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INFINITESIMAL ISOPECTRAL DEFORMATIONS OF THE GRASSMANNIAN OF 3-PLANES IN $\mathbb{R}^6$

Jacques GASQUI
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INFINITESIMAL ISOSPECTRAL DEFORMATIONS OF THE GRASSMANNIAN OF 3-PLANES IN $\mathbb{R}^6$

Jacques Gasqui, Hubert Goldschmidt

Abstract. — We study the real Grassmannian $G^{\mathbb{R}}_{n,n}$ of n-planes in $\mathbb{R}^{2n}$, with $n \geq 3$, and its reduced space. The latter is the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{n,n}$, which is the quotient of the space $G^{\mathbb{R}}_{n,n}$ under the action of its isometry which sends a n-plane into its orthogonal complement. One of the main results of this monograph asserts that the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{3,3}$ possesses non-trivial infinitesimal isospectral deformations; it provides us with the first example of an irreducible reduced symmetric space which admits such deformations. We also give a criterion for the exactness of a form of degree one on $\bar{G}^{\mathbb{R}}_{n,n}$ in terms of a Radon transform.

Résumé **(Déformations infinitésimales isospectrales de la grassmannienne des 3-plans dans $\mathbb{R}^6$)**

Ce mémoire a pour cadre la grassmannienne $G^{\mathbb{R}}_{n,n}$ des n-plans de $\mathbb{R}^{2n}$, avec $n \geq 3$, et son espace réduit $\bar{G}^{\mathbb{R}}_{n,n}$, qui est l'espace symétrique irréductible, quotient de $G^{\mathbb{R}}_{n,n}$ par l'involution envoyant un n-plan sur son orthogonal. Un de nos principaux résultats est la construction de déformations infinitésimales isospectrales non triviales sur $\bar{G}^{\mathbb{R}}_{3,3}$, obtenant ainsi le premier exemple d'espace symétrique irréductible réduit et non infinitésimalement rigide. Nous donnons aussi un critère d'exactitude pour les formes différentielles de degré 1 sur $\bar{G}^{\mathbb{R}}_{n,n}$, mettant en jeu la nullité d'une transformée de Radon.

CONTENTS

Introduction

We pursue our study of the infinitesimal isospectral deformations of Riemannian symmetric spaces of compact type undertaken in [**5**] and [**6**]. Let (X, g) be a Riemannian symmetric space of compact type. Let $\{g_t\}$ be family of Riemannian metrics on X, with $g_0 = g$. In [**14**], Guillemin proved that, if the family $\{g_t\}$ is an isospectral deformation of g (*i.e.*, if the spectrum of the Laplacian of the metric g_t is independent of t), then the corresponding infinitesimal deformation $h = \frac{d}{dt} g_{t|t=0}$ of the metric g belongs to the kernel $\mathcal{N}_2$ of a certain Radon transform defined on the space of symmetric 2-forms on X in terms of integration over the maximal flat totally geodesic tori of X. The infinitesimal deformation h of g is trivial if it can be written in the form $\frac{d}{dt}\varphi_t^* g_{|t=0}$, where $\{\varphi_t\}$ is one-parameter family of diffeomorphisms of X, or equivalently if it is a Lie derivative of the metric g; such Lie derivatives always belong to the kernel $\mathcal{N}_2$. Consequently, we define the space of infinitesimal isospectral deformations $I(X)$ of X to be the orthogonal complement of the space of Lie derivatives of the metric g in $\mathcal{N}_2$. If the space $I(X)$ vanishes, we say that the space (X, g) is infinitesimally rigid in the sense of Guillemin; under this assumption, an isospectral deformation of the metric g is trivial to first-order and the space X is infinitesimally spectrally rigid (*i.e.*, spectrally rigid to first-order).

The question of Guillemin rigidity for the spaces of rank one first arose in conjunction with the Blaschke problem. The Guillemin rigidity of these spaces which are not spheres was proved by Michel [**19**] for the real projective spaces and by Michel [**19**] and Tsukamoto [**22**] for the other projective spaces (see [**6**]).

The reduced space of X (called the adjoint space in [**6**]), which is constructed in [**16**, Chapter VII], plays a crucial role here; this symmetric space is covered by X and, when X is irreducible, it is not the cover of another symmetric space. We say that X is reduced if it is equal to its reduced space.

We showed that a product of irreducible symmetric spaces is not rigid in the sense of Guillemin (see Theorem 10.5 of [**6**]). Here we prove that an irreducible space which is infinitesimally rigid in the sense of Guillemin must necessarily be reduced (Theorem 1.4). In fact, if X is an irreducible space which is not reduced, then X always possesses an isometry which give rise to symmetric 2-forms which lie in the kernel $\mathcal{N}_2$ of our Radon transform and which are not Lie derivatives of the metric. Thus the relevant problem concerning infinitesimal isospectral deformations for our class of symmetric spaces may be formulated as follows: determine the space of infinitesimal isospectral deformations of an irreducible reduced space.

In [**5**] and [**6**], we began to address this problem for spaces of arbitrary rank and proved that an irreducible symmetric space, which is equal to a Grassmannian, is rigid in the sense of Guillemin if and only if it is reduced. In fact, the Grassmannians $G^{\mathbb{K}}_{m,n}$ of m-planes in $\mathbb{K}^{m+n}$, where $\mathbb{K}$ is a division algebra over $\mathbb{R}$, with $m \neq n$ and $m, n \geq 1$, are rigid. This generalizes the rigidity results for the projective spaces.

We say that a symmetric p-form u on X satisfies the Guillemin condition if, for every maximal flat totally geodesic torus Z contained in X and for all parallel vector fields ζ on Z, the integral

$$\int_Z u(\zeta, \zeta, \ldots, \zeta)\, dZ$$

vanishes, where dZ is the Riemannian measure of Z. The kernel $\mathcal{N}_p$ of the Radon transform for p-forms consists precisely of those forms satisfying the Guillemin condition. Thus the space X is rigid in the sense of Guillemin if and only if the only symmetric 2-forms on X satisfying the Guillemin condition are the Lie derivatives of the metric g. In [**14**], Guillemin proved that a symmetric 2-form, which is equal to the infinitesimal deformation of an isospectral deformation of g, satisfies the Guillemin condition. In [**12**] and [**13**], Grinberg studied the maximal flat Radon transform for functions on an irreducible space. This transform is known to be injective (*i.e.*, the kernel $\mathcal{N}_0$ vanishes) whenever X is one of the irreducible reduced spaces studied in [**6**]. Here we prove the converse of this result for an arbitrary irreducible space: the injectivity of this transform can only occur on a reduced space (Theorem 1.4).

In this monograph, we study the real Grassmannian $G^{\mathbb{R}}_{n,n}$ of n-planes in $\mathbb{R}^{2n}$, with $n \geq 3$, and its reduced space. The latter is the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{n,n}$, which is the quotient of the space $G^{\mathbb{R}}_{n,n}$ under the action of its isometry which sends a plane into its orthogonal complement. The first main result of this monograph asserts that the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{3,3}$ possesses non-trivial infinitesimal isospectral deformations (Theorem 6.2); it provides us with the first example of an irreducible reduced symmetric space which admits such deformations. In fact, we consider an explicit subspace $\mathcal{F}$ of the space of real-valued functions on $\bar{G}^{\mathbb{R}}_{3,3}$ of finite codimension, which is orthogonal to the space of constant functions, and construct an injective mapping

$$\mathcal{F} \longrightarrow I\big(\bar{G}^{\mathbb{R}}_{3,3}\big)$$

which we now describe.

The real Grassmannian $\widetilde{G}^{\mathbb{R}}_{n,n}$ of oriented n-planes in $\mathbb{R}^{2n}$, which is the universal covering manifold of $G^{\mathbb{R}}_{n,n}$, carries a symmetric n-form σ which is invariant under its group of isometries and which is therefore parallel; in fact, the form σ arises from the volume forms of the two canonical bundles of rank n on $\widetilde{G}^{\mathbb{R}}_{n,n}$. This form σ induces a symmetric n-form on $\bar{G}^{\mathbb{R}}_{n,n}$ and an injective mapping $*$ from the space of 1-forms on $\bar{G}^{\mathbb{R}}_{n,n}$ to the space of symmetric $(n-1)$-forms on $\bar{G}^{\mathbb{R}}_{n,n}$. We then show that a 1-form θ on $\bar{G}^{\mathbb{R}}_{n,n}$ satisfies the Guillemin condition if and only if the symmetric $(n-1)$-form $*\theta$ satisfies the Guillemin condition. When $n = 3$, the mapping $*$ sends the space of 1-forms on $\bar{G}^{\mathbb{R}}_{3,3}$ into the space of symmetric 2-forms on $\bar{G}^{\mathbb{R}}_{3,3}$. If f is a real-valued function on $\bar{G}^{\mathbb{R}}_{3,3}$, the symmetric 2-form $*df$ satisfies the Guillemin condition; if f is a non-zero element of $\mathcal{F}$, we prove that the 2-form $*df$ is not a Lie derivative of the metric of $\bar{G}^{\mathbb{R}}_{3,3}$ and thus gives rise to a non-zero element of $I\big(\bar{G}^{\mathbb{R}}_{3,3}\big)$. This construction of infinitesimal deformations is quite specific to this space. The rigidity problem for the other spaces $\bar{G}^{\mathbb{R}}_{n,n}$, with $n \geq 4$, remains open.

The other principal result of this monograph (Theorem 9.1) states that a 1-form on the irreducible symmetric space $\bar{G}^{\mathbb{R}}_{n,n}$ satisfying the Guillemin condition is exact. Here we prove it when $n = 3$; then the induction argument given in §2, Chapter VII of [**6**] provides us with the result for the other spaces $\bar{G}^{\mathbb{R}}_{n,n}$ (see Proposition 7.21 of [**6**]). For all the spaces which we had studied previously, the behavior of 1-forms and 2-forms with respect to the injectivity of the corresponding Radon transform is always the same. It is interesting to note that $\bar{G}^{\mathbb{R}}_{3,3}$ is the first example of a symmetric space for which we have injectivity of our Radon transform for functions and 1-forms and non-injectivity for 2-forms.

The harmonic analysis on the homogeneous space $\widetilde{G}^{\mathbb{R}}_{3,3}$ of the group $SO(6)$ plays an important role in the proofs of our two main results. We require an explicit description of the highest weight vectors of the isotypic components of the space of complex forms degree one on $\widetilde{G}^{\mathbb{R}}_{3,3}$. In §§6 and 7, we express these vectors in terms of certain functions and 1-forms on this space, which are introduced in §5 by means of the corresponding Stiefel manifold. Here we also need to know the multiplicities of these isotypic components; they are computed by means of branching laws which are to be found in §10. Our description allows us to tell which of these highest weight vectors arise from objects defined on the quotient spaces $G^{\mathbb{R}}_{3,3}$ and $\bar{G}^{\mathbb{R}}_{3,3}$ of $\widetilde{G}^{\mathbb{R}}_{3,3}$.

In order to demonstrate Theorem 6.2, we must determine when the symmetric 2-form $*df$, where f is a non-constant function on $\bar{G}^{\mathbb{R}}_{3,3}$, is a Lie derivative of the metric; for this result, we need only to consider the isotypic components corresponding to the irreducible representations of $SO(6)$ which appear in the decomposition of the space of functions on $\widetilde{G}^{\mathbb{R}}_{3,3}$.

Sections 7, 8 and 9 are devoted to the proof of our criterion for the exactness of forms of degree one on the space $\bar{G}^{\mathbb{R}}_{3,3}$ (Theorem 9.1). In particular, in §7 we complete our description of the isotypic components of the space of 1-forms on $\widetilde{G}^{\mathbb{R}}_{3,3}$. For the proof of this criterion, we are obliged to show that certain linear combinations of the highest weight vectors of these isotypic components satisfying the Guillemin condition are either exact or vanish. These verifications, which we carry out in §9, depend in a crucial way on results concerning polynomials in one variable which arise from the integration of the highest weight vectors over suitably chosen maximal flat tori of the space $\widetilde{G}^{\mathbb{R}}_{3,3}$. The properties of these polynomials are presented separately in §8, a section which can be read independently of the rest of this paper. In fact, we obtain a whole class of combinatorial identities; one of these is proved by means of the WZ theory described in [**20**].

The symmetrized covariant derivative of a symmetric $(p-1)$-form on a symmetric space X of compact type is a symmetric p-form satisfying the Guillemin condition. Verifying that the only symmetric p-forms which satisfy the Guillemin condition are precisely the symmetrized covariant derivatives of symmetric $(p-1)$-forms is an injectivity question for Radon transforms which unifies the problems considered above, namely, the injectivity question for the maximal flat Radon transform for functions on X, the problem concerning the exactness of 1-forms and the Guillemin rigidity problem for X. For the real projective spaces, this verification was carried out in all degrees (see §3, Chapter III of [**6**]). This monograph provides a further geometric motivation for this question when $p \geq 3$. In fact, if f is a non-constant function on $\bar{G}^{\mathbb{R}}_{n,n}$, when $n \geq 4$ it would be interesting to know whether the symmetric $(n-1)$-form $*df$, which satisfies the Guillemin condition, is a symmetrized covariant derivative of a symmetric $(n-2)$-form. Here we show that this does not hold when $n = 3$.

As we saw above, the existence of the non-trivial invariant symmetric 3-form on $\widetilde{G}^{\mathbb{R}}_{3,3}$ is a fundamental element in the construction of our space of non-zero infinitesimal deformations of $\bar{G}^{\mathbb{R}}_{3,3}$. This leads us to determine in §2 which simply-connected irreducible symmetric spaces admit invariant symmetric 3-forms and construct these forms for three classes of spaces. The symmetric 3-form σ on $\widetilde{G}^{\mathbb{R}}_{3,3}$ can be viewed as arising from a symmetric form on one of these classes of symmetric spaces, the special Lagrangian Grassmannians $SU(n)/SO(n)$, with $n \geq 3$. Such a space admits a non-trivial 3-form σ_3 invariant under its group of isometries and this form is unique up to a constant. In fact, in §11 we show that the Grassmannian $\widetilde{G}^{\mathbb{R}}_{3,3}$ is isometric to the special Lagrangian Grassmannian $SU(4)/SO(4)$ and that the symmetric 3-form σ on $\widetilde{G}^{\mathbb{R}}_{3,3}$ can be viewed as a constant multiple of the form σ_3 on $SU(4)/SO(4)$. Furthermore, we describe in §3 all the invariant symmetric forms on the space $\widetilde{G}^{\mathbb{R}}_{n,n}$. In view of this isometry, we investigate in [**7**] the infinitesimal spectral deformations of the reduced space of the special Lagrangian Grassmannian $SU(n)/SO(n)$, with $n \geq 3$, and prove the analogue of Theorem 6.2 for this space.

We have preferred to present the branching laws, needed to compute the multiplicities of the $SO(6)$-modules appearing in §6, in a separate section (§10) in a way that is essentially independent of the rest of this monograph. This is done for the convenience of the reader and to allow us to refer to them readily in our study of the Lagrangian Grassmannians.

Finally, in §12 we give an explicit construction due to Bryant of a certain space of symmetric 2-forms on the Grassmannian $\widetilde{G}^{\mathbb{R}}_{3,3}$. This space was originally introduced in [**5**] and does not appear elsewhere in this monograph. Various properties of forms belonging to this space, which are given in [**5**], are derived here directly from their definition. Also we may view §1 as a complement to §4, Chapter II of [**6**].

We would like to express our deep gratitude to H. Wilf for his verification of the identity of Lemma 8.1 and to M. Brion for providing us with proofs of Proposition 6.3 and the propositions presented in §10. We also wish to thank S. Helgason and M. Raïs for providing us with the requisite references for the results concerning invariant polynomials which can be found in §§2 and 3.

1. Symmetric spaces of compact type and the Guillemin condition

Let X be a differentiable manifold, whose tangent and cotangent bundles we denote by $T = T_X$ and $T^* = T^*_X$, respectively. Let $C^\infty(X)$ (resp. $C^\infty_{\mathbb{R}}(X)$) be the space of complex-valued (resp. real-valued) functions on X. Let $\mathbb{R}(X)$ denote the subspace of $C^\infty_{\mathbb{R}}(X)$ consisting of the constant functions on X. Let E be a vector bundle over X; we denote by $E_{\mathbb{C}}$ its complexification, by $\mathcal{E}$ the sheaf of sections of E over X and by $C^\infty(E) = C^\infty(X, E)$ the space of global sections of E over X. By $\bigotimes^k E$, $S^l E$, $\bigwedge^j E$, we shall mean the k-th tensor product, the l-th symmetric product and the j-th exterior product of the vector bundle E, respectively. We shall identify $S^k T^*$ and $\bigwedge^k T^*$ with sub-bundles of $\bigotimes^k T^*$ as in §1, Chapter I of [**6**]. In particular, if $\alpha, \beta \in T^*$, the symmetric product $\alpha \cdot \beta$ is identified with the element $\alpha \otimes \beta + \beta \otimes \alpha$ of $\bigotimes^2 T^*$. If u is a section of $S^p T^*$ over X, we consider the morphism of vector bundles

$$u^\flat : T \longrightarrow S^{p-1} T^*,$$

defined by

$$(u^\flat \xi)(\eta_1, \dots, \eta_{p-1}) = u(\xi, \eta_1, \dots, \eta_{p-1}),$$

for $\xi, \eta_1, \dots, \eta_{p-1} \in T$.

Let g be a Riemannian metric on X. We denote by $g^\sharp : T^* \to T$ the inverse of the isomorphism $g^\flat : T \to T^*$. For $p \geq 2$, we consider the trace mapping

$$\mathrm{Tr} : S^p T^* \longrightarrow S^{p-2} T^*;$$

its kernel $S^p_0 T^*$ consists of all traceless symmetric p-forms. If u is a section of $S^p T^*$ over X, we consider the morphism of vector bundles

$$\tilde{u} = u^\flat \cdot g^\sharp : T^* \longrightarrow S^{p-1} T^*.$$

We also consider the scalar products on the spaces $C^\infty(X)$, $C^\infty(T)$ and $C^\infty(S^2 T^*)$, defined in terms of the Riemannian measure of X and the scalar products on the vector bundles T and $S^2 T^*$ induced by the metric g. We denote by $C^\infty_{\mathbb{R},0}(X)$ the orthogonal complement of the subspace $\mathbb{R}(X)$ of $C^\infty_{\mathbb{R}}(X)$.

Let ∇ be the Levi-Civita connection of (X, g); if f is a real-valued function on X, we denote by $\mathrm{Hess}\, f = \nabla df$ the Hessian of f. The Killing operator

$$D_0 : \mathcal{T} \longrightarrow S^2 \mathcal{T}^*$$

of (X, g), which sends a vector field ξ into the Lie derivative $\mathcal{L}_\xi g$ of g along ξ of g along ξ, and the symmetrized covariant derivative

$$D^1 : \mathcal{T}^* \longrightarrow S^2 \mathcal{T}^*,$$

defined by

$$(D^1 \theta)(\xi, \eta) = \tfrac{1}{2}((\nabla \theta)(\xi, \eta) + (\nabla \theta)(\eta, \xi)),$$

for $\theta \in \mathcal{T}^*$, $\xi, \eta \in T$, are related by the formula

$$\tfrac{1}{2} D_0 \xi = D^1 g^\flat(\xi), \tag{1.1}$$

for $\xi \in \mathcal{T}$. We easily see that

$$D^1(f_1 df_2) = \tfrac{1}{2} df_1 \cdot df_2 + f_1 \operatorname{Hess} f_2, \tag{1.2}$$

for all $f_1, f_2 \in C^\infty(X)$. We also consider the divergence operator

$$\operatorname{div} : S^2\mathcal{T}^* \longrightarrow \mathcal{T}^*,$$

which is defined in §1, Chapter I of [**6**]; we recall that the formal adjoint of D_0 is equal to $2g^\sharp \cdot \operatorname{div} : S^2\mathcal{T}^* \to \mathcal{T}$. When X is compact, since the operator D_0 is elliptic, we therefore have the orthogonal decomposition

$$C^\infty(S^2T^*) = D_0 C^\infty(T) \oplus \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0 \,\} \tag{1.3}$$

given by the relation (1.11) of [**6**]; we denote by

$$P : C^\infty(S^2T^*) \longrightarrow \{\, h \in C^\infty(S^2T^*) \mid \operatorname{div} h = 0 \,\}$$

the projection determined by the decomposition (1.3).

We now suppose that X is a symmetric space of compact type. As $S^0T^*_{\mathbb{C}}$ is the trivial complex line bundle, we may identify $C^\infty(X)$ with $C^\infty(S^0T^*_{\mathbb{C}})$. We consider the subspace $\mathcal{N}_p$ of $C^\infty(S^pT^*)$ consisting of all symmetric p-forms satisfying the Guillemin condition; the complexification $\mathcal{N}_{p,\mathbb{C}}$ of $\mathcal{N}_p$ shall be viewed as the subspace of $C^\infty(S^pT^*_{\mathbb{C}})$ consisting of all complex symmetric p-forms satisfying the Guillemin condition. The space $\mathcal{N}_p$ is the kernel of the maximal flat Radon transform for symmetric p-forms on X defined in Chapter II of [**6**]. Below we shall be concerned with the injectivity of this Radon transform for functions on X.

We recall that $D_0C^\infty(T)$ is a subspace of $\mathcal{N}_2$ (see Lemma 2.10 of [**6**]). We define the space of infinitesimal isospectral deformations of g by

$$I(X) = \{\, h \in \mathcal{N}_2 \mid \operatorname{div} h = 0 \,\}.$$

From the decomposition (1.3), we obtain the orthogonal decomposition

$$\mathcal{N}_2 = D_0C^\infty(T) \oplus I(X); \tag{1.4}$$

moreover, the orthogonal projection of $\mathcal{N}_2$ onto $I(X)$ is equal to the restriction of the projection P to $\mathcal{N}_2$. Thus the vanishing of the space $I(X)$ is equivalent to the fact that the space X is rigid in the sense of Guillemin. Moreover if there exists a symmetric 2-form on X belonging to $\mathcal{N}_2$ which is not equal to a Lie derivative of the metric g, the space $I(X)$ does not vanish.

We know that there is a Riemannian symmetric pair (G, K) of compact type, where G is a compact, semi-simple Lie group and K is a closed subgroup of G, such that the space X is isometric to the homogeneous space G/K endowed with a G-invariant metric. We shall identify X with G/K. We shall denote by $\mathfrak{g}_0$ the Lie algebra of G and by B its Killing form. The pair (G, K) is associated to an orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$ of compact type, where θ is an involutive automorphism of $\mathfrak{g}_0$. The space $C^\infty(T)$ and the spaces $C^\infty(S^pT^*)$ and $C^\infty(S^pT^*_{\mathbb{C}})$ of symmetric p-forms on X inherit structures of G-modules from the action of G on X.

Since the connection ∇ is independent of the choice of the G-invariant metric g (see Corollary 4.3, Chapter IV of [**16**], by formula (1.1) we see that the spaces $\mathcal{N}_p$ and $D_0 C^\infty(T)$ depend only on the symmetric space X and not on the choice of metric g of X. Hence the vanishing of $\mathcal{N}_0$ or the Guillemin rigidity of X are properties of the symmetric space X which are independent of the choice of the G-invariant metric of X.

If X is irreducible, then the metric g is proportional to the G-invariant Riemannian metric on G/K induced by $-B$. Hence in this case, the space $I(X)$ does not depend on the choice of the G-invariant metric on X.

Let Σ be a finite group of isometries of X of order m; suppose that the elements of Σ commute with the action of G on X. If Σ' is a subgroup of Σ, then the space $C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma'}$ consisting of all Σ'-invariant sections of $C^\infty(S^pT^*_{\mathbb{C}})$ is a G-submodule of $C^\infty(S^pT^*_{\mathbb{C}})$; we denote by $C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma'\perp}$ the orthogonal complement of $C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma'}$ in $C^\infty(S^pT^*_{\mathbb{C}})$. If τ is an element of Σ and $\lambda \in \mathbb{C}$, we denote by $C^\infty(S^pT^*_{\mathbb{C}})^{\tau,\lambda}$ the G-submodule of $C^\infty(S^pT^*_{\mathbb{C}})$ consisting of all elements u of $C^\infty(S^pT^*_{\mathbb{C}})$ satisfying

$$\tau^* u = \lambda u.$$

We suppose that the group Σ acts freely on X; then the quotient $Y = X/\Sigma$ is a manifold and the natural projection $\varpi : X \to Y$ is an m-fold covering. Thus the metric g induces a Riemannian metric g_Y on Y such that $\varpi^* g_Y = g$. Clearly the space Y is locally symmetric and is a homogeneous space of G.

Let (G, K') be another Riemannian symmetric pair associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$. Assume that K is a subgroup of K' and that there exists a G-equivariant diffeomorphism $\varphi : Y \to G/K'$ which has the following property: when we identify X with G/K, the projection $\varphi \circ \varpi$ is equal to the natural projection $G/K \to G/K'$. Under these conditions, the space (Y, g_Y) is isometric to the symmetric space G/K' of compact type endowed with a G-invariant metric.

Let Z be a maximal flat totally geodesic torus of X. Then $\varpi(Z)$ is a flat torus of Y. On the other hand, if Z' is a maximal flat totally geodesic torus of Y, then $\varpi^{-1}(Z')$ is a totally geodesic flat torus of X. From these observations, it follows that $Z = \varpi^{-1}(Z')$, where $Z' = \varpi(Z)$; we also see that the rank of Y is equal to the rank of X and that the induced mapping $\varpi : Z \to Z'$ is a m-fold covering. Moreover, the torus Z is invariant under the group Σ. A parallel vector field ξ on Z is ϖ-projectable, *i.e.*, there exists a parallel vector field $\hat{\xi}$ on $Z' = \varpi(Z)$ such that $\varpi_* \xi(x) = \hat{\xi}(\varpi(x))$, for all $x \in Z$. Conversely, any parallel vector field on Z' is of the form $\varpi_*\xi$, for some parallel vector field ξ on Z. It follows that a parallel vector field on Z is invariant under all the elements of Σ.

We denote by $\{\tau_k\}_{1 \leq k \leq m}$ the m distinct elements of Σ. Then we see that there exists an open subset Z_0 of Z such that the sets $\tau_j(Z_0)$ and $\tau_k(Z_0)$ are disjoint for all $1 \leq j, k \leq m$, with $j \neq k$, and such that the complement of the union $\cup_{k=1}^m \tau_k(Z_0)$

in Z has measure zero. Therefore if f is a function on Z, we see that

$$\int_Z f\, dZ = \sum_{k=1}^{m} \int_{Z_0} \tau_k^* f\, dZ.$$

If u is a symmetric p-form on X and ξ is a parallel vector field on Z, from the previous relation we obtain the equality

$$\int_Z u(\xi, \xi, \ldots, \xi)\, dZ = \sum_{k=1}^{m} \int_{Z_0} (\tau_k^* u)(\xi, \xi, \ldots, \xi)\, dZ. \tag{1.5}$$

If u is invariant under Σ and $\hat{u}$ is the symmetric p-form on Y such that $u = \varpi^* \hat{u}$, and if $\hat{\xi}$ be the parallel vector field on Z' such that $\varpi_* \xi = \hat{\xi}$, then we have

$$u(\xi, \xi, \ldots, \xi) = \varpi^* \hat{u}(\hat{\xi}, \hat{\xi}, \ldots, \hat{\xi});$$

from (1.5), we now obtain the equality

$$\int_Z u(\xi, \xi, \ldots, \xi)\, dZ = m \int_{Z'} \hat{u}(\hat{\xi}, \hat{\xi}, \ldots, \hat{\xi})\, dZ'.$$

Let p be a given integer ≥ 3 and let σ be a non-zero symmetric p-form on X which is invariant under the group Σ. Then the symmetric form σ induces a symmetric p-form σ_Y on Y such that

$$\sigma = \varpi^* \sigma_Y.$$

We consider the morphism of vector bundles

$$\tilde{\sigma}_Y : T_Y^* \longrightarrow S^{p-1} T_Y^*$$

induced by the symmetric p-form σ_Y; if φ is a 1-form on Y, we have

$$\varpi^* \tilde{\sigma}_Y(\varphi) = \tilde{\sigma}(\varpi^* \varphi). \tag{1.6}$$

The following lemma is a direct consequence of Lemma 2.17 of [**6**] or the previous discussion.

LEMMA 1.1. — *Suppose that the quotient $Y = X/\Sigma$ is isometric to the symmetric space G/K'. Let $p \geq 3$ be a given integer and let σ be a symmetric p-form on X which is invariant under the group Σ. Suppose that the following condition holds: a 1-form φ on X satisfies the Guillemin condition if and only if the symmetric $(p-1)$-form $\tilde{\sigma}(\varphi)$ on X satisfies the Guillemin condition. Then a 1-form ψ on Y satisfies the Guillemin condition if and only if the symmetric $(p-1)$-form $\tilde{\sigma}_Y(\psi)$ on Y satisfies the Guillemin condition.*

Suppose that $p = 3$ and that the mapping

$$\tilde{\sigma} : T^* \longrightarrow S^2 T^*$$

is a monomorphism; then the mapping

$$\tilde{\sigma}_Y : T_Y^* \longrightarrow S^2 T_Y^*$$

is also a monomorphism. Assume that the following is true: if φ is a 1-form on X satisfying the Guillemin condition, the symmetric 2-form $\tilde{\sigma}(\varphi)$ also satisfies the Guillemin condition. Then if f is an element of $C^\infty_{\mathbb{R}}(X)$, the symmetric 2-form $\tilde{\sigma}(df)$ satisfies the Guillemin condition. Thus if P is the orthogonal projection corresponding to the decomposition (1.3) on the space X, the mapping

$$P_\sigma : P\tilde{\sigma}d : C^\infty_{\mathbb{R}}(X) \longrightarrow I(X)$$

is well-defined. Clearly, if f is an element of $C^\infty_{\mathbb{R}}(X)$, then $\tilde{\sigma}df$ is a Lie derivative of the metric if and only if $P_\sigma f = 0$. A subspace $\mathcal{A}$ of $C^\infty_{\mathbb{R},0}(X)$ satisfies the relation

$$D_0 C^\infty(T) \cap \tilde{\sigma} d C^\infty_{\mathbb{R}}(X) = \tilde{\sigma} d\mathcal{A} \tag{1.7}$$

if and only if the kernel of the mapping P_σ is equal to the subspace $\mathbb{R}(X) \oplus \mathcal{A}$ of $C^\infty_{\mathbb{R}}(X)$.

PROPOSITION 1.2. — *Suppose that the quotient $Y = X/\Sigma$ is isometric to the symmetric space G/K'. Let σ be a symmetric 3-form on X which is invariant under the group Σ; suppose that $\tilde{\sigma} : T^* \to S^2T^*$ is a monomorphism of vector bundles. Suppose that the following condition holds: a 1-form φ on X satisfies the Guillemin condition if and only if the symmetric 2-form $\tilde{\sigma}(\varphi)$ on X satisfies the Guillemin condition. Let $\mathcal{A}_Y$ be a finite-dimensional subspace of $C^\infty_{\mathbb{R},0}(Y)$ and let $\mathcal{F}_Y$ be the orthogonal complement of $\mathcal{F}'_Y = \mathbb{R}(Y) \oplus \mathcal{A}_Y$ in $C^\infty_{\mathbb{R}}(Y)$. If the subspace $\mathcal{A} = \varpi^*\mathcal{A}_Y$ of $C^\infty_{\mathbb{R},0}(X)$ satisfies the relation* (1.7), *then the following assertions hold:*

(i) *The symmetric space Y is not rigid in the sense of Guillemin*

(ii) *If f is a non-zero element of $\mathcal{F}_Y$, then the symmetric 2-form $\tilde{\sigma}_Y(df)$ on Y satisfies the Guillemin condition and is not a Lie derivative of the metric.*

(iii) *The relation*

$$D_0 C^\infty(T_Y) \cap \tilde{\sigma}_Y d C^\infty_{\mathbb{R}}(Y) = \tilde{\sigma}_Y d\mathcal{A}_Y$$

holds and the kernel of the mapping

$$P_{\sigma_Y} = P\tilde{\sigma}_Y d : C^\infty_{\mathbb{R}}(Y) \longrightarrow I(Y) \tag{1.8}$$

is the finite-dimensional space $\mathcal{F}'_Y$.

Proof. — According to Lemma 1.1, the mapping (1.8) is well-defined. If f is an element of $C^\infty_{\mathbb{R}}(Y)$, by (1.6) we have the equality $\varpi^*\tilde{\sigma}_Y(df) = \tilde{\sigma}(d\varpi^*f)$; according to (1.7), we easily see that the following assertions are equivalent:

(a) $P_{\sigma_Y} f = 0$;

(b) the symmetric 2-form $\tilde{\sigma}_Y(df)$ is a Lie derivative of the metric g_Y;

(c) the symmetric 2-form $\tilde{\sigma}(d\varpi^*f)$ is a Lie derivative of the metric g;

(d) $P_\sigma \varpi^* f = 0$;

(e) the function ϖ^*f belongs to the subspace $\mathbb{R}(X) \oplus \mathcal{A}$ of $C^\infty_{\mathbb{R}}(X)$;

(f) the function f belongs to $\mathcal{F}'_Y$.

The proposition is an immediate consequence of this observation. □

In fact, the orthogonal complement $\mathcal{F}_Y$ of $\mathcal{F}'_Y$ which appears in the preceding proposition consists of all functions f of $C^\infty_{\mathbb{R}}(Y)$ satisfying

$$\int_Y f\,dY = 0, \qquad \int_Y f\varphi\,dY = 0,$$

for all $\varphi \in \mathcal{A}_Y$, and we have the orthogonal decomposition

$$C^\infty_{\mathbb{R}}(Y) = \mathcal{F}'_Y \oplus \mathcal{F}_Y = \mathbb{R}(Y) \oplus \mathcal{A}_Y \oplus \mathcal{F}_Y.$$

We now consider an element τ of Σ of order $q \geq 2$; let λ be a primitive q-th root of unity. For $0 \leq k \leq q-1$, we consider the endomorphism μ_k of the G-module $C^\infty(S^pT^*_{\mathbb{C}})$ defined by

$$\mu_k(u) = \frac{1}{q}\sum_{r=1}^{q} \lambda^{rk} \cdot \tau^{q-r*}u,$$

for all $u \in S^pT^*_{\mathbb{C}}$, and we easily verify that

$$\tau^*\mu_k(u) = \lambda^k \mu_k(u), \tag{1.9}$$

for all $u \in S^pT^*_{\mathbb{C}}$, and hence that μ_k is a projection onto the subspace $C^\infty(S^pT^*_{\mathbb{C}})^{\tau,\lambda^k}$. Since we know that

$$1 + \lambda^k + \lambda^{2k} + \cdots + \lambda^{(q-1)k} = 0, \tag{1.10}$$

for $1 \leq k \leq q-1$, we have

$$\sum_{k=0}^{q-1} \mu_k(u) = u, \tag{1.11}$$

for all $u \in S^pT^*_{\mathbb{C}}$. Since τ has no fixed points, by (1.9) and (1.11) we know that the subspaces $C^\infty(S^pT^*_{\mathbb{C}})^{\tau,\lambda^k}$ are non-zero and we obtain the direct sum decomposition

$$C^\infty(S^pT^*_{\mathbb{C}}) = \bigoplus_{k=0}^{q-1} C^\infty(S^pT^*_{\mathbb{C}})^{\tau,\lambda^k}$$

of $C^\infty(S^pT^*_{\mathbb{C}})$ into G-submodules. Hence if Σ' is equal to the cyclic group of order q generated by τ, we have the equality $C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma'} = C^\infty(S^pT^*_{\mathbb{C}})^{\tau,1}$ and the decomposition

$$C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma'\perp} = \bigoplus_{k=1}^{q-1} C^\infty(S^pT^*_{\mathbb{C}})^{\tau,\lambda^k}. \tag{1.12}$$

PROPOSITION 1.3. — *Suppose that the quotient $Y = X/\Sigma$ is isometric to the symmetric space G/K'. Suppose that Σ is a product $\Sigma' \times \Sigma''$ of two subgroups, where Σ' is a cyclic group of order $q \geq 2$ generated by an isometry τ. Let λ be a primitive q-th root of unity. Then the following assertions hold:*

(i) *A complex symmetric p-form u on X which satisfies the relation $\tau^*u = \lambda u$ also satisfies the Guillemin condition, and we have the inclusion*

$$C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma'\perp} \subset \mathcal{N}_{p,\mathbb{C}}.$$

(ii) *The maximal flat Radon transform for functions on X is not injective.*

(iii) *The space X is not rigid in the sense of Guillemin.*

Proof. — Suppose that the subgroup Σ'' of Σ is of order r and let $\{\tau_1, \dots, \tau_r\}$ be the r distinct elements of Σ''. Let Z be a maximal flat totally geodesic torus of X and let Z_0 be an open subset of Z possessing the properties described above with respect to integration over Z. Let u be a symmetric p-form on X and ξ be a parallel vector field on Z. According to equality (1.5), we have

$$\int_Z u(\xi, \dots, \xi)\, dZ = \sum_{j=1}^{r} \int_{Z_0} \tau_j^*(u + \tau^* u + \tau^{2*} u + \dots + \tau^{q-1*} u)(\xi, \dots, \xi)\, dZ.$$

Hence if u is a complex symmetric p-form on X satisfying $\tau^* u = \lambda^k u$, with $1 \leq k \leq q-1$, from the equality (1.10) we infer that

$$\int_Z u(\xi, \dots, \xi)\, dZ = 0.$$

Therefore an arbitrary element of $C^\infty(S^p T^*_{\mathbb{C}})^{\Sigma'\perp}$ satisfies the Guillemin condition. Assertion (ii) is an immediate consequence of (i), with $p = 0$. We now construct an element of $C^\infty(S^2 T^*_{\mathbb{C}})^{\Sigma'\perp}$ which is not a Lie derivative of the metric. Let B be the sub-bundle of $\bigwedge^2 T^* \otimes \bigwedge^2 T^*$ consisting of all tensors satisfying the Bianchi identity. The differential operator D_1 of order 2 on X defined in Chapter I of [**6**] acts on $C^\infty(S^2 T^*)$ and takes its values in the space of sections of a quotient bundle of B; we recall that, if the restriction of an element h of $C^\infty(S^2 T^*)$ to an open subset V of X is a Lie derivative of the metric, then $D_1 h$ vanishes on V. Now let x be a point of X and U be a open neighborhood of x for which $U \cap \tau^k(U) = \varnothing$, for all $1 \leq k \leq q-1$. According to the argument given in the course of the proof of Proposition 2.22 of [**6**], we may choose an element h of $C^\infty(S^2 T^*)$ whose support is contained in U and which satisfies $(D_1 h)(x) \neq 0$; thus h is not a Lie derivative of the metric on any neighborhood of x. The symmetric 2-form

$$h' = \sum_{k=0}^{q-1} \lambda^{q-k} \cdot \tau^{k*} h$$

on X satisfies $\tau^* h' = \lambda h'$ and its restriction to U is equal to h. By (1.12), the symmetric 2-form h' belongs to $C^\infty(S^2 T^*_{\mathbb{C}})^{\Sigma'\perp}$, and therefore satisfies the Guillemin condition. Clearly, the element of $C^\infty(S^2 T^*)$ equal to the real part of h' also has these properties and its restriction to U is equal to h; thus the space X is not rigid in the sense of Guillemin. □

We now define the reduced space of the symmetric space X; this will provide us with examples of symmetric spaces X and Y satisfying the conditions considered above. In the following discussion based on §9, Chapter VII of [**16**], if (G', K') is a Riemannian symmetric pair associated with the orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$, we shall always endow the symmetric space G'/K' with the unique G'-invariant metric induced by $-B$. We may suppose that G is the identity component of the group of isometries of X, that K is the isotropy group of G at some point of X and that the metric g

of X is induced by $-B$. The fixed point set of the involutive automorphism θ of $\mathfrak{g}_0$ contains no non-trivial ideal of $\mathfrak{g}_0$. Let $\tilde{G}$ be the simply-connected Lie group with Lie algebra $\mathfrak{g}_0$ and let $\tilde{\theta}$ be the automorphism of $\tilde{G}$ determined by θ. If $\tilde{K}$ is the subgroup of $\tilde{G}$ equal to the set of fixed points of $\tilde{\theta}$, then $(\tilde{G}, \tilde{K})$ is a Riemannian symmetric pair associated with $(\mathfrak{g}_0, \theta)$ and $\tilde{G}/\tilde{K}$ is the simply-connected symmetric space of compact type, which is the universal covering space of X.

If $\tilde{Z}$ denotes the center of $\tilde{G}$ and S is a subgroup of $\tilde{Z}$, we consider the subgroup

$$K_S = \{\, a \in \tilde{G} \mid a^{-1}\tilde{\theta}(a) \in S \,\}$$

of $\tilde{G}$. Let $\varphi : \tilde{G} \to G$ is the natural projection and K^* be the subgroup $\varphi^{-1}(K)$ of G. Then there exists a $\tilde{\theta}$-invariant subgroup S of the center $\tilde{Z}$ of $\tilde{G}$ such that $G = \tilde{G}/S$. It is easily seen that K^* is $\tilde{\theta}$-invariant and that

$$K = K^*/S, \qquad \tilde{K}S \subset K^* \subset K_S,$$

and so $(\tilde{G}, K^*)$ is a Riemannian symmetric pair associated with $(\mathfrak{g}_0, \theta)$ and we have

$$X = \tilde{G}/K^*.$$

Also $(\tilde{G}, K_{\tilde{Z}})$ is a Riemannian symmetric pair and $\tilde{G}/K_{\tilde{Z}}$ is a symmetric space associated to the orthogonal symmetric pair $(\mathfrak{g}_0, \theta)$; moreover, X is a covering space of the symmetric space $\tilde{G}/K_{\tilde{Z}}$ *via* the natural projection $\tilde{G}/K^* \to \tilde{G}/K_{\tilde{Z}}$ obtained from the inclusion $K^* \subset K_{\tilde{Z}}$. The symmetric space $\tilde{G}/K_{\tilde{Z}}$ is called the *reduced* (or adjoint) space of X or of the simply-connected space $\tilde{G}/\tilde{K}$. If the symmetric space X is isometric to its reduced space, we say that X is reduced. In fact, any symmetric space X' associated to the orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$ covers the reduced space $\tilde{G}/K_{\tilde{Z}}$ and is covered by $\tilde{G}/\tilde{K}$ (see Theorem 9.1 and Corollary 9.3, Chapter VII of [**16**]). In §9, Chapter VII of [**16**], the symmetric space $\tilde{G}/K_{\tilde{Z}}$ is called the adjoint space of orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$.

According to the classification of the irreducible symmetric spaces of compact type (see §6 and Exercises C, Chapter X of [**16**]), if X is irreducible, the group of covering transformations of the covering mapping $X \to \tilde{G}/K_{\tilde{Z}}$ is an abelian group Σ of isometries of X commuting with the action of G and is equal either to a cyclic group or to a product of two cyclic groups of order 2. In fact, the group Σ is a product of two cyclic groups of order 2 only when X is the real Grassmannian $\widetilde{G}^{\mathbb{R}}_{n,n}$ of §3 or the group $Spin(2n)$, where in both cases n is an even integer ≥ 4 .

The n-sphere S^n, with $n \geq 2$, is an irreducible symmetric space of rank one, which is not reduced; its reduced space is the real projective space $\mathbb{RP}^n$. According to Proposition 1.3, the maximal flat Radon transform for functions on S^n is not injective and the sphere S^n is not rigid in the sense of Guillemin (see also Proposition 2.22 of [**6**]). Proposition 2.16 of [**6**] now gives us the first assertion of the next theorem. According to the observations made in the preceding paragraph and Proposition 1.3, if X is an irreducible symmetric space and if either the maximal flat Radon transform for functions on X is injective or X is rigid in the sense of Guillemin, then X must be reduced. Thus we have proved the following result:

THEOREM 1.4. — *Let X be an irreducible symmetric space of compact type.*

(i) *If X is rigid in the sense of Guillemin, the maximal flat Radon transform for functions on X is injective.*

(ii) *If the maximal flat Radon transform for functions on X is injective, then X is reduced.*

In [**13**], Grinberg conjectures that the converse of assertion (ii) of the preceding theorem holds and proposes an outline for a possible proof. In fact, this converse is known to hold for all the Grassmannians which are reduced; it also is true for the reduced space of a Grassmannian which is not reduced (see [**6**]).

If the space X is a product of irreducible symmetric spaces $X_1, \dots, X_m$, its reduced space is equal to the product of the reduced spaces of the X_j; from the observations made above, we infer that the group of covering transformations of the covering mapping $X \to \tilde{G}/K_{\tilde{Z}}$ is an abelian group Σ of isometries of X commuting with the action of G and is a product of cyclic subgroups. In fact, if X is simply-connected, by Proposition 5.5, Chapter VIII of [**16**], it may always be written as such a product. Then according to Proposition 1.3, we obtain the following generalization of Theorem 1.4,(ii):

THEOREM 1.5. — *Let X be a symmetric space of compact type which is equal to a product of irreducible spaces. If the maximal flat Radon transform for functions on X is injective, then X is reduced.*

If X is a product $X_1 \times X_2 \times \cdots \times X_m$ of m irreducible symmetric spaces, with $m \geq 2$, and if the factors X_1 and X_2 of this product are not equal to simple Lie groups, by Proposition 10.1 and Theorem 10.5 of [**6**], we know that the space X is not rigid in the sense of Guillemin.

2. Invariant symmetric forms on symmetric spaces

If V is a real vector space, we denote by $P(V)$ the algebra of real-valued polynomials on V and by $S(V) = \bigoplus_{k \geq 0} S^k V$ the symmetric algebra over V, where $S^k V$ is the k-th symmetric product of V. We shall identify the algebras $P(V)$ and $S(V^*)$ *via* the isomorphism of algebras $P(V) \to S(V^*)$, which associates to a homogeneous polynomial $q \in P(V)$ of degree p the unique element $\hat{q}$ of $S^p V^*$ determined by

$$\hat{q}(v, \dots, v) = q(v),$$

for all $v \in V$. If H is a group which acts on V, we consider the subalgebra

$$S(V)^H = \bigoplus_{k \geq 0} (S^k V)^H$$

of $S(V)$ consisting of all its H-invariant elements; in this equality, the component $(S^kV)^H$ is the subspace of S^kV consisting of all H-invariant elements of S^kV. In fact, the space $S(V^*)^H$ is identified with the algebra of all H-invariant polynomials on V.

Let X be a symmetric space of compact type. As in §1, we consider the Riemannian symmetric pair (G, K) of compact type, where G is a compact, semi-simple Lie group and K is a closed subgroup of G, such that the space X is isometric to the homogeneous space G/K endowed with a G-invariant metric; we shall identify X with G/K. We denote by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ the Lie algebras of G and K. The pair (G, K) is associated to an orthogonal symmetric Lie algebra $(\mathfrak{g}_0, \theta)$ of compact type, where θ is an involutive automorphism of $\mathfrak{g}_0$. We consider the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \tag{2.1}$$

of $\mathfrak{g}_0$, where $\mathfrak{p}_0$ is the K-submodule of $\mathfrak{g}_0$ equal to the eigenspace of θ corresponding to the eigenvalue -1. We identify $\mathfrak{p}_0$ with the tangent space of X at the coset x_0 of the identity element of G. If $p \geq 2$ is a given integer, an element q of $(S^p\mathfrak{p}_0^*)^K$ gives rise to a unique G-invariant symmetric p-form $\sigma(q)$ on X whose restriction to the tangent space of X at x_0 is equal to q, and every G-invariant symmetric p-form arises in this way. The restriction B_0 of the Killing form B of $\mathfrak{g}_0$ to its subspace $\mathfrak{p}_0$ is non-degenerate and is an element of $(S^2\mathfrak{p}_0^*)^K$. The symmetric 2-form $-\sigma(B_0)$ is a G-invariant Riemannian metric on X; if X is irreducible, it is equal to the metric of X induced by $-B$ considered in §1.

Let σ be a non-zero G-invariant symmetric p-form on X, with $p \geq 2$; clearly, σ is parallel, *i.e.*, we have $\nabla\sigma = 0$. The morphisms

$$\sigma^\flat : T \longrightarrow S^{p-1}T^*, \qquad \tilde{\sigma} : T^* \longrightarrow S^{p-1}T^*$$

induced by σ are G-equivariant; if X is an irreducible symmetric space and σ is non-zero, they are monomorphisms of vector bundles. If $p = 3$, we easily see that the form σ is traceless, *i.e.*, $\operatorname{Tr}\sigma = 0$; it follows directly that

$$\operatorname{Tr}\sigma^\flat(\xi) = 0,$$

for all $\xi \in T$, and so in this case we have G-equivariant morphisms

$$\sigma^\flat : T \longrightarrow S_0^2T^*, \qquad \tilde{\sigma} : T^* \longrightarrow S_0^2T^*.$$

Throughout the remainder of this section, we suppose that X is an irreducible simply-connected symmetric space. By the Chevalley restriction theorem (see, for example, Theorem 3.1.2 of [**24**]) and the classification of the invariants for finite reflection groups, if l is the rank of X, there exist l homogeneous algebraically independent generators $\{p_1, \dots, p_l\}$ of the algebra $S(\mathfrak{p}_0^*)^K$ of positive degree such that $S(\mathfrak{p}_0^*)^K$ is isomorphic to $\mathbb{R}[p_1, \dots, p_l]$. The degrees $\{d_1, \dots, d_l\}$ of the polynomials $\{p_1, \dots, p_l\}$ are independent of the choice of the algebraically independent generators; without any loss of generality, we shall suppose that $d_j \leq d_{j+1}$, for $1 \leq j \leq l-1$. Moreover, we have $d_1 = 2$ and $d_j > 2$, for $2 \leq j \leq l$; thus the space $(S^2\mathfrak{p}_0^*)^K$ is one-dimensional and the element p_1 of $S^2\mathfrak{p}_0^*$ is equal to a multiple of B_0. Furthermore, there exists a non-zero element of $(S^3\mathfrak{p}_0^*)^K$ if and only if $l \geq 2$ and $d_2 = 3$.

We now determine the space $S(\mathfrak{p}_0^*)^K$ for three classes of irreducible simply-connected symmetric spaces. Let n be a given integer ≥ 3. For $p \geq 2$, we consider the $SU(n)$-invariant homogeneous polynomial Q_p of degree p on the Lie algebra $\mathfrak{su}(n)$ defined by

$$Q_p(A) = (-i)^p \operatorname{Tr} A^p,$$

for $A \in \mathfrak{su}(n)$; we easily verify that Q_p is real-valued. The Killing form of $\mathfrak{su}(n)$ is equal to $-2nQ_1$. It is well-known that the algebra of all $SU(n)$-invariant polynomials on $\mathfrak{su}(n)$ is generated by the polynomials Q_p, with $2 \leq p \leq n$, and that these polynomials are algebraically independent. The polynomial Q_p induces a non-zero bi-invariant p-form σ'_p on the group $SU(n)$. The 2-form σ'_2 is a bi-invariant Riemannian metric on $SU(n)$ and, endowed with this metric, the simple group $SU(n)$ may be viewed as an irreducible symmetric space of compact type. If $G = SU(n) \times SU(n)$ and K is the subgroup $\{ (x, x) \mid x \in SU(n) \}$ of G, then (G, K) is a Riemannian symmetric pair and the homogeneous space G/K is diffeomorphic to $SU(n)$; moreover, the metric σ'_2 on $SU(n)$ determines a G-invariant metric on G/K (see §6, Chapter IV of [**16**]). In this case, we have $\mathfrak{k}_0 = \mathfrak{p}_0 = \mathfrak{su}(n)$, the polynomial Q_p belongs to $(S^p\mathfrak{p}_0^*)^K$ and the polynomials $Q_2, \dots, Q_n$ are algebraically independent generators of the space $S(\mathfrak{p}_0^*)^K$. Also the form σ'_p on $SU(n)$ is equal to $\sigma(Q_p)$, for $p \geq 2$.

Now let G be the group $SU(n)$ and let K be the subgroup $SO(n)$, which is equal to the set of fixed points of the involution s of G sending a matrix into its complex conjugate. Then (G, K) is a Riemannian symmetric pair. In the Cartan decomposition (2.1) of the Lie algebra $\mathfrak{g}_0$ of G corresponding to this involution, the K-submodule $\mathfrak{p}_0$ is the space of all symmetric purely imaginary $n \times n$ matrices of trace zero. Endowed with the G-invariant Riemannian metric $g_0 = -\sigma(B_0)$, the homogeneous space $X = G/K$ is an irreducible symmetric space of type AI called the special Lagrangian Grassmannian.

The restriction q_p of the G-invariant polynomial Q_p on $\mathfrak{g}_0 = \mathfrak{su}(n)$ to $\mathfrak{p}_0$ is K-invariant. It is well-known that the polynomials $q_2, \dots, q_n$ are algebraically independent generators of the algebra $S(\mathfrak{p}_0^*)^K$ (see, for example, [**10**, p. 560]. For $p \geq 2$, we consider the G-invariant p-form $\sigma_p = \sigma(q_p)$ on X. It is easily verified that

$$\sigma_3(\phi_1, \phi_2, \phi_3) = i \operatorname{Tr} (\phi_1 \cdot \phi_2 \cdot \phi_3),$$

for all $\phi_1, \phi_2, \phi_3 \in \mathfrak{p}_0$; the product on the right-hand side of this equality is the product of the elements of $\mathfrak{p}_0$ viewed as matrices. The metric g_0 is equal to the symmetric 2-form $2n \cdot \sigma_2$.

Next, we consider the $2n \times 2n$ matrix

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n is the unit matrix of order n. Let G be the group $SU(2n)$ and let K be the subgroup $Sp(n)$ of G, which is equal to the set of fixed points of the involution s of G sending an element $A \in G$ into $J_n \bar{A} J_n^{-1}$. Then (G, K) is a Riemannian symmetric pair. In the Cartan decomposition (2.1) of the Lie algebra $\mathfrak{g}_0$ of G corresponding to this involution, the K-submodule $\mathfrak{p}_0$ is the space of all $2n \times 2n$ matrices given by

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & -\bar{Z}_1 \end{pmatrix} \,\middle|\, Z_1 \in \mathfrak{su}(n),\ Z_2 \in \mathfrak{so}(n, \mathbb{C}) \right\}.$$

Endowed with the G-invariant Riemannian metric $g_0 = -\sigma(B_0)$, the homogeneous space $X = G/K$ is an irreducible symmetric space of type AII (see §2, Chapter X of [**16**]).

The restriction $\hat{q}_p$ of the G-invariant polynomial Q_p on $\mathfrak{g}_0 = \mathfrak{su}(n)$ to $\mathfrak{p}_0$ is K-invariant. It is well-known that the polynomials $\hat{q}_2, \dots, \hat{q}_n$ are algebraically independent generators of the algebra $S(\mathfrak{p}_0^*)^K$ (see, for example, [**10**, p. 560]). For $p \geq 2$, we consider the G-invariant p-form $\hat{\sigma}_p = \sigma(\hat{q}_p)$ on X. The metric g_0 is equal to the symmetric 2-form $4n \cdot \hat{\sigma}_2$.

For the three irreducible symmetric spaces

$$SU(n), \qquad SU(n)/SO(n), \qquad SU(2n)/Sp(n),$$

with $n \geq 3$, the degrees of the $n-1$ algebraically independent generators of the algebra $S(\mathfrak{p}_0^*)^K$ are given by $d_j = j + 1$, for $1 \leq j \leq n-1$. Therefore for any one of these symmetric spaces, the space $(S^3\mathfrak{p}_0^*)^K$ is one-dimensional. Hence each of these spaces admits an $SU(n)$-invariant symmetric 3-form, which is unique up to a constant; in fact, for the space $SU(n)$ (resp. $SU(n)/SO(n)$, $SU(2n)/Sp(n)$), the symmetric 3-form σ_3' (resp. σ_3, $\hat{\sigma}_3$) is a generator of $(S^3\mathfrak{p}_0^*)^K$.

We now return to the study of our irreducible symmetric space X and the algebra $S(\mathfrak{p}_0^*)^K$ associated to X.

If X is a simple Lie group, then we have $\mathfrak{p}_0 = \mathfrak{k}_0$ and, according to the tables of [**2**], there exists a non-zero element of $(S^3\mathfrak{p}_0^*)^K$ if and only if X is equal to $SU(n)$, with $n \geq 3$.

Now suppose that X is not a simple Lie group; then the Lie algebra $\mathfrak{g}_0$ is simple. By restricting a polynomial on the Lie algebra $\mathfrak{g}_0$ to its subspace $\mathfrak{p}_0$, we obtain a restriction mapping

$$S(\mathfrak{g}_0^*)^G \longrightarrow S(\mathfrak{p}_0^*)^K. \tag{2.2}$$

According to Proposition 7.4 and Theorem 7.5 of [**15**] and Theorem 3.4 of [**17**] (see also Propositions 2.1 and 3.1 of [**17**] and Theorem 10.3 of [**18**]), the mapping (2.2) is surjective if $(\mathfrak{g}_0, \mathfrak{p}_0)$ is not one of the pairs

$$(\mathfrak{e}_6, \mathfrak{so}(10) + \mathbb{R}), \qquad (\mathfrak{e}_6, \mathfrak{f}_4), \qquad (\mathfrak{e}_7, \mathfrak{e}_6 + \mathbb{R}), \qquad (\mathfrak{e}_8, \mathfrak{e}_7 + \mathfrak{su}(2)). \tag{2.3}$$

According to the tables of [**2**], we know that $S^3(\mathfrak{g}_0^*)^G$ is non-zero if and only if the complexification of $\mathfrak{g}_0$ is equal to $\mathfrak{a}_n$, with $n \geq 2$. Next, suppose that our space X is not one of the spaces (2.3) and possesses a non-zero element of $(S^3\mathfrak{p}_0^*)^K$. Then the rank l of the space X is ≥ 2 and the group G is equal to $SU(n)$, with $n \geq 3$. According to the classification of the irreducible symmetric spaces of compact type (see Chapter X of [**16**]), we know that X must be equal to one of the spaces $SU(n)/SO(n)$, $SU(2n)/Sp(n)$, with $n \geq 3$, or to one of the complex Grassmannians

$$SU(m+n)/S(U(m) \times U(n)), \tag{2.4}$$

with $m, n \geq 2$. If X is equal to the Grassmannian (2.4), for $k \geq 2$, we consider the restriction p_k of the polynomial Q_k of degree k on the Lie algebra $\mathfrak{su}(m+n)$ considered above to its subspace $\mathfrak{p}_0$. Then p_k vanishes when k is an odd integer, and, if $s = \min(m, n)$, the homogeneous polynomials $\{p_2, p_4, \dots, p_{2s}\}$ are an algebraically independent set of generators of the algebra $S(\mathfrak{p}_0^*)^K$ (see also [**10**, p. 561]). Thus in this case the space $S^3(\mathfrak{p}_0^*)^K$ vanishes.

Finally, suppose that $(\mathfrak{g}_0, \mathfrak{p}_0)$ is equal to one of the pairs (2.3). According to the table on pp. 796–797 of [**17**] (see also [**1**, p. 33]) and the Chevalley restriction theorem, there exists a homogeneous generator of $S(\mathfrak{p}_0^*)^K$ of degree 3 if and only if X is the space E_6/F_4 of type EIV. The rank of the space E_6/F_4 is equal to 2 and in this case we have $d_2 = 3$.

Thus we have completed the proof of the following result:

Proposition 2.1. — *Let X be an irreducible simply-connected symmetric space of compact type. The space $(S^3\mathfrak{p}_0^*)^K$ vanishes unless X is equal to one of the following spaces:*
- (i) *$SU(n)$, with $n \geq 3$;*
- (ii) *$SU(n)/SO(n)$, with $n \geq 3$;*
- (iii) *$SU(2n)/Sp(n)$, where $n \geq 3$;*
- (iv) *E_6/F_4.*

If X is equal to one of the spaces (i)–(iv)*, then the space $(S^3\mathfrak{p}_0^*)^K$ is one-dimensional.*

3. The real Grassmannians

Let $m, n \geq 1$ be given integers, with $m + n \geq 3$. We now suppose that X is the real Grassmannian $\widetilde{G}^{\mathbb{R}}_{m,n}$ of all oriented m-planes in $F = \mathbb{R}^{m+n}$. Let V be the canonical vector bundle (of rank m) over X whose fiber at $x \in X$ is the subspace of F determined by the oriented m-plane x. We denote by W the vector bundle of rank n over X whose fiber at $x \in X$ is the orthogonal complement W_x of V_x in F. Then we have a natural isomorphism of vector bundles

$$V^* \otimes W \longrightarrow T \tag{3.1}$$

over X. We may view X as a submanifold of $\bigwedge^m F$. In fact, the point $x \in X$ corresponds to the vector $v_1 \wedge \cdots \wedge v_m$ of $\bigwedge^m F$, where $\{v_1, \dots, v_m\}$ is a positively oriented orthonormal basis of the oriented m-plane x. The isomorphism (3.1) sends an element $\theta \in (V^* \otimes W)_x$ into the tangent vector $dx_t/dt|_{t=0}$ to X at x, where x_t is the point of X corresponding to the vector

$$(v_1 + t\theta(v_1)) \wedge \cdots \wedge (v_m + t\theta(v_m))$$

of $\bigwedge^m F$, for $t \in \mathbb{R}$.

Since the vector bundles V and W are sub-bundles of the trivial vector bundle over X whose fiber is F, the standard Euclidean scalar product on F induces by restriction positive definite scalar products g_1 and g_2 on the vector bundles V and W, respectively. If we identify the vector bundle V^* with V by means of the scalar product g_1, the isomorphism (3.1) gives rise to a natural isomorphism

$$V \otimes W \longrightarrow T \tag{3.2}$$

of vector bundles over X, which allows us to identify these two vector bundles and the vector bundle $\bigotimes^p T^*$ with $\bigotimes^p V^* \otimes \bigotimes^p W^*$, for $p \geq 1$. In fact, if $\theta_1 \in \bigotimes^p V^*$, $\theta_2 \in \bigotimes^p W^*$, we identify the element $\theta_1 \otimes \theta_2$ of $\bigotimes^p V^* \otimes \bigotimes^p W^*$ with the element u of $\bigotimes^p T^*$ determined by

$$u(v_1 \otimes w_1, v_2 \otimes w_2, \dots, v_p \otimes w_p) = \theta_1(v_1, v_2, \dots, v_p) \cdot \theta_2(w_1, w_2, \dots, w_p),$$

for $v_1, v_2, \dots, v_p \in V$ and $w_1, w_2, \dots, w_p \in W$. Then we have the inclusions

$$\begin{aligned} &S^pV^* \otimes S^pW^* \subset S^pT^*, \qquad && \textstyle\bigwedge^p V^* \otimes \bigwedge^p W^* \subset S^pT^*, \\ &S^pV^* \otimes \textstyle\bigwedge^p W^* \subset \bigwedge^p T^*, && \textstyle\bigwedge^p V^* \otimes S^pW^* \subset \bigwedge^p T^*. \end{aligned} \tag{3.3}$$

The scalar product g on T induced by the scalar product $g_1 \otimes g_2$ on the vector bundle $V \otimes W$ is a Riemannian metric on X. If x is a point of X and $v \in V_x$ and $w \in W_x$ are given vectors, we shall sometimes denote the vector $v \otimes w$ of T_x by $(x, v \otimes w)$. The sub-bundle S^2T^* of $\bigotimes^2 T^*$ admits the orthogonal decomposition

$$S^2T^* = (S^2V^* \otimes S^2W^*) \oplus (\textstyle\bigwedge^2 V^* \otimes \bigwedge^2 W^*). \tag{3.4}$$

Moreover, we know the metric g is a section of the bundle $S^2V^* \otimes S^2W^*$ and that an element h of S^2T^* belongs to the sub-bundle $\bigwedge^2 V^* \otimes \bigwedge^2 W^*$ if and only if

$$h(v \otimes w, v \otimes w) = 0,$$

for all $v \in V$ and $w \in W$.

We consider the standard basis $\{e_1, \dots, e_{m+n}\}$ of $\mathbb{R}^{m+n}$. The action of the group $G = SO(m+n)$ on $\mathbb{R}^{m+n}$ extends to an action on $\mathbb{C}^{m+n}$. The group G sends every oriented m-plane of $\mathbb{R}^{m+n}$ into another oriented m-plane. This gives rise to a transitive action of the group G on the Riemannian manifold (X, g) by isometries. The isotropy subgroup of the point x_0 of X corresponding to the vector $e_1 \wedge \cdots \wedge e_m$ of $\bigwedge^m F$ is equal to $K = SO(m) \times SO(n)$. We identify X with the Riemannian symmetric space G/K.

The involution τ of X, corresponding to the change of orientation of an m-plane of F, is an isometry of X which commutes with the action of G on X. For $x \in X$, the tangent space $T_{\tau(x)}$ is equal to $(V \otimes W)_x$, and it is easily verified that the mapping $\tau_* : T_x \to T_{\tau(x)}$ is equal to the identity mapping of $(V \otimes W)_x$. The group Σ' of isometries of X generated by τ, which is of order 2, acts freely on X. The quotient Riemannian manifold $Y' = X/\Sigma'$ endowed with the Riemannian metric induced by g admits X as a two-fold Riemannian covering and we identify it with the real Grassmannian $G^{\mathbb{R}}_{m,n}$ of all m-planes in $F = \mathbb{R}^{m+n}$; we denote by $\varpi' : X \to Y'$ the natural projection. The action of the group G passes to the quotient $G^{\mathbb{R}}_{m,n}$ and the group G acts transitively on this space. We identify $G^{\mathbb{R}}_{m,n}$ with the symmetric space G/K', where K' is the isotropy group of the image of x_0 in $G^{\mathbb{R}}_{m,n}$.

The oriented m-plane $x \in X$ gives us an orientation of V_x, which in turn induces an orientation of W_x: if $\{v_1, \dots, v_m\}$ is a positively oriented orthonormal basis of V_x, then the orientation of W_x is determined by an orthonormal basis $\{w_1, \dots, w_n\}$ of W_x satisfying

$$v_1 \wedge \cdots \wedge v_m \wedge w_1 \wedge \cdots \wedge w_n = e_1 \wedge \cdots \wedge e_{m+n}.$$

Then there is a natural diffeomorphism

$$\Psi : \widetilde{G}^{\mathbb{R}}_{m,n} \longrightarrow \widetilde{G}^{\mathbb{R}}_{n,m},$$

sending $x \in \widetilde{G}^{\mathbb{R}}_{m,n}$ into the n-plane W_x endowed with the orientation described above. For $x \in X$, we have $V'_{\Psi(x)} = W_x$ and $W'_{\Psi(x)} = V_x$. It is easily verified that the induced mapping $\Psi_* : (V \otimes W)_x \to (V' \otimes W')_{\Psi(x)}$ sends $v \otimes w$ into $-w \otimes v$, where $v \in V_x$ and $w \in W_x$; therefore Ψ is an isometry.

If E be a sub-bundle of S^pT^* or of its complexification $S^pT^*_{\mathbb{C}}$ which is invariant under the group G and the involution τ, we consider the G-submodules $C^\infty(E)$ of $C^\infty(S^pT^*_{\mathbb{C}})$ and

$$C^\infty(E)^{\tau,\varepsilon} = \{\, \theta \in C^\infty(E) \mid \tau^*\theta = \varepsilon\theta \,\}$$

of $C^\infty(E)$, where $\varepsilon = \pm 1$; in fact, the module $C^\infty(E)^{\tau,+1}$ is equal to the G-submodule $C^\infty(E)^{\Sigma'}$ of $C^\infty(E)$ consisting of all Σ'-invariant sections of E.

In this section, we henceforth suppose that $m = n$, with $n \geq 2$. The isometry Ψ always satisfies $\Psi^4 = \mathrm{id}$ and commutes with the involution τ; it is an involution only when n is even, and satisfies $\Psi^2 = \tau$ when n is odd. In fact, if x is a point of X and $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are positively oriented bases of V_x and W_x, respectively, then $\{w_1, \dots, w_n\}$ is a positively oriented basis of $V_{\Psi(x)}$; moreover, when n is even (resp. odd) integer, then $\{v_1, v_2, \dots, v_n\}$ (resp. $\{v_2, v_1, v_3, \dots, v_n\}$) is a positively oriented basis of the space $W_{\Psi(x)}$. Thus the group Σ of isometries of $\widetilde{G}^{\mathbb{R}}_{n,n}$ generated by Ψ and τ is of order 4. Let Σ_1 and Σ_2 be the cyclic subgroups of Σ generated by Ψ

and τ, respectively. When n is odd, Σ is equal to the cyclic group Σ_1 of order 4; on the other hand, when n is even, Σ is the product of the two cyclic subgroups Σ_1 and Σ_2 of order 2. Moreover, these isometries Ψ and τ commute with action of $G = SO(2n)$ on $\widetilde{G}^{\mathbb{R}}_{n,n}$. In §1, Chapter IV of [**6**], we considered the quotient $Y = \bar{G}^{\mathbb{R}}_{n,n}$ of $G^{\mathbb{R}}_{n,n}$ by the group of isometries of order 2 generated by this isometry Ψ of $G^{\mathbb{R}}_{n,n}$ endowed with the metric g_Y induced by the metric of $G^{\mathbb{R}}_{n,n}$; we saw that Y is a symmetric space of compact type of rank n and a homogeneous space of G. When $n \geq 3$, it is irreducible and equal to the adjoint space of X. In fact, the group Σ acts freely on X and the Riemannian manifold (Y, g_Y) is equal to the quotient X/Σ endowed with the metric induced by the metric g of X. Moreover, the natural projection $\varpi : X \to Y$ is a four-fold covering and the action of the group G on X passes to the quotient Y. When $n \geq 4$ is an even integer, we may also consider the symmetric space which is the quotient of $\widetilde{G}^{\mathbb{R}}_{n,n}$ by the group Σ_1 of order 2.

Since the isometries τ and Ψ commute with the action of G on X, a Killing vector field ξ on X satisfies the relations

$$\tau_*\xi = \xi, \qquad \Psi_*\xi = \xi \tag{3.5}$$

and is ϖ-projectable.

Let E be a vector bundle equal either to S^pT^* or its complexification $S^pT^*_{\mathbb{C}}$; we consider the G-submodules

$$C^\infty(E)^{\mathrm{ev}} = \{\, \theta \in C^\infty(E)^{\tau,+1} \mid \Psi^*\theta = \theta \,\},$$
$$C^\infty(E)^{\mathrm{odd}} = \{\, \theta \in C^\infty(E)^{\tau,+1} \mid \Psi^*\theta = -\theta \,\}$$

of $C^\infty(E)^{\tau,+1}$. Then we see that $C^\infty(E)^{\mathrm{ev}}$ is equal to the G-submodule $C^\infty(E)^{\Sigma}$ of $C^\infty(E)$ consisting of all Σ-invariant sections of E. Since the action of Ψ on $C^\infty(E)^{\tau,+1}$ is an involution, we have the decomposition

$$C^\infty(E)^{\tau,+1} = C^\infty(E)^{\mathrm{ev}} \oplus C^\infty(E)^{\mathrm{odd}}.$$

A symmetric p-form θ on X is invariant under the group Σ (resp. Σ') if and only if there is a symmetric p-form $\hat{\theta}$ on Y (resp. on Y') such that $\theta = \varpi^*\hat{\theta}$. Thus the projections ϖ and ϖ' induce isomorphisms of G-modules

$$\varpi^* : C^\infty(Y, S^pT^*_{Y,\mathbb{C}}) \longrightarrow C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma},$$
$$\varpi'^* : C^\infty(Y', S^pT^*_{Y',\mathbb{C}}) \longrightarrow C^\infty(S^pT^*_{\mathbb{C}})^{\Sigma'}.$$

If E is the trivial complex line bundle over X, as we identify $C^\infty(X)$ with $C^\infty(E)$, we may consider the corresponding G-submodules

$$C^\infty(X)^{\tau,\varepsilon} = C^\infty(E)^{\tau,\varepsilon},$$
$$C^\infty(X)^{\mathrm{ev}} = C^\infty(E)^{\mathrm{ev}}, \qquad C^\infty(X)^{\mathrm{odd}} = C^\infty(E)^{\mathrm{odd}}$$

of $C^\infty(X)$ and the isomorphism

$$\varpi^* : C^\infty(Y) \longrightarrow C^\infty(X)^{\mathrm{ev}} \tag{3.6}$$

of G-modules.

We identify the vector bundles $V^* \otimes W$ and $\mathrm{Hom}(V, W)$. If x is a point of X and A is an element of $(V^* \otimes W)_x$, we consider the matrix $\tilde{A}$ of A with respect to positively oriented orthonormal bases $\{v_1, \dots, v_n\}$ of V_x and $\{w_1, \dots, w_n\}$ of W_x; then the determinant $q'(A) = \det A$ of $\tilde{A}$ and, for $1 \leq k \leq n$, the coefficient $q_k(A)$ of the term of degree $n-k$ of the characteristic polynomial of the matrix ${}^t\tilde{A}\tilde{A}$ are independent of the choice of such bases. In fact, we have $q_n = (-1)^n q'^2$. The functions $q_1, \dots, q_{n-1}, q'$ on $(V^* \otimes W)_{x_0}$ are K-invariant homogeneous polynomials, with $\deg q' = n$ and $\deg q_k = 2k$. If we identify the tangent bundle T with $V^* \otimes W$ *via* the isomorphism (3.1), the functions q' and q_k on $V^* \otimes W$ determine a symmetric form σ' of degree n and a symmetric form σ_k of degree $2k$ on X by

$$\sigma'(\xi, \dots, \xi) = q'(\xi), \qquad \sigma_k(\xi, \dots, \xi) = q_k(\xi),$$

for all $\xi \in T$; clearly, these symmetric forms are G-invariant.

We consider the Cartan decomposition (2.1) corresponding to the Riemannian symmetric pair (G, K) and identify the K-modules $\mathfrak{p}_0$ and T_{x_0}. Then σ' and σ_k are the G-invariant symmetric forms on X corresponding to the polynomials q' and q_k on $\mathfrak{p}_0$, respectively. A suitable adaptation of the proof of Lemma 4.1 of [**9**] due to Raïs shows that $\{q_1, \dots, q_{n-1}, q'\}$ are algebraically independent generators of the algebra $S(\mathfrak{p}_0^*)^K$ (see also [**10**, pp. 562–563]).

The orientations of the spaces V_x and W_x, with $x \in X$, considered above, together with the scalar products g_1 on V and g_2 on W, determine sections ω_V of $\bigwedge^n V^*$ and ω_W of $\bigwedge^n W^*$. In fact, if $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ are positively oriented orthonormal bases of V_x^* and W_x^*, respectively, then

$$\omega_V = \alpha_1 \wedge \cdots \wedge \alpha_n, \qquad \omega_W = \beta_1 \wedge \cdots \wedge \beta_n.$$

Via the second inclusion of (3.3), with $p = n$, we view the section $\omega_V \otimes \omega_W$ of $\bigwedge^n V^* \otimes \bigwedge^n W^*$ as a section σ of $S^n T^*$ over X. Clearly, the symmetric n-form σ is a G-invariant section of $S_0^n T^*$. Then we easily verify that

$$\sigma = n!\, \sigma'. \tag{3.7}$$

For $x \in X$, we have

$$\begin{aligned} \omega_V(\tau(x)) &= -\omega_V(x), \qquad & \omega_W(\tau(x)) &= -\omega_W(x), \\ \omega_V(\Psi(x)) &= \omega_W(x), & \omega_W(\Psi(x)) &= (-1)^n \omega_V(x). \end{aligned}$$

It follows that

$$\tau^*\sigma = \sigma, \qquad \Psi^*\sigma = \sigma.$$

Thus the form σ is Σ-invariant and so induces a G-invariant symmetric traceless n-form σ_Y on Y such that

$$\sigma = \varpi^* \sigma_Y.$$

Then the mappings

$$\sigma^\flat : T \longrightarrow S^{n-1}T^*, \qquad \sigma_Y^\flat : T_Y \longrightarrow S^{n-1}T_Y^*$$

induced by σ and σ_Y are G-equivariant monomorphisms.

We consider the Hodge operators

$$* : V^* \longrightarrow \bigwedge^{n-1} V^*, \qquad * : W^* \longrightarrow \bigwedge^{n-1} W^*,$$

corresponding to the orientations of the spaces V_x and W_x, with $x \in X$, and to the scalar products g_1 and g_2; then the relations

$$\alpha \wedge *\alpha = |\alpha|^2 \omega_V, \qquad \beta \wedge *\beta = |\beta|^2 \omega_W$$

hold for all $\alpha \in V^*$ and $\beta \in W^*$. These Hodge operators determine a G-equivariant isomorphism of vector bundles

$$* = * \otimes * : V^* \otimes W^* \longrightarrow \bigwedge^{n-1} V^* \otimes \bigwedge^{n-1} W^*,$$

which, *via* the isomorphism (3.2) and the second inclusion of (3.3), with $p = n - 1$, may be viewed as a monomorphism of vector bundles

$$* = \tilde{\sigma} : T^* \longrightarrow S^{n-1}T^*. \tag{3.8}$$

We also consider the G-equivariant monomorphism of vector bundles

$$* = \tilde{\sigma}_Y : T_Y^* \longrightarrow S^{n-1}T_Y^*; \tag{3.9}$$

then if α is a section of T_Y^* over Y, by (1.6) we see that

$$\varpi^* {*\alpha} = * \varpi^* \alpha. \tag{3.10}$$

The Grassmannian $\widetilde{G}_{2,2}^{\mathbb{R}}$ is isometric to the product of 2-spheres $S^2 \times S^2$ endowed with the product metric $g_1 + g_2$, where g_j is the Riemannian metric of constant curvature 1 on the j-th factor of $S^2 \times S^2$. In this case, it is easily seen that σ corresponds to a constant multiple of the symmetric 2-form $g_1 - g_2$.

4. The Stiefel manifolds and the real Grassmannians

We view $\mathbb{R}^{m+n}$ as a subspace of the complex vector space $U = \mathbb{C}^{m+n}$; then $\{e_1, \dots, e_{m+n}\}$ is a basis of U. The action of the group G on $\mathbb{R}^{m+n}$ extends to U. The k-th symmetric product $S^k U^*$ of U^* and the k-th exterior product $\bigwedge^k U^*$ of U^* are G-submodules of the G-module $\bigotimes^k U^*$, which is the k-th tensor product of U^*. We view $S^2 U^*$ as the space all complex quadratic forms on $\mathbb{C}^{m+n}$. The subspace $S_0^2 U^*$ of $S^2 U^*$, consisting of all elements of $S^2 U^*$ which are traceless with respect to the standard quadratic form on $\mathbb{C}^{m+n}$, is an irreducible G-submodule of S^2U^*. In fact, an element θ of S^2U^* belongs to $S_0^2U^*$ if and only if

$$\sum_{j=1}^{m+n} \theta(e_j, e_j) = 0.$$

Let $S_{m,n}$ be the space of all real $(m+n) \times m$ matrices A satisfying ${}^tAA = I_m$. We view $S_{m,n}$ as the Stiefel manifold of all orthonormal m-frames in $\mathbb{R}^{m+n}$; the matrix A of $S_{m,n}$ determines the m-frame consisting of the m column vectors of A.

We consider $S_{m,n}$ as a submanifold of the space of all real $(m+n) \times m$ matrices $M_{m,n}$. For $1 \leq j \leq m+n$ and $1 \leq k \leq m$, let X_j^k be the function on $M_{m,n}$ which sends a matrix of $M_{m,n}$ into its (j,k)-th entry; we also consider the $\mathbb{R}^m$-valued function X_j (resp. the $\mathbb{R}^{m+n}$-valued function X^k) on $M_{m,n}$, which sends a matrix of $M_{m,n}$ into its j-th row (resp. its k-th column). Then we have $X_j = (X_j^1, \dots, X_j^m)$ and the functions $\{X_j^k\}$ form a coordinate system for $M_{m,n}$. If u is an element of $M_{m,n}$, we shall often write $u_j^k = X_j^k(u)$.

The orthogonal group $SO(m)$ acts on $S_{m,n}$ by right multiplication and we consider the quotient space $S_{m,n}/SO(m)$. The mapping

$$\rho : S_{m,n} \longrightarrow \widetilde{G}_{m,n}^{\mathbb{R}},$$

sending the element A of $S_{m,n}$ into the point of $\widetilde{G}_{m,n}^{\mathbb{R}}$ corresponding to the vector $X^1(A) \wedge \cdots \wedge X^m(A)$, induces by passage to the quotient a diffeomorphism

$$\bar{\rho} : S_{m,n}/SO(m) \longrightarrow \widetilde{G}_{m,n}^{\mathbb{R}}.$$

The group $G = SO(m+n)$ acts on $S_{m,n}$ by left multiplication; clearly, the mappings ρ and $\bar{\rho}$ are G-equivariant. A function f on $S_{m,n}$ which is invariant under the right action of $SO(m)$ determines a function $\tilde{f}$ on $\widetilde{G}_{m,n}^{\mathbb{R}}$ satisfying $\rho^*\tilde{f} = f$.

For $1 \leq j, l \leq m+n$, the real-valued function

$$f_{jl} = \langle X_j, X_l \rangle = \sum_{k=1}^{m} X_j^k X_l^k$$

on $S_{m,n}$ is invariant under the right action of $SO(m)$; we have $f_{jl} = f_{lj}$. The action of G on $S_{m,n}$ induces a structure of G-module on $C^\infty(S_{m,n})$. It is then easily verified that the mapping

$$\Phi_1 : S_0^2U^* \longrightarrow C^\infty(S_{m,n}),$$

which sends an element q of $S_0^2U^*$ into the function

$$\sum_{j,k=1}^{m+n} q(e_j, e_k) f_{jk}$$

on $S_{m,n}$ which is invariant under the right action of $SO(m)$, is a morphism of G-modules. Hence the image $\mathcal{H}$ of Φ_1 is a G-submodule of $C^\infty(S_{m,n})$. Since the function f_{12} belonging to $\mathcal{H}$ is non-zero and since $S_0^2U^*$ is an irreducible G-module, it follows that the mapping Φ_1 is injective and that $\mathcal{H}$ is an irreducible G-module. Thus the G-submodule

$$\tilde{\mathcal{H}} = \{\, f \in C^\infty(X) \mid \rho^* f \in \mathcal{H} \,\}$$

of $C^\infty(X)$ is isomorphic to $\mathcal{H}$ and therefore also to $S^2_0 U^*$.

For $1 \leq j, l \leq m+n$, it is easily verified that the vector field

$$\sum_{k=1}^{m} \Big(X^k_j \frac{\partial}{\partial X^k_l} - X^k_l \frac{\partial}{\partial X^k_j} \Big),$$

on $M_{m,n}$ is tangent to the submanifold $S_{m,n}$; we may therefore consider the vector field ξ_{jl} on $S_{m,n}$ which it determines. We consider the Lie algebra $\mathfrak{g}_0$ of G. For $1 \leq j, l \leq m+n$, we denote by E_{jl} the element of $\mathfrak{gl}(m+n, \mathbb{R})$ whose (j,l)-th entry is equal to 1 and all of whose other entries are equal to 0; then the matrix $A_{jl} = E_{lj} - E_{jl}$ belongs to the subalgebra $\mathfrak{g}_0$ of $\mathfrak{gl}(m+n, \mathbb{R})$. The action of the group G on $S_{m,n}$ induces a morphism of Lie algebras from $\mathfrak{g}_0$ to the Lie algebra of all vector fields on $S_{m,n}$. It is easily verified that this morphism sends the element A_{jl} of $\mathfrak{g}_0$ into the vector field ξ_{jl}, for $1 \leq j, l \leq m+n$. Since the actions of G and $SO(m)$ on $S_{m,n}$ commute, it follows that the vector field ξ_{jl} on $S_{m,n}$ is ρ-projectable; we denote by $\tilde{\xi}_{jl}$ the vector field on X induced by $\tilde{\xi}_{jl}$. Then the subspace of $C^\infty(T)$ generated by the vector fields $\{\tilde{\xi}_{jl}\}$ is equal to the G-module $\mathcal{K}$ of all Killing vector fields of X.

Let $\tilde{x}$ be a point of $S_{m,n}$ corresponding to the orthonormal m-frame $\{v_1, \dots, v_m\}$ in $\mathbb{R}^{m+n}$. If x is the point $\rho(\tilde{x})$ of X, then $\{v_1, \dots, v_m\}$ is a positively oriented orthonormal basis of V_x. Let w be a given unit vector of W_x and $1 \leq k \leq m$ be a given integer. Let $\gamma : [0, 2\pi] \to S_{m,n}$ be the closed path defined as follows: for $0 \leq t \leq 2\pi$, let $\gamma(t)$ be the matrix of $S_{m,n}$ corresponding to the orthonormal m-frame

$$\{v_1, \dots, v_{k-1}, \cos t \cdot v_k + \sin t \cdot w, v_{k+1}, \dots, v_m\}.$$

We have $\gamma(0) = \gamma(2\pi) = \tilde{x}$. The path $\rho \circ \gamma : [0, 2\pi] \to \widetilde{G}^{\mathbb{R}}_{m,n}$ is a closed geodesic of the Grassmannian $\widetilde{G}^{\mathbb{R}}_{m,n}$, and it is easily seen that the equality

$$\frac{d}{dt} \rho(\gamma(t))_{|t=0} = v_k \otimes w$$

holds among vectors of T_x. If f a function on $S_{m,n}$ which is invariant under the right action of the group $SO(m)$, we have

$$\begin{aligned} \langle d\tilde{f}, v_k \otimes w \rangle &= \frac{d}{dt} f(\gamma(t))_{|t=0}, \\ (\mathrm{Hess}\, \tilde{f})(v_k \otimes w, v_k \otimes w) &= \frac{d^2}{dt^2} f(\gamma(t))_{|t=0}. \end{aligned} \tag{4.1}$$

We consider the standard scalar product on the space $M_{m,n}$, given by

$$\langle u, v \rangle = \sum_{\substack{1 \leq k \leq m \\ 1 \leq j \leq m+n}} u^k_j v^k_j,$$

for $u, v \in M_{m,n}$, and the Riemannian metric $\tilde{g}$ on the submanifold $S_{m,n}$ of $M_{m,n}$ induced by this scalar product. Clearly, the metric $\tilde{g}$ is invariant under the right action of $SO(m)$ and under the left action of G. Therefore the metric $\tilde{g}$ induces a Riemannian metric g' on $\widetilde{G}^{\mathbb{R}}_{m,n}$ which is G-invariant.

For $u \in S_{m,n}$, we identify the tangent space $T_u(S_{m,n})$ to $S_{m,n}$ at the point u with the subspace

$$\left\{ v \in M_{m,n} \;\middle|\; \sum_{j=1}^{m+n} (u_j^k v_j^l + u_j^l v_j^k) = 0, \quad \text{for all } 1 \leq k, l \leq m \right\}$$

of $M_{m,n}$. The subspace $V_u(S_{m,n})$ of $T_u(S_{m,n})$ consisting of those vectors which are tangent to the fibers of ρ is then identified with the subspace

$$\{ v \in M_{m,n} \mid v = u \cdot v', \quad \text{with } v' \in \mathfrak{so}(m) \}$$

of $M_{m,n}$. We shall also consider the orthogonal complement $H_u(S_{m,n})$ of $V_u(S_{m,n})$ in $T_u(S_{m,n})$.

Let $M'_{m,n}$ be the subspace of $M_{m,n}$ consisting of those elements u satisfying $u_j^k = 0$, for all $m+1 \leq j \leq m+n$ and $1 \leq k \leq m$. We consider the point $\tilde{x}_0$ of $S_{m,n}$ corresponding to the m-frame $\{e_1, \dots, e_m\}$ of $\mathbb{R}^{m+n}$; then we have $x_0 = \rho(\tilde{x}_0)$. An element v of $M_{m,n}$ belongs to $T_{\tilde{x}_0}(S_{m,n})$ if and only if

$$v_j^k + v_k^j = 0,$$

for all $1 \leq j, k \leq m$. Then we verify that

$$V_{\tilde{x}_0}(S_{m,n}) = \left\{ v \in M'_{m,n} \;\middle|\; v_j^k + v_k^j = 0, \text{ for all } 1 \leq j, k \leq m \right\},$$

$$H_{\tilde{x}_0}(S_{m,n}) = \left\{ v \in M_{m,n} \;\middle|\; v_j^k = 0, \text{ for all } 1 \leq j, k \leq m \right\},$$

and that the restriction

$$\rho_* : H_{\tilde{x}_0}(S_{m,n}) \longrightarrow T_{x_0}$$

of the mapping ρ_* to the subspace $H_{\tilde{x}_0}(S_{m,n})$ is an isometry. Indeed, let w be a unit vector of W_{x_0} and $1 \leq k \leq m$ be a given integer; we consider the closed path γ in $S_{m,n}$ associated with the point $\tilde{x}_0$, the vector w and the integer k, which we defined above. Then the tangent vector $\dot{\gamma}(0)$ is the matrix belonging to $H_{\tilde{x}_0}(S_{m,n})$ whose l-th column is $\delta_{kl} w$, for $1 \leq l \leq m$, and we know that

$$\rho_* \dot{\gamma}(0) = e_k \otimes w.$$

Since the metrics g' and g on X are G-invariant, it follows that they are equal.

Let ϕ be an element of $SO(m)$ and R_ϕ be the diffeomorphism of $S_{m,n}$ induced by the right action of ϕ on $S_{m,n}$; we have $R_\phi(u) = u\phi$, for all $u \in S_{m,n}$. Then we see that

$$\rho \cdot R_\phi = \rho, \tag{4.2}$$

and hence that

$$R_{\phi *}V_u(S_{m,n}) = V_{u\phi}(S_{m,n}),$$

for $u \in S_{m,n}$. Since the metric $\tilde{g}$ is invariant under the right action of the group $SO(m)$, from the previous equality we deduce that

$$R_{\phi *}H_u(S_{m,n}) = H_{u\phi}(S_{m,n}), \tag{4.3}$$

for $u \in S_{m,n}$.

Let α be 1-form on $S_{m,n}$ which is invariant under the right action of the group $SO(m)$. We now show that α induces a 1-form $\tilde{\alpha}$ on X. Indeed, let ξ be elements of T_x, with $x \in X$. Choose a point $\hat{x}$ of $S_{m,n}$ satisfying $\rho(\hat{x}) = x$, and let $\hat{\xi}$ be the unique vector of $H_{\hat{x}}(S_{m,n})$ satisfying $\rho_*\hat{\xi} = \xi$; then according to the relations (4.2) and (4.3), the 1-form $\tilde{\alpha}$ on X determined by

$$\langle \xi, \tilde{\alpha} \rangle = \langle \hat{\xi}, \alpha \rangle$$

is well-defined. If f is a function on $S_{m,n}$ which is invariant under the right action of $SO(m)$, then $\beta = f\alpha$ is a 1-form on $S_{m,n}$ which is invariant under the right action of the group $SO(m)$, and we see that

$$\tilde{\beta} = \tilde{f}\tilde{\alpha}.$$

If $f = (f^1, \dots, f^m)$ is a $\mathbb{C}^m$-valued function and $\alpha = (\alpha^1, \dots, \alpha^m)$ is a $\mathbb{C}^m$-valued 1-form on $S_{m,n}$, where f^k is an element of $C^\infty(S_{m,n})$ and α^k is a complex 1-form on $S_{m,n}$, we define a complex 1-form $f \cdot \alpha$ on $S_{m,n}$ by

$$f \cdot \alpha = \sum_{k=1}^{m} f^k \alpha^k.$$

For $1 \leq j, k \leq m+n$, we view X_j and X_k as $\mathbb{R}^m$-valued functions on $S_{m,n}$. Then the 1-form

$$X_j \cdot dX_k$$

on $S_{m,n}$ is invariant under the right action of the group $SO(m)$. Clearly we have

$$df_{jk} = X_j \cdot dX_k + X_k \cdot dX_j.$$

We consider the 1-form

$$\alpha_{jk} = X_j \cdot dX_k - X_k \cdot dX_j$$

on $S_{m,n}$, which is invariant under the right action of the group $SO(m)$, and the 1-form $\tilde{\alpha}_{jk}$ on X induced by α_{jk}. We denote by T' the cotangent bundle of $S_{m,n}$; the action of G on $S_{m,n}$ induces a structure of G-module on $C^\infty(S_{m,n}, T'_{\mathbb{C}})$. We easily verify that the mapping

$$\Phi_2 : \textstyle\bigwedge^2 U^* \longrightarrow C^\infty(S_{m,n}, T'_{\mathbb{C}}),$$

which sends an element ω of $\bigwedge^2 U^*$ into the 1-form

$$\sum_{j,k=1}^{m+n} \omega(e_j, e_k)\alpha_{jk}$$

on $S_{m,n}$ which is invariant under the right action of $SO(m)$, is a morphism of G-modules. Hence its image $\mathcal{A}$ is a G-submodule of $C^\infty(S_{m,n}, T'_{\mathbb{C}})$. The form $\tilde{\alpha}_{12}$ on X is non-zero, and so the mapping Φ_2 is non-zero. Since the G-module $\bigwedge^2 U^*$ is irreducible, we know that $\mathcal{A}$ is an irreducible G-module and that the space $\tilde{\mathcal{A}}$ of 1-forms on X induced by the forms of $\mathcal{A}$ is an irreducible G-submodule of $C^\infty(T^*_{\mathbb{C}})$. Both of these G-modules are isomorphic to $\bigwedge^2 U^*$; moreover, the G-submodule $\tilde{\mathcal{A}}_0$ of $\tilde{\mathcal{A}}$ generated over $\mathbb{R}$ by the forms $\tilde{\alpha}_{jk}$ is isomorphic to $\bigwedge^2 \mathbb{R}^{m+n}$. Clearly, we have

$$\tilde{g}^\flat(\xi_{jk}) = \alpha_{jk},$$

for $1 \leq j, k \leq m+n$. From this equality and the definition of $\tilde{\alpha}_{jk}$, we obtain the relation

$$g^\flat(\tilde{\xi}_{jk}) = \tilde{\alpha}_{jk}, \tag{4.4}$$

for $1 \leq j, k \leq m+n$; therefore we have the equality

$$g^\flat(\mathcal{K}) = \tilde{\mathcal{A}}_0. \tag{4.5}$$

We henceforth suppose that $m \leq n$. For $1 \leq k \leq m$ and $\theta \in \mathbb{R}$, we consider the vectors

$$v_k(\theta) = \cos\theta \cdot e_{2k-1} + \sin\theta \cdot e_{2k}, \qquad w_k(\theta) = -\sin\theta \cdot e_{2k-1} + \cos\theta \cdot e_{2k}$$

of $\mathbb{R}^{m+n}$. Then for $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, the vectors

$$\{v_1(\theta_1), \dots, v_m(\theta_m), w_1(\theta_1), \dots, w_m(\theta_m)\}$$

form an orthonormal system of vectors of $\mathbb{R}^{m+n}$. We consider the mapping

$$\iota : \mathbb{R}^m \longrightarrow S_{m,n},$$

which sends $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$ into the point $\iota(\theta)$ of $S_{m,n}$ corresponding to the orthonormal m-frame $\{v_1(\theta_1), \dots, v_m(\theta_m)\}$ in $\mathbb{R}^{m+n}$. If $\{e'_1, \dots, e'_m\}$ is the standard basis of $\mathbb{R}^m$ and Λ is the lattice generated by the basis $\{2\pi e'_1, \dots, 2\pi e'_m\}$ of $\mathbb{R}^m$, the mapping ι induces by passage to the quotient a mapping

$$\iota : \mathbb{R}^m/\Lambda \longrightarrow S_{m,n}.$$

The image of the mappings ι is a flat torus $\tilde{Z}$ of $S_{m,n}$. The lattice Λ' of $\mathbb{R}^m$ generated by the basis

$$\{\pi e'_k + \pi e'_{k+1}, \pi e'_1 + (-1)^{m+1}\pi e'_m\}, \qquad \text{with } 1 \leq k \leq m-1,$$

of $\mathbb{R}^m$ contains the lattice Λ. Then it is easily seen that there is an injective mapping

$$\iota' : \mathbb{R}^m/\Lambda' \longrightarrow \widetilde{G}^{\mathbb{R}}_{m,n}$$

such that the diagram

$$\begin{array}{ccc} \mathbb{R}^m/\Lambda & \xrightarrow{\ \iota\ } & S_{m,n} \\ \big\downarrow & & \big\downarrow \rho \\ \mathbb{R}^m/\Lambda' & \xrightarrow{\ \iota'\ } & \widetilde{G}^{\mathbb{R}}_{m,n} \end{array}$$

is commutative. We also denote by ι' the mapping $\rho\circ\iota : \mathbb{R}^m \to \widetilde{G}^{\mathbb{R}}_{m,n}$. The image Z of the mappings ι' is a maximal flat totally geodesic torus of $\widetilde{G}^{\mathbb{R}}_{m,n}$, which is equal to $\rho(\tilde{Z})$. If f is a function on X, then we have the equalities

$$\begin{aligned}\int_Z f\,dZ &= \int_0^{\pi}\cdots\int_0^{\pi}\left(\int_0^{2\pi} f(\iota'(\theta))\,d\theta_m\right)d\theta_1\dots d\theta_{m-1}\\ &= \int_0^{\pi}\cdots\int_0^{\pi}\left(\int_0^{2\pi} f(\iota'(\theta))\,d\theta_1\right)d\theta_2\dots d\theta_m.\end{aligned}\tag{4.6}$$

For $1\le j,k\le m$, we consider the vector field $\partial/\partial\theta_k$ on $\mathbb{R}^m$ and the vector fields η_{jk} and $\tilde{\zeta}_k$ on $\tilde{Z}$ defined by

$$\begin{aligned}\eta_{jk}(\iota(\theta)) &= \cos\theta_j\cdot\big(\partial/\partial X^k_{2j-1}\big)(\iota(\theta)) + \sin\theta_j\cdot\big(\partial/\partial X^k_{2j}\big)(\iota(\theta)),\\ \tilde{\zeta}_k(\iota(\theta)) &= \cos\theta_k\cdot\big(\partial/\partial X^k_{2k}\big)(\iota(\theta)) - \sin\theta_k\cdot\big(\partial/\partial X^k_{2k-1}\big)(\iota(\theta)),\end{aligned}$$

for $\theta\in\mathbb{R}^m$. We now fix a point $\theta=(\theta_1,\dots,\theta_m)$ of $\mathbb{R}^m$. We easily verify that the tangent vectors

$$\{(\eta_{jk}-\eta_{kj})(\iota(\theta))\}_{1\le j<k\le m}$$

form a basis for the space $V_{\iota(\theta)}(S_{m,n})$; it follows that $\tilde{\zeta}_k(\iota(\theta))$ belongs to $H_{\iota(\theta)}(S_{m,n})$. Then we see that

$$\iota_*(\partial/\partial\theta_k)(\theta) = \tilde{\zeta}_k(\iota(\theta)),\tag{4.7}$$

for $\theta\in\mathbb{R}^m$; moreover, the vector fields $\{\tilde{\zeta}_1,\dots,\tilde{\zeta}_m\}$ form an orthonormal basis for the space of parallel vector fields on $\tilde{Z}$. For $1\le k\le m$ and $t\in\mathbb{R}$, let $\gamma_k(t)$ be the point of $S_{m,n}$ corresponding to the orthonormal m-frame

$$\{v_1,\dots,v_{k-1},\cos t\cdot v_k+\sin t\cdot w_k, v_{k+1},\dots,v_m\},$$

where $v_j=v_j(\theta_j)$ and $w_j=w_j(\theta_j)$, for $1\le j\le m$. Then we have $\gamma_k(0)=\iota(\theta)$ and the relation

$$\frac{d}{dt}\gamma_k(t)_{|t=0} = \tilde{\zeta}_k(\iota(\theta))$$

holds. We know that $\{v_1(\theta_1),\dots,v_m(\theta_m)\}$ is an orthonormal basis for $V_{\iota'(\theta)}$ and that $\{w_1(\theta_1),\dots,w_m(\theta_m)\}$ is an orthonormal system of vectors of $W_{\iota'(\theta)}$. As we have seen above, according to these observations the equality

$$\rho_*\tilde{\zeta}_k(\iota(\theta)) = (\iota'(\theta), v_k(\theta_k)\otimes w_k(\theta_k))$$

holds among vectors of $T_{\iota'(\theta)}$. It follows that the vector field $\tilde{\zeta}_k$ on $\tilde{Z}$ is ρ-projectable and determines a parallel vector field ζ_k on Z which satisfies

$$\zeta_k(\iota'(\theta)) = (\iota'(\theta), v_k(\theta_k)\otimes w_k(\theta_k)),\tag{4.8}$$

for all $\theta\in\mathbb{R}^m$; according to (4.7), we see that

$$\iota'_*(\partial/\partial\theta_k)(\theta) = \zeta_k(\iota'(\theta)),\tag{4.9}$$

for $\theta\in\mathbb{R}^m$. In fact, $\{\zeta_1,\dots,\zeta_m\}$ is an orthonormal basis for the space of parallel vector fields on Z.

If f is a function on $S_{m,n}$ which is invariant under the right action of $SO(m)$, we have

$$\iota^* f = \iota'^* \tilde{f},$$

and, hence by (4.7), we see that

$$\langle \zeta_k, d\tilde{f} \rangle(\iota'(\theta)) = \langle \tilde{\zeta}_k, df \rangle(\iota(\theta)) = \frac{\partial(\iota^* f)}{\partial \theta_k}(\theta), \tag{4.10}$$

for $\theta \in \mathbb{R}^m$. If α is a 1-form on $S_{m,n}$ which is invariant under the right action of $SO(m)$, by (4.7) and (4.9) we easily see that the relation

$$\iota^* \alpha = \iota'^* \tilde{\alpha} \tag{4.11}$$

holds among 1-forms on $\mathbb{R}^m$. If φ is a 1-form on X and $1 \leq j, k \leq m$, since the mapping ι' is totally geodesic, by (4.9) and the definition of the operator D^1 we have the equality

$$2\iota'^*(D^1\varphi)(\zeta_j, \zeta_k) = \frac{\partial}{\partial \theta_j} \langle \partial/\partial\theta_k, \iota'^*\varphi \rangle + \frac{\partial}{\partial \theta_k} \langle \partial/\partial\theta_j, \iota'^*\varphi \rangle \tag{4.12}$$

of functions on $\mathbb{R}^m$.

We now suppose that $m = n$, with $n \geq 2$. Let $\theta = (\theta_1, \dots, \theta_n)$ be a given element of $\mathbb{R}^n$ and consider the point $x = \iota'(\theta)$ of Z. According to (4.8), the tangent vectors $\zeta_k(x)$ belonging to $T_x = (V \otimes W)_x$ are of rank one, for $1 \leq k \leq n$. Therefore if u is an element of $S^p T_x^*$, with $p \geq 2$, which belongs to the subspace $(\bigwedge^p V^* \otimes \bigwedge^p W^*)_x$, we have

$$u(\zeta_j, \zeta_j, \zeta_{j_1}, \dots, \zeta_{j_{p-2}}) = 0, \tag{4.13}$$

for all $1 \leq j, j_1, \dots, j_{p-2} \leq n$. Now suppose that $n \geq 3$ and let φ be an element of T_x^*. Then $*\varphi$ belongs to $(\bigwedge^{n-1} V^* \otimes \bigwedge^{n-1} W^*)_x$; hence the relation (4.13) holds for $u = *\varphi$ and $p = n-1$. According to (4.8), we easily see that

$$\varphi(\zeta_j) = (*\varphi)(\zeta_{j_1}, \dots, \zeta_{j_{n-1}}), \tag{4.14}$$

whenever the integers $\{j, j_1, \dots, j_{n-1}\}$ are a permutation of $\{1, 2, \dots, n\}$. Since $\{\zeta_1, \dots, \zeta_n\}$ is an orthonormal basis for the space of parallel vector fields on the maximal flat totally geodesic torus Z, and since all maximal flat totally geodesic tori of X are conjugate under the action of $G = SO(2n)$ on X, from the relations (4.13) and (4.14) we deduce the following result:

LEMMA 4.1. — *Let X be the Grassmannian $\widetilde{G}^{\mathbb{R}}_{n,n}$, with $n \geq 3$. An element φ of $C^\infty(T^*)$ satisfies the Guillemin condition if and only if the symmetric $(n-1)$-form $*\varphi$ on X satisfies the Guillemin condition.*

According to Lemmas 1.1 and 4.1, we know that a 1-form φ on Y satisfies the Guillemin condition if and only if the symmetric $(n-1)$-form $*\varphi$ on Y satisfies the Guillemin condition.

Let u be the point of $S_{n,n}$ corresponding to an orthonormal n-frame $\{v_1,\dots,v_n\}$ of $\mathbb{R}^{2n}$ and let $\{w_1,\dots,w_n\}$ be an orthonormal n-frame which spans the orthogonal complement of the space generated by the vectors $\{v_1,\dots,v_n\}$ satisfying

$$v_1\wedge\cdots\wedge v_n\wedge w_1\wedge\cdots\wedge w_n = e_1\wedge\cdots\wedge e_{2n}.$$

If u' is the point of $S_{n,n}$ which corresponds to the orthonormal n-frame $\{w_1,\dots,w_n\}$, then we have $\Psi(\rho(u))=\rho(u')$ and we easily see that

$$\langle X_j(u),X_l(u)\rangle+\langle X_j(u'),X_l(u')\rangle=\delta_{jl}, \tag{4.15}$$

for $1\le j,l\le 2n$.

5. Functions and forms of degree one on the real Grassmannians

We now describe explicit functions and 1-forms on the real Grassmannian $X=\widetilde{G}^{\mathbb{R}}_{n,n}$ and the symmetric space $Y=\bar{G}^{\mathbb{R}}_{n,n}$, with $n\ge 3$. We consider the complexification $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}_0$ of the group $G=SO(2n)$. For $\mu\in\mathbb{C}$, we set

$$L(\mu)=\begin{pmatrix}0 & -i\mu\\ i\mu & 0\end{pmatrix}.$$

For $\mu_1,\mu_2,\dots,\mu_n\in\mathbb{C}$, we consider the $2n\times 2n$ matrix

$$L(\mu_1,\dots,\mu_n)=\begin{pmatrix}L(\mu_1) & 0 & \dots & 0\\ 0 & L(\mu_2) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & L(\mu_n)\end{pmatrix}$$

The subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}$ consisting of all $2n\times 2n$ matrices $L(\mu_1,\dots,\mu_n)$, where $\mu_1,\dots,\mu_n\in\mathbb{C}$ are purely imaginary, is the Lie algebra of a maximal torus of G. The complexification $\mathfrak{t}$ of $\mathfrak{t}_0$, which consists of all $2n\times 2n$ matrices $L(\mu_1,\dots,\mu_n)$, with $\mu_1,\dots,\mu_n\in\mathbb{C}$, is a Cartan subalgebra of the semi-simple Lie algebra $\mathfrak{g}$. For $1\le j\le n$, the linear form λ_j on $\mathfrak{t}$, which sends the element $L(\mu_1,\dots,\mu_n)$ of $\mathfrak{t}$, with $\mu_1,\dots,\mu_n\in\mathbb{C}$, into μ_j is purely imaginary on $\mathfrak{t}_0$. We set $\alpha_j=\lambda_j-\lambda_{j+1}$, for $1\le j\le n-1$, and $\alpha_n=\lambda_{n-1}+\lambda_n$. We choose the Weyl chamber of $(\mathfrak{g},\mathfrak{t})$ for which the system of simple roots of $\mathfrak{g}$ is equal to $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$. If Δ^+ is the corresponding system of positive roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, the unique element w_0 of the Weyl group of $\mathfrak{g}$ satisfying

$$w_0(\Delta^+)=-\Delta^+$$

is the automorphism $-\mathrm{id}$ when n is even, and is the automorphism determined by

$$w_0(\alpha_j)=-\alpha_j,\qquad w_0(\alpha_{n-1})=-\alpha_n,\qquad w_0(\alpha_n)=-\alpha_{n-1},$$

for $1\le j\le n-2$, when n is odd.

The highest weight of an irreducible G-module is a linear form

$$c_1\lambda_1 + c_2\lambda_2 + \cdots + \varepsilon c_n\lambda_n$$

on $\mathfrak{t}$, where $\varepsilon = \pm 1$ and $c_1, c_2, \ldots, c_n$ are integers satisfying

$$c_1 \geq c_2 \geq \cdots \geq c_n \geq 0.$$

The equivalence class of such an G-module is determined by this weight. We identify the dual Γ of G with the set of all such linear forms on $\mathfrak{t}$. If γ is an element of Γ, there exist integers $r_1, r_2, \ldots, r_{n-1} \geq 0$ and $s \in \mathbb{Z}$ such that

$$\gamma = \gamma_{r_1,r_2,\ldots,r_{n-1},s} = c_1\lambda_1 + c_2\lambda_2 + \cdots + c_n\lambda_n,$$

where

$$c_j = |s| + \sum_{k=j}^{n-1} r_k,$$

for $1 \leq j \leq n-1$, and $c_n = s$. The unique element $\bar{\gamma}$ of Γ determined by

$$w_0(\gamma) = -\bar{\gamma}$$

is given by

$$\bar{\gamma} = \begin{cases} \gamma = \gamma_{r_1,r_2,\ldots,r_{n-1},s} & \text{if } n \text{ is even,} \\ \gamma_{r_1,r_2,\ldots,r_{n-1},-s} & \text{if } n \text{ is odd.} \end{cases}$$

If γ is an element of Γ, we denote by $C^\infty_\gamma(X)$, $C^\infty_\gamma(S^pT^*_\mathbb{C})$ and $C^\infty_\gamma(Y)$ the isotypic components of the G-modules $C^\infty(X)$, $C^\infty(S^pT^*_\mathbb{C})$ and $C^\infty(Y)$, respectively, corresponding to γ. For $\gamma \in \Gamma$, we denote by $\mathcal{W}_\gamma$ the weight subspace of the G-module $C^\infty_\gamma(T^*_\mathbb{C})$ corresponding to its highest weight γ; we recall that the multiplicity of the G-module $C^\infty_\gamma(T^*_\mathbb{C})$ is equal to the dimension of the space $\mathcal{W}_\gamma$ (see §2, Chapter II of [**6**]). We write

$$C^\infty_\gamma(X)^{\mathrm{ev}} = C^\infty(X)^{\mathrm{ev}} \cap C^\infty_\gamma(X), \qquad C^\infty_\gamma(X)^{\mathrm{odd}} = C^\infty(X)^{\mathrm{odd}} \cap C^\infty_\gamma(S^pT^*_\mathbb{C}),$$
$$C^\infty_\gamma(S^pT^*_\mathbb{C})^\Sigma = C^\infty_\gamma(S^pT^*_\mathbb{C})^{\mathrm{ev}} = C^\infty(S^pT^*_\mathbb{C})^\Sigma \cap C^\infty_\gamma(S^pT^*_\mathbb{C}),$$
$$C^\infty_\gamma(S^pT^*_\mathbb{C})^{\mathrm{odd}} = C^\infty(S^pT^*_\mathbb{C})^{\mathrm{odd}} \cap C^\infty_\gamma(S^pT^*_\mathbb{C}).$$

The mapping (3.6) induces an isomorphism of G-modules

$$\varpi^* : C^\infty_\gamma(Y) \to C^\infty_\gamma(X)^{\mathrm{ev}}.$$

A linear form λ on $\mathfrak{t}$ is a weight of the G-module $C^\infty_\gamma(S^pT^*_\mathbb{C})$ if and only if $-\lambda$ is a weight of the complex conjugate $\overline{C^\infty_\gamma(S^pT^*_\mathbb{C})}$ of the space $C^\infty_\gamma(S^pT^*_\mathbb{C})$; therefore we have the equality

$$C^\infty_{\bar{\gamma}}(S^pT^*_\mathbb{C}) = \overline{C^\infty_\gamma(S^pT^*_\mathbb{C})} \tag{5.1}$$

of G-modules. In particular, when n is even, or when n is odd and γ is equal to $\gamma_{r_1,r_2,\ldots,r_{n-1},0}$, where $r_1, r_2, \ldots, r_{n-1} \geq 0$ are integers, the G-module $C^\infty_\gamma(S^pT^*_\mathbb{C})$ is invariant under complex conjugation.

If $k, l \geq 1$ are integers, let $M(k,l)$ be the space of all complex $k \times l$ matrices. For $1 \leq j \leq n$, we consider the $\mathbb{C}^n$-valued function

$$Z_j = X_{2j-1} + iX_{2j}$$

on $S_{n,n}$. For $1 \leq k \leq n$, let A_k be the $M(k,n)$-valued function on $S_{n,n}$ whose j-th row is equal to Z_j, for $1 \leq j \leq k$. We also consider the $M(n,n)$-valued function A'_n on $S_{n,n}$ whose j-th row is equal to Z_j, for $1 \leq j \leq n-1$, and whose n-th row is equal to $\bar{Z}_n$.

For $1 \leq k \leq n$, we consider the $M(k,k)$-valued function

$$B_k = A_k \cdot {}^t A_k$$

on $S_{n,n}$. We extend the scalar product of $\mathbb{R}^n$ to a bilinear form on $\mathbb{C}^n$; then for $1 \leq j, l \leq k$, it is easily seen that the (j,l)-th entry of the matrix-valued function B_k is equal to $\langle Z_j, Z_l \rangle$. For $1 \leq k \leq n-1$, the complex-valued function f_k on $S_{n,n}$, defined by

$$f_k = \det B_k = \det\big(\langle Z_j, Z_l \rangle_{1 \leq j,l \leq k}\big), \tag{5.2}$$

is clearly invariant under the right action of $SO(n)$. In particular, we have

$$f_1 = \langle Z_1, Z_1 \rangle = \langle X_1, X_1 \rangle - \langle X_2, X_2 \rangle + 2i\langle X_1, X_2 \rangle, \tag{5.3}$$

$$f_2 = \langle Z_1, Z_1 \rangle \langle Z_2, Z_2 \rangle - \langle Z_1, Z_2 \rangle^2. \tag{5.4}$$

For $1 \leq j, k \leq n$, the functions $\langle Z_j, Z_k \rangle$ and $\langle Z_j, \bar{Z}_k \rangle$ on $S_{n,n}$ belong to the irreducible G-module $\mathcal{H}$; in particular, we know that the function f_1 belongs to $\mathcal{H}$. We write $f'_1 = \langle Z_1, Z_2 \rangle$. The complex-valued functions $f_n = \det A_n$ and $f'_n = \det A'_n$ on $S_{n,n}$ are also invariant under the right action of $SO(n)$. Clearly, we have the equality

$$f_n^2 = \det B_n = \det(A_n \cdot {}^t A_n). \tag{5.5}$$

From the second formula of (4.1) and the definitions of the functions f_n and f'_n, if x is an arbitrary point of X, we verify directly that the equalities

$$(\mathrm{Hess}\, \tilde{f}_n)(v \otimes w, v \otimes w) = -\tilde{f}_n(x), \qquad (\mathrm{Hess}\, \tilde{f}'_n)(v \otimes w, v \otimes w) = -\tilde{f}'_n(x) \tag{5.6}$$

hold for all unit vectors $v \in V_x$ and $w \in W_x$.

If $r_1, \dots, r_{n-1} \geq 0$ and s are given integers, the complex-valued function $f_{r_1,\dots,r_{n-1},s}$ on $S_{n,n}$, defined by

$$f_{r_1,\dots,r_{n-1},s} = \begin{cases} f_n^s \cdot \prod_{k=1}^{n-1} f_k^{r_k} & \text{if } s \geq 0, \\ f_n'^{|s|} \cdot \prod_{k=1}^{n-1} f_k^{r_k} & \text{if } s < 0, \end{cases}$$

is invariant under the right action of $SO(n)$. According to Strichartz [**21**] and Grinberg [**11**], if γ is the element

$$\gamma^1_{r_1,\dots,r_{n-1},s} = \gamma_{2r_1,\dots,2r_{n-1},s}$$

of Γ, the function $\tilde{f}_{r_1,\dots,r_{n-1},s}$ on X induced by $f_{r_1,\dots,r_{n-1},s}$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$. We also know that $C^\infty_\gamma(X) = 0$ whenever $\gamma \in \Gamma$ is not of the form $\gamma^1_{r_1,\dots,r_{n-1},s}$, with $r_1, \dots, r_{n-1} \geq 0$ and $s \in \mathbb{Z}$. Since f_1 belongs to the irreducible G-module $\mathcal{H}$, it follows that

$$C^\infty_\gamma(X) = \tilde{\mathcal{H}}$$

when $\gamma = \gamma^1_{1,0,\dots,0} = \gamma_{2,0,\dots,0}$; in fact, the G-module $\tilde{\mathcal{H}}$ is isomorphic to the irreducible G-module $S^2_0 U^*$, with $U = \mathbb{C}^{2n}$. If $r_1, \dots, r_{n-1}, s \in \mathbb{Z}$, when one of the integers $r_1, \dots, r_{n-1}$ is < 0 we set $\tilde{f}_{r_1,\dots,r_{n-1},s} = 0$.

From the definitions of the functions f_{jl} and f_k, we infer that

$$\tau^* \tilde{f}_{jl} = \tilde{f}_{jl}, \qquad \tau^* \tilde{f}_k = \tilde{f}_k, \qquad \tau^* \tilde{f}_n = -\tilde{f}_n, \qquad \tau^* \tilde{f}'_n = -\tilde{f}'_n, \tag{5.7}$$

for $1 \leq j, l \leq 2n$ and $1 \leq k \leq n-1$. From the definitions of the functions f_{jl} and f_k and according to the relations (4.15) and (5.5), we obtain

$$\begin{aligned} \Psi^* \tilde{f}_{jl} &= -\tilde{f}_{jl}, & \Psi^* \tilde{f}_{jj} + \tilde{f}_{jj} &= 1, \\ \Psi^* \tilde{f}_k &= (-1)^k \tilde{f}_k, & \Psi^* \tilde{f}^2_n &= (-1)^n \tilde{f}^2_n, \end{aligned} \tag{5.8}$$

for $1 \leq j, l \leq 2n$ and $1 \leq k \leq n-1$, with $j \neq l$. Since the isometry Ψ commutes with the action of G on $\widetilde{G}^{\mathbb{R}}_{n,n}$ and since the functions $\tilde{f}_n$ and $\tilde{f}'_n$ are highest weight vectors of irreducible G-modules, we know that $\Psi^* \tilde{f}_n = c_n \tilde{f}_n$ and $\Psi^* \tilde{f}'_n = c'_n \tilde{f}'_n$, where c_n and c'_n are complex numbers.

Since the functions $\tilde{f}_n$ and $\tilde{f}'_n$ are highest weight vectors of irreducible G-modules of weight $\lambda_1 + \dots + \lambda_{n-1} + \lambda_n$ and $\lambda_1 + \dots + \lambda_{n-1} - \lambda_n$, respectively, the functions $\tilde{f}_{n-1}$ and $\tilde{f}_n \cdot \tilde{f}'_n$ are highest weight vectors of the irreducible G-module $\mathcal{B} = C^\infty_\gamma(X)$, where $\gamma = \gamma^1_{0,\dots,0,1,0}$; therefore we know that $\tilde{f}_n \cdot \tilde{f}'_n = c''_n \tilde{f}_{n-1}$, where c''_n is a non-zero complex number. According to (5.1), we see that the G-module $\mathcal{B}$ is invariant under conjugation and is therefore equal to the complexification of the subspace

$$\mathcal{B}_{\mathbb{R}} = \{ f \in \mathcal{B} \mid f = \bar{f} \}$$

of $C^\infty_{\mathbb{R}}(X)$.

If $r_1, \dots, r_{n-1} \geq 0$ and s are given integers and γ is the element $\gamma^1_{r_1,\dots,r_{n-1},s}$ of Γ, since the function $\tilde{f}_{r_1,\dots,r_{n-1},s}$ on X is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, according to the relations (5.7) we have the inclusion

$$C^\infty_\gamma(X) \subset C^\infty(X)^{\tau,\varepsilon}, \tag{5.9}$$

where $\varepsilon = (-1)^s$. From (5.7) and (5.8), we obtain the equality

$$C^\infty_\gamma(X)^{\mathrm{odd}} = \tilde{\mathcal{H}}, \tag{5.10}$$

with $\gamma = \gamma^1_{1,0,\dots,0}$.

We now suppose that n is odd and consider the element $\gamma = \gamma^1_{0,\dots,0,1,0}$ of Γ; by (5.7) and (5.8), we see that

$$C^\infty_\gamma(X)^{\mathrm{ev}} = \mathcal{B}. \tag{5.11}$$

Therefore $\mathcal{B}_Y = C^\infty_\gamma(Y)$ is an irreducible G-module isomorphic to $\mathcal{B}$ and invariant under conjugation; thus $\mathcal{B}_Y$ is equal to the complexification of the subspace

$$\mathcal{B}_{Y,\mathbb{R}} = \{\, f \in \mathcal{B}_Y \mid f = \bar f \,\}$$

of $C^\infty_{\mathbb{R}}(Y)$ and the mapping ϖ induces an isomorphism $\varpi^* : \mathcal{B}_{Y,\mathbb{R}} \to \mathcal{B}_{\mathbb{R}}$. The dimension of $\mathcal{B}$ or $\mathcal{B}_{Y,\mathbb{R}}$ is equal to

$$n(n+1)(2n+1)(2n-3)/3.$$

We no longer suppose that n is odd. We consider the mapping

$$\tilde\iota : \mathbb{R}^n \longrightarrow S_{n,n},$$

which sends $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ into the point $v = \tilde\iota(\theta)$ of $S_{n,n}$ determined by

$$\begin{aligned} X^k(v) &= -\sin\theta_k \cdot e_{2k-1} + \cos\theta_k \cdot e_{2k}, \\ X^n(v) &= (-1)^{[n/2]}(-\sin\theta_n \cdot e_{2n-1} + \cos\theta_n \cdot e_{2n}), \end{aligned}$$

for $1 \le k \le n-1$. For all $\theta \in \mathbb{R}^n$, we verify that

$$X^1(u) \wedge \cdots \wedge X^n(u) \wedge X^1(v) \wedge \cdots \wedge X^n(v) = e_1 \wedge \cdots \wedge e_{2n},$$

where $u = \iota(\theta)$ and $v = \tilde\iota(\theta)$; hence we obtain the relation

$$\Psi(\rho(\iota(\theta)) = \rho(\tilde\iota(\theta)).$$

Let $(\tilde e_1, \dots, \tilde e_n)$ be the standard basis of $\mathbb{C}^n$. For $\theta = (\theta_1, \dots, \theta_n)$, we easily verify that

$$\begin{aligned} Z_j(\iota(\theta)) &= e^{i\theta_j}\tilde e_j = -iZ_j(\tilde\iota(\theta)), \\ Z_n(\iota(\theta)) &= e^{i\theta_n}\tilde e_n = (-1)^{[n/2]+1} iZ_n(\tilde\iota(\theta)), \end{aligned}$$

for $1 \le j < n$. According to these relations, we see that

$$\begin{gathered} f_k(\iota(\theta)) = e^{2i(\theta_1 + \cdots + \theta_k)}, \\ f_n(\iota(\theta)) = e^{i(\theta_1 + \cdots + \theta_n)}, \qquad f'_n(\iota(\theta)) = e^{i(\theta_1 + \cdots + \theta_{n-1} - \theta_n)}, \end{gathered} \tag{5.12}$$

for $1 \le k < n$, and that

$$\begin{aligned} f_n(\tilde\iota(\theta)) &= (-1)^{[n/2]} \cdot i^n \cdot e^{i(\theta_1 + \cdots + \theta_n)}, \\ f'_n(\tilde\iota(\theta)) &= (-1)^{[n/2]+1} \cdot i^n \cdot e^{i(\theta_1 + \cdots + \theta_{n-1} - \theta_n)}. \end{aligned} \tag{5.13}$$

From the equalities (5.12) and (5.13) and observations made above concerning the functions $\tilde f_n$ and $\tilde f'_n$, we infer that

$$\begin{gathered} \Psi^* \tilde f_n = (-1)^{[n/2]} \cdot i^n \tilde f_n, \qquad \Psi^* \tilde f'_n = (-1)^{[n/2]+1} \cdot i^n \tilde f'_n, \\ \tilde f_n \cdot \tilde f'_n = (-1)^{n+1} \tilde f_{n-1}. \end{gathered} \tag{5.14}$$

If $1 \le k \le n$ and $1 \le i_1 < \cdots < i_k \le n$ are given integers, we consider the $M(k,k)$-valued function $A_k(i_1, i_2, \dots, i_k)$ on $S_{n,n}$ whose l-th column is equal to the

i_l-th column of the $M(k,n)$-valued function A_k, for $1 \leq l \leq k$, and the complex-valued function

$$D(i_1, i_2, \dots, i_k) = \det A_k(i_1, i_2, \dots, i_k)$$

on $S_{n,n}$.

According to Strichartz [**21**], we have the equality

$$f_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} D(i_1, i_2, \dots, i_k)^2, \tag{5.15}$$

for $1 \leq k < n$. Indeed, Strichartz shows that the function $\hat{f}_k$ on $S_{n,n}$ which is equal to the right hand side of the equality (5.15) is invariant under the right action of the group $SO(n)$ and the function on X induced by $\hat{f}_k$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, with $\gamma = 2(\lambda_1 + \cdots + \lambda_k)$. Since the function f_k possesses the same properties, the functions f_k and $\hat{f}_k$ differ by a constant. If $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, according to the description of the matrix $A_n(\iota(\theta))$ given above, we see that $D(i_1, i_2, \dots, i_k)(\iota(\theta))$ vanishes if one of the indices i_l is $> k$ and that

$$D(1, 2, \dots, k)(\iota(\theta)) = e^{i(\theta_1 + \cdots + \theta_k)};$$

from these observations and the first relation of (5.12), we infer that the equality (5.15) is true on the torus $\tilde{Z}$, and hence on all of $S_{n,n}$.

The 1-forms

$$\vartheta_{jk} = Z_j \cdot dZ_k - Z_k \cdot dZ_j, \qquad \vartheta_{\bar{j}\bar{k}} = \bar{Z}_j \cdot d\bar{Z}_k - \bar{Z}_k \cdot d\bar{Z}_j,$$
$$\vartheta_{j\bar{k}} = Z_j \cdot d\bar{Z}_k - \bar{Z}_k \cdot dZ_j$$

on $S_{n,n}$, with $1 \leq j, k \leq n$, belong to the G-module $\mathcal{A}$ and generate it over $\mathbb{C}$. We shall consider the 1-forms $\tilde{\vartheta}_{jk}$, $\tilde{\vartheta}_{\bar{j}\bar{k}}$ and $\tilde{\vartheta}_{j\bar{k}}$ on X induced by $\tilde{\vartheta}_{jk}$, $\tilde{\vartheta}_{\bar{j}\bar{k}}$ and $\tilde{\vartheta}_{j\bar{k}}$, respectively.

We again consider the vector space $U = \mathbb{C}^{2n}$. Let $\{\alpha_1, \dots, \alpha_{2n}\}$ be the basis of U^* dual to the basis $\{e_1, \dots, e_{2n}\}$ of U. Then the vectors

$$\beta_j = \alpha_{2j-1} + i\alpha_{2j}, \qquad \beta_{\bar{j}} = \alpha_{2j-1} - i\alpha_{2j}$$

of U^*, with $1 \leq j \leq n$, are vectors of the G-module U^* of weight λ_j and $-\lambda_j$, respectively. It is easily seen that

$$\begin{gathered} \Phi_1(\beta_j \cdot \beta_k) = \langle Z_j, Z_k \rangle, \qquad \Phi_1(\beta_{\bar{j}} \cdot \beta_{\bar{k}}) = \langle \bar{Z}_j, \bar{Z}_k \rangle, \\ \Phi_1(\beta_j \cdot \beta_{\bar{k}}) = \langle Z_j, \bar{Z}_k \rangle, \\ \Phi_2(\beta_j \wedge \beta_k) = \vartheta_{jk}, \qquad \Phi_2(\beta_{\bar{j}} \wedge \beta_{\bar{k}}) = \vartheta_{\bar{j}\bar{k}}, \\ \Phi_2(\beta_j \wedge \beta_{\bar{k}}) = \vartheta_{j\bar{k}}, \end{gathered} \tag{5.16}$$

for $1 \leq j, k \leq n$. Since $\beta_1 \wedge \beta_2$ and $\beta_1 \cdot \beta_1$ are highest weight vectors of the irreducible G-modules $\bigwedge^2 U^*$ and $S^2_0 U^*$, respectively, we see that ϑ_{12} is a highest weight vector of the irreducible G-module $\mathcal{A}$ and, once again, that $f_1 = \langle Z_1, Z_1 \rangle$ is a highest weight vector of the irreducible G-module $\mathcal{H}$. Therefore $\tilde{\vartheta}_{12}$ is a highest weight vector of the irreducible G-module $\tilde{\mathcal{A}}$. By (4.4), it is easily seen that the vector field

$$\xi_0 = \xi_{13} - \xi_{24} + i(\xi_{14} + \xi_{23})$$

on X satisfies the relation

$$g^\flat(\xi_0) = \tilde{\vartheta}_{12}.$$

If $\mathcal{K}_{\mathbb{C}}$ denotes the complexification of $\mathcal{K}$, by (4.5) it follows that ξ_0 is a highest weight vector of the irreducible G-module $\mathcal{K}_{\mathbb{C}}$. By (3.5), we also know that the equalities

$$(5.17)\qquad \begin{gathered} C^\infty_\gamma(T_{\mathbb{C}}) = \mathcal{K}_{\mathbb{C}}, \\ C^\infty_\gamma(T^*_{\mathbb{C}}) = C^\infty_\gamma(T^*_{\mathbb{C}})^\Sigma = g^\flat(\mathcal{K}_{\mathbb{C}}) = \tilde{\mathcal{A}}. \end{gathered}$$

of G-modules hold, where γ is the element $\gamma_{0,1,0,\dots,0} = \lambda_1 + \lambda_2$ of Γ, and that all these G-modules are irreducible and isomorphic to $\mathfrak{g}$. Since the element

$$A_{13} - A_{24} + i(A_{14} + A_{23})$$

of $\mathfrak{g}$ is a highest weight vector of the irreducible G-module $\mathfrak{g}$, we once more see that the vector field ξ_0 is a highest weight vector of the G-module $\mathcal{K}_{\mathbb{C}}$.

According to the description of the weight vectors of U^*, we easily see that the highest weight of the G-module $\bigwedge^2(S^2_0U^*)$ is $3\lambda_1 + \lambda_2$ and that its highest weight vectors are the non-zero multiples of $(\beta_1 \cdot \beta_1) \wedge (\beta_1 \cdot \beta_2)$. Thus $(\beta_1 \cdot \beta_1) \wedge (\beta_1 \cdot \beta_2)$ is a highest weight vector of an irreducible G-submodule of $\bigwedge^2(S^2_0U^*)$. Therefore by the first relation of (5.16), we know that $\langle Z_1, Z_1\rangle \wedge \langle Z_1, Z_2\rangle$ is a highest weight vector of an irreducible G-submodule of $\bigwedge^2\mathcal{H}$, whose highest weight is $3\lambda_1 + \lambda_2$. According to (5.10), the image of the morphism of G-modules

$$\chi_1 : \bigwedge^2\mathcal{H} \longrightarrow C^\infty(T^*_{\mathbb{C}}),$$

defined by

$$\chi_1(f \wedge f') = \tilde{f}\,d\tilde{f}' - \tilde{f}'d\tilde{f},$$

for $f, f' \in \mathcal{H}$, is a G-submodule of $C^\infty(T^*_{\mathbb{C}})^{\mathrm{ev}}$.

The G-modules $\mathcal{A}\otimes\mathcal{H}$ and $\mathcal{A}\otimes\mathcal{H}\otimes\mathcal{H}$ are isomorphic to the modules $\bigwedge^2U^* \otimes S^2_0U^*$ and $\bigwedge^2U^* \otimes S^2_0U^* \otimes S^2_0U^*$, respectively; the mappings

$$\chi_2 : \mathcal{A} \otimes \mathcal{H} \longrightarrow C^\infty(T^*_{\mathbb{C}}), \qquad \chi_3 : \mathcal{A} \otimes \mathcal{H} \otimes \mathcal{H} \longrightarrow C^\infty(T^*_{\mathbb{C}}),$$

defined by

$$\chi_2(\alpha \otimes f) = \tilde{f}\tilde{\alpha}, \qquad \chi_3(\alpha \otimes f \otimes f') = \tilde{f}\tilde{f}'\tilde{\alpha},$$

where $\alpha \in \mathcal{A}$ and $f, f' \in \mathcal{H}$, are morphisms of G-modules. According to (5.10) and (5.17), their images are G-modules satisfying

$$\chi_2(\mathcal{A} \otimes \mathcal{H}) \subset C^\infty(T^*_{\mathbb{C}})^{\mathrm{odd}}, \qquad \chi_3(\mathcal{A} \otimes \mathcal{H} \otimes \mathcal{H}) \subset C^\infty(T^*_{\mathbb{C}})^{\mathrm{ev}}.$$

We consider the G-modules

$$M_1 = \bigwedge^3U^* \otimes U^*, \qquad M_2 = \bigwedge^3U^* \otimes \bigwedge^2U^* \otimes U^*.$$

According to the description of the highest weight vectors of U^*, the elements

$$v_1^+ = \beta_1 \wedge \beta_2 \wedge \beta_3 \otimes \beta_1, \qquad v_1^- = \beta_1 \wedge \beta_2 \wedge \beta_{\bar{3}} \otimes \beta_1$$

of $\bigwedge^3 U^* \otimes U^*$ generate (over $\mathbb{C}$) the weight spaces of the G-module M_1 of weights $\varpi_1^+ = 2\lambda_1 + \lambda_2 + \lambda_3$ and $\varpi_1^- = 2\lambda_1 + \lambda_2 - \lambda_3$, respectively. Moreover, the elements

$$v_2^+ = \beta_1 \wedge \beta_2 \wedge \beta_3 \otimes \beta_1 \wedge \beta_2 \wedge \otimes \beta_1, \qquad v_2^- = \beta_1 \wedge \beta_2 \wedge \beta_{\bar{3}} \otimes \beta_1 \wedge \beta_2 \otimes \beta_1$$

of $\bigwedge^3 U^* \otimes \bigwedge^2 U^* \otimes U^*$ generate (over $\mathbb{C}$) the weight spaces of the G-module M_2 of weights $\varpi_2^+ = 3\lambda_1 + 2\lambda_2 + \lambda_3$ and $\varpi_2^- = 3\lambda_1 + 2\lambda_2 - \lambda_3$, respectively. Hence for $j = 1, 2$, the element v_j^+ of M_j is a highest weight vector of an irreducible $O(2n)$-submodule E_j of M_j. When $n > 3$, we know that E_j is an irreducible G-module. When $n = 3$, the G-module E_j decomposes into the direct sum of two irreducible G-modules E_j^+ and E_j^-, whose highest weights are ϖ_j^+ and ϖ_j^-, respectively; therefore, v_j^+ and v_j^- are highest weight vectors of the G-modules E_j^+ and E_j^-, respectively.

For $j \geq 1$, the mapping

$$\mu_j : \bigwedge^{j+1} U^* \otimes U^* \longrightarrow \bigwedge^j U^* \otimes S^2 U^*$$

defined by

$$(\mu_j v)(\xi_1, \dots, \xi_j, \eta_1, \eta_2) = (-1)^j (v(\xi_1, \dots, \xi_j, \eta_1, \eta_2) + v(\xi_1, \dots, \xi_j, \eta_2, \eta_1)),$$

for $v \in \bigwedge^{j+1} U^* \otimes U^*$ and $\xi_1, \dots, \xi_j, \eta_1, \eta_2 \in U$, is a morphism of G-modules. Then we easily verify that

$$\mu_j(\vartheta_1 \wedge \cdots \wedge \vartheta_{j+1} \otimes \vartheta) = \sum_{k=1}^{j+1} (-1)^{k+1} \vartheta_1 \wedge \cdots \wedge \widehat{\vartheta}_k \wedge \cdots \wedge \vartheta_{j+1} \otimes \vartheta_k \cdot \vartheta,$$

for $\vartheta_1, \dots, \vartheta_{j+1}, \vartheta \in U^*$. We consider the morphism of G-modules

$$\nu : \bigwedge^3 U^* \otimes \bigwedge^2 U^* \otimes U^* \longrightarrow \bigwedge^2 U^* \otimes S^2 U^* \otimes S^2 U^*$$

equal to the composition of the morphisms

$$\mathrm{id} \otimes \mu_1 : \bigwedge^3 U^* \otimes \bigwedge^2 U^* \otimes U^* \longrightarrow \bigwedge^3 U^* \otimes U^* \otimes S^2 U^*$$

and

$$\mu_2 \otimes \mathrm{id} : \bigwedge^3 U^* \otimes U^* \otimes S^2 U^* \longrightarrow \bigwedge^2 U^* \otimes S^2 U^* \otimes S^2 U^*.$$

The elements $\mu_2(v_1^+)$ and $\mu_2(v_1^-)$ of $\bigwedge^2 U^* \otimes S^2 U^*$ are given by

$$\mu_2(v_1^+) = \beta_2 \wedge \beta_3 \otimes \beta_1 \cdot \beta_1 - \beta_1 \wedge \beta_3 \otimes \beta_1 \cdot \beta_2 + \beta_1 \wedge \beta_2 \otimes \beta_1 \cdot \beta_3,$$
$$\mu_2(v_1^-) = \beta_2 \wedge \beta_{\bar{3}} \otimes \beta_1 \cdot \beta_1 - \beta_1 \wedge \beta_{\bar{3}} \otimes \beta_1 \cdot \beta_2 + \beta_1 \wedge \beta_2 \otimes \beta_1 \cdot \beta_{\bar{3}}.$$

We easily verify that the vectors $\nu(v_2^+)$ and $\nu(v_2^-)$ of $\bigwedge^2 U^* \otimes S^2U^* \otimes S^2U^*$ are given by

$$\nu(v_2^+) = \omega_1^+ \otimes \beta_1 \cdot \beta_1 - \omega_2^+ \otimes \beta_1 \cdot \beta_2, \qquad \nu(v_2^-) = \omega_1^- \otimes \beta_1 \cdot \beta_1 - \omega_2^- \otimes \beta_1 \cdot \beta_2,$$

where

$$\begin{aligned}
\omega_1^+ &= \beta_1 \wedge \beta_2 \otimes \beta_2 \cdot \beta_3 + \beta_2 \wedge \beta_3 \otimes \beta_1 \cdot \beta_2 - \beta_1 \wedge \beta_3 \otimes \beta_2 \cdot \beta_2,\\
\omega_2^+ &= \beta_1 \wedge \beta_2 \otimes \beta_1 \cdot \beta_3 + \beta_2 \wedge \beta_3 \otimes \beta_1 \cdot \beta_2 - \beta_1 \wedge \beta_3 \otimes \beta_1 \cdot \beta_2,\\
\omega_1^- &= \beta_1 \wedge \beta_2 \otimes \beta_2 \cdot \beta_3 + \beta_2 \wedge \beta_{\bar 3} \otimes \beta_1 \cdot \beta_2 - \beta_1 \wedge \beta_{\bar 3} \otimes \beta_2 \cdot \beta_2,\\
\omega_2^- &= \beta_1 \wedge \beta_2 \otimes \beta_1 \cdot \beta_{\bar 3} + \beta_2 \wedge \beta_{\bar 3} \otimes \beta_1 \cdot \beta_2 - \beta_1 \wedge \beta_{\bar 3} \otimes \beta_1 \cdot \beta_2.
\end{aligned}$$

From these formulas, we infer that the $\mu_2(v_1^+)$ and $\mu_2(v_1^-)$ are non-zero vectors of the G-submodule $\bigwedge^2 U^* \otimes S_0^2U^*$, while vectors $\mu_2(v_2^+)$ and $\mu_2(v_2^-)$ are non-zero vectors of the G-submodule $\bigwedge^2 U^* \otimes S_0^2U^* \otimes S_0^2U^*$. When $n = 3$, it follows that the G-submodules $\mu_2(E_1^+)$ and $\mu_2(E_1^-)$ are irreducible G-submodules of $\bigwedge^2 U^* \otimes S_0^2U^*$ and that the vectors $\mu_2(v_1^+)$ and $\mu_2(v_1^-)$ are highest weight vectors of the G-modules $\mu_2(E_1^+)$ and $\mu_2(E_1^-)$, respectively; moreover, the G-submodules $\nu(E_2^+)$ and $\nu(E_2^-)$ are irreducible G-submodules of $\bigwedge^2 U^* \otimes S_0^2U^* \otimes S_0^2U^*$ and that the vectors $\nu(v_2^+)$ and $\nu(v_2^-)$ are highest weight vectors of the G-modules $\nu(E_2^+)$ and $\nu(E_2^-)$, respectively.

We now suppose that $n = 3$ and we consider the vectors

$$w_1^+ = (\Phi_2 \otimes \Phi_1)\mu_2(v_1^+), \qquad w_1^- = (\Phi_2 \otimes \Phi_1)\mu_2(v_1^-)$$

of $\mathcal{A} \otimes \mathcal{H}$ and the vectors

$$w_2^+ = (\Phi_2 \otimes \Phi_1 \otimes \Phi_1)\nu(v_1^+), \qquad w_1^- = (\Phi_2 \otimes \Phi_1 \otimes \Phi_1)\nu(v_1^-)$$

of $\mathcal{A} \otimes \mathcal{H} \otimes \mathcal{H}$. Then we see that w_1^+ and w_1^- are highest weight vectors of the irreducible G-submodules

$$M^+ = (\Phi_2 \otimes \Phi_1)\mu_2(E_1^+), \qquad M^- = (\Phi_2 \otimes \Phi_1)\mu_2(E_1^-)$$

of $\mathcal{A} \otimes \mathcal{H}$, whose highest weights are ϖ_1^+ and ϖ_1^-, respectively, and that w_2^+ and w_2^- are highest weight vectors of the irreducible G-submodules

$$N^+ = (\Phi_2 \otimes \Phi_1 \otimes \Phi_1)\nu(E_2^+), \qquad N^- = (\Phi_2 \otimes \Phi_1 \otimes \Phi_1)\nu(E_2^-)$$

of $\mathcal{A} \otimes \mathcal{H} \otimes \mathcal{H}$, whose highest weights are ϖ_2^+ and ϖ_2^-, respectively. According to the relations (5.16) and the expressions for $\mu_2(v_1^+)$ and $\mu_2(v_1^-)$ given above, the elements

$$\psi^+ = \chi_2(w_1^+), \qquad \psi^- = \chi_2(w_1^-)$$

of $C^\infty(T^*_{\mathbb{C}})^{\mathrm{odd}}$ are equal to the 1-forms on X induced by the $SO(n)$-invariant 1-forms

$$\begin{aligned}
&\langle Z_1, Z_1\rangle \vartheta_{23} - \langle Z_1, Z_2\rangle \vartheta_{13} + \langle Z_1, Z_3\rangle \vartheta_{12},\\
&\langle Z_1, Z_1\rangle \vartheta_{2\bar 3} - \langle Z_1, Z_2\rangle \vartheta_{1\bar 3} + \langle Z_1, \bar Z_3\rangle \vartheta_{12}
\end{aligned}$$

on $S_{n,n}$, respectively. If the forms ψ^+ and ψ^- are non-zero, they are highest weight vectors of irreducible G-submodules $\chi_2(M^+)$ and $\chi_2(M^-)$ of $C^\infty(T^*_{\mathbb{C}})^{\mathrm{odd}}$ whose highest weights are ϖ_1^+ and ϖ_1^-, respectively. From the relations (5.16) and the expressions for $\nu(v_2^+)$ and $\nu(v_2^-)$ given above we infer that the elements

$$\varphi^+ = \chi_3(w_1^+), \qquad \varphi^- = \chi_3(w_1^-)$$

of $C^\infty(T^*_{\mathbb{C}})^{\mathrm{ev}}$ are equal to the 1-forms on X induced by the $SO(n)$-invariant 1-forms

$$(\langle Z_1, Z_1\rangle\langle Z_2, Z_3\rangle - \langle Z_1, Z_2\rangle\langle Z_1, Z_3\rangle)\vartheta_{12} - f_2\vartheta_{13},$$
$$(\langle Z_1, Z_1\rangle\langle Z_2, \bar{Z}_3\rangle - \langle Z_1, Z_2\rangle\langle Z_1, \bar{Z}_3\rangle)\vartheta_{12} - f_2\vartheta_{1\bar{3}}$$

on $S_{n,n}$, respectively. If the forms φ^+ and φ^- are non-zero, they are highest weight vectors of irreducible G-submodules $\chi_3(N^+)$ and $\chi_3(N^-)$ of $C^\infty(T^*_{\mathbb{C}})^{\mathrm{ev}}$ whose highest weights are ϖ_2^+ and ϖ_2^-, respectively.

6. Isospectral deformations of the real Grassmannian of 3-planes in $\mathbb{R}^6$

We now consider the real Grassmannian $X = \widetilde{G}^{\mathbb{R}}_{3,3}$, the symmetric space $Y = \bar{G}^{\mathbb{R}}_{3,3}$, and the group $G = SO(6)$ and its dual Γ. We also consider the monomorphisms $\sigma : T \to S^2T^*$ and $\sigma_Y : T_Y \to S^2T^*_Y$ induced by the symmetric 3-forms σ and σ_Y; they determine the monomorphisms

$$* : T^* \longrightarrow S^2T^*, \qquad * : T^*_Y \longrightarrow S^2T^*_Y$$

given by (3.8) and (3.9), with $n = 3$. In this case, by (5.8) and (5.14) we have

$$\Psi^*\tilde{f}_1 = -\tilde{f}_1, \qquad \Psi^*\tilde{f}_2 = \tilde{f}_2, \qquad \Psi^*\tilde{f}_3 = i\tilde{f}_3, \qquad \Psi^*\tilde{f}'_3 = -i\tilde{f}'_3, \tag{6.1}$$

$$\tilde{f}_2 = \tilde{f}_3 \cdot \tilde{f}'_3. \tag{6.2}$$

Most of this section is devoted to the proof of the following proposition:

Proposition 6.1. — *We have*

$$D_0 C^\infty(T) \cap *dC^\infty_{\mathbb{R}}(X) = *d\mathcal{B}_{\mathbb{R}}.$$

If P denotes the orthogonal projection corresponding to the decomposition (1.3) on the space Y, according to Lemmas 4.1 and 1.1 the mapping

$$P * d : C^\infty_{\mathbb{R}}(Y) \longrightarrow I(Y) \tag{6.3}$$

is well-defined. We denote by $\mathcal{F}_Y$ the orthogonal complement of the finite-dimensional space $\mathcal{F}'_Y = \mathbb{R}(Y) \oplus \mathcal{B}_Y$ in $C^\infty_{\mathbb{R}}(Y)$. From Propositions 1.2 and 6.1, we obtain:

Theorem 6.2. — *The symmetric space $Y = \bar{G}^{\mathbb{R}}_{3,3}$ is not rigid in the sense of Guillemin. If f is a non-zero element of $\mathcal{F}_Y$, then the symmetric 2-form $*df$ on Y satisfies the Guillemin condition and is not a Lie derivative of the metric. Moreover, the kernel of the mapping* (6.3) *is the finite-dimensional space $\mathcal{F}'_Y = \mathbb{R}(Y) \oplus \mathcal{B}_{Y,\mathbb{R}}$.*

According to remarks made in §5, we know that the dimension of the space $\mathcal{F}'_Y$ is equal to 85.

The remainder of this section is devoted to various results which are needed for the proof of Proposition 6.1.

If $r_1, r_2 \geq 0$ and s are integers, we consider the elements

$$\begin{aligned} \gamma^1_{r_1,r_2,s} &= (2r_1 + 2r_2 + |s|)\lambda_1 + (2r_2 + |s|)\lambda_2 + s\lambda_3, \\ \gamma^2_{r_1,r_2,s} &= (2r_1 + 2r_2 + |s| + 1)\lambda_1 + (2r_2 + |s| + 1)\lambda_2 + s\lambda_3, \\ \gamma^3_{r_1,r_2,s} &= (2r_1 + 2r_2 + |s| + 2)\lambda_1 + (2r_2 + |s| + 1)\lambda_2 + s\lambda_3, \\ \gamma^4_{r_1,r_2,s} &= (2r_1 + 2r_2 + |s| + 1)\lambda_1 + (2r_2 + |s|)\lambda_2 + s\lambda_3 \end{aligned} \tag{6.4}$$

of Γ. We note that, if γ is an arbitrary element of Γ, there exist integers $1 \leq j \leq 4$, and $r_1, r_2 \geq 0$ and s such that $\gamma = \gamma^j_{r_1,r_2,s}$.

The following proposition is a direct consequence of the results of §10 and Proposition 10.4.

Proposition 6.3. — *Let X be the Grassmannian $\widetilde{G}^{\mathbb{R}}_{3,3}$. The non-zero multiplicities of the $SO(6)$-modules $C^\infty_\gamma(T^*_{\mathbb{C}})$, with $\gamma \in \Gamma$, are given by the following table, where $r_1, r_2 \geq 0$ and s are integers and γ is an element of Γ:*

γ	Multiplicity
$\gamma^1_{r_1,r_2,s}$	3 if $r_1, r_2 \geq 1$ 2 if $r_1 = 0$ and $r_2 \geq 1$ 2 if $r_2 = 0$ and $r_1, \lvert s\rvert \geq 1$ 1 if $r_1 = r_2 = 0$ and $\lvert s\rvert \geq 1$ 1 if $r_2 = s = 0$ and $r_1 \geq 1$
$\gamma^2_{r_1,r_2,s}$	2 if $r_1 \geq 1$ 1 if $r_1 = 0$
$\gamma^3_{r_1,r_2,s}$	2
$\gamma^4_{r_1,r_2,s}$	2 if $r_2 \geq 1$ 1 if $r_2 = 0$ and $\lvert s\rvert \geq 1$

Let $r_1, r_2 \geq 0$ and s be given integers. We consider the sections

$$\varphi_1 = \tilde{f}_{r_1-1,r_2,s}\, d\tilde{f}_1, \qquad \varphi_2 = \tilde{f}_{r_1,r_2-1,s}\, d\tilde{f}_2,$$
$$\varphi_3 = \tilde{f}_{r_1,r_2,s-1}\, d\tilde{f}_3, \qquad \varphi_4 = \tilde{f}_{r_1,r_2,s+1}\, d\tilde{f}'_3,$$
$$\varphi_5 = \tilde{f}_{r_1,r_2-1,s+1}\, d\tilde{f}'_3$$

of $T^*_{\mathbb{C}}$, and the subspace $V_{r_1,r_2,s}$ of $C^\infty(T^*_{\mathbb{C}})$ which is generated (over $\mathbb{C}$) by the 1-forms $\varphi_1, \varphi_2, \varphi_3$ whenever $s \geq 1$, by the 1-forms $\varphi_1, \varphi_2, \varphi_4$ whenever $s \leq -1$, and by the 1-forms $\varphi_1, \varphi_2, \varphi_5$ whenever $s = 0$. Clearly, we have $V_{r_1,r_2,s} = \{0\}$ when $r_1 = r_2 = s = 0$.

By (6.2), we may consider the 1-form

$$\beta = \tilde{f}_3' d\tilde{f}_3 - \tilde{f}_3 d\tilde{f}_3' = d\tilde{f}_2 - 2\tilde{f}_3 d\tilde{f}_3',$$

and we have the relations

$$\tilde{f}_3 d\tilde{f}_2 - \tilde{f}_2 d\tilde{f}_3 = \tilde{f}_3^2 d\tilde{f}_3', \qquad \tilde{f}_3' d\tilde{f}_2 - \tilde{f}_2 d\tilde{f}_3' = \tilde{f}_3'^2 d\tilde{f}_3. \tag{6.5}$$

Thus when $r_1 = 0$ and $r_2 = s = 1$, we have the equality

$$\varphi_5 = \varphi_2 - \varphi_3.$$

LEMMA 6.4. — *Let $r_1, r_2 \geq 0$ and s be given integers, and suppose that*

$$r_1 + r_2 + |s| > 0.$$

(i) *The non-zero generators of the vector space $V_{r_1,r_2,s}$ form a basis of $V_{r_1,r_2,s}$. More precisely, the dimension and a basis of $V_{r_1,r_2,s}$ are given by the following table:*

Conditions on r_1, r_2 and s	dim $V_{r_1,r_2,s}$	Basis of $V_{r_1,r_2,s}$
$r_1, r_2, s \geq 1$	3	$\varphi_1, \varphi_2, \varphi_3$
$r_1 = 0,\ r_2, s \geq 1$	2	φ_2, φ_3
$r_2 = 0,\ r_1, s \geq 1$	2	φ_1, φ_3
$r_1 = r_2 = 0,\ s \geq 1$	1	φ_3
$s = 0,\ r_1, r_2 \geq 1$	3	$\varphi_1, \varphi_2, \varphi_5$
$r_1 = s = 0,\ r_2 \geq 1$	2	φ_2, φ_5
$r_2 = s = 0,\ r_1 \geq 1$	1	φ_1
$r_1, r_2 \geq 1,\ s \leq -1$	3	$\varphi_1, \varphi_2, \varphi_4$
$r_1 = 0,\ r_2 \geq 1,\ s \leq -1$	2	φ_2, φ_4
$r_2 = 0,\ r_1 \geq 1,\ s \leq -1$	2	φ_1, φ_4
$r_1 = r_2 = 0,\ s \leq -1$	1	φ_4

(ii) *When $s \neq 0$, or when $r_2 = s = 0$, an element φ of $V_{r_1,r_2,s}$ satisfying*

$$(D^1\varphi)(\zeta_j, \zeta_j) = 0,$$

for $j = 1, 2, 3$, vanishes identically.

Proof. — Let $r_1, r_2 \geq 0$ and s be given integers, and let a_1, a_2, a_3 be given complex numbers. We assume that $a_1 = 0$ whenever $r_1 = 0$, and that $a_2 = 0$ whenever $r_2 = 0$. We consider the function ψ on $\mathbb{R}^3$ defined by

$$\psi(\theta) = i \exp i((2r_1 + 2r_2 + |s|)\theta_1 + (2r_2 + |s|)\theta_2 + s\theta_3),$$

for $\theta \in \mathbb{R}^3$, and the parallel 1-forms

$$\begin{aligned}\Theta &= (2a_1 + 2a_2 + a_3)d\theta_1 + (2a_2 + a_3)d\theta_2 + a_3 d\theta_3,\\ \Theta' &= (2a_1 + 2a_2 + a_3)d\theta_1 + (2a_2 + a_3)d\theta_2 - a_3 d\theta_3\end{aligned}$$

on $\mathbb{R}^3$. We remark that the function $\partial\psi/\partial\theta_1$ is everywhere non-vanishing on $\mathbb{R}^3$; moreover when $s \neq 0$, the functions $\partial\psi/\partial\theta_2$ and $\partial\psi/\partial\theta_3$ are everywhere non-vanishing on $\mathbb{R}^3$. We first suppose that $s \geq 1$ and we consider the 1-form

$$\varphi = a_1 \tilde{f}_{r_1-1,r_2,s} d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,s} d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2,s-1} d\tilde{f}_3$$

belonging to $V_{r_1,r_2,s}$. If $\iota : \mathbb{R}^3 \to S_{3,3}$ is the mapping defined in §4, by the formulas (4.9) and (5.12) we obtain the relation

$$\iota'^* \varphi = \psi\Theta$$

among 1-forms on $\mathbb{R}^3$. According to formula (4.12), the relations

$$(D^1\varphi)(\zeta_j, \zeta_j) = 0,$$

with $j = 1, 2, 3$, imply that $\Theta = 0$, and we infer that the coefficients a_1, a_2, a_3 all vanish. Next, we suppose that $s \leq -1$ and we consider the 1-form

$$\varphi' = a_1 \tilde{f}_{r_1-1,r_2,s} d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,s} d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2,s+1} d\tilde{f}'_3$$

belonging to $V_{r_1,r_2,s}$. By formulas (4.9) and (5.12), we have the relation

$$\iota'^* \varphi' = \psi\Theta'$$

among 1-forms on $\mathbb{R}^3$. According to formula (4.12), the relations

$$(D^1\varphi')(\zeta_j, \zeta_j) = 0,$$

with $j = 1, 2, 3$, imply that $\Theta' = 0$, and we infer that the coefficients a_1, a_2, a_3 all vanish. Finally, we suppose that $s = 0$, that $r_1 + r_2 > 0$ and we also assume that $a_3 = 0$ whenever $r_2 = 0$. We consider the 1-form

$$\varphi'' = a_1 \tilde{f}_{r_1-1,r_2,0} d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,0} d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2-1,1} d\tilde{f}'_3$$

on X. By the formulas (4.9) and (5.12), we obtain the relation

$$\iota'^* \varphi'' = \psi\Theta' \tag{6.6}$$

among 1-forms on $\mathbb{R}^3$. Thus if the 1-form φ'' vanishes, we infer that the coefficients a_1, a_2, a_3 all vanish. Since the function $\partial\psi/\partial\theta_1$ is everywhere non-vanishing, according to formulas (4.12) and (6.6) the relation

$$(D^1\varphi'')(\zeta_1, \zeta_1) = 0$$

implies that $2a_1 + 2a_2 + a_3 = 0$. Thus if $r_2 = 0$, we see that the coefficient a_1 vanishes. The two assertions of the lemma are now direct consequences of the above argument. □

When $r_2 \geq 1$, the function $\partial\psi/\partial\theta_2$ is everywhere non-vanishing on $\mathbb{R}^3$, and so the relations

$$(D^1\varphi'')(\zeta_j, \zeta_j) = 0,$$

with $j = 1, 2$, imply that $a_1 = 0$ and $2a_2 + a_3 = 0$; then we see that $\Theta' = 2a_2 d\theta_3$. Moreover, by (4.12) we have

$$\iota'^*(D^1\varphi'')(\zeta_1, \zeta_3) = a_2 \frac{\partial\psi}{\partial\theta_1} = 2ia_2(r_1 + r_2)\psi,$$
$$\iota'^*(D^1\varphi'')(\zeta_2, \zeta_3) = a_2 \frac{\partial\psi}{\partial\theta_2} = 2ia_2 r_2\psi.$$

Now using the formulas (4.9) and (5.12), we verify that

$$\iota'^* d\tilde{f}_{r_1,r_2,0} = 2\psi((r_1 + r_2)d\theta_1 + r_2 d\theta_2).$$

If $D^1\varphi'' = c * d\tilde{f}_{r_1,r_2,0}$, with $c \in \mathbb{C}$, then by formulas (4.9) and (4.14) we obtain the relations

$$\iota'^*(D^1\varphi'')(\zeta_1, \zeta_3) = c\langle \partial/\partial\theta_2, \iota'^* d\tilde{f}_{r_1,r_2,0}\rangle = 2cr_2\psi,$$
$$\iota'^*(D^1\varphi'')(\zeta_2, \zeta_3) = c\langle \partial/\partial\theta_1, \iota'^* d\tilde{f}_{r_1,r_2,0}\rangle = 2c(r_1 + r_2)\psi.$$

From the preceding the equalities, we deduce that

$$ia_2 = c\frac{r_2}{r_1 + r_2} = c\frac{r_1 + r_2}{r_2}.$$

Hence when $r_1 > 0$, we infer that $c = 0$ and $a_2 = 0$, and so we see that $\varphi'' = 0$. On the other hand, when $r_1 = 0$, by (6.2) we know that

$$\varphi'' = a_2 \tilde{f}_2^{r_2 - 1}\beta, \qquad c = ia_2.$$

Thus we proved the following result:

LEMMA 6.5. — *Let $r_1 \geq 0$ and $r_2 \geq 1$ be given integers. Let φ be an element of $V_{r_1,r_2,0}$ satisfying*

$$D^1\varphi = c * d\tilde{f}_{r_1,r_2,0},$$

with $c \in \mathbb{C}$. Then:

(i) *If $r_1 \geq 1$, the element φ vanishes identically.*

(ii) *If $r_1 = 0$, there is an element $a \in \mathbb{C}$ such that*

$$\varphi = a\tilde{f}_2^{r_2-1}\beta, \qquad D^1\varphi = ia * d\tilde{f}_2^{r_2}.$$

We consider the matrix

$$A_0 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

of $O(3)$; we then consider the element $\phi_0 = (A_0, A_0)$ of the subgroup $S(O(3) \times O(3))$ of $G = SO(6)$.

The functions $\hat{f}_j = \iota^*\phi_0^* f_j$, with $j = 1,2,3$, and $\hat{f}'_1 = \iota^*\phi_0^* f'_1$ on $\mathbb{R}^3$ are given by

$$
\begin{aligned}
&\hat{f}_1(\theta) = \cos^2\theta_2 - \cos^2\theta_1, \qquad \hat{f}'_1(\theta) = i\cos\theta_1\cdot\sin\theta_1,\\
&\hat{f}_2(\theta) = \cos^2\theta_2\cdot(\sin^2\theta_1 - \sin^2\theta_3) + \cos^2\theta_1\cdot\sin^2\theta_3\\
&\qquad\quad = \cos^2\theta_1\cdot\sin^2\theta_2\cdot\sin^2\theta_3 + \sin^2\theta_1\cdot\cos^2\theta_2\cdot\cos^2\theta_3,\\
&\hat{f}_3(\theta) = \cos\theta_1\cdot\sin\theta_2\cdot\sin\theta_3 - i\sin\theta_1\cdot\cos\theta_2\cdot\cos\theta_3,
\end{aligned}
\tag{6.7}
$$

for $\theta \in \mathbb{R}^3$, and the function $\hat{f}'_3 = \iota^*\phi_0^* f'_3$ on $\mathbb{R}^3$ satisfies

$$
\hat{f}'_3 = \bar{\hat{f}}_3. \tag{6.8}
$$

Lemma 6.6. — *Let $r \geq 1$ be a given integer. Then the element $\varphi = \tilde{f}_2^r\beta$ of $V_{0,r+1,0}$ satisfies*

$$
(D^1\varphi)(\phi_{0*}\zeta_3, \phi_{0*}\zeta_3) \neq 0.
$$

Proof. — Using the formulas (6.7) and (6.8), we easily see that the equalities

$$
\begin{aligned}
\frac{\partial\hat{f}_2}{\partial\theta_3} &= 2\sin\theta_3\cdot\cos\theta_3\cdot(\cos^2\theta_1 - \cos^2\theta_2),\\
\langle\partial/\partial\theta_3, \iota'^*\phi_0^*\beta\rangle &= 2i\sin\theta_1\cdot\cos\theta_1\cdot\sin\theta_2\cdot\cos\theta_2
\end{aligned}
$$

hold on $\mathbb{R}^3$. According to formula (4.12), we have

$$
\iota'^*(D^1\phi_0^*\varphi)(\zeta_3,\zeta_3) = \frac{\partial}{\partial\theta_3}\langle\partial/\partial\theta_3, \iota'^*\phi_0^*\varphi\rangle = r\hat{f}_2^{r-1}\frac{\partial\hat{f}_2}{\partial\theta_3}\langle\partial/\partial\theta_3, \iota'^*\phi_0^*\beta\rangle.
$$

If $\theta = (\theta_1,\theta_2,\theta_3)$ is a point of $\mathbb{R}^3$ satisfying

$$
\cos\theta_j \neq 0, \qquad \sin\theta_j \neq 0, \qquad |\cos\theta_1| \neq |\cos\theta_2|,
$$

for $j = 1,2,3$, from the above relations and (6.7) we infer that the expression $(D^1\phi_0^*\varphi)(\zeta_3,\zeta_3)$ does not vanish at the point $\iota'(\theta)$ of X, and so we obtain the desired result. □

Lemma 6.7. — *Let r_1, r_2, s be integers satisfying*

$$
r_1, r_2 \geq 0, \qquad r_1 + r_2 + |s| > 0, \qquad (r_1,r_2,s) \neq (0,1,0). \tag{6.9}
$$

Let φ be an element of $V_{r_1,r_2,s}$ satisfying

$$
D^1\varphi = c * d\tilde{f}_{r_1,r_2,s},
$$

with $c \in \mathbb{C}$. Then we have $\varphi = 0$.

Proof. — According to our hypotheses, by (4.13) we know that

$$
(D^1\varphi)(\phi_*\zeta_j, \phi_*\zeta_j) = 0,
$$

for all $\phi \in G$ and $j = 1,2,3$. When $s \neq 0$, or when $r_2 = s = 0$, from Lemma 6.4,(ii) we infer that φ vanishes. Next, when $r_1, r_2 \geq 1$ and $s = 0$, Lemma 6.5,(i) tells us that φ vanishes. Finally, when $r_1 = s = 0$ and $r_2 \geq 2$, according to Lemma 6.5,(ii) we see that φ is a multiple of the 1-form $\tilde{f}_2^{r_2-1}\beta$; Lemma 6.6 then gives us the vanishing of φ. ◻

Let $r_1, r_2 \geq 0$ and s be given integers and consider the element $\gamma = \gamma^1_{r_1,r_2,s}$ of Γ. Let γ' be an arbitrary element of Γ, and let u be a highest weight vector of the isotypic component $C^\infty_{\gamma'}(S^pT^*_{\mathbb{C}})$. Since u is a real-analytic section of $S^pT^*_{\mathbb{C}}$ (see §7, Chapter II of [**6**]), the section $\tilde{f}_{r_1,r_2,s}u$ of $S^pT^*_{\mathbb{C}}$ is non-zero. Since the function $\tilde{f}_{r_1,r_2,s}$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, it follows that the section $\tilde{f}_{r_1,r_2,s}u$ is a highest weight vector of the isotypic component $C^\infty_{\gamma''}(S^pT^*_{\mathbb{C}})$, where $\gamma'' = \gamma' + \gamma^1_{r_1,r_2,s}$.

According to the preceding observation, we see that all the non-zero vectors of the space $V_{r_1,r_2,s}$ are highest weight vectors of the G-module $C^\infty_\gamma(T^*_{\mathbb{C}})$. From Lemma 6.4 and Proposition 6.3, we deduce the following result:

Lemma 6.8. — *Let $r_1, r_2 \geq 0$ and s be given integers and let γ be the element $\gamma^1_{r_1,r_2,s}$ of Γ. Then the weight subspace $\mathcal{W}_\gamma$ of $C^\infty_\gamma(T^*_{\mathbb{C}})$ is equal to $V_{r_1,r_2,s}$.*

By (4.9), (4.14) and (5.12), we see that the relations

$$\begin{aligned} \iota'^*(\mathrm{Hess}\,\tilde{f}_3)(\zeta_1,\zeta_2) &= \iota'^*(\mathrm{Hess}\,\tilde{f}'_3)(\zeta_1,\zeta_2) = -1,\\ \iota'^*(*d\tilde{f}_3)(\zeta_1,\zeta_2) &= \iota'^*\langle\zeta_3, d\tilde{f}_3\rangle = \langle\partial/\partial\theta_3, \iota'^*d\tilde{f}_3\rangle = i,\\ \iota'^*(*d\tilde{f}'_3)(\zeta_1,\zeta_2) &= \iota'^*\langle\zeta_3, d\tilde{f}'_3\rangle = \langle\partial/\partial\theta_3, \iota'^*d\tilde{f}'_3\rangle = -i. \end{aligned} \tag{6.10}$$

hold at the point 0 of $\mathbb{R}^3$. We consider the sections

$$h_1 = \mathrm{Hess}\,\tilde{f}_3 + \tilde{f}_3 g, \qquad h_2 = \mathrm{Hess}\,\tilde{f}'_3 + \tilde{f}'_3 g$$

of $T^*_{\mathbb{C}}$ and the elements $\gamma_1 = \gamma^1_{0,0,1}$ and $\gamma_2 = \gamma^1_{0,0,-1}$ of Γ. By (6.10), we know that the relations

$$\begin{aligned} \iota'^*h_1(\zeta_1,\zeta_2) &= \iota'^*(*id\tilde{f}_3)(\zeta_1,\zeta_2) = -1,\\ \iota'^*h_2(\zeta_1,\zeta_2) &= -\iota'^*(*id\tilde{f}'_3)(\zeta_1,\zeta_2) = -1, \end{aligned} \tag{6.11}$$

hold at $0 \in \mathbb{R}^3$. In §5, we saw that the functions $\tilde{f}_3$ and $\tilde{f}'_3$ are highest weight vectors of the irreducible G-modules $C^\infty_{\gamma_1}(X)$ and $C^\infty_{\gamma_2}(X)$, respectively. Therefore, since the differential operator Hess is homogeneous, the sections h_1 and h_2 are highest weight vectors of the G-modules $C^\infty_{\gamma_1}(S^2T^*_{\mathbb{C}})$ and $C^\infty_{\gamma_2}(S^2T^*_{\mathbb{C}})$, respectively.

Lemma 6.9. — (i) *We have the relations*

$$\mathrm{Hess}\,\tilde{f}_3 = -\tilde{f}_3 + *id\tilde{f}_3, \qquad \mathrm{Hess}\,\tilde{f}'_3 = -\tilde{f}'_3 - *id\tilde{f}'_3, \tag{6.12}$$

$$D^1\beta = *id\tilde{f}_2. \tag{6.13}$$

(ii) *For $\gamma = \gamma^1_{0,1,0}$, we have the inclusion*

$$*dC^\infty_\gamma(X) \subset D^1C^\infty(T^*_{\mathbb{C}}). \tag{6.14}$$

Proof. — According to (5.6), we see that

$$h_1(v \otimes w, v \otimes w) = h_2(v \otimes w, v \otimes w) = 0,$$

for all $v \in V$ and $w \in W$. From the decomposition (3.4), we infer that h_1 and h_2 are sections of the sub-bundle $(\bigwedge^2 V^* \otimes \bigwedge^2 W^*)_{\mathbb{C}}$ of $S^2 T^*_{\mathbb{C}}$. Since the mapping $* : T^* \to \bigwedge^2 V^* \otimes \bigwedge^2 W^*$ is an isomorphism, there exist unique sections α_1 and α_2 of $T^*_{\mathbb{C}}$ such that

$$h_1 = *\alpha_1, \qquad h_2 = *\alpha_2;$$

it follows that α_1 and α_2 are highest weight vectors of the G-modules $C^\infty_{\gamma_1}(T^*_{\mathbb{C}})$ and $C^\infty_{\gamma_2}(T^*_{\mathbb{C}})$, respectively. By Lemma 6.8, there are constants $a, b \in \mathbb{C}$ such that $\alpha_1 = ad\tilde{f}_3$ and $\alpha_1 = bd\tilde{f}'_3$. Then according to (6.11), we see that $a = i$ and $b = -i$, and so we obtain the equalities (6.12). By (1.2), we have

$$D^1\beta = \tilde{f}'_3 \operatorname{Hess} \tilde{f}_3 - \tilde{f}_3 \operatorname{Hess} \tilde{f}'_3;$$

the relation (6.13) is now a direct consequence of (6.12) and (6.2). As $\tilde{f}_2$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, with $\gamma = \gamma^1_{0,1,0}$, the inclusion (6.14) is a direct consequence of the identity (6.13). □

In order to prove Proposition 6.1, by formula (1.1) it suffices to show that

$$D^1 C^\infty(T^*_{\mathbb{C}}) \cap *dC^\infty(X) = *d\mathcal{B}.$$

Since the differential operators D^1 and $*d$ are homogeneous, according to Proposition 2.1 of [**6**] and Lemma 6.9,(ii) the preceding equality holds if and only if

$$D^1 C^\infty_\gamma(T^*_{\mathbb{C}}) \cap *dC^\infty_\gamma(X) = \{0\}, \tag{6.15}$$

for all $\gamma \in \Gamma$, with $\gamma \neq \gamma^1_{0,1,0}$. We now proceed to verify that (6.15) holds and, in the process, complete the proof of Proposition 6.1.

If $\gamma \in \Gamma$ is not of the form $\gamma^1_{r_1,r_2,s}$, where r_1, r_2, s are integers satisfying $r_1, r_2 \geq 0$ and $r_1 + r_2 + |s| > 0$, we know that $dC^\infty_\gamma(X) = \{0\}$, and so the equality (6.15) holds. Now let r_1, r_2, s be given integers satisfying (6.9). We consider the element $\gamma = \gamma^1_{r_1,r_2,s}$ of Γ, and suppose that the G-module M equal to the right-hand side of the equality (6.15) is non-zero. Then by Lemma 6.8, a highest weight vector h of M belongs to $D^1 V_{r_1,r_2,s}$. On the other hand, since the function $\tilde{f}_{r_1,r_2,s}$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, the symmetric 2-form h is a constant multiple of the highest weight vector $*d\tilde{f}_{r_1,r_2,s}$ of the irreducible G-module $*dC^\infty_\gamma(X)$. Lemma 6.7 tells us that h vanishes, which leads us to a contradiction; therefore the equality (6.15) holds, and so we have proved Proposition 6.1.

7. Forms of degree one

We pursue our study of the real Grassmannian $X = \widetilde{G}^{\mathbb{R}}_{3,3}$ and consider the objects which we associated with X and the group $G = SO(6)$ in §6. Here we present results on forms of degree one on X which are needed for the proof of our criterion for exactness of 1-forms on Y given by Theorem 9.1.

Let p, q be given integers; we remark that q is even and $p + q/2$ is even (resp. is odd) if and only if $2p + q \equiv 0 \bmod 4$ (resp. $2p + q \equiv 2 \bmod 4$), and we also note that q is odd and $p + (q+1)/2$ is even (resp. is odd) if and only if $2p + q \equiv 3 \bmod 4$ (resp. $2p + q \equiv 1 \bmod 4$).

Let $r_1, r_2 \geq 0$ and s be given integers. We now consider the element $\gamma = \gamma^1_{r_1,r_2,s}$ of Γ. We know that the inclusion (5.1), holds, with $\varepsilon = (-1)^s$. Since the function $\tilde{f}_{r_1,r_2,s}$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(X)$, according to (6.1) we have the equalities

$$C^\infty_\gamma(X) = \begin{cases} C^\infty_\gamma(X)^{\mathrm{ev}} & \text{if } 2r_1 + s \equiv 0 \bmod 4, \\ C^\infty_\gamma(X)^{\mathrm{odd}} & \text{if } 2r_1 + s \equiv 2 \bmod 4. \end{cases} \tag{7.1}$$

According to (5.7), we see that a non-zero 1-form on X equal to one of the forms φ_j, which we associated in §6 with the integers r_1, r_2, s, belongs to $C^\infty(T^*_{\mathbb{C}})^{\tau,\varepsilon}$, where $\varepsilon = (-1)^s$; according to (6.1), such a 1-form on X belongs to $C^\infty(T^*_{\mathbb{C}})^{\Sigma}$ (resp. to $C^\infty(T^*_{\mathbb{C}})^{\mathrm{odd}}$) if and only if $2r_1 + 3s \equiv 0 \bmod 4$ (resp. $2r_1 + 3s \equiv 2 \bmod 4$). Thus we have the inclusion

$$V_{r_1,r_2,s} \subset C^\infty(T^*_{\mathbb{C}})^{\tau,\varepsilon},$$

where $\varepsilon = (-1)^s$. If $r_1 + r_2 + |s| > 0$, we know that the inclusion

$$V_{r_1,r_2,s} \subset C^\infty(T^*_{\mathbb{C}})^{\Sigma}$$

holds whenever $2r_1 + 3s \equiv 0 \bmod 4$, and that the inclusion

$$V_{r_1,r_2,s} \subset C^\infty(T^*_{\mathbb{C}})^{\mathrm{odd}}$$

holds whenever $2r_1 + 3s \equiv 2 \bmod 4$. Proposition 6.3 tells us that

$$C^\infty_\gamma(T^*_{\mathbb{C}}) = \{0\}$$

when $\gamma = 0$. According to Lemma 6.9, we have the inclusion

$$C^\infty_\gamma(T^*_{\mathbb{C}}) \subset C^\infty(T^*_{\mathbb{C}})^{\tau,\varepsilon}, \tag{7.2}$$

with $\varepsilon = (-1)^s$. If $r_1 + r_2 + |s| > 0$, we know that

$$C^\infty_\gamma(T^*_{\mathbb{C}}) = \begin{cases} C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}} & \text{if } 2r_1 + 3s \equiv 0 \bmod 4, \\ C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{odd}} & \text{if } 2r_1 + 3s \equiv 2 \bmod 4. \end{cases} \tag{7.3}$$

We consider the element ϕ_0 of G defined in §6 and verify that

$$\begin{gathered} \iota^*\phi_0^*\langle Z_1, Z_3\rangle = \iota^*\phi_0^*\langle Z_1, \bar{Z}_3\rangle = \cos\theta_2 \cdot \sin\theta_2, \\ \iota^*\phi_0^*\langle Z_2, \bar{Z}_3\rangle = -\iota^*\phi_0^*\langle Z_2, \bar{Z}_3\rangle = \cos\theta_3 \cdot \sin\theta_3, \\ \iota^*\phi_0^*\vartheta_{12} = id\theta_1, \qquad \iota^*\phi_0^*\vartheta_{13} = \iota^*\phi_0^*\vartheta_{1\bar{3}} = d\theta_2, \\ \iota^*\phi_0^*\vartheta_{23} = -\iota^*\phi_0^*\vartheta_{2\bar{3}} = d\theta_3. \end{gathered} \tag{7.4}$$

Since the 1-form $g^\flat(\xi_0)$ is equal to $\tilde\vartheta_{12}$, according to (4.11) we see that

$$\iota'^*\phi_0^* g^\flat(\xi_0) = id\theta_1. \tag{7.5}$$

We consider the 1-forms φ^+, φ^-, ψ^+ and ψ^- on X which are defined in §5. We shall also consider the 1-form $\omega = \tilde f_1 d\tilde f_1' - \tilde f_1' d\tilde f_1$, which is the image of $f_1 \wedge f_1'$ under the mapping $\chi_1 : \bigwedge^2\mathcal{H} \to C^\infty(T^*_{\mathbb{C}})^{\mathrm{ev}}$.

Lemma 7.1. — *The sections ω, φ^+, φ^-, ψ^+ and ψ^- of $T^*_{\mathbb{C}}$ are non-zero.*

Proof. — Thus, by (4.11), (6.7) and (7.4) we see that the equalities

$$\begin{aligned}
\langle \partial/\partial\theta_1, \iota'^*\phi_0^*\omega\rangle &= i(2\cos^2\theta_1\cdot\cos^2\theta_2 - \cos^2\theta_1 - \cos^2\theta_2)\\
&= 2i(\cos^2\theta_1\cdot\sin^2\theta_2 + \sin^2\theta_1\cdot\cos^2\theta_2),\\
\langle \partial/\partial\theta_2, \iota'^*\phi_0^*\varphi^+\rangle &= \langle \partial/\partial\theta_2, \iota'^*\phi_0^*\varphi^-\rangle = -\hat f_2,\\
\langle \partial/\partial\theta_3, \iota'^*\phi_0^*\psi^+\rangle &= -\langle \partial/\partial\theta_3, \iota'^*\phi_0^*\psi^-\rangle = \hat f_1
\end{aligned} \tag{7.6}$$

hold on $\mathbb{R}^3$; since the functions $\hat f_1$ and $\hat f_2$ are given by (6.7), we see that the lemma is an immediate consequence of the preceding formulas. □

From Lemma 7.1 and comments made in §5, it follows that ω is a highest weight vector of the G-module $C^\infty_{\gamma^2_{1,0,0}}(T^*_{\mathbb{C}})^{\mathrm{ev}}$. Now let $r_1, r_2 \geq 0$ and s be given integers and consider the element $\gamma = \gamma^2_{r_1,r_2,s}$ of Γ. We consider the sections

$$\varphi_6 = \tilde f_{r_1,r_2,s} g^\flat(\xi_0), \qquad \varphi_7 = \tilde f_{r_1-1,r_2,s}\omega$$

of $T^*_{\mathbb{C}}$. According to the remark preceding Lemma 6.9, when $r_1 \geq 0$ (resp. when $r_1 \geq 1$) we know that φ_6 (resp. φ_7) is a highest weight vector of the G-module $C^\infty_\gamma(T^*_{\mathbb{C}})$. Since $g^\flat(\xi_0)$ and ω are elements of $C^\infty(T^*_{\mathbb{C}})^{\mathrm{ev}}$, if φ is a section of $T^*_{\mathbb{C}}$ equal to φ_6 or φ_7, according to (5.7) we see that

$$\tau^*\varphi = (-1)^s\varphi; \tag{7.7}$$

moreover, by (6.1) we also have the relations

$$\begin{aligned}
\Psi^*\varphi_8 &= \begin{cases} (-1)^{r_1} i^s \varphi_8 & \text{when } s \geq 0,\\ (-1)^{r_1+|s|} i^{|s|}\varphi_8 & \text{when } s < 0,\end{cases}\\
\Psi^*\varphi_9 &= \begin{cases} (-1)^{r_1-1} i^s \varphi_9 & \text{when } s \geq 0,\\ (-1)^{r_1+|s|-1} i^{|s|}\varphi_9 & \text{when } s < 0.\end{cases}
\end{aligned} \tag{7.8}$$

If $r_1 \geq 1$, from these relations we deduce the linear independence of the 1-forms φ_8 and φ_9.

Lemma 7.2. — *Let $r_1, r_2 \geq 0$ and s be given integers and let γ be the element $\gamma^2_{r_1,r_2,s}$ of Γ.*

(i) *If $r_1 = 0$, the section $\tilde f_{0,r_2,s} g^\flat(\xi_0)$ is a highest weight vector of the irreducible G-module $C^\infty_\gamma(T^*_{\mathbb{C}})$.*

(ii) *If* $r_1 \geq 1$, *the section* $\tilde{f}_{r_1,r_2,s}g^\flat(\xi_0)$ *and* $\tilde{f}_{r_1-1,r_2,s}\omega$ *form a basis of the space* $\mathcal{W}_\gamma$.

(iii) *We have inclusion*

$$C^\infty_\gamma(T^*_{\mathbb{C}}) \subset C^\infty(T^*_{\mathbb{C}})^{\tau,\varepsilon},$$

with $\varepsilon = (-1)^s$.

(iv) *Suppose that* $r_1 = 0$ *and that* s *is even. Then the relation* (7.3) *holds, with* $\gamma = \gamma^2_{0,r_2,s}$.

(v) *Suppose that* $r_1 \geq 1$ *and that* s *is even. Then the* G*-modules* $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$ *and* $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{odd}}$ *are irreducible. When* $2r_1 + s \equiv 0 \bmod 4$ (*resp.* $2r_1 + s \equiv 2 \bmod 4$), *the section* $\tilde{f}_{r_1,r_2,s}g^\flat(\xi_0)$ (*resp.* $\tilde{f}_{r_1-1,r_2,s}\omega$) *is a highest weight vector of the* G*-module* $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$.

Proof. — From the observations preceding the lemma and from Proposition 6.3, we deduce assertions (i) and (ii). Assertion (iii) is a direct consequence of (i) and (ii) and the relation (7.7). Assertions (iv)–(v) follow directly from the preceding assertions and the relations (7.8). ☐

According to Lemma 7.1 and the discussion which appears at the end of §5, we see that φ^+ and φ^- are highest weight vectors of the G-modules $C^\infty_{\gamma^3_{0,0,1}}(T^*_{\mathbb{C}})^{\mathrm{ev}}$ and $C^\infty_{\gamma^3_{0,0,-1}}(T^*_{\mathbb{C}})^{\mathrm{ev}}$, respectively. For $p \geq 0$, we consider the space $\mathcal{U}_p = \mathcal{W}_\gamma$, where $\gamma = \gamma^3_{0,p,0}$; by Proposition 6.3, we know that this space is two-dimensional. By the remark preceding Lemma 6.9, the sections $u_1 = \tilde{f}'_3\varphi^+$ and $u_2 = \tilde{f}_3\varphi^-$ of $T^*_{\mathbb{C}}$ are elements of $\mathcal{U}_1$. From (5.7) and (6.1), we obtain the relations

$$\tau^* u_j = -u_j, \qquad \Psi^* u_1 = -iu_1, \qquad \Psi^* u_2 = iu_2, \tag{7.9}$$

for $j = 1, 2$; therefore the sections u_1 and u_2 are linearly independent and so form a basis of the space $\mathcal{U}_1$. The endomorphism of $C^\infty(T^*_{\mathbb{C}})$, which sends u into $\tilde{f}_2 u$, induces an isomorphism $\mathcal{U}_0 \to \mathcal{U}_1$. By (5.7) and (6.1), the latter mapping commutes with the automorphisms τ^* and Ψ^* of the spaces $\mathcal{U}_p$, with $p = 0, 1$, induced by τ and Ψ. By (7.9), we see that $\tau^* u = -u$, for all $u \in \mathcal{U}_0$, and so the automorphism Ψ^* of $\mathcal{U}_0$ satisfies $\Psi^{*2} = -\mathrm{id}$. Hence by (7.9) we may choose non-zero elements φ'_0 and φ''_0 of $\mathcal{U}_0$ satisfying $\Psi^*\varphi'_0 = -i\varphi'_0$ and $\Psi^*\varphi''_0 = i\varphi''_0$. Therefore there exist non-zero constants $c', c'' \in \mathbb{C}$ such that $\tilde{f}_2\varphi'_0 = c' u_1$ and $\tilde{f}_2\varphi''_0 = c'' u_2$. Then the sections $\varphi^+_0 = (1/c') \cdot \varphi'_0$ and $\varphi^-_0 = (1/c'') \cdot \varphi''_0$ of $T^*_{\mathbb{C}}$ form a basis of $\mathcal{U}_0$ and satisfy the relations given by the following lemma:

Lemma 7.3. — *There exist sections* φ^+_0 *and* φ^-_0 *of* $T^*_{\mathbb{C}}$ *such that*

$$\varphi^+ = \tilde{f}_3\varphi^+_0, \qquad \varphi^- = \tilde{f}'_3\varphi^-_0.$$

Clearly, by (6.1) and (7.9) the vectors φ^+_0 and φ^-_0 of $C^\infty_{\gamma^3_{0,0,0}}(T^*_{\mathbb{C}})$ belong to the space $C^\infty(T^*_{\mathbb{C}})^{\tau,-1}$ and satisfy the relations

$$\Psi^*\varphi^+_0 = -i\varphi^+_0, \qquad \Psi^*\varphi^-_0 = i\varphi^-_0. \tag{7.10}$$

By (7.6) and (6.2), we have

$$\langle \partial/\partial\theta_2, \iota'^* \phi_0^* \varphi_0^+ \rangle = -\hat{f}_3', \qquad \langle \partial/\partial\theta_2, \iota'^* \phi_0^* \varphi_0^- \rangle = -\hat{f}_3. \tag{7.11}$$

LEMMA 7.4. — *Let $r_1, r_2 \geq 0$ and s be given integers and let γ be the element $\gamma^3_{r_1,r_2,s}$ of Γ.*

(i) *The section $\tilde{f}_{r_1,r_2,s}\varphi_0^+$ and $\tilde{f}_{r_1,r_2,s}\varphi_0^-$ form a basis of the vector space $\mathcal{W}_\gamma$.*

(ii) *We have inclusion*

$$C^\infty_\gamma(T^*_{\mathbb{C}}) \subset C^\infty(T^*_{\mathbb{C}})^{\tau,\varepsilon},$$

with $\varepsilon = (-1)^{s+1}$.

(iii) *Suppose that s is odd. The G-modules $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$ and $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{odd}}$ are irreducible. When $2r_1 + s \equiv 1 \bmod 4$ (resp. $2r_1 + s \equiv 3 \bmod 4$), the section $\tilde{f}_{r_1,r_2,s}\varphi_0^+$ (resp. $\tilde{f}_{r_1,r_2,s}\varphi_0^-$) is a highest weight vector of the G-module $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$.*

Proof. — From the observations preceding the lemma and from Proposition 6.3, we deduce assertion (i). Assertion (ii) is a direct consequence of (i) and the relations (5.7). Assertion (iii) follows directly from the preceding assertions and the relations (6.1) and (7.10). □

According to Lemma 7.1 and the discussion which appears at the end of §5, we see that $\chi_2(M^+)$ and $\chi_2(M^-)$ are irreducible G-submodules of $C^\infty_{\gamma^4_{0,0,1}}(T^*_{\mathbb{C}})^{\mathrm{odd}}$ and $C^\infty_{\gamma^4_{0,0,-1}}(T^*_{\mathbb{C}})^{\mathrm{odd}}$, respectively, and that ψ^+ and ψ^- are highest weight vectors of these modules. According to Proposition 6.3, the G-module $C^\infty_{\gamma^4_{0,0,s}}(T^*_{\mathbb{C}})$ is irreducible, for $s \in \mathbb{Z}$, with $|s| \geq 1$. Therefore, we have the equalities

$$C^\infty_{\gamma^4_{0,0,1}}(T^*_{\mathbb{C}}) = C^\infty_{\gamma^4_{0,0,1}}(T^*_{\mathbb{C}})^{\mathrm{odd}} = \chi_2(M^+),$$
$$C^\infty_{\gamma^4_{0,0,-1}}(T^*_{\mathbb{C}}) = C^\infty_{\gamma^4_{0,0,-1}}(T^*_{\mathbb{C}})^{\mathrm{odd}} = \chi_2(M^-).$$

Let $r_1, r_2 \geq 0$ and s be given integers. We now consider the sections

$$\varphi_8 = \tilde{f}_{r_1,r_2,s}\psi^+, \qquad \varphi_9 = \tilde{f}_{r_1,r_2,s}\psi^-$$

of $T^*_{\mathbb{C}}$. According to the remark preceding Lemma 6.9, we see that φ_8 is a highest weight vector of the G-module $C^\infty_{\gamma^+}(T^*_{\mathbb{C}})$, where γ^+ is the element of Γ equal to $\gamma^4_{r_1,r_2,s+1}$ when $s \geq 0$, and to $\gamma^4_{r_1,r_2+1,s+1}$ when $s \leq -1$; on the other hand, φ_9 is a highest weight vector of the G-module $C^\infty_{\gamma^-}(T^*_{\mathbb{C}})$, where γ^- is the element of Γ equal to $\gamma^4_{r_1,r_2+1,s-1}$ when $s \geq 1$, and to $\gamma^4_{r_1,r_2,s-1}$ when $s \leq 0$. Since ψ^+ and ψ^- are elements of $C^\infty(T^*_{\mathbb{C}})^{\mathrm{odd}}$, if φ is a section of $T^*_{\mathbb{C}}$ equal to φ_8 or to φ_9, according to (5.7) we see that

$$\tau^*\varphi = (-1)^s\varphi; \tag{7.12}$$

moreover, by (6.1) we also have the relation

$$\Psi^*\varphi = \begin{cases} (-1)^{r_1+1} i^s \varphi & \text{when } s \geq 0, \\ (-1)^{r_1+|s|+1} i^{|s|}\varphi & \text{when } s < 0. \end{cases} \tag{7.13}$$

If $p, q \geq 0$ are given integers, we easily deduce from (7.13) that the 1-forms $\tilde{f}_{r_1,p,s-1}\psi^+$ and $\tilde{f}_{r_1,q,s+1}\psi^-$ are linearly independent.

LEMMA 7.5. — *Let $r_1, r_2 \geq 0$ and s be given integers and let γ be the element $\gamma^4_{r_1,r_2,s}$ of Γ.*

(i) *A basis of the space $\mathcal{W}_\gamma$ is given by the following table:*

Conditions on r_2 and s	Basis of $\mathcal{W}_\gamma$
$r_2 \geq 1,\ s \geq 1$	$\tilde{f}_{r_1,r_2,s-1}\psi^+,\ \tilde{f}_{r_1,r_2-1,s+1}\psi^-$
$r_2 \geq 1,\ s = 0$	$\tilde{f}_{r_1,r_2-1,-1}\psi^+,\ \tilde{f}_{r_1,r_2-1,1}\psi^-$
$r_2 \geq 1,\ s \leq -1$	$\tilde{f}_{r_1,r_2-1,s-1}\psi^+,\ \tilde{f}_{r_1,r_2,s+1}\psi^-$
$r_2 = 0,\ s \geq 1$	$\tilde{f}_{r_1,0,s-1}\psi^+$
$r_2 = 0,\ s \leq -1$	$\tilde{f}_{r_1,0,s+1}\psi^-$

(ii) *We have inclusion*

$$C^\infty_\gamma(T^*_{\mathbb{C}}) \subset C^\infty(T^*_{\mathbb{C}})^{\tau,\varepsilon},$$

with $\varepsilon = (-1)^{s+1}$.

(iii) *Suppose that $r_2 = 0$ and $s \neq 0$. Then the G-module $C^\infty_\gamma(T^*_{\mathbb{C}})$ is irreducible. If s is odd and equal to $2l+1$, with $l \in \mathbb{Z}$, the equality*

$$C^\infty_\gamma(T^*_{\mathbb{C}}) = C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$$

holds when $s \geq 1$ and $r_1 + l$ is odd, or when $s \leq -1$ and $r_1 + l$ is even.

(iv) *Suppose that $r_2 \geq 1$ and that s is odd. Then the G-modules $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$ and $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{odd}}$ are irreducible.*

(v) *Suppose that s is odd. Then if the G-module $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$ is non-zero, a highest weight vector of this G-module is given by the following table:*

Conditions on r_1, r_2 and s	Highest weight vector
$s \geq 1$ and $2r_1 + s \equiv 3 \bmod 4$	$\tilde{f}_{r_1,r_2,s-1}\psi^+$
$r_2 \geq 1,\ s \geq 1$ and $2r_1 + s \equiv 1 \bmod 4$	$\tilde{f}_{r_1,r_2-1,s+1}\psi^-$
$r_2 \geq 1,\ s \leq -1$ and $2r_1 + s \equiv 3 \bmod 4$	$\tilde{f}_{r_1,r_2-1,s-1}\psi^+$
$s \leq -1$ and $2r_1 + s \equiv 1 \bmod 4$	$\tilde{f}_{r_1,r_2,s+1}\psi^-$

Proof. — From the observations preceding the lemma and from Proposition 6.3, we deduce assertion (i). Assertion (ii) is a direct consequence of (i) and the relation (7.12). Assertions (iii)–(iv) follow directly from (i), (ii) and the relations (7.13). ☐

8. A family of polynomials

In this section, we introduce a family of polynomials arising from trigonometric integrals. These integrals appear in the next section when we integrate various objects over maximal flat tori of $\widetilde{G}^{\mathbb{R}}_{3,3}$. The principal result of this section is a new combinatorial identity, formula (8.4), which is proved by means of the WZ algorithm (see [**8**]).

If $a \in \mathbb{Q}$ and $r \geq 0$ is an integer, we consider the rising factorial $(a)_r$, which is the element of $\mathbb{Q}$ defined inductively by $(a)_0 = 1$, $(a)_1 = a$, and $(a)_{r+1} = (a+r)\cdot(a)_r$ for $r \geq 1$. If m, r are integers, we define the binomial coefficient $\binom{m}{r}$ to be equal to zero whenever $r > m$, or whenever one of the integers m, r is negative.

Let $m, q \geq 0$ be given integers. We consider the polynomials $P_{m,q}$, $R_{m,q}$ and $\hat{R}_{m,q}$ of degree q belonging to $\mathbb{Q}[y]$ determined by

$$P_{m,q}(y) = \sum_{r=0}^{q} a_r y^r, \qquad R_{m,q}(y) = \sum_{r=0}^{q} b_r y^r, \qquad \hat{R}_{m,q}(y) = \sum_{r=0}^{q} b_{q-r} y^r,$$

where $a_0 = b_0 = 1$ and

$$a_r = \binom{q}{r} \prod_{j=1}^{r} \frac{2m+2j-1}{2m+2q-2j+1}, \qquad b_r = \frac{2m+2q-2r+1}{2m+2q+1} a_r,$$

for $1 \leq r \leq q$. In fact, we have

$$a_r = (-1)^r \binom{q}{r} \frac{\left(m+\frac{1}{2}\right)_r}{\left(\frac{1}{2}-m-q\right)_r},$$

for $0 \leq r \leq q$. We easily verify that

$$2R_{m,q}(y) = P_{m,q+1}(y) - (y-1)P_{m+1,q}(y). \tag{8.1}$$

Also we see that

$$\hat{R}_{m,q}(t) = t^q R_{m,q}(1/t), \tag{8.2}$$

for all $t \in \mathbb{R}$, with $t \neq 0$. For $0 \leq r \leq q$, we consider the polynomial $p_r \in \mathbb{Z}[y]$ of degree $2q$ defined by

$$p_r(y) = \binom{q}{r} (y+1)^{2r} \cdot (y-1)^{2(q-r)}$$

and the polynomial $Q_{m,q} \in \mathbb{Z}[y]$ of degree $2q$ defined by

$$Q_{m,q}(y) = \sum_{r=0}^{q} \binom{2m+q}{m+r} p_r(y).$$

Clearly, we have $p_0 = 1$ and $p_r(t) \geq 0$ for all $t \in \mathbb{R}$; it follows that

$$Q_{m,q}(t) > 0, \tag{8.3}$$

for all $t \in \mathbb{R}$. The following lemma tells us that all the coefficients of the polynomial $Q_{m,q}$ are positive, and hence also gives us the preceding inequality; its proof is due to H. Wilf (see [**8**]).

LEMMA 8.1. — *Let $m, q \geq 0$ be given integers. Then we have the equality*

$$Q_{m,q}(y) = \binom{2m+2q}{m+q} P_{m,q}(y^2) \tag{8.4}$$

among elements of $\mathbb{Z}[y]$.

Proof. — We denote by $\mathbb{N}$ the set of natural integers. We consider the indeterminate $x = y - 1$ and consider both sides of the equality (8.4) as polynomials in x. Upon computing the coefficients of x^n of these two polynomials, we see that it suffices to show that the identity

$$\sum_{r \geq 0} (-1)^r \binom{2m+2q}{m+q} \binom{2r}{n} \binom{q}{r} \frac{\left(m+\frac{1}{2}\right)_r}{\left(\frac{1}{2}-m-q\right)_r} = \sum_{r \geq 0} 2^{2q-n} \binom{2m+q}{m+r} \binom{q}{r} \binom{2r}{n+2r-2q} \tag{8.5}$$

holds for all $n \in \mathbb{N}$. Then standard techniques of WZ theory and the Zeilberger algorithm, as described in the book [**20**] and implemented by the EKHAD package for Maple, can be used to show that the identity (8.5) is indeed true. In fact, the functions $f_1 : \mathbb{N} \to \mathbb{Q}$ and $f_2 : \mathbb{N} \to \mathbb{Q}$, whose values at $n \in \mathbb{N}$ are equal to the left-hand and right-hand sides of (8.5), respectively, both satisfy the same recurrence of order 2; namely, if we set

$$\alpha_0(n) = (n-2q)(n+2m+1), \qquad \alpha_2(n) = 2(n+2)(n+2m+2),$$
$$\alpha_1(n) = 3n^2 + 7n + 4(m(n+1) - q(m+n) + 1) - 6q,$$

for $n \in \mathbb{N}$, we find that

$$\alpha_0(n) f_k(n) + \alpha_1(n) f_k(n+1) + \alpha_2(n) f_k(n+2) = 0,$$

for all $n \in \mathbb{N}$ and $k = 1, 2$. To complete the proof, we must show that the equality $f_1(n) = f_2(n)$ holds when $n = 0$ and $n = 1$. If each of these two cases, there is only one non-zero summand in the sum on the right-hand side of (8.5), namely the one corresponding to the summation index $r = q$, and so we see that

$$f_2(0) = 4^q \binom{2m+q}{m}, \qquad f_2(1) = q4^q \binom{2m+q}{m}.$$

The identity $f_1(n) = f_2(n)$, with $n = 0$ or $n = 1$, is now obtained by applying WZ theory to the sum on the left-hand side of (8.5); in fact, in both cases Zeilberger's algorithm returns a recurrence of order 1 in the index m. ∎

We continue to fix integers $m, q \geq 0$. From Lemma 8.1 and the identity (8.1), we deduce that

$$Q_{m,q+1}(y) - (y^2 - 1)Q_{m+1,q}(y) = 2\binom{2m+2q+2}{m+q+1} R_{m,q}(y^2). \tag{8.6}$$

For $s \geq 0$, the polynomial

$$H_s(y) = 2\sum_{k\geq 0}\binom{2s}{2k+1}y^k$$

of $\mathbb{Z}[y]$ has degree $s-1$ and satisfies the relation

$$(y+1)^{2s} - (y-1)^{2s} = yH_s(y^2),$$

belongs to $\mathbb{Z}[y]$; clearly, H_0 is equal to the zero polynomial. For $0 \leq r \leq q$, we consider the integer

$$c_r = \binom{2m+q+1}{m+r+1} - \binom{2m+q+1}{m+r} = \binom{2m+q+1}{m+q-r} - \binom{2m+q+1}{m+r}.$$

We verify that the integer c_r is positive when $2r < q$, and that $c_r = 0$ when $2r = q$. We set $q' = [q/2]$; we easily see that the elements $K_{m,q}$ and $L_{m,q}$ of $\mathbb{Z}[y]$, given by

$$K_{m,q} = \sum_{r=0}^{q} c_r p_r, \qquad L_{m,q}(y) = \sum_{r=0}^{q'} (y-1)^{2r} c_r \binom{q}{r} H_{q-2r}(y),$$

are related by

$$K_{m,q}(y) = -yL_{m,q}(y^2). \tag{8.7}$$

We notice that $K_{m,0} = L_{m,0} = 0$. Since the coefficients of the polynomials H_s and the coefficients c_r are positive, the polynomial $L_{m,q}$ is of degree $q-1$; moreover, when $q \geq 1$, it satisfies the inequality

$$L_{m,q}(t) > 0, \tag{8.8}$$

for all $t \in \mathbb{R}$.

We now consider the complex-valued functions u and v on $\mathbb{R}^2$ defined by

$$u(x,t) = t\sin x - i\cos x, \qquad v(x,t) = t\cos x + i\sin x,$$

with $x, t \in \mathbb{R}$; the functions u and v are related by

$$v(x,t) = \frac{\partial u}{\partial x}(x,t) = u(x + \pi/2, t),$$

for $x, t \in \mathbb{R}$; also the formula

$$v(x,t) = itu(x, 1/t)$$

holds for $x, t \in \mathbb{R}$, with $t \neq 0$. Let $r \geq 0$ be a given integer; we consider the functions $F^j_{r,q}$ on $\mathbb{R}$, with $0 \leq j \leq 5$, defined by

$$
\begin{aligned}
F^0_{r,q}(t) &= \frac{1}{2\pi} \int_0^{2\pi} (u^{q+r} \bar{u}^q)(x,t)\, dx, \\
F^1_{r,q}(t) &= \frac{1}{2\pi} \int_0^{2\pi} (u^{q+r} \bar{u}^q)(x,t) \sin x\, dx, \\
F^2_{r,q}(t) &= \frac{1}{2\pi} \int_0^{2\pi} (u^{q+r} \bar{u}^q)(x,t) \cos x\, dx, \\
F^3_{r,q}(t) &= \frac{1}{2\pi} \int_0^{2\pi} (u^{q+r} \bar{u}^q)(x,t) \sin x \cdot \cos x\, dx, \\
F^4_{r,q}(t) &= \frac{1}{2\pi} \int_0^{2\pi} (u^{q+r} \bar{u}^q v)(x,t)\, dx, \\
F^5_{r,q}(t) &= \frac{1}{2\pi} \int_0^{2\pi} (u^{q+r} \bar{u}^q v)(x,t) \sin x \cdot \cos x\, dx,
\end{aligned}
\tag{8.9}
$$

for $t \in \mathbb{R}$. Clearly, these functions $F^j_{r,q}$ are polynomials. We easily see that the functions $F^3_{0,q}$ and $F^4_{r,0}$ vanish identically. We shall require the following identities

$$F^0_{r+1,q}(t) = tF^1_{r,q}(t) - iF^2_{r,q}(t), \tag{8.10}$$

$$F^0_{r,q+1}(t) = tF^1_{r+1,q}(t) + iF^2_{r+1,q}(t), \tag{8.11}$$

which hold for all $t \in \mathbb{R}$; it follows that

$$2iF^2_{r+1,q} = F^0_{r,q+1} - F^0_{r+2,q}. \tag{8.12}$$

By making the change of variables $x \mapsto x + \pi/2$ in the integral defining the function $F^1_{r,q}$, we see that

$$
\begin{aligned}
F^1_{r,q}(x,t) &= \frac{1}{2\pi} \int_0^{2\pi} (v^{q+r} \bar{v}^q)(x,t) \cos x\, dx \\
&= \frac{1}{2\pi} i^r t^{2q+r} \int_0^{2\pi} (u^{q+r} \bar{u}^q)(x, 1/t) \cos x\, dx,
\end{aligned}
$$

for $t \in \mathbb{R}$, with $t \neq 0$. Therefore we have the equality

$$F^1_{r,q}(x,t) = i^r t^{2q+r} F^2_{r,q}(x, 1/t), \tag{8.13}$$

for $t \in \mathbb{R}$, with $t \neq 0$. By an integration by parts, we obtain

$$
\begin{aligned}
2\pi F^4_{r,q+1}(t) &= \frac{1}{r+1} \int_0^{2\pi} \left((u\bar{u})^{q+1} \frac{\partial u^{r+1}}{\partial x} \right)(x,t)\, dx \\
&= -2 \frac{q+1}{r+1} (t^2 - 1) \int_0^{2\pi} (u^{q+r+1} \cdot \bar{u}^q)(x,t) \sin x \cdot \cos x\, dx
\end{aligned}
$$

for all $t \in \mathbb{R}$; therefore we have

$$F^4_{r,q+1}(t) = -2\frac{q+1}{r+1}(t^2-1)F^3_{r+1,q}(t), \tag{8.14}$$

for all $t \in \mathbb{R}$. The equalities

$$\begin{aligned}\int_0^{2\pi} (u^{q+r+1} \cdot \bar{u}^{q+1} \cdot v^2)(x,t)\,dx \\ &= \int_0^{2\pi} ((u\bar{u})^q \cdot u^{r+1} \cdot v \cdot (\bar{u}v))(x,t)\,dx \\ &= \int_0^{2\pi} ((u\bar{u})^q \cdot u^{r+1} \cdot v)(x,t)((t^2-1)\sin x \cdot \cos x + it)\,dx \\ &= 2\pi((t^2-1)F^5_{r+1,q} + itF^4_{r+1,q}),\end{aligned}$$

with $t \in \mathbb{R}$, follow directly from the definitions of the functions which appear there. By an integration by parts, we obtain

$$\begin{aligned}2\pi(t^2-1)F^5_{r+1,q}(t) &= \frac{1}{2(q+1)}\int_0^{2\pi}\Big(u^{r+1}v\frac{\partial(u\bar{u})^{q+1}}{\partial x}\Big)(x,t)\,dx \\ &= \frac{\pi}{q+1}F^0_{r+2,q+1}(t) \\ &\quad - \frac{r+1}{2(q+1)}\int_0^{2\pi}(u^{q+r+1} \cdot \bar{u}^{q+1} \cdot v^2)(x,t)\,dx\end{aligned}$$

for all $t \in \mathbb{R}$. From the preceding relations, we immediately deduce that

$$(2q+r+3)(t^2-1)F^5_{r+1,q}(t) = F^0_{r+2,q+1}(t) - i(r+1)tF^4_{r+1,q}(t), \tag{8.15}$$

for all $t \in \mathbb{R}$.

In the next section, we shall need the following results in order to prove the positivity of certain integrals.

Proposition 8.2. — *Let $m, q \geq 0$ be given integers. Then there exist polynomials $\{Q^j_{m,q}\}_{0 \leq j \leq 4}$ of degree $2q$ and a polynomial $Q^5_{m,q}$ of degree $2q+2$ belonging to $\mathbb{Z}[y]$ satisfying the following properties:*

(i) *for $0 \leq j \leq 5$, we have*

$$Q^j_{m,q}(t) > 0, \tag{8.16}$$

for all $t \in \mathbb{R}$;

(ii) *the equalities*

$$F^0_{2m,q}(t) = \frac{1}{4^{m+q}}(t^2-1)^m \cdot Q^0_{m,q}(t), \tag{8.17}$$

$$F^1_{2m+1,q}(t) = \frac{1}{4^{m+q+1}}t(t^2-1)^m \cdot Q^1_{m,q}(t), \tag{8.18}$$

$$F^2_{2m+1,q}(t) = -\frac{i}{4^{m+q+1}}(t^2-1)^m \cdot Q^2_{m,q}(t), \tag{8.19}$$

$$F^3_{2m+2,q}(t) = -\frac{i}{4^{m+q+2}}t(t^2-1)^m \cdot Q^3_{m,q}(t), \tag{8.20}$$

$$F^4_{2m+1,q+1}(t) = \frac{i}{4^{m+q+2}}t(t^2-1)^{m+1} \cdot Q^4_{m,q}(t), \tag{8.21}$$

$$F^5_{2m+1,q}(t) = \frac{1}{4^{m+q+2}}(t^2-1)^m \cdot Q^5_{m,q}(t), \tag{8.22}$$

hold for all $t \in \mathbb{R}$.

Proof. — The polynomial $Q^0_{m,q} = Q_{m,q}$ and the polynomial $Q^4_{m,q}$, determined by

$$Q^4_{m,q}(y) = L_{m,q+1}(y^2),$$

both belong to $\mathbb{Z}[y]$ and are of degree $2q$. If p_r is the polynomial defined above, with $0 \leq r \leq q$, we define elements $Q^2_{m,q}$, and $Q^5_{m,q}$ of $\mathbb{Z}[y]$ by

$$Q^2_{m,q}(y) = \sum_{r=0}^{q} p_r(y)\left\{\binom{2m+q+1}{m+r+1}(y+1) - \binom{2m+q+1}{m+r}(y-1)\right\},$$

$$Q^5_{m,q}(y) = Q_{m,q+1}(y) - \sum_{r=0}^{q} p_r(y)\left\{\binom{2m+q+1}{m+r-1}(y-1)^2 + \binom{2m+q+1}{m+r+2}(y+1)^2\right\}.$$

If $r \geq 0$ is a given integer, we view the integrands of the right-hand sides of the equalities (8.9) as polynomials in e^{ix} and e^{-ix}; for $t \in \mathbb{R}$, the value of the integral $F^j_{r,q}(t)$ is equal to the constant term of the polynomial obtained from the corresponding integrand. In fact, we easily see that the equalities (8.17), (8.19) and (8.22) hold, that there exist a polynomial $Q^3_{m,q} \in \mathbb{Z}[y]$ such that (8.20) holds, and that the equality

$$F^4_{2m+1,q}(t) = -\frac{i}{4^{m+q+1}}(t^2-1)^{m+1} \cdot K_{m,q}(t),$$

holds for $t \in \mathbb{R}$. From the preceding equality and (8.7), we obtain the relation (8.21). The relation

$$Q^3_{m,q} = \frac{m+1}{q+1}Q^4_{m,q}, \tag{8.23}$$

is an immediate consequence of (8.21) and (8.14). If we set $Q^4_{m,-1} = 0$, the relation (8.15), together with the equalities (8.17), (8.21) and (8.22), implies that

$$(2m+2q+3)Q^5_{m,q}(y) = Q^0_{m+1,q+1}(y) + 4(2m+1)y^2 Q^4_{m,q-1}(y). \tag{8.24}$$

According to the relations (8.12), (8.17) and (8.19), we see that

$$2Q^2_{m,q}(y) = Q^0_{m,q+1}(y) - (y^2-1)Q^0_{m+1,q}(y).$$

Hence by (8.6), the polynomial $Q^2_{m,q}$ of $\mathbb{Z}[y]$ satisfies

$$Q^2_{m,q}(y) = \binom{2m+2q+2}{m+q+1} R_{m,q}(y^2) \tag{8.25}$$

and has degree $2q$. The polynomial $Q^1_{m,q}$ of degree $2q$ determined by

$$Q^1_{m,q}(y) = \binom{2m+2q+2}{m+q+1} \hat{R}_{m,q}(y^2)$$

belongs to $\mathbb{Z}[y]$ and, according to (8.2), satisfies the relation

$$Q^1_{m,q}(t) = t^{2q} Q^2_{m,q}(1/t),$$

for all $t \in \mathbb{R}$, with $t \neq 0$; from the equalities (8.19) and (8.13), we now obtain the relation (8.18). According to (8.3), (8.8) and (8.23), we see that the inequality (8.16) holds for $j = 0, 3, 4$; the relation (8.24) now implies that (8.16) also holds for $j = 5$. By (8.25) and the definition of the polynomials $R_{m,q}$ and $\hat{R}_{m,q}$, we know that all the coefficients of the polynomials $Q^1_{m,q}$ and $Q^2_{m,q}$ are positive, and thus the inequality (8.16) is true for $j = 1, 2$. □

9. Exactness of the forms of degree one

This section is devoted to the proof of the following theorem:

Theorem 9.1. — *A 1-form on the symmetric space $Y = \bar{G}^{\mathbb{R}}_{3,3}$ satisfies the Guillemin condition if and only if it is exact.*

From Theorem 9.1 and the remark following Lemma 4.1, we infer that a 1-form θ on the symmetric space $Y = \bar{G}^{R}_{3,3}$ is exact if and only if the symmetric 2-form $*\theta$ on Y satisfies the Guillemin condition. According to Proposition 7.21 of [**6**], from Theorem 9.1 we immediately deduce the following generalization:

Theorem 9.2. — *A 1-form on the symmetric space $\bar{G}^{\mathbb{R}}_{n,n}$, with $n \geq 3$, satisfies the Guillemin condition if and only if it is exact.*

We again consider the Grassmannian $X = \widetilde{G}^{\mathbb{R}}_{3,3}$ and the objects associated above with this space. Lemmas 6.4,(i), 6.9, 7.2, 7.4 and 7.5, and the relations (7.2) and (7.3) give us a complete description of the highest weight vectors of the non-zero isotypic components $C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}}$, with $\gamma \in \Gamma$. To prove Theorem 9.2, we need to verify that the equality

$$\mathcal{N}_{1,\mathbb{C}} \cap C^\infty_\gamma(T^*_{\mathbb{C}})^{\mathrm{ev}} = dC^\infty_\gamma(X)^{\mathrm{ev}} \tag{9.1}$$

holds for all $\gamma \in \Gamma$. If γ is an element of Γ, according to (5.1) we note that the equality (9.1) is true if and only if it holds with γ replaced by $\bar{\gamma}$; we therefore need only consider the equalities (9.1), with $\gamma = \gamma_{r_1,r_2,s}$ and $r_1, r_2, s \geq 0$. From Proposition 2.33 of [**6**], we then deduce the following result:

PROPOSITION 9.3. — *The assertion of Theorem 9.1 is equivalent to the fact that the following seven assertions all hold on the Grassmannian* $X = \widetilde{G}^{\mathbb{R}}_{3,3}$, *with* $n \geq 3$, *for all integers* $r_1, r_2 \geq 0$ *and* s:

(i) *Suppose that* $s \geq 2$ *and that* $2r_1 + s \equiv 0 \bmod 4$. *Let* a_1, a_2, a_3 *be elements of* $\mathbb{C}$; *if the* 1*-form*

$$a_1 \tilde{f}_{r_1-1,r_2,s} d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,s} d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2,s-1} d\tilde{f}_3$$

satisfies the Guillemin condition, then it is exact.

(ii) *Suppose that the integer* r_1 *is even and that* $r_2 \geq 1$. *Let* a_1, a_2, a_3 *be elements of* $\mathbb{C}$; *if the* 1*-form*

$$a_1 \tilde{f}_{r_1-1,r_2,0} d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,0} d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2-1,1} d\tilde{f}'_3$$

satisfies the Guillemin condition, then it is exact.

(iii) *Suppose that* $2r_1 + s \equiv 0 \bmod 4$. *Then the* 1*-form* $\tilde{f}_{r_1,r_2,s} g^\flat(\xi_0)$ *does not satisfy the Guillemin condition.*

(iv) *Suppose that* $2r_1 + s \equiv 0 \bmod 4$. *Then the* 1*-form* $\tilde{f}_{r_1,r_2,s}\omega$ *does not satisfy the Guillemin condition.*

(v) *Suppose that* $2r_1 + s \equiv 1 \bmod 4$. *Then the* 1*-form* $\tilde{f}_{r_1,r_2,s}\varphi_0^+$ *does not satisfy the Guillemin condition.*

(vi) *Suppose that* $2r_1 + s \equiv 3 \bmod 4$. *Then the* 1*-form* $\tilde{f}_{r_1,r_2,s}\varphi_0^-$ *does not satisfy the Guillemin condition.*

(vii) *Suppose that* $2r_1 + s \equiv 2 \bmod 4$. *Then the* 1*-forms* $\tilde{f}_{r_1,r_2,s}\psi^+$ *and* $\tilde{f}_{r_1,r_2,s}\psi^-$ *do not satisfy the Guillemin condition.*

We consider the maximal flat totally geodesic torus Z of $X = \widetilde{G}^{\mathbb{R}}_{3,3}$ and the parallel vector fields ζ_1, ζ_2, and ζ_3 on Z defined in §4. Let $r_1, r_2 \geq 0$ and s be given integers. We consider the function

$$\hat{f}_{r_1,r_2,s} = \iota^* \phi_0^* f_{r_1,r_2,s} = \iota'^* \phi_0^* \tilde{f}_{r_1,r_2,s}$$

on $\mathbb{R}^3$; we have $\hat{f}_{r_1,r_2,s} = \hat{f}_1^{r_1} \cdot \hat{f}_2^{r_2} \cdot \hat{f}_3^s$ for $s \geq 0$, and $\hat{f}_{r_1,r_2,s} = \hat{f}_1^{r_1} \cdot \hat{f}_2^{r_2} \cdot (\hat{f}'_3)^s$ for $s < 0$. We also define a function $\Phi_{r_1,r_2,s}$ on $\mathbb{R}^2$ by

$$\Phi_{r_1,r_2,s}(\theta_1, \theta_2) = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}_{r_1,r_2,s}(\theta_1, \theta_2, \theta_3)\, d\theta_3,$$

for $\theta_1, \theta_2 \in \mathbb{R}$; by (4.6), we may consider the integral

$$J_{r_1,r_2,s} = \int_Z \phi_0^* \tilde{f}_{r_1,r_2,s}\, dZ = 2\pi \int_0^\pi \int_0^\pi \Phi_{r_1,r_2,s}(\theta_1, \theta_2)\, d\theta_1\, d\theta_2.$$

By (6.7) and (6.8), we see that

$$\Phi_{r_1,r_2,s} = \bar{\Phi}_{r_1,r_2,-s}, \qquad J_{r_1,r_2,s} = \bar{J}_{r_1,r_2,-s}. \tag{9.2}$$

According to (7.5), (7.6), (7.11) and (6.2), we have the equalities

$$(9.3)\qquad \begin{aligned} -\langle \zeta_2, \phi_0^*(\tilde{f}_{r_1,r_2,s}\varphi_0^+)\rangle &= \begin{cases} \phi_0^*\tilde{f}_{r_1,r_2+1,s-1} & \text{when } s \geq 1, \\ \phi_0^*\tilde{f}_{r_1,r_2,s-1} & \text{when } s \leq 0, \end{cases} \\ -\langle \zeta_2, \phi_0^*(\tilde{f}_{r_1,r_2,s}\varphi_0^-)\rangle &= \begin{cases} \phi_0^*\tilde{f}_{r_1,r_2,s+1} & \text{when } s \geq 0, \\ \phi_0^*\tilde{f}_{r_1,r_2+1,s+1} & \text{when } s < 0, \end{cases} \end{aligned}$$

and

$$(9.4)\qquad \begin{gathered} \langle \zeta_1, \phi_0^*(\tilde{f}_{r_1,r_2,s}g^\flat(\xi_0))\rangle = i\phi_0^*\tilde{f}_{r_1,r_2,s}, \\ \langle \zeta_3, \phi_0^*(\tilde{f}_{r_1,r_2,s}\psi^+)\rangle = -\langle \zeta_3, \phi_0^*(\tilde{f}_{r_1,r_2,s}\psi^-)\rangle = \phi_0^*\tilde{f}_{r_1+1,r_2,s} \end{gathered}$$

on Z; moreover, the relation

$$(9.5)\qquad \langle \zeta_1, \phi_0^*(\tilde{f}_{r_1,r_2,s}\omega)\rangle(\iota'(\theta)) = 2i(\cos^2\theta_1 \cdot \sin^2\theta_2 + \sin^2\theta_1 \cdot \cos^2\theta_2)\hat{f}_{r_1,r_2,s}(\theta)$$

holds for all $\theta \in \mathbb{R}^3$.

We consider the real-valued functions A and B on $\mathbb{R}^2$ defined by

$$A(\theta_1,\theta_2) = \cos\theta_1 \cdot \sin\theta_2, \qquad B(\theta_1,\theta_2) = \sin\theta_1 \cdot \cos\theta_2,$$

for $\theta_1, \theta_2 \in \mathbb{R}$. Let $m, q \geq 0$ be given integers; if P is a polynomial of $\mathbb{R}[y]$ of degree q, there exists a unique real-valued function $\tilde{P}$ on $\mathbb{R}^2$ which is given by the expression

$$\tilde{P} = B^q \cdot \big(P \circ (A/B)\big)$$

at all points θ of $\mathbb{R}^2$ for which $B(\theta)$ is non-zero. We now consider the polynomials $\{Q^j_{m,q}\}_{0\leq j\leq 5}$ given by Proposition 8.2. Since the polynomial $Q^j_{m,q}$ is of degree $2q$, for $0 \leq j \leq 4$, and the polynomial $Q^5_{m,q}$ is of degree $2q+2$, the real-valued functions $\{\tilde{Q}^j_{m,q}\}_{0\leq j\leq 5}$ on $\mathbb{R}^2$ satisfy the relations

$$\tilde{Q}^j_{m,q} = B^{2q} \cdot \big(Q^j_{m,q} \circ (A/B)\big), \qquad \tilde{Q}^5_{m,q} = B^{2q+2} \cdot \big(Q^5_{m,q} \circ (A/B)\big)$$

at all points θ of $\mathbb{R}^2$ for which $B(\theta)$ is non-zero, with $0 \leq j \leq 4$. According to Proposition 8.2, these functions are non-zero and satisfy the inequalities

$$(9.6)\qquad \tilde{Q}^j_{m,q} \geq 0,$$

for $0 \leq j \leq 5$.

By (6.7), we may write

$$\begin{aligned} \hat{f}_1(\theta) &= \hat{f}_1(\theta_1,\theta_2) = B^2(\theta_1,\theta_2) - A^2(\theta_1,\theta_2), \\ \hat{f}_2(\theta) &= A^2(\theta_1,\theta_2) \cdot \sin^2\theta_3 + B^2(\theta_1,\theta_2) \cdot \cos^2\theta_3, \\ \hat{f}_3(\theta) &= A(\theta_1,\theta_2) \cdot \sin\theta_3 - iB(\theta_1,\theta_2) \cdot \cos\theta_3, \end{aligned}$$

for $\theta = (\theta_1,\theta_2,\theta_3) \in \mathbb{R}^3$. If $q, r \geq 0$ are given integers, at a point θ of $\mathbb{R}^2$, for which $B(\theta)$ is non-zero, we verify that

$$\Phi_{0,q,r}(\theta) = B^{2q+r}(\theta) \cdot F^0_{r,q}(t),$$

where $t = A(\theta)/B(\theta)$. Let $p, q, m \geq 0$ be given integers; then by Proposition 8.2, we obtain the relation

$$\Phi_{p,q,2m} = \frac{(-1)^p}{4^{m+q}}(A^2 - B^2)^{m+p} \cdot \tilde{Q}^0_{m,q}.$$

Thus the function $\Phi_{p,q,2m}$ is non-zero and real-valued; moreover if the integer $m + p$ is even, by (9.6) this function satisfies the inequality

$$(-1)^p \Phi_{p,q,2m} \geq 0. \tag{9.7}$$

From these remarks and the relations (9.2), we obtain the following:

LEMMA 9.4. — *If $p, q \geq 0$ and m are given integers, the function $\Phi_{p,q,2m}$ is non-zero and real-valued; moreover, if $m + p$ is an even integer, the inequality* (9.7) *holds.*

If Φ is a non-zero real-analytic real-valued function on $\mathbb{R}^2$ satisfying $\Phi \geq 0$, from Lemma 9.4 we infer that

$$(-1)^p \int_0^\pi \int_0^\pi \int_0^{2\pi} \Phi(\theta_1, \theta_2) \hat{f}_{p,q,2m}(\theta_1, \theta_2, \theta_3)\, d\theta_3\, d\theta_1\, d\theta_2 > 0,$$

for all integers $p, q \geq 0$ and m such that $m + p$ is even. In particular, we obtain the following:

LEMMA 9.5. — *For all integers $p, q \geq 0$ and m such that $m + p$ is even, we have*

$$(-1)^p J_{p,q,2m} > 0.$$

If $r_1, r_2 \geq 0$ and s are given integers, Lemma 9.5 implies that the function $\tilde{f}_{r_1,r_2,s}$ on X does not satisfy the Guillemin condition whenever $2r_1 + s \equiv 0 \bmod 4$. Since $\tilde{f}_{r_1,r_2,s}$ is a non-zero element of $C^\infty_\gamma(X)$, where γ is the element $\gamma^1_{r_1,r_2,s}$ of Γ, from the relations (5.9) and (7.1) and the preceding observation we obtain the injectivity of the maximal flat Radon transform for functions on X, which is also given by Proposition 7.17 of [**6**].

LEMMA 9.6. — *Let $r_1, r_2 \geq 0$ and s be given integers. Assertions* (iii)–(vii) *of Proposition 9.3 hold.*

Proof. — From the relations (9.3) and (9.4) and Lemma 9.5, we deduce that the assertions (iii), (v), (vi) and (vii) of Proposition 9.3 hold. On the other hand, according to the remark following Lemma 9.4, with

$$\Phi(\theta_1, \theta_2) = \cos^2\theta_1 \cdot \sin^2\theta_2 + \sin^2\theta_1 \cdot \cos^2\theta_2,$$

from relation (9.5) we obtain assertion (iv) of Proposition 9.3. □

We consider the functions v_1, v_2, v_3 on $\mathbb{R}^3$ defined by

$$v_1 = \cos^2\theta_1 \cdot \sin^2\theta_2 \cdot \cos^2\theta_3 + \sin^2\theta_1 \cdot \cos^2\theta_2 \cdot \sin^2\theta_3,$$
$$v_2 = -2\sin\theta_1 \cdot \cos\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_2,$$
$$v_3 = \cos\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_3 + i \sin\theta_1 \cdot \cos\theta_2 \cdot \sin\theta_3,$$

for $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$; we write $w_2 = \hat{f}_2$ and $w_3 = -i\hat{f}_3$. We then define functions on k_2 and k_3 on $\mathbb{R}^4$ by

$$k_2 = v_1(\theta)\lambda^2 + v_2(\theta)\lambda + w_2(\theta), \qquad k_3 = v_3(\theta)\lambda + w_3(\theta),$$

for $\theta \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. If $q, r \geq 0$ are given integers, we consider the function

$$\psi_{r,q} = k_2^{q+1} k_3^r \frac{\partial k_3}{\partial \theta_3}$$

on $\mathbb{R}^4$; then the function

$$\psi'_{r,q} = \frac{\partial \psi_{r,q}}{\partial \lambda}\Big|_{\lambda=0}$$

on $\mathbb{R}^3$ is given by

$$\psi'_{r,q} = (q+1) v_2 w_2^q w_3^r \frac{\partial w_3}{\partial \theta_3} + w_2^{q+1} \frac{\partial}{\partial \theta_3}(v_3 w_3^r).$$

We then see that function $\Psi_{r,q}$ on $\mathbb{R}^2$ defined by

$$\Psi_{r,q}(\theta_1, \theta_2) = \frac{1}{2(q+1)} \int_0^{2\pi} \psi'_{r,q}(\theta_1, \theta_2, \theta_3)\, d\theta_3,$$

for $(\theta_1, \theta_2) \in \mathbb{R}^2$, satisfies the relation

$$\Psi_{r,q} = \frac{1}{2} \int_0^{2\pi} w_2^q w_3^r \left(v_2 \frac{\partial w_3}{\partial \theta_3} - v_3 \frac{\partial w_2}{\partial \theta_3} \right) d\theta_3.$$

At a point θ of $\mathbb{R}^2$, for which $B(\theta)$ is non-zero, by means of the preceding formula we easily verify that

$$\frac{1}{2\pi} \Psi_{r,q}(\theta) = (-i)^r B^{2q+r+1}(\theta) \big\{ \hat{f}_1(\theta) \cdot F^5_{r,q}(t) + i(AB)(\theta) \cdot F^4_{r,q}(t) \big\} \tag{9.8}$$

where $t = A(\theta)/B(\theta)$.

Let $m, q \geq 0$ be given integers; we write $Q^4_{m,-1} = 0$. By Proposition 8.2 and (9.6), the real-valued function

$$\Psi'_{m,q} = \frac{2\pi}{4^{m+q+2}} \big(\tilde{Q}^5_{m,q} + 4A^2 \cdot B^2 \cdot \tilde{Q}^4_{m,q-1} \big)$$

on $\mathbb{R}^2$ is non-zero and satisfies $\Psi'_{m,q} \geq 0$. According to Proposition 8.2, from the equality (9.8) we obtain the relation

$$\Psi_{2m+1,q} = -i(B^2 - A^2)^{m+1} \cdot \Psi'_{m,q}.$$

Thus the function $i\Psi_{2m+1,q}$ is non-zero and real-valued; moreover, if $p \geq 0$ is an integer such that $m+p$ is odd, we see that

$$i\hat{f}_1^p \Psi_{2m+1,q} = (B^2 - A^2)^{m+p+1} \cdot \Psi'_{m,q} \geq 0. \tag{9.9}$$

For $\alpha \in \mathbb{R}$, we consider the element

$$\phi_{1,\alpha} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & 0 & 0 & -\sin\alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \sin\alpha & 0 & 0 & 0 & \cos\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

of $G = SO(6)$ and the automorphism τ_1 of $\mathbb{R}^3$ defined by

$$\tau_2(\theta_1, \theta_2, \theta_3) = (\theta_1, \theta_2, \theta_3 + \pi/2),$$

for $(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. Then it is easily verified that

$$\iota^* \phi_{1,\alpha}^* f_1 = \hat{f}_1$$

on $\mathbb{R}^3$ and that

$$(\iota^* \phi_{1,\alpha}^* f_j)(\tau_1 \theta) = k_j(\theta, \cos\alpha),$$

for $j = 2, 3$ and all $\theta \in \mathbb{R}^3$. By (4.10) and the preceding equalities, if $p, q, r \geq 0$ are given integers, we have

$$(9.10) \qquad \begin{gathered} \langle \zeta_3, \phi_{1,\alpha}^* d\tilde{f}_1 \rangle (\iota'(\theta)) = 0, \\ \langle \zeta_3, \phi_{1,\alpha}^* (\tilde{f}_1^p \tilde{f}_2^{q+1} \tilde{f}_3^r d\tilde{f}_3) \rangle (\iota'(\tau_1 \theta)) = \hat{f}_1^p(\theta) \cdot \psi_{r,q}(\theta, \cos\alpha), \end{gathered}$$

for $\theta \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. From the second equality of (9.10), we obtain

$$\frac{\partial}{\partial \alpha} \langle \zeta_3, \phi_{1,\alpha}^* (\tilde{f}_1^p \tilde{f}_2^{q+1} \tilde{f}_3^r d\tilde{f}_3) \rangle (\iota'(\tau_1 \theta))_{\big| \alpha = \frac{\pi}{2}} = -(\hat{f}_1^p \cdot \psi'_{r,q})(\theta),$$

for $\theta \in \mathbb{R}^3$. If $m \geq 0$ is a given integer, by (4.6) it follows that

$$\begin{aligned} \frac{\partial}{\partial \alpha} \int_Z \langle \zeta_3, \phi_{1,\alpha}^* (\tilde{f}_1^p \tilde{f}_2^{q+1} \tilde{f}_3^{2m+1} d\tilde{f}_3) \rangle \, dZ_{\big| \alpha = \frac{\pi}{2}} & \\ = -2(q+1) \int_0^\pi \int_0^\pi \hat{f}_1^p(\theta_1, \theta_2) \Psi_{2m+1,q}(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2. & \end{aligned}$$

Hence according to (9.9), if $m, p, q \geq 0$ are given integers such that $m+p$ is odd, there exists $\alpha_1 \in \mathbb{R}$ such that

$$\int_Z \langle \zeta_3, \phi_{1,\alpha_1}^* (\tilde{f}_1^p \tilde{f}_2^{q+1} \tilde{f}_3^{2m+1} d\tilde{f}_3) \rangle \, dZ \neq 0.$$

From the first equality of (9.10) and the preceding inequality, by (4.6) we deduce the following result:

LEMMA 9.7. — *Let $r_1 \geq 0$, $r_2 \geq 1$ and $s \geq 2$ be given integers; assume that $2r_1 + s \equiv 0 \bmod 4$. Let $a_1, a_3 \in \mathbb{C}$. If the 1-form*

$$a_1 \tilde{f}_{r_1 - 1, r_2, s} d\tilde{f}_1 + a_3 \tilde{f}_{r_1, r_2, s-1} d\tilde{f}_3$$

on X satisfies the Guillemin condition, then we have $a_3 = 0$.

We consider the function h_1 on $\mathbb{R}^4$ defined by

$$h_1(\theta, \lambda) = (\cos^2\theta_3 - \sin^2\theta_2)\lambda^2 - \hat{f}_1(\theta),$$

for $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$, and the functions h_2', h_2'', h_3 and h_4 on $\mathbb{R}^3$ defined by

$$\begin{aligned}
h_2'(\theta) &= \sin\theta_2 \cdot \cos\theta_2 \cdot \sin\theta_3 \cdot \cos\theta_3, \\
h_2''(\theta) &= \sin\theta_1 \cdot \cos\theta_1 \cdot \sin\theta_2 \cdot \cos\theta_2, \\
h_3(\theta) &= i(\cos\theta_1 \cdot \cos\theta_3 + \sin\theta_1 \cdot \sin\theta_3)e^{-i\theta_2}, \\
h_4(\theta) &= \sin\theta_1 \cdot \sin\theta_2 \cdot \sin\theta_3 + i\cos\theta_1 \cdot \cos\theta_2 \cdot \cos\theta_3,
\end{aligned}$$

for $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$; we write $h_2 = -2i(h_2' + h_2'')$. We verify that the relations

$$h_3 = \frac{\partial \hat{f}_3}{\partial \theta_3} + h_4, \qquad \frac{\partial \hat{f}_2}{\partial \theta_3}(\theta) = -2\sin\theta_3 \cdot \cos\theta_3 \cdot \hat{f}_1(\theta) \tag{9.11}$$

hold for $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. For $\alpha \in \mathbb{R}$, we consider the element

$$\phi_{2,\alpha} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\alpha & 0 & \cos\alpha \\ 0 & 0 & 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha & 0 & 0 & 0 \end{pmatrix}$$

of $G = SO(6)$ and the involution τ_2 of $\mathbb{R}^3$ defined by

$$\tau_2(\theta_1, \theta_2, \theta_3) = (\theta_3, \theta_2, \theta_1),$$

for $(\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. Then it is easily verified that

$$(\iota^*\phi_{2,\alpha}^* f_1)(\tau_2\theta) = h_1(\theta, \cos\alpha), \tag{9.12}$$

for all $\theta \in \mathbb{R}^3$, and that

$$\begin{gathered}
\tau_2^*\iota^*\phi_{2,\frac{\pi}{2}}^* f_2 = -\hat{f}_2, \qquad \tau_2^*\iota^*\phi_{2,\frac{\pi}{2}}^* f_3 = \hat{f}_3, \\
\frac{\partial}{\partial\alpha}(\tau_2^*\iota^*\phi_{2,\alpha}^* f_j)_{\big|\alpha=\frac{\pi}{2}} = -h_j
\end{gathered} \tag{9.13}$$

on $\mathbb{R}^3$, for $j = 2, 3$.

Let $p, q, r \geq 0$ be given integers. If f is a complex-valued function on X, by (4.10) and (9.12) we have the equality

$$\begin{aligned}
&\frac{\partial}{\partial\alpha}\langle\zeta_3, \phi_{2,\alpha}^*(f\tilde{f}_1^p d\tilde{f}_1)\rangle(\iota'(\tau_2\theta))_{\big|\alpha=\frac{\pi}{2}} \\
&\qquad = (-1)^{p+1} 2\sin\theta_1 \cdot \cos\theta_1 \cdot \hat{f}_1^p(\theta) \cdot \frac{\partial}{\partial\alpha}\big((\phi_{2,\alpha}^* f)(\iota'(\tau_2\theta))\big)_{\big|\alpha=\frac{\pi}{2}},
\end{aligned} \tag{9.14}$$

for all $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$; below we shall use this formula with $f = \tilde{f}_2^q \tilde{f}_3^r$. According to (9.13), we see that

$$\frac{\partial}{\partial \alpha}\big(\phi_{2,\alpha}^*(\tilde{f}_2^q \tilde{f}_3^r)(\iota'(\tau_2\theta))\big)\big|_{\alpha=\frac{\pi}{2}} = (-1)^{q+1}\big(rh_3 \hat{f}_2^q \hat{f}_3^{r-1} - qh_2 \hat{f}_2^{q-1} \hat{f}_3^r\big)(\theta), \tag{9.15}$$

for all $\theta \in \mathbb{R}^3$. Hence if h is the function on $\mathbb{R}^3$ defined by

$$h(\theta) = 2\sin\theta_1 \cdot \cos\theta_1 \cdot \hat{f}_1^p(\theta) \cdot \big(rh_3 \hat{f}_2^q \hat{f}_3^{r-1} - qh_2 \hat{f}_2^{q-1} \hat{f}_3^r\big)(\theta),$$

for all $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$, according to (9.14), (9.15) and (4.6) we have

$$\begin{aligned} \frac{\partial}{\partial \alpha} \int_Z \langle \zeta_3, \phi_{2,\alpha}^*(\tilde{f}_1^p \tilde{f}_2^q \tilde{f}_3^r d\tilde{f}_1)\rangle \, dZ_{\big|\alpha} \\ = (-1)^{p+q} \int_0^\pi \int_0^\pi \int_0^{2\pi} h(\theta_1, \theta_2, \theta_3)\, d\theta_3\, d\theta_1\, d\theta_2. \end{aligned} \tag{9.16}$$

We consider the functions J_1, J_1', J_1'', J_2' and J_2'' on $\mathbb{R}^2$ defined by

$$\begin{aligned} J_1(\theta_1, \theta_2) &= r \int_0^{2\pi} \big(h_3 \hat{f}_2^q \hat{f}_3^{r-1}\big)(\theta)\, d\theta_3, \\ J_1'(\theta_1, \theta_2) &= r \int_0^{2\pi} \big(h_4 \hat{f}_2^q \hat{f}_3^{r-1}\big)(\theta)\, d\theta_3, \\ J_1''(\theta_1, \theta_2) &= q \int_0^{2\pi} \sin\theta_3 \cdot \cos\theta_3 \cdot \big(\hat{f}_1 \hat{f}_2^{q-1} \hat{f}_3^r\big)(\theta)\, d\theta_3, \\ J_2'(\theta_1, \theta_2) &= q \int_0^{2\pi} \big(h_2' \hat{f}_2^{q-1} \hat{f}_3^r\big)(\theta)\, d\theta_3, \\ J_2''(\theta_1, \theta_2) &= q \int_0^{2\pi} \big(h_2'' \hat{f}_2^{q-1} \hat{f}_3^r\big)(\theta)\, d\theta_3, \end{aligned}$$

for $(\theta_1, \theta_2) \in \mathbb{R}^2$, where $\theta = (\theta_1, \theta_2, \theta_3)$. From the relations (9.11), we easily deduce that

$$J_1 = J_1' + 2J_1''. \tag{9.17}$$

Clearly, we have

$$\int_0^{2\pi} h(\theta_1, \theta_2, \theta_3)\, d\theta_3 = 2\sin\theta_1 \cdot \cos\theta_1 \cdot \big(\hat{f}_1^p \cdot (J_1 + 2i(J_2' + J_2''))\big)(\theta_1, \theta_2), \tag{9.18}$$

for all $(\theta_1, \theta_2) \in \mathbb{R}^2$.

If $\theta' = (\theta_1, \theta_2)$ is a point of $\mathbb{R}^2$ for which $B(\theta')$ is non-zero, we easily verify that

$$\cos\theta_1 \cdot h_4(\theta) = \cos\theta_2 \cdot \{\sin^2\theta_1 \cdot (t\sin\theta_3 - i\cos\theta_3) + i\cos\theta_3\},$$

where $\theta = (\theta_1, \theta_2, \theta_3)$ and $t = A(\theta')/B(\theta')$. Thus at this point θ' of $\mathbb{R}^2$, by (8.9) we obtain the relations

$$\begin{aligned}\frac{1}{2\pi}\sin\theta_1\cdot\cos\theta_1\cdot J_1'(\theta')\\ &= rB^{2q+r}(\theta')\cdot(\sin^2\theta_1\cdot F^0_{r,q}(t)+iF^2_{r-1,q}(t)),\\ &= rB^{2q+r}(\theta')\cdot(\sin^2\theta_1\cdot tF^1_{r-1,q}(t)+i\cos^2\theta_1\cdot F^2_{r-1,q}(t)),\end{aligned}\tag{9.19}$$

where $t = A(\theta')/B(\theta')$. At this point θ' of $\mathbb{R}^2$, we also see that

$$\frac{1}{2\pi}\sin\theta_1\cdot\cos\theta_1\cdot J_2'(\theta') = q(A\cdot B^{2q+r-1})(\theta')\cdot F^3_{r,q-1}(t),\tag{9.20}$$

where $t = A(\theta')/B(\theta')$.

If $k, m \geq 0$ are given integers, we easily verify that

$$\int_0^\pi\int_0^{2\pi}\cos^{2k}\theta_2\cdot\sin\theta_3\cdot\cos\theta_3\cdot\big(\hat f_2^q\hat f_3^{2m}\big)(\theta)\,d\theta_3\,d\theta_2 = 0,\tag{9.21}$$

$$\int_0^\pi\int_0^{2\pi}\sin\theta_2\cdot\cos^{2k+1}\theta_2\cdot\big(\hat f_2^q\hat f_3^{2m}\big)(\theta)\,d\theta_3\,d\theta_2 = 0,\tag{9.22}$$

where $\theta = (\theta_1, \theta_2, \theta_3)$. The equality (9.21) implies that

$$\int_0^\pi\int_0^{2\pi}\sin\theta_3\cdot\cos\theta_3\cdot(\hat f_1^p\hat f_2^q\hat f_3^{2m})(\theta)\,d\theta_3\,d\theta_2 = 0,\tag{9.23}$$

where $\theta = (\theta_1, \theta_2, \theta_3)$.

We now suppose that $r = 2m$, where m is an integer ≥ 1. According to (9.22) and (9.23), we have

$$\int_0^\pi(\hat f_1^p\cdot J_1'')(\theta_1,\theta_2)\,d\theta_2 = \int_0^\pi(\hat f_1^p\cdot J_2'')(\theta_1,\theta_2)\,d\theta_2 = 0,$$

for $\theta_1 \in \mathbb{R}$. Therefore from the formulas (9.16)–(9.18), we deduce that

$$\begin{aligned}\frac{\partial}{\partial\alpha}\int_Z\langle\zeta_3,\phi^*_{2,\alpha}(\tilde f_1^p\tilde f_2^q\tilde f_3^{2m}d\tilde f_1)\rangle\,dZ_{\big|\alpha=\frac{\pi}{2}}\\ = (-1)^{p+q}\int_0^\pi\int_0^\pi 2\sin\theta_1\cdot\cos\theta_1\cdot\big(\hat f_1^p\cdot(J_1'+2iJ_2')\big)(\theta_1,\theta_2)\,d\theta_1\,d\theta_2.\end{aligned}\tag{9.24}$$

By Proposition 8.2 and (9.6), we know that the real-valued function χ_1 on $\mathbb{R}^2$ defined by

$$\chi_1(\theta') = \sin^2\theta_1\big(A^2\cdot\tilde Q^1_{m-1,q}\big)(\theta') + \cos^2\theta_1\big(B^2\cdot\tilde Q^2_{m-1,q}\big)(\theta'),$$

for $\theta' = (\theta_1, \theta_2) \in \mathbb{R}^2$, is non-zero and everywhere ≥ 0. We consider the real-valued function χ_2 on $\mathbb{R}^2$ which is defined by

$$\chi_2 = A^2\cdot B^2\cdot\tilde Q^3_{m-1,q-1}$$

when $q \geq 1$ and which vanishes identically when $q = 0$; when $q \geq 1$, we know that the function χ_2 is non-zero and everywhere ≥ 0. From the relations (9.19) and (9.20) and Proposition 8.2, we deduce that the equalities

$$\sin\theta_1 \cdot \cos\theta_1 \cdot J_1' = \frac{m\pi}{4^{m+q-1}}(A^2 - B^2)^{m-1} \cdot \chi_1,$$
$$\sin\theta_1 \cdot \cos\theta_1 \cdot J_2' = -\frac{2iq\pi}{4^{m+q}}(A^2 - B^2)^{m-1} \cdot \chi_2$$

hold at the point $\theta' = (\theta_1, \theta_2) \in \mathbb{R}^2$. Hence the equality

$$\sin\theta_1 \cdot \cos\theta_1 \cdot \hat{f}_1^p \cdot (J_1' + 2iJ_2') = (-1)^p \frac{\pi}{4^{m+q-1}}(A^2 - B^2)^{m+p-1} \cdot (m\chi_1 + q\chi_2)$$

holds at the point $\theta' = (\theta_1, \theta_2)$ of $\mathbb{R}^2$. Thus whenever the integer $m + p$ is odd, the left-hand side of (9.24) does not vanish and so there exists $\alpha_2 \in \mathbb{R}$ such that the integral

$$\int_Z \langle \zeta_3, \phi^*_{2,\alpha_2}(\tilde{f}_1^p \tilde{f}_2^q \tilde{f}_3^{2m} d\tilde{f}_1)\rangle \, dZ$$

is non-zero. We have therefore proved the following result:

Lemma 9.8. — *Let $p, q \geq 0$ and $m > 0$ be given integers. If the integer $m+p$ is odd, the 1-form*

$$\tilde{f}_{p,q,2m} d\tilde{f}_1$$

on X does not satisfy the Guillemin condition.

Let $r_1, r_2 \geq 0$ and $s \geq 2$ be given integers, and let $a_1, a_2, a_3 \in \mathbb{C}$. Suppose that $2r_1 + s \equiv 0 \bmod 4$, and consider the 1-form

$$\vartheta = a_1 \tilde{f}_{r_1-1,r_2,s} d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,s} d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2,s-1} d\tilde{f}_3$$

on X. If $r_2 \geq 1$, we have the relation

$$\vartheta = a_1' \tilde{f}_{r_1-1,r_2,s} d\tilde{f}_1 + a_3' \tilde{f}_{r_1,r_2,s-1} d\tilde{f}_3 + \frac{a_2}{r_2} d\tilde{f}_{r_1,r_2,s},$$

where

$$a_1' = a_1 - \frac{r_1}{r_2}a_2, \qquad a_3' = a_3 - \frac{s}{r_2}a_2.$$

If the 1-form ϑ satisfies the Guillemin condition, from Lemma 9.8 we obtain the vanishing of the coefficient a_3'; then Lemma 9.8 tells us that $a_1' = 0$. On the other hand, when $r_2 = 0$ and $r_1 \geq 1$, we have

$$\vartheta = a_1'' \tilde{f}_{r_1-1,0,s} d\tilde{f}_1 + \frac{a_3}{s} d\tilde{f}_{r_1,0,s},$$

where

$$a_1'' = a_1 - \frac{r_1}{s}a_3;$$

if ϑ satisfies the Guillemin condition, Lemma 9.8 gives us the vanishing of the coefficient a_1''. When $r_1 = r_2 = 0$, the form ϑ is obviously exact. Therefore in all cases, if the 1-form ϑ satisfies the Guillemin condition, it is exact, and so we have shown that assertion (i) of Proposition 9.3 holds.

We consider the identity element I_3 and the matrix

$$A_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

of $SO(3)$; we then consider the element $\phi_3 = (A_3, I_3)$ of the subgroup

$$K = SO(3) \times SO(3)$$

of $G = SO(6)$. The functions $\check{f}_j = \iota^*\phi_3^* f_j$ and $\check{f}'_3 = \iota^*\phi_3^* f'_3$ on $\mathbb{R}^3$ are given by

$$\check{f}_1(\theta) = \cos^2\theta_2 - \sin^2\theta_1, \qquad \check{f}_3(\theta) = \cos(\theta_1+\theta_2)e^{i\theta_3}$$
$$\check{f}_2(\theta) = \cos^2(\theta_1+\theta_2), \qquad \check{f}'_3(\theta) = \cos(\theta_1+\theta_2)e^{-i\theta_3},$$

for $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. By (4.10) and the preceding equalities, if $p, q \geq 0$ are given integers, we have

$$\langle \zeta_3, \phi_3^* d\tilde{f}_1 \rangle(\iota'(\theta)) = \langle \zeta_3, \phi_3^* d\tilde{f}_2 \rangle(\iota'(\theta)) = 0,$$
$$\langle \zeta_3, \phi_3^*(\tilde{f}_1^p \tilde{f}_2^q \tilde{f}_3 d\tilde{f}'_3) \rangle(\iota'(\theta)) = -i(\check{f}_1^p \check{f}_2^q)(\theta) \cdot \cos^2(\theta_1+\theta_2),$$

for $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. Since the function $\check{f}_2$ is non-negative on $\mathbb{R}^3$, if $p, q \geq 0$ are given integers, by (4.6) it follows that

$$\int_Z \langle \zeta_3, \phi_3^*(\tilde{f}_1^p \tilde{f}_2^q d\tilde{f}_1) \rangle \, dZ = \int_Z \langle \zeta_3, \phi_3^*(\tilde{f}_1^p \tilde{f}_2^q d\tilde{f}_2) \rangle \, dZ = 0,$$
$$\frac{1}{i} \int_Z \langle \zeta_3, \phi_3^*(\tilde{f}_1^{2p} \tilde{f}_2^q \tilde{f}_3 d\tilde{f}'_3) \rangle \, dZ > 0.$$

The following lemma is an immediate consequence of the previous relations.

Lemma 9.9. — *Let $r_1 \geq 0$ and $r_2 \geq 1$ be given integers; assume that the integer r_1 is even. Let $a_1, a_2, a_3 \in \mathbb{C}$. If the 1-form*

$$a_1 \tilde{f}_{r_1-1,r_2,0} d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,0} d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2-1,1} d\tilde{f}'_3$$

on X satisfies the Guillemin condition, then we have $a_3 = 0$.

For $\alpha \in \mathbb{R}$, we consider the element

$$\phi_{4,\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & 0 & -\sin\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\alpha & 0 & \cos\alpha \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

of $G = SO(6)$. We consider the functions $\check{f}_{j,\alpha} = \tau_2^* \iota^* \phi_{4,\alpha}^* f_j$ on $\mathbb{R}^3$, with $j = 1, 2$, and the functions $\check{h}_1$ and $\check{h}_2$ on $\mathbb{R}^3$ defined by

$$\check{h}_1(\theta) = \sin 2\theta_2 \cdot (\cos 2\theta_1 + \cos 2\theta_3),$$
$$\check{h}_2(\theta) = \cos 2\theta_3 \cdot (\cos 2\theta_1 - \cos 2\theta_2) + 1 - \cos 2\theta_1 \cdot \cos 2\theta_2,$$

for $\theta \in \mathbb{R}^3$; we may also write

$$\hat{f}_1(\theta) = \tfrac{1}{2}(\cos 2\theta_2 - \cos 2\theta_1),$$

for $\theta \in \mathbb{R}^3$. Then it is easily verified that

$$\check{f}_{1,\alpha} = -\sin^2 \alpha \cdot \hat{f}_1 + \check{f}_{1,0}, \qquad \check{f}_{2,\alpha} = -\frac{\sin^2 \alpha}{4} \cdot \check{h}_2 + \frac{i}{2} \sin \alpha \cdot \check{h}_1 + \check{f}_{2,0}$$

as functions on $\mathbb{R}^3$, for $\alpha \in \mathbb{R}$.

Let $p, q \geq 0$ be given integers. According to (4.10), we have

$$\tau_2^* \iota'^* \langle \zeta_2, \phi_{4,\alpha}^* (\tilde{f}_1^p \tilde{f}_2^{q+1} d\tilde{f}_1) \rangle = \check{f}_{1,\alpha}^p \check{f}_{2,\alpha}^{q+1} \cdot \frac{\partial \check{f}_{1,\alpha}}{\partial \theta_2}, \tag{9.25}$$

for $\alpha \in \mathbb{R}$. Therefore there exists a function

$$P(t, \theta) = \sum_{k=0}^{N} \sigma_k(\theta) t^{N-k}$$

on $\mathbb{R}^4$, with $t \in \mathbb{R}$ and $\theta \in \mathbb{R}^3$, which is a polynomial of degree $N = 2(p + q + 2)$ in the variable t, such that

$$\langle \zeta_2, \phi_{4,\alpha}^* (\tilde{f}_1^p \tilde{f}_2^{q+1} d\tilde{f}_1) \rangle (\iota(\tau_2 \theta)) = \frac{(-1)^{q+1}}{2^{2q+p+1}} i(q+1) P(\sin \alpha, \theta)$$

for $\theta \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. By (4.6), for $0 \leq k \leq N$, we see that

$$\begin{aligned} \frac{1}{k!} \frac{\partial^k}{\partial \alpha^k} \int_Z \langle \zeta_2, \phi_{4,\alpha}^* (\tilde{f}_1^p \tilde{f}_2^{q+1} d\tilde{f}_1) \rangle \, dZ_{\big| \alpha = 0} \\ = \frac{(-1)^{q+1}}{2^{2q+p+1}} i(q+1) \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} \sigma_{N-k}(\theta) \, d\theta_1 \, d\theta_2 \, d\theta_3, \end{aligned} \tag{9.26}$$

where $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$. According to (9.25), the coefficient σ_1 of the polynomial P is given by

$$\sigma_1(\theta) = \sin^2 2\theta_2 \cdot (\cos 2\theta_1 - \cos 2\theta_2)^p \cdot (\cos 2\theta_1 + \cos 2\theta_3) \cdot \check{h}_2^q(\theta),$$

for $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$.

We consider the functions χ_1 and χ_2 on $\mathbb{R}^2$ defined by

$$\chi_1(x_1, x_2) = \cos x_1 - \cos x_2, \qquad \chi_2(x_1, x_2) = 1 - \cos x_1 \cdot \cos x_2,$$

for $(x_1, x_2) \in \mathbb{R}^2$, and the mapping $\chi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\chi(x_1, x_2) = (\chi_1(x_1, x_2), \chi_2(x_1, x_2)),$$

for $(x_1, x_2) \in \mathbb{R}^2$. Let $k, m \geq 0$ be given integers. We consider the function

$$\phi_{k,m} = \chi_1^{2k} \cdot \chi_2^m$$

on $\mathbb{R}^2$ and the function $\psi_{k,m}$ on $\mathbb{R}^2$ defined by

$$\psi_{k,m}(x_1, x_2) = \cos x_1 \cdot (\chi_1^{2k+1} \cdot \chi_2^m)(x_1, x_2),$$

for $(x_1, x_2) \in \mathbb{R}^2$. Then by the binomial theorem, there exist polynomials $P_{k,m}^+$, $P_{k,m}^-$, $Q_{k,m}^+$ and $Q_{k,m}^-$ belonging to $\mathbb{Z}[y_1, y_2]$ all of whose coefficients are non-negative such that the decompositions

$$\phi_{k,m} = \phi_{k,m}^+ - \phi_{k,m}^-, \qquad \psi_{k,m} = \psi_{k,m}^+ - \psi_{k,m}^-$$

hold, where the functions $\phi_{k,m}^+$, $\phi_{k,m}^-$, $\psi_{k,m}^+$ and $\psi_{k,m}^-$ on $\mathbb{R}^2$ are given by

$$\begin{aligned}
\phi_{k,m}^+(x_1, x_2) &= P_{k,m}^+(\cos^2 x_1, \cos^2 x_2), \\
\phi_{k,m}^-(x_1, x_2) &= P_{k,m}^-(\cos^2 x_1, \cos^2 x_2) \cdot \cos x_1 \cdot \cos x_2, \\
\psi_{k,m}^+(x_1, x_2) &= Q_{k,m}^+(\cos^2 x_1, \cos^2 x_2), \\
\psi_{k,m}^-(x_1, x_2) &= Q_{k,m}^-(\cos^2 x_1, \cos^2 x_2) \cdot \cos x_1 \cdot \cos x_2,
\end{aligned}$$

for $(x_1, x_2) \in \mathbb{R}^2$. In fact, the coefficient of y_1^k in $P_{k,m}^+$ is equal to 1, while the coefficient of $y_1 y_2^k$ in $Q_{k,m}^+$ is equal to $2k+1$. Thus the functions $\phi_{k,m}^+$ and $\psi_{k,m}^+$ are non-zero and satisfy the inequalities

$$\phi_{k,m}^+ \geq 0, \qquad \psi_{k,m}^+ \geq 0$$

on $\mathbb{R}^2$; moreover, since the integral over the interval $[0, 2\pi]$ of an odd power of $\cos x$ vanishes, we have the relations

$$(9.27) \qquad \begin{aligned}
\int_0^{2\pi} \phi_{k,m}(x_1, x_2)\, dx_1 &= \int_0^{2\pi} \phi_{k,m}^+(x_1, x_2)\, dx_1, \\
\int_0^{2\pi} \psi_{k,m}(x_1, x_2)\, dx_1 &= \int_0^{2\pi} \psi_{k,m}^+(x_1, x_2)\, dx_1,
\end{aligned}$$

for all $x_2 \in \mathbb{R}$.

We again consider the integers $p, q \geq 0$; we write $q' = [q/2]$ and let J_q be the set all integers r satisfying $0 \leq 2r + 1 \leq q$. For $\lambda, \mu \in \mathbb{R}$, we consider the functions $\Phi_q(\lambda, \mu)$, $\Psi_q(\lambda, \mu)$, $\Phi_q^+(\lambda, \mu)$ and $\Psi_q^+(\lambda, \mu)$ on $\mathbb{R}$ defined by

$$\Phi_q(\lambda, \mu)(x) = (\lambda \cos x + \mu)^q, \qquad \Psi_q(\lambda, \mu)(x) = \cos x \cdot (\lambda \cos x + \mu)^q,$$

$$\check{h}_2(\theta) \Phi_q^+(\lambda, \mu)(x) = \sum_{r=0}^{q'} \binom{q}{2r} \lambda^{2r} \mu^{q-2r} \cos^{2r} x,$$

$$\Psi_q^+(\lambda, \mu)(x) = \sum_{r \in J_q} \binom{q}{2r+1} \lambda^{2r+1} \mu^{q-2r-1} \cos^{2r+2} x,$$

for $x \in \mathbb{R}$. Since the integral over the interval $[0,2\pi]$ of an odd power of $\cos x$ vanishes, by the binomial theorem we easily verify that

$$\begin{aligned}\int_0^{2\pi} \Phi_q(\lambda,\mu)(x)\,dx &= \int_0^{2\pi} \Phi_q^+(\lambda,\mu)(x)\,dx,\\ \int_0^{2\pi} \Psi_q(\lambda,\mu)(x)\,dx &= \int_0^{2\pi} \Psi_q^+(\lambda,\mu)(x)\,dx,\end{aligned}\tag{9.28}$$

for all $\lambda, \mu \in \mathbb{R}$.

We define functions σ and σ^+ on $\mathbb{R}^3$ by

$$\begin{aligned}\sigma(x) &= (\cos x_1 - \cos x_2)^p \cdot \cos x_1 \cdot \Phi_q(\chi(x_1,x_2))(x_3)\\ &\qquad + (\cos x_1 - \cos x_2)^p \cdot \Psi_q(\chi(x_1,x_2))(x_3),\\ \sigma^+(x) &= (\cos x_1 - \cos x_2)^p \cdot \cos x_1 \cdot \Phi_q^+(\chi(x_1,x_2))(x_3)\\ &\qquad + (\cos x_1 - \cos x_2)^p \cdot \Psi_q^+(\chi(x_1,x_2))(x_3),\end{aligned}$$

for $x = (x_1,x_2,x_3) \in \mathbb{R}^3$; then by (9.28), we obtain the equality

$$\int_0^{2\pi} \sigma(x_1,x_2,x_3)\,dx_3 = \int_0^{2\pi} \sigma^+(x_1,x_2,x_3)\,dx_3,\tag{9.29}$$

for all $(x_1,x_2) \in \mathbb{R}^2$. Moreover, we have

$$\sigma_1(\theta) = \sin^2 2\theta_2 \cdot \sigma(2\theta),\tag{9.30}$$

for $\theta = (\theta_1,\theta_2,\theta_3) \in \mathbb{R}^3$.

We henceforth suppose that p is an odd integer which is equal to $2l+1$, with $l \geq 0$. We verify that

$$\begin{aligned}\cos x_1 \cdot \chi_1^p(x_1,x_2) \cdot \Phi_q^+(\chi(x_1,x_2))(x_3) &= \sum_{r=0}^{q'} \binom{q}{2r} \psi_{l+r,q-2r}(x_1,x_2)\cos^{2r} x_3,\\ \chi_1^p(x_1,x_2) \cdot \Psi_q^+(\chi(x_1,x_2))(x_3) &= \sum_{r\in J_q} \binom{q}{2r+1} \phi_{l+r+1,q-2r-1}(x_1,x_2)\cdot\cos^{2r+2} x_3,\end{aligned}$$

for $x_1, x_2, x_3 \in \mathbb{R}$. We introduce the function Θ on $\mathbb{R}^3$ defined by

$$\begin{aligned}\Theta(x) = \sum_{r\in J_q} &\binom{q}{2r+1} \phi^+_{l+r+1,q-2r-1}(x_1,x_2)\cdot\cos^{2r+2} x_3\\ &+ \sum_{r=0}^{q'} \binom{q}{2r} \psi^+_{l+r,q-2r}(x_1,x_2)\cos^{2r} x_3,\end{aligned}$$

for $x = (x_1,x_2,x_3) \in \mathbb{R}^3$. In fact, according to the properties of the polynomials $P^+_{k,m}$ and $Q^+_{k,m}$, there exists a non-zero polynomial $P' \in \mathbb{Z}[y_1,y_2,y_3]$ all of whose coefficients are non-negative such that

$$\Theta(x_1,x_2,x_3) = P'(\cos^2 x_1, \cos^2 x_2, \cos^2 x_3),$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$; moreover, the coefficient of $y_1 y_2^l$ in P' is equal to $2l + 1$. Thus the function Θ is non-zero and satisfies

$$\Theta \geq 0.$$

According to (9.27) and the preceding relations, we have

$$\int_0^{2\pi} \sigma^+(x_1, x_2, x_3)\, dx_1 = \int_0^{2\pi} \Theta(x_1, x_2, x_3)\, dx_1,$$

for $(x_2, x_3) \in \mathbb{R}^2$. By (9.29), we infer that

$$\int_0^{2\pi}\int_0^{2\pi} \sigma(x_1, x_2, x_3)\, dx_1\, dx_3 = \int_0^{2\pi}\int_0^{2\pi} \Theta(x_1, x_2, x_3)\, dx_1\, dx_3 \geq 0,$$

for $x_2 \in \mathbb{R}$; hence, since Θ is non-zero, we obtain the inequality

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sin^2 x_2 \cdot \sigma(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 > 0.$$

Therefore by (9.30), we have the relations

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sigma_1(\theta)\, d\theta_1\, d\theta_2\, d\theta_3 = \frac{1}{4}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sin^2 x_2 \cdot \sigma(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 > 0.$$

By the equality (9.26), with $k = N - 1$, the non-vanishing of the above integrals implies the existence of $\alpha_4 \in \mathbb{R}$ such that

$$\int_Z \langle \zeta_2, \phi_{4,\alpha_4}^*(\tilde{f}_1^p \tilde{f}_2^{q+1} d\tilde{f}_1)\rangle\, dZ \neq 0.$$

Thus we have proved the following:

LEMMA 9.10. — *Let $p, q \geq 1$ be given integers. If the integer p is odd, the 1-form*

$$\tilde{f}_{p,q,0}\, d\tilde{f}_1$$

on X does not satisfy the Guillemin condition.

Let $r_1 \geq 0$ and $r_2 \geq 1$ be given integers; suppose that r_1 is even. Let $a_1, a_2, a_3 \in \mathbb{C}$ and consider the 1-form

$$\vartheta = a_1 \tilde{f}_{r_1-1,r_2,0}\, d\tilde{f}_1 + a_2 \tilde{f}_{r_1,r_2-1,0}\, d\tilde{f}_2 + a_3 \tilde{f}_{r_1,r_2-1,1}\, d\tilde{f}_3'$$

on X. If ϑ satisfies the Guillemin condition, from Lemma 9.9 we obtain the vanishing of the coefficient a_3; then we have the relation

$$\vartheta = a_1' \tilde{f}_{r_1-1,r_2,0}\, d\tilde{f}_1 + \frac{a_2}{r_2} d\tilde{f}_{r_1,r_2,0},$$

where

$$a_1' = a_1 - \frac{r_1}{r_2} a_2.$$

If $r_1 = 0$, the form ϑ is obviously exact. On the other hand, when $r_1 \geq 2$, Lemma 9.10 tells us that $a_1' = 0$. Therefore in all cases, if the 1-form ϑ satisfies the Guillemin condition, it is exact, and so we have proved assertion (iii) of Proposition 9.3.

We have therefore shown that all the eight assertions of Proposition 9.3 hold, and so, according to Proposition 9.3, we have completed the proof of Theorem 9.1.

10. Branching laws

In this section, we present the branching laws and the results which give us the multiplicities of the $SO(6)$-modules announced in Proposition 6.3.

Let $n \geq 3$ be a given integer and consider the complex vector space $\mathbb{C}^n$ endowed with the symmetric bilinear form B defined by

$$B(x, y) = \sum_{j=1}^{n} x_j y_j,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are vectors of $\mathbb{C}^n$. We consider the group $G' = SL(n, \mathbb{C})$; its subgroup $K' = SO(n, \mathbb{C})$, which consists of all elements of G' preserving B, is the set of fixed points of the automorphism of $SL(n, \mathbb{C})$ sending the matrix A into the inverse of its transpose ${}^tA$. The k-th symmetric power $S^k\mathbb{C}^n$ of $\mathbb{C}^n$ and the l-th exterior power $\bigwedge^l \mathbb{C}^n$ of $\mathbb{C}^n$ are irreducible G'-modules; the space $S^k_0\mathbb{C}^n$, which consists of all elements of $S^k\mathbb{C}^n$ which lie in the kernel of the mapping $S^k\mathbb{C}^n \to S^{k-2}\mathbb{C}^n$ determined by B, is an irreducible K'-submodule of $S^k\mathbb{C}^n$.

If E is an G'-module, we denote by $E^{K'}$ the subspace of E consisting of all the K'-invariant elements of E. Then the multiplicity of an irreducible K'-module F in the decomposition of E viewed as an K'-module is equal to $\dim \mathrm{Hom}_{K'}(F, E)$. Moreover, if F is a G'-module viewed as a K'-module, the K'-module F is isomorphic to its contragredient module and so we have the equality

$$\dim \mathrm{Hom}_{K'}(F, E) = \dim (E \otimes F)^{K'}. \tag{10.1}$$

We consider the Lie algebra $\mathfrak{g}' = \mathfrak{sl}(n, \mathbb{C})$ of the group G' and its Cartan subalgebra $\mathfrak{t}'$, which consists of all diagonal matrices of $\mathfrak{g}'$. Let λ'_j be the linear form on $\mathfrak{t}'$ which sends the diagonal matrix of $\mathfrak{t}'$, with $a_1, \dots, a_n \in \mathbb{C}$ as its diagonal entries, into a_j. We write $\alpha'_j = \lambda'_j - \lambda'_{j+1}$, for $1 \leq j \leq n-1$. Then $\{\alpha'_1, \dots, \alpha'_{n-1}\}$ is a system of simple roots of $\mathfrak{g}'$ and the corresponding fundamental weights are

$$\varpi_j = \lambda'_1 + \cdots + \lambda'_j,$$

with $1 \leq j \leq n-1$; we remark that ϖ_j is the highest weight of the irreducible G'-module $\bigwedge^j \mathbb{C}^n$.

The highest weight of an irreducible G'-module is a linear form

$$a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1}$$

on $\mathfrak{t}'$, where $a_1, \dots, a_{n-1} \geq 0$ are integers. The equivalence class of such an G'-module is determined by this weight. We identify the dual Γ' of G' with the set of all such linear forms on $\mathfrak{t}'$.

A partition $\pi = (\pi_1, \dots, \pi_{n-1})$ is an $(n-1)$-tuple of integers satisfying

$$\pi_1 \geq \pi_2 \geq \cdots \geq \pi_{n-1} \geq 0.$$

We denote by P the set of all such partitions $\pi = (\pi_1, \dots, \pi_{n-1})$ for which $\pi_1 + \cdots + \pi_{n-1}$ is even. We say that a partition $\pi = (\pi_1, \dots, \pi_{n-1})$ is even if all the integers $\pi_1, \dots, \pi_{n-1}$ are even, and we denote by P_0 the subset of P consisting of all even partitions. For $1 \leq j \leq n-1$, we consider the subset

$$P_j = \{\, \pi = (\pi_1, \dots, \pi_{n-1}) \in P \mid \pi_j \text{ is even and } \pi_k \text{ is odd, for all } k \neq j \,\}$$

of P. When $n = 4$, we note that P is the disjoint union of the sets P_j, with $0 \leq j \leq 3$.

We associate with an element $\varpi = a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1}$ of Γ' the partition $\pi(\varpi) = (\pi_1, \dots, \pi_{n-1})$, where

$$\pi_j = a_1 + \cdots + a_{n-j},$$

for $1 \leq j \leq n-1$; in fact, this partition uniquely determines the element ϖ of Γ'. We consider an irreducible G'-module $E(\varpi)$ corresponding to $\varpi \in \Gamma'$; we shall also write

$$E(\pi(\varpi)) = E(\varpi).$$

Let $N_0(\varpi)$ be the integer which is equal to 1 if the partition $\pi(\varpi)$ is even and 0 otherwise; according to a result due to Cartan (see also [**10**, p. 550] and Theorem 3 of [**23**]), we know that

$$\dim E(\varpi)^{K'} = N_0(\varpi). \tag{10.2}$$

Let $\varpi = a_1\varpi_1 + \cdots + a_{n-1}\varpi_{n-1}$ be an element of Γ' and consider the partition $\pi(\varpi) = (\pi_1, \dots, \pi_{n-1})$ associated with ϖ. Let $k \geq 1$ be a given integer; then Pieri's formula (see Proposition 15.25,(i) and formula (A.7) of [**3**]) tells us that the G'-module $E(\varpi) \otimes S^k\mathbb{C}^n$ admits a decomposition

$$E(\varpi) \otimes S^k\mathbb{C}^n = \bigoplus_{\eta \in \Sigma(\varpi,k)} E(\eta) \tag{10.3}$$

into irreducible G'-submodules, where $\Sigma(\varpi, k)$ is the set of all partitions $\eta = (\eta_1, \dots, \eta_{n-1})$ defined as follows: a partition $\eta = (\eta_1, \dots, \eta_{n-1})$ belongs to $\Sigma(\varpi, k)$ if and only if there exist integers $\nu_1, \dots, \nu_n \geq 0$ satisfying the relations

$$\eta_j = \nu_j - \nu_n, \qquad \nu_j \geq \pi_j \geq \nu_{j+1},$$
$$\nu_1 + \cdots + \nu_n = \pi_1 + \cdots + \pi_{n-1} + k,$$

for $1 \leq j \leq n-1$. Each factor $E(\eta)$ appears in the sum on the right-hand side of (10.3) with multiplicity 1. We denote by $N_k(\varpi)$ the cardinality of the set $\Sigma'(\varpi, k) = \Sigma(\varpi, k) \cap P_0$ consisting of all even partitions of $\Sigma(\varpi, k)$.

From the relations (10.1) and (10.2) and the decomposition (10.3), for $k \geq 0$, we infer that the integer

$$\dim \mathrm{Hom}_{K'}(S^k\mathbb{C}^n, E(\varpi)) = \dim\,(E(\varpi) \otimes S^k\mathbb{C}^n)^{K'}$$

is equal to $N_k(\varpi)$. Therefore, for $k \geq 2$, the multiplicity $M_k(\varpi)$ of the irreducible K'-module $S_0^k\mathbb{C}^n$ in the decomposition of $E(\varpi)$ viewed as a K'-module is equal to

$$\dim \operatorname{Hom}_{K'}(S_0^k\mathbb{C}^n, E(\varpi)) = N_k(\varpi) - N_{k-2}(\varpi). \tag{10.4}$$

We write $\Sigma(\varpi) = \Sigma(\varpi, 2)$ and $\Sigma'(\varpi) = \Sigma'(\varpi, 2)$ and $M(\varpi) = M_2(\varpi)$; thus we have

$$M(\varpi) = \begin{cases} N_2(\varpi) - 1 & \text{if the partition } \pi(\varpi) \text{ is even,} \\ N_2(\varpi) & \text{otherwise.} \end{cases}$$

For $1 \leq j \leq n-1$, we consider the sequences

$$\xi^j = (\xi^j_1, \dots, \xi^j_{n-1}),$$

where $\xi^j_k = \pi_k$ for $k \neq j$ and $\xi^j_j = \pi_j + 2$; we also consider the sequence

$$\xi^n = (\pi_1 - 2, \dots, \pi_{n-1} - 2).$$

Then ξ^1 always belongs to $\Sigma(\varpi)$; moreover, for $2 \leq j \leq n-1$, the sequence ξ^j is an element of $\Sigma(\varpi)$ if and only if $\pi_{j-1} \geq \pi_j + 2$. On the other hand, the sequence ξ^n belongs to $\Sigma(\varpi)$ if and only if $\pi_{n-1} \geq 2$.

If all the integers $a_1, \dots, a_{n-1}$ are even, then $\Sigma(\varpi)$ is precisely the set of all partitions contained in $\{\xi^1, \dots, \xi^n\}$; from the previous observations, for $2 \leq j \leq n$, we infer that ξ^j belongs to $\Sigma(\varpi)$ if and only if $a_j \geq 2$. Therefore in this case, the integer $M(\varpi)$ is equal to the number of non-zero coefficients a_j.

We have just proved the second assertion of the following proposition; on the other hand, its first assertion is a direct consequence of the equalities (10.1) and (10.2).

Proposition 10.1. — *Let G' be the group $SL(n, \mathbb{C})$ and K' be the group $SO(n, \mathbb{C})$, with $n \geq 3$. Let $\varpi = a_1\varpi_1 + \dots + a_{n-1}\varpi_{n-1}$ be an element of Γ'. The multiplicity of the trivial K'-module in the decomposition of the G'-module $E(\varpi)$, viewed as a K'-module, is equal to 1 if all the coefficients a_j are even and to 0 otherwise. If all the coefficients a_j are even, the multiplicity $M(\varpi)$ of the K'-module $S_0^2\mathbb{C}^n$ in the decomposition of the G'-module $E(\varpi)$, viewed as a K'-module, is equal to the number of non-zero coefficients a_j.*

We now assume that the integer n is equal to 4. Let ϖ be an element of Γ' and consider the partition $\pi(\varpi) = (\pi_1, \pi_2, \pi_3)$ associated with ϖ. We consider the sequences

$$\begin{aligned} \eta^1 &= (\pi_1 + 2, \pi_2, \pi_3), & \eta^2 &= (\pi_1, \pi_2 + 2, \pi_3), \\ \eta^3 &= (\pi_1, \pi_2, \pi_3 + 2), & \eta^4 &= (\pi_1 - 2, \pi_2 - 2, \pi_3 - 2), \\ \eta^5 &= (\pi_1 + 1, \pi_2 + 1, \pi_3), & \eta^6 &= (\pi_1, \pi_2 + 1, \pi_3 + 1), \\ \eta^7 &= (\pi_1 + 1, \pi_2, \pi_3 + 1), & \eta^8 &= (\pi_1 - 1, \pi_2 - 1, \pi_3), \\ \eta^9 &= (\pi_1, \pi_2 - 1, \pi_3 - 1), & \eta^{10} &= (\pi_1 - 1, \pi_2, \pi_3 - 1) \end{aligned}$$

associated with the partition $\pi(\varpi)$. Then $\Sigma(\varpi)$ is a subset of the set of all partitions contained in $\{\eta^1, \dots, \eta^{10}\}$. In fact, η^1 always belongs to $\Sigma(\varpi)$; on the other hand η^5 belongs to $\Sigma(\varpi)$ if and only if $\pi_1 > \pi_2$, while η^6 belongs to $\Sigma(\varpi)$ if and only if $\pi_1 > \pi_2 > \pi_3$. Moreover, η^8 is an element of $\Sigma(\varpi)$ if and only if $\pi_2 > \pi_3 \geq 1$. Finally, any of the other sequences $\eta^j = (\eta^j_1, \eta^j_2, \eta^j_3)$ belongs to $\Sigma(\varpi)$ if and only if $\eta^j_1 \geq \eta^j_2 \geq \eta^j_3 \geq 0$.

We shall now assume that $\pi(\varpi)$ belongs to P; this assumption is equivalent to the fact that $a_1 + a_3$ is even. First, suppose that $\pi(\varpi)$ is even, *i.e.*, $\pi(\varpi)$ belongs to P_0; then from the description of $\Sigma(\varpi)$ given above, we infer that $\Sigma'(\varpi)$ is a subset of $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ and that the sequence η^1 always belongs to $\Sigma'(\varpi)$. On the other hand, the sequence η^2 belongs to $\Sigma'(\varpi)$ if and only if $\pi_1 \geq \pi_2 + 2$, the sequence η^3 belongs to $\Sigma'(\varpi)$ if and only if $\pi_2 \geq \pi_3 + 2$, and the sequence η^4 belongs to $\Sigma'(\varpi)$ if and only if $\pi_3 \geq 2$. In this case, we have $0 \leq M(\varpi) \leq 3$.

Next, we suppose that π_1 is even and that π_2, π_3 are odd, *i.e.*, $\pi(\varpi)$ belongs to P_1; then from the description of $\Sigma(\varpi)$ given above, we infer that $\Sigma'(\varpi)$ is a subset of $\{\eta^6, \eta^9\}$. The sequence η^6 belongs to $\Sigma'(\varpi)$ if and only if $\pi_2 > \pi_3$, and the sequence η^9 always belongs to $\Sigma'(\varpi)$. In this case, we have $1 \leq M(\varpi) \leq 2$.

We now suppose that π_2 is even and that π_1, π_3 are odd, *i.e.*, $\pi(\varpi)$ belongs to P_2; then from the description of $\Sigma(\varpi)$ given above, we infer that $\Sigma'(\varpi)$ is equal to $\{\eta^7, \eta^{10}\}$ and that $M(\varpi) = 2$.

Finally, we suppose that π_3 is even and that π_1, π_2 are odd, *i.e.*, $\pi(\varpi)$ belongs to P_3; then from the description of $\Sigma(\varpi)$ given above we infer that $\Sigma'(\varpi)$ is a subset of $\{\eta^5, \eta^8\}$. The sequence η^5 belongs to $\Sigma'(\varpi)$ if and only if $\pi_1 > \pi_2$, and the sequence η^8 belongs to $\Sigma'(\varpi)$ if and only if $\pi_3 \geq 1$. In this case, we have $1 \leq M(\varpi) \leq 2$.

From the preceding discussion, we obtain the following result:

PROPOSITION 10.2. — *Let G' be the group $SL(4,\mathbb{C})$ and K' be the group $SO(4,\mathbb{C})$. Let $\varpi = a_1\varpi_1 + a_2\varpi_2 + a_3\varpi_3$ be an element of Γ'; suppose that $a_1 + a_3$ is even. Then the partition $\pi = \pi(\varpi)$ belongs to P. The multiplicity $M(\varpi)$ of the K'-module $S^2_0\mathbb{C}^4$ in the decomposition of the G'-module $E(\varpi)$, viewed as a K'-module, satisfies the relations*

$$\begin{aligned} 0 \leq M(\varpi) \leq 3 &\quad \textit{if } \pi(\varpi) \in P_0, \\ M(\varpi) = 2 &\quad \textit{if } \pi(\varpi) \in P_2, \\ 1 \leq M(\varpi) \leq 2 &\quad \textit{if } \pi(\varpi) \in P_1 \cup P_3. \end{aligned}$$

If V is a real finite-dimensional vector space and q is a non-degenerate quadratic form on V with values in a one-dimensional vector space, we denote by $SL(V)$ the group of automorphisms of V whose determinants are equal to 1 and we consider the subgroup $SO(V,q)$ of $SL(V)$ consisting of those elements of $SL(V)$ which preserve the form q.

We once again suppose that the integer n is ≥ 3 and we now consider the real vector space U of dimension n which consists of the real vectors of $\mathbb{C}^n$. We shall identify the space $\mathbb{R}^n$ with U and view $\mathbb{C}^n$ as the complexification $U_{\mathbb{C}}$ of U; we denote by J the complex structure of $U_{\mathbb{C}}$.

The standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ is also a basis of U. The restriction of the symmetric bilinear form B to U is a the standard Euclidean scalar product g on U. We view the group $SL(U) = SL(n, \mathbb{R})$ as the subgroup of G' consisting of those matrices of G' with real entries; then the orthogonal group

$$SO(U) = SO(U, g) = SO(4)$$

is identified with the subgroup $SO(n) = SL(U) \cap K'$ of K'.

By means of the scalar product g, we shall identify the $SO(U)$-module U^* with U. The k-th tensor product $\bigotimes^k U^*$ of U^* is an $SO(U)$-module, and the k-th symmetric product $S^k U^*$ of U^* and the k-th exterior product $\bigwedge^k U^*$ of U^* are $SO(U)$-submodules of $\bigotimes^k U^*$. If M is an $SO(U)$-submodule of $\bigotimes^k U^*$, we identify the p-th symmetric power $S^p M^*$ of the coadjoint module $M^* = \mathrm{Hom}_{SO(U)}(M, \mathbb{R})$ with the $SO(U)$-module of all symmetric p-forms on M.

We view g as an element of $S^2 U^*$. The subspace $S^2_0 U^*$ of $S^2 U^*$, consisting of all elements of $S^2 U^*$ which are orthogonal to g, is an irreducible $SO(U)$-submodule of $S^2 U^*$. We identify $\mathrm{Hom}(U, U)$ with $U^* \otimes U^*$ *via* the scalar product g and consider the trace mapping

$$\mathrm{Hom}(U, U) \longrightarrow \mathbb{R},$$

which sends an endomorphism of U into its trace. Thus we obtain a monomorphism

$$\lambda : S^2 U^* \longrightarrow \mathrm{Hom}(U, U)$$

of $SO(U)$-modules, whose image consists of all self-adjoint endomorphisms of U; the image under λ of the submodule $S^2_0 U^*$ of $S^2 U^*$ is consists of all self-adjoint endomorphisms of U with trace zero.

We now suppose that the integer n is equal to 4. We endow U with the orientation corresponding to the volume element $\Omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ of $\bigwedge^4 U$. We identify the space $\bigwedge^2 U_{\mathbb{C}} = \bigwedge^2 \mathbb{C}^4$ with the complexification of $\bigwedge^2 U$ and denote by J the complex structure of $\bigwedge^2 U_{\mathbb{C}}$. We define a bilinear form Q on $\bigwedge^2 U_{\mathbb{C}}$ with values in the one-dimensional vector space $\bigwedge^4 U_{\mathbb{C}}$ by

$$Q(\xi_1, \xi_2) = \xi_1 \wedge \xi_2,$$

for $\xi_1, \xi_2 \in \bigwedge^2 U_{\mathbb{C}}$. We note that the restriction of Q to $\bigwedge^2 U$ takes its values in the space $\bigwedge^4 U$ and is a non-degenerate quadratic form on $\bigwedge^2 U$. The Hodge $*$ operator, corresponding to the given orientation of U and this quadratic form, is an involution of the vector space $\bigwedge^2 U$. The action of $SO(U)$ on $\bigwedge^2 U$ preserves the eigenspace $\bigwedge^2_+ U$ and $\bigwedge^2_- U$ of the Hodge $*$ operator corresponding to the eigenvalues $+1$ and -1, respectively, which are both three-dimensional. Then we have the decomposition

$$\bigwedge^2 U = \bigwedge^2_+ U \oplus \bigwedge^2_- U$$

of $\bigwedge^2 U$ into irreducible $SO(U)$-modules.

The vectors

$$\omega_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), \qquad \omega_4 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4),$$
$$\omega_2 = \frac{1}{\sqrt{2}}(e_2 \wedge e_3 + e_1 \wedge e_4), \qquad \omega_5 = \frac{1}{\sqrt{2}}(e_2 \wedge e_3 - e_1 \wedge e_4),$$
$$\omega_3 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_2 \wedge e_4), \qquad \omega_6 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_2 \wedge e_4)$$

of $\bigwedge^2 U$ form an orthonormal basis for $\bigwedge^2 U$, which diagonalizes the bilinear form Q; in fact, if we set $\varepsilon_j = +1$, for $j = 1, 2, 3$, and $\varepsilon_j = -1$, for $j = 4, 5, 6$, we have

$$Q(\omega_j, \omega_k) = \varepsilon_j \delta_{jk} \Omega,$$

for $1 \leq j, k \leq 6$. Moreover, the vectors $\{\omega_1, \omega_2, \omega_3\}$ form a basis of $\bigwedge^2_+ U$, while the vectors $\{\omega_4, \omega_5, \omega_6\}$ form a basis of $\bigwedge^2_- U$. We note that a change in the orientation of U simply permutes the factors $\bigwedge^2_+ U$ and $\bigwedge^2_- U$.

The $SO(U)$-module

$$\tilde{U} = \bigwedge^2_+ U \oplus J(\bigwedge^2_- U)$$

is a real subspace of $\bigwedge^2 U_{\mathbb{C}}$ which generates $\bigwedge^2 U_{\mathbb{C}}$ over $\mathbb{C}$. The scalar product g induces a scalar product on $\tilde{U}$ via the natural isomorphism of $SO(U)$-modules $\bigwedge^2 U \to \tilde{U}$. We consider the elements ω'_j of $\bigwedge^2 U_{\mathbb{C}}$ defined by

$$\omega'_j = \omega_j, \qquad \omega'_{j+3} = i\omega_{j+3},$$

for $j = 1, 2, 3$. Then the vectors $\{\omega'_1, \dots, \omega'_6\}$ form a basis of $\bigwedge^2 U_{\mathbb{C}}$ and also for its real subspace $\tilde{U}$; this basis diagonalizes the bilinear form Q and we have

$$Q(\omega'_j, \omega'_k) = \delta_{jk} \Omega,$$

for $1 \leq j, k \leq 6$. Thus if we view the restriction $\tilde{Q}$ of the bilinear form Q to the subspace $\tilde{U}$ as an ordinary quadratic form by means of the volume element Ω, this quadratic form $\tilde{Q}$ is positive definite; in fact, it is equal to the restriction of the scalar product on $\tilde{U}$ induced by g.

The space $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^p U_{\mathbb{C}}, \bigwedge^p U_{\mathbb{C}})$ carries a natural structure of $SO(U)$-module and we denote by J its complex structure. We view the $SO(U)$-module $\mathrm{Hom}(\bigwedge^p U, \bigwedge^p U)$ as a real subspace of $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^p U_{\mathbb{C}}, \bigwedge^p U_{\mathbb{C}})$ and the $SO(U)$-module $\mathrm{Hom}(\tilde{U}, \tilde{U})$ as a real subspace of $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^2 U_{\mathbb{C}}, \bigwedge^2 U_{\mathbb{C}})$. An endomorphism ϕ of $U_{\mathbb{C}}$ (over $\mathbb{C}$) extends to a derivation $\hat{\phi}$ of the exterior algebra of $U_{\mathbb{C}}$. The mapping

$$\tilde{\Psi} : \mathrm{Hom}_{\mathbb{C}}(U_{\mathbb{C}}, U_{\mathbb{C}}) \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\bigwedge^2 U_{\mathbb{C}}, \bigwedge^2 U_{\mathbb{C}}),$$

which sends an element ϕ of $\mathrm{Hom}_{\mathbb{C}}(U_{\mathbb{C}}, U_{\mathbb{C}})$ into the restriction of the mapping $\hat{\phi}$ to $\bigwedge^2 U_{\mathbb{C}}$, is a morphism of $SO(U)$-modules. The mappings

$$\Psi : S^2_0 U^* \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\textstyle\bigwedge^2 U_{\mathbb{C}}, \bigwedge^2 U_{\mathbb{C}}), \qquad \Psi' : S^2_0 U^* \longrightarrow \mathrm{Hom}(\textstyle\bigwedge^2 U, \bigwedge^2 U),$$

which send $h \in S^2_0 U^*$ into $\tilde{\Psi}(J\lambda(h))$ and $\tilde{\Psi}\lambda(h)$, respectively, are also morphisms of $SO(U)$-modules. Now let h be an arbitrary element of $S^2_0 U^*$. Clearly, we have

$$\Psi(h) = J\Psi'(h). \tag{10.5}$$

Because the trace of the endomorphism $\lambda(h)$ of U vanishes, we easily see that the relation

$$Q(\Psi(h)u, v) + Q(u, \Psi(h)v) = 0 \tag{10.6}$$

holds for all $u, v \in \bigwedge^2 U_{\mathbb{C}}$. If $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is the basis of U^* dual to the basis $\{e_1, e_2, e_3, e_4\}$ of U and h_1 is the element $\alpha_1 \otimes \alpha_1 - \alpha_3 \otimes \alpha_3$ of $S^2_0 U^*$, we easily see that

$$\begin{aligned} &\Psi'(h_1)(\omega_1) = \omega_4, && \Psi'(h_1)(\omega_2) = -\omega_5, && \Psi'(h_1)(\omega_3) = 0, \\ &\Psi'(h_1)(\omega_4) = \omega_1, && \Psi'(h_1)(\omega_5) = -\omega_2, && \Psi'(h_1)(\omega_6) = 0. \end{aligned}$$

Since the module $S^2_0 U^*$ is irreducible, from these formulas we obtain the inclusions

$$\Psi'(h)(\textstyle\bigwedge^2_+ U) \subset \bigwedge^2_- U, \qquad \Psi'(h)(\bigwedge^2_- U) \subset \bigwedge^2_+ U;$$

according to these inclusions and the relation (10.5), we see that

$$\Psi(h)(\textstyle\bigwedge^2_+ U) \subset J(\bigwedge^2_- U), \qquad \Psi(h)(J(\bigwedge^2_- U)) \subset \bigwedge^2_+ U, \tag{10.7}$$

and so $\Psi(h)$ belongs to the submodule $\mathrm{Hom}(\tilde{U}, \tilde{U})$ of $\mathrm{Hom}_{\mathbb{C}}(\bigwedge^2 U_{\mathbb{C}}, \bigwedge^2 U_{\mathbb{C}})$. Thus Ψ may be viewed as a morphism

$$\Psi : S^2_0 U^* \longrightarrow \mathrm{Hom}(\tilde{U}, \tilde{U})$$

of $SO(U)$-modules satisfying (10.7). We consider the elements

$$h_2 = \alpha_1 \otimes \alpha_2 + \alpha_2 \otimes \alpha_1, \qquad h_3 = \alpha_1 \otimes \alpha_1 - \alpha_3 \otimes \alpha_3$$

of $S^2_0 U^*$; then we easily verify that

$$\Psi(h_2) = \omega_2 \otimes \omega'_6 + \omega_3 \otimes \omega'_5, \qquad \Psi(h_3) = \omega_1 \otimes \omega'_4 - \omega_2 \otimes \omega'_5. \tag{10.8}$$

Since the spaces $S^2_0 U^*$ and $\mathrm{Hom}(\bigwedge^2_+ U, J(\bigwedge^2_- U))$ have the same dimension, the mapping

$$S^2_0 U^* \longrightarrow \mathrm{Hom}(\textstyle\bigwedge^2_+ U, J(\bigwedge^2_- U)),$$

which sends $h \in S^2_0 U^*$ into the restriction of $\Psi(h)$ to $\bigwedge^2_+ U$, is an isomorphism of $SO(U)$-modules. We write $V = \bigwedge^2_+ U$ and $W = J(\bigwedge^2_- U)$ and we identify the $SO(U)$-modules $\mathrm{Hom}(\bigwedge^2_+ U, J(\bigwedge^2_- U))$ and

$$V \otimes W = \textstyle\bigwedge^2_+ U \otimes J(\bigwedge^2_- U)$$

via the scalar product on $\bigwedge^2_+ U$ induced by g; from the above isomorphism, we therefore obtain an isomorphism of $SO(U)$-modules

$$\Psi : S^2_0 U^* \longrightarrow V \otimes W. \tag{10.9}$$

We consider the bases $\{\theta_1, \theta_2, \theta_3\}$ of V^* and $\{\theta_4, \theta_5, \theta_6\}$ of W^* dual to the bases $\{\omega_1, \omega_2, \omega_3\}$ of V and $\{\omega'_4, \omega'_5, \omega'_6\}$ of W, respectively. The symmetric 2-form g' and the symmetric 3-form σ' on the $SO(U)$-module $V \otimes W$, which are determined by

$$g'(v_1 \otimes w_1, v_2 \otimes w_2) = Q(v_1, v_2) \cdot Q(w_1, w_2\rangle,$$

$$\sigma'(v_1 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_3) = (\theta_1 \wedge \theta_2 \wedge \theta_3)(v_1, v_2, v_3) \cdot (\theta_4 \wedge \theta_5 \wedge \theta_6)(w_1, w_2, w_3),$$

for all $v_1, v_2, v_3 \in V$ and $w_1, w_2, w_3 \in W$, are clearly $SO(U)$-invariant. Thus the symmetric 2-form $\Psi^* g'$ and the symmetric 3-form $\Psi^* \sigma'$ on $S^2_0 U^*$ are $SO(U)$-invariant. According to (10.8), we see that

$$g'(\Psi(h_2), \Psi(h_2)) = 2, \qquad \sigma'(\Psi(h_2), \Psi(h_2), \Psi(h_3)) = -2. \tag{10.10}$$

An element ϕ of G' induces an automorphism ϕ' of $\bigwedge^2 U_{\mathbb{C}}$ which preserves the bilinear form Q. In fact, if G'' denotes the group of automorphisms of $\bigwedge^2 U_{\mathbb{C}}$ (over $\mathbb{C}$) of determinant one which preserve the bilinear form Q, the correspondence $\phi \mapsto \phi'$, where ϕ is an element of G', then gives rise to epimorphism

$$\Phi' : G' \longrightarrow G''.$$

If ϕ is an element of the subgroup $SL(U)$, then the automorphism ϕ' of $\bigwedge^2 U_{\mathbb{C}}$ preserves the subspace $\bigwedge^2 U$. If ϕ is an element of the subgroup $SU(4)$ of G', then the automorphism ϕ' of $\bigwedge^2 U_{\mathbb{C}}$ preserves both the bilinear form Q and the Hermitian scalar product on $\bigwedge^2 U_{\mathbb{C}}$, induced by the Hermitian scalar product on $U_{\mathbb{C}} = \mathbb{C}^4$; it follows that ϕ' also preserves the real subspace $\tilde{U}$ of $\bigwedge^2 U_{\mathbb{C}}$. Thus if we write

$$K'' = SO(\textstyle\bigwedge^2_+ U, \tilde{Q}) \times SO(J(\bigwedge^2_- U), \tilde{Q}),$$

the mapping Φ' induces by restriction epimorphisms

$$\Phi' : SU(4) \longrightarrow SO(\tilde{U}, \tilde{Q}), \qquad \Phi' : SO(4) \longrightarrow K''.$$

We note that the kernels of these two mappings are equal to $\{\pm I_4\}$, where I_4 denotes the identity matrix belonging to $SL(4, \mathbb{R})$.

Let $\{e'_1, \ldots, e'_6\}$ denote the standard basis of $\mathbb{C}^6$. The isomorphism

$$\iota : \textstyle\bigwedge^2 \mathbb{C}^4 \longrightarrow \mathbb{C}^6$$

which sends ω'_j into e'_j, for $1 \leq j \leq 6$, induces by restriction an isomorphism

$$\iota : \tilde{U} \longrightarrow \mathbb{R}^6.$$

We consider the decomposition $\mathbb{R}^6 = F_1 \oplus F_2$ of $\mathbb{R}^6$, where F_1 and F_2 are subspaces of $\mathbb{R}^6$ defined by

$$F_1 = \{\, (b_1, b_2, b_3, 0, 0, 0) \mid b_1, b_2, b_3 \in \mathbb{R} \,\},$$

$$F_2 = \{\, (0, 0, 0, b_4, b_5, b_6) \mid b_4, b_5, b_6 \in \mathbb{R} \,\}.$$

The image of $\bigwedge^2_+ U$ under the isomorphism ι is equal to F_1, while the image of $J(\bigwedge^2_- U))$ under the isomorphism ι is equal to F_2. Clearly, when $\tilde{U}$ is endowed with the scalar product induced by g, the isomorphism $\iota : \tilde{U} \to \mathbb{R}^6$ is an isometry.

The mapping ι induces an isomorphism

$$\iota : G'' \longrightarrow SO(6, \mathbb{C}),$$

which in turn induces by restriction isomorphisms

$$\iota : SO(\tilde{U}, \tilde{Q}) \longrightarrow SO(6), \qquad \iota : K'' \longrightarrow SO(3) \times SO(3).$$

Thus the epimorphism

$$\Phi = \iota \circ \Phi' : SL(4, \mathbb{C}) \longrightarrow SO(6, \mathbb{C})$$

give us by restrictions epimorphisms

$$\Phi : SU(4) \longrightarrow SO(6), \qquad \Phi : SO(4) \longrightarrow SO(3) \times SO(3).$$

We note that the kernels of these three epimorphisms are equal to $\{\pm I_4\}$. Therefore the epimorphism Φ induces an epimorphism

$$\Phi : SO(4, \mathbb{C}) \longrightarrow SO(3, \mathbb{C}) \times SO(3, \mathbb{C})$$

and gives rise to a commutative diagram

$$\begin{array}{ccc} SL(4, \mathbb{C})/\{\pm I_4\} & \longrightarrow & SO(6, \mathbb{C}) \\ \uparrow & & \uparrow \\ SO(4, \mathbb{C})/\{\pm I_4\} & \longrightarrow & SO(3, \mathbb{C}) \times SO(3, \mathbb{C}) \end{array}$$

whose horizontal arrows are isomorphisms induced by the morphisms Φ and whose vertical ones are inclusions.

We henceforth write

$$\tilde{G} = SO(6, \mathbb{C}), \qquad \tilde{K} = SO(3, \mathbb{C}) \times SO(3, \mathbb{C}).$$

We consider the Lie algebra $\mathfrak{g}$ of $\tilde{G}$, the Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and the linear forms $\lambda_1, \lambda_2, \lambda_3$ on $\mathfrak{t}$ introduced in §5. Then $\{\alpha_1, \alpha_2, \alpha_3\}$ is a system of simple roots of $\mathfrak{g}$, where

$$\alpha_1 = \lambda_1 - \lambda_2, \qquad \alpha_2 = \lambda_2 - \lambda_3, \qquad \alpha_3 = \lambda_2 + \lambda_3.$$

The corresponding fundamental weights are

$$\mu_1 = \lambda_1, \qquad \mu_2 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3), \qquad \mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3);$$

we note that λ_1 is the highest weight of the irreducible $\tilde{G}$-module $\mathbb{C}^4$.

The highest weight of an irreducible $\tilde{G}$-module is a linear form

$$c_1\lambda_1 + c_2\lambda_2 + \varepsilon c_3\lambda_3$$

on $\mathfrak{t}$, where $\varepsilon = \pm 1$ and $c_1, c_2, c_3 \geq 0$ are integers satisfying

$$c_1 \geq c_2 \geq c_3 \geq 0.$$

The equivalence class of such an $\tilde{G}$-module is determined by this weight. We identify the dual $\tilde{\Gamma}$ of $\tilde{G}$ with the set of all such linear forms on $\mathfrak{t}$.

We also consider an irreducible $\tilde{G}$-module (resp. $\tilde{K}$-module) E as an irreducible G'-module (resp. K'-module) Φ^*E *via* the epimorphism $\Phi : G' \to \tilde{G}$ (resp. $\Phi : K' \to \tilde{K}$). Now let E be an irreducible $\tilde{G}$-module and F be an irreducible $\tilde{K}$-module; then the multiplicity of the irreducible module F in the decomposition of E viewed as a $\tilde{K}$-module is equal to the multiplicity of the irreducible K'-module Φ^*F in the decomposition of the K'-module Φ^*E, and therefore also the dimension of the space $\mathrm{Hom}_{K'}(\Phi^*F, \Phi^*E)$.

If γ is an element of $\tilde{\Gamma}$, we consider an irreducible $\tilde{G}$-module V_γ corresponding to γ, and we shall denote by $\Phi(\gamma)$ the highest weight of the irreducible G'-module Φ^*V_γ. Then we have $\Phi(\lambda_1) = \varpi_2$ and, replacing the mapping Φ by the epimorphism $\Phi \circ \sigma$ if necessary, we may also suppose that

$$\Phi(\mu_2) = \varpi_1, \qquad \Phi(\mu_3) = \varpi_3.$$

If

$$\gamma = c_1\lambda_1 + c_2\lambda_2 + \varepsilon c_3\lambda_3$$

is an element of $\tilde{\Gamma}$, with $c_1 \geq c_2 \geq c_3 \geq 0$, then it follows that the element $\Phi(\gamma)$ of Γ' is given by

$$\Phi(\gamma) = (c_2 - \varepsilon c_3)\varpi_1 + (c_1 - c_2)\varpi_2 + (c_2 + \varepsilon c_3)\varpi_3.$$

Therefore the partition $\rho(\gamma) = \pi(\Phi(\gamma))$ associated with the G'-module $E(\varpi)$, where $\varpi = \Phi(\gamma)$, is equal to

$$\rho(\gamma) = (\rho_1(\gamma), \rho_2(\gamma), \rho_3(\gamma)) = (c_1 + c_2, c_1 + \varepsilon c_3, c_2 + \varepsilon c_3).$$

Clearly, the partition $\rho(\gamma)$ belongs to P and the mapping

$$\rho : \tilde{\Gamma} \longrightarrow P$$

is injective. Let $\pi = (\pi_1, \pi_2, \pi_3)$ be an arbitrary element of P. If we set

$$b_1 = \frac{1}{2}(\pi_1 + \pi_2 - \pi_3), \qquad b_2 = \frac{1}{2}(\pi_1 - \pi_2 + \pi_3), \qquad b_3 = \frac{\varepsilon}{2}(\pi_2 + \pi_3 - \pi_1),$$

where $\varepsilon = \pm 1$ is chosen so that $b_3 \geq 0$, we see that $\gamma' = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3$ is the unique element of $\tilde{\Gamma}$ satisfying $\rho(\gamma') = \pi$. Thus the mapping ρ is bijective.

We consider the subsets

$$\tilde{\Gamma}^1 = \{\, c_1\lambda_1 + c_2\lambda_2 + \varepsilon c_3\lambda_3 \in \tilde{\Gamma} \mid c_1 - c_2,\ c_2 - c_3 \text{ are even} \,\}$$
$$\tilde{\Gamma}^2 = \{\, c_1\lambda_1 + c_2\lambda_2 + \varepsilon c_3\lambda_3 \in \tilde{\Gamma} \mid c_1 - c_2 \text{ is even, } c_2 - c_3 \text{ is odd} \,\}$$
$$\tilde{\Gamma}^3 = \{\, c_1\lambda_1 + c_2\lambda_2 + \varepsilon c_3\lambda_3 \in \tilde{\Gamma} \mid c_1 - c_2 \text{ is odd, } c_2 - c_3 \text{ is odd} \,\}$$
$$\tilde{\Gamma}^4 = \{\, c_1\lambda_1 + c_2\lambda_2 + \varepsilon c_3\lambda_3 \in \tilde{\Gamma} \mid c_1 - c_2 \text{ is odd, } c_2 - c_3 \text{ is even} \,\}$$

of $\tilde{\Gamma}$; then $\tilde{\Gamma}$ is the disjoint union of these subsets. We easily verify that

$$\rho(\tilde{\Gamma}^{j+1}) \subset P_j,$$

for $j = 0, 1, 2, 3$. Since the subsets $\tilde{\Gamma}^k$ of $\tilde{\Gamma}$ and the subsets P_j of P are disjoint, the induced mapping

$$\rho : \tilde{\Gamma}^{j+1} \longrightarrow P_j$$

is bijective, for $j = 0, 1, 2, 3$.

If $\gamma \in \tilde{\Gamma}$, the subspace of V_γ consisting of all the $\tilde{K}$-invariant elements of V_γ is isomorphic to $E(\varpi)^{K'}$, where $\varpi = \Phi(\gamma)$. Hence by (10.2), we obtain the following result:

Proposition 10.3. — *Let γ be an element of the dual $\tilde{\Gamma}$ of the group $\tilde{G} = SO(6, \mathbb{C})$ and let V_γ be an irreducible $\tilde{G}$-module corresponding to γ. If $\tilde{K}$ is the subgroup $SO(3, \mathbb{C}) \times SO(3, \mathbb{C})$ of $\tilde{G}$, the dimension of the space of all the $\tilde{K}$-invariant elements of V_γ is equal to 1 if γ belongs to $\tilde{\Gamma}^1$, and to 0 otherwise.*

If F is the irreducible $\tilde{K}$-module equal to the complexification of $F_1 \otimes F_2$, then, by means of the isomorphism (10.9) of $SO(4)$-modules, we see that $\Phi^* F$ is equal to the K'-module equal to the complexification of $S^2_0 U^*$. Therefore the multiplicity of F in the decomposition of V_γ, with $\gamma \in \tilde{\Gamma}$, viewed as a $\tilde{K}$-module is equal to the multiplicity of $S^2_0\mathbb{C}^4$ in the decomposition of $E(\varpi)$ viewed as a K'-module, where $\varpi = \Phi(\gamma)$, and hence to the dimension of the space $\mathrm{Hom}_{K'}(S^2_0\mathbb{C}^4, E(\varpi))$.

Now let $1 \leq j \leq 4$, and $r_1, r_2 \geq 0$ and s be given integers, and consider the element $\gamma = \gamma^j_{r_1,r_2,s}$ of $\tilde{\Gamma}^j$ given by (6.4). We consider the element $\varpi = \Phi(\gamma)$ of Γ', the partition $\rho(\gamma) = \pi(\varpi)$ associated with the element ϖ, which belongs to P_{j-1}, and the subset $\Sigma'(\varpi)$ of P which we associated above with ϖ. We also consider the sequences η^k, with $1 \leq k \leq 10$, associated with the partition $\pi = \rho(\gamma)$ in the discussion preceding Proposition 10.2. From the proof of this proposition and this discussion, we deduce the following:

First, assume that $j = 1$. The sequence η^1 always belongs to $\Sigma'(\varpi)$, and the sequence η^3 belongs to $\Sigma'(\varpi)$ if and only if $r_1 \geq 1$. Moreover, if $r_2 \geq 1$, then η^2, η^4 belong to $\Sigma'(\varpi)$; when $r_2 = 0$, the sequence η^2 (resp. η^4) belongs to $\Sigma'(\varpi)$ if and only if $s \leq -1$ (resp. $s \geq 1$). Next, suppose that $j = 2$. The sequence η^6 belongs to $\Sigma'(\varpi)$ if and only if $r_1 \geq 1$, and the sequence η^9 always belongs to $\Sigma'(\varpi)$.

If $j = 3$, the set $\Sigma'(\varpi)$ is equal to $\{\eta^7, \eta^{10}\}$. Finally, suppose that $j = 4$. The sequence η^8 belongs to $\Sigma'(\varpi)$ if either $r_2 \geq 1$ or $s \geq 1$, while the sequence η^5 belongs to $\Sigma'(\varpi)$ if either $r_2 \geq 1$ or $s \leq -1$.

The following result is a consequence of these observations and Proposition 10.2:

Proposition 10.4. — *Let γ be an element of the dual $\tilde{\Gamma}$ of the group $\tilde{G} = SO(6, \mathbb{C})$ and let V_γ be an irreducible $\tilde{G}$-module corresponding to γ. If $\tilde{K}$ is the subgroup $SO(3, \mathbb{C}) \times SO(3, \mathbb{C})$ of $\tilde{G}$, the non-zero multiplicities of the $\tilde{K}$-module $(F_1 \otimes F_2)_{\mathbb{C}}$ in the decomposition of the $\tilde{G}$-module V_γ, viewed as a $\tilde{K}$-module, are given by the table of Proposition 6.3, where $r_1, r_2 \geq 0$ and s are integers and γ is an element of $\tilde{\Gamma}$.*

The vector spaces F_1 and F_2 are equal to the fibers of the vector bundles V and W over $X = \widetilde{G}^{\mathbb{R}}_{3,3}$ at the point x_0 considered in §3, respectively; the tangent space T_{x_0} of X at the point x_0 is a $SO(3) \times SO(3)$-module isomorphic to $V_{x_0} \otimes W_{x_0}$. Since the group $G = SO(6)$ is a real form of the group $\tilde{G} = SO(6, \mathbb{C})$ and the subgroup $SO(3) \times SO(3)$ of G is equal to $G \cap \tilde{K}$, from Proposition 10.4 we deduce the results of Proposition 6.3. The multiplicities of the G-modules $C^\infty_\gamma(X)$, which are given in §5, can also be obtained from Proposition 10.3.

11. The special Lagrangian Grassmannian $SU(4)/SO(4)$

Let G_1 be the group $SU(4)$ and let K_1 be the subgroup $SO(4)$, which is equal to the set of fixed points of the involution s_1 of G_1 sending a matrix into its complex conjugate. We consider the Riemannian symmetric pair (G_1, K_1) and the irreducible symmetric space $X_1 = G_1/K_1$, which is one of the special Lagrangian Grassmannians introduced in §2. In the Cartan decomposition

$$\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$$

of the Lie algebra $\mathfrak{g}_1$ of G_1 corresponding to the involution s_1, we know that $\mathfrak{k}_1$ is the Lie algebra of K_1 and that the K_1-submodule $\mathfrak{p}_1$ is the space of all symmetric purely imaginary 4×4 matrices of trace zero. As in §2, we identify the K_1-module $\mathfrak{p}_1$ with the tangent space of X_1 at the coset of the identity element x_1 of G_1.

We consider the space $U = \mathbb{R}^4$, with its standard Euclidean scalar product, the complexification $U_{\mathbb{C}}$ of U, with its standard basis; we also consider the complex structure J of $U_{\mathbb{C}}$ and the objects associated with U and $U_{\mathbb{C}}$ in §10, notably, the K_1-module $S^2_0 U^*$ and the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of U^*. For $1 \leq j, k \leq 4$, let $E_{jk} = (c_{lr})$ be the 4×4 matrix determined by $c_{jk} = 1$ and $c_{lr} = 0$ whenever $(l, r) \neq (j, k)$. The mapping

$$\mu : S^2_0 U^* \longrightarrow \mathfrak{p}_1,$$

which sends the element $\sum_{j,k=1}^4 a_{jk} \alpha_j \otimes \alpha_k$ of $S^2_0 U^*$, with $a_{jk} = a_{kj} \in \mathbb{R}$, into the 4×4 matrix

$$i \sum_{j,k=1}^{4} a_{jk} E_{jk},$$

is an isomorphism of K_1-modules. We also consider the isomorphism

$$\Psi : S^2_0 U^* \longrightarrow V \otimes W$$

of K_1-modules given by (10.9) and the isomorphism

$$\chi = \Psi \circ \mu^{-1} : \mathfrak{p}_1 \longrightarrow V \otimes W.$$

For $p \geq 2$, we consider the symmetric p-form σ_p on X_1 defined in §2; the symmetric 2-form σ_2 is a G_1-invariant metric on X_1. We also consider the symmetric 2-form g' and the symmetric 3-form σ' on $V \otimes W$ defined in §10, which are both K_1-invariant; the isomorphism χ therefore induces K_1-invariant symmetric forms $\chi^* g'$ and $\chi^* \sigma'$

on $\mathfrak{p}_1$. In §2, we saw that the spaces $(S^2\mathfrak{p}_1^*)^{K_1}$ and $(S^3\mathfrak{p}_1^*)^{K_1}$ are one-dimensional and are generated by σ_2 and σ_3, respectively. Therefore χ^*g' (resp. $\chi^*\sigma'$) is a multiple of σ_2 (resp. of σ_3). If h_2 and h_3 are the elements of $S^2_0U^*$ defined in §10, the elements ξ_2 and ξ_3 of $\mathfrak{p}_1$ defined by

$$\xi_2 = i(E_{12} + E_{21}), \qquad \xi_3 = i(E_{11} - E_{33})$$

satisfy

$$\xi_2 = \mu(h_2), \qquad \xi_3 = \mu(h_3).$$

By (10.10) and the definition of the forms σ_p, we have

$$(\chi^*g')(\xi_2,\xi_2) = 2 = \sigma_2(\xi_2,\xi_2), \tag{11.1}$$

$$(\chi^*\sigma')(\xi_2,\xi_2,\xi_3) = -2 = 2\sigma_3(\xi_2,\xi_2,\xi_3). \tag{11.2}$$

From these equalities, we now deduce that

$$\chi^*g' = \sigma_2, \qquad \chi^*\sigma' = 2\sigma_3. \tag{11.3}$$

We also consider the Grassmannian

$$X = \widetilde{G}^{\mathbb{R}}_{3,3} = SO(6)/SO(3)\times SO(3)$$

endowed with its metric g. If I_3 denotes the unit matrix of order 3, the element

$$S = \begin{pmatrix} -I_3 & 0 \\ 0 & I_3 \end{pmatrix}$$

of $O(6)$ determines an involution s of the group $SO(6)$ which sends the matrix A of $SO(6)$ into SAS; then the subgroup $SO(3)\times SO(3)$ is equal to the identity component of the set of fixed points of s. We consider the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

of the Lie algebra $\mathfrak{g}_0$ of $SO(6)$ corresponding to this involution; in fact, $\mathfrak{k}_0$ is the Lie algebra of $SO(3)\times SO(3)$ and $\mathfrak{p}_0$ is a subspace of $\mathrm{Hom}(\mathbb{R}^6,\mathbb{R}^6)$. In §1, Chapter IV of [**6**], an explicit isomorphism $(V\otimes W)_{x_0} \to \mathfrak{p}_0$ is defined, where x_0 is the point of X considered in §3.

The epimorphism

$$\Phi : SU(4) \longrightarrow SO(6)$$

defined in §10 induces an isomorphism

$$\Phi_* : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$$

and a diffeomorphism

$$\Phi : SU(4)/SO(4) \longrightarrow X.$$

It is easily verified that the diagram

$$\begin{array}{ccc} SU(4) & \xrightarrow{\Phi} & SO(6) \\ \big\downarrow{\scriptstyle s_1} & & \big\downarrow{\scriptstyle s} \\ SU(4) & \xrightarrow{\Phi} & SO(6) \end{array}$$

is commutative, and so the isomorphism Φ_* induces an isomorphism

$$\Phi_* : \mathfrak{p}_1 \longrightarrow \mathfrak{p}_0.$$

We know that $\Phi(x_1)$ is equal to the point x_0; thus the diffeomorphism Φ induces an isomorphism $\Phi_* : \mathfrak{p}_1 \to T_{x_0}$; we continue to identify T_{x_0} with $(V \otimes W)_{x_0}$ *via* the isomorphism (3.2).

We consider the K_1-submodule $\tilde{U}$ of $U_{\mathbb{C}}$ and the morphism

$$\Psi : S_0^2 U^* \longrightarrow \mathrm{Hom}(\tilde{U}, \tilde{U})$$

of K_1-modules defined in §10; as we saw above, we may identify the image of this morphism with $\bigwedge_+^2 U \otimes J(\bigwedge_-^2 U)$. We also recall that the isomorphism $\iota : \tilde{U} \to \mathbb{R}^6$ induces isomorphisms

$$\iota : {\textstyle\bigwedge_+^2} U \longrightarrow V_{x_0}, \qquad \iota : J({\textstyle\bigwedge_-^2} U) \longrightarrow W_{x_0}.$$

Then we see that the diagram

$$\begin{array}{ccc} S_0^2 U^* & \xrightarrow{\mu} & \mathfrak{p}_1 \\ \big\downarrow{\scriptstyle \Psi} & & \big\downarrow{\scriptstyle \Phi_*} \\ \mathrm{Hom}(\tilde{U}, \tilde{U}) & \xrightarrow{\iota} & \mathrm{Hom}(\mathbb{R}^6, \mathbb{R}^6) \end{array}$$

commutes, where the horizontal arrow ι is the mapping induced by the isomorphism $\iota : \tilde{U} \to \mathbb{R}^6$ of §10. From the commutativity of the preceding diagram and the relation (10.6), we infer that the diagram

$$\begin{array}{ccc} S_0^2 U^* & \xrightarrow{\mu} & \mathfrak{p}_1 \\ \big\downarrow{\scriptstyle \Psi} & & \big\downarrow{\scriptstyle \Phi_*} \\ \bigwedge_+^2 U \otimes J(\bigwedge_-^2 U) & \xrightarrow{\iota \otimes \iota} & (V \otimes W)_{x_0} \end{array}$$

commutes. Hence if σ is the symmetric 3-form on the Grassmannian X defined in §2, from the relations (11.3) we deduce that the equalities

$$\Phi^* g = \sigma_2, \qquad \Phi^* \sigma = 2\sigma_3 \tag{11.4}$$

hold on the symmetric space $X_1 = SU(4)/SO(4)$. Thus the diffeomorphism Φ is an isometry.

12. The complex quadric of dimension three

We return to the study of the Grassmannian $X = \widetilde{G}^{\mathbb{R}}_{m,n}$, with $m, n \geq 1$ and $m + n \geq 3$, which is a homogeneous space of the group $G = SO(m + n)$. If $\{v_1, \dots, v_{m+n}\}$ are elements of $\mathbb{R}^{m+n}$, there exists a unique real number

$$c = \det(v_1, \dots, v_{m+n})$$

such that

$$v_1 \wedge \cdots \wedge v_{m+n} = c \cdot e_1 \wedge \cdots \wedge e_{m+n}.$$

Let $p \geq 1$ be a given integer; suppose that $m = p$ and $n = p+1$. Let v be a vector of $\mathbb{R}^{2p+1}$; we consider the section θ_v of $\bigotimes^p T^*$ over X determined by

$$\theta_v(v_1 \otimes w_1, \dots, v_p \otimes w_p) = \det(v, v_1, w_1, \dots, v_p, w_p),$$

for $v_1, \dots, v_p \in V$ and $w_1, \dots, w_p \in W$. It is easily verified that θ_v is in fact a symmetric p-form on X, and that the mapping

$$\mathbb{R}^{2p+1} \longrightarrow C^\infty(S^p T^*),$$

which sends $v \in \mathbb{R}^{2p+1}$ into θ_v, is non-zero and G-equivariant; in fact, we have

$$\phi^* \theta_v = \theta_{\phi^{-1} v},$$

for all $v \in \mathbb{R}^{2p+1}$ and $\phi \in G$. Therefore the image of this mapping is a G-submodule of $C^\infty(S^p T^*)$ which is isomorphic to $\mathbb{R}^{2p+1}$. If τ is the involution of X, corresponding to the change of orientation of a p-plane of $\mathbb{R}^{p+1}$, clearly we have $\tau^*\theta_v = \theta_v$.

We henceforth suppose that $p = 2$. As in Chapter V of [**6**], we identify the Grassmannian $X = \widetilde{G}^{\mathbb{R}}_{2,3}$ with the complex quadric Q_3 of dimension 3 and view it as a Hermitian symmetric space and as a homogeneous space of the group $G = SO(5)$. If E is a sub-bundle of $S^p T^*$ or of $S^p T^*_{\mathbb{C}}$ which is invariant under the group G and the involution τ, we write

$$C^\infty(E)^{\mathrm{ev}} = C^\infty(E)^{\tau,+1}, \qquad C^\infty(E)^{\mathrm{odd}} = C^\infty(E)^{\tau,-1}.$$

We remark that this notation coincides with the one used in [**5**] or [**6**]. For this Grassmannian, we have an injective morphism of G-modules

$$\mathbb{R}^5 \longrightarrow C^\infty(S^2 T^*), \tag{12.1}$$

which sends $v \in \mathbb{R}^5$ into the symmetric 2-form $h_v = \theta_v$ on X; we also consider its complexification

$$\mathbb{C}^5 \longrightarrow C^\infty(S^2 T^*_{\mathbb{C}}),$$

which sends $v + iw$ into $h_{v+iw} = h_v + ih_w$, for all $v, w \in \mathbb{R}^5$. The existence of these two mappings is given by Proposition 9.1 of [**5**] or Proposition 6.25 of [**6**], while the explicit construction of the mapping (12.1) is due to Bryant.

Let v be a vector of $\mathbb{R}^5$. From the definition of the symmetric 2-form h_v, we infer directly that

$$h_v(v_j \otimes w_1, v_j \otimes w_2) = 0, \qquad h_v(v_1 \otimes w_1, v_2 \otimes w_2) = -h_v(v_1 \otimes w_2, v_2 \otimes w_1),$$

for all $v_1, v_2 \in V$, $w_1, w_2 \in W$ and $j = 1, 2$. According to §5, Chapter V of [**6**], it follows that the symmetric 2-form h_v is Hermitian and a section of the subbundle $(S^2T^*)^{+-}$ of S^2T^* introduced in [**4**] and in Chapter V of [**6**]. The subbundle $(S^2T^*)^{+-}$ is invariant under the group G and the involution τ; as we have seen above, h_v is an element of $C^\infty(S^2T^*)^{\mathrm{ev}}$. Thus we know that h_v is an element of $C^\infty((S^2T^*)^{+-})^{\mathrm{ev}}$.

We now consider the mapping

$$\iota' : \mathbb{R}^2 \longrightarrow X$$

defined in §4, whose image Z which is a maximal flat totally geodesic torus of X, and the vector fields ζ_1 and ζ_2 on Z. According to (4.8), we see that

$$h_v(\zeta_j, \zeta_k)(\iota'(\theta)) = \det(v, v_j(\theta_j), w_j(\theta_j), v_k(\theta_k), w_k(\theta_k)),$$

for all $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$ and $j, k = 1, 2$; in particular, we have

$$h_v(\zeta_1, \zeta_2) = \langle v, e_5 \rangle, \tag{12.2}$$

where $\langle\, , \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^5$. Let ϕ be the element of G determined by

$$\phi(e_3) = e_4, \qquad \phi(e_4) = e_5, \qquad \phi(e_5) = e_3, \qquad \phi(e_j) = e_j,$$

for $j = 1, 2$. When we identify X with the complex quadric Q_3, we see that the maximal flat totally geodesic torus $\psi(Z)$ of X is equal to the torus Z_0 of Q_3 considered in [**5**] and in §2, Chapter VI of [**6**]; in fact, the point $\tilde{\sigma}(\theta_1, \theta_2)$ of Z_0 defined there is equal to $\phi\iota'(\theta_1, \theta_2 + \pi/2)$, for $\theta_1, \theta_2 \in \mathbb{R}$; moreover, according to the relation (4.9) the vector fields ξ_0 and η_0 on Z_0 considered in [**5**] and [**6**] are given by

$$\xi_0 = \phi_*\zeta_1, \qquad \eta_0 = \phi_*\zeta_2. \tag{12.3}$$

Then by (12.2) and (12.3), we have

$$h_v(\xi_0, \eta_0) = h_{\phi^{-1}v}(\zeta_1, \zeta_2) = \langle \phi^{-1}v, e_5 \rangle = \langle v, e_3 \rangle. \tag{12.4}$$

Let ψ be the element of G determined by

$$\psi(e_2) = e_3, \qquad \psi(e_3) = e_2, \qquad \psi(e_j) = e_j,$$

for $j = 1, 4, 5$. From the formula (12.4), we infer that

$$(\psi^* h_v)(\xi_0, \eta_0) = h_{\psi^{-1}v}(\xi_0, \eta_0) = \langle \psi^{-1}v, e_3 \rangle = \langle v, e_2 \rangle. \tag{12.5}$$

We consider the complex-valued function $\tilde{f}_{0,1}$ on X, defined in [**4**, §2] or §7, Chapter V of [**6**], and the complex symmetric 2-form k^-, defined in [**4**, §2] or §3, Chapter VI of [**6**], which is a section of $(S^2T^*)^{+-}$ over X. We also consider certain objects introduced in [**4**, §9] and use results and notation found there (see also §7, Chapter V and §7, Chapter VI of [**6**]). We recall that $v_0 = e_1 - ie_2$ is a highest weight vector of the G-module $\mathbb{C}^5$; it follows that h_{v_0} is a highest weight of the G-module $C^\infty_{\gamma'_{0,0}}((S^2T^*)^{+-}_{\mathbb{C}})$.

According to Proposition 9.1 of [**5**] or Proposition 6.25 of [**6**], we know that the latter G-module is irreducible and so we have the equality

$$C^\infty_{\gamma'_{0,0}}((S^2T^*)^{+-}_{\mathbb{C}}) = C^\infty_{\gamma'_{0,0}}((S^2T^*)^{+-}_{\mathbb{C}})^{\mathrm{ev}};$$

another proof of this equality is given by Lemma 9.5 of [**5**] or Lemma 6.31 of [**6**]. According to [**4**, §9], we know that the G-module $C^\infty_{\gamma'_{0,1}}((S^2T^*)^{+-}_{\mathbb{C}})^{\mathrm{odd}}$ is irreducible and that the symmetric 2-form k^- is a highest weight vector of this module. Since $\tilde{f}_{0,1}$ is a highest weight vector of the G-module $C^\infty_{\gamma_{0,1}}(X)^{\mathrm{odd}}$, the symmetric 2-form $\tilde{f}_{0,1}h_{v_0}$ is also a highest weight vector of the G-module $C^\infty_{\gamma'_{0,1}}((S^2T^*)^{+-}_{\mathbb{C}})^{\mathrm{odd}}$. Therefore the two symmetric forms $\tilde{f}_{0,1}h_{v_0}$ and k^- differ by a non-zero constant. According to (12.5), we see that

$$(\psi^* h_{v_0})(\xi_0, \eta_0) = -i;$$

on the other hand, the relations (4.2) of [**5**] say that

$$(\psi^* k^-)(\xi_0, \eta_0) = -\frac{i}{2}\psi^* \tilde{f}_{0,1}.$$

Thus from the preceding formulas, we see that the forms k^- and h_{v_0} are related by

$$k^- = \tfrac{1}{2}\tilde{f}_{0,1} \cdot h_{v_0}; \tag{12.6}$$

thus the element h_0 of $C^\infty_{\gamma'_{0,0}}((S^2T^*)^{+-}_{\mathbb{C}})$ given by Lemma 9.4 of [**5**] or Lemma 6.30 of [**6**] is equal to $\frac{1}{2}h_{v_0}$.

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