{\vartheta} v \quad \text{if } v > \delta \\ v &= \tau_0 \text{ on } \partial\Omega \end{aligned} \tag{8.1}$$

Let us point out that the discontinuity of σ at $v = \delta$ is consistent with the fact that the gas is ionized.

Problem (8.1) is a Free Boundary Problem, because the region where $-\Delta v \neq 0$ is a-priori unknown. In order to trasform (8.1) in a Dirichlet Boundary Value problem, let us introduce the Heaviside function h :

$$h(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ 1 & \text{if } s > 0 \end{cases} \tag{8.2}$$

The reason to define $h(0) = 0$ will be clear later. However this choice will not effect the results we will find. Setting $u = v - \tau_0$, (8.1) becomes

$$-\Delta u = p_a(u), \quad x \in \Omega; \quad u = 0 \quad x \in \partial\Omega \tag{8.3}$$

where

$$\begin{aligned} p_a(u) &= h(u-a)q(u), \\ q(u) &= \frac{|e|^2}{\vartheta}(u + \tau_0), \\ a &= \delta - \tau_0 . \end{aligned} \tag{8.4}$$

In (8.3) the nonlinearity p_a has a simple discontinuity at $u = a$ and b,

$$b = \frac{(a + \tau_0)|e|^2}{\vartheta} > 0$$

is the size of the jump.

In general, let us consider a problem like

$$Lu = h(u-a)q(u), \quad x \in \Omega; \quad u = 0 \quad x \in \partial\Omega \tag{8.5}$$

where L stands for the linear second order, uniformly elliptic operator introduced in Section 6, $a > 0$, h denotes the Heaviside function (8.2) and q satisfies (q1):

(q1.1) $q \geq 0$, $q \in C(\mathbf{R})$, q is non-decreasing;
(q1.2) $q(s) \leq \alpha s + c_0$, with $\alpha < \lambda_1$ and c_0, a constant [1].

We set

$$b = q(a), \qquad T = [0, b]$$

and

$$\Omega_a = \{x \in \Omega : u(x) = a\}.$$

[1] hereafter $c_0, c_1, \cdots$ denote positive constants.

A solution of (8.5) is an $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that (8.5) holds a.e. in Ω. Let us explicitely point out that we do not exclude, in general, that $|\Omega_a| > 0$.

Setting $p(u) = p_a(u) = h(u-a)q(u)$, (8.5) becomes

$$Lu = p(u), \quad x \in \Omega; \;\; u = 0 \;\; x \in \partial\Omega. \tag{8.6}$$

By the maximum principle any non-zero solution u of (8.6) must satisfy $u(x) \geq 0$ on Ω and $u(x_0) > a$ at some $x_0 \in \Omega$. This will be referred as a *nontrivial solution* of (8.6).

By adding mu, $m > 0$, to both sides of (8.6), the nonlinearity becomes

$$p_m(u) = mu + p(u)$$

and is *strictly increasing*. Let $\hat{p}$ define the multivalued function

$$\hat{p}(s) = \begin{cases} p_m(s) & \text{if } s \neq a \\ T = [ma, b + ma] & \text{if } s = a \end{cases}$$

obtained by filling up the jump of p at $s = a$.

Let p^* denote the inverse of $\hat{p}$:

$$p^*(\omega) = s \;\text{ iff }\; \omega \in \hat{p}(s) \tag{8.7}$$

and set

$$P^*(\omega) = \int_0^\omega p^*(r)dr.$$

From the properties of p_m it follows that p^* is well defined on $\mathbf{R}$, is continuous and

$$p^*(\omega) = a \;\text{ iff }\; ma \leq \omega \leq p_m(a) = ma + b. \tag{8.8}$$

As for P^*, one has that $P^* \in C^1(\mathbf{R})$; moreover (q1.2) implies

$$P^*(\omega) \;\geq\; \frac{1}{2}\,\frac{1}{\alpha+m}\omega^2 - c_1|\omega| \tag{8.9}$$

$$P^*(\omega) \;\leq\; \frac{\omega^2}{2m}. \tag{8.10}$$

Let $E = L^2(\Omega)$ and let $G \in L(E, E)$ be the Green operator defined by setting

$$G(w) = u \Longleftrightarrow (L+m)u = w, \quad u \in H_0^1(\Omega) \cap H^2(\Omega)$$

For $w \in E$ we define

$$f(w) = \int_\Omega \left[P^*(w) - \frac{1}{2}wG(w)\right] dx.$$

Plainly, $f \in C^1(E, \mathbf{R})$.

Lemma 8.2 *Let $w \in E$ be such that $f'(w) = 0$. Then $u = G(w)$ is a solution of (8.6), in the sense that $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $Lu = p(u)$ a.e. in Ω.*

Proof. If $f'(w) = 0$ then $p^*(w) = G(w)$. By the definition of G, $u = G(w) \in H_0^1(\Omega) \cap H^2(\Omega)$ and satisfies $Lu + mu = w$. From $p^*(w) = u$ and (8.7) it follows that $w \in \hat{p}(u)$, and hence

$$Lu + mu \in \hat{p}(u).$$

For $x \in \Omega - \Omega_a$, namely when $u(x) \neq a$, one has $\hat{p}(u(x)) = mu(x) + p(u(x))$, and this implies

$$Lu(x) = p(u(x)) \quad (x \in \Omega - \Omega_a). \tag{8.11}$$

By a Theorem of Stampacchia [118] one has $Lu = 0$ a.e. on Ω_a. According to the fact that $h(0) = 0$, it follows that

$$p_m(u(x)) = ma + h(0)q(a) = ma \quad (x \in \Omega_a)$$

and therefore

$$Lu = p(u), \text{ a.e. in } \Omega_a.$$

This, jointly with (8.11), proves that u solves (8.6) in the sense specified above. ■

Remark 8.3 The idea of using a "dual" functional like f goes back to Clarke [57] and has been introduced to study Hamiltonian Systems. See [73]. The discussion outlined above follows [8]. ■

Lemma 8.4 *There exists $v_0 \in E$ such that $f(v_0) = \min_{u \in E} f(u)$. For the corresponding solution $u_0 = G(v_0)$, there results $|\Omega_a| = 0$.*

Proof. From (8.9) and the spectral properties of G it follows

$$f(w) \geq \frac{1}{2} \frac{1}{\alpha + m} |w|_2^2 - c_1 |w|_1 - \frac{1}{2} \frac{1}{\lambda_1 + m} |w|_2^2 . \tag{8.12}$$

Since $\alpha < \lambda_1$, (8.12) implies that f is coercive and bounded from below on E and, in a standard way, it follows that f attains its global minimum at some v_0. Let $u_0 = G(v_0)$; we claim that $\Omega_a = \{x \in \Omega : u_0(x) = a\}$ has zero

Lebesgue measure. To see this, let χ denote the characteristic function of Ω_a. There results

$$\begin{aligned}
\frac{d}{d\varepsilon}f(v_0+\varepsilon\chi) &= (f'(v_0+\varepsilon\chi) \mid \chi) \\
&= (p^*(v_0+\varepsilon\chi) \mid \chi) - \varepsilon\,(G(\chi) \mid \chi) - (G(v_0) \mid \chi) \\
&= \int_{\Omega_a} p^*(v_0+\varepsilon\chi)dx - \varepsilon\int_{\Omega}\chi G(\chi)dx - \int_{\Omega_a} u_0 dx \quad (8.13)
\end{aligned}$$

From $Lu_0 + mu_0 = v_0$ and $Lu_0(x) = 0$ a.e. in Ω_a, it follows that $v_0(x) = ma$, a.e. in Ω_a. Hence, taking $\varepsilon \in (0, b)$, one finds

$$ma \le v_0(x) + \varepsilon\chi(x) \le ma + b, \quad \text{a.e. in } \Omega_a \,.$$

Then, see (8.8), $p^*(v_0(x) + \varepsilon\chi(x)) = a$ a.e. in Ω_a and

$$\int_{\Omega_a} p^*(v_0(x)+\varepsilon\chi(x))\chi(x)dx = a|\Omega_a| = \int_{\Omega_a} u_0(x)dx. \tag{8.14}$$

Moreover, setting $z = G(\chi)$, it follows

$$(G(\chi) \mid \chi) = \int_{\Omega}\left[z \cdot Lz + m|z|^2\right]dx. \tag{8.15}$$

Inserting (8.14) and (8.15) into (8.13), one finds

$$\frac{d}{d\varepsilon}f(v_0+\varepsilon\chi) = -\varepsilon\left[\|z\|^2 + m|z|_2^2\right]. \tag{8.16}$$

If $|\Omega_a| > 0$, then $\|z\|^2, |z|_2^2 > 0$ and (8.16) yields

$$\frac{d}{d\varepsilon}f(v_0+\varepsilon\chi) < 0 \quad (0 < \varepsilon < b)$$

a contradiction, because v_0 is a minimum of f. ■

Remark 8.5 It is clear that the above arguments hold whenever v_0 is any local minimum of f. ■

If $a > 0$ then f has a local minimum at $w = 0$. Fixed b, when $a \gg 1$ then $w = 0$ will be the only critical point of f, see Remark 8.10-(i) below. On the contrary, next lemmas show that, under an appropriate relationship between a and b, f possesses a pair of non-trivial critical points: a negative global minimum and a Mountain-Pass critical point.

Let φ satisfy

$$L\varphi = \lambda_1\varphi, \ ; x \in \Omega; \quad \varphi = 0, \quad x \in \partial\Omega$$

and be such that $|\varphi|_\infty = 1$.

Lemma 8.6 *Suppose that*

$$\frac{b}{a} > 2\lambda_1 \frac{|\varphi|_1}{|\varphi|_2^2}. \tag{8.17}$$

Then, for all $m > 0$ sufficiently small, $f(b\varphi) < 0$.

Proof. Since $0 < b\varphi(x) \leq b$ and

$$p^*(\omega) \leq a, \text{ for } 0 \leq \omega \leq b,$$

it follows

$$\begin{aligned} f(b\varphi) &= \int_\Omega P^*(b\varphi(x))dx - \frac{1}{2}b^2\left(G(\varphi) \mid \varphi\right) \\ &\leq ba|\varphi|_1 - \frac{1}{2}\,\frac{b^2}{\lambda_1 + m}|\varphi|_2^2 < 0 \end{aligned} \tag{8.18}$$

whenever (8.17) holds and m is small enough. This proves the Lemma. ∎

The next Lemma shows that f has a Mountain-Pass geometry.

Lemma 8.7 *There exist $m^*, \rho, \alpha_0 > 0$ such that*

$$f(w) \geq \alpha_0, \quad \forall\, w \in E, \ |w|_{\frac{2N}{N+2}} = \rho.$$

Proof. Let $q = \frac{2N}{N+2}$. Since $\frac{1}{2^*} = \frac{1}{q} - \frac{2}{N}$, the Sobolev Embedding Theorem yields

$$|G(w)|_{2^*} \leq c_1 \|Gw\|_{2,q}.$$

From the elliptic theory we know that, for $w \in L^q(\Omega)$, there results

$$\|G(w)\|_{2,q} \leq c_2 |w|_q$$

Hence it follows

$$|G(w)|_{2^*} \leq c_3 |w|_q$$

and (note that $\frac{1}{2^*} + \frac{1}{q} = 1$)

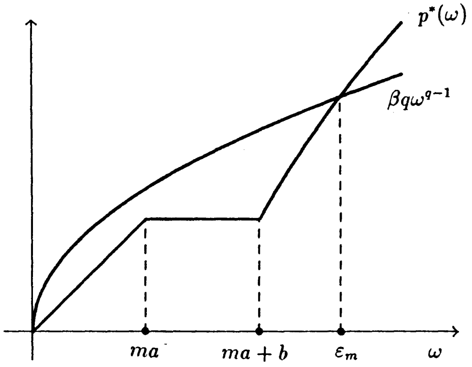
$$\int_\Omega wG(w)dx \leq |w|_q \cdot |G(w)|_{2^*} \leq c_3 |w|_q^2. \tag{8.19}$$

Next, we need to bound $\int_\Omega P^*(w)dx$ from below. For this, recall that p^* is non-decreasing and such that

$$p^*(\omega) = \begin{cases} \frac{\omega}{m} & \text{if } \omega \leq ma \\ a & \text{if } ma \leq \omega \leq ma + b \end{cases}$$

Then it is possible to find $\beta > 0$, $m > 0$, small enough, and $\varepsilon_m > 0$ such that

$$p^*(\omega) \geq \beta q \omega^{q-1}, \quad \forall\, \omega \geq \varepsilon_m .$$



As a consequence, one has

$$\int_\Omega P^*(w)dx \geq \int_{|w| \geq \varepsilon_m} \beta |w|^q dx.$$

Let

$$\beta_m = \beta \int_{|w| \leq \varepsilon_m} |w|^q dx.$$

Then, from

$$|w|_q^q = \int_{|w| \geq \varepsilon_m} |w|^q dx + \int_{|w| \leq \varepsilon_m} |w|^q dx = \int_{|w| \geq \varepsilon_m} |w|^q dx + \beta_m .$$

it follows

$$\int_\Omega P^*(w)dx \geq \beta |w|_q^q - \beta_m . \tag{8.20}$$

From (8.19) and (8.20) we deduce

$$f(w) \geq \beta |w|_q^q - \frac{1}{2} c_3 |w|_q^2 - \beta_m \tag{8.21}$$

Since $q < 2$, there exist $\rho, \alpha_0 > 0$ such that

$$\beta |w|_q^q - \frac{1}{2} c_3 |w|_q^2 \geq 2\alpha_0 \quad \forall\, |w|_q = \rho. \tag{8.22}$$

Finally,

$$\beta_m = \beta \int_{|w| \le \varepsilon_m} |w|^q dx \le \beta \varepsilon_m^q |\Omega| \to 0 \text{ as } m \to 0$$

jointly with (8.19) and (8.20) yield

$$f(w) \ge \alpha_0, \quad \forall\, |w|_q = \rho$$

provided m is small enough. This proves the Lemma. ∎

Finally we prove a weak form of (PS) condition (see Remark 3.2).

Lemma 8.8 *Let $w_n \in E$ be a $(PS)_c$ sequence ($c \in \mathbf{R}$). Then there exists $z \in E$ with $f(z) = c$, $f'(z) = 0$, such that $w_n \rightharpoonup z$.*

Proof. From (8.12) and $f(w_n) \to c$, it follows that $|w_n|_2 \le const.$, and, up to a subsequence, $w_n \rightharpoonup z$, for some $z \in E$. From $f'(w_n) \to 0$ and the compactness of G, it follows that $p^*(w_n) \to v := G(z)$, strongly in $L^2(\Omega)$ and a.e. in Ω. Let $\Gamma = \{x \in \Omega : v(x) = a\}$ and $\Omega' = \Omega - \Gamma$. Let us begin studying the convergence in Ω'. First, $p \in C(\mathbf{R} - \{a\})$ and $p^*(w_n) \to v$ a.e. in Ω, imply $w_n \to p(v)$ a.e. in Ω'. Plainly, $|\omega| \le |p^*(\omega)|$; this and the convergence of $p^*(w_n)$ in $L^2(\Omega)$ imply that there exists $h \in L^2(\Omega)$ and a subsequence of w_n such that (without relabeling) $|w_n| \le h$ a.e. in Ω. Then the Lebesgue Dominated Convergence Theorem yields: $w_n \to p(v)$ in $L^2(\Omega')$. Since one also has $w_n \rightharpoonup z$, one infers that $w_n \to z$ in $L^2(\Omega')$. Since p^* is asymptotically linear, it immediately follows

$$p^*(w_n) \to p^*(z) \text{ in } L^2(\Omega'), \text{ and } \int_{\Omega'} P^*(w_n)dx \to \int_{\Omega'} P^*(z)dx. \tag{8.23}$$

On the other side, for a.e. $x \in \Gamma$, one has that $z(x) = mv(x) = ma$ and hence $p^*(z(x)) = p^*(ma) = a = v(x)$. This and the first of (8.23) imply

$$p^*(z) = v, \text{ namely } f'(z) = 0.$$

In a quite similar way, taking into account the second of (8.23) and the specific form of $P^*(s)$ for $s \in T$, one finally finds that

$$\int_\Omega P^*(w_n)dx \to \int_\Omega P^*(z)dx$$

namely that $f(z) = c$. This completes the proof of the Lemma. ∎

We are now in position to state

Theorem 8.9 *Suppose that (q1) and (8.17) hold. Then (8.6) has two, distinct, positive solutions* $u_0 \neq u_1$. *Moreover* $\Omega_a(u_0) = \{x \in \Omega : u_0(x) = a\}$ *has zero Lebesgue measure.*

Proof. Let v_0 be the global minimum of f (Lemma 8.3). According to Lemma 8.5, $f(v_0) < 0$, whenever (8.17) holds. Hence $v_0 \neq 0$ and $u_0 = G(v_0)$ is a non-trivial (positive) solution of (8.6) (see Lemma 8.2 and the preceding discussion) and $|\Omega_a(u_0)| = 0$, see Lemma 8.3.

Lemmas 8.5, 8.6 and 8.7 allow us to apply the Mountain-Pass Theorem (see also Remark 3.2), yielding a second critical point $v_1 \in E$, with $f(v_1) > 0$. Hence $u_1 = G(v_1) \neq u_0$ is a second non-trivial, positive solution of (8.6). ∎

Remarks 8.10 (i) When $\Omega = B_R$, a ball of radius R, it is easy to check that $|\varphi|_1/|\varphi|_2^2$ is bounded above by a constant independent of R. Hence, fixed a and $b > 0$, (8.17) is satisfied whenever $R > 0$ is large enough, because the first eigenvalue of L on B_R, $\lambda_1(R)$, tends to zero as $R \to +\infty$. Dealing with large balls, q is assumed to be bounded, in such a way that (8.17) holds.
(ii) Fixed Ω and q, one has that $p(u) = h(u-a)q(u) < \lambda_1 u$, provided a is sufficiently large. It is plain that in such a case (8.6) has no positive solutions at all. ∎

When Ω inherits some symmetry, Theorem 8.9 can be greatly improved. Precisely, let Ω be symmetric with respect the plane $x_1 = 0$, say. Given $w(x)$, with $w(x) = 0$ on $\partial\Omega$, we denote by w^* the Steiner symmetrization of w with respect to x_1 (see, for ex., [40]); namely $w^*(x)$ which is even in x_1, non-increasing for $x_1 > 0$ and such that

$$meas\{x_1 : w^*(x) > c\} = meas\{x_1 : |w(x)| > c\}$$

for all $c > 0$.

Theorem 8.11 *Suppose that, in addition to the hypotheses of Theorem 8.9,* Ω *is Steiner symmetric. Then:*
(i) the critical points v_0, v_1 *are Steiner symmetric, and hence the same holds for the corresonding solutions:* $u_0 = u_0^*$, $u_1 = u_1^*$;
(ii) $\frac{\partial u_i}{\partial x_1} < 0$ *for* $x_1 > 0$ *and* $i = 0, 1$;
(iii) $|\Omega_a(u_i)| = 0$ *for* $i = 0, 1$.

Proof. Recall that the definition of w^* yields

$$\int_\Omega P^*(w)dx = \int_\Omega P^*(w^*)dx. \tag{8.24}$$

Moreover, from [1] it follows that

$$(G(w) \mid w) \leq (G(w^*) \mid w^*)\,. \tag{8.25}$$

From (8.24) and (8.25) we infer that a minimizing sequence w_n can be replaced by its symmetrization w_n^* to obtain a symmetric minimum $v_0 = v_0^*$.

As for the Mountain-Pass solution, let us note that the map $w \to w^*$, $w \in E = L^2(\Omega)$, is a contraction in E, in particular it is continuous. Let $\gamma \in C([0,1], E)$ be any path used to find the Mountain-Pass critical level (here the base points can be taken to be 0 and $b\varphi$). Then $\gamma^*(t) = (\gamma(t))^*$ is also a path and

$$\gamma^*(0) = 0, \quad \gamma^*(1) = (b\varphi)^* = b\varphi.$$

If $w^* \in \gamma^*$ corresponds to $w \in \gamma$, one has

$$\begin{aligned} f(w^*) &= \int_\Omega P^*(w^*) - \frac{1}{2}(G(w^*) \mid w^*) \\ &\leq \int_\Omega P^*(w) - \frac{1}{2}(G(w) \mid w) = f(w) \end{aligned}$$

and hence

$$\max_{\gamma^*} f \leq \max_{\gamma} f.$$

We can now use the version of the Mountain-Pass Theorem (with condition (PS) substituted by $(PS)_c^*$), as in Remark 3.2), yielding a sequence of symmetric w_n^* such that

$$f(w_n^*) \to c, \quad f'(w_n^*) \to 0.$$

By Lemma 8.8 one infers that $w_n^* \rightharpoonup \bar{w}$ and $f(\bar{w}) = c$, $f'(\bar{w}) = 0$. Then it follows

$$z_n^* := G(w_n^*) \to z := G(\bar{w}).$$

This shows that (8.6) has a second symmetric solution, proving (i).
Statements (ii) follows by the weak maximum principle applied to $\partial u_i / \partial x_1$, $x_1 > 0$; and (iii) follows from (i) and (ii). ∎

Theorems 8.9 and 8.11 are prompted for an application to the problem discussed in Example 8.1. To be specific, we will consider the case in which Ω is Steiner symmetric. Here one has:

$$a = \delta - \tau_0, \quad b = \frac{(a + \tau_0)|e|^2}{\vartheta}$$

and assumption (q1.2) holds whenever

$$|e|^2 < \lambda_1 \vartheta. \tag{8.26}$$

Since

$$\frac{b}{a} = \frac{|e|^2}{\vartheta} \cdot \frac{\delta}{\delta - \tau_0} > 2\lambda_1 \cdot \frac{|\varphi|_1}{|\varphi|_2^2}$$

is satisfied for $\delta \in (\tau_0, \tau_0 + \varepsilon)$, $\varepsilon > 0$, then Theorem 8.11 yields two distinct, symmetric solutions $u_0(a)$, $u_1(a)$. If

$$a > \bar{\tau} := \frac{\tau_0 |e|^2}{\lambda_1 \vartheta - |e|^2}$$

then Remark 8.10-(ii) shows that (8.3) has no positive solutions. Thus there exists a maximal interval (τ_0, τ_1) such that (8.3) has two solutions for all $\delta \in (\tau_0, \tau_1)$. As $\delta \downarrow \tau_0$, namely as $a \downarrow 0$, we claim that

$$u_0(a) \to \hat{u}, \quad u_1(a) \to 0$$

where $\hat{u}$ is the (unique) positive solution of

$$-\Delta u = \frac{|e|^2}{\vartheta}(u + \tau_0), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega. \tag{8.27}$$

To prove the claim, we first note that $p^*_{a_1}(\omega) \leq p^*_{a_2}(\omega)$, whenever $a_1 \leq a_2$, and thus

$$f_{a_1}(w) \leq f_{a_2}(w). \tag{8.28}$$

Here the subscript a highlights the dependence on the parameter a. We write $w_{0,a}$ (resp. $w_{1,a}$) to denote the minimum (resp. the Mountain-Pass) critical point of f_a, and set $u_{i,a} = G(w_{i,a})$, $i = 0, 1$.
For all $a \in (0, a_0)$, $a_0 > 0$ close to zero, (8.28) implies

$$f_a(w_{0,a}) \leq f_{a_0}(w_{0,a_0}) < 0$$

and thus there exists $m_0 < 0$ such that

$$f_a(w_{0,a}) \to m_0, \text{ as } a \downarrow 0$$

and $f'_a(w_{0,a}) = 0$. Then, arguments similar to those of Lemma 8.8 show that

$$\begin{aligned} w_{0,a} &\rightharpoonup \hat{w} \text{ weakly in } L^2, \\ f_0(\hat{w}) &= m_0\,, \\ f'_0(\hat{w}) &= 0. \end{aligned}$$

Since G is compact, then

$$u_{0,a} = G(w_{0,a}) \to \hat{u} = G(\hat{w})$$

and $\hat{u}$ is the unique solution of (8.27).

Concerning the Mountain-Pass critical point $w_{1,a}$, from

$$f(t\varphi) \le at|\varphi|_1 - \frac{1}{2}t^2(G(\varphi) \mid \varphi) \quad (0 \le t \le b)$$

we infer that

$$\max_{0\le t\le b} f(t\varphi) \le ca^2$$

for some constant $c > 0$ depending on φ. Hence $f_a(w_{1,a}) \to 0$ as $a \downarrow 0$, and arguments similar to the preceding ones yield

$$u_{1,a} = G(w_{1,a}) \to \tilde{u} = G(\tilde{w})$$

with $f_0(\tilde{w}) = 0$. It readily follows that $\tilde{w}$, and hence $\tilde{u}$ are zero.

In conclusion, we have shown

Theorem 8.12 *Let Ω be Steiner symmetric and (8.26) hold. Then there exists τ_1, $\tau_0 < \tau_1 < +\infty$ such that, setting $a = \delta - \tau_0$, there results*
(i) for $0 < a < \tau_1 - \tau_0$ problem (8.3) has two symmetric, positive solutions $u_{0,a}$, $u_{1,a}$;
(ii) if $a > \bar{\tau}$, then (8.3) has no positive solutions;
(iii) as $a \downarrow 0$, $u_{0,a} \to \hat{u}$ and $u_{1,a} \to 0$ in $H^1_o(\Omega)$, where $\hat{u}$ is the positive solution of (8.27);
(iv) meas$\{x \in \Omega : u_{i,a}(x) = a\} = 0$ for $i = 0, 1$.

Remark 8.13 Most of the preceding arguments can be carried over in much greater generality. For example, it is possible to handle boundary value problems of the type

$$Lu = p(u) + h(x) \quad x \in \Omega; \quad u = 0 \quad x \in \partial\Omega$$

where $p : \mathbf{R} \to \mathbf{R}$ satisfies:

1) p is measurable and there exists a set $A \subset \mathbf{R}$ with no finite accumulation points, such that $p \in C(\mathbf{R} - A)$;
2) there exists $m \ge 0$ such that $p(s) + ms$ is strictly increasing.

In particular, Lemma 8.2 holds and solutions of (8.6) can be found by looking critical points of a smooth functional f. According to Lemma 8.8, f satisfies (PS) condition in a form which enables us to use the Mountain-Pass or Linking Theorem. For more details and further applications, we refer to [8]. The material presented here is taken from [8] and [30]. Elliptic problems with discontinuous nonlinearities have been investigated,for example, in [59], [61], [121], [122]. ■

In spite of the great power of variational methods, there are problems in which it is more convenient to use nonvariational tools. Typical cases are global branching phenomena, studied by means of topological degree arguments. The following example illustrates a case where such an approach can be usefullly employed. For brevity, we will be sketchy, referring to [12] for some more detail.

It concerns the problem of finding the equilibria of a plasma confined in a toroidal cavity, which leads to a free boundary problem of the form:

Given $I > 0$, find $a \geq 0$, $\Omega_p \subset \Omega$ and $v \in C^1(\Omega) \cap C^2(\Omega - \Omega_p)$ such that

$$\begin{cases} -\Delta v &= \sigma(v), \quad v \geq 0, \quad \text{in } \Omega_p \\ v &= 0 \ \text{ on } \partial\Omega_p \\ -\Delta v &= 0 \ \text{ in } \Omega - \Omega_p \\ v &= -a \ \text{ on } \partial\Omega \\ -\int_{\partial\Omega} \frac{\partial v}{\partial n} &= I \end{cases} \tag{8.29}$$

Above, Ω_p is the region filled by the plasma (the physically relevant case is when $\Omega_p \subset\subset \Omega$), σ is a given function related to the electric field, $\mathbf{n}$ is the unit, outer normal at $\partial\Omega$, and I is the total current.

On σ we suppose that:

(σ1) $\sigma \in C^{0,\nu}(\mathbf{R}^+, \mathbf{R}^+)$ is non-decreasing;

(σ2) $\sigma(s) \leq \alpha s + \beta$, with $\alpha < \lambda_1$, $\beta > 0$[2];

(σ3) $\sigma(0+) = b > 0$

Let us observe that, dealing with a plasma, it is rather natural to take a nonlinearity σ such that $\sigma(s)$ is *strictly* positive for all $s \geq 0$.

As before, setting $p(s) = h(s)\sigma(s)$, (h denotes the Heaviside function),

[2] hereafter λ_1 denotes the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary conditions.

and $u = v + a$, (8.29) becomes

$$\begin{cases} (i) & -\Delta u = p(u-a) \text{ in } \Omega \\ (ii) & u = 0 \text{ on } \partial\Omega \\ (iii) & -\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} = I \end{cases} \tag{8.30}$$

With this notation, $\Omega_p = \{x \in \Omega : u(x) \geq a\}$ and $\Omega_p \subset\subset \Omega$ iff $a > 0$.

Let

$$\Gamma_a = \{x \in \Omega : u(x) = a\}.$$

Theorem 8.14 *Suppose Ω is Steiner symmetric and let σ verify ($\sigma 1-2-3$). Given $I > 0$, let b satisfy $b|\Omega| > I$.*

Then there exists $a > 0$ such that (8.30) has a symmetric solution $v \in C_0^1(\Omega) \cap C^2(\Omega - \Gamma_a)$ such that $|\Gamma_a| = 0$, corresponding to a region $\Omega_p \subset\subset \Omega$.

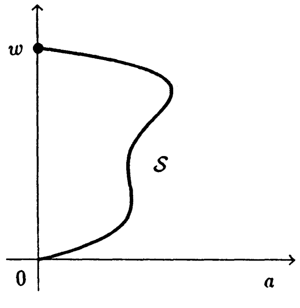
Proof. (Outline) First, we study the boundary value problem (8.30-i-ii): taking a as a parameter, we estabilish the existence of a bounded, connected branch $\mathcal{S}$ of solutions (a, u_a) bifurcating from $a = 0$, $u = 0$, with a behaviour like that indicated in the figure below.

For $(a, u) \in \mathcal{S}$ we define a real valued map h by setting

$$h(a,u) = -\int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} = \int_\Omega p(u-a)dx.$$

Plainly, h is continuous on $\mathcal{S}$, $h(0,0) = 0$. Moreover, if w is a positve solution of (8.30-i-ii) with $a = 0$, such that $(0, w) \in \mathcal{S}$, then

$$h(0,w) = \int_\Omega p(w)dx \geq b \cdot |\Omega|.$$



Since h is continuous, the assumption $b|\Omega| > I$ implies the existence of a $(\bar{a}, \bar{u}) \in \mathcal{S}$, $\bar{a} > 0$ such that $h(\bar{a}, \bar{u}) = I$. This proves that $\bar{u}$ is a symmetric solution of (8.30) corresponding to a region $\Omega_p \subset\subset \Omega$, because $\bar{a} > 0$. ∎

9 Vortex Theory

This section contains a short discussion of a classical problem, the existence of stationary vortex rings in an ideal fluid, which can be formulated as an elliptic problem with a discontinuous nonlinearity. We shall see that the variational tools developed so far can be adapted to handle such a problem.

A. FORMULATION OF THE PROBLEM. Consider an ideal (namely inviscid and with uniform density) fluid in $\mathbf{R}^3$ and assume it has a cylindrical symmetry. Let (r, θ, z) denote cylindrical coordinates and set

$$\Pi = \{(r, z) : r > 0\}.$$

In view of the symmetry and since the fluid is incompressible, there exists a *Stream Function*, referred to as the *Stokes* stream function, $\Psi : \bar{\Pi} \to \mathbf{R}$, $\Psi = \Psi(r, z)$, such that the velocity field $\mathbf{q}$ and the vorticity *curl* $\mathbf{q}$ have cylindrical components

$$\begin{aligned} \mathbf{q} &= \left(-\frac{1}{r}\frac{\partial \Psi}{\partial z}, 0, \frac{1}{r}\frac{\partial \Psi}{\partial r}\right) \\ \mathit{curl}\,\mathbf{q} &= \left(0, -\frac{1}{r}L\Psi, 0\right) \end{aligned}$$

where

$$L = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}.$$

Streamlines in any plane $\theta = const.$ are level curves of Ψ and $\Psi(r, z) = const.$ are the stream surfaces of the fluid.

Let $\omega = -r^{-1}L\Psi$. The laws of hydrodynamics require that $r^{-1}\omega$ be constant on each stream surface $\Psi = const.$, namely that

$$\frac{\omega}{r} = -\frac{1}{r^2}L\Psi = p(\Psi)$$

where $p : \mathbf{R}^+ \to \mathbf{R}^+$ is a given *vorticity function*.

A *vortex ring* is a axi-symmetric region $\mathcal{R}$ such that $\omega = 0$ on $\mathbf{R}^3 - \mathcal{R}$, while $\omega \neq 0$ on $\mathcal{R}$. In terms of the Stokes function Ψ, the existence of a vortex ring leads to find a set $A \subset \Pi$, the cross section of the vortex ring, or the *vortex core*, such that

$$\begin{cases} -L\Psi = 0 & \text{in } \Pi - A \\ -L\Psi = r^2 p(\Psi) & \text{in } A \end{cases} \tag{9.1}$$

We shall also require that $\Psi \in C^2(\Pi - \partial A) \cap C^1(\Pi)$. In order to add boundary conditions to (9.1), let us note that ∂A and, by the symmetry, the axis $\{r = 0\}$ are stream lines. We will set

$$\Psi = 0 \text{ on } \partial A; \text{ and } \Psi = -k \text{ on } \{r = 0\}. \tag{9.2}$$

where the constant $k \geq 0$ measures (up to the factor 2π) the flux rate between the boundary ∂A of the vortex core and the stream line $\{r = 0\}$. Finally, a condition at infinity is in order. We shall demand that

$$\mathbf{q} \to (0, 0, -W) \text{ as } r^2 + z^2 \to \infty \tag{9.3}$$

From the physical point of view, (9.3) means that the vortex moves upward, with respect to the fluid (assumed at rest at infinity) with *propagation speed* W. In terms of the Stokes function, (9.3) leads to require that

$$\frac{1}{r}\frac{\partial \Psi}{\partial z} \to 0, \quad \frac{1}{r}\frac{\partial \Psi}{\partial r} \to -W, \text{ as } r^2 + z^2 \to \infty. \tag{9.4}$$

The problem $(\mathcal{P})$ of finding a vortex ring can be now formulated as follows:

$(\mathcal{P})$ *given* $W > 0$, $k \geq 0$ *and the vorticity function* p, *to find a set* $A \subset \Pi$ *and* $\Psi \in C^2(\Pi - \partial A) \cap C^1(\Pi)$ *which solves (9.1-2-4).*

As in Section 8, it is convenient to make a change of variables, to obtain a Dirichlet boundary value problem. Let

$$\psi_0 = \frac{1}{2}Wr^2 + k \tag{9.5}$$

denote the stream function relative to the field $(0,0,-W)$. We introduce the *reduced stream function* ψ by setting

$$\psi(r,z) = \Psi(r,z) + \psi_0(r,z).$$

Since $L\psi_0 = 0$, (9.1) together with the boundary conditions (9.2) and (9.4) becomes

$$\begin{cases} -L\psi &= 0 \text{ in } \Pi - A \\ -L\psi &= r^2 p(\psi - \psi_0) \text{ in } A \\ \psi(0,z) &= 0 \\ \frac{1}{r}|\nabla\psi| &\to 0 \text{ as } r^2 + z^2 \to \infty \end{cases}$$

where

$$|\nabla\psi|^2 := \left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2.$$

If we extend p by $p(s) \equiv 0$ for $s \leq 0$, see Section 8, we finally find

$$\begin{cases} -L\psi &= r^2 p(\psi - \psi_0) \text{ in } \Pi \\ \psi(0,z) &= 0 \\ \frac{1}{r}|\nabla\psi| &\to 0 \text{ as } r^2 + z^2 \to \infty \end{cases} \tag{9.6}$$

If ψ is a solution of (9.6) then $\Psi = \psi - \psi_0$ solves $(\mathcal{P})$, and

$$A_\psi = \{(r,z) \in \Pi : \psi(r,z) > \psi_0(r,z)\}$$

is the corresponding vortex core. For brevity, we will say that ψ solves $(\mathcal{P})$.

Since $p \geq 0$, the maximum principle implies that any nontrivial solution ψ of $(\mathcal{P})$ is positive, $\psi(r,z) > \psi_0(r,z)$ for some $(r,z) \in \Pi$, and hence A_ψ will be not empty.

B. The Hill Spherical Vortex. When $k = 0$ and $p = h$, the Heaviside function (i.e $p(s) \equiv 1$ for $s > 0$), an explicit solution of $(\mathcal{P})$ was given by Hill, see [32]:

$$\psi_H(r,z) = \begin{cases} \frac{1}{2}Wr^2\left(\frac{5}{2} - \frac{3}{2}\frac{r^2+z^2}{a^2}\right) & \text{if } r^2+z^2 \leq a^2 \\ \frac{1}{2}Wr^2 \frac{a^3}{(r^2+z^2)^{3/2}} & \text{if } r^2+z^2 > a^2 \end{cases}$$

where the value of a, $a = \frac{15}{2}W$, is chosen in such a way that ψ_H is C^1 across the circle $r^2 + z^2 = a^2$.

The corresponding vortex core is a sphere of radius a and is referred to as the *Hill Spherical Vortex*. More recently, Amick and Fraenkel [32] have shown that ψ_H is the only solution of $(\mathcal{P})$, with $p = h$ and $k = 0$.

C. Bifurcation Results. The existence of small vortex rings bifurcating from the Hill vortex has been proved in [103] and [33]. Let us outline the arguments of the latter paper. Roughly, one assumes $p = h$ and takes k as a bifurcation parameter. Problem $(\mathcal{P})$ is approximated by Dirichlet problems $(\mathcal{P}_{n,R})$ on balls $B_R = \{(r,z) : r^2 + z^2 < R^2\}$ with piecewise linear vorticity functions p_n. Degree theoretical arguments provide the existence of a global branch $\Sigma_{n,R}$ of nontrivial solutions to $(\mathcal{P}_{n,R})$ emanating from ψ_H. Limiting procedures as $R \to \infty$ and $p_n \to p$ yield the existence of an unbounded branch Σ of nontrivial solutions of $(\mathcal{P})$ bifurcating from ψ_H. It is also shown that any solution $\psi \in \Sigma$ is even in z, $\partial\psi/\partial z < 0$ for $z > 0$ and that the vortex core A_ψ is bounded.

It is worth pointing out that in [33] no control on the behaviour of Σ for $k \to \infty$ is given; thus, in spite of its global nature, their resul provides the existence of solutions of $(\mathcal{P})$ for k small, only.

D. Global Results. The first global existence result to $(\mathcal{P})$ is due to Fraenkel and Berger [77]. Actually, they introduce a real parameter λ, and consider rather than $(\mathcal{P})$, the nonlinear eigenvalue problem

$$\begin{cases} -L\psi &= \lambda r^2 p(\psi - \psi_0) \text{ in } \Pi \\ \psi(0,z) &= 0 \\ \frac{1}{r}|\nabla\psi| &\to 0 \text{ as } r^2 + z^2 \to \infty \end{cases} \tag{9.7}$$

and prove

Theorem 9.1 *Suppose $p : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies (p4):*

(p4.1) p is non-decreasing and locally Lipschitz continuous;

(p4.2) $0 < p(s) \le c_1 + c_2 s^m$ for some $m > 0$, $c_1, c_2 > 0$ and $s > 0$.

Then there exists $\lambda \in \mathbf{R}$ and ψ solving (9.7) and such that

(i) ψ is even in z and $\partial\psi/\partial z < 0$ for $z > 0$;
(ii) the vortex core A_ψ is not empty and bounded;
(iii) $\psi \in C^2(\Pi - \partial A_\psi) \cap C^1(\Pi)$.

Proof. (outline) *Step 1.* Problem (9.7) is approximated by

$$-L\psi = \lambda r^2 p(\psi - \psi_0) \text{ in } B_R;\ \psi = 0 \text{ on } \partial B_R. \tag{9.8}$$

Let E_R denote the Hilbert space obtained as the closure of $C_0^\infty(B_R)$ under the norm

$$\|u\|_R^2 = \int_{B_R} \frac{1}{r}|\nabla u|^2 drdz.$$

Setting $P(u) = \int_0^u p(s)ds$, solutions ψ_R of (9.8) are found as

$$max\{\int_{B_R} P(u-\psi_0)rdrdz : u \in E_R, \|u\|_R^2 = 1\}.$$

Here λ enters as a Lagrange multiplier.

Step 2. Since ψ_R is a maximizer , one shows that $\psi_R = \psi_R^*$, the Steiner symmetrization of ψ_R, see Section 8. This fact permits to prove that there exists $\alpha, \beta > 0$, independent of R, such that

$$A_{\psi_R} \subset D := \left\{(r,z) \in \Pi : \frac{1}{\alpha} < r < \alpha,\ |z| < \beta\right\} \tag{9.9}$$

Step 3. The uniform bound (9.9) and the fact that $\|\psi_R\|_R = 1$ yield the existence of a $\bar{\psi} \in C^1(\bar{D})$ such that $\psi_R \to \bar{\psi}$ in $C^1(\bar{D})$ as $R \to \infty$. Such a $\bar{\psi}$ can be extended outside D to a solution of (9.7). ∎

A first existence result for $(\mathcal{P})$, namely when no parameter arises, has been obtained in [101] and, indipendently, in [21].

Theorem 9.2 *Let $W > 0$, $k > 0$ be given. Suppose that $p : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies, in addition to $(p4.1-2)$, the following assumptions:*

$(p4.3)$ $p(0) = 0$ *and is convex for* $s > 0$*;*

$(p4.4)$ *there exists* $\theta \in (0, \frac{1}{2})$ *such that* $P(s) \leq \theta s p(s)$*, for* $s > 0$.

Then $(\mathcal{P})$ has a solution ψ such that A_ψ is not empty and bounded. Moreover, ψ is even in z, $\partial\psi/\partial z < 0$ for $z > 0$ and $\psi \in C^2(\Pi - \partial A_\psi) \cap C^1(\Pi)$.

Proof. (Outline, see [21]) We consider again the approximated problems $(\mathcal{P}_R)$, taking $\lambda = 1$ in (9.8). Let $f_R : E_R \to \mathbf{R}$,

$$f_R(u) = \frac{1}{2}\|u\|_R^2 - \int_{B_R} P(u-\psi_0)rdrdz.$$

Assumptions $(p4.3-4)$ allow us to use the method discussed in Proposition 1.4, see also Theorem 2.9. Let us explicitely remark that for $k > 0$, p is flat in a neighbourhood of $s = 0$ and hence the arguments of Theorem 2.9 apply. As for the smoothness condition (the nonlinearity was assumed C^1 there), it can be overcome by a simple limiting process. A solution ψ_R of $(\mathcal{P}_R)$ is then found minimizing f_R constrained on

$$M_R = M_{f_R} = \{u \in E_R - \{0\} : \|u\|_R^2 = \int_{B_R} up(u-\psi_0)\, rdrdz\}.$$

Precisely, one shows

Lemma 9.3 *For all $R > 0$, $(\mathcal{P}_R)$ has a solution ψ_R such that $f_R(\psi_R) = \min_{M_R} f_R$. Moreover*
(i) $\psi_R = \psi_R^*$;
(ii) there exists $C > 0$ such that $\|\psi_R\|_R \le C$ for all $R > 0$.

Proof of Lemma 9.3. (i) Since $\|\psi_R^*\|_R \le \|\psi_R\|_R$ and $\int_{B_R} P(\psi_R - \psi_0) r dr dz = \int_{B_R} P(\psi_R^* - \psi_0) r dr dz$, one deduces that $f_R(\psi_R) \le f_R(\psi_R^*)$. Therefore, it suffices to show that $\psi_R^* \in M_R$. To see this, we note that

$$\|\psi_R^*\|_R^2 - \int_{B_R} \psi_R^* p(\psi_R^* - \psi_0) r dr dz \le \|\psi_R\|_R^2 - \int_{B_R} \psi_R p(\psi_R - \psi_0) r dr dz = 0.$$

The properties of M_R yield a $t^* \in (0,1]$ such that $t^* \psi_R^* \in M_R$. Since $f_R(tu)$ is increasing for $0 \le t \le 1$, $u \in M_R$ (see Theorem 2.9), it follows

$$f_R(t^* \psi_R^*) \le f_R(t^* \psi_R) \le f_R(\psi_R)$$

with strict inequality if $t^* < 1$. Since f_R achieves the minimum on M_R at ψ_R, then $t^* = 1$.
(ii) Fixed $R = R_0$, we extend ψ_{R_0} to all Π by setting $\psi_{R_0}(r,z) \equiv 0$ for $(r,z) \notin B_{R_0}$. Since $p(s) \equiv 0$, $\forall\, s \le 0$, one immediately finds that $\psi_{R_0} \in M_R$, for all $R \ge R_0$. Then, according to (2.10) of section 2, it follows

$$\|\psi_R\|_R^2 \le \left(\frac{1}{2} - \theta\right)^{-1} f_R(\psi_R) \le \left(\frac{1}{2} - \theta\right)^{-1} f_R(\psi_{R_0}) = \left(\frac{1}{2} - \theta\right)^{-1} f_{R_0}(\psi_{R_0}) \equiv C.$$

Proof of Theorem 9.2 completed. Lemma 9.3 allows us to repeat the limiting process sketched in Theorem 9.1, yielding a solution of $(\mathcal{P})$ with the required properties. ∎

Remarks 9.4 (i) The fact that ψ_R is symmetric would also follow from a more general result of Gidas, Ni and Nirenberg [80].
(ii) using the variational characterization of ψ_R it is possible to show that the approximated vortex core A_{ψ_R} is connected. Recently, it has been shown [131] that, for the *planar* vortex problem, the vortex core is connected. We do not know a proof of this fact for vortex rings in $\mathbf{R}^3$. ∎

The assumptions of Theorem 9.2 require that the vorticity function p is continuous and superlinear at infinity (condition (p4.4)). In particular, the case of $p = h$ and $k > 0$ cannot be covered by Theorem 9.2. A much more general, global existence result has been proved in [29]:

Theorem 9.5 *Suppose $p : \mathbf{R}^+ \to \mathbf{R}^+$ satisfies $(p4.1-2)$.*
Then the vortex problem $(\mathcal{P})$ has a solution ψ which satisfies (i)-(ii)-(iii) of Theorem 9.2.

For the sake of brevity we will not carry over the details of the proof, but rather we shall outline the main steps. See [29] for details.

Step 1: approximated problems. Let us suppose,to be specific, that the vorticity function p is bounded in such a way that one can exploit the arguments introduced in Section 8. Precisely, Theorem 8.11 and Remark 8.10 apply yielding the existence, for all $R > 0$ sufficiently large, of two symmetric solutions φ_R and ψ_R to $(\mathcal{P}_R)$. The former correspond to the global minimum, the latter to a Mountain-Pass critical point of the corresponding dual functional. Actually, solutions of $(\mathcal{P}_R)$ are found in [29] using a "direct" functional like f_R. However, this does not effect the arguments, because critical points of the dual functional give rise to critical points of f_R (in an appropriate sense, because it is in general not smooth), with the same variational characterization.

Step 2: estimates. The limiting process requires some care. First of all one remarks that the solutions corresponding to the minimum, φ_R, does not converge and has, indeed, the L^2-norm divergent. Consequently, any possible uniform estimate, like (iii) of Lemma 9.3, will not be an a-priori estimate related only to the fact that we are dealing with a solution of $(\mathcal{P}_R)$, but rather it shall depend on some specific feature of such a solution. Indeed, the idea here is to use the Mountain-Pass characterization of ψ_R to obtain a uniform estimate. Roughly, letting $c(R) = f_R(\psi_R)$, one verifies that $c(\cdot)$ is non-increasing, hence a.e. differentiable and there exists a sequence $R_n \to \infty$ such that $R_n c'(R_n) \to 0$. Then one shows that at each R where $c(R)$ is differentiable there results

$$\|\psi_R\|_R^2 \le a_1(c(R) + 2R|c'(R)| + a_2)$$

with a_1, a_2 constants independent of R. Plainly, this implies the required uniform estimate for ψ_R.

We shall see in more details the preceding arguments, discussing the existence of homoclinic orbits for a second order conservative system, see Section 10 below.

Step 3: limiting process. Once we have the preceding uniform bound one can repeat the limiting process as in Theorems 9.1 or 9.2, to find a solution of $(\mathcal{P})$ with the required properties. ∎

E. Limiting Behaviour of the Vortex Core. Consider a planar ideal fluid filling a *bounded* region $\Omega \subset \mathbf{R}^2$. In such a case, the problem of finding a stream function Ψ and a vortex core $A \subset \Omega$ can be formulated as a free boundary problem of the form

$$\begin{cases} -\Delta\Psi(x) &= 0 & x \in \Omega - A \\ -\Delta\Psi(x) &= \lambda p(\Psi), \quad \Psi > 0 & x \in A \\ \Psi(x) &= 0 & x \in \partial A \\ \Psi(x) &= -\Psi_0(x) & x \in \partial\Omega \end{cases} \tag{9.10}$$

Above, $x = (x_1, x_2) \in \mathbf{R}^2$, Δ is the usual Laplace operator in $\mathbf{R}^2$ and $\Psi_0(x)$ is a given function which is the counterpart of ψ_0. Let us point out that the real parameter λ introduced in (9.10) is, in contrast to (9.7), *prescribed*: actually, we are interested in the behaviour of the solutions of (9.10) as $\lambda \to \infty$. Limiting process of planar vortices has been studied in [46] when λ is a-priori unknown, as in Theorem 9.1. Below, we prefer to follow the approach of [31] where λ is a prescribed parameter, which is the natural setting in this kind of limiting questions.

As usual, it is convenient to transform (9.10) into a Dirichlet problem. Let q_0 be the solution of

$$\begin{cases} -\Delta q_0 &= 0 \text{ in } \Omega \\ q_0 &= \Psi_0 \text{ on } \partial\Omega \end{cases} \tag{9.11}$$

We set $\psi = \Psi + q_0$ and extend p to all $\mathbf{R}$ putting $p(s) \equiv 0$ for $s \le 0$. Then (9.10) becomes

$$\begin{cases} -\Delta\psi &= \lambda p(\psi - q_0) \text{ in } \Omega \\ \psi &= 0 \text{ on } \partial\Omega \end{cases} \tag{9.12}$$

Let us suppose that p satisfies $(p4.1 - 2 - 3 - 4)$. Then arguments similar to those of Theorem 9.2, leads to show that, for each λ (9.12) possesses a positive solution ψ_λ, with corresponding vortex core $A_\lambda = \{x \in \Omega : \psi_\lambda > q_0\}$. Precisely, letting $f_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda\int_\Omega P(u - q_0)dx, \quad u \in H_0^1(\Omega)$ and $M_\lambda = \{u \in H_0^1(\Omega) - \{0\} : \|u\|^2 = \lambda\int_\Omega up(u - q_0)dx\}$, ψ_λ is found as $min_{M_\lambda} f_\lambda$. This variational characterization leads to show that A_λ is connected (see Remark 9.4-ii). Moreover there results

$$c(\lambda) = f_\lambda(\psi_\lambda) \to 0 \text{ as } \lambda \to \infty. \tag{9.13}$$

Let us remark that (9.13) follows by means of a comparison between (9.12) and a model problem

$$\begin{cases} -\Delta\psi &= \lambda\bar{p}(\psi - \bar{q}_0) \text{ in } B \subset \Omega \\ \psi &= 0 \text{ on } \partial B \end{cases} \tag{9.14}$$

where B is a ball, $\bar{p}(s) = a_0 s^m, \quad m > 1, \quad (s > 0)$ and $\bar{q}_0 = max_\Omega q_0$. If $\bar{f}_\lambda$ and $\bar{M}_\lambda$ denote the functional (resp. the manifold) corresponding to (9.14), one shows that $\bar{c}(\lambda) := min_{\bar{M}_\lambda} \bar{f}_\lambda \to 0$ as $\lambda \to \infty$. Choosing a_0 in such a way that $a_0 s^m < p(s)$, one has $c(\lambda) \le \bar{c}(\lambda)$ and (9.13) follows. Since $c(\lambda) \ge (\frac{1}{2} - \theta)\|\psi_\lambda\|^2$ (cfr. (2.10)), there results

$$\|\psi_\lambda\| \to 0 \text{ as } \lambda \to \infty. \tag{9.15}$$

The above arguments allow us to control the asymptotic behaviour of stream function ψ_λ and the vortex core A_λ. Let G denote the Green operator of $-\Delta$ in Ω with zero Dirichlet boundary conditions and let

$$h(\lambda) = \lambda \int_{A_\lambda} p(\psi_\lambda - q_0)dx.$$

Theorem 9.6 *Let $\Psi_0 > 0$ on $\partial\Omega$ be smooth and suppose p satisfies* ($p4.1 - 2 - 3 - 4$). *Then (9.12) has a positive solution ψ_λ such that, as $\lambda \to \infty$:*
(i) diam $(A_\lambda) \to 0$;
(ii) let $x(\lambda)$ be any point in A_λ; then

$$\frac{\psi_\lambda(\cdot)}{h(\lambda)} - G(\cdot, x(\lambda)) \to 0 \text{ in } H_0^{1,p}(\Omega),\ 1 \le p < 2.$$

The arguments to deduce (i) and (ii) from (9.13-14) are the same as those employed in [46]. ■

10 Homoclinic Orbits

In this section we will investigate the existence of homoclinic orbits for a class of second order conservative systems

$$\ddot{q} + V'(q) = 0. \tag{10.1}$$

Assuming that $V'(0) = 0$, we say that a solution q of (10.1) is a homoclinic orbit (to the equilibrium $q = 0$) if $q \in C^2(\mathbf{R}, \mathbf{R}^n)$ solves (10.1), $q(t) \not\equiv 0$ and

$$q(t) \to 0, \ \ \dot{q}(t) \to 0 \ \ (t \to \pm\infty).$$

Recently, homoclinic orbits have been faced by critical point theory and existence and multiplicity results (when V depends on time in a periodic fashion) have been obtained; see Remark 10.6 below. In the sequel we will follow [11] showing how the approach used to prove Theorem 9.5 applies here, too. The potentials we deal with, have a strict local maximum at $x = 0$ and are negative in a bounded, deleted neighbourhood of $x = 0$. Precisely, we consider a system of the type (10.1) with

$$V(x) = -\frac{1}{2}\langle Ax, x\rangle + W(x) \tag{10.2}$$

and assume $(V1)$:

$(V1.1)$ A is a symmetric and positive definite constant matrix;

$(V1.2)$ $W \in C^1(\mathbf{R}^n, \mathbf{R})$, $W'(x) = o(|x|)$ as $x \to 0$;

$(V1.3)$ there exists a bounded open set $\Omega \subset \mathbf{R}^n$ such that:
 (i) $0 \in \Omega$,

(ii) $V(x) < 0$ in $\Omega - \{0\}$, and
(iii) $V = 0$ on $\partial\Omega$;

$(V1.4)$ $\langle V'(x), x\rangle > 0$ on $\partial\Omega$.

Our goal will be to show that (10.1) has a homoclinic orbit whenever V has the form (10.2) and satisfies $(V1)$. As anticipated before, the procedure we will use is similar to that employed in Theorem 9.5: the homoclinic will be found as limit of solutions of approximating Dirichlet boundary value problems on intervals $[-T, T]$ as $T \to \infty$.

A. Approximating Problems. First of all, it is convenient to replace V with a bounded potential. Precisely, by $(V1.3)$ there exists $R > 0$ such that $\Omega \subset B_R$; moreover, using $(V1.4)$ we can find $U_0 > 0$ and $U \in C^1(\mathbf{R}^n, \mathbf{R})$ such that

$$U(x) = V(x) \text{ for all } x \in \bar{\Omega} \tag{10.3}$$
$$U(x) > 0 \text{ for all } x \notin \bar{\Omega} \tag{10.4}$$
$$U(x) = U_0 \text{ for all } |x| \geq R. \tag{10.5}$$

Let $E_T = H_0^1([-T, T], \mathbf{R}^n)$ endowed with the norm

$$\|q\|_T^2 = \int_{-T}^{T} |\dot{q}|^2 dt.$$

Consider the functional $f_T \in C^1(E_T, \mathbf{R})$ defined by setting

$$f_T(u) = \frac{1}{2}\|u\|_T^2 - \phi_T(u),$$

where

$$\phi_T(u) = \int_{-T}^{T} U(u(t))dt.$$

Critical points of f_T give rise to solutions of

$$\ddot{q}(t) + U'(q(t)) = 0, \ -T < t < T; \quad q(-T) = q(T) = 0. \tag{10.6}$$

Since U is bounded, then plainly f_T is bounded from below, coercive and satisfies (PS). Moreover one has

Lemma 10.1 *For all $T > 0$ sufficiently large, f_T has two nontrivial critical points: a global minimum w_T and a Mountain-Pass critical point v_T, such that $f_T(w_T) < 0 < f_T(v_T)$. Moreover $\int_{-T}^{T} |w_T|^2 dt \to \infty$ as $T \to \infty$.*

Proof. Since Ω is bounded and (10.4) holds, there exists $\bar{z} \in H_0^1([-1,1],\mathbf{R}^n)$ such that

$$\bar{a} := \int_{-1}^{1} U(\bar{z}(t))dt > 0.$$

Letting $z_T(t) := \bar{z}(\frac{t}{T})$ and $a_1 = \frac{1}{2}\|\bar{z}\|_{T=1}^2$, there results

$$f_T(z_T) = \frac{1}{2T}\int_{-1}^{1} |\dot{\bar{z}}|^2 dt - T\int_{-1}^{1} U(\bar{z}(t))dt = \frac{1}{T}a_1 - T\bar{a}. \tag{10.7}$$

From (10.7) it follows that for $T > 0$ large enough, f_T achieves its global minimum at some w_T, with $f_T(w_T) < 0$. In addition, $(V1.1-2)$ and (10.3) immediately imply that $u = 0$ is a strict local minimum for f_T. Hence the Mountain-Pass Theorem applies and yields a second, nontrivial critical point v_T such that

$$c(T) := f_T(v_T) = \inf_{\gamma\in\Gamma_T}\sup_{0\le t\le 1} f_T(\gamma(t))$$

where

$$\Gamma_T = \{\gamma \in C([0,1],E_T) : \gamma(0) = 0,\ \gamma(1) = z_T\}.$$

Finally, from the definition of U, there exists $a > 0$ such that $U(x) \le a|x|^2$ and hence there results

$$a\int_{-T}^{T} |w_T|^2 dt \ge \int_{-T}^{T} U(w_T)dt \ge -f_T(w_T).$$

This and (10.7) immediately imply that $\int_{-T}^{T} |w_T|^2 dt \to \infty$ as $T \to \infty$, and the proof of the Lemma is completed. ∎

B. Uniform Estimates. The following Lemma provides the estimate, uniform in T, we need to show that v_T converges to a homoclinic orbit of (10.1). The proof will also indicate in a more detailed way the kind of the arguments sketched in the proof of Theorem 9.5, Step 2.

Lemma 10.2 *There exists a constant $a^* > 0$ and a sequence $T_n \to \infty$ such that for the Mountain-Pass critical points $v_n = v_{T_n}$ one has*

$$\|v_n\|_{T_n}^2 \le a^*. \tag{10.8}$$

Proof. The proof will be divided into several steps. In the following it is understood that T is large enough, in such a way that Lemma 10.1 holds true.

Step 1. First of all we remark that any $u \in E_T$ can be extended to a function in $H^{1,2}(\mathbf{R},\mathbf{R}^n)$, simply by setting $u(t) \equiv 0$ for all $|t| > T$. It immediately follows that $\Gamma_{T_1} \subset \Gamma_{T_2}$ whenever $T_1 \geq T_2$ and hence there results

$$c(T_2) \leq c(T_1), \ \ \forall\, T_1 \leq T_2 \,.$$

As a consequence, $c(T)$ is a.e. differentiable and there results

$$\int_{T_0}^{\infty} |c'(T)|dT \leq c(T_0) - \liminf_{T\to\infty} c(T) \leq c(T_0) < \infty.$$

Thus there exists a sequence $T_n \to \infty$ such that

$$T_n c'(T_n) \to 0. \tag{10.9}$$

Step 2. For $s < 1$ and close to 1 and $u \in E_T$, respectively $v \in E_{sT}$, we set $u_s(t) := u(\frac{t}{s})$ and $v^s(t) := v(st)$. The maps $u \to u_s$, and $v \to v^s$ define an isomorphism between E_T and E_{sT} as well as between Γ_T and Γ_{sT}. Then one readily infers

$$c(sT) = \inf_{\gamma\in\Gamma_T} \sup_{u\in\gamma[0,1]} f_{sT}(u_s). \tag{10.10}$$

Step 3. Let $T > 0$ be such that $c(\cdot)$ is differentiable at T. According to (10.10) there exists $\gamma^* \in \Gamma_T$ such that

$$\sup_{u\in\gamma^*[0,1]} f_{sT}(u_s) \leq c(sT) + \varepsilon(1-s) \ \ (0 < s < 1, \ \ 0 < \varepsilon < \frac{1}{5}). \tag{10.11}$$

Let $\hat{u} \in \gamma^*[0,1]$ be such that

$$f_T(\hat{u}) > c(T) - \varepsilon(1-s). \tag{10.12}$$

We claim that

$$\|\hat{u}\|_T^2 \leq \kappa(T) \tag{10.13}$$

where

$$\kappa(T) = c(T) + T|c'(T)| + 1.$$

Indeed, combining (10.11) and (10.12) we find

$$f_{sT}(\hat{u}_s) - f_T(\hat{u}) \leq c(sT) - c(T) + 2\varepsilon(1-s). \tag{10.14}$$

By a direct computation, one has

$$\|\hat{u}_s\|_{sT}^2 = \frac{1}{s}\|\hat{u}\|_T^2$$

as well as

$$\begin{aligned}\phi_{sT}(\hat{u}_s) &= \int_{-sT}^{sT} U(\hat{u}(\frac{t}{s}))dt = \\ &= s \cdot \int_{-T}^{T} U(\hat{u}(t))dt = s \cdot \phi_T(\hat{u}).\end{aligned}$$

Then it follows

$$f_{sT}(\hat{u}_s) - f_T(\hat{u}) = (1-s)\left(\frac{1}{2}\|\hat{u}_s\|_{sT}^2 + \frac{1}{s}\phi_{sT}(\hat{u}_s)\right). \tag{10.15}$$

Inserting (10.15) into (10.14) we infer, for $s \uparrow 1$,

$$\frac{1}{2}\|\hat{u}_s\|_{sT}^2 + \frac{1}{s}\phi_{sT}(\hat{u}_s) \le \frac{c(sT) - c(T)}{1-s} + 2\varepsilon \le T|c'(T)| + 3\varepsilon.$$

and this in turn implies

$$\begin{aligned}\frac{1}{2}\|\hat{u}_s\|_{sT}^2 + \frac{1}{2s}\|\hat{u}_s\|_{sT}^2 &\le \frac{1}{s}\phi_{sT}(\hat{u}_s) + T|c'(T)| + 3\varepsilon \le \\ &\le \frac{1}{s}c(sT) + T|c'(T)| + 4\varepsilon.\end{aligned} \tag{10.16}$$

Letting $s \uparrow 1$, one has

$$\|\hat{u}_s\|_{sT}^2 = \frac{1}{s}\|\hat{u}\|_T^2 \to \|\hat{u}\|_T^2$$

and hence (10.16) implies

$$\|\hat{u}\|_T^2 \le c(T) + T|c'(T)| + 5\varepsilon \le \kappa(T)$$

proving (10.13).

Step 4. The properties of $c(T)$ imply that there exists $a^* > 0$ such that $\kappa(T) \le a^*$ for all $T > 0$ sufficiently large. For $\varepsilon^* > 0$, let us define

$$N_{\varepsilon^*} = \{u \in E_T : \|u\|_T^2 \le a^*, \quad |f_T(u) - c(T)| \le \varepsilon^*\}.$$

If we show that for all $\varepsilon^* > 0$ and small enough there results

$$\inf_{u \in N_{\varepsilon^*}} \|f_T'(u)\| = 0 \tag{10.17}$$

the lemma will follow. Indeed, in such a case, there exists a sequence $u_k \in N_\delta$ such that $f_T'(u_k) \to 0$. Since (PS) holds, then, up to a subsequence, u_k

converges to some u_T in E_T, with $f_T'(u_T) = 0$ and $\|u_T\|_T^2 \le a^*$. Taking a sequence $T_n \to \infty$ where $c(\cdot)$ is differentiable, the lemma follows.

It remains to prove (10.17). For this, we can argue by contradiction. Let $\varepsilon^* > 0$ be such that

$$\|f_T'(u)\| > \varepsilon^*, \quad \forall\, u \in N_{\varepsilon^*} .$$

By the Deformation Lemma (cfr. Lemma 2.5) we can find an ε, with $0 < \varepsilon < \varepsilon_0 := min\{\varepsilon^*, c(T), \frac{1}{5}\}$, and a homeomorphism $\sigma : E_T \to E_T$ such that

$$\begin{array}{lll} \sigma(u) & = u & \text{if} |f_T(u) - c(T)| \ge \varepsilon_0 \\ f_T(\sigma(u)) & \le f_T(u) & \text{for all} u \qquad\qquad (10.18) \\ f_T(\sigma(u)) & \le c(T) - \varepsilon & \text{for all} u \in N_{\varepsilon^*} \cap f_T^{c(T)+\varepsilon} \qquad (10.19) \end{array}$$

Let $\gamma^* \in \Gamma_T$ be such that (10.11) holds (it is understood that $\varepsilon < \varepsilon_0$). Then for all $u \in \gamma^*[0,1]$ one has

$$f_T(u) \le f_{sT}(u_s) \le c(sT) + \varepsilon(1-s) \ \le c(T) + \varepsilon. \tag{10.20}$$

If there is a $u \in \gamma^*[0,1]$ such that $f_T(u) \ge c(T) - \varepsilon(1-s)$ then (10.13) implies that $u \in N_{\varepsilon^*}$ (indeed, one plainly has $|f_T(u) - c(T)| < \varepsilon^*$), and (10.19) yields $f_T(\sigma(u)) \le c(T) - \varepsilon(1-s)$. This and (10.18) imply that $sup_{0\le\tau\le1} f_T(\sigma(\gamma^*(\tau))) \le c(T) - \varepsilon(1-s)$, a contradiction with the definition of $c(T)$. This proves (10.17) and completes the proof of the Lemma. ■

C. Existence of Homoclinics. According to Lemma 10.2, let us consider a sequence of critical points $v_n := v_{T_n}$ which give rise to solutions of (10.6). As a consequence, the energy

$$h_n = \frac{1}{2}|\dot{v}_n(t)|^2 + U(v_n(t)) \tag{10.21}$$

is independent of t. Integrating (10.21) on the interval $[-T_n, T_n]$ and recalling that v_n satisfy the estimate (10.8), we find

$$\begin{aligned} 2|T_n h_n| &\le \frac{1}{2}\|v_n\|_{T_n}^2 + |\int_{-T_n}^{T_n} U(v_n)dt| \\ &\le a^* + |\int_{-T_n}^{T_n} U(v_n)dt|. \end{aligned} \tag{10.22}$$

Since $|\int_{-T_n}^{T_n} U(v_n)dt| \le c(T_n)$ and $c(T_n) \le const.$ (see point (a) in the proof of Lemma 10.2), then (10.22) implies ($|T_n h_n| \le const.$, and hence) $h_n \to 0$. Since $v_n(\pm T_n) = 0$ and $U(0) = 0$, (10.21) yields

$$h_n = \frac{1}{2}|\dot{v}_n(\pm T_n)|^2.$$

Let us remark that, actually, $h_n > 0$, otherwise $\dot{v}_n(\pm T_n) = 0$ and $v_n(\pm T_n) = 0$ would imply $v_n(t) \equiv 0$, a contradiction.

Lemma 10.3 *There exists $\delta > 0$ such that $\|v_n\|_\infty \geq \delta > 0$.*

Proof. From the preceding remarks it follows

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}|v_n(t)|^2 &= |\dot{v}_n(t)|^2 - \langle U'(v_n(t)), v_n(t)\rangle \\ &= 2h_n - 2U(v_n(t)) - \langle U'(v_n(t)), v_n(t)\rangle \\ &> -2U(v_n(t)) - \langle U'(v_n(t)), v_n(t)\rangle. \qquad (10.23)\end{aligned}$$

Using $(V1.1-2)$ and (10.3), we can find a $\delta > 0$ such that $B_\delta \subset \Omega$ and

$$\begin{aligned}-2U(x) - \langle U'(x), x\rangle &= \\ 2\langle A(x), x\rangle - W(x) - \langle W'(x), x\rangle &\geq 0 \quad \forall\, x \in B_\delta\,. \qquad (10.24)\end{aligned}$$

Since $\frac{d^2}{dt^2}|v_n(t)|^2 \leq 0$ at points t^* where $\|v_n\|_\infty = |v_n(t^*)|$, (10.23) and (10.24) imply that $\|v_n\|_\infty \geq \delta$, as required. ∎

In the sequel it is convenient to make a rescaling of time and define $y_n(t) = v_n(t - \tau_n)$ in such a way that $y_n(0) = \delta$. Plainly, y_n are solutions of (10.6) in an open interval I_n, with $|I_n| = 2T_n$, $y_n(t) \equiv 0$ for all $t \notin I_n$, satisfy the energy relationship (10.21) and are such that

$$\int_{-\infty}^{\infty} |\dot{y}_n(t)|^2 dt \leq c^*. \qquad (10.25)$$

From (10.25) it immediately follows that, up to a subsequence, $y_n \to y$ in $L^\infty_{loc}(\mathbf{R}, \mathbf{R}^n)$. Moreover $\dot{y}_n \to \dot{y}$ weakly in $L^2(\mathbf{R}, \mathbf{R}^n)$ and one has

$$\int_{-\infty}^{\infty} |\dot{y}(t)|^2 dt \leq const. \qquad (10.26)$$

Moreover, since y_n solve (10.6) and U is bounded, then one easily finds that, again up to a subsequence, $y_n \to y$ in $W^{2,\infty}_{loc}(\mathbf{R}, \mathbf{R}^n)$. Hence y is a weak and, by regularity, strong solution of (10.6). In addition, (10.21) and the preceding discussion yield

$$\frac{1}{2}|\dot{y}(t)|^2 + U(y(t)) = \lim_{n\to\infty}\left(\frac{1}{2}|\dot{y}_n(t)|^2 + U(y_n(t))\right) = \lim_{n\to\infty} h_n = 0. \qquad (10.27)$$

From (10.27) it follows that $U(y(t)) \leq 0$ and thus, according to (10.3-4), $y(t) \in \bar{\Omega}$ and hence $U(y(t)) = V(y(t))$, namely y is a solution of (10.1). Let us also remark that $y(t) \not\equiv 0$, because $y(0) = \lim_{n\to\infty} |y_n(0)| = \delta$.

In order to show that y is a homoclinic orbit, it remains to show that $y(t) \to 0$ as $t \to \pm\infty$. This follows from the next lemma.

Lemma 10.4 $y(t) \in L^2(\mathbf{R}, \mathbf{R}^n)$.

We will not carry over the details of the proof of this lemma, which is rather technical, but limit ourselves to outline the arguments. For more details, we refer to [11]. Using the assumptions on V, one can find positive $\rho, \varepsilon, \delta$, and subsets B_ρ, C_ρ and D_ρ of Ω such that:

$$W(x) \leq \varepsilon |x|^2 \quad \forall \quad x \in B_\rho = \{|x| < \rho\} \tag{10.28}$$

$$\langle V'(x), x \rangle \geq \delta > 0 \quad \forall \quad x \in D_\rho = \{x \in \bar{\Omega} : dist(x, \partial\Omega) < \rho\} \tag{10.29}$$

and $C_\rho = \Omega - \{B_\rho \cup D_\rho\}$. In the sequel the dependence on ρ will be understood. Let τ_B, respectively τ_C, τ_D denote the interval $\{t : y(t) \in B\}$ (resp. $\in C, \in D$). From $(V1.1)$ we infer there exists $\alpha > 0$ such that $\langle A(x), x \rangle \geq \alpha |x|^2$. Then, taking into account (10.28), one finds

$$\begin{aligned} \frac{1}{2}\alpha \int_{\tau_B} |y(t)|^2 dt &\leq \frac{1}{2}\int_{\tau_B} \langle A(y(t)), y(t) \rangle dt = \\ &= \frac{1}{2}\int_{\tau_B} |\dot{y}(t)|^2 dt + \int_{\tau_B} W(y(t)) dt \leq \\ &\leq const. + \varepsilon \int_{\tau_B} |y(t)|^2 dt. \end{aligned} \tag{10.30}$$

Taking ρ possibly smaller, we can suppose that $\varepsilon < \frac{1}{2}\alpha$ and (10.30) yields

$$\int_{\tau_B} |y(t)|^2 dt < a_1 . \tag{10.31}$$

Next, since $V(x) \leq -\beta < 0$ on C, then one immediately has

$$\beta \cdot meas(\tau_C) \leq \int_{\tau_C} -V(y(t)) dt < a_2 . \tag{10.32}$$

Finally, using (10.29), one shows that τ_D does not contain any unbounded interval J; otherwise, from (10.1) it would follow (let, for example $J = [t^*, +\infty)$)

$$\begin{aligned} &\int_J |\dot{y}(t)|^2 dt + |y(t^*)||\dot{y}(t^*)| + \limsup_{t \to \infty} |y(t)||\dot{y}(t)| \geq \\ &\geq \int_J \langle V'(y(t)), y(t) \rangle dt. \end{aligned} \tag{10.33}$$

Since (10.27) and $U \leq const.$ imply that both $|y(t)|$ and $|\dot{y}(t)|$ are bounded ($t \in \mathbf{R}$), then (10.33) and (10.29) give rise to a contradiction. Actually, with some more care, one shows that $meas(\tau_D)$ is finite. This, (10.31) and (10.32) imply that $\int_{-\infty}^{\infty} |y(t)|^2 dt < \infty$, and the proof of the Lemma is complete. ■

We are finally in position to prove

Theorem 10.5 *Suppose that V has the form (10.2) and satisfies* ($V1$). *Then (10.1) possesses a homoclinic orbit.*

Proof. As anticipated before, it remains only to show that $y(t), \dot{y}(t) \to 0$ as $t \to \pm\infty$. Indeed, integrating

$$|y(t)| \leq |y(s)| + |\int_s^t \dot{y}(r)dr|$$

on $[t-\frac{1}{2}, t+\frac{1}{2}]$ and using the Hölder inequality, one infers

$$\begin{aligned} |y(t)| &\leq \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |y(s)|ds + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\int_s^t \dot{y}(r)dr\ |ds \\ &\leq \left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |y(s)|^2 ds\right)^{1/2} + \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\dot{y}(r)|dr\ ds \\ &\leq 2\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |y(s)|^2 + |\dot{y}(s)|^2 ds\right)^{1/2}. \end{aligned}$$

This, jointly with (10.26), Lemma 10.4 and (10.27), show that $\lim_{t\to\pm\infty} y(t) = 0$, as well as $\lim_{t\to\pm\infty} |\dot{y}(t)| = 0$, proving the theorem. ■

Remark 10.6 A result quite similar to Theorem 10.5 has been found by completely different methods in [112]. In a preceding paper [111] the existence of homoclinic orbits was proved under the additional assumption that $W \leq \theta\langle W'(x), x\rangle$ for some $0 < \theta < \frac{1}{2}$ and all $x \neq 0$. Homoclinics for first order Hamiltonian systems are discussed in [86] still under a super-quadraticity assumption on the Hamiltonian. Dealing with time depending potentials: $V(t+T, x) = V(t, x)$, the existence of multiple homoclinic orbits have been proved in [66] and [114] for first order *convex* Hamiltonian systems, and in [67] for second order systems with a super-quadratic potential. ■

Remark 10.7 Similar arguments allow us to prove the existence of a homoclinic orbit for a class of Potentials V which possess a singularity at some $p \in \Omega - \{0\}$. Roughly, we can handle the case when $V \in C^1(\mathbf{R}^n - \{p\}, \mathbf{R})$, satisfies the preceding assumptions and behaves like $-\frac{1}{|x-p|^a}$, with $a \geq 2$, near the singularity $x = p$. For results on homoclinics for other, different classes of singular potentials we refer to [124] and, as far as a multiplicity result is concerned, [49]. ■

11 Conservative Systems with Singular Potentials (I)

These last 3 sections deal with the existence of periodic motions to a class of Lagrangian systems with Singular Potentials arising in Celestial Mechanics. One of the main purposes will be to provide a suitable variational framework for this kind of problems.

Min-max methods and Linking Theorems, such as those discussed in Sections 1-5, have been usefully employed to find nonlinear oscillations of a broad class of first order Hamiltonian systems like

$$\begin{cases} \dot{p} = & -H_q(p,q) \\ \dot{q} = & H_p(p,q) \end{cases} \tag{11.1}$$

when the Hamiltonian $H(p,q) = H : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is smooth.

Referring the reader to the recent books [73] [98] and [120] as well as to the extensive bibliography therein, let us limit ourselves to a short overview, only.

a. *Solutions with prescribed period.* These results deal with the existence of periodic solutions of (11.1) having a prescribed period $T > 0$ and cover both the case in which H is super-quadratic, as well as that one of sub- quadratic or quadratic growth. See, for example [3, 107].

The question of the minimality of the period has also been studied, see, for example, [22, 58, 74, 81]. These latter results are based upon a remarkable procedure, the *Dual Variational Principle* [57], which is widely discussed in Ekeland's book [73].

b. *Solutions with prescribed energy.* When (11.1) is an autonomous system, the Hamiltonian H is a constant of the motion and it makes sense to seek periodic solutions having prescribed energy: $H(p,q) = h$. The case in which H is smooth and $H^{-1}(h)$ is a compact hypersurface in $\mathbf{R}^{2n}$ has been extensively investigated. Let $\Omega_h \subset \mathbf{R}^{2n}$ be bounded and such that $H^{-1}(h) = \partial\Omega_h$. If Ω_h is convex (resp., star-shaped with respect to $(p,q) = (0,0)$, i.e. radially diffeomorphic to the unit sphere $\{|p|^2 + |q|^2 = 1\}$) the existence of one closed orbit of (11.1) such that $H(p(t), q(t)) = h$ has been estabilished in [130] (resp., [108]). Variational tools in connections with Symplectic Geometry have permitted to greatly improve such a result. For example, if $H^{-1}(h)$ is a compact hypersurface in $\mathbf{R}^{2n}$ and $p \cdot H_p > 0$ for all $(p,q) \in H^{-1}(h)$, $p \neq 0$, then (11.1) has a periodic solution with energy h [87].

c. *Multiple trajectories on an energy surface.* If $H \in C^2$, $H(0,0) = H'(0,0) = 0$ and $H''(0,0)$ is positive definite, Weinstein [129] and Moser [99] have shown there exist n distinct modes of vibration on the surface $H = \varepsilon$, for any $\varepsilon > 0$, small enough, improving the celebrated *Lyapunov Center Theorem* . A remarkable extension in the large of such a local result is reported below and is due to Ekeland and Lasry [75] (see also [23] for a different proof, and [45] for an improvement).

Suppose that Ω_h is bounded, convex and let

$$r^2 = \inf_{H^{-1}(h)} \{|p|^2 + |q|^2\}, \quad R^2 = \sup_{H^{-1}(h)} \{|p|^2 + |q|^2\}$$

If $R^2 < 2r^2$ then (11.1) possesses n geometrically distinct periodic orbits with energy h.

The proof makes use of the Lusternik-Schnirelman theory for S^1 invariant functionals in connection with the *Dual Variational Principle.*

In contrast with the results outlined above, which deal with regular Hamiltonians, defined on all of $\mathbf{R}^{2n}$, we will discuss in the sequel Conservative Systems like

$$\ddot{q} + V'(q) = 0 \tag{11.2}$$

where the potential V is, roughly, of the form $V(x) \simeq -\frac{1}{|x|^\alpha}$, with $\alpha > 0$.

Conservative Systems with those singular potentials include, for example, Kepler's Problem and (see Section 13 below) the N-body Problem, which are a classical subject of Celestial Mechanics. See also Examples 11.3 for other examples.

It is worth pointing out that the point of view from which these problems have been studied in Celestial Mechanics is quite different from ours. For example a typical question has been to estabilish stability or other qualitative properties of orbits. These matters have been usually studied by means of perturbation techniques, which give rise to rather precise results but, in general, they work for specific classes of problems, such as the Kepler one, or the restricted 3-body problem.

On the other side, systems like (11.2) have a variational structure and it is natural to attempt to handle them by means of the Calculus of Variations in the Large. Our point of view will be the usual one in Nonlinear Functional Analysis: we will try to provide a general framework which allows us to understand the common features of various classes of potentials, mainly depending upon their behaviour at the singularity and at infinity. The results will be global in nature and will cover a large class of gravitational-like potentials, including the classical ones as particular cases.

Postponing to section 12 the study of periodic motions of (11.2) with prescribed energy, let us begin discussing the problem

(P_T) Given $T > 0$, find T-periodic solutions of (11.2).

Remark 11.1 Problem (P_T) makes sense even if $V = V(t,x)$ and $V(t+T,x) = V(t,x)$ (in such a case V' stands for $\frac{\partial V}{\partial x}$). For the sake of notations, the t-dependence will be always understood hereafter. ■

Set $\Omega = \mathbf{R}^n - \{0\}$ and let us consider a potential $V \in C^1(\mathbf{R} \times \Omega, \mathbf{R})$[1]. Since the potentials we deal with have a singularity at $x = 0$, the meaning of *solution* has to be clarified.

Let

$$E = H^{1,2}(S_T, \mathbf{R}^n).$$

The norm and the scalar product in E will br denoted, respectively, by $\|\cdot\|$ and $(\cdot|\cdot)$. For $u \in E$ we set $\mathcal{C}_u = \{t \in [0,T] : u(t) = 0\}$. Let

$$L(u, \dot{u}) = \frac{1}{2}|\dot{u}|^2 - V(u).$$

denote the Langrangian. We say that $u \in E$ is a *generalized* solution of (P_T) whenever the following conditions are satisfied:

(i) $u \in E$ and $\int_0^T L(u, \dot{u})dt < +\infty$;

(iii) $\mathcal{C}_u$ has zero measure;

(iii) for all $t \notin \mathcal{C}_u$ $u \in C^2$ and solves (11.2).

[1]The case in which $\Omega = \mathbf{R}^n - K$, K compact, can also be handled, see Remark 11.9

When V is autonomous, we shall also require:
(iv) there is a constant ℓ such that $\frac{1}{2}|\dot{u}(t)|^2 + V(u(t)) \equiv \ell$ for all $t \notin \mathcal{C}_u$.

Using a terminology borrowed from Celestial Mechanics, we will say that u is a non-collision (collision, resp.) orbit whenever $\mathcal{C}_u = \emptyset$ (resp. $\neq \emptyset$).

We shall work on the Hilbert space E. For future refence we set $W = \{w \in E : \int_0^T w(t)dt = 0\}$ and $E = \mathbf{R}^n \oplus W$. Correspondingly, let us write $u = \xi + w$, with $\xi = \int_0^T u(t)dt$ and $w \in W$. Recall that for $w \in W$, $|\dot{w}|_2^2$ is a norm equivalent to the $H^{1,2}$ one. Moreover one trivially has

$$|w|_\infty \le \sqrt{T}\,|\dot{w}|_2 \tag{11.3}$$

for all $w \in W$.

The Variational principle we will employ is nothing but the classical principle of the *Least Action.* Let

$$\Lambda = \{u \in E : u(t) \neq 0 \ \ \forall\, t \in [0,T]\},$$

and define functionals $f, \psi : \Lambda \to \mathbf{R}$ by setting

$$\psi(u) = \int_0^T V(u)dt\,,$$

$$f(u) = \int_0^T L(u,\dot{u})dt = \frac{1}{2}|\dot{u}|_2^2 - \psi(u).$$

One immediately cheks that $f \in C^1(\Lambda, \mathbf{R})$ and that any $u \in \Lambda$ such that $f'(u) = 0$ is a non-collision solution of (P_T). In other words, such an u is a classical solution of (11.2) which does not cross (or fall into) the singularity $x = 0$.

In order to highlight the kind of problems one meets, let us take the model potential $V(x) = -|x|^{-\alpha}$, with $\alpha > 0$, so that the functional f becomes

$$f(u) = \int_0^T \left[\frac{1}{2}|\dot{u}|^2 + \frac{1}{|u|^\alpha}\right] dt\,.$$

Plainly, $\inf_{u\in\Lambda} f(u) = 0$ but f does not achieve the minimum on Λ and hence one needs to look for critical points of f by means of min-max techniques. In particular, since $cat(\Lambda) = +\infty$ (see Example 2.2-(iv)), it is natural to try to use the Lusternik-Schnirelman theory. Nevertheless, in order to apply those tools, one has to overcome two main difficulties: first of all, since f is defined on the open dense subset $\Lambda \subset E$, one has to control the behaviour of f on the boundary $\partial\Lambda$; second, the (PS) condition does not hold, in general: for any sequence $x_m \in \mathbf{R}^n - \{0\}$ such that $|x_m| \to +\infty$, one has $f(x_m) \to 0$ as well as $f'(x_m) \to 0$.

In particular, because of the former problem, it is convenient to distinguish in the sequel two cases, according to the behaviour of V at $x = 0$: in subsection A we will deal with potentials $V(x) \simeq -|x|^{-\alpha}$ near $x = 0$, with $\alpha \geq 2$ (referred to as *Strongly Attractive Potentials* or as *Strong Forces*); in subsection B we will be concerned with the case $0 < \alpha < 2$ (*Weakly Attractive Potentials* or *Weak Forces*) which includes, among other examples, the Kepler problem. We anticipate that the results on the Strong Forces, apart of being interesting in itself because of applications to physically interesting cases (see Examples 11.3), will serve as a basic tool for the Weakly Attractive Potentials. In a last subsection C we will deal with perturbation results.

A. STRONGLY ATTRACTIVE POTENTIALS. Let us start with a key lemma which relates the behaviour of V at $x = 0$ with the behaviour of f at $\partial\Lambda$.
Lemma 11.2 *Suppose that V satisfies*

(SF) $\quad \exists\, a, \rho > 0$ *such that* $V(x) \leq -a|x|^{-2}$, $\forall\, 0 < |x| < \rho$ [2].

Let $u_m \in \Lambda$ be a sequence such that $u_m \to u$, weakly in E and uniformly, and let $u \in \partial\Lambda$. Then one has

$$\psi(u_m) \to -\infty \ .$$

Proof. We will prove the lemma under the stronger assumption that $V(x) \leq -a|x|^{-2}$ for all $x \neq 0$ (the general case can be handled with small changes), so that one has

$$\int_0^T V(u_m(t)dt \leq -a \cdot \int_0^T \frac{1}{|u_m(t)|^2}dt \ . \tag{11.4}$$

We will show the last integral diverges whenever $u_m \to u \in \partial\Lambda$ (weakly and uniformly). Without loss of generality, we can assume that $u \in \partial\Lambda - \{0\}$: otherwise, $u_m \to 0$ uniformly on $[0, T]$ and the result follows trivially. Let s_0, $s_1 \in [0, T]$ be such that $u(s_0) = 0$, whereas $u(s_1) \neq 0$. There results:

$$\begin{aligned}[\log|u_m(t)|]_{s_0}^{s_1} &\leq \int_{s_0}^{s_1} \frac{|\dot{u}_m(t)|}{|u_m(t)|}dt \leq \\ &\leq \left[\int_0^T |\dot{u}_m|^2\right]^{\frac{1}{2}} \cdot \left[\int_0^T \frac{1}{|u_m|^2}\right]^{\frac{1}{2}} \ . \end{aligned} \tag{11.5}$$

[2]It is understood that assumption (SF), as well as the following $(V2.1)$, etc., holds uniformly in t.

Since $u_m \to u$ weakly and $u_m(s_0) \to 0$, $u_m(s_1) \to u(s_1) \neq 0$, (11.5) readily implies that $\int_0^T |u_m|^{-2} \to +\infty$ and the lemma follows from (11.4). ■

Before going on, let us indicate some problems of Celestial Mechanics in which Strong Forces arise.

Examples 11.3 (i) The relativistic correction, in static conditions, to a gravitational potential like $-\frac{1}{|x|} + \gamma U$, U smooth, leads to a potential of the form (see [95]) $V(x) = (1+a)(-\frac{1}{|x|} + \gamma U) - b(-\frac{1}{|x|} + \gamma U)^2$, where a, b are constants of the order of c^{-2}, c being the speed of light. Plainly, such a V satisfies (SF).

(ii) Consider a particle $x \in \mathbf{R}^3$ attracted by a solid body S according to Newton's law. Let I_i, $i = 1,2,3$, denote the moments of inertia of S. The body S attracts the particle x with a force which can be obtained integrating over S the attraction forces (see [90, Chapter V]) of each element dS. The corresponding potential is given by (up to constants)

$$V(x) = -\frac{1}{|x|} - \frac{Ax \cdot x}{|x|^5} + O(|x|^{-4})$$

where A is a matrix depending on I_i. The remainder term $O(|x|^{-4})$ and A are zero if and only if the solid is spherical (i.e. if $I_1 = I_2 = I_3$). If this is not the case, V satisfies (SF).■

As anticipated before, the (PS) condition does not hold at level $c = 0$. A condition which guarantees that this is the only level where (PS) fails, is given in the following lemma.

Lemma 11.4 *Suppose that V satisifies (SF) and*

$(V2.1)$ $\quad V(x) \to 0$ *and* $V'(x) \to 0$ *as* $|x| \to \infty$.

Then $(PS)_c$ holds, whenever $c > 0$.

Proof. Let u_m be a sequence such that

$$f(u_m) = \frac{1}{2}|\dot{u}_m|_2^2 - \psi(u_m) \to c > 0 \tag{11.6}$$

and

$$f'(u_m) \to 0\ . \tag{11.7}$$

Since $\sup_\Omega V < +\infty$, then ψ is bounded above on Λ and from (11.6) one infers that

$$|\dot{u}_m|_2^2 \le a_1\,[3]\ . \tag{11.8}$$

[3] Here and in the following sections 12 and 13, $a_1, a_2, \ldots$ denote positive constants.

Letting $u_m = \xi_m + w_m$, with $\xi_m \in \mathbf{R}^n$ and $w_m \in W$, one immediately deduces from (11.8) there exists $\bar{w} \in W$ such that

$$w_m \to \bar{w}, \quad \text{weakly in E,} \tag{11.9}$$

up to a subsequence.

Moreover, we claim that $|\xi_m| \leq a_2$. Otherwise, $|\xi_m| \to +\infty$, (11.3) and (11.8) imply that

$$|u_m(t)| \to +\infty, \quad \text{uniformly,} \tag{11.10}$$

and hence, from $(V2.1)$

$$\int_0^T V'(u_m(t)) \cdot w_m(t)\, dt \to 0 \tag{11.11}$$

as well as

$$\psi(u_m) \to 0\,. \tag{11.12}$$

Then, using (11.7), (11.9) and (11.11) it follows

$$|\dot{w}_m|_2^2 = (f'(u_m)|w_m) + \int_0^T V'(u_m(t)) \cdot w_m(t)\, dt \to 0.$$

and this, jointly with (11.12), yields

$$f(u_m) = \frac{1}{2}|\dot{w}_m|_2^2 - \psi(u_m) \to 0,$$

a contradiction.

Since $|\xi_m| \leq a_2$ and recalling (11.9), it follows that $u_m = \xi_m + w_m \to \bar{\xi} + \bar{w}$ (up to a subsequence), for some $\bar{\xi} \in \mathbf{R}^n$. From (11.6) and Lemma 11.2 one has that $\bar{u} := \bar{\xi} + \bar{w} \in \Lambda$ and, by a standard argument, one shows that $u_m \to \bar{u}$ strongly, completing the proof. ■

Remark 11.5 The preceding proof actually shows that every (PS) sequence $u_m = \xi_m + w_m$, such that $|\xi_m|$ is bounded, has a converging subsequence. ■

Since the (PS) condition does not hold at the level $c = 0$ we need to evaluate the topology of sublevels f^ε, $\varepsilon > 0$, small enough. This will be done by means of the Lusternik-Schnirelman category (see Section 2).

For $r > 0$ let $\Sigma(r) = \{x \in \mathbf{R}^n : \ |x| \geq r\}$. Plainly, $cat(\Sigma(r); \Lambda) = 2$.

Lemma 11.6 *Let V satisfy $(V2.1)$ (actually, $V(x) \to 0$ as $|x| \to +\infty$ would suffice) and*

$(V2.2)$ $\quad V(x) < 0$ *for all* $x \in \Omega$

Then there exists $\varepsilon^* > 0$ such that $cat(f^\varepsilon; \Lambda) = 2$ for all $0 < \varepsilon \leq \varepsilon^*$.

Proof. We start showing:

(a) Let $u = \xi + w$ be such that $f(u) \leq 1$. Then

$$|w|_\infty < \sqrt{2T}\ .$$

To see this, it suffices to note that $(V2.2)$ implies $\psi(u) < 0\ \forall u \in \Lambda$ and hence from $f(u) \leq 1$ it follows

$$\frac{1}{2}|\dot{w}|_2^2 < \frac{1}{2}|\dot{w}|_2^2 - \psi(u) = f(u) \leq 1.$$

Thus $|w|_\infty \leq \sqrt{T} \cdot |\dot{w}|_2 < \sqrt{2T}$.

Next, we claim:

(b) For all $r > 0$, $\exists\, \varepsilon(r) > 0$ such that $\forall\, 0 < \varepsilon \leq \varepsilon(r)$ there results

$$f(\xi + w) \leq \varepsilon \Rightarrow |\xi| > r.$$

Indeed, otherwise, there exists $r_0 > 0$ and a sequence $u_m = \xi_m + w_m \in \Lambda$ such that

$$f(u_m) \leq \frac{1}{m} \tag{11.13}$$

$$|\xi_m| \leq r_0 \tag{11.14}$$

From (11.13), point (a) above and (11.14) it follows

$$|u_m(t)| \leq r_0 + \sqrt{2T} \equiv a_3$$

Letting $V_0 = \sup\{V(x)\ :\ |x| \leq a_3\} < 0$, we infer

$$f(u_m) \geq -\int_0^T V(u_m(t))dt \geq -TV_0\ ,$$

in contradiction with (11.13), whenever m is large enough. This proves (b).

To complete the proof of the Lemma, let $r^* = \sqrt{2T} + 1$. According to (a) and (b), there exists $\varepsilon^* > 0$ such that for all $u = \xi + w \in f^\varepsilon$ one has:

$$|\xi| > r^* \text{ and } |w|_\infty < \sqrt{2T}, \tag{11.15}$$

whenever $\varepsilon \leq \varepsilon^*$. For $(s, u) \in [0, 1] \times f^\varepsilon$, $(\varepsilon \leq \varepsilon^*)$, $u = \xi + w$, let $\varphi(s, u) = \xi + sw$. From (11.15) it follows that

$$|\xi + sw(t)| \geq |\xi| - |w|_\infty > r^* - \sqrt{2T} = 1\ .$$

Therefore $\varphi(s,u) \in \Lambda$ for all $u \in f^\varepsilon$, $\varepsilon \le \varepsilon^*$. Hence φ defines a deformation on Λ such that $\varphi(1,u) = \Sigma(r^*)$ and Lemma 2.1-(iii) implies that $cat(f^\varepsilon;\Lambda) \le cat(\Sigma(r^*);\Lambda) = 2$. Since, as a consequence of $(V2.1)$, $\Sigma(r) \subset f^\varepsilon$, whenever $r > 0$ is sufficiently large, there results $cat(f^\varepsilon;\Lambda) \ge 2$, too, and the lemma follows. ∎

We are now in position to state:

Theorem 11.7 *Suppose $V \in C^1(\mathbf{R}\times\Omega,\mathbf{R})$ is T-periodic in t and satisfies (SF) and $(V2)$*[4]*. Then problem (P_T) has infinitely many non collision solutions.*

Proof. As recalled before, $cat(\Lambda) = +\infty$, hence $\mathcal{A}_k = \{A \in \Lambda : cat(A;\Lambda) \ge k\}$ is not empty for each integer k and we can define the min-max levels (see (2.1))

$$c_k = \inf_{A\in\mathcal{A}_k}\left[\sup\{f(u) : u \in A\}\right].$$

According to Lemma 11.5, $c_k \ge \varepsilon^*$ for all $k \ge 3$, otherwise there would exists $A \subset f^{\varepsilon^*}$ such that $A \in \mathcal{A}_3$. Then $cat(f^{\varepsilon^*};\Lambda) \ge cat(A;\Lambda) \ge 3$, a contradiction. Since $(PS)_c$ holds for all $c > 0$ (Lemma 11.3), it follows that any c_k, $k \ge 3$, is a critical level for f. Let us explicitely point out that the arguments of Section 2 can be carried over in the present case, because of (SF). Indeed, as a consequence of Lemma 11.2 one has that $f^a \cap \partial\Lambda = \emptyset$ for all a. ∎

Assumption (V2.2) can be weakened.

Theorem 11.8 *Suppose $V \in C^1(\mathbf{R}\times\Omega,\mathbf{R})$ is T-periodic in t and satisfies (SF), $(V2.1)$ and*

$(V2.2')$ *There exists $R > 0$ such that $V(x) < 0$ for all $|x| > R$.*

Then problem (P_T) has infinitely many non-collision solutions.

Proof. (Sketch) Only Lemma 11.6 needs to be substituted. Roughly, arguing as in points (a) and (b) therein, one shows there exists $r^* > 0$ such that $|\xi| \ne r^*$, whenever $u = \xi + w \in f^\varepsilon$, $\varepsilon > 0$ small. It follows that $f^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$, where

$$\Gamma_1^\varepsilon = \{u = \xi + w \in f^\varepsilon : |\xi| < r^*\},$$
$$\Gamma_2^\varepsilon = \{u = \xi + w \in f^\varepsilon : |\xi| > r^*\}.$$

[4]hereafter (V2) means both (V2.1) and (V2.2)

Note that both Γ_1^ε and Γ_2^ε are invariant with respect to the steepest descent flow (see the Deformation lemma in Section 2). Plainly, the arguments of Lemma 11.6 can be still used to show that

$$cat(\Gamma_2^\varepsilon; \Lambda) = 2. \tag{11.16}$$

As for Γ_1^ε, we can assume without loss of generality that

$$cat(\Gamma_1^\varepsilon; \Lambda) < +\infty. \tag{11.17}$$

Otherwise, taking into account that, according to Remark 11.5, (PS) condition holds on Γ_1^ε, we would already find infinitely many critical points in Γ_1^ε.

From (11.16) and (11.17) it follows that $k_0 := cat(f^\varepsilon; \Lambda) < +\infty$. As in Theorem 11.7, we infer that $c_k > \varepsilon > 0$ for all $k > k_0$ and this suffices to apply the Lusternik-Schnirelman theory, showing that any c_k, $k > k_0$ is a critical level for f. ∎

Remark 11.9 The arguments outlined before closely follow the paper [13], where the Morse Theory has been used instead of the Lusternik-Schnirelman one. Among other results contained therein, let us recall here the following ones:

(a) The existence of infinitely many non-collision solutions of (P_T) can be proved substituting (V2.2') with:

$(V2.2'')$ there exists $R > 0$ such that $V'(x) \cdot x < 0$ for all $|x| > R$.

(b) It is possible to show that (P_T) possesses a sequence of solutions u_k such that $f(u_k) \to +\infty$.

(c) It is possible to handle potentials V which are defined on an open set $\Omega = \mathbf{R}^n - K$, K being compact. In such a case one has to work on the loop space

$$\Lambda_K = \{u \in E : u(t) \notin K, \ \forall\, t \in [0,T]\}$$

It is easy to see that $cat(\Lambda_K) = +\infty$ and the preceding arguments can be carried over with minor changes. ∎

Remarks 11.10 (i) In the case of planar systems (i.e when $n = 2$) the existence of infinitely many solutions can be proved in a more direct way. Indeed, in such a case, Λ equals the disjoint union of infinitely many components $\Lambda_j = \{u \in \Lambda : i(u) = j$, where i denotes the index of the loop u with respect to 0. The functional f is coercive on each Λ_j, $j > 0$, and solutions $u_j$can be found as $\min_{u\in\Lambda_j} f(u)\}$. See [82], where a condition like (SF) has been introduced. See also [56].

(ii) The existence of *one* T-periodic solution for n-dimensional systems has been proved in [84], indipendently from [13] and using different methods. ■

Among possible extensions of Theorems 11.7 and 11.8, it is worth mentioning a result [97] dealing with

$$\ddot{q} + aq + V'(t,q) = 0 \,. \tag{11.18}$$

Theorem 11.11 *Suppose $V \in C^1(\mathbf{R}\times\Omega,\mathbf{R})$ is T-periodic in t and satisfies (SF) and*

$$\exists\, c, R > 0 \;:\; V(x) \leq c, \;\; V'(x)\cdot x \leq c, \; \forall\, |x| > R^5. \tag{11.19}$$

Moreover, let $a < (\frac{\pi}{T})^2$. Then problem (11.18) has infinitely many T-periodic solutions.

Referring to [97] for the proof and for a complete discussion, we limit ourselves to some remarks, only. First, when $a > 0$, the corresponding functional f is no more bounded from below; however, under the assumptions listed above, one can still prove that $cat(f^\lambda;\Lambda)$ is finite for all $\lambda \in \mathbf{R}$. Second, if V merely satisfies (11.19) instead of $(V2.1)$, one proves there exists λ_0 such that $(PS)_c$ holds for all $c \geq \lambda_0$. Let us point out that examples show that if $a \geq (\frac{\pi}{T})^2$ then $cat(f^\lambda;\Lambda)$ can be infinite; and (PS) can possibly fail (for any range of c) if (11.19) is violated. ■

If V is autonomous, i.e. $V = V(x)$, any point of $X = \{x \in \Omega : V'(x) = 0\}$ is a (trivial) solution of (P_T) and the preceding existence results require some more discussion. For simplicity we will deal with the case in which V satisfies $(V2.2)$.

Theorem 11.12 *Suppose $V \in C^1(\Omega,\mathbf{R})$ is autonomous and satisfies (SF) and $(V2)$. Moreover, let $X = \{x \in \Omega : V'(x) = 0\}$ be compact. Then problem (P_T) has infinitely many nontrivial solutions.*

Proof. Let $b \in \mathbf{R}$ be such that $b > max_X f$. Without loss of generality we can suppose $K_b = \emptyset$. Recall that, from Lemma 11.6, there results

$$cat(f^{\varepsilon^*};\Lambda) = 2.$$

Moreover $(PS)_c$ holds for all $c > 0$ (Lemma 11.4). Then Remark 2.6-(ii) applies with $a = \varepsilon^*$ and therefore $k_1 := cat(f^b;\Lambda) < +\infty$. For all $k > k_1$

[5] actually, condition (11.19) is slightly stronger than the assumptions in [97]

there results $c_k > b$ and hence each c_k carries a critical point u of f, with $u \notin X$, which gives rise to a nontrivial solution of (11.2). ■

Remark 11.13 When V is autonomous the existence of infinitely many nontrivial solutions can be proved directly, once one knows the existence of *one* nontrivial solution. Indeed, let u_1 be a non-constant, periodic solutions of (P_T), and let $T' \leq T$ be its *minimal* period. The preceding theorems (applied with $\frac{T'}{2}$ instead of T) yield the existence of a periodic solution u_2 of (11.2) with period $\frac{T'}{2}$ (in particular T-periodic). Plainly, $u_2 \neq u_1$ because their minimal periods are different. Repeating this argument, one finds infinitely many nontrivial solutions of (P_T). ■

B. Weakly Attractive Potentials. Here we will be concerned with potentials which do not satisfy (SF). In such a case, Lemmas 11.2 and 11.4 do not hold any more and, in particular, f might attain finite values at $\partial\Lambda$. For example, in the case of Kepler's problem, namely when $V(x) = -|x|^{-1}$, any $u(t) \equiv \xi \cdot t^{2/3}$ near $t = 0$ ($\xi \in \mathbf{R}^n - \{0\}$) belongs to $\partial\Lambda$ and is such that $f(u) < +\infty$. Moreover, collision orbits could arise (it suffices to think again to Kepler's problem).

In order to overcome these problems, one can use an approximation argument, taking advantage of the results of subsection A.

Let us consider the perturbed potential (for simplicity, we will deal with autonomous potentials, only)

$$V_\delta(x) = V(x) - \frac{\delta}{|x|^2} \quad (\delta > 0) \tag{11.20}$$

and the corresponding functional

$$f_\delta(u) = \frac{1}{2}|\dot{u}|_2^2 - \int_0^T V_\delta(u)dt \,.$$

Let $c_k(\delta)$ denote the min-max levels

$$c_k(\delta) = \inf_{A \in \mathcal{A}_k} [\sup\{f_\delta(u) : u \in A\}] \,.$$

Let $V \in C^1(\Omega, \mathbf{R})$ satisfy $(V2)$ and

$$V(x) \to -\infty \text{ as } |x| \to 0 \,. \tag{11.21}$$

Then Lemmas 11.2, 11.4 and 11.6, jointly with the arguments of Theorem 11.7, yield, for all $\delta > 0$ and $k \geq 3$, the existence of critical points $u_{\delta,k} \in \Lambda$ of f_δ such that

$$f_\delta(u_\delta) = c_k(\delta) \,.$$

In the sequel we will fix $k \geq 3$, and set $u_\delta = u_{\delta,k}$. We will also denote by c_k the min-max level $c_k(0)$, corresponding to f, as well as $\bar{c} = c_k(1)$. Plainly, for all $0 < \delta < 1$ one has $f < f_\delta < f_1$, and hence:

$$c_k \leq c_k(\delta) \leq \bar{c}\,. \tag{11.22}$$

Next, we bound from below c_k by means of Lemma 11.6, yielding

$$c_k \geq \varepsilon^* > 0\,. \tag{11.23}$$

Let us point out that Lemma 11.6 only requires assumptions $(V2)$, not (SF), and therefore applies to our potential V.

Putting together (11.22) and (11.23) we find

$$0 < \varepsilon^* \leq f_\delta(u_\delta) \leq \bar{c}\,. \tag{11.24}$$

Let $u_\delta = \xi_\delta + w_\delta$, with $w_\delta \in W$. From the right-hand side of (11.24) it follows

$$\frac{1}{2}|\dot{w}_\delta|^2 \leq \bar{c} \tag{11.25}$$

$$-\int_0^T V_\delta(u_\delta(t))dt \leq \bar{c}\,. \tag{11.26}$$

Moreover one has:

$$|\xi_\delta| \leq b \tag{11.27}$$

for some $b \in \mathbf{R}$. Otherwise from

$$|u_\delta(t)| \geq |\xi_\delta| - |w_\delta|_\infty \geq |\xi_\delta| - \sqrt{T}|\dot{w}_\delta|_2 \geq |\xi_\delta| - \sqrt{2\bar{c}T},$$

one infers that $|u_\delta(t)| \to \infty$, uniformly. Since

$$f_\delta(u_\delta) = \frac{1}{2}|\dot{w}_\delta|_2^2 - \int_0^T V_\delta(u_\delta(t))dt = \frac{1}{2}\int_0^T V_\delta'(u_\delta(t))\cdot w_\delta(t)dt - \int_0^T V_\delta(u_\delta(t))dt\,,$$

$(V2.1)$ implies that $f_\delta(u_\delta) \to 0$, in contradiction with the right-hand side of (11.24).

From (11.25) and (11.27) it follows that u_δ converges (up to a subsequence) weakly in E and strongly in L^∞ to $u^* \in E$. We claim that u^* is a generalized solution of (P_T). To see this, let us remark that u_δ satisfy

$$\ddot{u}_\delta + V_\delta'(u_\delta) = 0 \tag{11.28}$$

together with the energy relationship

$$\frac{1}{2}|\dot{u}_\delta(t)|^2 + V_\delta(u_\delta(t)) \equiv h_\delta. \tag{11.29}$$

From (11.22) it follows

$$\bar{c} \geq f_\delta(u_\delta) \geq -\int_0^T V(u_\delta)dt. \tag{11.30}$$

Let $J = \mathcal{C}_{u^*}$. Since $V(x) < 0$, (11.30) implies

$$-\int_J V(u_\delta)dt \leq \bar{c}. \tag{11.31}$$

Since $u_\delta \to 0$ uniformly in J, (11.31) immediately implies that $|J| = 0$. Furthermore, it is easy to see that $u_\delta \to u^*$ in $C^2([0,T] - J, \mathbf{R}^n)$ and that

$$\ddot{u}^* + V'(u^*) = 0, \quad \forall\, t \in [0,T] - J.$$

Finally, $f(u^*) < +\infty$ and condition (iv) readily follows from (11.24) and (11.29), respectively. ■

The existence of generalized solutions of (P_T) has been proved in [38] by a different min-max procedure. Actually, an additional argument, on the line of Remark 11.13, leads to show:

Theorem 11.14 *Let $V \in C^1(\Omega, \mathbf{R})$ satisfy $(V2)$ and (11.21). Then (P_T) has infinitely many generalized solutions.*

Remark 11.15 Theorem 11.14 deals with *autonomous* Potentials. When V depends on time the preceding arguments lead to possibly obtain the existence of *one* generalized solution of (P_T), only. The existence of infinitely many generalized solutions of (P_T) for time depending potentials has not yet been obtained and it would be an interesting question to pursue. In this direction, a multiplicity result will be discussed in subsection C below by means of perturbation techniques. ■

An important problem to be investigated is the regularity of the collision orbits and their behaviour near the singularity. An interesting result in this direction has been recently found by Coti Zelati and Serra [68] who have shown that, for a large class of Potentials, Theorem 11.14 can be greatly improved.

Theorem 11.16 *Suppose $V(x) = -|x|^{-\alpha} + U(x)$ with $\alpha > 0$ and $U \in C^1(\mathbf{R}^n, \mathbf{R})$. Then the collision set $\mathcal{C}_u$ of any generalized solution u of (P_T) is at most finite.*
If, in addition, U satisfies:

$$\begin{cases} U(x) < 0, \ \forall\, x \in \mathbf{R}^n; \\ U(x) \text{ and } U'(x) \to 0 \text{ as } |x| \to +\infty \\ \exists\, r > 0 \text{ and } \varphi \in C^1([0,r], \mathbf{R}) \text{ such that } U(x) = \varphi(|x|) \text{ for } 0 < |x| \leq r. \end{cases}$$

Then:
(i) if $1 < \alpha < 2$, there exists a non collision orbit of (P_T);
(ii) if $\alpha = 1$, there exists a generalized solution u of (P_T) with at most a collision t_0, and

$$u(t + t_0) = u(t_0 - t).$$

For a slightly more general statement and proofs, we refer to [68]; see also [125].

Let us remark that statement (ii) generalizes what happens in the case of Kepler's Problem: collisions enter the singularity in finite time; the system being reversible, $u(-t)$ is a solution whenever u is so, thus a periodic collision orbit will be obtained reversing the time in a trajectory falling into $x = 0$. The fact that for perturbed Kepler's Problem *any* collision orbit inherits such a property would be interesting to be proved.

C. PERTURBATION RESULTS. When V has the form

$$V(x) = -\frac{1}{|x|^\alpha} + \varepsilon U(x) \tag{11.32}$$

with $\alpha > 0$ and U smooth, perturbation arguments can yield the existence of multiple non collision orbits for $|\varepsilon|$ small. An interesting feature is that these results do not depend on the fact that V is a *Strongly* or *Weakly Attractive Potential*.

The main abstract tool is a result proved in [17] (see also [60], [70]) dealing with perturbation in Critical Point Theory. Indeed, letting

$$f_0(u) = \int_0^T \{\frac{1}{2}|\dot{u}|^2 + \frac{1}{|u|^\alpha}\}dt \tag{11.33}$$

and

$$\phi(u) = \int_0^T U(u)dt, \tag{11.34}$$

the functional f_ε corresponding to V given by (11.32) has the form

$$f_\varepsilon(u) = f_0(u) + \varepsilon\phi(u) \tag{11.35}$$

and one can try to find critical points of f_ε near those of f_0, provided $|\varepsilon|$ is small.

Let us remark that critical points of f_0 arise in manifolds because f_0 is O(n)-invariant [6].

Let E be a Hilbert space, $\mathcal{D}$ an open subset of E and $f \in C^2(\mathcal{D}, \mathbf{R})$. A subset $Z \subset \mathcal{D}$ is said a *non-degenerate critical manifold* of f if:

(a) Z is a compact, connected C^1 manifold and $f'(z) = 0, \forall\, z \in Z$;
(b) $T_z Z = Ker[f''(z)], \forall\, z \in Z$.

Hereafter, $T_x M$ denotes the tangent space to the manifold M at $x \in M$.

The following result is a particular case of Theorem 2.1 of [17], and is sufficient for our applications.

Theorem 11.17 *Suppose $f_0, \phi \in C^2(\Lambda, \mathbf{R})$ and let Z be a nondegenerate critical manifold of f_0. In addition, let f_0'' be a Fredholm operator of index zero for all $z \in Z$. Then $\exists\ \bar\varepsilon > 0$ and a neighborhood $\mathcal{U}$ of Z such that $\forall\ 0 < |\varepsilon| < \bar\varepsilon$ the perturbed functional $f_0 + \varepsilon\phi$ has at least $cat(Z)$ critical points in $\mathcal{U}$.*

Proof. (Sketch) Let N_z (resp. R_z) denote the Kernel (resp. the Range) of $f_0''(z)$. Since, for all $z \in Z$, $f_0''(z)$.is a Fredholm map of index 0, then one has

$$E = N_z \oplus R_z \quad (z \in Z).$$

Using the Implicit Function Theorem and the fact that Z is compact, one finds a neighborhood $\mathcal{U}$ of Z in E and maps $P, Q \in C^2(\mathcal{U}, E)$ such that, $\forall\, u \in \mathcal{U}$, one has $u = Pu + Qu$ and

$$Pu \in Z, \quad Qu \in R_{Pu}.$$

Let

$$Z_\varepsilon = \{u \in \mathcal{U} : f_\varepsilon'(u) \in N_{Pu}\}.$$

One proves that for ε sufficiently small, Z_ε is diffeomorphic to Z and (hence)

$$(T_u Z_\varepsilon)^\perp \not\equiv T_{Pu} Z, \forall\, u \in Z_\varepsilon. \tag{11.36}$$

[6] Actually, in some cases (such as when $\alpha = 1$) the manifold of critical points inherits other symmetries.

Let $u \in Z_\varepsilon$ be a critical point of f_ε constrained on Z_ε. Then

$$f'_\varepsilon(u) \in (T_u Z_\varepsilon)^\perp. \tag{11.37}$$

By the definition of Z_ε and since Z is non-degenerate (see (b) of the preceding definition) there results

$$f'_\varepsilon(u) \in N_{Pu} = T_{Pu} Z.$$

This, jointly with (11.36) and (11.37), implies that $f'_\varepsilon(u) = 0$. In other words, seeking critical points of f_ε in $\mathcal{U}$ is equivalent to look for critical points of f_ε constrained on Z_ε. Since Z_ε is compact, the Lusternik-Schnirelman theory applies and f_ε has at least $cat(Z_\varepsilon) = cat(Z)$ critical points in $\mathcal{U}$, whenever $|\varepsilon|$ is small enough. ∎

Remark 11.18 If f_ε is invariant under the action of a group $\mathcal{G}$ which acts freely on Z, one will obtain the existence of at least $cat(Z/\mathcal{G})$ critical points near Z. ∎

Theorem 11.17 is prompted for seeking T-periodic solutions of

$$\ddot{q} + \alpha \frac{q}{|q|^{\alpha+2}} + \varepsilon\, U'(q) = 0 \tag{11.38}$$

In the sequel f_0, ϕ and f_ε are given by (11.33), (11.34) and (11.35), respectively.

Set $\omega = \frac{2\pi}{T}$, and let $R > 0$ be such that $\omega^2 R^{2+\alpha} = \alpha$.

Consider the set

$$Z = \{z(t) = R[\xi e^{i\omega t} + \bar{\xi} e^{-i\omega t}] \;:\; \xi \in \mathbf{C}^n,\ \xi \cdot \xi = \frac{1}{2}, \xi \cdot \bar{\xi} = 0\}$$

where $\bar{\xi}$ denotes the complex conjugate of $\xi \in \mathbf{C}^n$ and $\cdot$ stands for the scalar product in $\mathbf{C}^n$. One immediately verifies that any $z \in Z$ is a solution of the unperturbed equation

$$\ddot{q} + \alpha \frac{q}{|q|^{\alpha+2}} = 0, \tag{11.39}$$

hence a critical point of f_0. Moreover one has

Lemma 11.19 *If $\alpha \neq 1$ then Z is a nondegenerate critical manifold of f_0.*

Proof. It remains to prove that condition (b) above holds true. For the sake of brevity, we shall be sketchy; for details we refer to [14, Lemma 2.1].

First, it is immediate to check that $v \in N_z$ (we are keeping the notation introduced in the proof of Theorem 11.17) whenever v is T-periodic and satisfies

$$-\ddot{v} = \frac{\alpha}{|z|^{\alpha+2}}[v - (\alpha+2)\frac{v \cdot z}{|z|^2}] \tag{11.40}$$

If v_k denote the Fourier coefficients of v ($v_{-k} = \bar{v}_k$), one proves that (11.40) yields

$$\begin{cases} v_{k+2} \cdot \xi = \frac{k^2-1}{(k^2+2)^2-1} v_k \cdot \bar{\xi} & \forall\, k \in \mathbf{Z} \\ v_{-3} \cdot \bar{\xi} = 0 & \\ \gamma(k) v_k \cdot \bar{\xi} = 0 & \forall\, k \in \mathbf{Z} - \{-1,-3\} \end{cases} \tag{11.41}$$

where

$$\gamma(k) = \frac{k+1}{k+3}\,[k^2 + 2k + \alpha - 1].$$

Now, if $\alpha > 0$, $\alpha \neq 1$, then $\gamma(k) \neq 0$ for all $k \in \mathbf{Z} - \{-1,-3\}$ and (11.41) yields that $v \in N_z$ provided

$$\begin{cases} v_k = 0, & \forall\, k \in \mathbf{Z} - \{\pm 1\} \\ v_k \cdot \xi + v_{-k} \cdot \bar{\xi} = 0 & \forall\, k \in \mathbf{Z} \\ v_k \cdot \bar{\xi} = 0 & \forall\, k \in \mathbf{Z} \end{cases} \tag{11.42}$$

Since the preceding condition defines nothing but the tangent space $T_z Z$, it follows that $N_z = T_z Z$ and hence Z is a non-degenerate critical manifold for f_0. ∎

We are now in position to prove:

Theorem 11.20 *Let V be of the form (11.32) where U is of class C^2 and $\alpha > 0$, $\alpha \neq 1$. Then*
(i) Equation (11.38) has at least 3 T-periodic solution near Z, provided $|\varepsilon|$ is small enough;
(ii) If U is autonomous, then (11.38) has at least n ($n \geq 3$) distinct T-periodic solutions near Z, provided $|\varepsilon|$ is small enough.

Proof. (i) Since Z is diffeomorphic to $T_1 S^{n-1}$, and $cat(T_1 S^{n-1}) \geq 3$ (see [14, Appendix] for a proof), the result follows from Theorem 11.17.
(ii) If U is autonomous, f_ε is invariant under the O(2), too, and we can apply Theorem 11.17, jointly with Remark 11.18. Since it is known that $cat(Z/O(2)) = cat(T_1 S^{n-1}/O(2)) \geq n$ ($n \geq 3$), see [88, Prop.2.3.3] [7], then f_ε

[7] Actually, for the correct statement see ref. [Al 1, pag.151] cited in [88].

possesses at least n critical points near Z. They correspond to geometrically distinct T-periodic solutions of (11.38) because they are close to Z and hence, by continuity, they have *minimal* period T. ■

Remarks 11.21 (i) Instead of Z, one can find periodic solutions of (11.38) near each $Z_m = \{z(t) = R[\xi e^{im\omega t} + \bar{\xi} e^{-im\omega t}] \ : \xi \in \mathbf{C}^n, \xi \cdot \xi = \frac{1}{2}, \xi \cdot \bar{\xi} = 0\}$, $m \geq 1$ integer. Indeed, if $m^2\omega^2 R^{2+\alpha} = \alpha$, Z_m is a critical manifold of f_0 and it is non-degenerate provided $\alpha \neq 1$.
(ii) The perturbation result stated in Theorem 11.20 is in fact a bifurcation result, in the sense that solutions of (11.38) 'branch off' from those of the unperturbed problem (11.39) satisfying an *averaging* condition. Roughly, the possible 'bifurcation points' are the $z \in Z$ which are the critical points on Z of the averaged potential $\int_0^T U(z(t))dt$. See [14, Section 5] for more details.
(iii) Theorem 11.20 can be extended to cover a class of Potentials of the form $V(x) = \chi(|x|) + \varepsilon U(x)$ where $\chi \in C^2(]0, +\infty[, \mathbf{R})$. See [14, CRAS Note]. ■

The preceding discussion does not cover the case of the perturbed Kepler problem

$$\ddot{q} + \frac{q}{|q|^3} + \varepsilon U'(q) = 0 \tag{11.43}$$

Indeed, in such a case, the Kepler equation

$$\ddot{q} + \frac{q}{|q|^3} = 0 \tag{11.44}$$

has, in addition to the circular orbits given by Z, elliptic orbits and collision orbits and Z is merely a submanifold of a larger manifold of solutions.

In order to overcome this difficulty, we shall assume that U is *even* in x. If this is the case, let us consider the subspace

$$E_* = \{u \in E : u(t + \frac{T}{2}) = -u(t)\}$$

and set

$$\Lambda_* = E_* \cap \Lambda.$$

In the sequel f_ε denotes the same functional as before, with $\alpha = 1$.

Lemma 11.22 *Let U be even in x. Then $u \in \Lambda_*$ is a critical point of f_ε whenever u is a critical point of f_ε on E_*.*

Proof. Since U is even, one immediately verifies that $f'_\varepsilon \in E_*$. Thus, if $u \in \Lambda_*$ is such that $f'_\varepsilon(u) \in (E_*)^\perp$ then $f'_\varepsilon(u) = 0$. ■

After remarking that $Z \subset \Lambda_*$, the preceding Lemma allows us to work on E_*. We consider again Z with $\alpha = 1$; if R satisfies $\omega^2 R^3 = 1$, then any $z \in Z$ is a solution of (11.44). As in Lemma 11.19 one still shows that $v \in N_z \cap E_*$ provided (11.41) holds true. In the present case $\gamma(k) = k(k+1)(k+2)/(k+3)$ and $\gamma(k) \neq 0$ for all $k \in \mathbf{Z} - \{-1, -3\}$, k even. Since $v \in E_*$ then $v_{2k} = 0$ and one infers that the Fourier coefficients of v satisfy (11.42). This shows that Z is non-degenerate critical manifold for f_0 on E_*. Then Theorem 11.17 applies and yields:

Theorem 11.23 *Let U be of class C^2 and be even. Then:*
(i) the perturbed Kepler problem (11.43) has at least 3 T- periodic solution near Z, provided $|\varepsilon|$ is small enough;
(ii) If U is autonomous, then (11.43) has at least n ($n \geq 3$) distinct T-periodic solutions near Z, provided $|\varepsilon|$ is small enough.

Remark 11.24 As in Remark 11.21-(ii), the critical points of the averaged Potential $\int_0^T U(z)dt$ are the possible bifurcations of periodic solutions of (11.43). ∎

Remark 11.25 Existence of noncollision orbits for potentials verifying a condition like $-a|x|^{-\alpha} \leq V(x) \leq -b|x|^{-\alpha}$ has been studied in [71]. It is shown that such a solution exists provided that, roughly, $b - a$ is small depending on α and $\alpha > 1$ (when $\alpha = 1$ the potential V is assumed to be even, like in Theorem 11.23 above. For other results of this kind, see [115] and [126]. ∎

Remark 11.26 All the preceding results deal with *second order* Conservative Systems. Extension to first order Hamiltonian Systems would be an interesting question to pursue. A result in this direction is discussed in [18] by means of perturbation techniques like the preceding ones, jointly with a *local* Dual Variational Principle. Applications to the restricted 3-body problem are also given. ∎

12 Conservative Systems with Singular Potentials (II)

In this section we will deal with *autonomous* Conservative Systems

$$\ddot{q} + V'(q) = 0, \tag{12.1}$$

with singular potentials V, looking for periodic solutions q of (12.1) having prescribed energy $h \in \mathbf{R}$:

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h. \tag{12.2}$$

As in the preceding section, we will be concerned with attractive potentials $V(x)$ which behave like $-|x|^{-\alpha}$ ($\alpha > 0$) near $x = 0$ and thus solutions of (12.1) can be possibly *generalized* solutions, according to the definition given in Section 11. In the sequel, a (generalized) periodic solution of (12.1) satisfying (12.2) will referred to as a (generalized) solution of *Problem* (P_h).

The material is divided into 3 subsections. The first two are taken from [15] and contain a Variational Principle and, respectively, the main existence results. The last one is concerned with perturbation results.

We will keep the notation introduced in section 11, with $T = 1$. So $\Omega = \mathbf{R}^n - \{0\}$,

$$E = H^{1,2}(S^1, \mathbf{R}^n), \quad \Lambda = \{u \in E \; : u(t) \neq 0 \;\; \forall\, t \in S^1\},$$

and so on.

A. A Variational Principle. We begin introducing a slightly unusual Variational Principle. Let $V \in C^2(\Omega, \mathbf{R})$ and define $F \in C^2(\Lambda, \mathbf{R})$ by setting

$$F(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 dt \cdot \int_0^1 [h - V(u)]dt. \tag{12.3}$$

The functional F can be seen as the geometric mean of the (averaged) Kinetic and Potential energies.

Lemma 12.1 *Let $u \in \Lambda$ be such that $F'(u) = 0$ and $F(u) > 0$. Let*

$$\omega^2 = \frac{\int_0^1 [h - V(u)]dt}{\frac{1}{2}|\dot{u}|_2^2} \tag{12.4}$$

Then $q(t) := u(\omega t)$ is a solution of (P_h).

Proof. If $F'(u) = 0$ then

$$\left(\int_0^1 \dot{u} \cdot \dot{v} dt\right)(h + \psi(u)) - \frac{1}{2}|\dot{u}|_2^2 \cdot \int_0^1 V'(u) \cdot v dt = 0, \quad \forall\, v \in E.$$

Using (12.4), it follows

$$\omega^2 \int_0^1 \dot{u} \cdot \dot{v} dt - \int_0^1 V'(u) \cdot v dt = 0, \quad \forall\, v \in E.$$

Hence u is a (weak, and by regularity, strong) solution of

$$\omega^2 \ddot{u} + V'(u) = 0 \tag{12.5}$$

and $q(t) = u(\omega t)$ solves (12.1). Moreover, from (12.5) it follows that

$$\omega^2 \frac{1}{2} |\dot{u}(t)|^2 + V(u(t)) \equiv c,$$

for some constant c. Integrating over $[0, 1]$ we find

$$\omega^2 \frac{1}{2} |\dot{u}|_2^2 + \int_0^1 V(u(t)) dt = c.$$

Taking into account (12.4) there results $c = h$ and q satisfies (12.2). ∎

The above principle has been used in [127] to prove the existence of brake orbits for symmetric conservative systems with smooth potentials $V \in C^2(\mathbf{R}^n, \mathbf{R})$. For a much more general result on brake orbits, see, for example, [109].

It will be clear later on (see Remark 12.11) that it is convenient to modify the Variational Principle illustrated in Lemma 12.1. For this purpose, we will follow the procedure discussed in Proposition 1.4.

Let

$$G(u) = \int_0^1 [V(u) + \frac{1}{2} V'(u) \cdot u] dt$$

and consider the set

$$M_h = \{u \in \Lambda \ : G(u) = h\}.$$

Since

$$(F'(u)|u) = |\dot{u}|_2^2 \cdot [h - G(u)]$$

then

$$(F'(u)|u) = 0 \, \forall \, u \in M_h. \tag{12.6}$$

Let us remark that $F_{|M_h}$ has the form

$$F(u) = \frac{1}{4}|\dot{u}|_2^2 \cdot \int_0^1 V'(u) \cdot u dt \ \ (u \in M_h). \tag{12.7}$$

For future reference we point out :
Remark 12.2 If $u_m \to u$ weakly in E and uniformly, and $u_m, u \in \Lambda$, then $G(u_m) \to G(u)$ and $G'(u_m) \to G'(u)$. ■

The following lemma plays the same role of Proposition 1.4.

Lemma 12.3 *Let $M_h \neq \emptyset$ and suppose*
(V3.1) $\quad 3V'(x) \cdot x + V''(x)x \cdot x \neq 0 \ \ \forall \, x \in \Omega$.
Then M_h is a C^1 manifold of codimension one in E and any critical point $u \in M_h$ of F constrained on M_h is a critical point of F.

If, in addition, V satisfies
(V3.2) $\quad V'(x) \cdot x > 0 \ \ \forall \, x \in \Omega,$
then $F(u) = 0$, $u \in M_h$, if and only if $u(t) \equiv$constant.

Proof. Using (V3.1) one infers

$$(G'(u)|u) = \int_0^1 [3V'(u) \cdot u + V''(u)u \cdot u]dt \neq 0 \ \ \forall \, u \in M_h. \tag{12.8}$$

Taking into account (12.6) and (12.8), the first statement follows as in Proposition 1.4. Moreover, if (V3.2) holds true, then (12.7) immediately implies that $F(u) = 0$ iff $|\dot{u}|_2 = 0$, namely iff $u(t)$ is identically constant. ■

From Lemma 12.3 we deduce a modified Variational Principle.

Lemma 12.4 *Suppose that $M_h \neq \emptyset$ and that (V3.1-2) hold. Let $u \in M_h$ be a non constant critical point of $F_{|M_h}$ and let ω be given by (12.4). Then $q(t) = u(\omega t)$ is a solution of (P_h).*

Remarks 12.5 (i) When V is the model potential $V(x) = -|x|^{-\alpha}$, then $G(u) = (\frac{\alpha}{2} - 1)\int_0^1 |u|^{-\alpha} dt$. hence if $\alpha > 2$, then $M_h \neq \emptyset$ whenever $h > 0$, while if $\alpha < 2$, then $M_h \neq \emptyset$ whenever $h < 0$.
(ii) Being modelled on Λ, M_h is possibly not closed in E. For example, when $\alpha = 1$, and $h < 0$, M_h contains all the set S of noncollision solutions of Kepler's problem, having energy h, namely the circular and the elliptic ones. Plainly, the collision orbits with the same energy h belong to the closure of S but not to M_h. On the contrary, it is possible to show that for a class of Potentials verifying the (SF) condition, M_h is closed in E (see [15, Lemma 3.3]). ■

B. Existence Results. Remarks 12.5 suggest that it is again convenient to distinguish between the Weakly and the Strongly Attractive Potentials. For the sake of brevity, we will discuss below the Weakly Attractive case, only. For the case of Strong Forces we refer to [15, Section 3] and [43].

We shall suppose that $V \in C^2(\Omega, \mathbf{R}^n)$ satisfies, in addition to $(V3.1-2)$ the following:
(V3.3) $\exists\, \alpha, \beta \in (0,2)$ and $r > 0$ such that

$$\begin{cases} i) \ V'(x)\cdot x & \geq & -\alpha V(x) & \forall\, x \in \Omega \\ ii) \ V'(x)\cdot x & \leq & -\beta V(x) & \forall\, 0 < |x| \leq r \end{cases}$$

(V3.4) $\liminf_{|x|\to+\infty} [V(x) + \frac{1}{2}V'(x)\cdot x] \geq 0$.

As a first consequence of $(V3.2-3)$ let us show that there exists $a_1 > 0$ such that

$$V(x) \leq -\frac{a_1}{|x|^\alpha}, \quad \forall\, 0 < |x| \leq r. \tag{12.9}$$

To see this, let $|y| = r$ and define $\varphi = \varphi_y : (0,1] \to \mathbf{R}$ by setting $\varphi(s) = V(sy)$. By $(V3.2)$ and $(V3.3-(ii))$ it follows that $V(x) < 0$ for all $0 < |x| \leq r$, namely that $\varphi(s) < 0$. Using $(V3.3-(i))$ one finds

$$\varphi'(s) = V'(sy)\cdot y \geq -\frac{\alpha}{s} V(sy) = -\frac{\alpha}{s}\varphi(s).$$

Since $\varphi(s) < 0$, it follows that

$$\frac{\varphi'(s)}{\varphi(s)} \leq -\frac{\alpha}{s}$$

and hence

$$\varphi(s) \leq -|\varphi(1)| s^{-\alpha}. \tag{12.10}$$

Let $x \in \Omega$, $|x| \leq r$, and set $y = r\frac{x}{|x|}$. Then $V(x) = V(\frac{|x|}{r}y) = \varphi(\frac{|x|}{r})$ and (12.9) follows from (12.10).

Lemma 12.6 *If $(V3)$ holds then $M_h \neq \emptyset$ for all $h < 0$. Moreover, for all $k \in \mathbf{N}$, the class $\mathcal{X}_k = \{X \subset M_h \ : cat(X; M_h) \geq k\}$ is not empty. In particular, there results $cat(M_h) = +\infty$.*

Proof. For all $u \in \Lambda$ the map $s \to G(su)$ is strictly increasing in view of (12.8). Moreover, $(V3.4)$ implies

$$\liminf_{s\to+\infty} G(su) \geq 0.$$

For $s \to 0$, $(V3.3-(ii))$ and (12.9) yield

$$\begin{aligned} G(su) &= \int_0^1 [V(su) + \frac{1}{2}V'(su)\cdot su]dt \\ &\leq (1-\frac{\beta}{2})\int_0^1 V(su)dt \leq -a_1(1-\frac{\beta}{2})\int_0^1 \frac{1}{s^\alpha|u|^\alpha}dt = -a_2 s^{-\alpha}. \end{aligned}$$

and thus $G(su) \to -\infty$ as $s \to 0$. Since $h < 0$, it follows that for any $u \in M_h$ there exists a unique $s(u) \in \mathbf{R}$ such that $s(u)u \in M_h$. In addition, (12.8) implies that $s(u)$ is continuous. For all $k \in \mathbf{N}$, let $Y \subset \Lambda$ be compact and such that $cat(Y;\Lambda) \geq k$ (the existence of such an Y is guaranteed by the result of [76], see Example 2.2-(iv)). Set $\sigma(u) = s(u)u$ and $X = \sigma(Y)$. Then the properties of the category (see Lemma 2.1) yield

$$cat(X; M_h) \geq cat(X;\Lambda) \geq cat(Y,\Lambda) \geq k$$

and $X \in \mathcal{X}_k$, proving the lemma. ∎

Taking into account Remark 12.5-(ii), we need to control the behaviour of F at ∂M_h. For this we will proceed as in the preceding section and consider the perturbed potential

$$V_\delta(x) = V(x) - \frac{\delta}{|x|^2}.$$

Let us denote by F_δ the functional defined in (12.3) with V_δ instead of V. The following remark is in order

Remark 12.7 Since V and V_δ differ by a term which is homogeneous of degree -2, it follows immediately that

$$V_\delta(x) + \frac{1}{2}V_\delta'(x)\cdot x = V(x) + \frac{1}{2}V'(x)\cdot x.$$

and therefore $(F_\delta' u|u) = (F'(u)|u)$. In other words the manifold M_h remains unaffected by the change of V with V_δ. Moreover, since V_δ plainly satisfies $(V3.1-2)$, then Lemma 12.3 applies and any critical point $u_\delta \in M_h$ of F_δ constrained on M_h such that $F_\delta(u_\delta) > 0$, gives rise to a solution of

$$\omega_\delta^2 \ddot{u}_\delta + V'(u_\delta) + 2\delta \frac{u_\delta}{|u_\delta|^4} = 0 \tag{12.11}$$

$$\frac{1}{2}\omega_\delta^2 |\dot{u}_\delta|^2 + V(u_\delta) - \frac{\delta}{|u_\delta|^2} = h \tag{12.12}$$

where

$$\omega_\delta^2 = \frac{\int_0^1 [h - V_\delta(u_\delta)]dt}{\frac{1}{2}|\dot{u}_\delta|_2^2}. \tag{12.13}$$

Our strategy will be the following one: first, we will use the Lusternik-Schnirelman theory to find critical points u_δ of F_δ on M_h which are solutions of approximated problems (12.11-12); the second step will consist in showing that $u_\delta \to z$, $\omega_\delta \to \omega_0 \neq 0$ and that z gives rise to a generalized solution of (P_h).

Let us begin proving that F_δ satisfies (PS) on M_h. For this, two Lemmas are in order. The former highlights the advantage of working with a modified Potential satisfying (SF).

Lemma 12.8 *Let $\delta > 0$. Then any sublevel $F_\delta^b = \{u \in M_h \ : F_\delta(u) \le b\}$ is closed.*

Proof. We will prove a slightly stronger statement: for any sequence $\{u_m\} \subset F_\delta^b$ such that $u_m \to v$, weakly in E and uniformly in $[0,1]$, one has that $v \in F_\delta^b$. To prove this it suffices, according to Remark 12.2, to show that $v \in \Lambda$. We argue by contradiction and suppose that $v \in \partial\Lambda$. First, let us consider the case in which $v(t) \equiv 0$. Then $u_m \to 0$ uniformly and using $(V3.3-(ii))$ and (12.9) it follows

$$\begin{aligned} h &= \int_0^1 [V(u_m) + \frac{1}{2}V'(u_m)\cdot u_m]dt \le \\ &\le (1-\frac{\beta}{2})\int_0^1 V(u_m)dt \le -a_3 \int_0^1 |u_m|^{-\alpha}dt. \end{aligned}$$

Since $\int_0^1 |u_m|^{-\alpha}dt \to +\infty$, the preceding inequality gives rise to a contradiction.

Let $u_m \in \partial\Lambda - \{0\}$. In particular $u_m(t)$ is not identically constant (the only constant loop in $\partial\Lambda$ is 0) and hence

$$0 < |\dot{v}|_2^2 \leq \liminf_{m\to\infty} |\dot{u}_m|_2^2 . \tag{12.14}$$

Since V_δ satisfies condition (SF), then Lemma 11.2 and (12.14) yield

$$F_\delta(u_m) = \frac{1}{2}|\dot{u}_m|_2^2 \cdot \int_0^1 [h - V_\delta(u_m)]dt \to +\infty,$$

while $u_m \in F_\delta^b$. ■

Lemma 12.9 *If* $\{u_m\} \subset F_\delta^b$ *then* $\exists\ a_4 > 0$ *such that* $|u_m|_2^2 \leq a_4$ *and* $|u_m|_\infty \leq a_4$.

Proof. Since $u_m \in M_h$ and using $(V3.3-(i))$, then it follows

$$h = \int_0^1 [V(u_m) + \frac{1}{2}V'(u_m)\cdot u_m]dt \geq (\frac{1}{2} - \frac{1}{\alpha}) \int_0^1 V'(u_m)\cdot u_m dt,$$

and thus

$$\int_0^1 V'(u_m)\cdot u_m dt \geq a_5 \equiv \frac{2\alpha h}{\alpha - 2}\ (>0)$$

Then, from $u_m \in F_\delta^b$ we infer

$$\begin{aligned} b \geq F_\delta(u_m) &= \frac{1}{4}|\dot{u}_m|_2^2 \int_0^1 V_\delta'(u_m)\cdot u_m dt \geq \\ &\geq \frac{1}{4}|\dot{u}_m|_2^2 \int_0^1 V'(u_m)\cdot u_m dt \geq \frac{a_5}{4}|\dot{u}_m|_2^2 , \end{aligned}$$

proving the upper bound for $|u_m|_2^2$.

Let $u_m = \xi_m + w_m$, where $\xi_m = \int_0^1 u_m dt$. From the preceding step we infer that $w_m \to \bar{w}$ uniformly (up to a subsequence). If $|u_m|_\infty \to +\infty$ then, as in the proof of Lemma 11.4, it follows that $|\xi_m| \to +\infty$ and hence $|u_m(t)| \to +\infty$ uniformly in t. Then $(V3.4)$ implies

$$\liminf_{m\to\infty} G(u_m) \geq 0,$$

while $G(u_m) = h < 0$, because $u_m \in M_h$.
The contradiction proves the upper bound for $|u_m|_\infty$. ■

We are now in position to prove

Lemma 12.10 *F_δ satisfies (PS) on M_h.*

Proof. Let u_m be any PS-sequence. By Lemma 12.9 it follows that $\|u_m\| \leq a_6$ and thus, along a subsequence, $u_m \to \bar{u}$, weakly and uniformly. Lemma 12.8 implies that $\bar{u} \in \Lambda$. At this point, one shows in a standard way that $F'_\delta(u_m) \to 0$ implies that u_m strongly converges to $\bar{u}$, up to a subsequence. ∎

Remark 12.11 In general, F_δ does not satisfy (PS) (on Λ). To see this, let us consider a potential $V(x) = -|x|^{-\alpha}$ with $0 < \alpha < 1$ and let $u_m = r_m e^{i2\pi t}$ with $r_m \in \mathbf{R}$ (we use complex notation). If $r_m \to 0$, one has

$$F_\delta(u_m) = 2\pi^2 r_m^2 (h + r_m^{-\alpha} + \delta r_m^{-2}) \to 2\pi^2\delta.$$

Moreover there results

$$\begin{aligned}(F'_\delta(u_m)|v) &= \left(\int_0^1 \dot{u}_m \cdot \dot{v}dt\right)\left(h + r_m^{-\alpha} + \delta r_m^{-2}\right) + \\ &+ 2\pi^2 r_m^2(-\alpha r_m^{-\alpha-2} - 2\delta r_m^{-4}) \int_0^1 u_m \cdot v dt.\end{aligned}$$

If v_1 denotes the first Fourier component of v, it follows with straigth calculation that

$$(F'_\delta(u_m)|v) = 4\pi^2 r_m v_1 \left(h + (1 - \frac{\alpha}{2}) r_m^{-\alpha}\right) \to 0$$

whenever $0 < \alpha < 1$. Hence $(PS)_c$ does not hold, with $c = 2\pi^2\delta$, because such a u_m is a (PS) sequence whose limit is $0 \notin \Lambda$. Similarly, one can show that condition $(PS)_c$ is violated along any sequence like $u_m = r_m e^{i2\pi k t}$ with $c = 2k^2\pi^2\delta$. It is worth pointing out how this example shows that here there is an additional advantage of working on the manifold M_h. Actually, not only we can deal with a functional bounded from below, but also any problem concerning the (PS) condition is eliminated: indeed, the sequences where (PS) fails, do not belong to the manifold. We anticipate that for a class of symmetric potentials satisfying $V(x) \simeq -|x|^{-\alpha}$, with $1 \leq \alpha < 2$ the corresponding F_δ satisfies (PS). We will see this in the next section, discussing a class of N-body problems. ∎

As a consequence of the preceding Lemmas we have

Lemma 12.12 *There exists (infinitely many) $u_\delta \in \Lambda$ satisfying (12.11-12), with ω_δ given by (12.13).*

Proof. Since (PS) holds true, the Lusternik-Schnirelman theory applies (here the completeness of the manifold is substituted by Lemma 12.8, which plainly suffices). We will refer to Theorem 2.3. Let Let $c_{k,\delta}$ denote the k-th min-max critical level: $c_{k,\delta} = \inf_{X\in\mathcal{X}_k}\max_X F_\delta$. We claim that $c_{k,\delta} > 0$ whenever $k \geq 3$. Indeed, otherwise we have $c_{1,\delta} = \cdots = c_{k_0,\delta} = 0$ for some $k_0 \geq 3$. Then Theorem 2.3-(ii) implies that $cat(K_0; M_h) \geq k_0$. But, according to Lemma 12.3-(ii), $u \in K_0$ iff $u(t)$ is identically constant. In other words one has that $K_0 \equiv S^{n-1}$ and hence $cat(K_0; M_h) = 2$. The contradiction proves that $c_{k,\delta} > 0$ for all $k \geq 3$. Then the arguments of Lemma 12.1 show that each critical point at level $c_{k,\delta}$ gives rise to a solution of (12.11-12). ∎

Remark 12.13 Lemma 12.12 has an interest in itself. Indeed, it provides the existence of a noncollision orbit with energy $h < 0$ for the relativistic correction of Keplerian potentials, discussed in Example 11.3-(i). ∎

In the sequel we fix k, say $k = 3$ and set $c_\delta = c_{3,\delta}$, $u_\delta = u_{3,\delta}$.

We now provide the estimates to perform the limiting process as $\delta \to 0$.

Lemma 12.14 *(i)* $\|u_\delta\| \leq a_7$ *and* $\exists\, z \in E$ *such that* $u_\delta \to z$ *weakly in* E *and uniformly;*
(ii) $z(t) \not\equiv 0$ *and* $V(z(t)) \not\equiv h$*;*
(iii) $\exists\, \omega_0 > 0$ *such that* $\omega_\delta \to \omega_0$.

Proof. (i) Let $\bar{c} = c_{\delta=1}$. Since $V_\delta(x) \geq V(x) - |x|^{-2}(= V_{\delta=1}(x))$, one immediately has $c_\delta \leq \bar{c}$ and using Lemma 12.9 the result follows.
(ii) If $V(z(t)) \equiv h$ then $z(t) \neq 0$ for all t and $V(u_\delta)$ (resp. $V'(u_\delta)\cdot u_\delta$) uniformly converges to $V(z)$ (resp. $V'(z)\cdot z$). As a consequence one has that $G(u_\delta) \to G(z)$. Since $u_\delta \in M_h$ then $G(u_\delta) = h$ and we infer

$$h = G(z) = \int_0^1 V(z)dt + \frac{1}{2}\int_0^1 V'(z)\cdot z dt = h + \frac{1}{2}\int_0^1 V'(z)\cdot z dt.$$

Hence $\int_0^1 V'(z)\cdot z dt = 0$, in contradiction with $(V3.2)$.

Let us show that $z(t) \not\equiv 0$. If not $u_\delta \to 0$ uniformly and as in the proof of Lemma 12.8 we have: $h = G(u_\delta) \leq (1 - \frac{\beta}{2})\int_0^1 V(u_\delta)dt$, a contradiction, because the last integral $\to -\infty$.
(iii) We will show that $\exists\, \underline{\omega}, \bar{\omega} > 0$ such that $\underline{\omega} \leq \omega_\delta \leq \bar{\omega}$. By (ii) there exists a closed interval J, with $|J| > 0$ such that $z(t) \neq 0, V(z(t)) \neq h$ for all $t \in J$. Integrating (12.12) on J we obtain

$$\frac{1}{2}\omega_\delta^2 \int_J |\dot{u}_\delta|^2 dt + \int_J V_\delta(u_\delta)dt = h|J|.$$

By (i), $\int_J |\dot{u}_\delta|^2 dt \le a_7$ and thus

$$\frac{1}{2}\omega_\delta^2 \ge \frac{1}{a_7}\int_J [h - V_\delta(u_\delta)]dt. \tag{12.15}$$

Since $V_\delta(u_\delta) \to V(z)$ and, by (12.12), $V_\delta(u_\delta) \le h$ then

$$\int_J [h - V(z)]dt = \lim_{\delta \to 0} \int_J [h - V_\delta(u_\delta)]dt \ge 0,$$

and the definition of J imply that $h - \int_J V(z)dt > 0$. Then (12.15) yields, for δ close to 0:

$$\frac{1}{2}\omega_\delta^2 \ge a_8 > 0.$$

Finally, from (12.13) and $c_\delta \le \bar{c}$ it follows

$$\omega_\delta^2 = \frac{\int_0^1 [h - V_\delta(u_\delta]dt}{\frac{1}{2}|\dot{u}_\delta|_2^2} = \frac{c_\delta}{\frac{1}{4}|\dot{u}_\delta|_2^4} \le \frac{\bar{c}}{\frac{1}{4}|\dot{u}_\delta|_2^4}$$

Let $y_\delta(t) := u_\delta(\omega_\delta t)$. If $\omega_\delta \to +\infty$, the preceding inequality implies that both $|\dot{u}_\delta|_2 \to 0$ and $|\dot{y}_\delta|_2 \to 0$, thus $z \equiv \xi$, with $\xi \ne 0$, $V(\xi) \ne h$, according to point (ii) above. From the definition of y_δ it follows that

$$h \equiv \frac{1}{2}|\dot{y}_\delta(t)|^2 + V_\delta(y_\delta(t))$$

Hence

$$h = \frac{1}{2}|\dot{y}_\delta|_2^2 + \int_0^1 V_\delta(y_\delta)dt \to \int_0^1 V(\xi)dt = V(\xi),$$

a contradiction. ■

Using Lemma 12.14, an argument quite similar to that already discussed in the proof of Theorem 11.14 leads to show:

Theorem 12.15 *Suppose that $V \in C^2(\Omega, \mathbf{R})$ satisfies $(V3)$. Then for all $h < 0$ there exists a generalized periodic solution of (P_h).*

Remark 12.16 It would be possible to prove that the collision set C_u is at most finite (for a class of potentials like those discussed in Theorem 11.16). A more general result similar to Theorem 11.16, however, it is not yet known. ■

From (12.2) it follows that any possible solution q of (P_h) satisfies $V(q(t)) \le h$. Therefore it makes sense to find an existence result in which the assumptions are made in $V(x) \le h$, only. Let D_h denote the connected component of $\{x \in \Omega\ : V(x) \le h\}$ such that $0 \in \bar{D}_h$.

Theorem 12.17 *Let $h < 0$ be given. Suppose that $\bar{D}_h$ is compact and that $(V3.1-2-3)$ hold true for all $x \in \bar{D}_h$. Moreover, let us assume that $V \in C^4$ in a neighbourhood of $\partial\bar{D}_h$ and that $\max_{x\in\partial\bar{D}_h}[V''(x)x \cdot x] < 0$. Then (P_h) possesses a generalized periodic solution.*

We do not carry over the details of the proof, referring to [15, Theorem 5.1][1]. Roughly, it is possible to extend V in $\Omega - D_h$ in such a way that the extended potential $\hat{V}$ verifies $(V3)$ for all $x \in \Omega$ and satisfies $\hat{V}(x) > h$ on $\Omega - D_h$. Then the solution obtained by means of Theorem 12.15 is confined in D_h and hence solves (P_h).

The following example illustrates possible applications of Theorem 12.17.

Example 12.18 Let $U : \mathbf{R}^n \to \mathbf{R}$ be smooth and let $h < 0$ be given. Let us consider the Kepler problem

$$\ddot{q} + \frac{q}{|q|^3} + U'(q) = 0 \tag{12.16}$$

where U is smooth on $\mathbf{R}^n$. If we look for solutions of (12.16) in the form $q(t) = k^\mu y(kt)$, $k \in \mathbf{N}$, we are led to

$$k^{2+3\mu}\ddot{y} + \frac{y}{|y|^3} + k^{2\mu}U'(k^\mu y) = 0.$$

For $\mu = -\frac{2}{3}$ the preceding equation becomes

$$\ddot{y} + \frac{y}{|y|^3} + k^{-4/3}U'(k^{-2/3}y) = 0. \tag{12.17}$$

Given any $h < 0$, one immediately verifies that Theorem 12.15 applies, provided k is sufficiently large (depending on h), yielding a solution y of (12.17) with energy h. As a consequence, $q(t) = k^{-2/3}y(kt)$ (has the same energy, and) is a periodic solution of (12.16). ∎

C. Perturbation Results. The abstract Theorem 11.17 can also be used to study perturbed problems like

$$\ddot{q} + \alpha\frac{q}{|q|^{\alpha+2}} + \varepsilon U'(q) = 0 \tag{12.18}$$

$$\frac{1}{2}|\dot{q}|^2 + \frac{1}{|q|^\alpha} + \varepsilon U(q) = h \tag{12.19}$$

[1]Let us point out that there is a misprint at pag. 359, line 12 of [15]: instead of $S(\xi) < 1$ there should be $S(\xi) < \frac{1}{3}$

Since the arguments are close to those of Section 11.C, we will be sketchy. For more details we refer to [10].

Let F_0, $F_\varepsilon \in C^2(\Lambda, \mathbf{R})$ be defined by setting

$$\begin{aligned} F_0(u) &= \frac{1}{2}|u|_2^2 \cdot \int_0^1 [h + \frac{1}{|u|^\alpha}]dt \\ F_\varepsilon(u) &= \frac{1}{2}|u|_2^2 \cdot \int_0^1 [h + \frac{1}{|u|^\alpha} - \varepsilon U(u)]dt \end{aligned}$$

Hereafter it is understood that $U \in C^2(\mathbf{R}^n, \mathbf{R})$.

Let $R > 0$ be such that

$$(\frac{\alpha}{2} - 1)R^{-\alpha} = h.$$

Then, setting

$$Z = \{z(t) = R[\xi e^{i2\pi t} + \bar{\xi} e^{-i2\pi t}] \ : \xi \in \mathbf{C}^n, \xi \cdot \xi = \frac{1}{2}, \ \xi \cdot \bar{\xi} = 0\}$$

one has that $F_0'(z) = 0$, $\forall\ z \in Z$. To show that Z is a non- degenerate critical manifold one argues as in Lemma 11.19. In the present case, $v \in Ker[F_0''(z)] = N_z$ whenever

$$-\frac{\alpha}{2}R^{-\alpha}\ddot{v} + \frac{\alpha}{2\pi^2}R^{-\alpha-2}\ddot{z}\int_0^1 \dot{z} \cdot \dot{v}dt - 2\pi^2 R^2 V_0''(z)v = 0$$

where $V_0(x) = -|x|^{-\alpha}$.

Using Fourier series, it follows that $N_z = T_z Z$ provided $\alpha \neq 1$. If $\alpha = 1$, one works on the space of anti-periodic functions E_* (see notation introduced in Section 11.C) and proves that Z is non-degenerate for F_0 constrained on E_*. As a consequence, one deduces the following results, which are the counterpart of Theorems 11.20-(ii) and 11.23-(ii).

Theorem 12.19 *Suppose that $U \in C^2(\mathbf{R}^n, \mathbf{R})$. If $\alpha > 0$, $\alpha \neq 1$, then (12.18-19) has, for ε small, at least n closed (non-collision) orbits near Z. The same conclusion holds if $\alpha = 1$ and U is even.*

Remarks 12.20 (i) We can repeat, with obvious changes, what has been pointed out in Remark 11.21. In particular, the preceding result can be interpreted as a bifurcation result, the possible bifurcation of periodic solutions being the critical points of the averaged functional $\int_0^1 U(z(t))dt$ on Z.

(ii) Theorem 12.19 can be extended to perturbed Lagrangian systems with Lagrangian of the form

$$L(q,\dot{q}) = \frac{1}{2}|\dot{q}|^2 - \chi(|q|) + \varepsilon L_1(q,\dot{q}) \tag{12.20}$$

where $\chi \in C^2(0,+\infty)$ and

$$\begin{aligned} L_1(q,\dot{q}) &= \frac{1}{2}\mathcal{B}(q,\dot{q}) - U(q) \\ \mathcal{B}(q,\dot{q}) &= \sum_{i,j} b_{i,j}\dot{q}_i\dot{q}_j,\ b_{i,j} = b_{j,i} \in C^2(\mathbf{R}^n,\mathbf{R}) \\ U &\in C^2(\mathbf{R}^n,\mathbf{R}) \end{aligned}$$

Systems of this type model, for example, a class of restricted N-body problems (see [10, Section 4]).
(iii) The case $\alpha = 1$ in Theorem 12.19 deals with perturbed Kepler's problems. With respect to Example 12.18, it is now assumed that U is even. On the other part, Theorem 12.19 contains a multiplicity result and permits to locate the solutions. In particular, it is worth pointing out that the solutions found in Theorem 12.19 are not collisions, being close to Z.
(iv) Periodic solutions of autonomous perturbed Kepler's problem have been investigated, among other things, in [100] from the bifurcation point of view. Using a regularizing transformation, it is discussed the branching off from orbits which can possibly be different from *circular*, but the bifurcation is proved under the assumption that the averaged potential is a Morse function. ■
(v) Concerning the existence of multiple geometrically distinct periodic orbits of (P_h), some results have been obtained, under the condition that $a|x|^{-\alpha} \leq V(x) \leq b|x|^{-\alpha}$ and a, b are sufficiently close. See [9, 47, 48, 126]. However, a result like the Ekeland-Lasry Theorem (see Section 11) is far to be proved.
(vi) The case in which $V(x) \simeq -|x|^{-\alpha} - |x|^{-\beta}$, with $0 < \alpha < 1 < \beta$ has been discussed in [7]. ■

13 A class of N-body problems

The variational framework and the arguments of the Sections 11 and 12 can be adapted to obtain the existence of periodic motions for a class of mechanical systems, including the N-body problem. Such a topic will be discussed in this last section. For the sake of conciseness we will deal with periodic solutions with prescribed energy, only. For the existence of generalized solutions with prescribed period the reader is referred to [39] for a 3-body problem and to [65] for a class of symmetric N-body; see also [50, 116] for some results concerning the existence of non-collision orbits.

Our model is the classical N-body problem which describes the motion of N bodies $x_1, \cdots, x_N$ in $\mathbf{R}^3$ with masses $m_1, \cdots, m_N$, under the mutual gravitational attraction. The trajectories of these bodies are the solutions of the Conservative System

$$m_i \ddot{x}_i + V_{x_i}(x_1, \cdots, x_N) = 0, \quad (1 \le i \le N)$$

where

$$V(x_1, \cdots, x_N) = -\frac{1}{2} \sum_{i \neq j} \frac{m_i m_j}{|x_i - x_j|}.$$

Motivated by this problem, we let $X = (x_1, \cdots, x_N) \in \mathbf{R}^{3N}$ (one could take $x_i \in \mathbf{R}^d$ for any $d > 0$ as well; all the following results remain unaffected), $\Omega = \mathbf{R}^3 - \{0\}$, and consider potentials of the form

$$V(X) = \frac{1}{2} \sum_{i \neq j} V_{i,j}(x_i - x_j)$$

where, for all $1 \le i, j \le N$, $V_{i,j} \in C^1(\Omega, \mathbf{R})$ satisfies $(V4)$:

(V4.1) $V_{i,j}(x) = V_{j,i}(x) \ \forall\, x \in \Omega$;
(V4.2) $\exists\, \alpha \in [1,2)$, $\beta \in (0,2)$ and $r > 0$ such that

$$V'_{i,j}(x) \cdot x \ \geq \ -\alpha V_{i,j}(x) > 0 \ \ \forall\, x \in \Omega \tag{13.1}$$

$$V'_{i,j}(x) \cdot x \ \leq \ -\beta V_{i,j}(x) \ \ \forall\, 0 < |x| \leq r \tag{13.2}$$

(V4.3) $V_{i,j}(x) \to 0$ as $|x| \to +\infty$.

Here and always in the sequel we will use the same notation for scalar products and euclidean norm, both in $\mathbf{R}^3$ and in $\mathbf{R}^{3N}$.

Given $h < 0$, we look for periodic solutions $Q = (q_1, \cdots, q_N)$ of

$$\ddot{Q} + V'(Q) \ = \ 0 \tag{13.3}$$

$$\frac{1}{2}|\dot{Q}(t)|^2 + V(Q(t)) \ \equiv \ h \tag{13.4}$$

As in Sections 11 and 12, by a solution we mean a generalized solution.

Plainly, $V_{i,j}(x) = -|x|^{-1}$ satisfies $(V4)$, with $\alpha = \beta = 1$.

Let us introduce the functional set up. We put $E^N = H^{1,2}(S^1, \mathbf{R}^3) \times \cdots \times H^{1,2}(S^1, \mathbf{R}^3)$ (N times), $E^N_* = \{u \in E^N \ : u(t + \frac{1}{2}) = -u(t)\}$ and

$$\Lambda^N_* = \{u = (u_1, \cdots, u_N) \in E^N_* \ : u_i(t) \neq u_j(t),\ \forall\, t,\ \forall\, i \neq j\}.$$

As a norm in E^N_* we will take

$$\|u\|^2 = \sum_{i=1}^{N} |\dot{u}_i|_2^2$$

The corresponding scalar product will be denoted by $(\cdot|\cdot)$. Let us recall that, for all $1 \leq i \leq N$, $|\dot{u}_i|_2 \geq 4|u_i|_\infty$ and hence

$$\|u\| \geq a_1 |u(t)| \ \ \forall\, t \in [0,1]. \tag{13.5}$$

As in Section 12, closed orbits of (13.3-4) will be found using a limiting procedure.

Let

$$R(X) = \sum_{i \neq j} \frac{1}{|x_i - x_j|^2}$$

and set, for $\delta > 0$,

$$V_\delta(X) = V(X) - \delta R(X).$$

Define $F_\delta \in C^1(\Lambda_*^N, \mathbf{R})$, by

$$F_\delta(u) = \frac{1}{2}\|u\|^2 \cdot \int_0^1 [h - V_\delta(u)]dt.$$

We will use the following Variational Principle which plays the role of Lemma 12.1.

Lemma 13.1 *For all $\delta > 0$, let $u_\delta \in \Lambda_*^N$ be a critical point of F_δ such that $F_\delta(u_\delta) > 0$. Then, letting*

$$\omega_\delta^2 = \frac{\int_0^1 [h - V_\delta(u_\delta)]dt}{\frac{1}{2}\|u_\delta\|^2} \ (> 0) \tag{13.6}$$

and $Q_\delta(t) = u_\delta(\omega_\delta t)$, one has:

$$\ddot{Q}_\delta + V_\delta'(Q_\delta) = 0 \tag{13.7}$$

$$\frac{1}{2}|\dot{Q}_\delta(t)|^2 + V_\delta(Q_\delta(t)) \equiv h \tag{13.8}$$

Proof. As in Lemma 12.1, one has

$$\omega_\delta^2(u_\delta|v) - \int_0^1 V_\delta'(u_\delta) \cdot v dt = 0 \ \forall\, v \in E_*^N. \tag{13.9}$$

From $(V4.1)$ it follows that $V(X) = V(-X)$ and hence, as in Lemma 11.22, (13.8) is satisfied for all $v \in E$. This shows that (13.7) holds. The energy relationship (13.8) follows as in lemma 12.1. ∎

Critical points of F_δ at a positive level will not be found as constrained critical points on M_h (no assumption like $(V3.1)$ is made here), but rather using directly the Mountain-Pass Theorem.

First, let us remark, for future references, that $(V4.2)$ implies (see the arguments in Section 12, before Lemma 12.6) there exist constants $a_2, a_3 > 0$ such that (taking r possibly smaller)

$$\begin{cases} i) & V_{i,j}(x) \geq -a_2|x|^{-\beta} \ \forall\, 0 < |x| \leq r \\ ii) & V_{i,j}(x) \leq -a_3|x|^{-\alpha} \ \forall\, 0 < |x| \leq r, \end{cases} \tag{13.10}$$

as well as

$$V'(X) \cdot X \geq -\alpha V(X) > 0 \tag{13.11}$$

$$V'(X) \cdot X \leq -\beta V(X), \ \forall\, 0 < |x_i - x_j| \leq r. \tag{13.12}$$

Moreover (13.10-ii) yields

$$V(X) \leq -\frac{a_4}{|X|^\alpha} \;\forall\, 0 < |x_i - x_j| \leq r. \tag{13.13}$$

The following Lemma shows that F_δ inherits the Mountain-Pass geometry.

Lemma 13.2 *There exists δ_0 such that F_δ satisfies $(A2)$ for all $0 < \delta \leq \delta_0$.*

Proof. From (13.13) and (13.5) it follows there results, for $\|u\|$ small enough,

$$\begin{aligned} F_\delta(u) &\geq \frac{1}{2}\|u\|^2 \cdot \int_0^1 [h - V(u)]dt \geq \\ &\geq \frac{1}{2}\|u\|^2 \cdot \int_0^1 [h + a_4|u|^{-\alpha}]dt \geq \frac{h}{2}\|u\|^2 + a_5\|u\|^{2-\alpha}. \end{aligned}$$

Since $\alpha < 2$ it follows there exist $a, \rho > 0$ such that

$$F_\delta(u) \geq a \quad \forall\, u \in \Lambda_*^N,\ \|u\| = \rho,\ \forall\, \delta > 0,$$

proving $(A2.1)$.

Let $\xi, \eta \in \mathbf{R}^3$ be such that $|\xi| = |\eta| = 1$ and $\xi \cdot \eta = 0$. Set

$$z_k(t) = \xi \cos(2\pi(t + \frac{k}{N})) + \eta \sin(2\pi(t + \frac{k}{N})) \ \ (1 \leq k \leq N)$$

and $z(t) = (z_1(t), \cdots, z_N(t))$. Let us remark that

$$|z_i(t) - z_j(t)| \equiv d \ \text{(independent of t)} \tag{13.14}$$

and hence

$$R(z) = \sum_{i \neq j} \frac{1}{|z_i(t) - z_j(t)|^2} \equiv d'. \tag{13.15}$$

From (13.14), (13.15) and $(V4.3)$ if follows, $\forall\, i \neq j$:

$$\begin{aligned} V_{i,j}(sz_i - sz_j) &\to 0 \\ R(sz) &\to 0 \end{aligned}$$

as $s \to +\infty$. Then one has

$$\liminf_{s \to +\infty} \int_0^1 V_\delta(sz) \geq 0.$$

Since $h < 0$ and $\|z\| > 0$ we deduce that

$$F_\delta(sz) \to -\infty \quad (s \to +\infty)$$

and any $u_1 = s_1 z$, with $s_1 > 0$ and sufficiently large, satisfies $F_\delta(u_1) < 0$.

On the other side, (13.10-i) and (13.15) imply, for $s \downarrow 0$,

$$V_\delta(sz) \geq -a_6 s^{-\beta} - \delta a_7 s^{-2}$$

Then one has

$$F_\delta(sz) \leq \frac{s^2}{2}\|z\|^2(h + a_7 s^{-\beta} + \delta a_6 s^{-2}) = a_8 h s^2 + a_9 s^{2-\beta} + \delta a_{10}.$$

Since $\beta < 2$, it follows that there exist $\delta_0 > 0$ and $s_0 > 0$ such that $\|s_0 z\| < \rho$, $F_\delta(s_0 z) < a$, $\forall\, 0 < \delta \leq \delta_0$. This suffices for $(A2.3)$ with $u_0 = s_0 z$. ∎

Next we are concerned with the (PS) condition.. Let $u_k \in \Lambda_*^N$ be such that

$$\begin{aligned} F_\delta(u_k) &\to c > 0, && (13.16)\\ F_\delta'(u_k) &\to 0. && (13.17)\end{aligned}$$

We set $\sigma_k = (F_\delta'(u_k)|u_k)$.

Lemma 13.3 $\|u_k\| \leq a_{11}$.

Proof. Using (13.11) we deduce

$$\begin{aligned}\sigma_k &= \|u_k\|^2 \int_0^1 [h - V(u_k) - \frac{1}{2}V'(u_k)\cdot u_k]dt \leq \\ &\leq \|u_k\|^2 \int_0^1 [h - (1 - \frac{\alpha}{2})V(u_k)]dt \qquad (13.18)\end{aligned}$$

From (13.16) one has $b \geq F_\delta(u_k)$ for some $b > 0$, and hence

$$-\frac{1}{2}\|u_k\|^2 \int_0^1 V(u_k)dt \leq b - \frac{1}{2}h\|u_k\|^2. \qquad (13.19)$$

Inserting (13.18) into (13.19) we find

$$\sigma_k \leq h\|u_k\|^2 - (1 - \frac{\alpha}{2})h\|u_k\|^2 + a_{12} = \frac{\alpha}{2}h\|u_k\|^2 + a_{12}.$$

Since $h < 0$ and $|\sigma_k| \leq \|u_k\|\ \|F_\delta'(u_k)\|$, the Lemma follows. ∎

Next Lemma prevents phenomena like those seen in Remark 12.11. It is just here where condition (13.1) with $\alpha \geq 1$ plays the role.

Lemma 13.4 *There exists $b_0 > 0$ such that $|u_k|_\infty \geq b_0$.*

Proof. By contradiction, let $|u_k|_\infty \to 0$. We set

$$\begin{aligned} \psi_k &= -\int_0^1 V(u_k)dt, \\ A_k &= \frac{1}{2}\|u_k\|^2[h+\psi_k], \\ B_k &= \|u_k\|^2\int_0^1 R(u_k)dt, \\ C_k &= \int_0^1 V'(u_k)\cdot u_k dt. \end{aligned}$$

Since $F_\delta(u_k) = A_k + \frac{1}{2}\delta B_k$, we will reach a contradiction with (13.16) whenever we show that $\limsup A_k \leq 0$ and $B_k \to 0$.

Step 1: $\limsup A_k \leq 0$. Since $|u_k|_\infty \to 0$, it follows from (13.12) that $C_k \leq \beta\psi_k$, and hence

$$\sigma_k = \|u_k\|^2[h+\psi_k - \frac{1}{2}C_k] \geq \|u_k\|^2[h+(1-\frac{\beta}{2})\psi_k]. \tag{13.20}$$

Then it turns out that

$$A_k \leq \frac{1}{2}\frac{\sigma_k}{h+(1-\frac{\beta}{2})\psi_k}\cdot[h+\psi_k].$$

Since $\sigma_k \to 0$ and $\psi_k \to +\infty$ (because $|u_k|_\infty \to 0$), it follows that $\limsup A_k \leq 0$.

Step 2: $B_k \to 0$. Define r_k, R_k, t_k, T_k by setting

$$\begin{aligned} r_k &= \min_{t\in[0.1]}|u_k(t)| = u_k(t_k) \\ R_k &= |u_k|_\infty = u_k(T_k) \end{aligned}$$

There results

$$\begin{aligned} log\frac{R_k}{r_k} &= \int_{t_k}^{T_k}\frac{d}{ds}\log|u_k| \leq \int_0^1\frac{|\dot{u}_k|}{|u_k|}dt \leq \\ &\leq |\dot{u}_k|_2\left[\int_0^1\frac{1}{|u_k|^2}\right]^{1/2} \leq a_{13}\|u_k\|\cdot\left[\int_0^1\frac{1}{|u_k|^2}\right]^{1/2}. \end{aligned} \tag{13.21}$$

Since $|u_k|_\infty \to 0$ then $h+\psi_k >$ for k large. Then $F_\delta(u_k) > \frac{1}{2}\delta B_k$ and if $\frac{R_k}{r_k} \to +\infty$, then (13.21) would imply $F_\delta(u_k) \to +\infty$, a contradiction. This shows that

$$\frac{R_k}{r_k} \leq a_{14}. \tag{13.22}$$

Using again (13.20) and (13.13) we infer

$$\sigma_k \geq \|u_k\|^2[h + a_{15}\int_0^1 |u_k|^{-\alpha}dt].$$

Let us remark that $h + a_{15}\int_0^1 |u_k|^{-\alpha}dt > 0$ because $|u_k|_\infty \to 0$. Therefore

$$\|u_k\| \leq \frac{\|F_\delta'(u_k)\|}{h + a_{15}\int_0^1 |u_k|^{-\alpha}dt}$$

thus

$$B_k \leq \frac{\|F_\delta'(u_k)\|^2}{(h + a_{15}\int_0^1 |u_k|^{-\alpha}dt)^2}\int_0^1 |u_k|^{-2}dt.$$

Since $r_k \leq |u_k(t)| \leq R_k$, it follows

$$B_k \leq \|F_\delta'(u_k)\|^2 \frac{r_k^{-2}}{[h + a_{15}R_k^{-\alpha}]^2}.$$

and (13.21), $\alpha \geq 1$, jointly with $\|F_\delta'(u_k)\| \to 0$ imply that $B_k \to 0$, as required. This completes the proof of the Lemma. ■

We are now in position to prove

Lemma 13.5 *F_δ satisfies $(PS)_c$ for all $c > 0$.*

Proof. From Lemma 13.3 it follows that $u_k \to u^*$ weakly and uniformly, along a subsequence. Lemma 13.4 implies that $u^*(t) \not\equiv 0$. If $u^* \in \partial\Lambda_*^N$ then $\int_0^1 V_\delta(u_k)dt \to -\infty$, because V_δ satisfies the (SF) condition. Moreover $u^* \not\equiv$ const. and $\liminf \|u_k\| \geq \|u^*\| > 0$. Hence $F_\delta(u_k) \to +\infty$, a contradiction, which proves that $u^* \in \Lambda_*^N$. It is now a standard procedure to show that u_k strongly converges (up to a subsequence) to u^*. ■

As a consequence, we can state

Lemma 13.6 *For all $0 < \delta < \delta_0$, F_δ possesses a critical point $u_\delta \in \Lambda_*^N$ such that $F_\delta(u_\delta) > 0$.*
Moreover there exist $a, A > 0$ such that $a \leq \|u_\delta\| \leq A$, for all $0 < \delta \leq \delta_0$.

Proof. Lemmas 13.2 and 13.5 allow us to use the Mountain- Pass Theorem, in a slightly modified version (see [16] for more details), finding a critical point $u_\delta \in \Lambda_*^N$ of F_δ such that $c_\delta = F_\delta(u_\delta) > 0$. The min-max characterization of c_δ easily implies $\|u_\delta\| \leq A$. Moreover, if $\|u_\delta\| \to 0$ then $|u_\delta|_\infty \to 0$ and, from (13.12),

$$h = \int_0^1 [V(u_\delta) + \frac{1}{2}V'(u_\delta)\cdot u_\delta]dt \leq (1 - \frac{\beta}{2})\int_0^1 V(u_\delta)dt.$$

Using (13.13), the right hand side tends to $-\infty$, a contradiction. ∎

According to Lemma 12.1, $Q_\delta(t) = u_\delta(\omega_\delta t)$ solves (13.7-8), with ω_δ given by (13.6). The same arguments of Lemma 12.13 show that $\omega_\delta \to \omega_0 \neq 0$ and $u_\delta \to z$ in E. Setting $Q(t) = z(\omega_0 t)$ it follows that Q is a generalized solution of (13.3-4).

In conclusion one has

Theorem 13.7 *Let $V(X) = \frac{1}{2}\sum_{i\neq j} V_{i,j}(x_i - x_j)$ and $V_{i,j}$ satisfy $(V4)$. Then for all $h < 0$ problem (13.3-4) has a generalized periodic solution.*

Remark 13.8 With respect to the results of Section 12, Theorem 13.7 deals with potentials such that, roughly, $V_{i,j} \simeq -|x|^{-\alpha}$, with $1 \leq \alpha < 2$. On the other side, we do not assume any condition like $(V3.1)$. It would be possible to find an existence result for (13.3-4) assuming that $V_{i,j}$ satisfy $(V3)$ and $(V4.1)$. ∎

Remark 13.9 As indicated in Remark 12.20-(ii), using the perturbation techniques like those discussed in Sections 11.C and 12.C, one can prove the existence of multiple noncollision orbits for a class of *restricted* N-body problems. See [10, Section 4]. See also [64] concerning the existence of T-periodic solutions. ∎

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