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COMPARING BUSHNELL–KUTZKO AND SÉCHERRE'S CONSTRUCTIONS OF TYPES FOR GL_N AND ITS INNER FORMS WITH YU'S CONSTRUCTION

by Arnaud Mayeux & Yuki Yamamoto

Dedicated to Colin J. Bushnell

Abstract. — Let F be a non-Archimedean local field with odd residual characteristic, A be a central simple F-algebra, and G be the multiplicative group of A. To construct types for complex supercuspidal representations of G, simple types by Sécherre and Yu's construction are already known. In this paper, we compare these constructions. In particular, we show essentially tame supercuspidal representations of G defined by Bushnell–Henniart are nothing but tame supercuspidal representations defined by Yu.

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Résumé (*Comparer les constructions de Bushnell–Kutzko et Sécherre de types pour GL_N et ses formes intérieurs à la construction de Yu*). — Soit F un corps local non-archimédien de caractéristique résiduelle impaire, A une F-algèbre centrale simple, et G le groupe multiplicatif de A. Afin de construire des types pour les représentations supercuspidales complexes de G, on dispose des constructions de Sécherre et Yu. Dans cet article, nous comparons ces constructions. En particulier, nous montrons que les représentations essentiellement modérées introduites par Bushnell-Henniart sont des représentations modérées de Yu.

1. Introduction

Let F be a non-Archimedean local field such that the residual characteristic p is odd, and let G be the group of F-points of a connected reductive group defined over F. The aim of type theory is to classify, up to some natural equivalence, the smooth irreducible complex representations of G in terms of representations of compact open subgroups. For complex supercuspidal representations of G, some constructions of types are known.

For example, Bushnell–Kutzko [7] constructed types, called simple types, for any irreducible supercuspidal representations when $G = \mathrm{GL}_N(F)$. Sécherre [26], [27], [28], and Sécherre–Stevens [29] extended the construction of simple types to any irreducible supercuspidal representations of any inner form G of $\mathrm{GL}_N(F)$.

For an arbitrary reductive group G, Yu's construction [31] (cf. also [1] for a similar pioneering method) provides some supercuspidal representations. In his paper [31, p580], Yu wrote "*In particular, our method should yield all supercuspidal representations when p is large enough relative to the type of G.*", Yu also wrote "*it is possible that our method yields all supercuspidal representations that deserve to be called tame*". Yu's expectations are now theorems by works of Kim [19] and Fintzen [12]. More precisely, Yu's construction yields all supercuspidal representations if p does not divide the order of the Weyl group of G by [12], a condition that guarantees that all tori of G split over a tamely ramified field extension of F.

It is a natural question whether there exists some relationship between these constructions of types. A natural motivation being to unify or generalize these constructions by taking advantage of each theory. In his paper [31, p581], Yu wrote "*However, the real difficulty in the wild case is that considerably different* (authors note: than his) *constructions should be involved as revealed in the GL_n case by the work of Bushnell, Corwin, and Kutzko.*" The goal of the present article is to carefully compare Bushnell–Kutzko and Sécherre's constructions to Yu's one.

From now on, let A be a finite dimensional central simple F-algebra, and let V be a simple left A-module. Then $\mathrm{End}_A(V)$ is a central division F-algebra.

Let D be the opposite algebra of $\mathrm{End}_A(V)$. Then V is also a right D-module, and we have $A \cong \mathrm{End}_D(V)$. Let G be the multiplicative group of A. Then we have $G \cong \mathrm{GL}_m(D)$, which is the group of F-points of an inner form of GL_N.

We introduce our main theorem. In Yu's construction, from a tuple $\Psi = (x, (G^i)_{i=0}^d, (\mathbf{r}_i)_{i=0}^d, (\mathbf{\Phi}_d)_{i=0}^d, \rho)$, which is called a Yu datum of G, one constructs some open subgroups ${}^\circ K^d(\Psi)$, $K^d(\Psi)$ in G and an irreducible representation $\rho_d(\Psi)$ of $K^d(\Psi)$, which are precisely defined in §3.

Theorem 1.1 (Theorem 10.6). — *Let (J, λ) be a simple type for an essentially tame supercuspidal representation π, and let $(\tilde{J}, \Lambda)$ be an extension of (J, λ) such that $\pi \cong \text{c-Ind}_{\tilde{J}}^G \Lambda$. Then there exists a Yu datum $\Psi = (x, (G^i)_{i=0}^d, (\mathbf{r}_i)_{i=0}^d, (\mathbf{\Phi}_d)_{i=0}^d, \rho)$ satisfying the following conditions:*

1. $J = {}^\circ K^d(\Psi)$.
2. $\tilde{J} \subset K^d(\Psi)$.
3. $\rho_d(\Psi) \cong \text{c-Ind}_{\tilde{J}}^{K^d(\Psi)} \Lambda$.

Theorem 1.2 (Theorem 11.8). — *Conversely, let $\Psi = \big(x, (G^i), (\mathbf{r}_i), (\mathbf{\Phi}_i), \rho\big)$ be a Yu datum of G. Then there exists a tame simple type (J, λ) and a maximal extension $(\tilde{J}, \Lambda)$ of (J, λ) such that we have the following.*

1. ${}^\circ K^d(\Psi) = J$.
2. $K^d(\Psi) \supset \tilde{J}$.
3. $\rho_d(\Psi) \cong \text{c-Ind}_{\tilde{J}}^{K^d(\Psi)} \Lambda$.

By these theorems, we obtain the following corollary.

Corollary 1.3 (Corollary 11.9). — *For any inner form G of $\mathrm{GL}_N(F)$, the set of essentially tame supercuspidal representations of G is equal to the set of tame supercuspidal representations of G defined by Yu* [31].

In particular, for $G = \mathrm{GL}_N(F)$, the statements of the above theorems are as follows:.

Theorem 1.4. — *Let $G = \mathrm{GL}_N(F)$. Then $\tilde{J} = K^d(\Psi)$ in the statement of (2) in Theorem 1.1 and Theorem 1.2, and $\rho_d(\Psi) \cong \Lambda$ in the statement of (3) in these theorems.*

We sketch the outline of this paper and the main steps to prove Theorems 1.1 and 1.2. First, in §2 and 3, we recall constructions of types. We explain simple types of G by Sécherre in §2 and Yu's construction of tame supercuspidal representations in §3. Next, in §4-9, we prepare ingredients to compare these two constructions. A class of simple types corresponding to Yu's type is defined in §4. In §5, we determine tame twisted Levi subgroups in G. For some tame twisted Levi subgroup G' in G and some "nice" x in the enlarged Bruhat–Tits building $\mathscr{B}^E(G', F)$, we obtain another description of Moy–Prasad filtration on $G'(F)$ attached to x, using hereditary orders, in §6. Then we can compare the groups that the types are defined as representations of. In §7, we discuss generic elements and generic characters. We relate them to some defining sequence of some simple stratum. In §8, we show some lemmas on simple types of depth zero. These lemmas are used to take "depth-zero" parts of types. In §9, we represent a simple character with a tame simple stratum as a product of characters. This factorization is needed to construct generic characters. Finally, in §10 and 11, we prove the main theorem. In §9, from tame Sécherre data, which are used to construct tame simple types, we construct Yu data. By comparing these types constructed by these data and confirming some kind of match between the two, we show that tame simple types can be constructed from Yu's types. Conversely, we also show that Yu's types are constructed from tame simple types in §11. In §12, we briefly discuss the wild case.

Remark 1.5. — The first version of this work, containing all main ideas and arguments, appeared publicly in June 2017. This brings together and extends previous works of the authors that are not intended to be published in journals. These works are precisely:

- A. Mayeux: *Représentations supercuspidales: comparaison des construction de Bushnell–Kutzko et Yu*, arXiv:1706.05920, 2017, unpublished.
- A. Mayeux: *Comparison of Bushnell–Kutzko and Yu's constructions of supercuspidal representations*, arXiv:2001.06259, 2020, unpublished.
- Y. Yamamoto: *Comparison of types for inner forms of GL_N*, arXiv: 2005.02622, 2020, unpublished.
- The parts about the comparison of our PhD theses defended at Paris in 2019 and Tokyo in 2022.

The present paper covers all the mathematical content of all unpublished works listed above. Our present paper is mathematically self-contained in the sense that it does not rely on the previously mentioned unpublished works.

Notation. — If X is a scheme over a base scheme S and if $T \to S$ is a morphism of schemes, then X_T denotes $X \times_S T$ and is seen as a scheme over T.

Let $G \to S$ be a group scheme acting on a scheme X/S. The functor of fixed points, by definition, sends a scheme T over S to $\{x \in \mathrm{Hom}_T(T, X_T) | x \text{ is } G_T\text{-equivariant}\}$, where T is endowed with the trivial action of G_T. This S-functor is denoted by X^G. Note that for any scheme T over S, we have $X^G(T) \subset X(T)$. It is known that this functor is representable by a scheme in many cases (cf., e.g., [9, Exp. 12 Prop. 9.2]).

In this paper, we consider smooth representations over $\mathbb{C}$. We fix a non-Archimedean local field F such that the residual characteristic p is odd. For a finite-dimensional central division algebra D over F, let $\mathfrak{o}_D$ be the ring of integers, $\mathfrak{p}_D$ be the maximal ideal of $\mathfrak{o}_D$, and let k_D be the residual field of D. We fix a smooth, additive character $\psi : F \to \mathbb{C}^\times$ of conductor $\mathfrak{p}_F$. For a finite field extension E/F, let v_E be the unique surjective valuation $E \to \mathbb{Z} \cup \{\infty\}$.

Moreover, for any element β in some algebraic extension field of F, we put $\mathrm{ord}(\beta) = e(F[\beta]/F)^{-1} v_{F[\beta]}(\beta)$.

If K is a field and G is a K-group scheme, then $\underline{\mathrm{Lie}}(G)$ denotes the Lie algebra functor, and we put $\mathrm{Lie}(G) = \underline{\mathrm{Lie}}(G)(K)$. If a K-group scheme is denoted by a capital letter G, the Lie algebra functor of G is denoted by the same small Gothic letter $\mathfrak{g}$. We also denote by $\mathrm{Lie}^*(G)$ or $\mathfrak{g}^*(K)$ the dual of $\mathrm{Lie}(G) = \mathfrak{g}(K)$. For connected reductive group G over F, we denote by $\mathscr{B}^E(G, F)$ (resp. $\mathscr{B}^R(G, F)$) the enlarged Bruhat–Tits building (resp. the reduced Bruhat–Tis building) of G over F defined in [4], [5]. If $x \in \mathscr{B}^E(G, F)$, we denote by $[x]$ the image of x via the canonical surjection $\mathscr{B}^E(G, F) \to \mathscr{B}^R(G, F)$. The group $G(F)$ acts on $\mathscr{B}^E(G, F)$ and $\mathscr{B}^R(G, F)$. For $x \in \mathscr{B}^E(G, F)$, let $G(F)_x$ (reps. $G(F)_{[x]}$) denote the stabilizer of $x \in \mathscr{B}^E(G, F)$ (resp. $[x] \in \mathscr{B}^R(G, F)$). We denote by $\tilde{\mathbb{R}}$ the totally ordered commutative monoid $\mathbb{R} \cup \{r+ \mid r \in \mathbb{R}\}$. When G splits over some tamely ramified extension of F, for $x \in \mathscr{B}^E(G, F)$ let $\{G(F)_{x,r}\}_{r \in \tilde{\mathbb{R}}_{\geq 0}}$, $\{\mathfrak{g}(F)_{x,r}\}_{r \in \tilde{\mathbb{R}}}$ and $\{\mathfrak{g}^*(F)_{x,r}\}_{r \in \tilde{\mathbb{R}}}$ be the Moy–Prasad filtration [23], [24] on $G(F)$, $\mathfrak{g}(F)$ and $\mathfrak{g}^*(F)$, respectively. Here, we have $\mathfrak{g}^*(F)_{x,r} = \{X^* \in \mathfrak{g}^*(F) \mid X^*(\mathfrak{g}(F)_{x,(-r)+}) \subset \mathfrak{p}_F\}$ for $r \in \mathbb{R}$. If G is a torus, Moy–Prasad filtrations are independent of x, and then we omit x.

Let G be a group, H be a subgroup in G and λ be a representation of H. Then we put ${}^gH = gHg^{-1}$ for $g \in G$ and we define a ${}^gH$-representation ${}^g\lambda$ as ${}^g\lambda(h) = \lambda(g^{-1}hg)$ for $h \in {}^gH$. Moreover, we also put

$$I_G(\lambda) = \{g \in G \mid \mathrm{Hom}_{H \cap {}^gH}(\lambda, {}^g\lambda) \neq 0\}.$$

2. Simple types by Sécherre

We recall the theory of simple types from [26], [27], [28], [29]. In this section, we can omit the assumption that the residual characteristic of F is odd.

2.1. Lattices, hereditary orders. — Let D be a finite-dimensional central division F-algebra. Let V be a right D-module with $\dim_F V < \infty$. We put $A = \mathrm{End}_D(V)$, and then A is a central simple F-algebra. Moreover, there exists $m \in \mathbb{Z}_{>0}$ such that $A \cong \mathrm{M}_m(D)$. Let G be the multiplicative group of A, and then G is isomorphic to $\mathrm{GL}_m(D)$. We also put $d = (\dim_F D)^{1/2}$ and $N = md$.

An $\mathfrak{o}_D$-submodule $\mathcal{L}$ in V is called a lattice if and only if $\mathcal{L}$ is a compact open submodule.

Definition 2.1 ([26, Définition 1.1]). — For $i \in \mathbb{Z}$, let $\mathcal{L}_i$ be a lattice in V. We say that $\mathcal{L} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ is an $\mathfrak{o}_D$-sequence if

1. $\mathcal{L}_i \supset \mathcal{L}_j$ for any $i < j$, and
2. there exists $e \in \mathbb{Z}_{>0}$ that $\mathcal{L}_{i+e} = \mathcal{L}_i \mathfrak{p}_D$ for any i.

The number $e = e(\mathcal{L})$ is called the $\mathfrak{o}_D$-period of $\mathcal{L}$. An $\mathfrak{o}_D$-sequence $\mathcal{L}$ is called an $\mathfrak{o}_D$-chain if $\mathcal{L}_i \supsetneq \mathcal{L}_{i+1}$ for every i. An $\mathfrak{o}_D$-chain $\mathcal{L}$ is called uniform if $[\mathcal{L}_i : \mathcal{L}_{i+1}]$ is constant for any i.

An $\mathfrak{o}_F$-subalgebra $\mathfrak{A}$ in A is called a hereditary $\mathfrak{o}_F$-order if every left and right ideal in $\mathfrak{A}$ is $\mathfrak{A}$-projective.

We explain the relationship between $\mathfrak{o}_D$-sequences in V and hereditary $\mathfrak{o}_F$-orders in A from [26, 1.2]. Let $\mathcal{L} = (\mathcal{L}_i)$ be an $\mathfrak{o}_D$-sequence in V. For $i \in \mathbb{Z}$, we put

$$\mathfrak{P}_i(\mathcal{L}) = \{a \in A \mid a\mathcal{L}_j \subset \mathcal{L}_{i+j},\, j \in \mathbb{Z}\}.$$

Then $\mathfrak{A} = \mathfrak{P}_0(\mathcal{L})$ is a hereditary $\mathfrak{o}_F$-order with the radical $\mathfrak{P}(\mathfrak{A}) = \mathfrak{P}_1(\mathcal{L})$. For every hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in V, there exists an $\mathfrak{o}_D$-chain $\mathcal{L}$ in V such that $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$. If $\mathcal{L}$ is a uniform $\mathfrak{o}_D$-chain, $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$ is called principal.

For any $\mathfrak{o}_D$-chain $\mathcal{L} = (\mathcal{L}_i)$, let $\mathfrak{K}(\mathcal{L})$ be the set of $g \in G$ such that there exists $n \in \mathbb{Z}$ satisfying $g\mathcal{L}_i = \mathcal{L}_{i+n}$ for any i. For the hereditary $\mathfrak{o}_F$-order $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$, let $\mathfrak{K}(\mathfrak{A})$ be the set of $g \in G$ such that $g\mathfrak{A}g^{-1} = \mathfrak{A}$. Then we have $\mathfrak{K}(\mathfrak{A}) = \mathfrak{K}(\mathcal{L})$ and $\mathfrak{K}(\mathfrak{A})$ is compact modulo center.

For $g \in \mathfrak{K}(\mathfrak{A})$, there exists a unique element $n \in \mathbb{Z}$ such that $g\mathfrak{A} = \mathfrak{P}(\mathfrak{A})^n$. The map $g \mapsto n$ induces a group homomorphism $v_{\mathfrak{A}} : \mathfrak{K}(\mathfrak{A}) \to \mathbb{Z}$. Let $\mathbf{U}(\mathfrak{A})$ be the kernel of $v_{\mathfrak{A}}$. Then we have $\mathbf{U}(\mathfrak{A}) = \mathfrak{A}^\times$, and $\mathbf{U}(\mathfrak{A})$ is the unique maximal compact open subgroup in $\mathfrak{K}(\mathfrak{A})$. We put $\mathbf{U}^0(\mathfrak{A}) = \mathbf{U}(\mathfrak{A})$ and $\mathbf{U}^n(\mathfrak{A}) = 1 + \mathfrak{P}(\mathfrak{A})^n$ for $n \in \mathbb{Z}_{>0}$. We also put $e(\mathfrak{A}|\mathfrak{o}_F) = v_{\mathfrak{A}}(\varpi_F)$, and then we have $e(\mathfrak{A}|\mathfrak{o}_F) = de(\mathcal{L})$ for an $\mathfrak{o}_D$-chain $\mathcal{L}$ in V such that $\mathfrak{A} = \mathfrak{A}(\mathcal{L})$.

Let E be an extension field of F in A. Since A is a central simple F-algebra, the centralizer $B = \mathrm{Cent}_A(E)$ of E in A is a central simple E-algebra. On the other hand, V is equipped with an E-vector space structure via $E \subset A$. Since the actions of E and D in V are compatible, V is also equipped with a right $D \otimes_F E$-module structure, and then we have $B = \mathrm{Cent}_A(E) = \mathrm{End}_{D\otimes_F E}(V)$.

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A. The order $\mathfrak{A}$ is called E-pure if we have $E^\times \subset \mathfrak{K}(\mathfrak{A})$.

Proposition 2.2 ([2, Theorem 1.3]). — *For an E-pure hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A, the subring $\mathfrak{B} = \mathfrak{A} \cap B$ in B is a hereditary $\mathfrak{o}_E$-order in B with the radical $\mathfrak{Q} = \mathfrak{P}(\mathfrak{A}) \cap B$.*

For any finite extension field E of F, we put $A(E) = \mathrm{End}_F(E)$, and then $A(E)$ is a central simple F-algebra. The field E is canonically embedded in $A(E)$ as a maximal subfield. By [7, 1.2], there exists a unique E-pure hereditary $\mathfrak{o}_F$-order $\mathfrak{A}(E) = \{x \in A(E) \mid x(\mathfrak{p}_E^i) \subset \mathfrak{p}_E^i,\, i \in \mathbb{Z}\}$ in $A(E)$, which is associated with the $\mathfrak{o}_F$-chain $(\mathfrak{p}_E^i)_{i\in\mathbb{Z}}$. Then we have $v_{\mathfrak{A}(E)}(\beta) = v_E(\beta)$ for $\beta \in E^\times$.

For $\beta \in \bar{F}$, we put $n_F(\beta) = -v_{F[\beta]}(\beta) = -v_{\mathfrak{A}(F[\beta])}(\beta)$, as in [26, 2.3.3].

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A with the radical $\mathfrak{P}$. For nonnegative integer i, j with $\lfloor j/2 \rfloor \leq i \leq j$, the map $1 + x \mapsto x$ induces the group isomorphism

$$\mathbf{U}^{i+1}(\mathfrak{A})/\mathbf{U}^{j+1}(\mathfrak{A}) \cong \mathfrak{P}^{i+1}/\mathfrak{P}^{j+1}.$$

If i and j are as above and $c \in \mathfrak{P}^{-j}$, we can define a character ψ_c of $\mathbf{U}^{i+1}(\mathfrak{A})$ as

$$\psi_c(1 + x) = \psi \circ \mathrm{Trd}_{A/F}(cx)$$

for $1 + x \in \mathbf{U}^{i+1}(\mathfrak{A})$. We have $\psi_c = \psi_{c'}$ if and only if $c - c' \in \mathfrak{P}^{-i}$.

2.2. Strata, defining sequences of simple strata. —

Definition 2.3 ([28, §2.1, Remarque 4.1]). — 1. A 4-tuple $[\mathfrak{A}, n, r, \beta]$ is called a stratum in A if $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A, n and r are nonnegative integer with $n \geq r$, and $\beta \in \mathfrak{P}(\mathfrak{A})^{-n}$.

2. A stratum $[\mathfrak{A}, n, r, \beta]$ is called pure if the following hold:
   (a) $E = F[\beta]$ is a field.
   (b) $\mathfrak{A}$ is E-pure.
   (c) $n > r$.
   (d) $v_{\mathfrak{A}}(\beta) = -n$.
3. A stratum $[\mathfrak{A}, n, r, \beta]$ is called simple if one of the following holds:
   (a) $n = r = 0$ and $\beta \in \mathfrak{o}_F$.
   (b) $[\mathfrak{A}, n, r, \beta]$ is pure, and $r < -k_0(\beta, \mathfrak{A})$, where $k_0(\beta, \mathfrak{A}) \in \mathbb{Z} \cup \{-\infty\}$ is defined as in [28, §2.1] such that $k_0(\beta, \mathfrak{A}) = -\infty$ if and only if $\beta \in F$, and $v_{\mathfrak{A}}(\beta) \leq k_0(\beta, \mathfrak{A})$ for $\beta \notin F$.

Remark 2.4. — In [28, §2.1], simple strata are assumed be pure. By adding strata satisfying (3)(a) to simple strata, we can regard simple types of depth zero as coming from simple strata.

Definition 2.5. — Strata $[\mathfrak{A}, n, r, \beta]$ and $[\mathfrak{A}, n, r, \beta']$ in A are called equivalent if $\beta - \beta' \in \mathfrak{P}(\mathfrak{A})^{-r}$.

Theorem 2.6 ([28, Théorème 2.2]). — *Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum. Then there exists $\gamma \in A$ such that $[\mathfrak{A}, n, r, \gamma]$ is simple and equivalent to $[\mathfrak{A}, n, r, \beta]$.*

For $\beta \in \bar{F}$, we put $k_F(\beta) = k_0(\beta, \mathfrak{A}(F[\beta]))$ as in [26, 2.3.3].

Proposition 2.7 ([26, Proposition 2.25]). — *Suppose $E = F[\beta]$ can be embedded in A. We fix an embedding $E \hookrightarrow A$. Let $\mathfrak{A}$ be an E-pure hereditary $\mathfrak{o}_F$-order in A. Then we have $k_0(\beta, \mathfrak{A}) = e(\mathfrak{A}|\mathfrak{o}_F)e(E/F)^{-1}k_F(\beta)$.*

The following lemma is used later.

Lemma 2.8. — *Let E/F be a field extension in A, and let $\mathfrak{A}$ be an E-pure hereditary $\mathfrak{o}_F$-order in A. Then, we have $k_0(\gamma,\mathfrak{A}) = e(\mathfrak{A}|\mathfrak{o}_F)e(E/F)^{-1}k_0(\gamma,\mathfrak{A}(E))$ for any $\gamma \in E$.*

Proof. — First, by Proposition 2.7 we have

$$k_0(\gamma,\mathfrak{A}(E)) = e(\mathfrak{A}(E)|\mathfrak{o}_F)e(F[\gamma]/F)^{-1}k_F(\gamma).$$

On the other hand, we also have $e(\mathfrak{A}(E)|\mathfrak{o}_F) = e(E/F)$ by definition of $\mathfrak{A}(E)$. Then we obtain

$$\begin{aligned} e(\mathfrak{A}|\mathfrak{o}_F)e(E/F)^{-1}k_0(\gamma,\mathfrak{A}(E)) &= e(\mathfrak{A}|\mathfrak{o}_F)e(E/F)^{-1}e(E/F)e(F[\gamma]/F)^{-1}k_F(\gamma) \\ &= e(\mathfrak{A}|\mathfrak{o}_F)e(F[\gamma]/F)^{-1}k_F(\gamma) \\ &= k_0(\gamma,\mathfrak{A}), \end{aligned}$$

where the last equality also follows from Proposition 2.7. □

Definition 2.9. — An element $\beta \in \bar{F}$ is called minimal if $\beta \in F$ or $k_F(\beta) = -v_F(\beta)$.

Definition 2.10. — Let $[\mathfrak{A},n,r,\beta]$ be a simple stratum in A. A sequence $([\mathfrak{A},n,r_i,\beta_i])_{i=0}^{s}$ is called a defining sequence of $[\mathfrak{A},n,r,\beta]$ if

1. $\beta_0 = \beta, r_0 = r$.
2. $r_{i+1} = -k_0(\beta_i,\mathfrak{A})$ for $i = 0,1,\ldots,s-1$.
3. $[\mathfrak{A},n,r_{i+1},\beta_{i+1}]$ is simple and equivalent to $[\mathfrak{A},n,r_{i+1},\beta_i]$ for $i = 0,1,\ldots,s-1$.
4. β_s is minimal over F.

By Theorem 2.6, for any simple stratum $[\mathfrak{A},n,r,\beta]$, there exists a defining sequence of $[\mathfrak{A},n,r,\beta]$, as in the case A is split over F.

2.3. Simple characters. — Let $[\mathfrak{A},n,0,\beta]$ be a simple stratum in A. Then we can define compact open subgroups $J(\beta,\mathfrak{A})$ and $H(\beta,\mathfrak{A})$ in $\mathbf{U}(\mathfrak{A})$ as in [26, §3]. The subgroup $H(\beta,\mathfrak{A})$ in $\mathbf{U}(\mathfrak{A})$ is also contained in $J(\beta,\mathfrak{A})$. For $i \in \mathbb{Z}_{\geq 0}$, we put $J^i(\beta,\mathfrak{A}) = J(\beta,\mathfrak{A}) \cap \mathbf{U}^i(\mathfrak{A})$ and $H^i(\beta,\mathfrak{A}) = H(\beta,\mathfrak{A}) \cap \mathbf{U}^i(\mathfrak{A})$.

Lemma 2.11 ([26, §3.3]). — *Let $[\mathfrak{A},n,0,\beta]$ be a simple stratum in A. If β is not minimal, let $([\mathfrak{A},n,r_i,\beta_i])_{i=0}^{s}$ be a defining sequence of $[\mathfrak{A},n,0,\beta]$.*

1. *$J^i(\beta,\mathfrak{A})$ is normalized by $\mathfrak{K}(\mathfrak{A}) \cap B^\times$ for any $i \in \mathbb{Z}_{\geq 0}$.*
2. *We have $J(\beta,\mathfrak{A}) = \mathbf{U}(\mathfrak{B})J^1(\beta,\mathfrak{A})$, where $\mathfrak{B} = \mathfrak{A} \cap B$.*
3. *If β is minimal, we have*

$$J^1(\beta,\mathfrak{A}) = \mathbf{U}^1(\mathfrak{B})\mathbf{U}^{\lfloor (n+1)/2 \rfloor}(\mathfrak{A}),\ H^1(\beta,\mathfrak{A}) = \mathbf{U}^1(\mathfrak{B})\mathbf{U}^{\lfloor n/2 \rfloor+1}(\mathfrak{A}).$$

4. *If β is not minimal, we have*

$$J^t(\beta,\mathfrak{A}) = J^t(\beta_1,\mathfrak{A}),\ H^{t'+1}(\beta,\mathfrak{A}) = H^{t'+1}(\beta_1,\mathfrak{A}),$$

where $t = \lfloor(-k_0(\beta,\mathfrak{A})+1)/2\rfloor$ *and* $t' = \lfloor -k_0(\beta,\mathfrak{A})/2\rfloor$. *Moreover, we also have*

$$J^1(\beta,\mathfrak{A}) = \mathbf{U}^1(\mathfrak{B})J^t(\beta_1,\mathfrak{A}),\ H^1(\beta,\mathfrak{A}) = \mathbf{U}^1(\mathfrak{B})H^{t'+1}(\beta_1,\mathfrak{A}).$$

Definition 2.12 ([26, Proposition 3.47]). — Let $[\mathfrak{A},n,0,\beta]$ be a simple stratum. We put $q = -k_0(\beta,\mathfrak{A})$. Let $0 \le t < q$ and we put $t' = \max\{t, \lfloor q/2\rfloor\}$. If β is not minimal over F, we fix a defining sequence $([\mathfrak{A},n,r_i,\beta_i])_{i=0}^{s}$ of $[\mathfrak{A},n,0,\beta]$. The set of simple characters $\mathscr{C}(\beta,t,\mathfrak{A})$ consists of characters θ of $H^{t+1}(\beta,\mathfrak{A})$ satisfying the following conditions:

1. $\mathfrak{K}(\mathfrak{A})\cap B^\times$ normalizes θ.
2. $\theta|_{H^{t+1}(\beta,\mathfrak{A})\cap\mathbf{U}(\mathfrak{B})}$ factors through $\mathrm{Nrd}_{B/E}$.
3. If β is minimal over F, we have $\theta|_{H^{t+1}(\beta,\mathfrak{A})\cap\mathbf{U}^{\lfloor n/2\rfloor+1}(\mathfrak{A})} = \psi_\beta$.
4. If β is not minimal over F, there exists $\theta' \in \mathscr{C}(\beta_1,t',\mathfrak{A})$ such that $\theta|_{H^{t'+1}(\beta,\mathfrak{A})} = \psi_{\beta-\beta_1}\theta'$.

Remark 2.13. — This definition is well defined and independent of the choice of a defining sequence by [26, Définition 3.45, Proposition 3.47]. Moreover, for any simple stratum $[\mathfrak{A},n,0,\beta]$ the set $\mathscr{C}(\beta,0,\mathfrak{A})$ is nonempty by [26, Corollaire 3.35, Définition 3.45].

We recall the properties of $\mathscr{C}(\beta,0,\mathfrak{A})$ from [27]. For $\theta\in\mathscr{C}(\beta,0,\mathfrak{A})$, there exists an irreducible $J^1(\beta,\mathfrak{A})$-representation η_θ containing θ, unique up to isomorphism. We call η_θ the Heisenberg representation of θ. We have $\dim\eta_\theta = \big(J^1(\beta,\mathfrak{A}) : H^1(\beta,\mathfrak{A})\big)^{1/2}$. Moreover, there exists an extension κ of η_θ to $J(\beta,\mathfrak{A})$ such that $I_G(\kappa) = J^1B^\times J^1$. We call κ a β-extension of η_θ. If κ is a β-extension of η_θ, then any β-extension of η_θ is the form $\kappa\otimes(\chi\circ\mathrm{Nrd}_{B/E})$, where χ is trivial on $1+\mathfrak{p}_E$ and $\chi\circ\mathrm{Nrd}_{B/E}$ is regarded a character of $J(\beta,\mathfrak{A})$ via the isomorphism $J(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A}) \cong \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$.

2.4. Maximal simple types. — We state the definition of maximal simple types. Recall that for a simple stratum $[\mathfrak{A},n,0,\beta]$ we put $E = F[\beta]$, $B = \mathrm{Cent}_A(E)$ and $\mathfrak{B} = \mathfrak{A}\cap B$. Since B is a central simple E-algebra, there exist $m_E\in\mathbb{Z}$ and a division E-algebra D_E such that $B\cong \mathrm{M}_{m_E}(D_E)$.

Definition 2.14 ([28, §2.5, §4.1]). — A pair (J,λ) consisting a compact open subgroup J in G and an irreducible J-representation λ is called a maximal simple type if there exists a simple stratum $[\mathfrak{A},n,0,\beta]$ and irreducible J-representations κ and σ satisfying the following assertions:

1. $\mathfrak{B}$ is a maximal hereditary $\mathfrak{o}_E$-order in A, that is, $\mathfrak{B}\cong\mathrm{M}_{m_E}(\mathfrak{o}_{D_E})$.
2. $J = J(\beta,\mathfrak{A})$.
3. κ is a β-extension of η_θ for some $\theta\in\mathscr{C}(\beta,0,\mathfrak{A})$.

4. σ is trivial on $J^1(\beta,\mathfrak{A})$, and when we regard σ as a $\mathrm{GL}_{m_E}(k_{D_E})$-representation via the isomorphism
$$J(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A}) \cong \mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) \cong \mathrm{GL}_{m_E}(k_{D_E}),$$
σ is a cuspidal representation of $\mathrm{GL}_{m_E}(k_{D_E})$.
5. $\lambda \cong \kappa \otimes \sigma$.

Remark 2.15. — Let (J,λ) be a maximal simple type associated with a simple stratum $[\mathfrak{A},0,0,\beta]$. Then we have $E=F[\beta]=F$, $B=\mathrm{Cent}_A(F)=A$ and $\mathfrak{B}=\mathfrak{A}\cap A=\mathfrak{A}$. Since (J,λ) is a maximal simple type, $\mathfrak{A}$ is a maximal hereditary $\mathfrak{o}_F$-order in A. Moreover, we have $J(\beta,\mathfrak{A})=\mathbf{U}(\mathfrak{A})$ and $H^1(\beta,\mathfrak{A})=\mathbf{U}^1(\mathfrak{A})$. Let κ and σ be as in Definition 2.14. Since we have $\mathscr{C}(\beta,0,\mathfrak{A})=\{1\}$, there exists a character χ of $F^\times$ trivial on $1+\mathfrak{p}_F$ such that $\kappa=\chi\circ\mathrm{Nrd}_{A/F}$. Then $\kappa\otimes\sigma$ is trivial on $\mathbf{U}^1(\mathfrak{A})$ and cuspidal as a $\mathrm{GL}_m(k_D)$-representation. Therefore, $(J,\lambda)=(\mathbf{U}(\mathfrak{A}),\kappa\otimes\sigma)$ is nothing but the maximal simple type of level 0, defined in [28, §2.5].

Theorem 2.16 ([14, Theorem 5.5(ii)] and [28, Théorème 5.21]). — *Let π be an irreducible representation of G. Then π is supercuspidal if and only if there exists a maximal simple type (J,λ) such that $\lambda\subset\pi|_J$.*

We recall the construction of irreducible supercuspidal representations of G from maximal simple types. Let (J,λ) be a maximal simple type associated with a simple stratum $[\mathfrak{A},n,0,\beta]$. Let κ and σ be as in Definition 2.14. Since $\mathfrak{B}$ is maximal, we have $\mathfrak{K}(\mathfrak{B})=\mathfrak{K}(\mathfrak{A})\cap B^\times$ by [27, Lemme 1.6], and then $\mathfrak{K}(\mathfrak{B})$ normalizes $J(\beta,\mathfrak{A})$.

We fix $g\in\mathfrak{K}(\mathfrak{B})$ with $v_{\mathfrak{B}}(g)=1$. Since g normalizes $J(\beta,\mathfrak{A})$, we can consider the twist ${}^g\sigma$ of σ by g. Let l_0 be the smallest positive integer such that ${}^{g^{l_0}}\sigma\cong\sigma$. Then $\tilde{J}(\lambda)=I_G(\lambda)$ is the subgroup in G generated by J and g^{l_0}.

Theorem 2.17 ([28, Théorème 5.2], [29, Corollary 5.22]). —
1. *For any maximal simple type (J,λ), there exists an extension Λ of λ to $\tilde{J}(\lambda)$.*
2. *Let $(\tilde{J}(\lambda),\Lambda)$ be as above. Then* $\text{c-Ind}_{\tilde{J}(\lambda)}^G\Lambda$ *is irreducible and supercuspidal.*
3. *For any irreducible supercuspidal representation π of G, there exists an extension $(\tilde{J}(\lambda),\Lambda)$ of a maximal simple type (J,λ) such that $\pi=$* $\text{c-Ind}_{\tilde{J}(\lambda)}^G\Lambda$.

2.5. Concrete presentation of open subgroups. — Above we defined open subgroups $H^1(\beta,\mathfrak{A})$, $J(\beta,\mathfrak{A})$ and $\tilde{J}(\lambda)$. In this section, we define another subgroup $\hat{J}(\beta,\mathfrak{A})$ and obtain the concrete presentation of some groups, which is used later.

Definition 2.18. — Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum with $\mathfrak{B}$ is maximal. Then we put $\hat{J}(\beta, \mathfrak{A}) = \mathfrak{K}(\mathfrak{B})J(\beta, \mathfrak{A})$.

Remark 2.19. —

1. Since $\mathfrak{K}(\mathfrak{B})$ normalizes $J(\beta, \mathfrak{A})$, the set $\hat{J}(\beta, \mathfrak{A})$ is also a subgroup in G. We have $\mathfrak{K}(\mathfrak{B}) \cap J(\beta, \mathfrak{A}) = \mathbf{U}(\mathfrak{B})$, and then
$$\hat{J}(\beta, \mathfrak{A})/J(\beta, \mathfrak{A}) \cong \mathfrak{K}(\mathfrak{B})/\mathbf{U}(\mathfrak{B}) \cong \mathbb{Z}.$$
2. Let (J, λ) be a maximal simple type associated with $[\mathfrak{A}, n, 0, \beta]$. Then we have $\tilde{J}(\lambda) \subset \hat{J}(\beta, \mathfrak{A})$. The group $\hat{J}(\beta, \mathfrak{A})$ only depends on $[\mathfrak{A}, n, 0, \beta]$, while $\tilde{J}(\lambda)$ also depends on λ in general.
3. In the condition in (2), furthermore suppose $G = \mathrm{GL}_N(F)$. In this case, the group $\mathfrak{K}(\mathfrak{B})$ is generated by $\mathbf{U}(\mathfrak{B})$ and $E^\times$, which are contained in $I_G(\lambda) = \tilde{J}(\lambda)$. Then we have $\mathfrak{K}(\mathfrak{B}) \subset \tilde{J}(\lambda)$ and $\hat{J}(\lambda) = \mathfrak{K}(\mathfrak{B})J(\beta, \mathfrak{A}) = \tilde{J}(\lambda) \subset \hat{J}(\beta, \mathfrak{A})$, which implies that $\tilde{J}(\lambda) = \hat{J}(\beta, \mathfrak{A})$ is independent of the choice of λ for $G = \mathrm{GL}_N(F)$ case.

We describe $H^1(\beta, \mathfrak{A}), J(\beta, \mathfrak{A})$ and $\hat{J}(\beta, \mathfrak{A})$ concretely, using a defining sequence $([\mathfrak{A}, n, r_i, \beta_i])_{i=0}^s$ of $[\mathfrak{A}, n, 0, \beta]$. We put $B_{\beta_i} = \mathrm{Cent}_A(F[\beta_i])$ for $i = 0, \ldots, s$.

Lemma 2.20. — *Let $[\mathfrak{A}, n, 0, \beta]$ be a maximal simple stratum of A and $([\mathfrak{A}, n, r_i, \beta_i])_{i=0}^s$ be a defining sequence of $[\mathfrak{A}, n, 0, \beta]$. Then we have following concrete presentations of groups:*

1. $H^1(\beta, \mathfrak{A}) = \big(B_{\beta_0}^\times \cap \mathbf{U}^{\lfloor \frac{r_0}{2} \rfloor + 1}(\mathfrak{A})\big) \cdots \big(B_{\beta_s}^\times \cap \mathbf{U}^{\lfloor \frac{r_s}{2} \rfloor + 1}(\mathfrak{A})\big)\mathbf{U}^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A})$.
2. $J(\beta, \mathfrak{A}) = \mathbf{U}(\mathfrak{B})\big(B_{\beta_1}^\times \cap \mathbf{U}^{\lfloor \frac{r_1+1}{2} \rfloor}(\mathfrak{A})\big) \cdots \big(B_{\beta_s}^\times \cap \mathbf{U}^{\lfloor \frac{r_s+1}{2} \rfloor}(\mathfrak{A})\big)\mathbf{U}^{\lfloor \frac{n+1}{2} \rfloor}(\mathfrak{A})$.
3. $\hat{J}(\beta, \mathfrak{A}) = \mathfrak{K}(\mathfrak{B})\big(B_{\beta_1}^\times \cap \mathbf{U}^{\lfloor \frac{r_1+1}{2} \rfloor}(\mathfrak{A})\big) \cdots \big(B_{\beta_s}^\times \cap \mathbf{U}^{\lfloor \frac{r_s+1}{2} \rfloor}(\mathfrak{A})\big)\mathbf{U}^{\lfloor \frac{n+1}{2} \rfloor}(\mathfrak{A})$.

Proof. — We show (1) by induction on the length s of a defining sequence. When $s = 0$, that is, β is minimal over F, then $H^1(\beta, \mathfrak{A}) = \mathbf{U}^1(\mathfrak{B})\mathbf{U}^{\lfloor n/2 \rfloor + 1}(\mathfrak{A})$. Since we have $\mathbf{U}^1(\mathfrak{B}) = 1 + (B \cap \mathfrak{P}) = B \cap (1 + \mathfrak{P}) = B \cap \mathbf{U}^1(\mathfrak{A})$ and $r_0 = 0$, the equality in (1) for minimal β holds. Suppose $s > 0$, that is, β is not minimal over F. Then $H^1(\beta, \mathfrak{A}) = \mathbf{U}^1(\mathfrak{B})H^{\lfloor r_1/2 \rfloor + 1}(\beta_1, \mathfrak{A})$. By the induction hypothesis, we have

$$H^1(\beta_1, \mathfrak{A}) = \mathbf{U}^1(\mathfrak{B}_{\beta_1})\big(B_{\beta_2}^\times \cap \mathbf{U}^{\lfloor \frac{r_2}{2} \rfloor + 1}(\mathfrak{A})\big) \cdots \big(B_{\beta_s}^\times \cap \mathbf{U}^{\lfloor \frac{r_s}{2} \rfloor + 1}(\mathfrak{A})\big)\mathbf{U}^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}).$$

Since $r_1 < r_2 < \ldots < r_s < n$, we have $\lfloor r_1/2 \rfloor + 1 \leq \lfloor r_2/2 \rfloor + 1 \leq \ldots \leq \lfloor r_s/2 \rfloor + 1 \leq \lfloor n/2 \rfloor + 1$ and

$$B_{\beta_2}^\times \cap \mathbf{U}^{\lfloor \frac{r_2}{2} \rfloor + 1}(\mathfrak{A}), \cdots, B_{\beta_s}^\times \cap \mathbf{U}^{\lfloor \frac{r_s}{2} \rfloor + 1}(\mathfrak{A}), \mathbf{U}^{\lfloor \frac{n}{2} \rfloor + 1}(\mathfrak{A}) \subset \mathbf{U}^{\lfloor \frac{r_1}{2} \rfloor + 1}(\mathfrak{A}).$$

Therefore, we obtain

$$\begin{aligned} H^{\lfloor \frac{r_1}{2} \rfloor +1}(\beta_1, \mathfrak{A}) &= \big(\mathbf{U}^1(\mathfrak{B}_{\beta_1}) \cap \mathbf{U}^{\lfloor \frac{r_1}{2} \rfloor +1}(\mathfrak{A})\big)\big(B_{\beta_2}^{\times} \cap \mathbf{U}^{\lfloor \frac{r_2}{2} \rfloor +1}(\mathfrak{A})\big) \\ &\qquad \cdots \big(B_{\beta_s}^{\times} \cap \mathbf{U}^{\lfloor \frac{r_s}{2} \rfloor +1}(\mathfrak{A})\big)\mathbf{U}^{\lfloor \frac{n}{2} \rfloor +1}(\mathfrak{A}) \\ &= \big(B_{\beta_1}^{\times} \cap \mathbf{U}^{\lfloor \frac{r_1}{2} \rfloor +1}(\mathfrak{A})\big) \cdots \big(B_{\beta_s}^{\times} \cap \mathbf{U}^{\lfloor \frac{r_s}{2} \rfloor +1}(\mathfrak{A})\big)\mathbf{U}^{\lfloor \frac{n}{2} \rfloor +1}(\mathfrak{A}), \end{aligned}$$

and the equality in (1) for nonminimal β also holds.

Similarly, we can show that

$$J^1(\beta, \mathfrak{A}) = \mathbf{U}^1(\mathfrak{B})\big(B_{\beta_1}^{\times} \cap \mathbf{U}^{\lfloor \frac{r_1+1}{2} \rfloor}(\mathfrak{A})\big) \cdots \big(B_{\beta_s}^{\times} \cap \mathbf{U}^{\lfloor \frac{r_s+1}{2} \rfloor}(\mathfrak{A})\big)\mathbf{U}^{\lfloor \frac{n+1}{2} \rfloor}(\mathfrak{A}).$$

Then (2) and (3) are deduced from the fact $J(\beta, \mathfrak{A}) = \mathbf{U}(\mathfrak{B})J^1(\beta, \mathfrak{A})$ and $\hat{J}(\beta, \mathfrak{A}) = \mathfrak{K}(\mathfrak{B})J(\beta, \mathfrak{A})$. □

3. Yu's construction of types for tame supercuspidal representations

In this section, we recall how to construct Yu's types from [31]. Let G be a connected reductive group over F.

3.1. Admissible sequences. —

Definition 3.1. — Let $(G^i) = (G^0, \dots, G^d)$ be a sequence of subgroup schemes in G over F. We call (G^i) is a tame twisted Levi sequence if $G^0 \subset G^1 \subset \cdots \subset G^d = G$, and there exists a tamely ramified extension E of F such that $G^i \times_F E$ is a split Levi subgroup in $G \times_F E$ for $i = 0, \dots, d$.

Let $\vec{G} = (G^0, \dots, G^d)$ be a tame twisted Levi sequence in G. Then there exist a maximal torus T in G^0 over F and a tamely ramified, finite Galois extension E over F such that $T \times_F E$ is split. For $i = 0, \dots, d$, we put $\Phi_i = \Phi(G^i, T; E) \cup \{0\}$. For $\alpha \in \Phi_d \setminus \{0\} = \Phi(G, T; E)$, we denote by G_α the root subgroup in G_E defined by α. Let $G_\alpha = T$ if $\alpha = 0$. Let $\mathfrak{g}_\alpha$ be the Lie algebra of G_α, which is a Lie subalgebra in $\mathfrak{g}_E$, and let $\mathfrak{g}_\alpha^*$ be its dual.

Let $\vec{\mathbf{r}} = (\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_d) \in \tilde{\mathbb{R}}^{d+1}$. Then we can define a map $f_{\vec{\mathbf{r}}} : \Phi_d \to \tilde{\mathbb{R}}$ by $f_{\vec{\mathbf{r}}}(\alpha) = \mathbf{r}_i$ if $i = \min\{j \mid \alpha \in \Phi_j\}$.

A sequence $\vec{\mathbf{r}} = (\mathbf{r}_0, \dots, \mathbf{r}_d) \in \tilde{\mathbb{R}}^{d+1}$ is called an admissible sequence if and only if there exists $\nu \in \{0, \dots, d\}$ such that

$$0 \le \mathbf{r}_0 = \ldots = \mathbf{r}_\nu, \ \frac{1}{2}\mathbf{r}_\nu \le \mathbf{r}_{\nu+1} \le \ldots \le \mathbf{r}_d.$$

Let x be in the apartment $A(G, T, E) \subset \mathscr{B}^E(G, E)$. Then we can determine the filtrations $\{G_\alpha(E)_{x,r}\}_{r \in \tilde{\mathbb{R}}_{\ge 0}}$ on $G_\alpha(E)$, $\{\mathfrak{g}_\alpha(E)_{x,r}\}_{r \in \tilde{\mathbb{R}}}$ on $\mathfrak{g}_\alpha(E)$, and $\{\mathfrak{g}_\alpha^*(E)_{x,r}\}_{r \in \tilde{\mathbb{R}}}$ on $\mathfrak{g}_\alpha^*(E)$.

We denote by $\vec{G}(E)_{x,\vec{\mathbf{r}}}$ the subgroup in $G(E)$ generated by $G_\alpha(E)_{x, f_{\vec{\mathbf{r}}}(\alpha)}$ $(\alpha \in \Phi_d)$.

By taking $x \in A(G,T,E) \cap \mathscr{B}^E(G,E)$, we can determine a valuation on the root datum of (G,T,E) in the sense of [4]. By restricting this valuation, we can also define a valuation on the root datum of (G^i,T,E). Then we can determine $x_i \in \mathscr{B}^E(G^i,E)$ by the valuation, uniquely up to $X^*(G^i) \otimes \mathbb{R}$. When we take x_i in such a way, we can determine an affine, $G^i(E)$-equivalent embedding $j_i : \mathscr{B}^E(G^i,E) \to \mathscr{B}^E(G,E)$ such that $j_i(x_i) = x$. This embedding depends on the choice of x. We identify x_i with x via j_i.

Now $E|F$ is a tamely ramified Galois extension. To consider subgroups in $G(F)$, we also assume $x \in \mathscr{B}^E(G,E)^{\mathrm{Gal}(E/F)}$, that is, $x \in \mathscr{B}^E(G,F)$. Then we can determine the Moy–Prasad filtration [23], [24] on $G^i(F)$, $\mathfrak{g}^i(F)$ and $(\mathfrak{g}^i)^*(F)$ by x. We put $\vec{G}(F)_{x,\vec{\mathbf{r}}} = \vec{G}(E)_{x,\vec{\mathbf{r}}} \cap G(F)$.

Proposition 3.2 ([31, 2.10]). — *The group $\vec{G}(F)_{x,\vec{\mathbf{r}}}$ is independent of the choice of T. If $\vec{\mathbf{r}}$ is increasing with $\mathbf{r}_0 > 0$, then we have*

$$\vec{G}(F)_{x,\vec{\mathbf{r}}} = G^0(F)_{x,\mathbf{r}_0} G^1(F)_{x,\mathbf{r}_1} \cdots G^d(F)_{x,\mathbf{r}_d}.$$

3.2. Generic elements, generic characters. — Let r and r' be two elements in $\tilde{\mathbb{R}}_{>0}$ with $r \leq r' \leq 2r$. We put $G(F)_{x,r:r'} = G(F)_{x,r}/G(F)_{x,r'}$ and $\mathfrak{g}(F)_{x,r:r'} = \mathfrak{g}(F)_{x,r}/\mathfrak{g}(F)_{x,r'}$. Then we have a group isomorphism $G(F)_{x,r:r'} \cong \mathfrak{g}(F)_{x,r:r'}$, cf.[31, Corollary 2.4].

Remark 3.3. — The above isomorphism is often called the "Moy–Prasad isomorphism". Let us mention that in [22, Theorem 4.3] and [21], Moy–Prasad-like isomorphisms, called "congruent isomorphisms", are proved for group schemes using dilatations of schemes. In [18, Theorem 13.5.1], the authors prove Moy–Prasad isomorphisms using congruent isomorphisms. We refer to [10] for a survey on the theory of algebraic dilatations, including references to pioneering works such as [32].

Let S be a subgroup of $G(F)$ between $G(F)_{x,r/2+}$ and $G(F)_{x,r+}$, and let $\mathfrak{s}$ be the sublattice of $\mathrm{Lie}(G)$ between $\mathfrak{g}(F)_{x,r/2+}$ and $\mathfrak{g}(F)_{x,r+}$ such that $\mathfrak{s}/\mathfrak{g}(F)_{x,r+} \cong S/G(F)_{x,r+}$.

Definition 3.4. — A character Φ of $S/G(F)_{x,r+}$, with respect to ψ, is realized by $X^* \in \mathrm{Lie}^*(G)_{x,-r}$ if Φ is equal to

$$S/G(F)_{x,r+} \cong \mathfrak{s}/\mathfrak{g}(F)_{x,r+} \xrightarrow{X^*} F \xrightarrow{\psi} \mathbb{C}^\times .$$

Let G' be a tame twisted Levi subgroup in G. The Lie algebra $\underline{\mathrm{Lie}}(G')$ and its dual $\underline{\mathrm{Lie}^*}(G')$ are equipped with canonical adjoint actions of the group scheme G'. Then the functor of fixed point $(\underline{\mathrm{Lie}^*}(G'))^{G'}$ is representable by a scheme (cf. Notation). We now consider $(\underline{\mathrm{Lie}^*}(G'))^{G'}(F)$ as a subset of $\underline{\mathrm{Lie}^*}(G')(F) = \mathrm{Lie}^*(G')$.

To define G-generic characters of depth r of G', we define G-generic elements of depth r in $(\underline{\mathrm{Lie}}^*(G'))^{G'}(F)$. For this, following [31, §8], as corrected in [17, Rem.4.1.3] and [13, Def. 2.1], we consider the conditions **GE0**, **GE1** and **GE2**.

We start with **GE0**.

Definition 3.5. — Let $X^* \in (\underline{\mathrm{Lie}}^*(G'))^{G'}(F)$. We say X^* satisfies **GE0** with depth r if for some (equivalently, every by [13, Lemma 2.3]) point $x \in \mathscr{B}^E(G', F)$, we have $X^* \in \mathrm{Lie}^*(G')_{x,-r}$.

Let E be a finite, tamely ramified extension of F and T be an F-torus in G' such that $T \times_F E$ is maximal and split. Let $\alpha \in \Phi(G, T; \bar{F})$. Then the derivation $\mathrm{d}\check{\alpha}$ is an $\bar{F}$-linear map from $\underline{\mathrm{Lie}}(\mathbb{G}_m)(\bar{F}) \cong \bar{F}$ to $\mathrm{Lie}(T \times_F \bar{F})$. We obtain $H_\alpha = \mathrm{d}\check{\alpha}(1)$ as an element in $\mathrm{Lie}(T \times_F \bar{F})$.

Here, we recall the condition **GE1**. Let $X^* \in (\underline{\mathrm{Lie}}^*(G'))^{G'}(F)$. Then we can regard $X^* \in \mathrm{Lie}^*(G')$ as above. We put $X^*_{\bar{F}} = X^* \otimes_F 1 \in \mathrm{Lie}^*(G') \otimes_F \bar{F} = \mathrm{Lie}^*(G' \times_F \bar{F})$. Since $T \subset G'$, we have $H_\alpha \in \mathrm{Lie}(G' \times_F \bar{F}) = \mathrm{Lie}(G') \otimes_F \bar{F}$. Therefore, we obtain $X^*_{\bar{F}}(H_\alpha) \in \bar{F}$.

Definition 3.6. — Let $X^* \in (\underline{\mathrm{Lie}}^*(G'))^{G'}(F)$. We say X^* satisfies **GE1** with depth r if $\mathrm{ord}\left(X^*_{\bar{F}}(H_\alpha)\right) = -r$ for all $\alpha \in \Phi(G, T; \bar{F}) \setminus \Phi(G', T; \bar{F})$.

We also have to consider the condition **GE2** defined in [31, §8]. However, in our case, if **GE1** holds, then **GE2** automatically holds. We use the notion of torsion prime of a root datum as defined in [31, §7]

Proposition 3.7 ([31, Lemma 8.1]). — *If the residual characteristic of F is not a torsion prime for the root datum of G, then* **GE1** *implies* **GE2**.

Proposition 3.8 ([30, Corollary 1.13]). — *If a root datum is type A, then the set of torsion primes for the datum is empty.*

From these propositions, we obtain the following corollary.

Corollary 3.9. — *If the root datum of G is type A, then* **GE1** *implies* **GE2**.

Definition 3.10. — Let $X^* \in (\underline{\mathrm{Lie}}^*(G'))^{G'}(F) \subset \mathrm{Lie}^*(G')$. The linear form X^* is called G-generic of depth r if and only if conditions **GE0**, **GE1** and **GE2** hold.

Eventually, we can define generic characters.

Definition 3.11. — Let $r \in \mathbb{R}_{>0}$. A character Φ of $G'(F)$ is called G-generic of depth r relative to x if $\Phi|_{G'(F)_{x,r+}}$ is trivial, $\Phi|_{G'(F)_{x,r}}$ is nontrivial, and there exists a G-generic element of depth r $X^* \in (\underline{\mathrm{Lie}}^*(G'))^{G'}(F)$ such that Φ is realized by X^* when Φ is regarded as a character of $G'(F)_{x,r:r+}$.

3.3. Yu data. — Let $d \in \mathbb{Z}_{\geq 0}$.

A 5-tuple $\Psi = \big(x, (G^i)_{i=0}^d, (\mathbf{r}_i)_{i=0}^d, (\mathbf{\Phi}_i)_{i=0}^d, \rho\big)$ is called a Yu datum if Ψ satisfies the following conditions.

- The sequence $(G^i)_{i=0}^d$ is a tame twisted Levi sequence such that $Z(G^i)/Z(G)$ is anisotropic for $i = 0, \ldots, d$ and
$$G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^d = G.$$
- We have $x \in \mathscr{B}^E(G^0, F) \cap A(G, T, E)$, where T is a maximal F-torus in G that splits over some tamely ramified extension E of F.
- For $i = 0, \ldots, d$, the number $\mathbf{r}_i \in \mathbb{R}$ such that
$$0 = \mathbf{r}_{-1} < \mathbf{r}_0 < \ldots < \mathbf{r}_{d-1} \leq \mathbf{r}_d.$$
- For $i = 0, \ldots, d-1$, the character $\mathbf{\Phi}_i$ of $G^i(F)$ is G^{i+1}-generic relative to x of depth $\mathbf{r}_i$. If $\mathbf{r}_{d-1} \neq \mathbf{r}_d$, the character $\mathbf{\Phi}_d$ of $G^d(F)$ is of depth $\mathbf{r}_d$. If $\mathbf{r}_{d-1} = \mathbf{r}_d$, the character $\mathbf{\Phi}_d$ of $G^d(F)$ is trivial.
- The irreducible representation ρ of $G^0(F)_{[x]}$ is trivial on $G^0(F)_{x,0+}$ but nontrivial on $G^0(F)_x$, and $\text{c-Ind}_{G^0(F)_{[x]}}^{G^0(F)} \rho$ is irreducible and supercuspidal.

3.4. Yu's construction. — In this section, we construct Yu's type by using some data from a Yu datum. Let $\Psi = \big(x, (G^i)_{i=0}^d, (\mathbf{r}_i)_{i=0}^d, (\mathbf{\Phi}_i)_{i=0}^d, \rho\big)$ be a Yu datum.

First, Yu constructed subgroups in G, which some representations are defined over.

Definition 3.12. — For $i = 0, \ldots, d$, let $\mathbf{s}_i = \mathbf{r}_i/2$. Put $\mathbf{s}_{-1} = 0$.

1. $K_+^i = G^0(F)_{x,0+} G^1(F)_{x,\mathbf{s}_0+} \cdots G^i(F)_{x,\mathbf{s}_{i-1}+}$
$= (G^0, \ldots, G^i)(F)_{x,(0+,\mathbf{s}_0+,\ldots,\mathbf{s}_{i-1}+)}$.
2. ${}^\circ K^i = G^0(F)_{x,0} G^1(F)_{x,\mathbf{s}_0} \cdots G^i(F)_{x,\mathbf{s}_{i-1}}$
$= G^0(F)_{x,0}(G^1, \ldots, G^i)(F)_{x,(\mathbf{s}_0,\ldots,\mathbf{s}_{i-1})}$.
3. $K^i = G^0(F)_{[x]} G^1(F)_{x,\mathbf{s}_0} \cdots G^i(F)_{x,\mathbf{s}_{i-1}} = G^0(F)_{[x]} {}^\circ K^i$. Recall that $G^0(F)_{[x]}$ denotes the stabiliser of $[x]$ in $G^0(F)$.

Proposition 3.13. — *For any $i = 0, \ldots, d$, the groups K_+^i and ${}^\circ K^i$ are compact, and K^i is compact modulo center.*

Yu also defined subgroups in $G(F)$, which "fill the gap" between subgroups defined as above.

Definition 3.14. — For $i = 1, \ldots, d$,

1. $J^i = (G^{i-1}, G^i)(F)_{x,(\mathbf{r}_{i-1},\mathbf{s}_{i-1})}$,
2. $J_+^i = (G^{i-1}, G^i)(F)_{x,(\mathbf{r}_{i-1},\mathbf{s}_{i-1}+)}$.

Note that, in general, J^i is different from $G^i(F)_{x,\mathbf{s}_{i-1}}$. Then, we have $K^iJ^{i+1} = K^{i+1}$ and $K^i_+J^{i+1}_+ = K^{i+1}_+$ for $i = 0,\dots,d-1$.

Next, Yu defined characters $\hat{\mathbf{\Phi}}_i$ of K^d_+. The Lie algebra $\mathfrak{g}(F)$ of $G(F)$ is equipped with a canonical $G(F)$-action. In particular, $Z(G^i)^\circ(F)$ acts on $\mathfrak{g}(F)$ by restricting the $G(F)$-action. Then $Z(G^i)^\circ(F)$-fixed part of $\mathfrak{g}(F)$ is equal to the Lie algebra $\mathfrak{g}^i(F)$ of $G^i(F)$. Moreover, we have a decomposition $\mathfrak{g}(F) = \mathfrak{g}^i(F) \oplus \mathfrak{n}^i(F)$ as a $Z(G^i)^\circ(F)$-representation. This decomposition is well behaved on the Moy–Prasad filtration; we have $\mathfrak{g}(F)_{x,s} = \mathfrak{g}^i(F)_{x,s} \oplus \mathfrak{n}^i(F)_{x,s}$ for any $s \in \tilde{\mathbb{R}}$, where $\mathfrak{n}^i(F)_{x,s} \subset \mathfrak{n}^i(F)$. Let $\pi_i : \mathfrak{g}(F) = \mathfrak{g}^i(F) \oplus \mathfrak{n}^i(F) \to \mathfrak{g}^i(F)$ be the projection. Then π_i induces $\mathfrak{g}(F)_{x,\mathbf{s}_i+:\mathbf{r}_i+} \to \mathfrak{g}^i(F)_{x,\mathbf{s}_i+:\mathbf{r}_i+}$, and we obtain a group homomorphism

$$\tilde{\pi}_i : G(F)_{x,\mathbf{s}_i+} \longrightarrow G(F)_{x,\mathbf{s}_i+:\mathbf{r}_i+} \xrightarrow{\pi_i} G^i(F)_{x,\mathbf{s}_i+:\mathbf{r}_i+}.$$

Here, Yu defined a character $\hat{\mathbf{\Phi}}_i$ of K^d_+ as

$$\hat{\mathbf{\Phi}}_i|_{K^d_+\cap G^i(F)} = \mathbf{\Phi}_i,$$
$$\hat{\mathbf{\Phi}}_i|_{K^d_+\cap G(F)_{x,\mathbf{s}_i+}} = \mathbf{\Phi}_i \circ \tilde{\pi}_i,$$

where K^d_+ is generated by $K^d_+ \cap G^i(F)$ and $K^d_+ \cap G(F)_{x,\mathbf{s}_i+}$ as we have

$$K^d_+ \cap G^i(F) = G^0(F)_{x,0+}G^1(F)_{x,\mathbf{s}_0+}\cdots G^i(F)_{x,\mathbf{s}_{i-1}+} = K^i_+,$$
$$K^d_+ \cap G(F)_{x,\mathbf{s}_i+} = G^{i+1}(F)_{x,\mathbf{s}_i+}\cdots G^d(F)_{x,\mathbf{s}_{d-1}+}.$$

Using $\hat{\mathbf{\Phi}}_i$, Yu constructed a representation ρ_j of K_j for $j = 0,\dots,d$.

Lemma 3.15 ([31, §4]). — *Let $0 \le i \le d-1$. There is an irreducible representation $\tilde{\mathbf{\Phi}}_i$ of $K^i \ltimes J^{i+1}$ such that*

1. $\tilde{\mathbf{\Phi}}_i|_{1\ltimes J^{i+1}_+}$ *is $\hat{\mathbf{\Phi}}_i|_{J^{i+1}_+}$-isotypic, and*
2. $\tilde{\mathbf{\Phi}}_i|_{K^i_+\ltimes 1}$ *is $\mathbf{1}$-isotypic.*

Lemma 3.16 ([31, §4]). — *Let $0 \le i \le d-1$. Let $\inf(\mathbf{\Phi}_i)$ be the inflation of $\mathbf{\Phi}_i|_{K^i}$ to $K^i \ltimes J^{i+1}$, and let $\tilde{\mathbf{\Phi}}_i$ be as in Lemma 3.15. Then the $K^i \ltimes J^{i+1}$-representation $\inf(\mathbf{\Phi}_i) \otimes \tilde{\mathbf{\Phi}}_i$ factors through $K^i \ltimes J^{i+1} \to K^iJ^{i+1} = K^{i+1}$.*

Definition 3.17. — We denote by $\mathbf{\Phi}'_i$ the K^{i+1}-representation $\inf(\mathbf{\Phi}_i) \otimes \tilde{\mathbf{\Phi}}_i$.

To obtain ρ_j constructed by Yu, we use a little different way from Yu, by Hakim–Murnaghan.

Lemma 3.18 ([15, 3.4]). — 1. *For $i = 1, \dots, d-1$, we have $K^i \cap J^{i+1} = G^i(F)_{x,\mathbf{r}_i} \subset J^i$.*

2. *For $i = 0, \dots, d-1$, let μ be a K^i-representation that is trivial on $K^i \cap J^{i+1}$. Then we can obtain the inflation $\inf_{K^i}^{K^{i+1}} \mu$ of μ to K^{i+1} via $K^{i+1}/J^{i+1} \cong K^i/(K^i \cap J^{i+1})$. The representation $\inf_{K^i}^{K^{i+1}} \mu$ is trivial on J^{i+1} and also trivial on $K^{i+1} \cap J^{i+2}$ if $i < d-1$.*
3. *If i, μ is as in (ii) and $i \le j \le d$, then we can also obtain the inflation $\inf_{K^i}^{K^j} \mu$ of μ to K^j as $\inf_{K^i}^{K^j} \mu = \inf_{K^{j-1}}^{K^j} \circ \cdots \circ \inf_{K^i}^{K^{i+1}} \mu$.*

Definition 3.19 ([15, 3.4]). — For $0 \le i < j \le d$, we put $\kappa_i^j = \inf_{K^{i+1}}^{K^j} \mathbf{\Phi}'_i$. For $0 \le j \le d$, we put $\kappa_{-1}^j = \inf_{K^0}^{K^j} \rho$ and $\kappa_j^j = \mathbf{\Phi}_j|_{K^j}$. And also, for $-1 \le i \le d$ we put $\kappa_i = \kappa_i^d$.

Proposition 3.20. — *Let $0 \le j \le d$. The representation ρ_j constructed by Yu is isomorphic to*

$$\kappa_{-1}^j \otimes \kappa_0^j \otimes \cdots \otimes \kappa_j^j.$$

In particular,

$$\rho_d \cong \kappa_{-1} \otimes \kappa_0 \otimes \cdots \otimes \kappa_d.$$

Proof. — The representation ρ_j is constructed in [31] on page 592. Yu inductively constructs two representations: ρ_j and ${\rho_j}'$.

Let us show by induction on j that ${\rho_j}' = \kappa_{-1}^j \otimes \kappa_0^j \otimes \cdots \otimes \kappa_{j-1}^j$ and $\rho_j = \kappa_{-1}^j \otimes \kappa_0^j \otimes \cdots \otimes \kappa_j^j$ If $j = 0$, then by definition the representation ρ'_0 constructed by Yu is ρ and ρ_0 is $\rho'_0 \otimes (\mathbf{\Phi}_0|_{K^0})$. We have $\kappa_{-1}^0 = \rho$ and $\kappa_0^0 = \mathbf{\Phi}_0|_{K^0}$. So the case $j = 0$ is complete. Assume that $\rho'_{j-1} = \kappa_{-1}^{j-1} \otimes \kappa_0^{j-1} \cdots \otimes \kappa_{j-2}^{j-1}$ and $\rho_{j-1} = \kappa_{-1}^{j-1} \otimes \kappa_0^{j-1} \otimes \cdots \otimes \kappa_{j-1}^{j-1}$. Then by definition ρ'_j is equal to $\inf_{K^{j-1}}^{K^j}(\rho'_{j-1}) \otimes \mathbf{\Phi}'_{j-1}$. By definition $\mathbf{\Phi}'_{j-1}$ is equal to κ_{j-1}^j. Moreover,

$$\begin{aligned}
\inf_{K^{j-1}}^{K^j}(\rho'_{j-1}) &= \inf_{K^{j-1}}^{K^j}(\kappa_{-1}^{j-1} \otimes \kappa_0^{j-1} \otimes \cdots \otimes \kappa_{j-2}^{j-1}) \\
&= \inf_{K^{j-1}}^{K^j}(\kappa_{-1}^{j-1}) \otimes \inf_{K^{j-1}}^{K^j}(\kappa_0^{j-1}) \otimes \cdots \otimes \inf_{K^{j-1}}^{K^j}(\kappa_{j-2}^{j-1}) \\
&= \kappa_{-1}^j \otimes \kappa_0^j \otimes \cdots \otimes \kappa_{j-2}^j
\end{aligned}$$

Consequently ${\rho_j}' = \kappa_{-1}^j \otimes \kappa_0^j \otimes \cdots \otimes \kappa_i^j \otimes \cdots \otimes \kappa_{j-1}^j$. Finally, by Yu's definition, ρ_j is equal to $\rho'_j \otimes \mathbf{\Phi}_j|_{K^j}$, and thus $\rho_j = \kappa_{-1}^j \otimes \kappa_0^j \otimes \cdots \otimes \kappa_j^j$, as required. □

Therefore, we obtain ρ_j constructed by Yu. Spice recently found a problem in the proof of the following statement. This was recently discussed and corrected by Fintzen [11], so that we are allowed to do not worry about this problem.

Theorem 3.21 ([31, 15.1]). — *The compactly induced representation $\text{c-Ind}_{K^j}^{G^j(F)} \rho_j$ of $G^j(F)$ is irreducible and supercuspidal.*

For later use, we recall the following proposition on the dimension of representation space of κ_i.

PROPOSITION 3.22. — *Let $0 \leq i < j \leq d$. Then the dimension of κ_i^j is equal to the dimension of $\mathbf{\Phi}'_i$, which is also equal to $(J^{i+1} : J_+^{i+1})^{1/2}$.*

Proof. — By definition κ_i^j is an inflation of $\mathbf{\Phi}_i$; consequently, these representations have equal dimensions. The representation $\mathbf{\Phi}'_i$ is the unique representation of K^{i+1} whose inflation to $K^i \ltimes J^{i+1}$ is $\tilde{\mathbf{\Phi}}_i$. Thus, the dimension of $\mathbf{\Phi}'_i$ is equal to $\tilde{\mathbf{\Phi}}_i$. The representation $\tilde{\mathbf{\Phi}}_i$ is constructed in [31, 11.5] and is the pull back of the Weil representation of $Sp(J^{i+1}/J_+^{i+1}) \ltimes (J^{i+1}/N_i)$ where $N_i = \ker(\hat{\mathbf{\Phi}}_i)$. Thus, the dimension of $\tilde{\mathbf{\Phi}}_i$ is $[J^{i+1} : J_+^{i+1}]^{\frac{1}{2}}$. □

4. Tame simple strata

In this section, we consider the class of simple strata corresponding to some Yu datum. We fix a uniformizer ϖ_F of F.

DEFINITION 4.1. —
1. A pure stratum $[\mathfrak{A}, n, r, \beta]$ is called tame if $E = F[\beta]$ is a tamely ramified extension of F.
2. A simple type (J, λ) associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ is called tame if $[\mathfrak{A}, n, 0, \beta]$ is tame.

REMARK 4.2. —
1. By [6, (2.6.2)(4)(b), 2.7 Proposition], the above definition is independent of the choice of simple strata.
2. Essentially tame supercuspidal representations, defined in [6, 2.8], are G-representations containing some tame simple types.

As explained in §2, any simple strata has a defining sequence. Actually, if a simple stratum $[\mathfrak{A}, n, 0, \beta]$ is tame, then we can show the existence of a "nice" defining sequence of $[\mathfrak{A}, n, 0, \beta]$. To discuss such a defining sequence, we state several related propositions.

LEMMA 4.3. — *Let $\mathfrak{A}$ be an hereditary $\mathfrak{o}_F$-order in $A \cong \mathrm{M}_N(F)$, and let E be a field in A such that $E^\times \subset \mathfrak{K}(\mathfrak{A})$. Let β be an element in E, then*

$$v_{\mathfrak{A}}(\beta) e(E|F) = e(\mathfrak{A}|\mathfrak{o}_F) v_E(\beta). \tag{1}$$

Proof. — Let ϖ_E denote a uniformizer element in E. Since $E^\times \subset \mathfrak{K}(\mathfrak{A})$, the elements ϖ_E, ϖ_F and β are in $\mathfrak{K}(\mathfrak{A})$. Thus, Equality [7, 1.1.3] is valid for these elements and we use it in the following equalities. On the one hand

$$\beta^{e(E|F)}\mathfrak{A} = \varpi_E^{v_E(\beta)e(E|F)}\mathfrak{A} = \varpi_F^{v_E(\beta)}\mathfrak{A}. \tag{2}$$

On the other hand

$$\beta^{e(E|F)}\mathfrak{A} = \mathfrak{P}^{v_{\mathfrak{A}}(\beta)e(E|F)}. \tag{3}$$

Moreover by definition of $e(\mathfrak{A}|\mathfrak{o}_F)$ (see [7, 1.1.2]), we have

$$\varpi_F^{v_E(\beta)}\mathfrak{A} = \mathfrak{P}^{e(\mathfrak{A}|\mathfrak{o}_F)v_E(\beta)}. \tag{4}$$

Equalities 2 , 3 and 4 show that

$$\mathfrak{P}^{v_{\mathfrak{A}}(\beta)e(E|F)} = \mathfrak{P}^{e(\mathfrak{A}|\mathfrak{o}_F)v_E(\beta)}. \tag{5}$$

Consequently, $v_{\mathfrak{A}}(\beta)e(E|F) = e(\mathfrak{A}|\mathfrak{o}_F)v_E(\beta)$ and equality 1 holds as required. □

The following is analogous to [7, 2.2.3]; the main differences are that the tameness condition is assumed and a maximality condition is removed.

Proposition 4.4. — *Assume $A \cong \mathrm{M}_N(F)$ for some N. Let $[\mathfrak{A}, n, r, \beta]$ be a tame simple stratum with $r > 0$. Let $[\mathfrak{B}_\beta, r, r-1, b]$ be a simple stratum, where $\mathfrak{B}_\beta = \mathfrak{A} \cap \mathrm{Cent}_A(\beta)$. Then we have $F[\beta + b] = F[\beta, b]$ and $[\mathfrak{A}, n, r, \beta + b]$ is a pure stratum with* $k_0(\beta + b, \mathfrak{A}) = \begin{cases} -r = k_0(b, \mathfrak{B}_E) & \text{if } b \notin E \\ k_0(\beta, \mathfrak{A}) & \text{if } b \in E. \end{cases}$

Before proving the proposition let us set $E = F[\beta]$. Also set $B_E = \mathrm{End}_E(V) = \mathrm{Cent}_A(\beta)$. Let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$. Set $\mathfrak{B}_E = \mathfrak{B}_\beta = \mathfrak{A} \cap B_E$ and $\mathfrak{Q}_E = \mathfrak{P} \cap \mathfrak{B}_E$. Thus, $\mathfrak{B}_E$ is an $\mathfrak{o}_E$-hereditary order in B_E and $\mathfrak{Q}_E$ is the Jacobson radical of $\mathfrak{B}_E$. We now prove the proposition.

Proof. — Let $s : A \to B_E$ be the tame corestriction that is the identity when restricted to B_E; we recall that such maps exist by [7, Remark (1.3.8) (ii)].

Let $\mathcal{L} = \{L_i\}_{i\in\mathbb{Z}}$ be an $\mathfrak{o}_F$-lattice chain such that

$$\mathfrak{A} = \{x \in A | x(L_i) \subset L_i, i \in \mathbb{Z}\}.$$

By definition [7, 2.2.1],

$$\mathfrak{K}(\mathfrak{A}) = \{x \in G | x(L_i) \in \mathcal{L}, i \in \mathbb{Z}\}$$

and

$$\mathfrak{K}(\mathfrak{B}_E) = \{x \in G_E \mid x(L_i) \in \mathcal{L}, i \in \mathbb{Z}\}.$$

Thus,

$$\mathfrak{K}(\mathfrak{B}_E) \subset \mathfrak{K}(\mathfrak{A}). \tag{6}$$

The stratum $[\mathfrak{B}_E, r, r-1, b]$ is simple, and thus the definition of a simple stratum shows that

$$E[b]^\times \subset \mathfrak{K}(\mathfrak{B}_E). \tag{7}$$

Put $E_1 = E[b] = F[\beta, b]$. Equations 6 and 7 imply that $E_1^\times \subset \mathfrak{K}(\mathfrak{A})$. This allows us to use the machinery of [7, 1.2] for $\mathfrak{A}$ and E_1.

Set $B_{E_1} = \mathrm{End}_{E_1}(V)$ and $\mathfrak{B}_{E_1} = \mathfrak{A} \cap \mathrm{End}_{E_1}(V)$. Proposition [7, 1.2.4] implies that $\mathfrak{B}_{E_1}$ is an $\mathfrak{o}_{E_1}$-hereditary order in B_{E_1}. Let $A(E_1)$ be the algebra $\mathrm{End}_F(E_1)$ and let $\mathfrak{A}(E_1)$ be the $\mathfrak{o}_F$-hereditary order in $A(E_1)$ defined by $\mathfrak{A}(E_1) = \{x \in \mathrm{End}_F(E_1) \mid x(\mathfrak{p}^i_{E_1}) \subset \mathfrak{p}^i_{E_1}, i \in \mathbb{Z}\}$. Let W be the F-span of an $\mathfrak{o}_{E_1}$-splitting basis of the $\mathfrak{o}_{E_1}$-lattice chain $\mathcal{L}$. Proposition [7, 1.2.8] shows that the (W, E_1)-decomposition of A restricts to an isomorphism $\mathfrak{A} \simeq \mathfrak{A}(E_1) \otimes_{\mathfrak{o}_{E_1}} \mathfrak{B}$ of $(\mathfrak{A}(E_1), \mathfrak{B}_{E_1})$-bimodules. Similarly, we have a decomposition $\mathfrak{B}_E \simeq \mathfrak{B}_E(E_1) \otimes_{\mathfrak{o}_{E_1}} \mathfrak{B}_{E_1}$. Set $B_E(E_1) = \mathrm{End}_E(E_1)$ and $\mathfrak{B}_E(E_1) = B_E(E_1) \cap \mathfrak{A}(E_1)$. Also set $n(E_1) = \dfrac{n}{e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1})}$ and $r(E_1) = \dfrac{r}{e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1})}$. Let us prove the following equalities:

$$v_{\mathfrak{A}(E_1)}(\beta) = -n(E_1), \tag{8}$$

$$v_{\mathfrak{B}_E(E_1)}(b) = -r(E_1). \tag{9}$$

Let us prove that the equation 8 holds. By definition of E_1, the element β is inside E_1 and thus $v_{\mathfrak{A}(E_1)}(\beta) = v_{E_1}(\beta)$. Thus, Lemma 4.3 shows that

$$v_{\mathfrak{A}}(\beta) e(E_1|F) = e(\mathfrak{A}|\mathfrak{o}_F) v_{\mathfrak{A}(E_1)}(\beta). \tag{10}$$

Proposition [7, 1.2.4] gives us the equality

$$e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1}) = \frac{e(\mathfrak{A}|\mathfrak{o}_F)}{e(E_1|F)}. \tag{11}$$

Because $[\mathfrak{A}, n, r, \beta]$ is a simple stratum, n is equal to $-v_{\mathfrak{A}}(\beta)$, consequently using equations 10 and 11, the following sequence of equality holds.

$$v_{\mathfrak{A}(E_1)}(\beta) = \frac{v_{\mathfrak{A}}(\beta) e(E_1|F)}{e(\mathfrak{A}|\mathfrak{o}_F)} = \frac{v_{\mathfrak{A}}(\beta)}{e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1})} = \frac{-n}{e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1})} = -n(E_1)$$

This concludes the proof of the equality 8, and the equality 9 is easily proven in the same way. Proposition [7, 1.4.13] gives

$$\begin{cases} k_0(\beta, \mathfrak{A}(E_1)) = \dfrac{k_0(\beta, \mathfrak{A})}{e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1})} \\ k_0(b, \mathfrak{B}_E(E_1)) = \dfrac{k_0(b, \mathfrak{B}_E)}{e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1})}. \end{cases}$$

Consequently, $[\mathfrak{A}(E_1), n(E_1), r(E_1), \beta]$ and $[\mathfrak{B}_E(E_1), r(E_1), r(E_1) - 1, b]$ are simple strata and satisfy the hypothesis of the proposition [7, 2.2.3]. Consequently, $[\mathfrak{A}(E_1), n, r-1, \beta + b]$ is simple, and the field $F[\beta + b]$ is equal to the field $F[\beta, b]$. Moreover, [7, 2.2.3] implies that

$$k_0(\beta + b, \mathfrak{A}(E_1)) = \begin{cases} -r(E_1) = k_0(b, \mathfrak{B}_E(E_1)) & \text{if } b \notin E \\ k_0(\beta, \mathfrak{A}(E_1)) & \text{if } b \in E. \end{cases}$$

The valuation $v_{\mathfrak{A}(E_1)}(\beta + b)$ is equal to $-n(E_1)$, and the same argument as before shows that $v_{\mathfrak{A}}(\beta + b) = -n$. The proposition [7, 1.4.13] shows that

$k_0(\beta+b,\mathfrak{A}) = k_0(\beta+b,\mathfrak{A}(E_1))e(\mathfrak{B}_{E_1}|\mathfrak{o}_{E_1})$. Thus

$$k_0(\beta+b,\mathfrak{A}) = \begin{cases} -r = k_0(b,\mathfrak{B}_E) & \text{if } b \notin E \\ k_0(\beta,\mathfrak{A}) & \text{if } b \in E. \end{cases}$$

This completes the proof. □

Compare Proposition 4.4 with [8, Proposition 4.2].

Proposition 4.5 ([6, 3.1 Corollary]). — *Let E be a finite, tamely ramified extension of F and let $\beta \in E$ such that $E = F[\beta]$. Let $[\mathfrak{A}(E), n, r, \beta]$ be a pure stratum in $A(E)$ with $r = -k_F(\beta) < n$. Then there exists $\gamma \in E$ such that $[\mathfrak{A}(E), n, r, \gamma]$ is simple and equivalent to $[\mathfrak{A}(E), n, r, \beta]$. Moreover, if $\iota : E \hookrightarrow A$ is an F-algebra inclusion and $[\mathfrak{A}, n', r', \iota(\beta)]$ is a pure stratum of A with $r' = -k_0(\iota(\beta), \mathfrak{A})$, then $[\mathfrak{A}, n', r', \iota(\gamma)]$ is simple and equivalent to $[\mathfrak{A}, n', r', \beta]$.*

Proposition 4.6. — *Assume $A \cong \mathrm{M}_N(F)$ for some N. Let $[\mathfrak{A}, n, r, \beta]$ be a tame, pure stratum of A with $r = -k_0(\beta, \mathfrak{A})$. Let $\gamma \in E = F[\beta]$ such that $[\mathfrak{A}, n, r, \gamma]$ is simple and equivalent to $[\mathfrak{A}, n, r, \beta]$. Then $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is simple.*

Proof. — Using an argument similar to the one in the proof of Proposition 4.4, it is enough to prove the proposition in the case where $F[\beta]$ is a maximal subfield of the algebra $A = \mathrm{End}_F(V)$. So let $[\mathfrak{A}, n, r, \beta]$ be a tame pure stratum such that $F[\beta]$ is a maximal subfield of A and $k_0(\beta, \mathfrak{A}) = -r$. Let γ be in $F[\beta]$ such that $[\mathfrak{A}, n, r, \gamma]$ is simple. The stratum $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is pure in the algebra $\mathrm{End}_{F[\gamma]}(V)$ because it is equivalent to a simple one by [7, 2.4.1]. Moreover $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is tame pure, so Proposition 4.5 shows that there exists a simple stratum $[\mathfrak{B}_\gamma, r, r-1, \alpha]$ equivalent to $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$, such that $F[\gamma][\alpha] \subset F[\gamma][\beta-\gamma]$. By Proposition 4.4, $[\mathfrak{A}, n, r-1, \gamma+\alpha]$ is simple, and $F[\gamma+\alpha]$ is equal to the field $F[\gamma, \alpha]$. Set $\mathfrak{Q}_\gamma = \mathrm{rad}(\mathfrak{B}_\gamma) = \mathfrak{B}_\gamma \cap \mathfrak{P}$. The equivalence $[\mathfrak{B}_\gamma, r, r-1, \alpha] \sim [\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ shows that $\alpha \equiv \beta - \gamma \pmod{\mathfrak{Q}_\gamma^{-(r-1)}}$. This implies $\gamma + \alpha \equiv \beta \pmod{\mathfrak{P}^{-(r-1)}}$. We deduce that $[\mathfrak{A}, n, r-1, \gamma+\alpha]$ and $[\mathfrak{A}, n, r-1, \beta]$ are two simple strata equivalent. Indeed, the first is simple by construction, and the second is simple by hypothesis because $k_0(\beta, \mathfrak{A}) = -r$. The definitions shows that $F[\gamma+\alpha] \subset F[\beta]$, and [7, 2.4.1] shows that $[F[\gamma+\alpha] : F] = [F[\beta] : F]$. Thus, $F[\gamma+\alpha] = F[\beta]$. The trivial inclusions $F[\gamma+\alpha] \subset F[\gamma, \alpha] \subset F[\beta]$ then show that $F[\gamma+\alpha] = F[\gamma, \alpha] = F[\beta]$. We have thus obtained that the following three assertions hold:

- The stratum $[\mathfrak{B}_\gamma, r, r-1, \alpha]$ is a simple stratum in $\mathrm{End}_{F[\gamma]}(V)$.
- The field $F[\gamma][\alpha]$ is a maximal subfield of the $F[\gamma]$-algebra $\mathrm{End}_{F[\gamma]}(V)$.
- $[\mathfrak{B}_\gamma, r, r-1, \alpha] \sim [\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$

Consequently, by [7, 2.2.2], $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is simple, as required. □

By these propositions, we obtain the following proposition needed in our case.

PROPOSITION 4.7. — *Let* $[\mathfrak{A}, n, r, \beta]$ *be a tame pure stratum of* A *with* $r = -k_0(\beta, \mathfrak{A})$. *Then there exists an element* γ *in* $F[\beta]$ *satisfying the following conditions:*

1. *The stratum* $[\mathfrak{A}, n, r, \gamma]$ *is simple and equivalent to* $[\mathfrak{A}, n, r, \beta]$.
2. $\beta - \gamma$ *is minimal over* $F[\gamma]$.
3. *The equality* $v_{\mathfrak{A}}(\beta - \gamma) = k_0(\beta, \mathfrak{A})$ *holds.*

Proof. — By Proposition 4.5, there exists γ satisfying (1). We show that γ also satisfy (2) and (3). We apply Proposition 4.6 to the case $A = A(E)$. Then the stratum $[\mathfrak{B}', -k_0(\beta, \mathfrak{A}(E)), -k_0(\beta, \mathfrak{A}(E)) - 1, \beta - \gamma]$ is simple, where $\mathfrak{B}' = \mathrm{Cent}_{A(E)}(\gamma) \cap \mathfrak{A}(E)$. Since this stratum is simple, $\beta - \gamma$ is minimal over $F[\gamma]$ and (2) is satisfied. To obtain (3), we calculate $v_{\mathfrak{A}}(\beta - \gamma)$ and $k_0(\beta, \mathfrak{A})$. First, we have

$$v_E(\beta - \gamma) = v_{\mathfrak{o}_E}(\beta - \gamma) = v_{\mathfrak{B}'}(\beta - \gamma) = -\left(-k_0\left(\beta, \mathfrak{A}(E)\right)\right) = k_0\left(\beta, \mathfrak{A}(E)\right).$$

Then we obtain

$$v_{\mathfrak{A}}(\beta - \gamma) = \frac{e(\mathfrak{A}|\mathfrak{o}_F)}{e(E/F)} v_E(\beta - \gamma) = \frac{e(\mathfrak{A}|\mathfrak{o}_F)}{e(E/F)} k_0\left(\beta, \mathfrak{A}(E)\right) = k_0(\beta, \mathfrak{A})$$

and (3) is also satisfied. □

5. Tame twisted Levi subgroups of *G*

The set of lattice functions in G are tame twisted Levi subgroups.

Let E/F be a field extension, and let W be a right $D \otimes_F E$-module such that $\dim_E(W) < \infty$. Then we can define an E-scheme $\underline{\mathrm{Aut}}_{D \otimes_F E}(W)$ as

$$\underline{\mathrm{Aut}}_{D \otimes_F E}(W)(C) = \mathrm{Aut}_{D \otimes_F C}(W \otimes_E C)$$

for an E-algebra C.

Here, let V be a right D-module and let E/F be a field extension in $\mathrm{End}_D(V)$. Then V can be equipped with the canonical right $D \otimes_F E$-module structure.

Let $E'/E/F$ be a field extension in $\mathrm{End}_D(V)$ such that E' is a tamely ramified extension of F. We put $G = \underline{\mathrm{Aut}}_D(V)$, $H = \mathrm{Res}_{E/F} \underline{\mathrm{Aut}}_{D \otimes_F E}(V)$ and $H' = \mathrm{Res}_{E'/F} \underline{\mathrm{Aut}}_{D \otimes_F E'}(V)$, where the functor Res is the Weil restriction. Then H' is a closed subscheme in H and H is a closed subscheme in G.

To show that (H', H, G) is a tame twisted Levi sequence, we fix a maximal torus in G. We take a maximal subfield L in $\mathrm{End}_D(V)$ such that L is a tamely ramified extension of E'. We put $T = \mathrm{Res}_{L/F} \underline{\mathrm{Aut}}_{D \otimes_F L}(V)$.

For a finite field extension E_0/F, we put $X_{E_0} = \mathrm{Hom}_F(E_0, \bar{F})$. Let $\tilde{L}$ be the Galois closure of L/F in $\bar{F}$ and let C be an $\tilde{L}$-algebra. Then we have an E-algebra isomorphism

$$E \otimes_F C \cong \prod_{\sigma \in X_E} C_\sigma$$
$$l \otimes a \mapsto (\sigma(l)a)_\sigma\,,$$

where $C_\sigma = C$ is equipped with an E-algebra structure by the inclusion $l \mapsto \sigma(l)$ from E to C. The canonical inclusion $E \otimes_F C \hookrightarrow E' \otimes_F C$ induced from $E \subset E'$ is the map $(l_\sigma)_{\sigma \in X_E} \mapsto (l_{\sigma'|_E})_{\sigma' \in X_{E'}}$. We also have $V \otimes_F C \cong V \otimes_E E \otimes_F C \cong V \otimes_E (\prod_{\sigma \in X_E} C_\sigma) \cong \bigoplus_{\sigma \in X_E} V \otimes_E C_\sigma$. We put $V_\sigma := V \otimes_E C_\sigma$ for any $\sigma \in X_E$.

We need the following lemma.

Lemma 5.1. — *Let t_0 be a positive integer, and let R_t be a (noncommutative) ring for $t = 1, \ldots, t_0$. We put $R := \prod_{t=1}^{t_0} R_t$ and we regard R_t as an R-submodule in $R \cong \bigoplus_{t=1}^{t_0} R_t$. Let V be a right R-module. We put $V_t := V \cdot 1_t$, where $1_t \in R_t \subset R$ is the identity element in R_t. Then we have $V = \bigoplus_{t=1}^{t_0} V_t$ and an isomorphism*

$$\mathrm{Aut}_R(V) \cong \prod_{t=1}^{t_0} \mathrm{Aut}_{R_t}(V_t)$$
$$f \mapsto (f|_{V_t})_t.$$

The inverse of this isomorphism is the map $(f_t)_t \mapsto \bigoplus_{t=1}^{t_0} f_t$.

Proposition 5.2. — *Let V, $L/E'/E/F$, H and H' be as above. Moreover, let C be an extension field of $\tilde{L}$.*

1. *For $\sigma \in X_E$, we have $V_\sigma = \bigoplus_{\sigma' \in X_{E'}, \sigma'|_E = \sigma} V_{\sigma'}$.*
2. *We have a C-group scheme isomorphism*

$$H \times_F C \cong \prod_{\sigma \in X_E} \underline{\mathrm{Aut}}_{D \otimes_F C_\sigma}(V_\sigma).$$

3. *We have a commutative diagram of C-group schemes:*

$$\begin{array}{ccc} H' \times_F C & \xrightarrow{\cong} & \prod_{\sigma' \in X_{E'}} \underline{\mathrm{Aut}}_{D \otimes_F C_{\sigma'}}(V_{\sigma'}) \\ \downarrow & & \downarrow \\ H \times_F C & \xrightarrow{\cong} & \prod_{\sigma \in X_E} \underline{\mathrm{Aut}}_{D \otimes_F C_\sigma}(V_\sigma)\,, \end{array}$$

Proof. — Since we can regard V as a right $D \otimes_F E$-module, the group $V \otimes_F C$ is equipped with a canonical right $D \otimes_F E \otimes_F C$-module structure. Moreover, by the canonical isomorphism $V \otimes_F C \cong V \otimes_E E \otimes_F C$ we can also equip $V \otimes_E (E \otimes_F C)$ with a right $D \otimes_F (E \otimes_F C)$-module structure. This action on $V \otimes_E E \otimes_F C$ is as follows: For $v \in V$, $z \in D$ and $b_1, b_2 \in E \otimes_F C$, we have

$(v \otimes_E b_1) \cdot (z \otimes_F b_2) = (vz) \otimes_E b_1 b_2$. Let $\sigma \in X_E$. Let $1_\sigma \in C_\sigma \subset \bigoplus_{\tau \in X_E} C_\tau$ be the identity element in C_σ, and we regard 1_σ as an element in $E \otimes_F C$ by the $E \otimes_F C$-module isomorphism $E \otimes_F C \cong \prod_{\tau \in X_E} C_\tau \cong \bigoplus_{\tau \in X_E} C_\tau$. Similarly, for $\sigma' \in X_{E'}$, we define $1_{\sigma'} \in C_{\sigma'} \subset E' \otimes_F C$.

We have $(V \otimes_F C) \cdot (1 \otimes_F 1_\sigma) = (V \otimes_E (E \otimes_F C)) \cdot (1 \otimes_F 1_\sigma) = (V \cdot 1) \otimes_E ((\prod_{\tau \in X_E} C_\tau) \cdot 1_\sigma) = V \otimes_E C_\sigma = V_\sigma$. By the same argument, we also obtain $V_{\sigma'} = (V \otimes_F C) \cdot (1 \otimes_F 1_{\sigma'})$ for $\sigma' \in X_{E'}$. Since $1_\sigma = \sum_{\sigma' \in X_E, \sigma'|_E = \sigma} 1_{\sigma'}$, we obtain

$$\begin{aligned} V_\sigma &= (V \otimes_F C) \cdot (1 \otimes_F 1_\sigma) \\ &= (V \otimes_{E'} (E' \otimes_F C)) \cdot \left(1 \otimes_F \left(\sum_{\sigma' \in X_E, \sigma'|_E = \sigma} 1_{\sigma'}\right)\right) \\ &= V \otimes_{E'} \left(\bigoplus_{\sigma' \in X_E, \sigma'|_E = \sigma} C_{\sigma'}\right) \\ &= \bigoplus_{\sigma' \in X_E, \sigma'|_E = \sigma} (V \otimes_{E'} C_{\sigma'}) = \bigoplus_{\sigma' \in X_E, \sigma'|_E = \sigma} V_{\sigma'}, \end{aligned}$$

which is the equality in (1).

To show (2), let R be a C-algebra. Then we have

$$\begin{aligned} H \times_F C(R) &= \mathrm{Res}_{E/F} \underline{\mathrm{Aut}}_{D \otimes_F E}(V)(R) = \underline{\mathrm{Aut}}_{D \otimes_F E}(V)(E \otimes_F R) \\ &= \mathrm{Aut}_{D \otimes_F E \otimes_E E \otimes_F R}(V \otimes_E E \otimes_F R) \\ &\cong \mathrm{Aut}_{D \otimes_F (E \otimes_F C) \otimes_C R}(V \otimes_E (E \otimes_F C) \otimes_C R). \end{aligned}$$

Here, we have a ring isomorphism $D \otimes_F (E \otimes_F C) \otimes_C R \cong \prod_{\sigma \in X_E} D \otimes_F C_\sigma \otimes_C R$, and the identity element in $D \otimes_F C_\sigma \otimes_C R$ is $1 \otimes_F 1_\sigma \otimes_C 1$. Moreover, $(V \otimes_E (E \otimes_F C) \otimes_C R) \cdot (1 \otimes_F 1_\sigma \otimes_C 1) = V \otimes_E C_\sigma \otimes_C R = V_\sigma \otimes R$. Then, by Lemma 5.1, we obtain $\mathrm{Aut}_{D \otimes_F (E \otimes_F C) \otimes_C R}(V \otimes_E (E \otimes_F C) \otimes_C R) \cong \prod_{\sigma \in X_E} \mathrm{Aut}_{D \otimes_F C_\sigma \otimes_C R}(V_\sigma \otimes_C R) = \left(\prod_{\sigma \in X_E} \underline{\mathrm{Aut}}_{D \otimes_F C_\sigma}(V_\sigma)\right)(R)$, which completes the proof of (2).

(3) is the result from (2) and the fact that there exists a canonical inclusion $H' \subset H$. □

Remark 5.3. — We describe the right vertical morphism in (3). First, the isomorphism

$$\begin{aligned} H \times_F C(R) &= \mathrm{Aut}_{D \otimes_F (\prod_{\sigma \in X_E} C_\sigma) \otimes_C R}\left(\bigoplus_{\sigma \in X_E} (V_\sigma \otimes_C R)\right) \\ &\cong \prod_{\sigma \in X_E} \mathrm{Aut}_{D \otimes_F C_\sigma \otimes_C R}(V_\sigma \otimes_C R) = \left(\prod_{\sigma \in X_E} \underline{\mathrm{Aut}}_{D \otimes_F C_\sigma}(V_\sigma)\right)(R) \end{aligned}$$

is given as $f \mapsto (f|_{V_\sigma \otimes_C R})_\sigma$ by Lemma 5.1. Moreover, the monomorphism $H' \times_F C \to H \times_F C$ induced by $H \subset H'$ is as follows: For a C-algebra R, $H' \times_F C(R) = H'(R) = \mathrm{Aut}_{D\otimes_F E' \otimes_F R}(V \otimes_F R) \subset \mathrm{Aut}_{D\otimes_F E\otimes_F R}(V \otimes_F R) = H(R) = H' \times_F C(R)$. Therefore, by Proposition 5.2 (1), the monomorphism

$$\prod_{\sigma' \in X_{E'}} \underline{\mathrm{Aut}}_{D\otimes_F C_{\sigma'}}(V_{\sigma'}) \hookrightarrow \prod_{\sigma \in X_E} \underline{\mathrm{Aut}}_{D\otimes_F C_\sigma}(V_\sigma)$$

is given as $(f_{\sigma'})_{\sigma' \in X_{E'}} \mapsto (\prod_{\sigma' \in X_E, \sigma'|_E = \sigma} f_{\sigma'})_{\sigma \in X_E}$.

We put $I_1 = \{1, \dots, [E : F]\}, I_2 = \{1, \dots, [E' : E]\}$ and $I_3 = \{1, \dots, [L : E']\}$. Let $\sigma_1, \sigma_2, \dots, \sigma_{[E:F]}$ be distinct elements in $X_E = \mathrm{Hom}_F(E, \bar{F})$. For $i \in I_1$, let $\sigma_{i,1}, \sigma_{i,2}, \dots, \sigma_{i,[E':E]}$ be distinct elements in $\mathrm{Hom}_F(E', \bar{F})$ whose restrictions to E are equal to σ_i. For $(i,j) \in I_1 \times I_2$, let $\sigma_{i,j,1}, \sigma_{i,j,2}, \dots, \sigma_{i,j,[L:E']}$ be distinct elements in $\mathrm{Hom}_F(L, \bar{F})$ whose restrictions to E' are equal to $\sigma_{i,j}$. Then we have

$$\begin{aligned} X_{E'} &= \mathrm{Hom}_F(E', \bar{F}) = \{\sigma_{i,j} | (i,j) \in I_1 \times I_2\} \\ X_L &= \mathrm{Hom}_F(L, \bar{F}) = \{\sigma_{i,j,k} | (i,j,k) \in I_1 \times I_2 \times I_3\} \end{aligned}$$

as L/F is separable. For $(i,j,k) \in I_1 \times I_2 \times I_3$ and an $\tilde{L}$-algebra C, we put $C_{i,j,k} := C_{\sigma_{i,j,k}}$ and $V_{i,j,k} := C_{\sigma_{i,j,k}}$. We can similarly define an E-algebra C_i, an E'-algebra $C_{i,j}$, a $D \otimes_F C_i$-module V_i and a $D \otimes_F C_{i,j}$-module $V_{i,j}$.

Proposition 5.4. — 1. *Let C be an extension field of $\tilde{L}$. Then we have a commutative diagram of C-schemes:*

$$\begin{array}{ccc} T \times_F C & \xrightarrow{\cong} & \prod_{i,j,k} \underline{\mathrm{Aut}}_{D\otimes_F C_{i,j,k}}(V_{i,j,k}) \\ \downarrow & & \downarrow \\ H' \times_F C & \xrightarrow{\cong} & \prod_{i,j} \underline{\mathrm{Aut}}_{D\otimes_F C_{i,j}}\left(\bigoplus_k V_{i,j,k}\right) \\ \downarrow & & \downarrow \\ H \times_F C & \xrightarrow{\cong} & \prod_{i} \underline{\mathrm{Aut}}_{D\otimes_F C_{i}}\left(\bigoplus_{j,k} V_{i,j,k}\right) \\ \downarrow & & \downarrow \\ G \times_F C & \xrightarrow{\cong} & \underline{\mathrm{Aut}}_{D\otimes_F C}\left(\bigoplus_{i,j,k} V_{i,j,k}\right). \end{array}$$

2. *We have a commutative diagram of C-vector spaces:*

$$\begin{array}{ccc}
\mathrm{Lie}(T \times_F C) & \xrightarrow{\cong} & \prod_{i,j,k} \mathrm{End}_{D\otimes_F C_{i,j,k}}(V_{i,j,k}) \\
\downarrow & & \downarrow \\
\mathrm{Lie}(H' \times_F C) & \xrightarrow{\cong} & \prod_{i,j} \mathrm{End}_{D\otimes_F C_{i,j}}\left(\bigoplus_k V_{i,j,k}\right) \\
\downarrow & & \downarrow \\
\mathrm{Lie}(H \times_F C) & \xrightarrow{\cong} & \prod_{i} \mathrm{End}_{D\otimes_F C_{i}}\left(\bigoplus_{j,k} V_{i,j,k}\right) \\
\downarrow & & \downarrow \\
\mathrm{Lie}(G \times_F C) & \xrightarrow{\cong} & \mathrm{End}_{D\otimes_F C}\left(\bigoplus_{i,j,k} V_{i,j,k}\right),
\end{array}$$

where the vertical morphisms are all monomorphisms.

3. *Let $c \in L$, and let $m_c \in \mathrm{Lie}(T) = \mathrm{End}_{D\otimes L}(V)$ be the map $v \mapsto cv$ for $v \in V$. We put $m_{c,C} = m_c \otimes_F 1 \in \mathrm{Lie}(T) \otimes_F C = \mathrm{Lie}(T \times_F C)$. When we regard $m_{c,C}$ as an element in $\mathrm{End}_{D\otimes_F C}\left(\bigoplus_{i,j,k} V_{i,j,k}\right)$ via the morphisms in (2), for $v_{i,j,k} \in V_{i,j,k}$ we have $m_{c,C}(v_{i,j,k}) = v_{i,j,k}\cdot\sigma_{i,j,k}(c)$.*

Proof. — (1) is the result from Proposition 5.2. By taking the Lie algebra of (1), we obtain (2).

To show (3), let $v \in V$, $l \in L$ and $b \in C$. Then $m_{c,C}(v \otimes_L l \otimes_F b) = m_{c,C}(lv\otimes_F b) = clv\otimes_F b = v\otimes_L(c\otimes_F 1\cdot l\otimes_F b) \in V\otimes_L(L\otimes_F C)$. Here, let $v \in V$ and $a \in C_{i,j,k}$. Then we have $v\otimes_L a \in V\otimes_L C_{i,j,k} = V_{i,j,k} \subset V\otimes_L(L\otimes_F C)$ and $m_{c,C}(v\otimes_L a) = v\otimes_L((c\otimes_F 1)\cdot a)$. Since $(c\otimes_F 1)\cdot a = \sigma_{i,j,k}(c)a$ by the L-algebra structure in $C_{i,j,k}$, we obtain $m_{c,C}(v\otimes_L a) = v\otimes_L \sigma_{i,j,k}(c)a = (v\otimes_L a)\cdot\sigma_{i,j,k}(c)$. Since $V_{i,j,k}$ is generated by elements of the form $v\otimes_L a$ for some $v \in V$ and $a \in C_{i,j,k}$, we obtain $m_{c,C}(v_{i,j,k}) = v_{i,j,k}\cdot\sigma_{i,j,k}(c)$ for $v_{i,j,k} \in V_{i,j,k}$. □

Corollary 5.5. — *The sequence (H', H, G) is a tame twisted Levi sequence. Moreover, $Z(H')/Z(G)$ is anisotropic.*

Proof. — We put $C = \tilde{L}$, which is a finite, tamely ramified Galois extension of F. Since L is a maximal F-subfield in A, the right $D\otimes_F L$-module V is simple. Then for any $(i,j,k) \in I_1 \times I_2 \times I_3$ and C-algebra $\tilde{C}$, we have

$$\begin{aligned}
\underline{\mathrm{End}}_{D\otimes_F C_{i,j,k}}(V_{i,j,k})(\tilde{C}) &= \mathrm{End}_{D\otimes_F C_{i,j,k}\otimes_C \tilde{C}}(V\otimes_L C_{i,j,k}\otimes_C \tilde{C}) \\
&\cong \mathrm{End}_{D\otimes_F L\otimes_L \tilde{C}}(V\otimes_L \tilde{C}) \\
&\cong \mathrm{End}_{D\otimes_F L}(V)\otimes_L \tilde{C} \\
&\cong L\otimes_L \tilde{C} \cong \tilde{C} = \underline{\mathrm{End}}_C(C)(\tilde{C}).
\end{aligned}$$

Therefore, we have $\underline{\mathrm{End}}_{D\otimes_F C_{i,j,k}}(V_{i,j,k}) \cong \underline{\mathrm{End}}_C(C)$ as C-schemes. We also have

$$\begin{aligned}
\prod_{i,j,k} \underline{\mathrm{End}}_{D\otimes_F C_{i,j,k}}(V_{i,j,k}) &\cong \prod_{i,j,k} \underline{\mathrm{End}}_C(C), \\
\prod_{i,j} \underline{\mathrm{End}}_{D\otimes_F C_{i,j}}\left(\bigoplus_k V_{i,j,k}\right) &\cong \prod_{i,j} \underline{\mathrm{End}}_C\left(C^{\oplus|I_3|}\right), \\
\prod_{i} \underline{\mathrm{End}}_{D\otimes_F C_{i}}\left(\bigoplus_{j,k} V_{i,j,k}\right) &\cong \prod_{i} \underline{\mathrm{End}}_C\left(C^{\oplus(|I_2|\times|I_3|)}\right), \\
\underline{\mathrm{End}}_{D\otimes_F C}\left(\bigoplus_{i,j,k} V_{i,j,k}\right) &\cong \underline{\mathrm{End}}_C\left(C^{\oplus(|I_1|\times|I_2|\times|I_3|)}\right).
\end{aligned}$$

By taking the multiplicative group, we obtain

$$\begin{aligned}
T\times_F C &\cong \prod_{i,j,k} \underline{\mathrm{Aut}}_{D\otimes_F C_{i,j,k}}(V_{i,j,k}) \cong \mathbb{G}_m{}^{\times(|I_1|\times|I_2|\times|I_3|)}, \\
H'\times_F C &\cong \prod_{i,j} \underline{\mathrm{Aut}}_{D\otimes_F C_{i,j}}\left(\bigoplus_k V_{i,j,k}\right) \cong \mathrm{GL}_{|I_3|}{}^{\times(|I_1|\times|I_2|)}, \\
H\times_F C &\cong \prod_{i} \underline{\mathrm{Aut}}_{D\otimes_F C_{i}}\left(\bigoplus_{j,k} V_{i,j,k}\right) \cong \mathrm{GL}_{|I_2|\times|I_3|}{}^{\times|I_1|}, \\
G\times_F C &\cong \underline{\mathrm{Aut}}_{D\otimes FC}\left(\bigoplus_{i,j,k} V_{i,j,k}\right) \cong \mathrm{GL}_{|I_1|\times|I_2|\times|I_3|}.
\end{aligned}$$

Therefore, $H'\times_F C$ and $H\times_F C$ are Levi subgroups in $G\times_F C$ with a split maximal torus $T\times_F C$. Since C is a finite, tamely ramified Galois extension of F, the sequence (H', H, G) is a tame twisted Levi sequence.

Moreover, we have $(Z(H')/Z(G))(F) = E'^{\times}/F^{\times}$, which is compact. Then $Z(H')/Z(G)$ is anisotropic. □

Let $C = \bar{F}$. For distinct elements $(i', j', k'), (i'', j'', k'') \in I_1 \times I_2 \times I_3$, we define the root $\alpha_{(i',j',k'),(i'',j'',k'')} \in \Phi(G, T; \bar{F})$ as

$$\alpha_{(i',j',k'),(i'',j'',k'')} : \prod_{i,j,k} \mathrm{Aut}_{D\otimes_F \bar{F}_{i,j,k}}(V_{i,j,k}) \to \bar{F}^{\times}; (t_{i,j,k})_{i,j,k} \mapsto t_{i',j',k'} t^{-1}_{i'',j'',k''}.$$

Therefore,we have

$$\begin{aligned}
\Phi(H, T; \bar{F}) &= \left\{\alpha_{(i',j',k'),(i'',j'',k'')} \in \Phi(G, T; \bar{F}) | i' = i''\right\} \\
\Phi(H', T; \bar{F}) &= \left\{\alpha_{(i',j',k'),(i'',j'',k'')} \in \Phi(G, T; \bar{F}) | i' = i'', j' = j''\right\},
\end{aligned}$$

and we obtain

$$\Phi(H,T;\bar{F}) \setminus \Phi(H',T;\bar{F}) = \left\{\alpha_{(i',j',k'),(i'',j'',k'')} \in \Phi(G,T;\bar{F}) | i'=i'', j' \neq j''\right\}.$$

Moreover, the coroot $\check{\alpha}_{(i',j',k'),(i'',j'',k'')}$ with respect to $\alpha_{(i',j',k'),(i'',j'',k'')}$ is as follows:

$$\check{\alpha}_{(i',j',k'),(i'',j'',k'')} : \bar{F}^{\times} \to \prod_{i,j,k} \mathrm{Aut}_{D\otimes_F \bar{F}_{i,j,k}}(V_{i,j,k}) \cong \prod_{i,j,k} \bar{F}^{\times}; t \mapsto (t_{i,j,k})_{i,j,k},$$

where

$$t_{i,j,k} = \begin{cases} t & ((i,j,k)=(i',j',k')), \\ t^{-1} & ((i,j,k)=(i'',j'',k'')), \\ 1 & \text{otherwise.} \end{cases}$$

Then we have $\mathrm{d}\check{\alpha}_{(i',j',k'),(i'',j'',k'')}(u) = (u_{i,j,k})_{i,j,k}$ where

$$u_{i,j,k} = \begin{cases} u & ((i,j,k)=(i',j',k')), \\ -u & ((i,j,k)=(i'',j'',k'')), \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, we determine the set of tame twisted Levi subgroup G' in G with $Z(G')/Z(G)$ anisotropic.

LEMMA 5.6. — *Let G' be a tame twisted Levi subgroup of $G = \underline{\mathrm{Aut}}_D(V)$. Suppose $Z(G')/Z(G)$ is anisotropic. Then there exists a finite, tamely ramified extension E of F such that $G' \cong \mathrm{Res}_{E/F}\, \underline{\mathrm{Aut}}_{D\otimes_F E}(V)$.*

Proof. — Let F^{tr} be the maximal tamely ramified extension of F. Since G' is a tame twisted Levi subgroup in G, $G'_{F^{\mathrm{tr}}}$ is a Levi subgroup in $G_{F^{\mathrm{tr}}} \cong \underline{\mathrm{Aut}}_{D\otimes F^{\mathrm{tr}}}(V \otimes F^{\mathrm{tr}})$. There exists a one-to-one relationship between Levi subgroups in $G_{F^{\mathrm{tr}}}$ and direct decompositions of $V \otimes F^{\mathrm{tr}}$ as a right $D \otimes F^{\mathrm{tr}}$-module. Let $V \otimes F^{\mathrm{tr}} = \bigoplus_{k=1}^{j} V_k$ be the corresponding decomposition with $G'_{F^{\mathrm{tr}}}$. Then we have $G'_{F^{\mathrm{tr}}} = \prod_{k=1}^{j} \underline{\mathrm{Aut}}_{D\otimes F^{\mathrm{tr}}}(V_k)$. We remark that the right-hand-side group is the multiplicative group of $\underline{\mathrm{End}}_{D\otimes F^{\mathrm{tr}}}(V_k)$ with a $Gal(F^{\mathrm{tr}}/F)$-action defined by its F-structure. Let Z_k be the center of $\underline{\mathrm{End}}_{D\otimes F^{\mathrm{tr}}}(V_k)$, which is F^{tr}-isomorphic to $\underline{\mathrm{End}}_{F^{\mathrm{tr}}}(F^{\mathrm{tr}})$. Then $Z(G')_{F^{\mathrm{tr}}}$ is the multiplicative group of $Z = \prod_{k=1}^{j} Z_k$, equipped with the same $Gal(F^{\mathrm{tr}}/F)$-action. Therefore, we consider the structure of Z_k. Let $\mathbf{1}_k$ be (the F^{tr}-rational point corresponding to) the identity element in Z_k. Since the $\mathrm{Gal}(F^{\mathrm{tr}}/F)$-action to Z preserves the F-algebra structure, the set $\{\mathbf{1}_k \mid k=1,\dots,j\}$ is $\mathrm{Gal}(F^{\mathrm{tr}}/F)$-invariant. Then by changing the indices if necessary, we may assume there exist integers $0 = n_0 < n_1 < \cdots < n_l = j$ such that $\mathrm{Gal}(F^{\mathrm{tr}}/F)$ acts on the set $\{\mathbf{1}_{n_{i-1}+1}, \dots, \mathbf{1}_{n_i}\}$ transitively for $l = 1, \dots, i$. We put $Y_i = \prod_{k=n_{i-1}+1}^{n_i} Z_k$. Since $a \in F^{\mathrm{tr}}, b \in Z$ and $\gamma \in \mathrm{Gal}(F^{\mathrm{tr}}/F)$ we have $\gamma(ab) = \gamma(a)\gamma(b)$ and $\{\mathbf{1}_{n_{i-1}+1}, \dots, \mathbf{1}_{n_i}\}$ is $\mathrm{Gal}(F^{\mathrm{tr}}/F)$-invariant, Y_i is also $\mathrm{Gal}(F^{\mathrm{tr}}/F)$-invariant. Then Y_i is defined over F. Let X_i be the Galois descent of Y_i to F. Let

$\mathrm{Gal}(F^{\mathrm{tr}}/F_i)$ be the stabilizer of $\mathbf{1}_{n_i}$. The fields F_i is tamely ramified, and finite-dimensional over F since $\mathrm{Gal}(F^{\mathrm{tr}}/F)/\mathrm{Gal}(F^{\mathrm{tr}}/F_i)$ is $\mathrm{Gal}(F^{\mathrm{tr}}/F)$-isomorphic to the finite set $\{\mathbf{1}_{n_{i-1}+1}, \dots, \mathbf{1}_{n_i}\}$.

We show X_i is isomorphic to $\mathrm{Res}_{F_i/F}\,\underline{\mathrm{End}}_{F_i}(F_i)$. If this follows, then we can show the multiplicative group of X_i is $\mathrm{Res}_{F_i/F}\,\mathbb{G}_m$ and $Z(G') = \prod_{i=1}^{l} \mathrm{Res}_{F_i/F}\,\mathbb{G}_m$.

Any F^{tr}-rational point of Y_i is uniquely represented as the form $\sum_{k'=n_{i-1}+1}^{n_i} a_{k'}\mathbf{1}_{k'}$, where $a_{k'} \in F^{\mathrm{tr}}$. Suppose $z = \sum_{k'=n_{i-1}+1}^{n_i} a_{k'}\mathbf{1}_{k'}$ is stabilized by $\mathrm{Gal}(F^{\mathrm{tr}}/F)$. For any $\gamma \in \mathrm{Gal}(F^{\mathrm{tr}}/F_i)$, we have $z = \gamma(z) = \sum_{k'=n_{i-1}+1}^{n_i-1} \gamma(a_{k'})\gamma(\mathbf{1}_{k'}) + \gamma(a_{n_i})\mathbf{1}_{n_i}$. Then we have $\gamma(a_{n_i}) = a_{n_i}$, that is, $a_{n_i} \in F_i$. For $n_{i-1} < k' < n_i$, we pick $\gamma_{k'} \in \mathrm{Gal}(F^{\mathrm{tr}}/F)$ such that $\gamma_{k'}(\mathbf{1}_{n_i}) = \mathbf{1}_{k'}$. Then we have

$$z = \gamma_{k'}(z) = \sum_{k''=n_{i-1}+1}^{n_i-1} \gamma_{k''}(a_{k''})\gamma_{k'}(\mathbf{1}_{k'}) + \gamma_{k'}(a_{n_i})\mathbf{1}_{k'},$$

whence $a_{k'} = \gamma_{k'}(a_{n_i})$. Therefore, any F-rational point of X_i is the form

$$\sum_{k'=n_{i-1}+1}^{n_i-1} \gamma_{k'}(a_{n_i})\mathbf{1}_{k'} + a_{n_i}\mathbf{1}_{n_i},$$

where $a_{n_i} \in F_i$, and the ring structure of $X_i(F)$ is isomorphic to F_i. Since the ring structure of $X_i(C)$ is isomorphic to $X_i(F) \otimes C$ for any F-algebra C, we obtain $X_i \cong \mathrm{Res}_{F_i/F}\,\underline{\mathrm{End}}_{F_i}(F_i)$.

We have shown $Z(G') = \prod_{i=1}^{l} \mathrm{Res}_{F_i/F}\,\mathbb{G}_m$. Since $Z(G')/Z(G)$ is anisotropic and $Z(G) = \mathbb{G}_m$, we have $l = 1$ and $Z(G') = \mathrm{Res}_{E/F}\,\mathbb{G}_m$, where we put $E = F_1$.

The field E can be regarded as a F-subfield in A via $X \subset \underline{\mathrm{End}}_D(V)$. We put $H = \underline{\mathrm{Aut}}_{D\otimes E}(V)$. Then H is a tame twisted Levi subgroup in G, and we have $Z(H) = Z(G')$. Since there exists a one-to-one relationship between subtori in G defined over F and Levi subgroups in G defined over F, we obtain $G' = H$. □

6. Embeddings of buildings for Levi sequences of G

6.1. Lattice functions in V. — First, we recall the lattice functions in V and their properties from [3].

Definition 6.1. — The map $\mathcal{L}$ from $\mathbb{R}$ to the set of $\mathfrak{o}_D$-lattices in V is a lattice function in V if we have the following.

1. We have $\mathcal{L}(r)\varpi_D = \mathcal{L}(r + (1/d))$ for some uniformizer ϖ_D of D and $r \in \mathbb{R}$.
2. $\mathcal{L}$ is decreasing, that is, $\mathcal{L}(r) \supset \mathcal{L}(r')$ if $r \leq r'$.
3. $\mathcal{L}$ is left-continuous, where the set of lattices in V is equipped with the discrete topology.

The set of lattice functions in V is denoted by $\mathrm{Latt}^1(V)$. The groups G and $\mathbb{R}$ act on $\mathrm{Latt}^1(V)$ by $(g \cdot \mathcal{L})(r) = g \cdot (\mathcal{L}(r))$ and $(r' \cdot \mathcal{L})(r) = \mathcal{L}(r + r')$ for $g \in G$, $r, r' \in \mathbb{R}$ and $\mathcal{L} \in \mathrm{Latt}^1(V)$. These actions are compatible, and then $\mathrm{Latt}(V) := \mathrm{Latt}^1(V)/\mathbb{R}$ is equipped with the canonical G-action. The G-sets $\mathrm{Latt}^1(V)$ and $\mathrm{Latt}(V)$ are also equipped with an affine structure. Then there exists a canonical G-equivariant, affine isomorphism $\mathscr{B}^E(G, F) \to \mathrm{Latt}^1(V)$. This isomorphism induces a G-equivariant, affine isomorphism $\mathscr{B}^R(G, F) \to \mathrm{Latt}(V)$.

We construct lattice functions from $\mathfrak{o}_D$-sequences. Let $c \in \mathbb{R}$ and let $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ be an $\mathfrak{o}_D$-sequence with period e. Then

$$\mathcal{L}(r) = \mathcal{L}_{\lceil de(r-c) \rceil},\ r \in \mathbb{R}$$

is a lattice function in V.

Proposition 6.2. — *Let $\mathcal{L}$ be a lattice function in V. The following assertions are equivalent.*

1. *$\mathcal{L}$ is constructed from an $\mathfrak{o}_D$-chain.*
2. *There exists $c \in \mathbb{R}$ and $e \in \mathbb{Z}_{>0}$ such that the set of discontinuous points of $\mathcal{L}$ is equal to $c + (de)^{-1}\mathbb{Z}$.*

Moreover, if (1) (and (2)) holds, e is equal to the period of some $\mathfrak{o}_D$-chain, which $\mathcal{L}$ is constructed from.

Proof. — First, suppose $\mathcal{L}$ is constructed from an $\mathfrak{o}_D$-chain. Then there exists $c \in \mathbb{R}$ and an $\mathfrak{o}_D$-chain $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ with period e such that $\mathcal{L}(r) = \mathcal{L}_{\lceil de(r-c) \rceil}$ for $r \in \mathbb{R}$. Since $(\mathcal{L}_i)$ is an $\mathfrak{o}_D$-chain, the set of discontinuous points of $\mathcal{L}$ is equal to $c + (de)^{-1}\mathbb{Z}$, whence (2) holds.

Conversely, suppose (2) holds. For $i \in \mathbb{Z}$, we put $\mathcal{L}_i = \mathcal{L}(c + (de)^{-1}i)$. Since $\mathcal{L}$ is not right-continuous at $r = c + (de)^{-1}i$, we have

$$\mathcal{L}_i = \mathcal{L}(c + (de)^{-1}i) \supsetneq \mathcal{L}(c + (de)^{-1}(i+1)) = \mathcal{L}_{i+1}.$$

Moreover, we also have

$$\begin{aligned}\mathcal{L}_{i+e} &= \mathcal{L}(c + (de)^{-1}(i+e)) = \mathcal{L}(c + (de)^{-1}i + d^{-1}) = \mathcal{L}(c + (de)^{-1}i)\varpi_D \\ &= \mathcal{L}_i \varpi_D.\end{aligned}$$

Then $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ is an $\mathfrak{o}_D$-chain with period e.

Let $\mathcal{L}'$ be the lattice function constructed from $c \in \mathbb{R}$ and the $\mathfrak{o}_D$-chain $(\mathcal{L}_i)$. We show $\mathcal{L} = \mathcal{L}'$. For $i \in \mathbb{Z}$, we have $\mathcal{L}'(c + (de)^{-1}i) = \mathcal{L}_i = \mathcal{L}(c + (de)^{-1}i)$ and $\mathcal{L} = \mathcal{L}'$ on $c + (de)^{-1}\mathbb{Z}$. For $r \in \mathbb{R}$, there exists $i \in \mathbb{Z}$ such that $r \in (c + (de)^{-1}(i-1), c + (de)^{-1}i]$. Since the set of discontinuous points of $\mathcal{L}$ is $c + (de)^{-1}\mathbb{Z}$, then $\mathcal{L}|_{(c+(de)^{-1}(i-1), c+(de)^{-1}i]}$ is continuous and

$$\mathcal{L}(r) = \mathcal{L}(c + (de)^{-1}i) = \mathcal{L}_i = \mathcal{L}_{\lceil de(r-c) \rceil} = \mathcal{L}'(r).$$

Therefore $\mathcal{L} = \mathcal{L}'$ is the lattice function constructed from the $\mathfrak{o}_D$-chain $(\mathcal{L}_i)$ of period e. The last assertion follows from the above argument. □

Conversely, for any lattice function $\mathcal{L}$ there exists an $\mathfrak{o}_D$-chain $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ such that $\{\mathcal{L}(r) \mid r \in \mathbb{R}\} = \{\mathcal{L}_i \mid i \in \mathbb{Z}\}$, unique up to translation. Since $\mathcal{L}(r + (1/d)) = \mathcal{L}(r)\varpi_D$ for $r \in \mathbb{R}$, the period of $(\mathcal{L}_i)$ is equal to the number of discontinuous points of $\mathcal{L}$ in $[0, 1/d)$.

6.2. Comparison of filtrations: hereditary orders and Moy–Prasad filtration. — Let x be an element in $\mathscr{B}^E(G,F)$, corresponding to a lattice function $\mathcal{L}$ via $\mathscr{B}^E(G,F) \cong \mathrm{Latt}^1(V)$. We can define a filtration $(\mathfrak{a}_{x,r})_{r\in\mathbb{R}}$ in A associated with x as

$$\mathfrak{a}_{x,r} = \mathfrak{a}_{\mathcal{L},r} = \{a \in A \mid a\mathcal{L}(r') \subset \mathcal{L}(r + r'),\ r' \in \mathbb{R}\}$$

for $r \in \mathbb{R}$. We also put $\mathfrak{a}_{x,r+} = \bigcup_{r<r'} \mathfrak{a}_{x,r'}$. Then we can define a hereditary $\mathfrak{o}_F$-order $\mathfrak{A} = \mathfrak{a}_{x,0}$ associated with x. The radical of $\mathfrak{A}$ is equal to $\mathfrak{P} = \mathfrak{a}_{x,0+}$. We also put $\mathbf{U}_0(x) = \mathfrak{A}^\times$, and $\mathbf{U}_r(x) = 1 + \mathfrak{a}_{x,r}$ for $r \in \mathbb{R}_{>0}$.

Proposition 6.3 ([3, Appendix A]). — *Let $x \in \mathscr{B}^E(G,F)$.*

1. *When we identify A with the Lie algebra $\mathfrak{g}(F)$ of G, we have $\mathfrak{a}_{x,r} = \mathfrak{g}(F)_{x,r}$ for $r \in \mathbb{R}$.*
2. *For $r \geq 0$, we have $\mathbf{U}_r(x) = G(F)_{x,r}$.*

Suppose $\mathcal{L}$ is constructed from an $\mathfrak{o}_D$-chain. Then there exist $c \in \mathbb{R}$ and an $\mathfrak{o}_D$-chain $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ with period e such that $\mathcal{L}(r) = \mathcal{L}_{\lceil de(r-c)\rceil}$. Since $\mathcal{L}_{i+e} = \mathcal{L}_i\varpi_D$ for $i \in \mathbb{Z}$, we have $\mathcal{L}_{i+de} = \mathcal{L}_i\varpi_F$, and then $e(\mathfrak{A}|\mathfrak{o}_F) = de$.

Proposition 6.4. — *Let $x, \mathcal{L}$ be as above, and let $r \in \mathbb{R}$.*

1. *We have $\mathfrak{P}^{\lceil r\rceil} = \mathfrak{g}(F)_{x,r/e(\mathfrak{A}|\mathfrak{o}_F)}$.*
2. *Suppose $r \geq 0$. Then $\mathbf{U}^{\lceil r\rceil}(\mathfrak{A}) = G(F)_{x,r/e(\mathfrak{A}|\mathfrak{o}_F)}$.*
3. *We have $\mathfrak{K}(\mathfrak{A}) = G(F)_{[x]}$.*

Proof. — We show (1). By Proposition 6.3 (1), it suffices to show $\mathfrak{P}^{\lceil r\rceil} = \mathfrak{a}_{\mathcal{L},r/e(\mathfrak{A}|\mathfrak{o}_F)}$. We put $n = \lceil r\rceil$. Suppose $a \in \mathfrak{a}_{x,r/e(\mathfrak{A}|\mathfrak{o}_F)}$. For $n' \in \mathbb{Z}$, we put $r_{n'} = c + e(\mathfrak{A}|\mathfrak{o}_F)^{-1}n'$. Then we have $\mathcal{L}(r_{n'}) = \mathcal{L}_{\lceil de(r_{n'}-c)\rceil} = \mathcal{L}_{n'}$, and $\mathcal{L}\left(e(\mathfrak{A}|\mathfrak{o}_F)^{-1}r + r_{n'}\right) = \mathcal{L}_{\lceil de(e(\mathfrak{A}|\mathfrak{o}_F)^{-1}r + r_{n'} - c)\rceil} = \mathcal{L}_{n'+\lceil n\rceil}$. Since $a \in \mathfrak{a}_{\mathcal{L},r/e(\mathfrak{A}|\mathfrak{o}_F)}$, in particular,

$$a\mathcal{L}_{n'} = a\mathcal{L}(r_{n'}) \subset \mathcal{L}(e(\mathfrak{A})|\mathfrak{o}_F)^{-1}r + r_{n'}) = \mathcal{L}_{n+n'}$$

for $n' \in \mathbb{Z}$. Since $\{a \in A \mid a\mathcal{L}_{n'} \subset \mathcal{L}_{n+n'},\ n' \in \mathbb{Z}\} = \mathfrak{P}^n$, we have $a \in \mathfrak{P}^n$.

Conversely, suppose $a \in \mathfrak{P}^n$. For $r' \in \mathbb{R}$, we have $\mathcal{L}(r') = \mathcal{L}_{\lceil de(r'-c)\rceil}$ and $\mathcal{L}(e(\mathfrak{A}|\mathfrak{o}_F)^{-1}r + r') = \mathcal{L}_{\lceil r+de(r'-c)\rceil}$. Since $\lceil r + de(r'-c)\rceil < r + de(r'-c) + 1$ and $\lceil de(r'-c)\rceil \geq de(r'-c)$, we have

$$\lceil r + de(r'-c)\rceil - \lceil de(r'-c)\rceil < r + de(r'-c) + 1 - de(r'-c) = r + 1.$$

Since $\lceil r + de(r' - c)\rceil - \lceil de(r' - c)\rceil \in \mathbb{Z}$, we also have $\lceil r + de(r' - c)\rceil - \lceil de(r' - c)\rceil \leq \lceil r\rceil$. When we put $n' = \lceil de(r' - c)\rceil$, we have $n + n' \geq \lceil r + de(r' - c)\rceil$. Therefore,

$$a\mathcal{L}(r') = a\mathcal{L}_{\lceil de(r'-c)\rceil} = a\mathcal{L}_{n'} \subset \mathcal{L}_{n+n'} \subset \mathcal{L}_{\lceil r+de(r'-c)\rceil} = \mathcal{L}(e(\mathfrak{A}|\mathfrak{o}_F)^{-1}r + r')$$

for $r' \in \mathbb{R}$, which implies $a \in \mathfrak{a}_{\mathcal{L},r/e(\mathfrak{A}|\mathfrak{o}_F)}$. Thus (1) holds.

To show (2), it is enough to show $\mathbf{U}^{\lceil r\rceil}(\mathfrak{A}) = \mathbf{U}_{r/e(\mathfrak{A}|\mathfrak{o}_F)}(x)$ by Proposition 6.3 (2). Therefore (2) follows from (1).

(3) is a corollary of [3, I Lemma 7.3], as $\mathcal{L}$ is constructed from an $\mathfrak{o}_D$-chain. □

Proposition 6.5. — *Let $x \in \mathscr{B}^E(G, F)$ correspond with a lattice function constructed from an $\mathfrak{o}_D$-chain, and let $n \in \mathbb{Z}$.*

1. (a) $\mathfrak{P}^n = \mathfrak{g}(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)}$
   (b) $\mathfrak{P}^{n+1} = \mathfrak{g}(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)+}$
   (c) $\mathfrak{P}^{\lfloor(n+1)/2\rfloor} = \mathfrak{g}(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)}$
   (d) $\mathfrak{P}^{\lfloor n/2\rfloor+1} = \mathfrak{g}(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)+}$.
2. *Suppose $n \geq 0$. Then we have*
   (a) $\mathbf{U}^n(\mathfrak{A}) = G(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)}$
   (b) $\mathbf{U}^{n+1}(\mathfrak{A}) = G(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)+}$
   (c) $\mathbf{U}^{\lfloor(n+1)/2\rfloor}(\mathfrak{A}) = G(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)}$
   (d) $\mathbf{U}^{\lfloor n/2\rfloor+1}(\mathfrak{A}) = G(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)+}$.

Proof. — We show (1), and (2) can be shown in the same way as (1).

(a) follows from Proposition 6.4 (1). (c) also follows from Proposition 6.4 (1) and the fact $\lfloor(n+1)/2\rfloor = \lceil n/2\rceil$ for $n \in \mathbb{Z}$.

We show (b). For $r \in (n, n+1]$, we have $\lceil r\rceil = n + 1$. Then we have

$$\begin{aligned}\mathfrak{g}(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)+} &= \bigcup_{n/e(\mathfrak{A}|\mathfrak{o}_F)<r'} \mathfrak{g}(F)_{x,r'} \\ &= \bigcup_{n/e(\mathfrak{A}|\mathfrak{o}_F)<r'\leq(n+1)/e(\mathfrak{A}|\mathfrak{o}_F)} \mathfrak{P}^{\lceil r'e(\mathfrak{A}|\mathfrak{o}_F)\rceil} \\ &= \mathfrak{P}^{n+1}.\end{aligned}$$

To show (d), we consider two cases. First, suppose $n \in 2\mathbb{Z}$. Then we have $\mathfrak{P}^{\lfloor n/2\rfloor+1} = \mathfrak{P}^{(n/2)+1} = \mathfrak{g}(F)_{x,((n/2)+1)/e(\mathfrak{A}|\mathfrak{o}_F)}$ by (a). Since $n/2 \in \mathbb{Z}$, for any $r \in (n/2, (n/2) + 1]$, we have $\lceil r\rceil = (n/2) + 1$ and $\mathfrak{g}(F)_{x,r/e(\mathfrak{A}|\mathfrak{o}_F)} = \mathfrak{P}^{\lceil r\rceil} =$

$\mathfrak{P}^{(n/2)+1}$. Therefore,

$$\begin{aligned}\mathfrak{g}(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)+} &= \bigcup_{n/2e(\mathfrak{A}|\mathfrak{o}_F)<r'} \mathfrak{g}(F)_{x,r'} \\ &= \bigcup_{n/2e(\mathfrak{A}|\mathfrak{o}_F)<r'\leq((n/2)+1)/e(\mathfrak{A}|\mathfrak{o}_F)} \mathfrak{P}^{\lceil r'e(\mathfrak{A}|\mathfrak{o}_F)\rceil} \\ &= \mathfrak{P}^{(n/2)+1} = \mathfrak{P}^{\lfloor n/2\rfloor+1}.\end{aligned}$$

Next, suppose $n\in\mathbb{Z}\setminus 2\mathbb{Z}$. Then we have $\lfloor n/2\rfloor+1=(n+1)/2=\lceil(n+1)/2\rceil$ and $\mathfrak{P}^{\lfloor n/2\rfloor+1}=\mathfrak{g}(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)}$ by (b). Since $\lceil r\rceil=(n+1)/2$ for $r\in(n/2,(n+1)/2]$, we obtain

$$\begin{aligned}\mathfrak{g}(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)+} &= \bigcup_{n/2e(\mathfrak{A}|\mathfrak{o}_F)<r'} \mathfrak{g}(F)_{x,r'} \\ &= \bigcup_{n/2e(\mathfrak{A}|\mathfrak{o}_F)<r'\leq(n+1)/2e(\mathfrak{A}|\mathfrak{o}_F)} \mathfrak{P}^{\lceil r'e(\mathfrak{A}|\mathfrak{o}_F)\rceil} \\ &= \mathfrak{P}^{(n+1)/2} = \mathfrak{P}^{\lceil n/2\rceil+1}.\end{aligned}$$

□

Let (H',H,G) be a tame twisted Levi sequence. Then there exists a tower $E'/E/F$ of tamely ramified extensions in A such that $H'=\mathrm{Res}_{E'/F}\,\underline{\mathrm{Aut}}_{D\otimes_F E'}(V)$ and $H=\mathrm{Res}_{E/F}\,\underline{\mathrm{Aut}}_{D\otimes_F E}(V)$. We put $B=\mathrm{Cent}_A(E)$ and $B'=\mathrm{Cent}_A(E')$. There exist a division E-algebra D_E and a right D_E-module W such that $B\cong\mathrm{End}_{D_E}(W)$. Similarly, there exist a division E'-algebra $D_{E'}$ and a right $D_{E'}$-module W' such that $B'\cong\mathrm{End}_{D_{E'}}(W')$. Since $E'/E/F$ is a tower of tamely ramified extensions, we have canonical identifications

$$\begin{aligned}\mathscr{B}^E(H,F) &\cong \mathscr{B}^E(\underline{\mathrm{Aut}}_{D\otimes E}(V),E)\cong\mathscr{B}^E(\underline{\mathrm{Aut}}_{D_E}(W),E),\\ \mathscr{B}^E(H',F) &\cong \mathscr{B}^E(\underline{\mathrm{Aut}}_{D\otimes E'}(V),E')\cong\mathscr{B}^E(\underline{\mathrm{Aut}}_{D_{E'}}(W'),E').\end{aligned}$$

Let $x\in\mathscr{B}^E(H',F)\cong\mathscr{B}^E(\underline{\mathrm{Aut}}_{D_{E'}}(W'),E')$ and let $\mathcal{L}$ be the corresponding lattice function in W' with x.

Proposition 6.6. — *The following assertions are equivalent.*

1. $[x]$ *is a vertex in* $\mathscr{B}^R(H',F)$.
2. *The hereditary* $\mathfrak{o}_{E'}$*-order* $\mathfrak{B}'$ *associated with* x *is maximal.*
3. $\mathcal{L}$ *is constructed from an* $\mathfrak{o}_{D_{E'}}$*-chain of period 1.*

Proof. — The element $[x]$ is a vertex if and only if the stabilizer $\mathrm{Stab}_{H'(F)}(x)$ of x in $H'(F)$ is a maximal compact subgroup in $H'(F)$. Since $\mathcal{L}$ is identified with x via the $H'(F)$-isomorphism $\mathrm{Latt}^1(W')\cong\mathscr{B}^E(H',F)$, we have $\mathrm{Stab}_{H'(F)}(x)=\mathrm{Stab}_{\mathrm{Aut}_{D_{E'}}(W')}(\mathcal{L})=\mathbf{U}(\mathfrak{B}')$. The group $\mathbf{U}(\mathfrak{B}')$ is a maximal compact subgroup in $H'(F)$ if and only if $\mathfrak{B}'$ is maximal, which implies the equivalence of (1) and (2).

To show the equivalence of (2) and (3), let $(\mathcal{L}_i)$ be an $\mathfrak{o}_{D_{E'}}$-chain in W' such that $\{\mathcal{L}(r) \mid r \in \mathbb{R}\} = \{\mathcal{L}_i \mid i \in \mathbb{Z}\}$. Since $\mathfrak{B}' = \{b' \in B' \mid b'\mathcal{L}(r) \subset \mathcal{L}(r), r \in \mathbb{R}\} = \{b' \in B' \mid b'\mathcal{L}_i \subset \mathcal{L}_i, i \in \mathbb{Z}\}$, the hereditary $\mathfrak{o}_{E'}$-order $\mathfrak{B}'$ is maximal if and only if the period of $(\mathcal{L}_i)$ is equal to 1. Since the period of $(\mathcal{L}_i)$ is also equal to the number of discontinuous points of $\mathcal{L}$ in $[0, 1/d_{E'})$, where $d_{E'} = (\dim_{E'} D_{E'})^{1/2}$, (2) holds if and only if there exists a unique discontinuous point c in $[0, 1/d_{E'})$. Here, since $\mathcal{L}(r + (1/d_{E'})) = \mathcal{L}(r)\varpi_{D_{E'}}$, $\mathcal{L}$ is discontinuous at $c \in \mathbb{R}$ if and only if $\mathcal{L}$ is discontinuous at the unique element c' in $(c + d_{E'}^{-1}\mathbb{Z}) \cap [0, 1/d_{E'})$. Therefore, (2) holds if and only if the discontinuous points of $\mathcal{L}$ is equal to $c + d_{E'}^{-1}\mathbb{Z}$ for some $c \in \mathbb{R}$, which is also equivalent to (3) by Proposition 6.2. □

We fix an $H'(F)$-equivalent, affine embedding $\iota_{H/H'} : \mathscr{B}^E(H', F) \hookrightarrow \mathscr{B}^E(H, F)$ and an $H(F)$-equivalent, affine embedding $\iota_{G/H} : \mathscr{B}^E(H, F) \hookrightarrow \mathscr{B}^E(G, F)$. We also put $\iota_{G/H'} = \iota_{G/H} \circ \iota_{H/H'}$.

Proposition 6.7. — *Let $x \in \mathscr{B}^E(H, F)$.*

1. *The canonical identification $\mathscr{B}^E(H, F) \cong \mathscr{B}^E(\underline{\mathrm{Aut}}_{D_E}(W), E)$ and $\iota_{G/H}$ induce*
$$j : \mathscr{B}^R(\underline{\mathrm{Aut}}_{D_E}(W), E) \hookrightarrow \mathscr{B}^R(G, F),$$
which is equal to j_E^{-1} in [3, II-Theorem 1.1].
2. *Let $(\mathfrak{a}_{\iota_{G/H}(x),r})_{r\in\mathbb{R}}$ be the filtration in A associated with $\iota_{G/H}(x)$, and let $(\mathfrak{b}_{x,r})_{r\in\mathbb{R}}$ be the filtration in B associated with x. Then*
$$\mathfrak{b}_{x,r} = B \cap \mathfrak{a}_{\iota_{G/H}(x),r/e(E/F)}.$$
3. *The hereditary $\mathfrak{o}_F$-order $\mathfrak{a}_{\iota_{G/H}(x),0}$ is E-pure.*

Proof. — Since $\mathscr{B}^E(H, F) \cong \mathscr{B}^E(\underline{\mathrm{Aut}}_{D_E}(W), E)$ and $\iota_{G/H}$ are $H(F)$-equivalent and affine, they induce the $H(F)$-equivalent, affine embedding
$$j : \mathscr{B}^R(\underline{\mathrm{Aut}}_{D_E}(W), E) \cong \mathscr{B}^R(H', F) \hookrightarrow \mathscr{B}^R(G, F).$$
However, $H(F)$-equivalent, affine embedding $\mathscr{B}^R(\underline{\mathrm{Aut}}_{D_E}(W), E) \hookrightarrow \mathscr{B}^R(G, F)$ is unique. Since j and j_E^{-1} are $H(F)$-equivalent and affine, we obtain $j = j_E^{-1}$. The remainder assertions are results from [3, II-Theorem 1.1]. □

Proposition 6.8. — *Let $x \in \mathscr{B}^E(H', F)$ such that $[x]$ is a vertex.*

1. *The corresponding lattice function $\mathcal{L}$ in W with $\iota_{H/H'}(x)$ is constructed from a uniform $\mathfrak{o}_{D_E}$-chain. In particular, the hereditary $\mathfrak{o}_E$-order $\mathfrak{B}$ in B associated with $\mathcal{L}$ is principal.*
2. *Let $\mathfrak{B}'$ be the hereditary $\mathfrak{o}_{E'}$-order in B' associated with x. Then $\mathfrak{B}$ is the unique E'-pure hereditary $\mathfrak{o}_E$-order in B such that $\mathfrak{B}' = B' \cap \mathfrak{B}$.*

Proof. — By Proposition 6.6, the corresponding lattice function in W' with x is constructed from an $\mathfrak{o}_{D_{E'}}$-chain with period 1. Since an $\mathfrak{o}_{D_{E'}}$-chain with period 1 is uniform, (1) follows from Proposition 6.7 and [3, II-Proposition 5.4]. The claim (2) follows from Proposition 6.7 and [27, Lemme 1.6]. □

We regard $\mathscr{B}^E(H',F)$ as a subset in $\mathscr{B}^E(H,F)$ via $\iota_{H/H'}$, and $\mathscr{B}^E(H,F)$ as a subset in $\mathscr{B}^E(G,F)$ via $\iota_{G/H}$.

PROPOSITION 6.9. — *Let $x \in \mathscr{B}^E(H',F)$ such that $[x]$ is a vertex. Let $\mathfrak{A}$ be the hereditary $\mathfrak{o}_F$-order in A associated with $x \in \mathscr{B}^E(G,F)$ and let $\mathfrak{P}$ be the radical of $\mathfrak{A}$. We put $\mathfrak{h}(F) = \mathrm{Lie}(H) = B$.*

1. *Let $n \in \mathbb{Z}$.*
   (a) $B \cap \mathfrak{P}^n = \mathfrak{h}(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)}$.
   (b) $B \cap \mathfrak{P}^{n+1} = \mathfrak{h}(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)+}$.
   (c) $B \cap \mathfrak{P}^{\lfloor (n+1)/2 \rfloor} = \mathfrak{h}(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)}$.
   (d) $B \cap \mathfrak{P}^{\lfloor n/2 \rfloor + 1} = \mathfrak{h}(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)+}$.
2. *Let $n \in \mathbb{Z}_{\geq 0}$.*
   (a) $B^\times \cap \mathbf{U}^n(\mathfrak{A}) = H(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)}$.
   (b) $B^\times \cap \mathbf{U}^{n+1}(\mathfrak{A}) = H(F)_{x,n/e(\mathfrak{A}|\mathfrak{o}_F)+}$.
   (c) $B^\times \cap \mathbf{U}^{\lfloor (n+1)/2 \rfloor}(\mathfrak{A}) = H(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)}$.
   (d) $B^\times \cap \mathbf{U}^{\lfloor n/2 \rfloor + 1}(\mathfrak{A}) = H(F)_{x,n/2e(\mathfrak{A}|\mathfrak{o}_F)+}$.

Proof. — By [1, Proposition 1.9.1], we have $B \cap \mathfrak{g}_{x,r} = \mathfrak{h}(F) \cap \mathfrak{g}_{x,r} = \mathfrak{h}_{x,r}$ for $r \in \tilde{\mathbb{R}}$ and $B^\times \cap G(F)_{x,r} = H(F) \cap G(F)_{x,r} = H(F)_{x,r}$ for $r \in \tilde{\mathbb{R}}_{\geq 0}$. On the other hand, $x \in \mathscr{B}^E(G,F)$ is constructed from an $\mathfrak{o}_D$-chain by Proposition 6.8. Then we can apply Proposition 6.5 and assertions follow. □

LEMMA 6.10. — *Let $x \in \mathscr{B}^E(H',F)$ such that $[x]$ is a vertex. Let $\mathfrak{B}$ be the hereditary $\mathfrak{o}_E$-order in B with $x \in \mathscr{B}^E(H,F)$, and let $\mathfrak{Q}$ be the radical of $\mathfrak{B}$.*

1. *For $r \in \mathbb{R}$, we have $\mathfrak{Q}^{\lceil r \rceil} = \mathfrak{h}(F)_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)} = \mathfrak{b}_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)}$.*
2. *For $r \in \mathbb{R}_{\geq 0}$, we have $\mathbf{U}^{\lceil r \rceil}(\mathfrak{B}) = H(F)_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)}$.*
3. *Let $r \in \mathbb{R}_{\geq 0}$. If $H(F)_{x,r} \neq H(F)_{x,r+}$, then $n = re(\mathfrak{B}|\mathfrak{o}_E)e(E/F)$ is an integer, and we have*
   (a) $H(F)_{x,r} = \mathbf{U}^n(\mathfrak{B})$.
   (b) $H(F)_{x,r+} = \mathbf{U}^{n+1}(\mathfrak{B})$.
   (c) $H(F)_{x,r/2+} = \mathbf{U}^{\lfloor n/2 \rfloor + 1}(\mathfrak{B})$.

Proof. — We show (1), and (2) follows from (1). Let $(\mathfrak{a}_{x,r})$ be the filtration in A with x, and let $(\mathfrak{b}_{x,r})$ be the filtration in B with x. Since $[x] \in \mathscr{B}^R(H',F)$ is a vertex, by Proposition 6.8 (1) $x \in \mathscr{B}^E(H,F)$ is constructed from an $\mathfrak{o}_{D_E}$-chain. Then by Proposition 6.3 and Proposition 6.4 we have $\mathfrak{Q}^{\lceil r\rceil} = \mathfrak{b}_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)}$. On the other hand, by Proposition 6.7 (2), we also have $\mathfrak{b}_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)} = B \cap \mathfrak{a}_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)}$. Since $\mathfrak{a}_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)} = \mathfrak{g}(F)_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)}$ by Proposition 6.3, we obtain $\mathfrak{Q}^{\lceil r\rceil} = \mathfrak{h}(F) \cap \mathfrak{g}(F)_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)} = \mathfrak{h}(F)_{x,r/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)}$, where the last equality follows from [1, Proposition 1.9.1].

To show (3), let $r \in \mathbb{R}_{\geq 0}$ and suppose $H(F)_{x,r} \neq H(F)_{x,r+}$. If $re(\mathfrak{B}|\mathfrak{o}_E)e(E/F) \notin \mathbb{Z}$, then $(re(\mathfrak{B}|\mathfrak{o}_E)e(E/F), \lceil re(\mathfrak{B}|\mathfrak{o}_E)e(E/F)\rceil]$ is nonempty and

$$\begin{aligned} H(F)_{x,r+} &= \bigcup_{r<r'} H(F)_{x,r'} \\ &= \bigcup_{re(\mathfrak{B}|\mathfrak{o}_E)e(E/F)<r'\leq\lceil re(\mathfrak{B}|\mathfrak{o}_E)e(E/F)\rceil} H(F)_{x,r'/e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)} \\ &= \mathbf{U}^{\lceil re(\mathfrak{B}|\mathfrak{o}_E)e(E/F)\rceil}(\mathfrak{B}) \\ &= H(F)_{x,r}, \end{aligned}$$

which is a contradiction. Therefore, we have $n = re(\mathfrak{B}|\mathfrak{o}_E)e(E/F) \in \mathbb{Z}$. We put $G' = \underline{\mathrm{Aut}}_{D\otimes_F E}(V)$. Then we can regard x as an element in $\mathscr{B}^E(G', E)$, and for any $r' \in \mathbb{R}_{\geq 0}$ we have $\mathbf{U}^{\lceil r'\rceil}(\mathfrak{B}) = G'(E)_{x,r'/e(\mathfrak{B}|\mathfrak{o}_E)}$ by Proposition 6.4 (2). Therefore, we obtain $H(F)_{x,r'} = \mathbf{U}^{\lceil r'e(\mathfrak{B}|\mathfrak{o}_E)e(E/F)\rceil}(\mathfrak{B}) = G'(E)_{x,r'e(E/F)}$ and $H(F)_{x,r'+} = G'(E)_{x,r'e(E/F)+}$ for $r' \in \mathbb{R}$. Then by Proposition 6.5 (2)-(a), (b) and (d),

$$\begin{aligned} \mathbf{U}^n(\mathfrak{B}) =& G'(E)_{x,re(E/F)} = H(F)_{x,r}, \\ \mathbf{U}^{n+1}(\mathfrak{B}) =& G'(E)_{x,re(E/F)+} = H(F)_{x,r+} \\ \mathbf{U}^{\lfloor n/2\rfloor+1}(\mathfrak{B}) =& G'(E)_{x,re(E/F)/2+} = H(F)_{x,r/2+}, \end{aligned}$$

which completes the proof of (3). □

7. Generic elements and generic characters of G

In this section, we discuss generic elements and generic characters, using descriptions of tame twisted Levi subgroups in G, given in §5. Moreover, we relate minimal elements to generic characters using standard representatives, a notion used by Howe [16] in the GL_N tame case.

7.1. Standard representatives. — In this section, we introduce and discuss standard representatives. Certain results appear in [16]. Since we can not find detailed proofs, we give a complete exposition. We fix a uniformizer ϖ_F of F. Let E be a finite, tamely ramified extension of F. Then we can consider the subgroup C_E of "standard representatives" in $E^\times$. We recall the construction of C_E.

Lemma 7.1. — *There exists a uniformizer ϖ_E of E and a root of unity $z \in E$, of order prime to p, such that $\varpi_E^e z = \varpi_F$.*

Proof. — Let ϖ be a uniformizer of E. Let $q = p^f$ be the number of elements in the residue field of E. Let μ_{q-1} denote the group of $(q-1)$-th roots of unity in E. Then [25, Chapter 2 Proposition 5.7] shows that there exists an isomorphism $f : E^\times \simeq \varpi^{\mathbb{Z}} \times \mu_{q-1} \times G'$, where $G' = 1+\mathfrak{p}_E$ is a multiplicatively denoted group. Each element of G' has an e-th root. Indeed, [25, Chapter 2 Proposition 5.7] shows that $1+\mathfrak{p}_E$ is isomorphic to an additive group $\mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}_p^d$ or to an additive group $\mathbb{Z}_p^{\mathbf{N}}$. The image of ϖ_F by f is (e, z, g), where $(e, z, g) \in \varpi_E^{\mathbb{Z}} \times \mu_{q-1} \times G'$; that is, $\varpi_F = \varpi^e zg$. Let r be in G' such that $r^e = g$. Then $r\varpi$ is a uniformizer of E and $\varpi_F = (r\varpi)^e z$. So $\varpi_E = r\varpi$ has the required property. □

Definition 7.2. — Let E/F be a finite, tamely ramified extension, and let $\varpi_E \in E$ be as above. We denote by C_E the subgroup in $E^\times$ that is generated by ϖ_E and roots of unity in E with order prime to p.

Proposition 7.3. — *The group C_E is well defined, i.e., it is independent of the choice of ϖ_E used to define it.*

Proof. — Let ϖ_1 and ϖ_2 be two uniformizers of E and z_1, z_2 be two roots of unity of order prime to p such that $\varpi^e z_1 = \varpi_F$ and $\varpi_2^e z_2 = \varpi_F$. Let C^1 be the group generated by ϖ_1 and the root of unity of order prime to p. Let C^2 be the group generated by ϖ_2 and the root of unity of order prime to p. By symmetry, it is enough show that $C^1 \subset C^2$. It is also enough to show that $\varpi_1 \in C_2$. The equation $\varpi_1^e z_1 = \varpi_F$ implies that $\varpi_1^e \in C_2$, and thus there exists a root of unity z of order prime to p such that $\varpi_1^e = \varpi_2^e z$. We have $(\varpi_1\varpi_2^{-1})^e = z$. Let o_z be the order of z, which is an integer prime to p. We have $(\varpi_1\varpi_2^{-1})^{eo_z} = 1$. The integer eo_z is prime to p, indeed, $e = e(E|F)$ is prime to p because E/F is a tamely ramified extension and o_z is prime to p. Consequently $\varpi_1\varpi_2^{-1}$ is a root of unity of order prime to p. This implies that $\varpi_1 \in C_2$, as required. □

Then C_E depends only the choice of ϖ_F, which we already fixed. We recall properties of C_E.

Proposition 7.4. — *Let E/F be a finite, tamely ramified extension.*

1. *Let $c \in E^\times$. Then there exists a unique* $\mathrm{sr}(c) \in C_E$*, called the standard representative of c, such that* $\mathrm{sr}(c) \in c(1 + \mathfrak{p}_E)$.
2. *For any $c \in E^\times$, the standard representative* $\mathrm{sr}(c)$ *is the unique element in C_E such that* $\mathrm{ord}\,(\mathrm{sr}(c) - c) > \mathrm{ord}(c)$.
3. *Let E'/E be also a finite, tamely ramified extension. Then we have an inclusion $C_E \subset C_{E'}$ as groups.*
4. *Let $s \in C_E$. Let $\sigma_1, \sigma_2 \in \mathrm{Hom}_F(E, \bar{F})$ such that $\sigma_1(s) \neq \sigma_2(s)$. Then we have* $\mathrm{ord}\,(\sigma_1(s) - \sigma_2(s)) = \mathrm{ord}(s)$.

Proof. — We have $E^\times \simeq C_E \times (1+\mathfrak{p}_E)$,and (1) is a consequence. The element $sr(c)$ is the unique element in C_E such that $c = sr(c) \times (1+y)$, with $y \in \mathfrak{p}_E$. Thus, $sr(c)$ is the unique element in C_E such that $c - sr(c) \in sr(c)\mathfrak{p}_E$. Thus, (2) holds, remarking that $sr(c)$ and c have the same valuation. Recall that the groupsC_E and $C_{E'}$ are independent of the choices of uniformizers used to define them. Let ϖ_E be a uniformizer of E and z a root of unity of order prime to p in E such that $\varpi_E^{e(E|F)} z = \varpi_F$. Because E'/E is tamely ramified, there exists a uniformizer $\varpi_{E'} \in E'$ and a root of unity w of order prime to p in E' such that $\varpi_{E'}^{e(E'|E)} w = \varpi_E$. Elevating to the power $e(E|F)$, we have $\varpi_{E'}^{e(E'|E)e(E|F)} w^{e(E|F)} = \varpi_E^{e(E|F)}$. We thus get $\varpi_{E'}^{e(E|F)} w^{e(E|F)} z = \varpi_F$. The element $w^{e(E|F)} z$ is a root of unity of order prime to p. Consequently, $C_{E'}$ is the group generated by $\varpi_{E'}$ and the roots of unity of order prime to p in E'. The equation $\varpi_{E'}^{e(E'|E)} w = \varpi_E$ shows that ϖ_E is inside $C_{E'}$. Trivially, the roots of unity of order prime to p in E are inside the roots of unity of order prime to p in E'. Consequently C_E is inside $C_{E'}$, as required for (3).

We now prove the claim (4) when the extension E/F is Galois. Let $\sigma \in Gal(E/F)$, and let ϖ_E be an element such that $\varpi_E^e z = \varpi_F$ for z a root of unity in E of order prime to p. Let o_z be the order of z. It is enough to show that z and ϖ_E are mapped in C_E by σ. The equality $(\sigma(z))^{o_z} = 1$ shows that $\sigma(z)$ is a root of unity of order prime to p and thus inside C_E. The equality $\sigma(\varpi_E)^e \sigma(z) = \varpi_F$ together with Proposition 7.3 show that we can use $\sigma(\varpi_E)$ to define C_E, and thus $\sigma(\varpi_E)$ is inside C_E. The element $\sigma_1(s)$ is in C_E, so $sr(\sigma_1(s)) = \sigma_1(s)$. Consequently, $v_E(\sigma_1(s) - \sigma_2(s)) = v_E(\sigma_1(s))$; indeed, assume $v_E(\sigma_1(s) - \sigma_2(s)) \neq v_E(\sigma_1(s))$, then $v_E(\sigma_1(s) - \sigma_2(s)) > \sigma_1(s)$, and so $\sigma_2(s) = sr(\sigma_1(s)) = \sigma_1(s)$ by the previous criterion (2), this is a contradiction. This proves (4) for Galois extensions. For general E, since E/F is tamely ramified, then E/F is separable, and we can take the Galois closure $\tilde{E}$ of E in $\bar{F}$. Then $\tilde{E}/F$ is a finite Galois extension. Let $\tilde{\sigma}_1, \tilde{\sigma}_2 \in \mathrm{Hom}_F(\tilde{E}, \bar{F})$ be a extension of σ_1, σ_2, respectively. By (3), $s \in C_E \subset C_{\tilde{E}}$. We also have $\tilde{\sigma}_1(s) = \sigma_1(s) \neq \sigma_2(s) = \tilde{\sigma}_2(s)$. Therefore, by applying (4) for $\tilde{E}/F$ we have

$$\mathrm{ord}(s) = \mathrm{ord}\left(\tilde{\sigma}_1(s) - \tilde{\sigma}_2(s)\right) = \mathrm{ord}\left(\sigma_1(s) - \sigma_2(s)\right),$$

which is what we wanted. □

By using standard representatives, we can judge whether or not some element in E is minimal.

Proposition 7.5. — *Let E/F be a finite, tamely ramified extension, and let $c \in E^\times$ such that $E = F[c]$. Then the following assertions are equivalent.*

1. *c is minimal over F.*
2. $E = F[\mathrm{sr}(c)]$.

3. *Put* $\mathrm{ord}(c) = -r$. *For all morphisms of F-algebras $\sigma \neq \sigma'$ from E to $\overline{F}$, we have*

$$\mathrm{ord}(\sigma(c) - \sigma'(c)) = -r.$$

Proof. — We need a Lemma.

LEMMA 7.6. — *Let E/F be a finite unramified extension. Let $z \in E$ be a root of unity of order prime to p. Then, z generates E/F if and only if $z + \mathfrak{p}_E$ generates the residual field extension k_E/k_F.*

Proof. — If z generates E over F, then z generates $\mathfrak{o}_E$ over $\mathfrak{o}_F$ by [25, 7.12]. Thus, z generates the residual field extension k_E/k_F. Let us check the reverse implication. Assume that $z + \mathfrak{p}_E$ generates k_E/k_F. The field extension E/F is unramified, so $[k_E : k_F] = [E : F]$. Let $P_z \in F[X]$ be the minimal polynomial of z and d its degree; clearly P_z is in $\mathfrak{o}_F[X]$. It is enough to show that $d = [E : F]$. We have $d \leq [E : F]$. The reduction $\mod \mathfrak{p}_E$ of P_z is of degree d and annihilates $z + \mathfrak{p}_E$, a generator of k_E/k_F, and thus $[k_E : k_F] \leq d$. So $[k_E : k_F] \leq d \leq [E : F]$. So $d = [E : F]$, and this concludes the proof. □

We now prove Proposition 7.5. Let us prove that (1) implies (2). Assume that c is minimal over F. Let us remark that the definition of $sr(c)$ implies trivially that $F[sr(c)] \subset E$. Let E^{nr} denote the maximal unramified extension contained in E. To prove the opposite inclusion $E \subset F[sr(c)]$, it is enough to show that $E^{\mathrm{nr}} \subset F[sr(c)]$ and $E \subset E^{\mathrm{nr}}[sr(c)]$. Put $v = v_E(c)$, $e = e(E|F)$. The valuation of $\varpi_F^{-v} c^e$ is equal to 0, consequently by Proposition 7.1 (2) we have $v_E(sr(\varpi_F^{-v} c^e) - \varpi_F^{-v} c^e) > 0$, and so $sr(\varpi_F^{-v} c^e) + \mathfrak{p}_E = \varpi_F^{-v} c^e + \mathfrak{p}_E$. We have $sr(\varpi_F^{-v} c^e) = \varpi_F^{-v} sr(c)^e$, and this is a root of unity of order prime to p. The definition of being minimal implies that $\varpi_F^{-v} sr(c)^e + \mathfrak{p}_E$ generates k_E/k_F. So $\varpi_F^{-v} sr(c)^e$ generates E^{nr} by Lemma 7.6. So $E^{\mathrm{nr}} \subset F[sr(c)]$. We have $v_E(c) = v_E(sr(c))$, so $\gcd(v_E(sr(c)), e) = 1$. Let a and b be integers such that $a v_E(sr(c)) + be = 1$. Thus, $v_E(sr(c)^a \varpi_F^b) = 1$ and so $E^{\mathrm{nr}}[sr(c)^a \varpi_F^b] = E$ because a finite totally ramified extension is generated by an arbitrary uniformizer. So $E^{\mathrm{nr}}[sr(c)] = E$ and (i) hold. We have thus shown that $E^{\mathrm{nr}} \subset F[sr(c)]$ and $E \subset E^{\mathrm{nr}}[sr(c)]$ and so (i) implies (ii).

Let us prove that (2) implies (1). Assume that $F[sr(c)] = E$. We start by showing that e is prime to v. The field E^{nr} is generated over F by the roots of unity of order prime to p contained in E. Let $d = \gcd(v, e)$ and $b = \frac{e}{d}$. Let ϖ_E be a uniformizer in E such that $\varpi_E^e z = \varpi_F$ with z a root of unity of order prime to p. The element $sr(c)$ is in C_E and so $sr(c) = \varpi_E^v w$ with w a root of unity of order prime to p in E. Equalities $sr(c)^b = (\varpi_E^e)^{\frac{v}{gcd(v,e)}} w^b = (\varpi_F z^{-1})^{\frac{v}{gcd(v,e)}} w^b$ show that $sr(c)^b$ is contained in E^{nr}. By hypothesis, the element $sr(c)$ generates E over F and so generates E over E^{nr}. Consequently, the field E is generated by an element whose b-th power is in E^{nr}. Therefore, the inequality $[E : E^{\mathrm{nr}}] \leq b$ holds. The extension E^{nr} is the maximal unramified extension contained in E, so $[E : E^{\mathrm{nr}}] = e$. Thus, the inequality $e \leq b \leq \frac{e}{d}$ holds. This implies $d = 1$ and so v is prime to e. Let us prove that $\varpi_F^{-v} c^e + \mathfrak{p}_E$ generates the residue field extension k_E over k_F. Because $\varpi_F^{-v} c^e + \mathfrak{p}_E = \varpi_F^{-v} sr(c)^e + \mathfrak{p}_E$, it is equivalent to show that $x + \mathfrak{p}_E$ generates k_E over k_F, where $x = \varpi_F^{-v} sr(c)^e$. The element $sr(c)$ generates E over F by hypothesis; that is, $E = F[sr(c)]$. So the inequality $[E : F[x]] \leq e$ holds; indeed, E is generated over $F[x]$ by the element $sr(c)$ whose e-th power is in $F[x]$. Because x is a root of unity of order prime to p, the field $F[x]$ is a subset of E^{nr}, so $[E : E^{\mathrm{nr}}] \leq [E : F[x]]$. Consequently, the identity $e = [E : E^{\mathrm{nr}}] \leq [E : F[x]] \leq e$ holds. Because $F[x] \subset E^{\mathrm{nr}}$, the previous identity implies that $F[x] = E^{\mathrm{nr}}$. Thus by 7.6 the element $x + \mathfrak{p}_E$ generates k_E over k_F. So c is minimal over F.

Let us prove that (2) is equivalent to (3). Let $\sigma \neq \sigma'$ be two morphisms of F-algebras from E to $\overline{F}$. Put

$$\begin{aligned} A &= \sigma(c) - \sigma'(c) \\ B &= \sigma(sr(c)) - \sigma'(sr(c)). \end{aligned}$$

We have $\operatorname{ord}(A) \geq -r$ and $\operatorname{ord}(B) \geq -r$. We have

$$\begin{aligned} \operatorname{ord}(A - B) &= \operatorname{ord}\big(\sigma(c) - \sigma'(c) - (\sigma(sr(c)) - \sigma'(sr(c)))\big) \\ &= \operatorname{ord}\big(\sigma(c) - \sigma(sr(c)) - (\sigma'(c) - \sigma'(sr(c)))\big) \\ &= \operatorname{ord}\big(\sigma(c) - sr(\sigma(c)) - (\sigma'(c) - sr(\sigma'(c)))\big) \\ &> -r \end{aligned}$$

because $\operatorname{ord}\big(\sigma(c) - sr(\sigma(c))\big) > -r$ and $\operatorname{ord}\big(\sigma'(c) - sr(\sigma'(c))\big) > -r$, by definition of standard representatives. Let us prove (2) $\Rightarrow$ (3). If (2) holds, then using Proposition 7.1 we have $\sigma(sr(c)) \neq \sigma'(sr(c))$ and $\operatorname{ord}(B) = \operatorname{ord}(\sigma(sr(c)) - \sigma'(sr(c))) = -r$, because $\sigma(sr(c))$ and $\sigma'(sr(c))$ are both in $C_{\overline{E}}$. So $\operatorname{ord}(A) \geq -r, \operatorname{ord}(B) = -r, \operatorname{ord}(A - B) > -r$; this implies $\operatorname{ord}(A) = -r$. Let us prove (3)$\Rightarrow$ (2). We assume that (3) holds and we want to prove that $sr(c)$ generates E/F. It is enough to show that $\sigma(sr(c)) \neq \sigma'(sr(c))$ for any σ, σ' as before. By hypothesis $\operatorname{ord}(A) = -r$; because of $\operatorname{ord}(B) \geq -r$ and $\operatorname{ord}(A - B) > -r$, we deduce $\operatorname{ord}(B) = -r$, in particular $\sigma(sr(c)) \neq \sigma'(sr(c))$ as required. This finishes the proof. □

LEMMA 7.7. — *Let E/F be a finite, tamely ramified extension and let $c, c' \in E^\times$ such that $c^{-1}c' \in 1 + \mathfrak{p}_E$. Then c is minimal relative to E/F if and only if c' is minimal relative to E/F.*

Proof. — It suffices to show that if c is minimal relative to E/F, then c' is also minimal relative to E/F. Suppose c is minimal relative to E/F. In particular, E is generated by c over F. Then we have $E = F[\mathrm{sr}(c)]$ by Proposition 7.5. Since $\mathrm{sr}(c) \in c(1 + \mathfrak{p}_E) = c'(1 + \mathfrak{p}_E)$, we have $\mathrm{sr}(c') = \mathrm{sr}(c)$ by Proposition (1).

If E is also generated by c' over F, then we can apply Proposition 7.5 and c' is minimal relative to E/F. Thus it is enough to show $E = F[c']$.

We put $\mathrm{Hom}_F(E, \bar{F}) = \{\tau_1, \dots, \tau_{[E:F]}\}$. We have $\tau_i \neq \tau_j$ for distinct $i, j \in \{1, \dots, [E : F]\}$ as E/F is separable. Since $E = F[\mathrm{sr}(c)]$, if $i \neq j$ we have $\tau_i(\mathrm{sr}(c)) \neq \tau_j(\mathrm{sr}(c))$ and $\mathrm{ord}\,(\tau_i(\mathrm{sr}(c')) - \tau_j(\mathrm{sr}(c'))) = \mathrm{ord}(c')$ by Proposition 7.1 (4). On the other hand, since $\mathrm{ord}(\mathrm{sr}(c') - c') > \mathrm{ord}(c')$ by Proposition 7.1 we have

$$\mathrm{ord}\,(\tau_i(\mathrm{sr}(c') - c')) = \mathrm{ord}\,(\mathrm{sr}(c') - c') > \mathrm{ord}(c').$$

For $i \neq j$, we obtain

$$\begin{aligned}&\mathrm{ord}(\tau_i(c') - \tau_j(c'))\\ &= \mathrm{ord}\Big(\big(\tau_i(\mathrm{sr}(c')) - \tau_j(\mathrm{sr}(c'))\big) - \big(\tau_i(\mathrm{sr}(c') - c')\big) + \big(\tau_j(\mathrm{sr}(c') - c')\big)\Big),\end{aligned}$$

and then

$$\mathrm{ord}(\tau_i(c') - \tau_j(c')) = \mathrm{ord}\,(\tau_i(\mathrm{sr}(c')) - \tau_j(\mathrm{sr}(c'))) = \mathrm{ord}(c') \in \mathbb{R}.$$

In particular, we have $\tau_i(c') \neq \tau_j(c')$. Since $\mathrm{Hom}_F(E, \bar{F}) = \{\tau_1, \dots, \tau_{[E:F]}\}$, the element c' generates E over F. □

7.2. Concrete description of GE1 for G. — Let $E'/E/F$ be a tamely ramified field extension in A. We put

$$H = \mathrm{Res}_{E/F}\, \underline{\mathrm{Aut}}_{D\otimes_F E}(V),\ H' = \mathrm{Res}_{E'/F}\, \underline{\mathrm{Aut}}_{D\otimes_F E'}(V).$$

Then (H', H, G) is a tame twisted Levi sequence by Corollary 5.5. And we also have a natural isomorphism $\mathrm{Lie}(H') \cong \mathrm{End}_{D\otimes_F E'}(V)$. For $c \in \mathrm{End}_{D\otimes_F E'}(V)$, we can define $X_c^* \in \mathrm{Lie}^*(H')$ as

$$X_c^*(z) = \mathrm{Tr}_{E'/F} \circ \mathrm{Trd}_{\mathrm{End}_{D\otimes_F E'}(V)/E'}(cz),$$

for $z \in \mathrm{Lie}(H') \cong \mathrm{End}_{D\otimes_F E'}(V)$, where cz is a product of c and z as elements in $\mathrm{End}_{D\otimes_F E'}(V)$. Since E'/F is separable, $\mathrm{Tr}_{E'/F}$ is surjective, and there exists $e' \in E'$ such that $\mathrm{Tr}_{E'/F}(e') \neq 0$. Here, suppose $c \neq 0$. Since the map $(c, z) \mapsto \mathrm{Trd}_{\mathrm{End}_{D\otimes_F E'}(V)/E'}(cz)$ is a nondegenerate bilinear form on $\mathrm{End}_{D\otimes_F E'}(V)$, there exists $z \in \mathrm{End}_{D\otimes_F E'}(V)$ such that $\mathrm{Trd}_{\mathrm{End}_{D\otimes_F E'}(V)/E'}(cz) = e'$. In this case, we have $X_c^*(z) \neq 0$. Then, the map $c \mapsto X_c^*$ gives an isomorphism

$$\mathrm{Lie}(\mathrm{Res}_{E'/F}\, \underline{\mathrm{Aut}}_{D\otimes_F E'}(V)) \cong \mathrm{Lie}^*(\mathrm{Res}_{E'/F}\, \underline{\mathrm{Aut}}_{D\otimes_F E'}(V)).$$

Since $\mathrm{Trd}_{A/F}|_{\mathrm{End}_{D\otimes_F E'}(V)} = \mathrm{Tr}_{E'/F} \circ \mathrm{Trd}_{\mathrm{End}_{D\otimes_F E'}(V)/E'}$, we also have

$$X_c^*(z) = \mathrm{Trd}_{A/F}(cz).$$

For any $h \in H'(F)$ and $z \in \mathrm{Lie}(H')$, we have

$$X_c^*(hzh^{-1}) = \mathrm{Trd}_{A/F}(chzh^{-1}) = \mathrm{Trd}_{A/F}(h^{-1}chz) = X_{h^{-1}ch}^*(z).$$

Then the linear form X_c^* is invariant under $H'(F)$-conjugation if and only if $c = h^{-1}ch$ for any $h \in H'(F) = \mathrm{Aut}_{D\otimes_F E'}(V)$, that is, $c \in \mathrm{Cent}\,(\mathrm{End}_{D\otimes_F E'}(V)) = E'$. Therefore, X_c^* belongs to $(\underline{\mathrm{Lie}^*}(H'))^{H'}(F)$.

Let $c \in E'^{\times}$. We denote by $X_{c,\bar{F}}^*$ the image of X_c^* in $(\underline{\mathrm{Lie}^*}(H'))(\bar{F})$.

To describe $X_{c,\bar{F}}^*(H_\alpha)$ concretely, we use the notations in §5.

Proposition 7.8. — *Let $c \in E'^{\times}$ and $\alpha = \alpha_{(i',j',k'),(i'',j'',k'')} \in \Phi(G,T;\bar{F})$. Then we have $X_{c,\bar{F}}^*(H_\alpha) = \sigma_{i',j'}(c) - \sigma_{i'',j''}(c)$.*

Proof. — Let $z = \sum_i z_i \otimes_F a_i \in \mathrm{Lie}^*(G) \otimes_F \bar{F} \cong \mathrm{Lie}\left(G \times_F \bar{F}\right)$. Then we have

$$\begin{aligned} X_{c,\bar{F}}^*(z) &= \sum_i \mathrm{Trd}_{A/F}(cz_i) \otimes_F a_i = \sum_i \mathrm{Tr}_{A\otimes_F \bar{F}/\bar{F}}(cz_i \otimes_F a_i) \\ &= \mathrm{Tr}_{A\otimes_F \bar{F}/\bar{F}}\left((c\otimes_F 1)\sum_i z_i \otimes_F a_i\right) \\ &= \mathrm{Tr}_{.(\mathrm{End}_{D\otimes\bar{F}}(\bigoplus_{i,j,k} V_{i,j,k}))/\bar{F}}(m_{c,\bar{F}}z), \end{aligned}$$

where $\mathrm{End}_{D\otimes\bar{F}}(\bigoplus_{i,j,k} V_{i,j,k}) \cong \mathrm{M}_{|I_1|\times|I_2|\times|I_3|}\left(\mathrm{End}_{D\otimes\bar{F}}(V\otimes_L \bar{F})\right) \cong \mathrm{M}_{[L:F]}(\bar{F})$. Then, to calculate $\mathrm{Tr}_{A\otimes\bar{F}/\bar{F}}(m_{c,\bar{F}}H_\alpha)$ we consider the value $m_{c,\bar{F}} \circ H_\alpha(v_{i,j,k})$ for some $v_{i,j,k} \in V_{i,j,k} \setminus \{0\}$. By construction of H_α and Proposition 5.4 (iii), we obtain

$$m_{c,\bar{F}} \circ H_\alpha(v_{i,j,k}) = \begin{cases} v_{i',j',k'} \cdot \sigma_{i',j',k'}(c) & ((i,j,k) = (i',j',k'))\,, \\ v_{i'',j'',k''} \cdot (-\sigma_{i'',j'',k''}(c)) & ((i,j,k) = (i'',j'',k''))\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $X_c^*(H_\alpha) = \mathrm{Tr}_{A\otimes\bar{F}/\bar{F}}(m_{c,\bar{F}}H_\alpha) = \sigma_{i',j',k'}(c) - \sigma_{i'',j'',k''}(c)$. Since $c \in E'$, we have $\sigma_{i',j',k'}(c) = \sigma_{i',j'}(c)$ and $\sigma_{i'',j'',k''}(c) = \sigma_{i'',j''}(c)$, which complete the proof. □

7.3. General elements of G. —

Lemma 7.9. — *Suppose $\mathfrak{A}$ is a principal hereditary $\mathfrak{o}_F$-order in A with its radical $\mathfrak{P}$ and $e = e(\mathfrak{A}|\mathfrak{o}_F)$. Therefore, we have $\mathrm{Trd}_{A/F}(\mathfrak{P}^n) = \mathfrak{p}_F^{\lceil n/e \rceil}$ for any $n \in \mathbb{Z}$.*

Proof. — Since $\mathrm{Trd}_{A/F}$ is invariant by $A^\times$-conjugation, we may assume

$$\mathfrak{A} = \begin{pmatrix} \mathrm{M}_{md/e}(\mathfrak{o}_D) & \cdots & \mathrm{M}_{md/e}(\mathfrak{o}_D) \\ \vdots & \ddots & \vdots \\ \mathrm{M}_{md/e}(\mathfrak{p}_D) & \cdots & \mathrm{M}_{md/e}(\mathfrak{o}_D) \end{pmatrix}.$$

First, we show the lemma when A is split. In this case, we have

$$\mathfrak{P}^n = \begin{pmatrix} \mathrm{M}_{N/e}\left(\mathfrak{p}_F^{\lceil n/e \rceil}\right) & & * \\ & \ddots & \\ * & & \mathrm{M}_{N/e}\left(\mathfrak{p}_F^{\lceil n/e \rceil}\right) \end{pmatrix},$$

where each block in * is contained in $\mathrm{M}_{N/e}(F)$. Then $\mathrm{Tr}_{A/F}\left(\mathfrak{P}^n\right) \subset \mathfrak{p}_F^{\lceil n/e \rceil}$. To obtain the converse inclusion, let $b \in \mathfrak{p}_F^{\lceil n/e \rceil}$. Let a be an element in A with the $(1,1)$-entry b, and other entries 0. Then $a \in \mathfrak{P}^n$ and $\mathrm{Tr}_{A/F}(a) = b$.

In the general case, we take a maximal unramified extension E/F in D. Then $A \otimes_F E$ is split, and the subring $\mathfrak{A}_E := \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$ in $A \otimes_F E$ is a hereditary $\mathfrak{o}_E$-order with $e(\mathfrak{A}_E|\mathfrak{o}_E) = e(\mathfrak{A}|\mathfrak{o}_F) = e$. Let $\mathfrak{P}_E$ be the radical of $\mathfrak{A}_E$. Then ${\mathfrak{P}_E}^n = \mathfrak{P}^n \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$ and $\mathrm{Tr}_{A\otimes_F E/E}({\mathfrak{P}_E}^n) = \mathfrak{p}_E^{\lceil n/e \rceil}$ by the split case. Since $\mathrm{Trd}_{A/F}(A) = \mathrm{Tr}_{A\otimes_F E/E}\left(A \otimes_F 1\right) = F$, we have

$$\mathrm{Trd}_{A/F}(\mathfrak{P}^n) \subset \mathfrak{p}_E^{\lceil n/e \rceil} \cap F = \mathfrak{p}_F^{\lceil n/e \rceil},$$

where the last equality follows from the assumption E/F is unramified.

To obtain the converse inclusion, let $b \in \mathfrak{p}_F^{\lceil n/e \rceil}$. Since E/F is unramified, we have $\mathrm{Tr}_{E/F}\left(\mathfrak{p}_E^{\lceil n/e \rceil}\right) = \mathfrak{p}_F^{\lceil n/e \rceil}$, and there exists $b' \in \mathfrak{p}_E^{\lceil n/e \rceil}$ such that $\mathrm{Tr}_{E/F}(b') = b$. Let a be an element in $A \cong \mathrm{M}_m(D)$ with the $(1,1)$-entry b', and other entries 0. Then $a \in \mathrm{Cent}_A(E) \cong \mathrm{M}_m(E)$, and

$$\mathrm{Trd}_{A/F}(a) = \mathrm{Tr}_{E/F} \circ \mathrm{Tr}_{\mathrm{Cent}_A(E)/E}(a) = \mathrm{Tr}_{E/F}(b') = b.$$

Therefore, it suffices to check $a \in \mathfrak{P}^n$. We have

$$\mathfrak{P}^n = \begin{pmatrix} \mathrm{M}_{md/e}\left(\mathfrak{p}_D^{\lceil nd/e \rceil}\right) & & * \\ & \ddots & \\ ** & & \mathrm{M}_{N/e}\left(\mathfrak{p}_D^{\lceil nd/e \rceil}\right) \end{pmatrix},$$

Then $a \in \mathfrak{P}^n$ if and only if $b' \in \mathfrak{p}_D^{\lceil nd/e \rceil}$. However, $b' \in \mathfrak{p}_E^{\lceil n/e \rceil} \subset \mathfrak{p}_D^{\lceil n/e \rceil d} \subset \mathfrak{p}_D^{\lceil nd/e \rceil}$, where $\lceil n/e \rceil d \geq \lceil nd/e \rceil$ since $nd/e \leq \lceil n/e \rceil d \in \mathbb{Z}$. □

Proposition 7.10. — *Let $c \in E'^{\times}$. We put $r = -\operatorname{ord}(c) = \operatorname{ord}(c^{-1})$.*

1. $X_c^* \in \mathrm{Lie}^*(H')_{x,-r}$ *for any* $x \in \mathscr{B}^E(H', F)$.
2. X_c^* *is H-generic of depth r if and only if c is minimal relative to E'/E.*

Proof. — We show (1). By [13, Lemma 2.3], it is enough to show $X_c^* \in \mathrm{Lie}^*(H')_{x,-r}$ for some $x \in \mathscr{B}^E(H', F)$. Then, we may assume $[x]$ is a vertex in $\mathscr{B}^R(H', F)$. By Proposition 6.6, the corresponding hereditary $\mathfrak{o}_{E'}$-order $\mathfrak{B}'$ in $B' := \mathrm{End}_{D\otimes_F E'}(V) \cong \mathrm{M}_{n_{E'}}(D_{E'})$ is maximal, where $D_{E'}$ is a central division E'-algebra and $n_{E'} \in \mathbb{N}$. Therefore, we may also assume $\mathfrak{B}' = \mathrm{M}_{n_{E'}}(\mathfrak{o}_{D_{E'}})$ by taking an isomorphism $B' \cong \mathrm{M}_{n_{E'}}(D_{E'})$ of $D_{E'}$-modules.

We will describe $\mathfrak{h}'(F)_{x,r+} = \mathrm{Lie}^*(H')_{x,r+}$ concretely. Let $\mathfrak{Q}'$ be the radical of $\mathfrak{B}'$. By Proposition 6.10 (1), we have $\mathrm{Lie}(H')_{x,s} = \mathfrak{h}'(F)_{x,s} = \mathfrak{Q}'^{\lceil se(\mathfrak{B}'|\mathfrak{o}_{E'})e(E'/F)\rceil}$ for $s \in \mathbb{R}$. Here, we have $re(\mathfrak{B}')e(E'/F) = \mathrm{ord}(c^{-1})e(\mathfrak{B}'|\mathfrak{o}_{E'})e(E'/F) = v_{E'}(c^{-1})e(\mathfrak{B}'|\mathfrak{o}_{E'}) = v_{\mathfrak{B}'}(c^{-1}) \in \mathbb{Z}$. Then, for any sufficiently small $\varepsilon > 0$, we have $\mathfrak{Q}'^{\lceil(r+\varepsilon)e(\mathfrak{B}'|\mathfrak{o}_{E'})e(E'/F)\rceil} = (\mathfrak{Q}')^{v_{\mathfrak{B}'}(c^{-1})+\lceil\varepsilon e(\mathfrak{B}'|\mathfrak{o}_{E'})e(E'/F)\rceil} = (\mathfrak{Q}')^{v_{\mathfrak{B}'}(c^{-1})+1} = c^{-1}\mathfrak{Q}'$. Therefore, we obtain $\mathfrak{h}'(F)_{x,r+} = c^{-1}\mathfrak{Q}'$.

By the definition of $\mathrm{Lie}^*(H')_{x,-r}$, to show (1) it is enough to show that $X_c^*(\mathfrak{h}'(F)_{x,r+}) \subset \mathfrak{p}_F$. Here, we have $X_c^*(\mathfrak{h}'(F)_{x,r+}) = \mathrm{Tr}_{E'/F} \circ \mathrm{Trd}_{B'/E'}(c \cdot c^{-1}\mathfrak{Q}') = \mathrm{Tr}_{E'/F} \circ \mathrm{Trd}_{B'/E'}(\mathfrak{Q}')$. Since $[x]$ is a vertex, the hereditary order $\mathfrak{B}'$ is principal, and we have $\mathrm{Trd}_{B'/E'}(\mathfrak{Q}') = \mathfrak{p}_{E'}$ by Lemma 7.9. Moreover, $\mathrm{Tr}_{E'/F}(\mathfrak{p}_{E'}) = \mathfrak{p}_F$ since E'/F is tame. Therefore, we obtain $X_c^*(\mathfrak{h}'(F)_{x,-r+}) = \mathrm{Tr}_{E'/F} \circ \mathrm{Trd}_{B'/E'}(\mathfrak{Q}') = \mathrm{Tr}_{E'/F}(\mathfrak{p}_{E'}) = \mathfrak{p}_F$ and complete the proof of (1).

To show (2), first suppose X_c^* is H-generic of depth r.

We will show $E' = E[c]$. We fix an embedding $\sigma_i : E \to \bar{F}$. Then we have $\mathrm{Hom}_E(E', \bar{F}) = \{\sigma_{i,j} \mid j \in I_2\}$. Since E/F is separable, to show $E' = E[c]$ it suffices to show $\sigma_{i,j}(c) \neq \sigma_{i,j'}(c)$ for any distinct $j, j' \in I_2$. We fix $k \in I_3$ and we put $\alpha = \alpha_{(i,j,k),(i,j',k)} \in \Phi(G, T; \bar{F})$. Then $\alpha \in \Phi(H, T; \bar{F}) \setminus \Phi(H', T; \bar{F})$. Since X_c^* is H-generic of depth r, we have $-r = \mathrm{ord}\left(X_{c,\bar{F}}^*(H_\alpha)\right) = \mathrm{ord}\,(\sigma_{i,j}(c) - \sigma_{i,j'}(c))$, where the last equality follows from Proposition 7.8. In particular, we have $\mathrm{ord}\,(\sigma_{i,j}(c) - \sigma_{i,j'}(c)) \in \mathbb{R}$. Then $\sigma_{i,j}(c) - \sigma_{i,j'}(c) \neq 0$, that is, $\sigma_{i,j}(c) \neq \sigma_{i,j'}(c)$. Therefore, we obtain $E' = E[c]$. Moreover, we already know $\mathrm{ord}\,(\sigma_{i,j}(c) - \sigma_{i,j'}(c)) = -r$. Then, by Proposition 7.5, the element c in E' is minimal.

Conversely, suppose c is minimal relative to E'/E. In particular, we have $E' = E[c]$. By Corollary 3.9, to show that X_c^* is H-generic it suffices to check X_c^* satisfies **GE1**. Let $\alpha = \alpha_{(i,j,k),(i',j',k')} \in \Phi(H, T; F) \setminus \Phi(H', T; F)$. Then we have $i = i'$ and $j \neq j'$. We equip $\bar{F}$ with E-structure via σ_i. Then we have $\mathrm{Hom}_E(E', \bar{F}) = \{\sigma_{i,j} \mid j \in I_2\}$. Since c is minimal over E and $E' = E[c]$, we have $\mathrm{ord}\,(\sigma_{i,j}(c) - \sigma_{i,j'}(c)) = -r$ by Proposition 7.5. Therefore, $X_c^*(H_\alpha) = \mathrm{ord}\,(\sigma_{i,j}(c) - \sigma_{i,j'}(c)) = -r$, which implies X_c^* is H-generic of depth r. □

7.4. General characters of G. — In this section, we discuss smooth characters of G. Let χ be a smooth character of G. Let $\mathfrak{A}$ be a principal hereditary $\mathfrak{o}_F$-order. The goal of this subsection is to prove the following proposition.

Proposition 7.11. — *Suppose χ is trivial on $\mathbf{U}^{n+1}(\mathfrak{A})$ but not trivial on $\mathbf{U}^n(\mathfrak{A})$ for some $n \in \mathbb{Z}_{\geq 0}$. Then there exists $c \in F$ such that $v_F(c) = -n/e(\mathfrak{A}|\mathfrak{o}_F)$ and*

$$\chi|_{\mathbf{U}^{\lfloor n/2\rfloor+1}(\mathfrak{A})}(1+y) = \psi \circ \mathrm{Trd}_{A/F}(cy)$$

for $y \in \mathfrak{P}^{\lfloor n/2\rfloor+1}$.

To prove Proposition 7.11, we need some preliminary. We put $e = e(\mathfrak{A}|\mathfrak{o}_F)$. If Proposition 7.11 holds for some χ, n and $\mathfrak{A}$, then it also holds for any G-conjugation of $\mathfrak{A}$ and the same χ, n as above. Therefore, we may assume

$$\mathfrak{A} = \begin{pmatrix} \mathrm{M}_{md/e}(\mathfrak{o}_D) & \cdots & \mathrm{M}_{md/e}(\mathfrak{o}_D) \\ \vdots & \ddots & \vdots \\ \mathrm{M}_{md/e}(\mathfrak{p}_D) & \cdots & \mathrm{M}_{md/e}(\mathfrak{o}_D) \end{pmatrix}.$$

Lemma 7.12. — *Suppose χ is trivial on $\mathbf{U}^{e(n+1)}(\mathfrak{A})$. Then χ is also trivial on $\mathbf{U}^{en+1}(\mathfrak{A})$.*

Proof. — Since χ factors through $\mathrm{Nrd}_{A/F}$, it is enough to show that

$$\mathrm{Nrd}_{A/F}\left(\mathbf{U}^{e(n+1)}(\mathfrak{A})\right) = \mathrm{Nrd}_{A/F}\left(\mathbf{U}^{en+1}(\mathfrak{A})\right).$$

We can deduce it from the following lemma. □

Lemma 7.13. — *We have $\mathrm{Nrd}_{A/F}(1+\mathfrak{P}^n) = 1 + \mathfrak{p}_F^{\lceil n/e \rceil}$.*

Proof. — First we show the lemma when A is split. In this case, we have

$$1+\mathfrak{P}^n = \begin{pmatrix} 1+\mathrm{M}_{N/e}\left(\mathfrak{p}_F^{\lceil n/e \rceil}\right) & & * \\ & \ddots & \\ ** & & 1+\mathrm{M}_{N/e}\left(\mathfrak{p}_F^{\lceil n/e \rceil}\right) \end{pmatrix},$$

where each block in $**$ is equal to $\mathrm{M}_{N/e}\left(\mathfrak{p}_F^{\lceil n/e \rceil}\right)$ or $\mathrm{M}_{N/e}\left(\mathfrak{p}_F^{\lceil n/e \rceil+1}\right)$. Then any element a in $1+\mathfrak{P}^n$ are upper triangular modulo $\mathfrak{p}_F^{\lceil n/e \rceil}$, and $\det_{A/F}(a)$ is 1 modulo $\mathfrak{p}_F^{\lceil n/e \rceil}$, whence $\det_{A/F}(1+\mathfrak{P}^n) \subset 1+\mathfrak{p}_F^{\lceil n/e \rceil}$. To obtain the converse inclusion, let $1+b \in 1+\mathfrak{p}_F^{\lceil n/e \rceil}$. Let a be an element in A with the $(1,1)$-entry b, and other entries 0. Then $1+a \in 1+\mathfrak{P}^n$ and $\det_{A/F}(1+a) = 1+b$.

In the general case, we take a maximal unramified extension E/F in D. Then $A \otimes_F E$ is split, and the subring $\mathfrak{A}_E := \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$ in $A \otimes_F E$ is a hereditary $\mathfrak{o}_E$-order with $e(\mathfrak{A}_E|\mathfrak{o}_E) = e(\mathfrak{A}|\mathfrak{o}_F) = e$. Let $\mathfrak{P}_E$ be the radical of $\mathfrak{A}_E$. Then $\mathfrak{P}_E^n = \mathfrak{P}^n \otimes_{\mathfrak{o}_F} \mathfrak{o}_E$ and $\det_{A\otimes_F E/E}(1+\mathfrak{P}_E^n) = 1+\mathfrak{p}_E^{\lceil n/e \rceil}$ by the split case. Since $\mathrm{Nrd}_{A/F}(A^\times) = \det_{A\otimes_F E/E}\left((A\otimes_F 1)^\times\right) = F^\times$, we have

$$\mathrm{Nrd}_{A/F}(1+\mathfrak{P}^n) \subset \left(1+\mathfrak{p}_E^{\lceil n/e \rceil}\right) \cap F^\times = 1+\mathfrak{p}_F^{\lceil n/e \rceil},$$

where the last equality follows from the assumption E/F is unramified.

To obtain the converse inclusion, let $1+b \in 1+\mathfrak{p}_F^{\lceil n/e \rceil}$. Since E/F is unramified, we have $\mathrm{N}_{E/F}\left(1+\mathfrak{p}_E^{\lceil n/e \rceil}\right) = 1+\mathfrak{p}_F^{\lceil n/e \rceil}$, and there exists $b' \in \mathfrak{p}_E^{\lceil n/e \rceil}$ such that $\mathrm{N}_{E/F}(1+b') = 1+b$. Let a be an element in $A \cong \mathrm{M}_m(D)$ with the $(1,1)$-entry b', and other entries 0. Then $1+a \in \mathrm{Cent}_A(E) \cong \mathrm{M}_m(E)$, and

$$\mathrm{Nrd}_{A/F}(1+a) = \mathrm{N}_{E/F} \circ \det{}_{\mathrm{Cent}_A(E)/E}(1+a) = \mathrm{N}_{E/F}(1+b') = 1+b.$$

Therefore, it suffices to check $a \in \mathfrak{P}^n$. We have

$$\mathfrak{P}^n = \begin{pmatrix} \mathrm{M}_{md/e}\left(\mathfrak{p}_D^{\lceil nd/e \rceil}\right) & & * \\ & \ddots & \\ ** & & \mathrm{M}_{N/e}\left(\mathfrak{p}_D^{\lceil nd/e \rceil}\right) \end{pmatrix}.$$

Then $a \in \mathfrak{P}^n$ if and only if $b' \in \mathfrak{p}_D^{\lceil nd/e \rceil}$. However, $b' \in \mathfrak{p}_E^{\lceil n/e \rceil} \subset \mathfrak{p}_D^{\lceil n/e \rceil d} \subset \mathfrak{p}_D^{\lceil nd/e \rceil}$, where $\lceil n/e \rceil d \geq \lceil nd/e \rceil$ since $nd/e \leq \lceil n/e \rceil d \in \mathbb{Z}$. □

Proposition 7.14. — *Suppose $n > 0$. Furthermore, assume χ is trivial on $\mathbf{U}^{en+1}(\mathfrak{A})$ but not on $\mathbf{U}^{en}(\mathfrak{A})$. Then there exists $c \in F$ with $v_F(c) = -n$ such that*

$$\chi|_{\mathbf{U}^{en}(\mathfrak{A})}(1+y) = \psi \circ \mathrm{Trd}_{A/F}(cy)$$

for $y \in \mathfrak{P}^{en}$.

Proof. — We have $\mathbf{U}^{en}(\mathfrak{A})/\mathbf{U}^{en+1}(\mathfrak{A}) \cong \mathfrak{P}^{en}/\mathfrak{P}^{en+1}$; we can regard any smooth character $\mathbf{U}^{en}(\mathfrak{A})/\mathbf{U}^{en+1}(\mathfrak{A})$ as a smooth character of $\mathfrak{P}^{en}/\mathfrak{P}^{en+1}$. For any smooth character ϕ of $\mathfrak{P}^{en}/\mathfrak{P}^{en+1}$, there exists $c_0 \in \mathfrak{P}^{-en}$, unique up to modulo $\mathfrak{P}^{-en+1}$, such that $\phi(y) = \psi \circ \mathrm{Trd}_{A/F}(c_0 y)$ for any $y \in \mathfrak{P}^{en}$. In particular, there exists $c_0 \in \mathfrak{P}^{-en}$ such that $\chi(1+y) = \psi \circ \mathrm{Trd}_{A/F}(c_0 y)$ for any $y \in \mathfrak{P}^{en}$. Since χ is not trivial on $\mathbf{U}^{en}(\mathfrak{A})$, we have $c_0 \notin \mathfrak{P}^{-en+1}$. By the uniqueness of c_0, it suffices to show that $c_0 + \mathfrak{P}^{-en+1}$ contains some element c in F with $v_F(c) = -n$.

Here, let $g \in \mathfrak{K}(\mathfrak{A})$ and $y \in \mathfrak{P}^{-en}$. Since χ is a character of G, we have $\chi(1+y) = \chi\left(g(1+y)g^{-1}\right)$. However, we have $g(1+y)g^{-1} = 1 + gyg^{-1}$ and $gyg^{-1} \in \mathfrak{P}^{en}$ since $g \in \mathfrak{K}(\mathfrak{A})$. Then we obtain

$$\begin{aligned}\chi\left(g(1+y)g^{-1}\right) &= \chi\left(1+gyg^{-1}\right) \\ &= \psi \circ \mathrm{Trd}_{A/F}(c_0 gyg^{-1}) \\ &= \psi \circ \mathrm{Trd}_{A/F}(g^{-1}c_0 gy).\end{aligned}$$

Since $g^{-1}c_0 g \in \mathfrak{P}^{-en}$, we have $c_0 + \mathfrak{P}^{-en+1} = g^{-1}c_0 g + \mathfrak{P}^{-en+1}$ by the uniqueness of c_0. We take $t \in F^\times$ such that $v_F(t) = n$. Then we have

$$tc_0 + \mathfrak{P} = t(c_0 + \mathfrak{P}^{-en+1}) = tg^{-1}c_0 g + t\mathfrak{P}^{-en+1} = g^{-1}(tc_0)g + \mathfrak{P}$$

for $g \in \mathfrak{K}(\mathfrak{A})$. If we put $c' = tc_0$, then $c', g^{-1}c'g \in \mathfrak{A}$ and $c' + \mathfrak{P} = g^{-1}c'g + \mathfrak{P}$. Since $c_0 \in \mathfrak{P}^{-en} \setminus \mathfrak{P}^{-en+1}$, we have $c' \in t\left(\mathfrak{P}^{en} \setminus \mathfrak{P}^{-en+1}\right) = \mathfrak{A} \setminus \mathfrak{P}$. Therefore, we obtain $\overline{c'} = \overline{g^{-1}c'g}$ for $g \in \mathfrak{K}(\mathfrak{A})$, where for $a \in \mathfrak{A}$ we denote by $\overline{a}$ the image of a in $\mathfrak{A}/\mathfrak{P}$. By the form of $\mathfrak{A}$, we have an isomorphism $\mathfrak{A}/\mathfrak{P} \cong \mathrm{M}_{md/e}(k_D)^{\frac{e}{d}}$

as

$$\mathfrak{A}/\mathfrak{P} \cong \begin{pmatrix} \mathrm{M}_{md/e}(k_D) & & \\ & \ddots & \\ & & \mathrm{M}_{md/e}(k_D) \end{pmatrix} \ni \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_{e/d} \end{pmatrix}$$
$$\mapsto (b_1, \dots, b_{e/d}) \in \prod_{i=1}^{e/d} \mathrm{M}_{md/e}(k_D).$$

Here, let $g \in \mathbf{U}(\mathfrak{A})$. Then $g \in \mathfrak{A}$ and we have $\overline{c'} = \overline{g}^{-1} \cdot \overline{c'} \cdot \overline{g}$. Since $\mathbf{U}(\mathfrak{A}) \to (\mathfrak{A}/\mathfrak{P})^\times$ is surjective, $\overline{c'} \in Z(\mathfrak{A}/\mathfrak{P}) \cong Z\big(\prod_{i=1}^{e/d} \mathrm{M}_{md/e}(k_D)\big) = \prod_{i=1}^{e/d} k_D$. Let $(b_1, \dots, b_{e/d})$ be the image of $\overline{c'}$ in $\prod_{i=1}^{e/d} k_D$.

We take $g \in \mathfrak{K}(\mathfrak{A})$ with $v_{\mathfrak{A}}(g) = -1$. Then $\overline{g^{-1}c'g} = (b_2, \dots, b_{e/d}, \tau(b_1))$, where $\tau \in \mathrm{Gal}(k_D/k_F)$ is a generator. Since $\overline{c'} = \overline{g^{-1}c'g}$, we have $b_1 = b_2 = \cdots = b_{e/d} = \tau(b_1)$. Since τ is a generator of $\mathrm{Gal}(k_D/k_F)$, the element b_1 is stabilized by $\mathrm{Gal}(k_D/k_F)$, that is, $b_1 \in k_F$. Therefore, $\overline{c'} \in k_F \subset \prod_{i=1}^{e/d} k_D$. We take a lift a of b_1 to $\mathfrak{o}_F$. Since $\overline{c'} \neq 0$, we have $b_1 \neq 0$ and then $a \in \mathfrak{o}_F^\times$. Therefore, $c = t^{-1}a$ satisfies the desired condition. □

Lemma 7.15. — *Let $c \in F^\times$ such that $v_F(c) = -n < 0$. Then there exists a smooth character θ of $A^\times$ such that*

$$\theta|_{\mathbf{U}^{\lfloor en/2 \rfloor +1}(\mathfrak{A})}(1+y) = \psi \circ \mathrm{Trd}_{A/F}(cy)$$

for $y \in \mathfrak{P}^{en}$.

Proof. — Since $v_{\mathfrak{A}}(c) = -en$, the 4-tuple $[\mathfrak{A}, en, 0, c]$ is a simple stratum. Then we can take an element θ in $\mathscr{C}(c, 0, \mathfrak{A})$, which is nonempty by Remark 2.13. Since θ is simple, $\theta|_{\mathrm{Cent}_A(F[c])^\times \cap H^1(c,\mathfrak{A})}$ can be extended to a character of $\mathrm{Cent}_A(F[c])^\times$. However, we have $F[c] = F$ and then $\mathrm{Cent}_A(F[c]) = A$. Therefore, θ can be extended to a character of $A^\times$. Since θ is simple and $c \in F$ is minimal over F, we have

$$\theta|_{\mathbf{U}^{\lfloor en/2 \rfloor +1}(\mathfrak{A})}(1+y) = \psi_c(1+y) = \psi \circ \mathrm{Trd}_{A/F}(cy)$$

for $y \in \mathfrak{P}^{\lfloor en/2 \rfloor} + 1$. □

Let us start the proof of Proposition 7.11.

Proof. — First, if $n = 0$, then $c = 1$ satisfies the condition. Then we may assume $n > 0$.

If $n \notin e\mathbb{Z}$ and χ is trivial on $\mathbf{U}^{n+1}(\mathfrak{A})$, then χ is also trivial in $\mathbf{U}^n(\mathfrak{A})$ by Lemma 7.12, which is a contradiction. Then $n \in e\mathbb{Z}$. Let i_0 be the smallest integer satisfying $\lfloor n/2 \rfloor + 1 \leq ei_0$. Since $n \geq 1$, we have $i_0 \geq 1$. For $i = i_0, \dots, n/e$, we construct $c_i \in F$ and a character θ_i of $F^\times$ such that $\theta_i|_{\mathbf{U}^{\lfloor ei/2 \rfloor +1}(\mathfrak{A})} = \psi_{c_i}$,and $\chi \cdot \big(\prod_{j=i}^{n/e} \theta_j\big)^{-1}$ is trivial on $\mathbf{U}^{ei}(\mathfrak{A})$, by downward induction.

Let $i = n/e$. Since χ is not trivial on $\mathbf{U}^n(\mathfrak{A})$, then there exists $c_{n/e} \in F$ such that $v_F(c_{n/e}) = -n$, and χ is equal to $\psi_{c_{n/e}}$ by Proposition 7.14. Then we take a character $\theta_{n/e}$ of $F^\times$ as in Lemma 7.15 for c_i, and $\chi \cdot {\theta_{n/e}}^{-1}$ is trivial on $\mathbf{U}^n(\mathfrak{A}) = \mathbf{U}^{ei}(\mathfrak{A})$.

Let $i_0 \leq i < n/e$ and suppose we construct c_j and θ_i for $i < j \leq n/e$. Since $\chi \cdot \left(\prod_{j=i+1}^{n/e} \theta_j\right)^{-1}$ is trivial on $\mathbf{U}^{e(i+1)}(\mathfrak{A})$ by induction hypothesis, it is also trivial on $\mathbf{U}^{ei+1}(\mathfrak{A})$ by Lemma 7.12. If $\chi \cdot \left(\prod_{j=i+1}^{n/e} \theta_j\right)^{-1}$ is also trivial on $\mathbf{U}^{ei}(\mathfrak{A})$, then we put $c_i = 0$ and $\theta_i = 1$, whence c_i and θ_i satisfy the condition. Otherwise, there exists $c_i \in F$ such that $v_F(c_i) = -i$ and $\chi \cdot \left(\prod_{j=i+1}^{n/e} \theta_j\right)^{-1}$ is equal to ψ_{c_i} on $\mathbf{U}^{ei}(\mathfrak{A})$ by Proposition 7.14. Then we take a character θ_i of $F^\times$ as Lemma 7.15 for c_i, and $\chi \cdot \left(\prod_{j=i}^{n/e} \theta_j\right)^{-1}$ is trivial on $\mathbf{U}^{ei}(\mathfrak{A})$.

Therefore, $\chi \cdot \left(\prod_{i=i_0}^{n/e} \theta_i\right)^{-1}$ is trivial on $\mathbf{U}^{ei_0}(\mathfrak{A})$. By Lemma 7.12, it is also trivial on $\mathbf{U}^{e(i_0-1)+1}(\mathfrak{A})$. Since i_0 is the smallest integer satisfying $\lfloor n/2 \rfloor + 1 \leq ei_0$, we have $e(i_0 - 1) < \lfloor n/2 \rfloor + 1$, that is, $e(i_0 - 1) + 1 \leq \lfloor n/2 \rfloor + 1$. Then $\mathbf{U}^{\lfloor n/2 \rfloor + 1}(\mathfrak{A}) \subset \mathbf{U}^{e(i_0-1)+1}(\mathfrak{A})$, whence $\chi \cdot \left(\prod_{i=i_0}^{n/e} \theta_i\right)^{-1}$ is trivial on $\mathbf{U}^{\lfloor n/2 \rfloor + 1}(\mathfrak{A})$. This implies χ is equal to $\prod_{i=i_0}^{n/e} \theta_i$ on $\mathbf{U}^{\lfloor n/2 \rfloor + 1}(\mathfrak{A})$. For $i = i_0, \ldots, n/e$, we have $\lfloor ei/2 \rfloor + 1 \leq \lfloor e(n/e)/2 \rfloor + 1 = \lfloor n/2 \rfloor + 1$. By construction of θ_i, the restriction of θ_i to $\mathbf{U}^{\cdot \lfloor n/2 \rfloor + 1}(\mathfrak{A}) \subset \mathbf{U}^{\lfloor ei/2 \rfloor + 1}$ is equal to ψ_{c_i}. Then χ is equal to $\prod_{i=i_0}^{n/e} \psi_{c_i} = \psi_{\left(\sum_{i=i_0}^{n/e} c_i\right)}$ on $\mathbf{U}^{\cdot \lfloor n/2 \rfloor + 1}(\mathfrak{A})$. We put $c = \sum_{i=i_0}^{n/e} c_i$. Since $v_F(c_{n/e}) = -n$ and $v_F(c_i) \geq -i > -n$ for $i = i_0, \ldots, (n/e) - 1$, we have $v_F(c) = -n$, which completes the proof. □

8. Some lemmas on maximal simple types

In this section, we show some lemmas which are used when we take the "depth-zero" part of Sécherre's datum or Yu's datum.

LEMMA 8.1. — *Let Λ, Λ' be extensions of a maximal simple type (J, λ) to $\tilde{J} = \tilde{J}(\lambda)$. Then there exists a character χ of $\tilde{J}(\lambda)/J$ such that $\Lambda' \cong \chi \otimes \Lambda$.*

Proof. — Since $\Lambda|_J = \lambda = \Lambda'|_J$ is irreducible, we have $\mathrm{Hom}_J(\Lambda, \Lambda') \cong \mathbb{C}$. The group $\tilde{J}$ acts on $\mathrm{Hom}_J(\Lambda, \Lambda') \cong \mathbb{C}$ as the character χ of $\tilde{J}$ by

$$g \cdot f := \Lambda'(g) \circ f \circ \Lambda(g^{-1}) = \chi(g) f$$

for $g \in \tilde{J}$ and $f \in \mathrm{Hom}_J(\Lambda, \Lambda')$. Since f is a J-homomorphism, χ is trivial on J. We take a nonzero element f in $\mathrm{Hom}_J(\Lambda, \Lambda')$. Then for $g \in \tilde{J}$ we have

$$\Lambda'(g) \circ f = f \circ (\chi(g)\Lambda(g)) = f \circ (\Lambda \otimes \chi(g))$$

and an $\tilde{J}$-isomorphism $\Lambda' \cong \Lambda \otimes \chi$. □

If a maximal simple type (J,λ) is associated with a simple stratum $[\mathfrak{A},n,0,\beta]$, we put $\hat{J}=\hat{J}(\beta,\mathfrak{A})$ as in Definition 2.18.

LEMMA 8.2. — *Let $(J=\mathbf{U}(\mathfrak{A}),\lambda)$ be a simple type of depth zero, where $\mathfrak{A}$ is a maximal hereditary $\mathfrak{o}_F$-order in A, and let $(\tilde{J},\Lambda)$ be a maximal extension of (J,λ). We put $\rho=\mathrm{Ind}_{\tilde{J}}^{\hat{J}}\Lambda$.*

1. $\text{c-Ind}_{\hat{J}}^{G}\rho$ *is irreducible and supercuspidal.*
2. ρ *is irreducible.*
3. ρ *is trivial on* $\mathbf{U}^1(\mathfrak{A})$*.*

Proof. — Since $(\tilde{J},\Lambda)$ is a maximal extension of a simple type of depth zero, $\text{c-Ind}_{\tilde{J}}^{G}\Lambda$ is irreducible and supercuspidal. However, by the transitivity of compact induction, we also have $\text{c-Ind}_{\tilde{J}}^{G}\Lambda=\text{c-Ind}_{\hat{J}}^{G}\text{c-Ind}_{\tilde{J}}^{\hat{J}}\Lambda=\text{c-Ind}_{\hat{J}}^{G}\rho$, which implies (1).

Since $\text{c-Ind}_{\hat{J}}^{G}\rho$ is irreducible, ρ is also irreducible, that is, (2) holds.

To show (3), we consider the Mackey decomposition of $\mathrm{Res}_{J}^{\hat{J}}\mathrm{Ind}_{\tilde{J}}^{\hat{J}}\Lambda$. We have

$$\mathrm{Res}_{J}^{\hat{J}}\mathrm{Ind}_{\tilde{J}}^{\hat{J}}\Lambda=\bigoplus_{g\in J\backslash\hat{J}/\tilde{J}}\mathrm{Ind}_{J\cap{}^{g}\tilde{J}}^{J}\mathrm{Res}_{J\cap{}^{g}\tilde{J}}^{{}^{g}\tilde{J}}{}^{g}\Lambda=\bigoplus_{i=0}^{i-1}{}^{h^i}\lambda,$$

where $l=(\hat{J}:\tilde{J})$ and $h\in\hat{J}$ such that the image of h in $\hat{J}/J\cong\mathbb{Z}$ is 1. Since $h\mathbf{U}^1(\mathfrak{A})h^{-1}=\mathbf{U}^1(\mathfrak{A})$, the representation ${}^{h^i}\lambda$ is trivial on $\mathbf{U}^1(\mathfrak{A})$ for $i=0,\dots,l-1$. Therefore ρ is also trivial on $\mathbf{U}^1(\mathfrak{A})$. □

LEMMA 8.3. — *Let $[\mathfrak{A},n,0,\beta]$ be a simple stratum with $\mathfrak{B}$ maximal. Let σ^0 be an irreducible cuspidal representation of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$, and let $(\tilde{J}^0,\tilde{\sigma}^0)$ be a maximal extension of $(\mathbf{U}(\mathfrak{B}),\sigma^0)$ in $\mathfrak{K}(\mathfrak{B})$. We put $\rho=\text{c-Ind}_{\tilde{J}^0}^{\mathfrak{K}(\mathfrak{B})}\tilde{\sigma}^0$. We denote by $\tilde{\sigma}$ the representation $\tilde{\sigma}^0$ as a representation of $\tilde{J}=\tilde{J}^0J^1(\beta,\mathfrak{A})$ via the isomorphism $\tilde{J}^0/\mathbf{U}^1(\mathfrak{B})\cong\tilde{J}/J^1(\beta,\mathfrak{A})$. Then $\text{c-Ind}_{\tilde{J}}^{\hat{J}(\beta,\mathfrak{A})}\tilde{\sigma}$ is the representation ρ regarded as a representation of $\hat{J}=\hat{J}(\beta,\mathfrak{A})$ via $\mathfrak{K}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})\cong\hat{J}(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A})$.*

Proof. — Since $(\mathbf{U}(\mathfrak{B}),\sigma^0)$ is a simple type of $B^\times$ of depth zero, ρ is trivial on $\mathbf{U}^1(\mathfrak{B})$ by Lemma 8.2 (3). Then we can regard ρ as a $\hat{J}(\beta,\mathfrak{A})$-representation.

Since $\rho=\text{c-Ind}_{\tilde{J}^0}^{\mathfrak{K}(\mathfrak{B})}\tilde{\sigma}^0$, the dimension of ρ is equal to $(\mathfrak{K}(\mathfrak{B}):\tilde{J}^0)\dim\tilde{\sigma}^0$. On the other hand, the dimension of $\text{c-Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}$ is equal to $(\hat{J}:\tilde{J})\dim\tilde{\sigma}$. Since $\tilde{\sigma}$ is an extension of $\tilde{\sigma}^0$, we have $\dim\tilde{\sigma}^0=\dim\tilde{\sigma}$. Moreover, we also have $\mathfrak{K}(\mathfrak{B})/\tilde{J}^0\cong\hat{J}/\tilde{J}$ and $(\mathfrak{K}(\mathfrak{B}):\tilde{J}^0)=(\hat{J}:\tilde{J})$ as $\hat{J}=\mathfrak{K}(\mathfrak{B})J(\beta,\mathfrak{A})=\mathfrak{K}(\mathfrak{B})\tilde{J}$ and $\mathfrak{K}(\mathfrak{B})\cap\tilde{J}=\tilde{J}^0$. Since ρ is irreducible by 8.2 (2), it is enough to show that there exists a nonzero $\hat{J}$-homomorphism $\rho\to\text{c-Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}$.

First, since $\hat{J}$ is compact modulo center in G and $\tilde{J}^0$ contains the center of G, for any subgroups $J' \subset J''$ between $\hat{J}$ and $\tilde{J}^0$ we have $\mathrm{Ind}_{J'}^{J''} = \text{c-}\mathrm{Ind}_{J'}^{J''}$. By the Frobenius reciprocity, $\mathrm{Hom}_{\tilde{J}}\left(\mathrm{Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}, \tilde{\sigma}\right) \neq 0$. Restricting these representations to $\tilde{J}^0$, we have $\mathrm{Hom}_{\tilde{J}^0}\left(\mathrm{Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}, \tilde{\sigma}^0\right) \neq 0$. Using the Frobenius reciprocity, we have $\mathrm{Hom}_{\mathfrak{K}(\mathfrak{B})}\left(\mathrm{Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}, \mathrm{Ind}_{\tilde{J}^0}^{\mathfrak{K}(\mathfrak{B})}\tilde{\sigma}^0\right) \neq 0$. Since $\mathfrak{K}(\mathfrak{B})$ is compact modulo center, every $\mathfrak{K}(\mathfrak{B})$-representation of finite length with a central character is semisimple and $\mathrm{Hom}_{\mathfrak{K}(\mathfrak{B})}\left(\mathrm{Ind}_{\tilde{J}^0}^{\mathfrak{K}(\mathfrak{B})}\tilde{\sigma}^0, \mathrm{Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}\right) \neq 0$.

Here, since $J^1(\beta,\mathfrak{A})$ is normal in $\hat{J}$, and $\tilde{\sigma}$ is trivial on $J^1(\beta,\mathfrak{A})$, the restriction of $\mathrm{Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}$ to $J^1(\beta,\mathfrak{A})$ is also trivial. Then, if we extend $\mathrm{Ind}_{\tilde{J}^0}^{\mathfrak{K}(\mathfrak{B})}\tilde{\sigma}^0 = \rho$ to $\hat{J} = \mathfrak{K}(\mathfrak{B})J^1(\beta,\mathfrak{A})$ as trivial on $J^1(\beta,\mathfrak{A})$, there exists a nonzero $\hat{J}$-homomorphism $\rho \to \mathrm{Ind}_{\tilde{J}}^{\hat{J}}\tilde{\sigma}$. □

The following lemma guarantees the existence of extensions of β-extensions for simple characters.

Lemma 8.4. — *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum of A with $\mathfrak{B}$ maximal. Let $\theta \in \mathscr{C}(\beta, 0, \mathfrak{A})$, and let κ be a β-extension of the Heisenberg representation η_θ of θ to $J(\beta,\mathfrak{A})$.*

1. *There exists an extension $\hat{\kappa}$ of κ to $\hat{J}(\beta,\mathfrak{A})$.*
2. *Let $\hat{\kappa}'$ be another extension of η_θ to $\hat{J}(\beta,\mathfrak{A})$. Then there exists a character χ of $\hat{J}(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A})$ such that $\hat{\kappa}' \cong \hat{\kappa} \otimes \chi$.*

Proof. — We fix $g \in \mathfrak{K}(\mathfrak{B})$ with $v_{\mathfrak{B}}(g) = 1$. Since $\mathfrak{K}(\mathfrak{B}) \subset B^\times \subset I_G(\kappa)$ and $\mathfrak{K}(\mathfrak{B})$ normalizes $J(\beta,\mathfrak{A})$, we can take a $J(\beta,\mathfrak{A})$-isomorphism $f : {}^g\kappa \to \kappa$. The group $\hat{J}(\beta,\mathfrak{A})/J(\beta,\mathfrak{A})$ is a cyclic group generated by the image of g, and then we can define $\hat{\kappa}$ as

$$\hat{\kappa}(g^l u) = f^l \circ \kappa(u)$$

for $l \in \mathbb{Z}$ and $u \in J(\beta,\mathfrak{A})$. It is enough to show $\hat{\kappa}$ is a group homomorphism. Let $g_1, g_2 \in \hat{J}(\beta,\mathfrak{A})$. Then there exist $l_1, l_2 \in \mathbb{Z}$ and $u_1, u_2 \in J(\beta,\mathfrak{A})$ such that $g_i = g^{l_i}u_i$ for $i = 1, 2$. We have $g_1 g_2 = g^{l_1+l_2}(g^{-l_2}u_1 g^{l_2})u_2$ with $g^{-l_2}u_1 g^{l_2} \in J(\beta,\mathfrak{A})$. Therefore, we obtain

$$\begin{aligned}\hat{\kappa}(g_1 g_2) &= f^{l_1+l_2} \circ \kappa(g^{-l_2}u_1 g^{l_2}) \circ \kappa(u_2)\\ &= f^{l_1} \circ \kappa(u_1) \circ f^{l_2} \circ \kappa(u_2) = \hat{\kappa}(g_1) \circ \hat{\kappa}(g_2),\end{aligned}$$

whence (1) holds.

Let $\hat{\kappa}'$ be another extension of η_θ to $\hat{J}(\beta,\mathfrak{A})$. Then we have $\mathrm{Hom}_{J^1(\beta,\mathfrak{A})}(\hat{\kappa}, \hat{\kappa}') = \mathrm{Hom}_{J^1(\beta,\mathfrak{A})}(\eta_\theta, \eta_\theta) \cong \mathbb{C}$. The group $\hat{J}(\beta,\mathfrak{A})$ acts on $\mathrm{Hom}_{J^1(\beta,\mathfrak{A})}(\hat{\kappa}, \hat{\kappa}') \cong \mathbb{C}$. Then as in the proof of Lemma 8.1, we obtain χ, and (2) also holds. □

The following proposition is one of the key points to construct a Yu datum from a Sécherre datum.

Proposition 8.5. — *Let (J,λ) be a maximal simple type associated to a simple stratum $[\mathfrak{A},n,0,\beta]$. Let $\theta\in\mathscr{C}(\beta,0,\mathfrak{A})$ be a subrepresentation in λ, and let η_θ be the Heisenberg representation of θ. For any extension Λ of λ to $\tilde{J}$ and any extension $\hat{\kappa'}$ of η_θ to $\hat{J}$, there exists an irreducible $\mathfrak{K}(\mathfrak{B})$-representation ρ such that the following hold.*

1. *$\rho|_{\mathbf{U}(\mathfrak{B})}$ is trivial on $\mathbf{U}^1(\mathfrak{B})$ and cuspidal as a representation of $\mathbf{U}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})$.*
2. $\text{c-Ind}_{\mathfrak{K}(\mathfrak{B})}^{B^\times}\rho$ *is irreducible and supercuspidal.*
3. *Regarding ρ as a $\hat{J}$-representation via the isomorphism $\mathfrak{K}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B})\cong\hat{J}/J^1$, the representation $\hat{\kappa'}\otimes\rho$ is isomorphic to* $\text{c-Ind}_{\tilde{J}}^{\hat{J}}\Lambda$.

Proof. — Let $\lambda=\kappa\otimes\sigma$ be a decomposition as in Definition 2.14. We take an extension $\hat{\kappa}$ of κ to $\hat{J}$, which exists by Lemma 8.4 (1). Then there exists a character χ_1 of $\hat{J}/J^1(\beta,\mathfrak{A})$ such that $\hat{\kappa}\cong\hat{\kappa}'\otimes\chi_1$ by Lemma 8.4. Let $\tilde{\sigma}$ be an extension of σ to $\tilde{J}$. Then the $\tilde{J}$-representations Λ and $\hat{\kappa}'\otimes\chi_1\otimes\tilde{\sigma}$ are extensions of λ. By Lemma 8.1, there exists a character χ_2 of $\tilde{J}$ such that $\Lambda\cong\hat{\kappa}'\otimes\chi_1\otimes\tilde{\sigma}\otimes\chi_2$. Since χ_2 is trivial on J and $\hat{J}/J\cong\mathbb{Z}$, we can extend χ_2 to $\hat{J}$. Let J' be a subgroup in $\mathfrak{K}(\mathfrak{B})$ corresponding to $\tilde{J}$ via the isomorphism $\mathfrak{K}(\mathfrak{B})/\mathbf{U}(\mathfrak{B})\cong\hat{J}/J$. Then $(J',\tilde{\sigma}\otimes\chi_1\chi_2)$ is a maximal extension of the depth zero simple type $(\mathbf{U}(\mathfrak{B}),\sigma)$. Therefore, we obtain a $\mathfrak{K}(\mathfrak{B})$-representation $\rho=\text{c-Ind}_{J'}^{\mathfrak{K}(\mathfrak{B})}(\tilde{\sigma}\otimes\chi_1\chi_2)$. Regarding ρ as a $\hat{J}$-representation, ρ is equal to $\text{c-Ind}_{\tilde{J}}^{\hat{J}}(\tilde{\sigma}\otimes\chi_1\chi_2)$ by Lemma 8.3. Then we have

$$\hat{\kappa'}\otimes\rho=\hat{\kappa'}\otimes\text{c-Ind}_{\tilde{J}}^{\hat{J}}(\tilde{\sigma}\otimes\chi_1\chi_2)\cong\text{c-Ind}_{\tilde{J}}^{\hat{J}}(\hat{\kappa'}\otimes\tilde{\sigma}\otimes\chi_1\chi_2)\cong\text{c-Ind}_{\tilde{J}}^{\hat{J}}\Lambda.$$

Therefore, ρ satisfies the desired conditions by Lemma 8.2. □

Conversely, the following proposition is used to construct Sécherre data from Yu data.

Proposition 8.6. — *Let $(x,(G^i),(\mathbf{r}_i),(\mathbf{\Phi}_i),\rho)$ be a Yu datum of $G\cong\mathrm{GL}_m(D)$.*

1. *Then $[x]$ is a vertex in $\mathscr{B}^R(G^0,F)$.*
2. *There exists a simple type $(G^0(F)_x,\sigma)$ of depth zero and a maximal extension $(\tilde{J},\tilde{\sigma})$ of $(G^0(F)_x,\sigma)$ such that $\rho\cong\mathrm{Ind}_{\tilde{J}}^{G^0(F)_{[x]}}\tilde{\sigma}$.*

Proof. — In the beginning, G^0 is a tame twisted Levi subgroup in G with $Z(G^0)/Z(G)$ anisotropic. Then there exists a tamely ramified field extension E_0/F in $A\cong\mathrm{M}_m(D)$ such that $G^0(F)$ is the multiplicative group of $\mathrm{Cent}_A(E_0)$. Since $\mathrm{Cent}_A(E_0)$ is a central simple E_0-algebra, there exists $m_{E_0}\in\mathbb{Z}_{>0}$ and a division E_0-algebra D_{E_0} such that $\mathrm{Cent}_A(E_0)\cong\mathrm{M}_{m_{E_0}}(D_{E_0})$.

By our assumption, $\pi:=\text{c-Ind}_{G^0(F)_{[x]}}^{G^0(F)}\rho$ is an irreducible and supercuspidal representation of depth zero. Then there exists $y\in\mathscr{B}^E(G^0,F)$ and an irreducible $G^0(F)_y$-representation σ such that $[y]$ is a vertex and $(G^0(F)_y,\sigma)$ is a $[G^0(F), \pi]_{G^0(F)}$-type. Since vertices in $\mathscr{B}^R(G^0, F)$ are permuted transitively by the action of $G^0(F)$, we may assume $G^0(F)_y \supset G^0(F)_x$.

We show that $\mathrm{Ind}_{G^0(F)_x}^{G^0(F)_y} \mathrm{Res}_{G^0(F)_x}^{G^0(F)_{[x]}} \rho$ has a nonzero $G^0(F)_{y,0+}$-fixed part. Since $G^0(F)_x \cap G^0(F)_{y,0+} \subset G^0(F)_{x,0+}$, the representation ρ is trivial on $G^0(F)_x \cap G^0(F)_{y,0+}$. Then $\mathrm{Ind}_{G^0(F)_x \cap G^0(F)_{y,0+}}^{G^0(F)_{y,0+}} \mathrm{Res}_{G^0(F)_x \cap G^0(F)_{y,0+}}^{G^0(F)_{[x]}} \rho$ has a nonzero $G^0(F)_{y,0+}$-fixed part by the Frobenius reciprocity. However,

$$\begin{aligned} &\mathrm{Ind}_{G^0(F)_x \cap G^0(F)_{y,0+}}^{G^0(F)_{y,0+}} \mathrm{Res}_{G^0(F)_x \cap G^0(F)_{y,0+}}^{G^0(F)_{[x]}} \rho \\ &\subset \mathrm{Res}_{G^0(F)_{y,0+}}^{G^0(F)_y} \mathrm{Ind}_{G^0(F)_x}^{G^0(F)_y} \mathrm{Res}_{G^0(F)_x}^{G^0(F)_{[x]}} \rho \end{aligned}$$

by the Mackey decomposition. Therefore, $\mathrm{Ind}_{G^0(F)_x}^{G^0(F)_y} \mathrm{Res}_{G^0(F)_x}^{G^0(F)_{[x]}} \rho$ has a nonzero $G^0(F)_{y,0+}$-fixed part.

Since $\mathrm{Ind}_{G^0(F)_x}^{G^0(F)_y} \mathrm{Res}_{G^0(F)_x}^{G^0(F)_{[x]}} \rho \subset \mathrm{Res}_{G^0(F)_y}^{G^0} \text{c-}\mathrm{Ind}_{G^0(F)_{[x]}}^{G^0} \rho = \mathrm{Res}_{G^0(F)_y}^{G^0} \pi$ by the Mackey decomposition, we may also assume $\sigma \subset \mathrm{Ind}_{G^0(F)_x}^{G^0(F)_y} \mathrm{Res}_{G^0(F)_x}^{G^0(F)_{[x]}} \rho$ by $G^0(F)_{[x]}$-conjugation if necessary by [14, Theorem 5.5(ii)]. By the Frobenius reciprocity, $\mathrm{Res}_{G^0(F)_x}^{G^0(F)_y} \sigma$ is a subrepresentation of $\mathrm{Res}_{G^0(F)_x}^{G^0(F)_{[x]}} \rho$, which is trivial on $G^0(F)_{x,0+}$. Therefore, σ has a nonzero $G^0(F)_{x,0}G^0(F)_{y,0+}$-fixed part. Since the image of $G^0(F)_{x,0}$ in $G^0(F)_y/G^0(F)_{y,0+}$ is a parabolic subgroup of $G^0(F)_y/G^0(F)_{y,0+}$, and σ is cuspidal when we regard σ as a $G^0(F)_y/G^0(F)_{y,0+}$-representation, we have $G^0(F)_{x,0}G^0(F)_{y,0+} = G^0(F)_y$, which implies $[x] = [y]$, that is, (1) holds.

To show (2), let $(\tilde{J}, \tilde{\sigma})$ be the unique extension of $(G^0(F)_x, \sigma)$ such that $\pi \cong \text{c-}\mathrm{Ind}_{\tilde{J}}^{G} \tilde{\sigma}$.

We show the $G^0(F)_{x,0+}$-fixed part in π is contained in $\mathrm{Ind}_{\tilde{J}}^{G^0(F)_{[x]}} \tilde{\sigma}$. By the Mackey decomposition, we have

$$\pi \cong \bigoplus_{g \in G^0(F)_{x,0+} \backslash G / \tilde{J}} \mathrm{Ind}_{G^0(F)_{x,0+} \cap {}^g\tilde{J}}^{G^0(F)_{x,0+}} \mathrm{Res}_{G^0(F)_{x,0+} \cap {}^g\tilde{J}}^{{}^g\tilde{J}} {}^g\tilde{\sigma}.$$

We put $\tau(g) = \mathrm{Ind}_{G^0(F)_{x,0+} \cap {}^g\tilde{J}}^{G^0(F)_{x,0+}} \mathrm{Res}_{G^0(F)_{x,0+} \cap {}^g\tilde{J}}^{{}^g\tilde{J}} {}^g\tilde{\sigma}$. Suppose $\tau(g)$ has a nonzero $G^0(F)_{x,0+}$-fixed part. Then $\mathrm{Hom}_{G^0(F)_{x,0+} \cap {}^g\tilde{J}}(\mathbf{1}, {}^g\tilde{\sigma}) \neq 0$ by the Frobenius reciprocity. Here, since $[x]$ is a vertex, $G^0(F)_x$ is a maximal compact open subgroup. Therefore, we may assume $G^0(F)_x = \mathrm{GL}_{m_{E_0}}(\mathfrak{o}_{D_{E_0}})$ by $G^0(F)$-conjugation if necessary. Then there exist $k, k' \in G^0(F)_x$ and a diagonal matrix g' such that the (i,i)-coefficient of g' is $\varpi_{D_{E_0}}^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m$, and such that $g = kg'k'$. Since $G^0(F)_{x,0+}$ is normal in $G^0(F)_x$ and $G^0(F)_x \subset \tilde{J}$, the condition $\mathrm{Hom}_{G^0(F)_{x,0+} \cap {}^g\tilde{J}}(\mathbf{1}, {}^g\tilde{\sigma}) \neq 0$ holds if and only if $\mathrm{Hom}_{{}^{(g')^{-1}}G^0(F)_{x,0+} \cap \tilde{J}}(\mathbf{1}, \tilde{\sigma}) \neq 0$. Therefore, σ has a nonzero $G^0(F)_{x,0+}\big({}^{(g')^{-1}}G^0(F)_{x,0+} \cap \tilde{J}\big)$-fixed part. If

$a_i > a_{i+1}$ for some i, the image of ${}^{(g')^{-1}}G^0(F)_{x,0+} \cap \tilde{J}$ in $G^0(F)_x/G^0(F)_{x,0+}$ is a proper parabolic subgroup, which is a contradiction since σ is cuspidal. Then $g' \in D_{E_0}^{\times} \subset G^0(F)_{[x]}$ and $g = kg'k' \in G^0(F)_{[x]}$. Therefore, the $G^0_{x,0+}$-fixed part in π is contained in $\bigoplus_{g \in G^0(F)_{x,0+} \backslash G^0(F)_{[x]}/\tilde{J}} \tau(g) = \text{c-Ind}_{\tilde{J}}^{G^0(F)_{[x]}} \tilde{\sigma}$.

Then we have $\rho \subset \mathrm{Ind}_{\tilde{J}}^{G^0(F)_{[x]}} \tilde{\sigma}$. Since these representations are irreducible, we obtain $\rho = \mathrm{Ind}_{\tilde{J}}^{G^0(F)_{[x]}} \tilde{\sigma}$. □

9. Factorization of tame simple characters

Let $[\mathfrak{A}, n, 0, \beta]$ be a tame simple stratum of A. If $n = 0$, suppose $\beta \in \mathfrak{o}_F^{\times}$. By Proposition 4.7, there exists a defining sequence $([\mathfrak{A}, n, r_i, \beta_i])_{i=0,1,\ldots,s}$ of $[\mathfrak{A}, n, 0, \beta]$ such that

1. $F[\beta_i] \supsetneq F[\beta_{i+1}]$,
2. $\beta_i - \beta_{i+1}$ is minimal over $F[\beta_{i+1}]$,
3. $v_{\mathfrak{A}}(\beta_i - \beta_{i+1}) = k_0(\beta_i, \mathfrak{A}) = -r_{i+1}$

for $i = 0, 1, \ldots, s-1$.

We put $E_i = F[\beta_i]$. Let B_i be the centralizer of E_i in A. Let $c_i = \beta_i - \beta_{i+1}$ for $i = 0, \ldots, s-1$ and let $c_s = \beta_s$.

Proposition 9.1. — *Let $0 \leq t < -k_0(\beta, \mathfrak{A})$. Let $\theta \in \mathscr{C}(\beta, t, \mathfrak{A})$. Then for $i = 0, 1, \ldots, s$ there exists a smooth character ϕ_i of $E_i^{\times}$ such that we have $\theta = \prod_{i=0}^{s} \theta^i$, where the characters θ^i of $H^{t+1}(\beta, \mathfrak{A})$ are defined as in the following:*

1. $\theta^i|_{B_i^{\times} \cap H^{t+1}(\beta, \mathfrak{A})} = \phi_i \circ \mathrm{Nrd}_{B_i/E_i}$, *and*
2. $\theta^i|_{H^{t_i+1}(\beta, \mathfrak{A})} = \psi_{c_i}$, *where* $t_i = \max\{t, \lfloor -v_{\mathfrak{A}}(c_i)/2 \rfloor\}$.

Proof. — We show this proposition by induction on the length s of the defining sequence.

First, suppose that $s = 0$, that is, β is minimal over F. We have $\theta = \theta^0$. Then it is enough to show that θ satisfies (1) and (2). Since θ is simple, $\theta | B_0^{\times} \cap H^{t+1}(\beta, \mathfrak{A})$ factors through Nrd_{B_0/E_0}. Then there exists a character ϕ_0 of $E_0^{\times}$ such that $\theta = \phi_0 \circ \mathrm{Nrd}_{B_0/E_0}$, whence (1) holds. We have $v_{\mathfrak{A}}(c_0) = v_{\mathfrak{A}}(\beta) = -n$ and $t_0 = \max\{t, \lfloor -v_{\mathfrak{A}}(c_0)/2 \rfloor\} \geq \lfloor n/2 \rfloor$. Then we have $H^{t_0+1}(\beta, \mathfrak{A}) \subset H^{\lfloor n/2 \rfloor + 1}(\beta, \mathfrak{A})$. Since θ is simple, we have $\theta|_{H^{t_0+1}}(\beta, \mathfrak{A}) = \psi_\beta = \psi_{c_0}$, whence (2) also holds.

Next, suppose that $s > 0$, that is, β is not minimal over F. We put $t' = \max\{t, \lfloor -k_0(\beta, \mathfrak{A})/2 \rfloor\}$. Since $k_0(\beta, \mathfrak{A}) = v_{\mathfrak{A}}(c_0)$, we have $t' = \max\{t, \lfloor -v_{\mathfrak{A}}(c_0)/2 \rfloor\} = t_0$. Since θ is simple, there exists $\theta' \in \mathscr{C}(\beta, t', \mathfrak{A})$ such that $\theta|_{H^{t'+1}(\beta, \mathfrak{A})} = \psi_{c_0} \theta'$. By induction hypothesis, for $i = 1, \ldots, s$, there exist a smooth character ϕ_i of $E_i^{\times}$ and a smooth character θ'^i of $H^{t'+1}(\beta_1, \mathfrak{A}) = H^{t'+1}(\beta, \mathfrak{A})$ such that $\theta'^i|_{B_i^{\times} \cap H^{t'+1}(\beta, \mathfrak{A})} = \phi_i \circ \mathrm{Nrd}_{B_i/E_i}$ and $\theta'^i|_{H^{t'_i+1}(\beta, \mathfrak{A})} = \psi_{c_i}$, where $t'_i = \max\{t', \lfloor -v_{\mathfrak{A}}(c_i)/2 \rfloor\}$. Here, we have $r_1 < \ldots < r_s < n$, whence we obtain $-v_{\mathfrak{A}}(c_0) < \ldots < -v_{\mathfrak{A}}(c_{s-1}) < -v_{\mathfrak{A}}(c_s)$. Since $t' = \max\{t, \lfloor -v_{\mathfrak{A}}(c_0)/2 \rfloor\}$ and $-v_{\mathfrak{A}}(c_0) < -v_{\mathfrak{A}}(c_i)$, we have $t'_i = t_i$. We want to extend θ'^i to a character θ^i of $H^{t+1}(\beta, \mathfrak{A})$ as $\theta^i|_{B_i^\times \cap H^{t+1}(\beta,\mathfrak{A})} = \phi_i \circ \mathrm{Nrd}_{B_i/E_i}$. Suppose we obtain θ^i in such a way. Then θ^i satisfies (1) by construction of θ^i, and (2) since $H^{t_i+1}(\beta, \mathfrak{A}) = H^{t'_i+1}(\beta, \mathfrak{A})$ and $\theta^i|_{H^{t_i+1}(\beta,\mathfrak{A})} = \theta'^i|_{H^{t_i+1}(\beta,\mathfrak{A})} = \psi_{c_i}$.

We have $B_i^\times \cap H^{t+1}(\beta, \mathfrak{A}) \cap H^{t'+1}(\beta, \mathfrak{A}) = B_i^\times \cap H^{t'+1}(\beta, \mathfrak{A})$, whence restrictions of θ'^i and $\phi_i \circ \mathrm{Nrd}_{B_i/E_i}$ to $B_i^\times \cap H^{t+1}(\beta, \mathfrak{A}) \cap H^{t_i+1}(\beta, \mathfrak{A})$ are equal. Let $b_1, b_2 \in B_i^\times \cap H^{t+1}(\beta, \mathfrak{A})$ and $h'_1, h'_2 \in H^{t'+1}(\beta, \mathfrak{A})$ with $b_1 h'_1 = b_2 h'_2$. Then $b_1^{-1} b_2 = h'_1 (h'_2)^{-1} \in B_i^\times \cap H^{t+1}(\beta, \mathfrak{A}) \cap H^{t_i+1}(\beta, \mathfrak{A})$ and $\phi_i \circ \mathrm{Nrd}_{B_i/E_i}(b_1^{-1} b_2) = \theta'^i(h'_1 (h'_2)^{-1})$. Therefore, we also have

$$\phi_i \circ \mathrm{Nrd}_{B_i/E_i}(b_1) \theta'^i(h'_1) = \phi_i \circ \mathrm{Nrd}_{B_i/E_i}(b_2) \theta'^i(h'_2).$$

Then θ^i is well defined as a map from $H^{t+1}(\beta, \mathfrak{A})$ to $\mathbb{C}^\times$.

We show $\psi_{c_i}(b^{-1} h b) = \psi_{c_i}(h)$ for $b \in B_i^\times \cap H^{t+1}(\beta, \mathfrak{A})$ and $h \in H^{t_i+1}(\beta, \mathfrak{A})$. By definition of ψ_{c_i}, we have

$$\psi_{c_i}(b^{-1} h b) = \mathrm{Trd}_{A/F}(c_i (b^{-1} h b - 1)).$$

Since $c_i \in E_i$ and $b \in B_i = \mathrm{Cent}_A(E_i)$, we have

$$c_i (b^{-1} h b - 1) = c_i b^{-1} (h - 1) b = b^{-1} c_i (h - 1) b.$$

Therefore, we obtain

$$\psi_{c_i}(b^{-1} h b) = \mathrm{Trd}_{A/F}(b^{-1} c_i (h - 1) b) = \mathrm{Trd}_{A/F}(c_i (h - 1)) = \psi_{c_i}(h).$$

To show θ^i is a character, let $h_1, h_2 \in H^{t+1}(\beta, \mathfrak{A})$. Then there exist $b_1, b_2 \in B_i^\times \cap H^{t+1}(\beta, \mathfrak{A})$ and $h'_1, h'_2 \in H^{t_i+1}(\beta, \mathfrak{A})$ such that $h_1 = b_1 h'_1$ and $h_2 = b_2 h'_2$. Therefore, we have

$$\begin{aligned}
\theta^i(h_1 h_2) &= \theta^i(b_1 h'_1 b_2 h'_2) \\
&= \theta^i\left((b_1 b_2)(b_2^{-1} h'_1 b_2 h'_2)\right) \\
&= (\phi_i \circ \mathrm{Nrd}_{B_i/E_i})(b_1)(\phi_i \circ \mathrm{Nrd}_{B_i/E_i})(b_2) \psi_{c_i}(b_2^{-1} h'_1 b_2) \psi_{c_i}(h'_2) \\
&= (\phi_i \circ \mathrm{Nrd}_{B_i/E_i})(b_1) \psi_{c_i}(h'_1)(\phi_i \circ \mathrm{Nrd}_{B_i/E_i})(b_2) \psi_{c_i}(h'_2) \\
&= \theta^i(b_1 h'_1) \theta^i(b_2 h'_2) = \theta^i(h_1) \theta^i(h_2).
\end{aligned}$$

We put $\theta^0 = \theta \prod_{i=1}^s (\theta^i)^{-1}$. To complete the proof, it is enough to show that θ^0 satisfies (1) and (2).

To see (1), we show the restrictions of θ and θ^i $(i = 1, \ldots, s)$ to $B_0^\times \cap H^{t+1}(\beta, \mathfrak{A})$ factor through Nrd_{B_0/E_0}. Since θ is simple, $\theta|_{B_0^\times \cap H^{t+1}(\beta,\mathfrak{A})}$ factors through Nrd_{B_0/E_0}. We already have $\theta|_{B_i^\times \cap H^{t+1}(\beta,\mathfrak{A})} = \phi_i \circ \mathrm{Nrd}_{B_i/E_i}$. Since $B_0^\times \subset B_i^\times$, we have $\theta_{B_0^\times \cap H^{t+1}(\beta,\mathfrak{A})} = \phi_i \circ (\mathrm{Nrd}_{B_i/E_i}|_{B_0^\times})$. However, the equation $\mathrm{Nrd}_{B_i/E_i}|_{B_0^\times} = \mathrm{N}_{E_0/E_i} \circ \mathrm{Nrd}_{B_0/E_0}$ holds. Then $\theta^i|_{B_0^\times \cap H^{t+1}(\beta,\mathfrak{A})}$ factors through

Nrd_{B_0/E_0}. Therefore, $\theta^0|_{B_0^\times \cap H^{t+1}(\beta,\mathfrak{A})}$ also factors through Nrd_{B_0/E_0}, and there exists a character ϕ_0 of $E_0^\times$ such that $\theta^0|_{B_0^\times \cap H^{t+1}(\beta,\mathfrak{A})} = \phi_0 \circ \mathrm{Nrd}_{B_0/E_0}$.

By restricting $\theta^0 = \theta \prod_{i=1}^s (\theta^i)^{-1}$ to $H^{t_0+1}(\beta,\mathfrak{A}) = H^{t'+1}(\beta,\mathfrak{A})$, we have

$$\begin{aligned}\theta^0|_{H^{t_0+1}(\beta,\mathfrak{A})} &= (\theta|_{H^{t'+1}(\beta,\mathfrak{A})} \prod_{i=1}^s (\theta^i|_{H^{t'+1}(\beta,\mathfrak{A})})^{-1} \\ &= \psi_{c_0}\theta' \prod_{i=1}^s (\theta'^i)^{-1} = \psi_{c_0}\theta'\theta'^{-1} = \psi_{c_0}.\end{aligned}$$

Therefore, (2) also holds, and we complete the proof. □

10. Construction of a Yu datum from a Sécherre datum

Let $[\mathfrak{A}, n, 0, \beta]$ be a tame simple stratum, $(J(\beta,\mathfrak{A}), \lambda)$ be a maximal simple type with $[\mathfrak{A}, n, 0, \beta]$, and let $(\tilde{J}(\lambda), \Lambda)$ be a maximal extension of $(J(\beta,\mathfrak{A}), \lambda)$. We construct a Yu datum from the data of $[\mathfrak{A}, n, 0, \beta]$ and λ.

We put $\mathfrak{P} = \mathfrak{P}(\mathfrak{A})$. Let $([\mathfrak{A}, n, r_i, \beta_i])_{i=0}^s$, E_i, B_i and c_i be as in §9. For $i = 0, 1, \ldots, s$, we put $G^i = \mathrm{Res}_{E_i/F} \underline{\mathrm{Aut}}_{D\otimes E_i}(V)$ and $\mathbf{r}_i = -\mathrm{ord}(c_i)$. If $\beta_s \in F$, we put $d = s$. If $\beta_s \notin F$, we put $d = s+1$, $G^d = G$ and $\mathbf{r}_d = \mathbf{r}_s$. Then $(G^0, \ldots, G^d)$ is a tame twisted Levi sequence by Corollary 5.5. We also put $\mathbf{r}_{-1} = 0$. For $i = -1, 0, 1, \ldots, d$, we put $\mathbf{s}_i = \mathbf{r}_i/2$.

Proposition 10.1. — *We fix a $G^{i-1}(F)$-equivalent and affine embedding*

$$\iota_i : \mathscr{B}^E(G^{i-1}, F) \hookrightarrow \mathscr{B}^E(G^i, F)$$

for $i = 1, \ldots, d$ and we put $\tilde{\iota}_i = \iota_i \circ \cdots \circ \iota_1$. We also put $\tilde{\iota}_0 = \mathrm{id}_{\mathscr{B}^E(G^0,F)}$.

1. *There exists $x \in \mathscr{B}^E(G^0, F)$ such that $[x]$ is a vertex and*
   (a) $G^0(F)_{[x]} = \mathfrak{K}(\mathfrak{B}_0)$,
   (b) $G^0(F)_x = B_0^\times \cap \mathbf{U}(\mathfrak{A}) = \mathbf{U}(\mathfrak{B}_0)$,
   (c) $G^0(F)_{x,0+} = B_0^\times \cap \mathbf{U}^1(\mathfrak{A})$,
   (d) $\mathfrak{g}^0(F)_x = B_0 \cap \mathfrak{A} = \mathfrak{B}_0$, *and*
   (e) $\mathfrak{g}^0(F)_{x,0+} = B_0 \cap \mathfrak{P}$.
2. *For $i = 1, \ldots, d$, we have*
   (a) $G^i(F)_{\tilde{\iota}_i(x),\mathbf{s}_{i-1}} = B_i^\times \cap \mathbf{U}^{\lfloor(-v_{\mathfrak{A}}(c_{i-1})+1)/2\rfloor}(\mathfrak{A})$,
   (b) $G^i(F)_{\tilde{\iota}_i(x),\mathbf{s}_{i-1}+} = B_i^\times \cap \mathbf{U}^{\lfloor -v_{\mathfrak{A}}(c_{i-1})/2\rfloor+1}(\mathfrak{A})$,
   (c) $G^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_{i-1}} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_{i-1})}(\mathfrak{A})$,
   (d) $G^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_{i-1}+} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_{i-1})+1}(\mathfrak{A})$,
   (e) $\mathfrak{g}^i(F)_{\tilde{\iota}_i(x),\mathbf{s}_{i-1}} = B_i \cap \mathfrak{P}^{\lfloor(-v_{\mathfrak{A}}(c_{i-1})+1)/2\rfloor}$,

(f) $\mathfrak{g}^i(F)_{\tilde{\iota}_i(x),\mathbf{s}_{i-1}+} = B_i \cap \mathfrak{P}^{\lfloor -v_{\mathfrak{A}}(c_{i-1})/2\rfloor+1}$,
(g) $\mathfrak{g}^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_{i-1}} = B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_{i-1})}$, *and*
(h) $\mathfrak{g}^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_{i-1}+} = B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_{i-1})+1}$.

3. *For $i = 0, \dots, s$, we have*
   (a) $G^i(F)_{\tilde{\iota}_i(x),\mathbf{s}_i+} = B_i^\times \cap \mathbf{U}^{\lfloor -v_{\mathfrak{A}}(c_i)/2\rfloor+1}(\mathfrak{A})$,
   (b) $G^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_i} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_i)}(\mathfrak{A})$,
   (c) $G^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_i+} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_i)+1}(\mathfrak{A})$,
   (d) $\mathfrak{g}^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_i} = B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)}(\mathfrak{A})$, *and*
   (e) $\mathfrak{g}^i(F)_{\tilde{\iota}_i(x),\mathbf{r}_i+} = B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)+1}(\mathfrak{A})$.

Proof. — We find $x \in \mathscr{B}^E(G^0, F)$. Since B is a central simple E_0-algebra, there exists a division E_0-algebra D_{E_0} and a right D_{E_0}-module W_0 such that $B \cong \mathrm{End}_{D_{E_0}}(W_0)$. Since $\mathfrak{B}_0$ is a maximal hereditary $\mathfrak{o}_{E_0}$-order in B_0, there exists an $\mathfrak{o}_{D_{E_0}}$-chain $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ in W_0 of period 1 such that $\mathfrak{B}_0$ is the hereditary $\mathfrak{o}_{E_0}$-order associated with $(\mathcal{L}_i)_{i\in\mathbb{Z}}$. Let $x \in \mathscr{B}^E(G^0, F) \cong \mathscr{B}^E(\underline{\mathrm{Aut}}_{D_{E_0}}(W_0), E_0)$ be an element that corresponds to a lattice function constructed from $(\mathcal{L}_i)_{i\in\mathbb{Z}}$. Then by Proposition 6.6 $[x]$ is a vertex in $\mathscr{B}^E(G^0, F)$. Therefore, by Proposition 6.4 (3) we have (1)-(a).

To show the remainder assertion, we show $\mathfrak{A}$ is the hereditary $\mathfrak{o}_F$-order in A associated with $\tilde{\iota}_d(x)$. Since $[\mathfrak{A}, n, 0, \beta]$ is a stratum, $\mathfrak{A}$ is $E = F[\beta]$-pure. Moreover, we have $\mathfrak{A} \cap B_0 = \mathfrak{B}_0$ by definition of $\mathfrak{B}_0$. Therefore, by Proposition 6.8 (2) $\mathfrak{A}$ is associated with $\tilde{\iota}_d(x)$. Since $v_{\mathfrak{A}}(c_i) \in \mathbb{Z}_{\geq 0}$ and $v_{\mathfrak{A}}(c_i) = \mathrm{ord}(c_i)e(\mathfrak{A}|\mathfrak{o}_F)$, the remaining assertions follow from Proposition 6.9. □

In the following, we regard $\mathscr{B}^E(G^0, F), \dots, \mathscr{B}^E(G^{d-1}, F)$ as subsets in $\mathscr{B}^E(G, F)$ via $\tilde{\iota}_1, \dots, \tilde{\iota}_d$.

Proposition 10.2. —
1. $H^1(\beta, \mathfrak{A}) = K_+^d$,
2. $J(\beta, \mathfrak{A}) = {}^\circ K^d$,
3. $\hat{J}(\beta, \mathfrak{A}) = K^d$.

Proof. — We show (1). We have $r_i = -v_{\mathfrak{A}}(c_{i-1}) = -\mathrm{ord}(c_{i-1})e(\mathfrak{A}|\mathfrak{o}_F) = -e(\mathfrak{A}|\mathfrak{o}_F)\mathbf{r}_{i-1}$ for $i = 1, \dots, s$ and $n = -v_{\mathfrak{A}}(c_s) = -e(\mathfrak{A}|\mathfrak{o}_F)\mathbf{r}_s$. We have $G^0(F)_{x,0+} = B_0^\times \cap \mathbf{U}^1(\mathfrak{A})$ by Proposition 10.1 (1)-(c). For $i = 1, \dots, s$ we have $B_i^\times \cap \mathbf{U}^{\lfloor r_i/2\rfloor+1}(\mathfrak{A}) = B_i^\times \cap \mathbf{U}^{\lfloor -v_{\mathfrak{A}}(c_{i-1})/2\rfloor+1}(\mathfrak{A}) = G^i(F)_{x,\mathbf{s}_{i-1}+}$ by Proposition 10.1 (2)-(b). We also have $B_s^\times \cap \mathbf{U}^{\lfloor n/2\rfloor+1}(\mathfrak{A}) = G^s(F)_{x,\mathbf{s}_s+}$. If $d = s+1$, by comparing Lemma 2.20 (1) and Definition 3.12 (1) of K_+^d we have $H^1(\beta, \mathfrak{A}) = K_+^d$. If $d = s$, we have $H^1(\beta, \mathfrak{A}) = K^d\mathbf{U}^{\lfloor n/2\rfloor+1}(\mathfrak{A})$, and it suffices to show

$K^d \supset \mathbf{U}^{\lfloor n/2 \rfloor+1}(\mathfrak{A})$. However, since $\mathbf{s}_{s-1} < \mathbf{s}_s$ we have

$$\mathbf{U}^{\lfloor n/2 \rfloor+1}(\mathfrak{A}) = G^s(F)_{x,\mathbf{s}_s+} \subset G^s(F)_{x,\mathbf{s}_{s-1}+} \subset K^d.$$

(2) is similarly shown as (1), using Proposition 10.1 (1)-(b), (2)-(a), Lemma 2.20 (2) and Definition 3.12 (2) instead of Proposition 10.1 (1)-(c), (2)-(b), Lemma 2.20 (1) and Definition 3.12 (1), respectively. Since $J(\beta, \mathfrak{A}) = {}^\circ K^d$ and $\mathfrak{K}(\mathfrak{B}_0) = G^0(F)_{[x]}$ by Proposition 10.1 (1)-(a), we obtain

$$\hat{J}(\beta, \mathfrak{A}) = \mathfrak{K}(\mathfrak{B}) J(\beta, \mathfrak{A}) = G^0(F)_{[x]} {}^\circ K^d = K^d,$$

whence (3) holds. □

Let $\theta \in \mathscr{C}(\beta, 0, \mathfrak{A})$ be the unique character of $H^1(\beta, \mathfrak{A})$ in λ. Then we can take characters ϕ_i of $E_i^\times$ for $i = 0, 1, \ldots, s$ and define characters θ^i as in Proposition 9.1. We put $\mathbf{\Phi}_i = \phi_i \circ \mathrm{Nrd}_{B_i/E_i}$. If $d = s+1$, we put $\mathbf{\Phi}_d = 1$.

Proposition 10.3. — *For $i = 0, 1, \ldots, d-1$, the character $\mathbf{\Phi}_i$ is G^{i+1}-generic relative to x of depth $\mathbf{r}_i$. If $s = d$, then $\mathbf{\Phi}_d$ is of depth $\mathbf{r}_d$.*

Proof. — First we show that the restriction of $\mathbf{\Phi}_i$ to $G^i(F)_{x,\mathbf{s}_i+}$ is equal to ψ_{c_i} for $i = 0, \ldots, s$. We have

$$\begin{aligned} B_i^\times \cap H^1(\beta, \mathfrak{A}) &= G^i(F) \cap K^d_+ \\ &= G^0(F)_{x,0+} G^1(F)_{x,\mathbf{s}_0+} \cdots G^i(F)_{x,\mathbf{s}_{i-1}+}, \end{aligned}$$

and $G^i(F)_{x,\mathbf{s}_i+} \subset B_i^\times \cap H^1(\beta, \mathfrak{A})$, as we have $s_i > s_{i-1}$ and then

$$G^i(F)_{x,\mathbf{s}_i+} \subset G^i(F)_{x,\mathbf{s}_{i-1}+} \subset G^0(F)_{x,0+} G^1(F)_{x,\mathbf{s}_0+} \cdots G^i(F)_{x,\mathbf{s}_{i-1}+}.$$

To show $G^i(F)_{x,\mathbf{s}_i+} \subset H^{t_i+1}(\beta, \mathfrak{A})$, where $t_i = \max\{0, \lfloor -v_{\mathfrak{A}}(c_i)/2 \rfloor\} = \lfloor -v_{\mathfrak{A}}(c_i)/2 \rfloor$, we consider two cases. If $i < d$, we have

$$\begin{aligned} H^{t_i+1}(\beta, \mathfrak{A}) &= H^1(\beta, \mathfrak{A}) \cap \mathbf{U}^{t_i+1}(\mathfrak{A}) \\ &= K^d_+ \cap G(F)_{x,\mathbf{s}_i+} \\ &= G^{i+1}(F)_{x,\mathbf{s}_i+} \cdots G^d(F)_{x,\mathbf{s}_{d-1}+}, \end{aligned}$$

and $G^i(F)_{x,\mathbf{s}_i+} \subset H^{t_i+1}(\beta, \mathfrak{A})$ since

$$G^i(F)_{x,\mathbf{s}_i+} \subset G^{i+1}(F)_{x,\mathbf{s}_i+} \subset G^{i+1}(F)_{x,\mathbf{s}_i+} \cdots G^d(F)_{x,\mathbf{s}_{d-1}+}.$$

Otherwise, that is, if $i = s = d$, we also have

$$H^{t_d+1}(\beta, \mathfrak{A}) = K^d_+ \cap G(F)_{x,\mathbf{s}_d+} = G^d(F)_{x,\mathbf{s}_d+}.$$

Therefore, $G^i(F)_{x,\mathbf{s}_i+} \subset \left(B_i^\times \cap H^1(\beta, \mathfrak{A})\right) \cap H^{t_i+1}(\beta, \mathfrak{A})$, and we obtain

$$\begin{aligned} \mathbf{\Phi}_i|_{G^i(F)_{x,\mathbf{s}_i+}} &= \theta^i|_{G^i(F)_{x,\mathbf{s}_i+}} \\ &= \psi_{c_i}|_{G^i(F)_{x,\mathbf{s}_i+}}. \end{aligned}$$

In particular, $\mathbf{\Phi}_i$ is trivial on

$$\mathbf{U}^{-v_{\mathfrak{A}}(c_i)+1}(\mathfrak{A}) \cap G^i(F)_{x,\mathbf{s}_i+} = G(F)_{x,\mathbf{r}_i+} \cap G^i(F)_{x,\mathbf{s}_i+} = G^i(F)_{x,\mathbf{r}_i+}.$$

Note that all c_i have negative valuation. Next, we show $\mathbf{\Phi}_i$ is not trivial on $G^i(F)_{x,\mathbf{r}_i}$. We have $G^i(F)_{x,\mathbf{r}_i} = \mathbf{U}(\mathfrak{B}_i) \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_i)}(\mathfrak{A}) = B_i \cap \left(1 + \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)}\right) = 1 + \left(B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)}\right)$. Then

$$\begin{aligned}\mathbf{\Phi}_i(G^i(F)_{x,\mathbf{r}_i}) &= \psi_{c_i}\left(1 + (B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)})\right) = \psi \circ \mathrm{Trd}_{A/F}\left(c_i(B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)})\right) \\ &= \psi \circ \mathrm{Trd}_{A/F}\left(B_i \cap \mathfrak{A}\right) = \psi \circ \mathrm{Tr}_{E_i/F} \circ \mathrm{Trd}_{B_i/E_i}(\mathfrak{B}_i).\end{aligned}$$

Since $\mathfrak{B}_i$ is a hereditary $\mathfrak{o}_{E_i}$-order in B_i, we have $\mathrm{Trd}_{B_i/E_i}(\mathfrak{B}_i) = \mathfrak{o}_{E_i}$. Moreover, since E_i/F is tamely ramified, $\mathrm{Tr}_{E_i/F}(\mathfrak{o}_{E_i}) = \mathfrak{o}_F$. Therefore, $\mathbf{\Phi}_i$ is not trivial on $G^i(F)_{x,\mathbf{r}_i}$, as ψ is not trivial on $\mathfrak{o}_F$. In particular, we completed the proof when $i = s = d$ and we may assume $i < d$ in the following.

Finally, let $X^*_{c_i} \in (\underline{\mathrm{Lie}}^*(G^i))^{G^i}(F)$ as §7.2. Since c_i is minimal relative to E_i/E_{i+1}, the element $X^*_{c_i}$ is G^{i+1}-generic of depth $\mathbf{r}_i$ by Proposition 7.10. Then, to complete the proof it suffices to show that $\mathbf{\Phi}_i|_{G^i(F)_{x,\mathbf{r}_i:\mathbf{r}_i+}}$ is realized by $X^*_{c_i}$. The isomorphism $G^i(F)_{x,\mathbf{r}_i:\mathbf{r}_i+} \cong \mathfrak{g}^i(F)_{x,\mathbf{r}_i:\mathbf{r}_i+}$ is induced from $1 + y \mapsto y$. Therefore, when we regard $\psi \circ X^*_{c_i}$ as a character of $G^i(F)_{x,\mathbf{r}_i:\mathbf{r}_i+}$, for $1 + y \in G^i(F)_{x,\mathbf{r}_i}$ we have

$$\left(\psi \circ X^*_{c_i}\right)(1+y) = \psi \circ X^*_{c_i}(y) = \psi \circ \mathrm{Trd}_{A/F}(c_i y) = \psi_{c_i}(1+y) = \mathbf{\Phi}_i(1+y).$$

□

Then we have a 4-tuple $\left(x, (G^i), (\mathbf{r}_i), (\mathbf{\Phi}_i)\right)$. As in §3.4, we can define characters $\hat{\mathbf{\Phi}}_i$ of K^d_+.

Proposition 10.4. — *For $i = 0, 1, \ldots, s$, we have $\hat{\mathbf{\Phi}}_i = \theta^i$.*

Proof. — Recall the definition of $\hat{\mathbf{\Phi}}_i$. The character $\hat{\mathbf{\Phi}}_i$ is defined as

$$\begin{aligned}\hat{\mathbf{\Phi}}_i|_{K^d_+ \cap G^i(F)}(g) &= \mathbf{\Phi}_i(g), \\ \hat{\mathbf{\Phi}}_i|_{K^d_+ \cap G(F)_{x,\mathbf{s}_i+}}(1+y) &= \mathbf{\Phi}_i(1 + \pi_i(y)).\end{aligned}$$

Since $\left(K^d_+ \cap G^i(F)\right)\left(K^d_+ \cap G(F)_{x,\mathbf{s}_i+}\right) = K^d_+$, it is enough to show that $\hat{\mathbf{\Phi}}_i$ is equal to θ^i on $K^d_+ \cap G^i(F)$ and $K^d_+ \cap G(F)_{x,\mathbf{s}_i+}$.

We have that $K^d_+ \cap G^i(F) = B_i^\times \cap H^1(\beta, \mathfrak{A})$ and $K^d_+ \cap G(F)_{r,\mathbf{s}_i+} = H^{t_i+1}(\beta, \mathfrak{A})$, where $t_i = \lfloor -v_{\mathfrak{A}}(c_i)/2 \rfloor$.

If $g \in B_i^\times \cap H^1(\beta, \mathfrak{A})$, then $\hat{\mathbf{\Phi}}_i(g) = \mathbf{\Phi}_i(g) = \phi_i \circ \mathrm{Nrd}_{B_i/E_i}(g) = \theta^i(g)$.

Suppose $1 + y \in H^{t_i+1}(\beta, \mathfrak{A})$. Then $\pi_i(y) \in \mathfrak{g}^i(F)_{x,\mathbf{s}_i+} = B \cap \mathfrak{P}_i^{t_i+1}$ and $1 + \pi_i(y) \in B_i^\times \cap H^{t_i+1}(\beta, \mathfrak{A})$. Therefore, we have $\hat{\mathbf{\Phi}}_i(1+y) = \mathbf{\Phi}_i(1 + \pi_i(y)) = \theta^i(1 + \pi_i(y)) = \psi \circ \mathrm{Trd}_{A/F}(c_i \pi_i(y))$.

Here, we show if $n \in \mathfrak{n}^i(F)$, then $\mathrm{Trd}_{A/F}(c_i n) = 0$. Since c_i is in the center of $B_i^\times$, the linear automorphism $z \mapsto c_i z$ of A is also a $Z(G^i)(F)$-automorphism. Then $c_i\mathfrak{g}^i(F)$ is a trivial $Z(G^i)(F)$-representation and $c_i\mathfrak{n}^i(F) \cong \mathfrak{n}^i(F)$ is a $Z(G^i)(F)$-representation that does not contain any trivial subquotient. Therefore, we have $c_i\mathfrak{g}^i(F) \subset \mathfrak{g}^i(F)$ and $c_i\mathfrak{n}^i(F) \subset \mathfrak{n}^i(F)$. On the other hand, $\mathrm{Trd}_{A/F}$ is a $Z(G^i)(F)$-homomorphism from $\mathfrak{g}(F)$ to the trivial representation F. Since $\mathfrak{n}^i(F)$ does not have any trivial quotient, $\mathrm{Trd}_{A/F}(\mathfrak{n}^i(F)) = 0$. In particular, $\mathrm{Trd}_{A/F}(c_i n) = 0$ as $c_i n \in c_i\mathfrak{n}^i(F) \subset \mathfrak{n}^i(F)$.

Since $\pi_i : \mathfrak{g}^i(F) \oplus \mathfrak{n}^i(F) \to \mathfrak{g}^i(F)$ is the projection, $y - \pi_i(y) \in \mathfrak{n}^i(F)$. Therefore, we have $\mathrm{Trd}_{A/F}(c_i y) = \mathrm{Trd}_{A/F}\left(c_i\left(y - \pi_i(y)\right)\right) + \mathrm{Trd}_{A/F}\left(c_i\pi_i(y)\right) = \mathrm{Trd}_{A/F}\left(c_i\pi_i(y)\right)$ and

$$\hat{\mathbf{\Phi}}_i(1+y) = \psi \circ \mathrm{Trd}_{A/F}(c_i\pi_i(y)) = \psi \circ \mathrm{Trd}_{A/F}(c_i y) = \psi_{c_i}(1+y) = \theta^i(1+y).$$

□

Proposition 10.5. — *The representation $\kappa_0 \otimes \cdots \otimes \kappa_d$ is an extension of η_θ to K^d (cf. Definition 3.19 for the definition of κ_i).*

Proof. — We put $\hat{\kappa}' = \kappa_0 \otimes \cdots \otimes \kappa_d$. By [15, Lemma 3.27], $\kappa_i|_{K_+^d}$ contains $\hat{\mathbf{\Phi}}_i$ for $i = 0, \ldots, d$. If $d = s+1$, then $\hat{\mathbf{\Phi}}_d = \mathbf{\Phi}_d = 1$ and

$$\prod_{i=0}^{d} \hat{\mathbf{\Phi}}_i = \prod_{i=0}^{s} \hat{\mathbf{\Phi}}_i = \prod_{i=0}^{s} \theta^i = \theta$$

by Proposition 9.1 and Proposition 10.4. Then $\hat{\kappa}'$ contains θ as a K_+^d-representation. Since η_θ is the unique irreducible $J^1(\beta, \mathfrak{A})$-representation that contains θ, the $J^1(\beta, \mathfrak{A})$-representation $\hat{\kappa}'$ contains η_θ.

Then it suffices to show that the dimension of $\hat{\kappa}$ is equal to the dimension of η_θ. The dimension of η_θ is $(J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A}))^{1/2}$. On the other hand, for $i = 0, \ldots, d-1$ the dimension of κ_i is $(J^{i+1} : J_+^{i+1})^{1/2}$ (cf. Proposition 3.22), and the dimension of κ_d is 1. Then the dimension of $\hat{\kappa}'$ is $\prod_{i=1}^{d}(J^i : J_+^i)^{1/2}$, and it suffices to show that $(J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A})) = \prod_{i=1}^{d}(J^i : J_+^i)$. Here, $H^1(\beta, \mathfrak{A}) = K_+^d = K_+^0 J_+^1 \cdots J_+^d = G^0(F)_{x,0+} J_+^1 \cdots J_+^d$. Since $G^i(F)_{x,\mathbf{s}_i} J^{i+1} = G^{i+1}(F)_{x,\mathbf{s}_i}$ for $i = 0, \ldots, d-1$, we also have

$$\begin{aligned} G^0(F)_{x,0+} J^1 \cdots J^d &= G^0(F)_{x,0+} G^0(F)_{x,\mathbf{s}_0} J^1 \cdots J^d \\ &= G^0(F)_{x,0+} G^1(F)_{x,\mathbf{s}_0} J^2 \cdots J^d = \cdots \\ &= G^0(F)_{x,0+} G^1(F)_{x,\mathbf{s}_0} \cdots G^d(F)_{x,\mathbf{s}_{d-1}} \\ &= G(F)_{x,0+} \cap K^d = \mathbf{U}^1(\mathfrak{A}) \cap J(\beta, \mathfrak{A}) = J^1(\beta, \mathfrak{A}). \end{aligned}$$

Since $G^0(F)_{x,0+}\cap(J^1\cdots J^d)=G^0(F)_{x,\mathbf{r}_0}\subset J^1_+$ (e.g., using [31, Lemma 13.2]), we have $\big(G^0(F)_{x,0+}J^1_+\cdots J^d_+\big)\cap(J^1\cdots J^d)=J^1_+\cdots J^d_+$, and

$$\begin{aligned}J^1(\beta,\mathfrak{A})/H^1(\beta,\mathfrak{A})&=\big(G^0(F)_{x,0+}J^1\cdots J^d\big)\,/\,\big(G^0(F)_{x,0+}J^1_+\cdots J^d_+\big)\\&\cong(J^1\cdots J^d)/(J^1_+\cdots J^d_+).\end{aligned}$$

Then it is enough to show $\big((J^1\cdots J^d):(J^1_+\cdots J^d_+)\big)=\prod_{i=1}^d(J^i:J^i_+)$. Let us prove this by induction on d. If $d=1$, this is trivial. Let us assume that this is true for $d-1$. It is now enough to show that $[J^d:J^d_+]=\frac{[J^1\cdots J^d:J^1_+\cdots J^d_+]}{[J^1\cdots J^{d-1}:J^1_+\cdots J^{d-1}_+]}$. The following fact will be useful.

Fact: let $G'\subset G$ be groups and let H be a normal subgroup of G. Let ι be the injective morphism of group $G'/(G'\cap H)\hookrightarrow G/H$. As G-set, G/HG' and $(G/H)/\iota(G'/(G'\cap H))$ are isomorphic.

Because $J^1_+\cdots J^d_+$ is a normal subgroup of $J^1\cdots J^d$, we can apply the previous fact to $G=J^1\cdots J^d$, $G'=J^1\cdots J^{d-1}$, $H=J^1_+\cdots J^d_+$. Using the fact that $H\cap G'=J^1_+\cdots J^{d-1}_+$, we deduce that, as $J^1\cdots J^d$-sets, $J^1\cdots J^d/J^1\cdots J^{d-1}J^d_+$ and $(J^1\cdots J^d/J^1_+\cdots J^d_+)/\iota(J^1\cdots J^{d-1}/J^1_+\cdots J^{d-1}_+)$ are isomorphic. Let X be this $J^1\cdots J^d$-set. The set X is a fortiori a J^d-set. The group J^d acts transitively on $X=J^1\cdots J^d/J^1\cdots J^{d-1}J^d_+$, and the stabiliser of $(J^1\cdots J^{d-1}J^d_+)\in J^1\cdots J^d/J^1\cdots J^{d-1}J^d_+$ is $J^1\cdots J^{d-1}J^d_+\cap J^d$. The group $J^1\cdots J^{d-1}J^d_+\cap J^d$ is equal to J^d_+. Consequently,

$$[J^d:J^d_+]=\#(X)=\frac{[J^1\cdots J^d:J^1_+\cdots J^d_+]}{[J^1\cdots J^{d-1}:J^1_+\cdots J^{d-1}_+]},$$

as required. Therefore, we obtain $\hat\kappa'|_{J^1(\beta,\mathfrak{A})}=\eta_\theta$. □

THEOREM 10.6. — *Let (J,λ) be a maximal simple type associated to a tame simple stratum $[\mathfrak{A},n,0,\beta]$. Let $(\tilde J,\Lambda)$ be a maximal extension of (J,λ). Then there exists a Yu datum $\big(x,(G^i)_{i=0}^d,(\mathbf{r}_i)_{i=0}^d,(\mathbf{\Phi}_i)_{i=0}^d,\rho\big)$ such that*

1. *$\hat J(\beta,\mathfrak{A})=K^d$, and*
2. *$\rho_d\big(x,(G^i),(\mathbf{r}_i),(\mathbf{\Phi}_i),\rho\big)\cong\text{c-Ind}_{\tilde J}^{\hat J(\beta,\mathfrak{A})}\Lambda$.*

Proof. — In the above argument, we can take a 4-tuple $\big(x,(G^i),(\mathbf{r}_i),(\mathbf{\Phi}_i)\big)$ from a Sécherre datum. Therefore, it is enough to show that we can take an irreducible $G^0(F)_{[x]}$-representation ρ such that the Yu datum $\big(x,(G^i),(\mathbf{r}_i),(\mathbf{\Phi}_i),\rho\big)$ satisfies the desired conditions.

Let η be the unique $J^1(\beta,\mathfrak{A})$-subrepresentation in $\lambda|_{J^1(\beta,\mathfrak{A})}$. Then $\kappa_0\otimes\cdots\otimes\kappa_d$ is an extension of η to $K^d=\hat J(\beta,\mathfrak{A})$ by Proposition 10.5. Therefore, there exists an irreducible $\mathfrak{K}(\mathfrak{B})$-representation ρ such that ρ is trivial on $\mathbf{U}^1(\mathfrak{B})$ but not trivial on $\mathbf{U}(\mathfrak{B})$, the representation $\text{c-Ind}_{\mathfrak{K}(\mathfrak{B})}^{B^\times}\rho$ is irreducible and supercuspidal, and $\text{c-Ind}_{\tilde J}^{\hat J(\beta,\mathfrak{A})}\Lambda\cong\rho\otimes\kappa_0\otimes\cdots\otimes\kappa_d$ by Proposition 8.5. Since we have equalities of groups $B^\times = G^0(F)$, $\mathfrak{K}(\mathfrak{B}) = G^0(F)_{[x]}$, $\mathbf{U}(\mathfrak{B}) = G^0(F)_x$ and $\mathbf{U}^1(\mathfrak{B}) = G^0(F)_{x+}$, then the 5-tuple $\big(x, (G^i), (\mathbf{r}_i), (\mathbf{\Phi}_i), \rho\big)$ is a Yu datum satisfying the condition in the theorem. □

11. Construction of a Sécherre datum from a Yu datum

Let $(x, (G^i)_{i=0}^d, (\mathbf{r}_i)_{i=0}^d, (\mathbf{\Phi}_i)_{i=0}^d, \rho)$ be a Yu datum.

First, since G^i are tame twisted Levi subgroups in G with $Z(G^i)/Z(G)$ anisotropic, there exist tamely ramified field extensions E_i/F in A such that

$$G^i \cong \mathrm{Res}_{E_i/F} \underline{\mathrm{Aut}}_{D\otimes_F E_i}(V)$$

by Lemma 5.6. Since $G^0 \subsetneq \ldots \subsetneq G^d$, we can choose $E_0 \supsetneq \ldots \supsetneq E_d = F$. We put $B_i = \mathrm{Cent}_A(E_i)$.

Since $\text{c-Ind}_{G^0(F)_{[x]}}^{G^0(F)} \rho$ is supercuspidal, $[x]$ is a vertex in $\mathscr{B}^E(G^0, F)$ by Proposition 8.6. Let $\mathfrak{B}_0$ be the hereditary $\mathfrak{o}_{E_0}$-order in B_0 associated with x. Then the hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ associated with $x \in \mathscr{B}^E(G, F)$ is E_0-pure and principal, and $\mathfrak{A} \cap B_0 = \mathfrak{B}_0$ by Proposition 6.8. We also put $\mathfrak{P} = \mathfrak{P}(\mathfrak{A})$.

To obtain a simple stratum, we need an element $\beta \in E_0$. We will take β by using information from characters $(\mathbf{\Phi}_i)_i$. For $c_i \in E_i = \mathrm{Lie}(Z(G^i))$, let $X^*_{c_i} \in \mathrm{Lie}^*(Z(G^i))$ be as in §7.2. We put $s = \sup\{i \mid \mathbf{\Phi}_i \neq 1\}$.

Proposition 11.1. — *Suppose $s \geq 0$.*

1. *For $i = 0, \ldots, d$, the hereditary $\mathfrak{o}_{E_i}$-order in B_i associated with $x \in \mathscr{B}^E(G^i, F)$ is equal to $\mathfrak{B}_i = B_i \cap \mathfrak{A}$.*
2. *There exists $c_i \in \mathrm{Lie}(Z(G^i))_{-\mathbf{r}_i}$ such that $\mathbf{\Phi}_i|_{G^i(F)_{x,\mathbf{r}_i/2+:\mathbf{r}_i+}}$ is realized by $X^*_{c_i}$ for $i = 0, \ldots, d-1$.*
3. *If $s = d$, then there also exists $c_s \in \mathrm{Lie}(Z(G))_{-\mathbf{r}_s}$ such that $\mathbf{\Phi}_s|_{G(F)_{x,\mathbf{r}_s/2+:\mathbf{r}_s+}}$ is realized by $X^*_{c_s}$.*
4. *For $i = 0, \ldots, s$, we have $\mathbf{r}_i = -\mathrm{ord}(c_i)$.*
5. *For $i = 0, \ldots, d-1$, the element c_i is minimal relative to E_i/E_{i+1}. In particular, we have $E_i = E_{i+1}[c_i]$.*

Proof. — We show (1). First, we have $\mathfrak{B}_i \cap B_0 = \mathfrak{A} \cap B_i \cap B_0 = \mathfrak{A} \cap B_0 = \mathfrak{B}_0$. Moreover, for $g \in E_i^\times$ we also have

$$g\mathfrak{B}_i g^{-1} = g(\mathfrak{A} \cap B_i)g^{-1} = g\mathfrak{A}g^{-1} \cap gB_ig^{-1} = \mathfrak{A} \cap B_i = \mathfrak{B}_i,$$

as $\mathfrak{A}$ is E_0-pure and $E_0 \subset B_i$. Therefore, (1) holds by Proposition 6.8 (2).

Next, we show (2), and (3) is similarly shown. Since $\mathbf{\Phi}_i$ is trivial on $G^i(F)_{x,\mathbf{r}_i+}$ but not on $G^i(F)_{x,\mathbf{r}_i}$, we have $G^i(F)_{x,\mathbf{r}_i} \neq G^i(F)_{x,\mathbf{r}_i+}$ in particular. Then $n_i = \mathbf{r}_i e(\mathfrak{B}_i|\mathfrak{o}_{E_i})e(E_i/F)$ is a nonnegative integer, and we have $G^i(F)_{x,\mathbf{r}_i} = \mathbf{U}^n(\mathfrak{B})$ and $G^i(F)_{x,\mathbf{r}_i+} = \mathbf{U}^{n+1}(\mathfrak{B})$, by Lemma 6.10 (3). On the other hand, a character $\psi \circ \mathrm{Tr}_{E_i/F}$ of E_i is with conductor $\mathfrak{p}_{E_i}$ since E_i/F is tamely ramified. Therefore, we can apply Proposition 7.11 for $\mathfrak{B}_i, n$ and

$\psi \circ \mathrm{Tr}_{E_i/F}$ as $\mathfrak{B}_i$ is principal by (1) and Proposition 6.8 (1). Thus there exists $c_i \in E_i$ such that

$$\begin{aligned}\mathbf{\Phi}_i(1+y) &= (\psi \circ \mathrm{Tr}_{E_i/F}) \circ \mathrm{Trd}_{B_i/E_i}(c_i y) = \psi \circ (\mathrm{Tr}_{E_i/F} \circ \mathrm{Trd}_{B_i/E_i})(c_i y) \\ &= \psi \circ X^*_{c_i}(y)\end{aligned}$$

for $1+y \in \mathbf{U}^{\lfloor n_i/2 \rfloor + 1}(\mathfrak{B}_i) = G^i(F)_{x,\mathbf{r}_i/2+}$. Then (2) holds.

We have $v_{E_i}(c_i) = -n_i/e(\mathfrak{B}_i|\mathfrak{o}_{E_i}) = -\mathbf{r}_i e(E_i/F)$ by Proposition 7.11, and

$$\mathrm{ord}(c_i) = v_{E_i}(c_i)/e(E_i/F) = -\mathbf{r}_i,$$

whence (4) holds.

To show (5), let $c'_i \in E_i^\times$ such that $X^*_{c'_i}$ is G^{i+1}-generic of depth $\mathbf{r}_i$ and the restriction of $\mathbf{\Phi}_i$ to $G^i(F)_{x,\mathbf{r}_i:\mathbf{r}_i+}$ is realized by $X^*_{c'_i}$. In particular, we have

$$\begin{aligned}(\psi \circ \mathrm{Tr}_{E_i/F}) \circ \mathrm{Trd}_{B_i/E_i}(c_i y) = \mathbf{\Phi}_i(1+y) &= \psi \circ X_{c'_i}(y) \\ &= (\psi \circ \mathrm{Tr}_{E_i/F}) \circ \mathrm{Trd}_{B_i/E_i}(c'_i y)\end{aligned}$$

for $y \in \mathfrak{Q}_i^{n_i}$, where $\mathfrak{Q}_i$ is the radical of $\mathfrak{B}_i$. Then we have $c_i - c'_i \in \mathfrak{Q}_i^{-n_i+1} \cap E_i \subset c_i(\mathfrak{Q}_i \cap E_i) = c_i \mathfrak{p}_{E_i}$ and $c_i^{-1} c'_i \in 1 + \mathfrak{p}_{E_i}$. Thus $(c'_i)^{-1} c_i \in 1 + \mathfrak{p}_{E_i}$. On the other hand, c'_i is minimal relative to E_i/E_{i+1} by Proposition 7.10. Therefore, by Lemma 7.7 c_i is also minimal relative to E_i/E_{i+1}. □

Therefore, if $s \geq 0$, we can take c_i for $i = 0, 1, \ldots, s$. We put $\beta_i = \sum_{j=i}^{s} c_j$ for $i = 0, 1, \ldots, s$, $\beta = \beta_0$ and $n = -v_{\mathfrak{A}}(\beta)$. Since

$$\begin{aligned}v_{\mathfrak{A}}(c_i) &= -e(\mathfrak{A}|\mathfrak{o}_F)\,\mathrm{ord}(c_i) = -e(\mathfrak{A}|\mathfrak{o}_F)\mathbf{r}_i < -e(\mathfrak{A}|\mathfrak{o}_F)\mathbf{r}_j = -e(\mathfrak{A}|\mathfrak{o}_F)\,\mathrm{ord}(c_j) \\ &= v_{\mathfrak{A}}(c_j)\end{aligned}$$

for $i, j = 0, 1, \ldots, s$ with $i > j$, we have $n = -v_{\mathfrak{A}}(\beta_i)$ for any $i = 0, 1, \ldots, s$. We also put $r_i = -v_{\mathfrak{A}}(c_{i-1})$ for $i = 1, \ldots, s$ and $r_0 = 0$.

Proposition 11.2. — *Suppose $s \geq 0$.*

1. $E_i = F[\beta_i]$ *for* $i = 0, 1, \ldots, s$. *In particular,* $[\mathfrak{A}, n, 0, \beta]$ *is a simple stratum.*
2. $([\mathfrak{A}, n, r_i, \beta_i])_{i=0}^{s}$ *is a defining sequence of* $[\mathfrak{A}, n, 0, \beta]$.

Proof. — First, suppose $\mathfrak{A} = \mathfrak{A}(E_0)$. We will show this proposition by downward induction on i.

If $i = s$, then $\beta_s = c_s$ is minimal over F. Therefore, for any $r' \in \{0, 1, \ldots, n-1\}$, the stratum $[\mathfrak{A}, n, r', \beta_s]$ is simple. The equation $E_s = F[\beta_s]$ trivially holds. If $s = 0$, then $([\mathfrak{A}, n, r_i, \beta_i])_{i=0}^{0}$ is a defining sequence of $[\mathfrak{A}, n, 0, \beta]$, and this proposition holds. If $s > 0$, we have $r_s = -v_{\mathfrak{A}}(c_{s-1}) < -v_{\mathfrak{A}}(c_s)$. We prove by downward induction on i_0 that $([\mathfrak{A}, n, r_{j+i_0}, \beta_{j+i_0}])_{j=0}^{s-i_0}$ is a defining sequence of a simple stratum $[\mathfrak{A}, n, r_{i_0}, \beta_{i_0}]$. For $i_0 = s$, the stratum $[\mathfrak{A}, n, r_s, \beta_s]$ is simple, and $([\mathfrak{A}, n, r_{i+s}, \beta_{i+s}])_{i=0}^{0}$ is a defining sequence of $[\mathfrak{A}, n, r_s, \beta_s]$.

Let $i_0 \in \{0, 1, \ldots, s-1\}$ and suppose that $E_i = F[\beta_i]$ and that $([\mathfrak{A}, n, r_{j+i}, \beta_{j+i}])_{j=0}^{s-i}$ is a defining sequence of a simple stratum $[\mathfrak{A}, n, r_i, \beta_i]$ for any integer i with $i_0 < i \leq s$. The element c_{i_0} is minimal over E_{i_0+1}. Since $r_{i_0+1} = -v_{\mathfrak{A}}(c_{i_0})$, a 4-tuple $[\mathfrak{B}_{\beta_{i_0}+1}, r_{i_0+1}, r_{i_0+1} - 1, c_{i_0}]$ is a simple stratum, where $\mathfrak{B}_{\beta_{i_0+1}} = \mathfrak{A} \cap \mathrm{Cent}_{A(E_0)}(\beta_{i_0+1})$. Moreover, $c_{i_0} \notin E_{i_0+1} = F[\beta_{i_0+1}]$. Therefore, by Proposition 4.4, we have $F[\beta_{i_0}] = F[\beta_{i_0+1}, c_{i_0}] = E_{i_0+1}[c_{i_0}]$ and $[\mathfrak{A}, n, r_{i_0+1}, \beta_{i_0}]$ is a pure stratum with $k_0(\beta_{i_0}, \mathfrak{A}) = -r_{i_0+1}$, where $F[\beta_{i_0+1}, c_{i_0}] = E_{i_0+1}[c_{i_0}]$ follows from our induction hypothesis. If $i_0 > 0$, we have $r_{i_0} = -v_{\mathfrak{A}}(c_{i_0-1}) < -v_{\mathfrak{A}}(c_{i_0}) = r_{i_0+1}$ and $[\mathfrak{A}, n, r_{i_0}, \beta_{i_0}]$ is a simple stratum. Since $([\mathfrak{A}, n, r_{j+i_0+1}, \beta_{j+i_0+1}])_{j=0}^{s-i_0-1}$ is a defining sequence of a simple stratum $[\mathfrak{A}, n, r_{i_0+1}, \beta_{i_0+1}]$ by our induction hypothesis, $([\mathfrak{A}, n, r_{j+i_0}, \beta_{j+i_0}])_{j=0}^{s-i_0}$ is also a defining sequence of a simple stratum $[\mathfrak{A}, n, r_{i_0}, \beta_{i_0}]$. If $i_0 = 0$, then $[\mathfrak{A}, n, 0, \beta]$ is simple, and we can show $([\mathfrak{A}, n, r_i, \beta_i])_{i=0}^{s}$ is also a defining sequence of a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in the same way as above. Then the proposition for $\mathfrak{A} = \mathfrak{A}(E_0)$ case holds.

We will show the proposition in general case. Since $\beta_i \in E_i \subset E_0$ for $i = 0, \ldots, s$, we can regard β_i as in $A(E_0)$. Then (1) follows from the proposition for $\mathfrak{A} = \mathfrak{A}(E_0)$ case. Moreover, if we put $n' = -v_{\mathfrak{A}(E_0)}(\beta)$, $r'_0 = 0$ and $r'_i = -v_{\mathfrak{A}(E_0)}(c_{i-1})$ for $i = 1, \ldots, s$, then $([\mathfrak{A}(E_0), n', r'_i, \beta_i])_{i=0}^{s}$ is a defining sequence of a simple type $[\mathfrak{A}(E_0), n', 0, \beta]$ by the proposition for $\mathfrak{A} = \mathfrak{A}(E_0)$ case. Since for $c \in E_0$ we have $v_{\mathfrak{A}}(c) = e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}v_{\mathfrak{A}(E_0)}(c)$; we also have

$$n = -v_{\mathfrak{A}}(\beta) = -e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}v_{\mathfrak{A}(E_0)}(\beta) = e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}n'$$

and

$$r_i = -v_{\mathfrak{A}}(c_{i-1}) = -e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}v_{\mathfrak{A}(E_0)}(c_{i-1}) = e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}r'_i$$

for $i = 1, \ldots, s$. Since $([\mathfrak{A}(E_0), n', r'_i, \beta_i])_{i=0}^{s}$ is a defining sequence of a simple type $[\mathfrak{A}(E_0), n', 0, \beta]$, we have $r'_i = -k_0(\beta_{i-1}, \mathfrak{A}(E_0))$ for $i = 1, \ldots, s$. We also have $k_0(c, \mathfrak{A}) = e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}k_0(c, \mathfrak{A}(E_0))$ by Lemma 2.8, whence

$$\begin{aligned} r_i &= e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}r'_i = -e(\mathfrak{A}|\mathfrak{o}_F)e(E_0/F)^{-1}k_0(\beta_{i-1}, \mathfrak{A}(E_0)) \\ &= -k_0(\beta_{i-1}, \mathfrak{A}) \end{aligned}$$

for $i = 1, \ldots, s$. Then by Proposition 4.5 strata $[\mathfrak{A}, n, r_i, \beta_i]$ are simple and equivalent to $[\mathfrak{A}, n, r_i, \beta_{i-1}]$ for $i = 1, \ldots, s$. Therefore, (2) holds. □

Then we have a simple stratum $[\mathfrak{A}, n, 0, \beta]$ with a defining sequence $([\mathfrak{A}, n, r_i, \beta_i])_{i=0}^{s}$ if $s \geq 0$. If $s = -\infty$, we take a simple stratum $[\mathfrak{A}, 0, 0, \beta]$ with $\mathfrak{A}$ maximal and $c_0 = \beta_0 = \beta \in \mathfrak{o}_F^{\times}$, and then we can define subgroups $H^1(\beta, \mathfrak{A})$ and $J(\beta, \mathfrak{A})$ in G for any case. Moreover, since $\mathfrak{B}_0$ is maximal, we also can define $\hat{J}(\beta, \mathfrak{A}) = \mathfrak{K}(\mathfrak{B}_0)J(\beta, \mathfrak{A})$.

PROPOSITION 11.3. —

1. *We have*
   (a) $G^0(F)_{[x]} = \mathfrak{K}(\mathfrak{B}_0)$,
   (b) $G^0(F)_x = B_0^\times \cap \mathbf{U}(\mathfrak{A}) = \mathbf{U}(\mathfrak{B}_0)$,
   (c) $G^0(F)_{x,0+} = B_0^\times \cap \mathbf{U}^1(\mathfrak{A})$,
   (d) $\mathfrak{g}^0(F)_x = B_0 \cap \mathfrak{A} = \mathfrak{B}_0$, *and*
   (e) $\mathfrak{g}^0(F)_{x,0+} = B_0 \cap \mathfrak{P}$.
2. *For* $i = 1, \ldots, d$, *we have*
   (a) $G^i(F)_{x,\mathbf{s}_{i-1}} = B_i^\times \cap \mathbf{U}^{\lfloor(-v_{\mathfrak{A}}(c_{i-1})+1)/2\rfloor}(\mathfrak{A})$,
   (b) $G^i(F)_{x,\mathbf{s}_{i-1}+} = B_i^\times \cap \mathbf{U}^{\lfloor -v_{\mathfrak{A}}(c_{i-1})/2\rfloor+1}(\mathfrak{A})$,
   (c) $G^i(F)_{x,\mathbf{r}_{i-1}} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_{i-1})}(\mathfrak{A})$,
   (d) $G^i(F)_{x,\mathbf{r}_{i-1}+} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_{i-1})+1}(\mathfrak{A})$,
   (e) $\mathfrak{g}^i(F)_{x,\mathbf{s}_{i-1}} = B_i \cap \mathfrak{P}^{\lfloor(-v_{\mathfrak{A}}(c_{i-1})+1)/2\rfloor}$,
   (f) $\mathfrak{g}^i(F)_{x,\mathbf{s}_{i-1}+} = B_i \cap \mathfrak{P}^{\lfloor -v_{\mathfrak{A}}(c_{i-1})/2\rfloor+1}$,
   (g) $\mathfrak{g}^i(F)_{x,\mathbf{r}_{i-1}} = B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_{i-1})}$, *and*
   (h) $\mathfrak{g}^i(F)_{x,\mathbf{r}_{i-1}+} = B_i \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_{i-1})+1}$.
3. *For* $i = 0, \ldots, s$, *we have*
   (a) $G^i(F)_{x,\mathbf{s}_i+} = B_i^\times \cap \mathbf{U}^{\lfloor -v_{\mathfrak{A}}(c_i)/2\rfloor+1}(\mathfrak{A})$,
   (b) $G^i(F)_{x,\mathbf{r}_i} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_i)}(\mathfrak{A})$,
   (c) $G^i(F)_{x,\mathbf{r}_i+} = B_i^\times \cap \mathbf{U}^{-v_{\mathfrak{A}}(c_i)+1}(\mathfrak{A})$,
   (d) $\mathfrak{g}^i(F)_{x,\mathbf{r}_i} = B \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)}$, *and*
   (e) $\mathfrak{g}^i(F)_{x,\mathbf{r}_i+} = B \cap \mathfrak{P}^{-v_{\mathfrak{A}}(c_i)}$.

Proof. — Similar to the proof of Proposition 10.1. □

PROPOSITION 11.4. —

1. $K_+^d = H^1(\beta, \mathfrak{A})$.
2. ${}^\circ K^d = J(\beta, \mathfrak{A})$.
3. $K^d = \hat{J}(\beta, \mathfrak{A})$.

Proof. — Similar to the proof of Proposition 10.2. □

Next, we construct a simple character in $\mathscr{C}(\beta, 0, \mathfrak{A})$ from $(\mathbf{\Phi}_i)_i$.

LEMMA 11.5. — *Suppose $s \geq 0$. For $i = 0, 1, \ldots, s$, the following assertions hold.*

1. $\hat{\mathbf{\Phi}}_i|_{B_i^\times \cap H^1(\beta,\mathfrak{A})}$ *factors through* Nrd_{B_i/E_i}.
2. $\hat{\mathbf{\Phi}}_i|_{H^{t_i+1}(\beta,\mathfrak{A})} = \psi_{c_i}$*, where* $t_i = \lfloor -v_{\mathfrak{A}}(c_i)/2 \rfloor$.
3. $H^{t_i+1}(\beta,\mathfrak{A}) = H^{t_i+1}(\beta_i,\mathfrak{A})$ *is normalized by* $B_i^\times \cap \mathfrak{K}(\mathfrak{A})$.
4. *For any* $g \in B_i^\times \cap \mathfrak{K}(\mathfrak{A})$ *and* $h \in H^1(\beta,\mathfrak{A}) \cap {}^g H^1(\beta,\mathfrak{A})$*, we have* $\hat{\mathbf{\Phi}}_i(g^{-1}hg) = \hat{\mathbf{\Phi}}_i(h)$.

Proof. — We have $B_i^\times \cap H^1(\beta,\mathfrak{A}) = G^i(F) \cap K^d$. By construction of $\hat{\mathbf{\Phi}}_i$ we have $\hat{\mathbf{\Phi}}_i|_{B_i^\times \cap H^1(\beta,\mathfrak{A})} = \hat{\mathbf{\Phi}}_i|_{G^i(F)\cap K^d} = \mathbf{\Phi}_i$. The map $\mathbf{\Phi}_i$ is a character of $G^i(F)$, and then $\mathbf{\Phi}_i$ factors through Nrd_{B_i/E_i}, and (1) holds.

We also have $H^{t_i+1}(\beta,\mathfrak{A}) = K^d \cap G(F)_{x,\mathbf{s}_i+}$. Since $\mathbf{\Phi}_i|_{G^i(F)_{x,\mathbf{s}_i+:\mathbf{r}_i+}}$ is realized by $X^*_{c_i}$ by Proposition 11.1 (2) or (3), we have

$$\mathbf{\Phi}_i(1+y) = \psi \circ \mathrm{Tr}_{E_i/F} \circ \mathrm{Trd}_{B_i/E_i}(c_i y) = \psi \circ \mathrm{Trd}_{A/F}(c_i y)$$

for $y \in B_i \cap \mathfrak{P}^{t_i+1} = \mathfrak{g}^i(F)_{x,\mathbf{s}_i+}$. We recall that $\pi_i : \mathfrak{g}(F) = \mathfrak{g}^i(F) \oplus \mathfrak{n}^i(F) \to \mathfrak{g}^i(F)$ is the projection and

$$\hat{\mathbf{\Phi}}_i(1+y) = \mathbf{\Phi}_i(1+\pi_i(y)) = \psi \circ \mathrm{Trd}_{A/F}(c_i \pi_i(y))$$

for $1 + y \in K^d \cap G(F)_{x,\mathbf{s}_i+} = H^{t_i+1}(\beta,\mathfrak{A})$. However, we also can show $\mathrm{Trd}_{A/F}(c_i\pi_i(y)) = \mathrm{Trd}_{A/F}(c_i y)$ as in the proof of Proposition 10.4. In conclusion, for $1+y \in H^{t_i+1}(\beta,\mathfrak{A})$ we obtain $\hat{\mathbf{\Phi}}_i(1+y) = \psi \circ \mathrm{Trd}_{A/F}(c_i y) = \psi_{c_i}(y)$, and (2) holds.

Let $g \in B_i^\times \cap \mathfrak{K}(\mathfrak{A})$. We check that g normalizes $H^{t_i+1}(\beta,\mathfrak{A})$. We consider two cases. First, suppose $i < d$. Then we have $H^{t_i+1}(\beta,\mathfrak{A}) = G^{i+1}(F)_{x,\mathbf{s}_i+} \cdots G^d(F)_{x,\mathbf{s}_{d-1}+}$. Thus it suffices to show g normalizes $G^j(F)_{x,\mathbf{s}_{j-1}+}$ for $j = i+1, \ldots, d$. However, we have

$$\begin{aligned} gG^j(F)_{x,\mathbf{s}_{j-1}+}g^{-1} &= g\left(B_j^\times \cap \mathbf{U}^{t_{j-1}+1}(\mathfrak{A})\right)g^{-1} \\ &= (gB_j^\times g^{-1}) \cap (g\mathbf{U}^{t_{j-1}+1}(\mathfrak{A})g^{-1}). \end{aligned}$$

Since $g \in B_i^\times \subset B_j^\times$ we have $gB_j^\times g^{-1} = B_j^\times$. Moreover, we also have $g\mathbf{U}^{t_{j-1}+1}(\mathfrak{A})g^{-1} = \mathbf{U}^{t_{j-1}+1}(\mathfrak{A})$ as $g \in \mathfrak{K}(\mathfrak{A})$. Therefore, we obtain $gG^j(F)_{x,\mathbf{s}_j+}g^{-1} = B_j^\times \cap \mathbf{U}^{t_{j-1}+1}(\mathfrak{A}) = G^i(F)_{x,\mathbf{s}_j+}$. Next, suppose $i = d = s$. Then we have $H^{t_s+1}(\beta,\mathfrak{A}) = G^d(F)_{x,\mathbf{s}_s+} = \mathbf{U}^{t_s+1}(\mathfrak{A})$. Since $g \in \mathfrak{K}(\mathfrak{A})$, we obtain

$$gH^{t_s+1}(\beta,\mathfrak{A})g^{-1} = g\mathbf{U}^{t_s+1}(\mathfrak{A})g^{-1} = \mathbf{U}^{t_s+1}(\mathfrak{A}) = H^{t_s+1}(\beta,\mathfrak{A}).$$

Therefore, we obtain (3).

Here, let g be as above and $h \in H^1(\beta,\mathfrak{A})$. Since

$$H^1(\beta,\mathfrak{A}) = \left(B_i^\times \cap H^1(\beta,\mathfrak{A})\right) H^{t_i+1}(\beta,\mathfrak{A}),$$

we have $h = bh'$ for some $b \in B_i^\times \cap H^1(\beta, \mathfrak{A})$ and $h' \in H^{t_i+1}(\beta, \mathfrak{A})$. By the above argument, we have $h' \in H^{t_i+1}(\beta, \mathfrak{A}) = gH^{t_i+1}(\beta, \mathfrak{A})g^{-1}$, and h' is an element in $H^1(\beta, \mathfrak{A}) \cap gH^1(\beta, \mathfrak{A})g^{-1}$. Then, $h \in H^1(\beta, \mathfrak{A}) \cap gH^1(\beta, \mathfrak{A})g^{-1}$ if and only if $b \in H^1(\beta, \mathfrak{A}) \cap gH^1(\beta, \mathfrak{A})g^{-1}$. Suppose $h \in H^1(\beta, \mathfrak{A}) \cap gH^1(\beta, \mathfrak{A})g^{-1}$. Therefore, we obtain

$$\begin{aligned}\hat{\mathbf{\Phi}}_i(g^{-1}hg) &= \hat{\mathbf{\Phi}}_i\left((g^{-1}bg)(g^{-1}h'g)\right) \\ &= \hat{\mathbf{\Phi}}_i(g^{-1}bg)\hat{\mathbf{\Phi}}_i(g^{-1}h'g) = \mathbf{\Phi}_i(g^{-1}bg)\psi_{c_i}(g^{-1}h'g).\end{aligned}$$

Here, since $\mathbf{\Phi}_i$ is a character of $G^i(F) = B_i^\times$ and $g \in B_i^\times$, we have $\mathbf{\Phi}_i(g^{-1}bg) = \mathbf{\Phi}_i(b)$. Moreover, since c_i is an element in E_i, which is the center of B_i, we also have

$$\begin{aligned}\psi_{c_i}(g^{-1}h'g) &= \psi \circ \mathrm{Trd}_{A/F}(c_i g^{-1}h'g) = \psi \circ \mathrm{Trd}_{A/F}(g^{-1}c_i h'g) \\ &= \psi \circ \mathrm{Trd}_{A/F}(c_i h') = \psi_{c_i}(h').\end{aligned}$$

Therefore, we obtain $\hat{\mathbf{\Phi}}_i(g^{-1}hg) = \mathbf{\Phi}_i(b)\psi_{c_i}(h') = \hat{\mathbf{\Phi}}_i(bh') = \hat{\mathbf{\Phi}}_i(h)$, which implies (4). □

Proposition 11.6. — *We have* $\prod_{i=0}^d \hat{\mathbf{\Phi}}_i \in \mathscr{C}(\beta, 0, \mathfrak{A})$.

Proof. — If $s = -\infty$, then $\mathbf{\Phi}_d = 1$ and $\hat{\mathbf{\Phi}}_d = 1$, and then $\prod_{i=0}^d \hat{\mathbf{\Phi}}_i = 1 \in \mathscr{C}(\beta, 0, \mathfrak{A})$. Therefore, we assume $s \in \mathbb{Z}$. If $d = s+1$, then $\mathbf{\Phi}_d = 1$ and $\hat{\mathbf{\Phi}}_i = 1$, and we have $\prod_{j=i}^d \hat{\mathbf{\Phi}}_j = \prod_{j=i}^s \hat{\mathbf{\Phi}}_j$ for $i = 0, \ldots, s$. Thus we show $\bar{\theta}_i := \prod_{j=i}^s \hat{\mathbf{\Phi}}_j|_{H^{t_j+1}(\beta, \mathfrak{A})} \in \mathscr{C}(\beta_i, \lfloor r_i/2 \rfloor, \mathfrak{A})$ by downward induction on $i = 0, \ldots, s$.

First, suppose $i = s$. Since $\beta_s = c_s$ is minimal over F, we need to check (1), (2) and (3) in Definition 2.12. (2) is already shown as Lemma 11.5 (1). Since $-v_{\mathfrak{A}}(c_s) = -v_{\mathfrak{A}}(\beta_s) = n$, we have $H^{t_s+1}(\beta, \mathfrak{A}) = \mathbf{U}^{\lfloor n/2 \rfloor + 1}(\mathfrak{A})$, and (3) is also shown as Lemma 11.5 (2). Let $g \in B_i^\times \cap \mathfrak{K}(\mathfrak{A})$ and $h \in H^{t_s+1}(\beta, \mathfrak{A})$. Then $g^{-1}hg \in H^{t_s+1}(\beta, \mathfrak{A})$ by Lemma 11.5 (3), and $\hat{\mathbf{\Phi}}_i(g^{-1}hg) = \hat{\mathbf{\Phi}}_i(h)$ by Lemma 11.5 (4), which implies (1). Therefore, $\hat{\mathbf{\Phi}}_s \in \mathscr{C}(\beta_s, t_s, \mathfrak{A})$.

Next, suppose $0 < i < s$. Since $k_0(\beta_{i-1}, \mathfrak{A}) = v_{\mathfrak{A}}(c_{i-1}) = -r_i > -n = v_{\mathfrak{A}}(\beta_{i-1})$, the element β_{i-1} is not minimal over F, and then we need to check (1), (2) and (4) in Definition 2.12.

To show (1), let $g \in B_{i-1}^\times \cap \mathfrak{K}(\mathfrak{A})$ and $h \in H^{t_{i-1}+1}(\beta, \mathfrak{A})$. Then $g^{-1}hg \in H^{t_{i-1}+1}(\beta, \mathfrak{A})$ by Lemma 11.5 (3). For $j = i-1, \ldots, s$, we have $g \in B_{i-1}^\times \cap \mathfrak{K}(\mathfrak{A}) \subset B_j^\times \cap \mathfrak{K}(\mathfrak{A})$. Therefore, by Lemma 11.5 (4) we have $\hat{\mathbf{\Phi}}_j(g^{-1}hg) = \hat{\mathbf{\Phi}}_j(h)$ and $\bar{\theta}_{i-1}(g^{-1}hg) = \prod_{j=i-1}^s \hat{\mathbf{\Phi}}_j(g^{-1}hg) = \prod_{j=i-1}^s \hat{\mathbf{\Phi}}_j(h) = \bar{\theta}_{i-1}(h)$, whence (1) holds.

For $j = i-1, \ldots, s$, the restriction of $\hat{\mathbf{\Phi}}_j$ to $B_j^\times \cap H^{t_{i-1}+1}(\beta, \mathfrak{A})$ factors through Nrd_{B_j/E_j}. Since $\mathrm{Nrd}_{B_j/E_j}|_{B_{i-1}^\times} = \mathrm{N}_{E_{i-1}/E_j} \circ \mathrm{Nrd}_{B_{i-1}/E_{i-1}}$, the restriction of $\hat{\mathbf{\Phi}}_j$

to $B_{i-1}^{\times}\cap H^{t_{i-1}+1}(\beta,\mathfrak{A})$ factors through $\mathrm{Nrd}_{B_{i-1}/E_{i-1}}$. Then the character $\bar{\theta}_{i-1}=\prod_{j=i-1}^{s}\hat{\mathbf{\Phi}}_j|_{B_{i-1}^{\times}\cap H^{t_{i-1}+1}(\beta,\mathfrak{A})}$ also factors through $\mathrm{Nrd}_{B_{i-1}/E_{i-1}}$, and (2) holds.

We show (4). We put $r'_{i-1}=0$ and $r'_j=r_i$ for $j=i,\dots,s$. Then the sequence $([\mathfrak{A},n,r'_{(i-1)+i'},\beta_{(i-1)+i'}])_{i'=0}^{s-i+1}$ is a defining sequence of $[\mathfrak{A},n,0,\beta_{i-1}]$. Since $-k_0(\beta_{i-1},\mathfrak{A})=r_i$, we have $\max\{\lfloor r_{i-1}/2\rfloor,\lfloor -k_0(\beta_{i-1},\mathfrak{A})/2\rfloor\}=\lfloor r_i/2\rfloor=t_{i-1}$. Then $\bar{\theta}_{i-1}|_{H^{t_{i-1}+1}(\beta,\mathfrak{A})}=\bar{\theta}_i\hat{\mathbf{\Phi}}_{i-1}|_{H^{t_{i-1}+1}(\beta,\mathfrak{A})}$. The character $\bar{\theta}_i$ is an element in $\mathscr{C}(\beta_i,\lfloor r_i/2\rfloor,\mathfrak{A})$ by induction hypothesis. On the other hand, $\hat{\mathbf{\Phi}}_{i-1}|_{H^{t_{i-1}+1}(\beta,\mathfrak{A})}=\psi_{c_{i-1}}$ by Lemma 11.5 (2). Therefore, (4) is shown, and we complete the proof. □

We put $\theta=\prod_{i=0}^{d}\hat{\Phi}_i$, and let η_θ be the Heisenberg representation of θ.

Proposition 11.7. — *$\kappa_0\otimes\cdots\otimes\kappa_d$ is an extension of η_θ to K^d.*

Proof. — Similar to the proof of Proposition 10.5. □

Theorem 11.8. — *Let $\big(x,(G^i)_{i=0}^d,(\mathbf{r}_i)_{i=0}^d,(\mathbf{\Phi}_i)_{i=0}^d,\rho\big)$ be a Yu datum. Then there exists a maximal, tame simple type (J,λ) associated with $[\mathfrak{A},n,0,\beta]$ and a maximal extension $(\tilde{J},\Lambda)$ of (J,λ) such that*

1. *$\hat{J}:=\hat{J}(\beta,\mathfrak{A})=K^d$, and*
2. *$\rho_d=\text{c-Ind}_{\tilde{J}}^{\hat{J}}\Lambda$.*

Proof. — We can construct a tame simple stratum $[\mathfrak{A},n,0,\beta]$ and a simple character $\theta\in\mathscr{C}(\beta,\mathfrak{A})$ as above. We take a β-extension κ of η_θ and an extension $\hat{\kappa}$ of κ to $\hat{J}$ by Lemma 8.4 (1). On the other hand, let κ_i be the representation of K^d as in Section 3 for $i=-1,0,\dots,d$. By Proposition 11.7, the representation $\hat{\kappa}'=\kappa_0\otimes\cdots\otimes\kappa_d$ is an extension of a β-extension $^\circ\lambda$ of η_θ to K^d. Then by Lemma 8.4 (2), there exists a character χ of $\hat{J}/J^1(\beta,\mathfrak{A})$ such that $\hat{\kappa}'\cong\hat{\kappa}\otimes\chi$. The representation κ_{-1} is the extension of ρ to K^d, trivial on $K^d\cap G(F)_{x,0+}=J^1(\beta,\mathfrak{A})$.

We construct "depth-zero part" σ of a simple type from ρ. By Lemma 8.6, there exists a depth-zero simple type $(G^0(F)_x,\sigma^0)$ of $G^0(F)$ and a maximal extension $(\tilde{J}^0,\tilde{\sigma}^0)$ such that $\rho\cong\mathrm{Ind}_{\tilde{J}^0}^{G^0(F)_{[x]}}\tilde{\sigma}^0$. We put $\tilde{J}=\tilde{J}^0J=\tilde{J}^0J^1(\beta,\mathfrak{A})$. Since $J^1(\beta,\mathfrak{A})\cap G^0(F)=G^0(F)_{x,0+}$, we have $\tilde{J}^0/G^0(F)_{x,0+}\cong\tilde{J}/J^1(\beta,\mathfrak{A})$ and we can extend $\tilde{\sigma}^0$ to $\tilde{J}$ as $\tilde{\sigma}$, which is trivial on $J^1(\beta,\mathfrak{A})$. We put $\sigma=\mathrm{Res}_J^{\tilde{J}}\tilde{\sigma}$. The representation σ is an extension of σ^0 to J, trivial on $J^1(\beta,\mathfrak{A})$. Since $(G^0(F)_x,\sigma^0)$ is a maximal simple type of depth zero, and χ is a character of $\hat{J}$ trivial on $J^1(\beta,\mathfrak{A})$, the $J(\beta,\mathfrak{A})/J^1(\beta,\mathfrak{A})$-representation $\sigma\otimes\chi$ is cuspidal, and then $(J,\sigma\otimes\chi\otimes\kappa)$ is a simple type. By construction of $\tilde{J}$ and $\tilde{\sigma}$, the pair $(\tilde{J},\tilde{\sigma}\otimes\mathrm{Res}_{\tilde{J}}^{\hat{J}}(\chi\otimes\hat{\kappa}))$ is a maximal extension of $(J,\sigma\otimes\chi\otimes\kappa)$. We put $\Lambda=\tilde{\sigma}\otimes\mathrm{Res}_{\tilde{J}}^{\hat{J}}(\chi\otimes\hat{\kappa})$.

The representation κ_{-1} is the extension of ρ as κ_{-1} is trivial on $K^0_+ J^1 \cdots J^d = J^1(\beta, \mathfrak{A})$, that is, the representation κ_{-1} is ρ regarded as a representation of $K^d = \hat{J}$ via $K^0/K^0_+ = \mathfrak{K}(\mathfrak{B})/\mathbf{U}^1(\mathfrak{B}) \cong K^d/(K^0_+ J^1 \cdots J^d) = \hat{J}/J^1(\beta, \mathfrak{A})$. Then we have $\kappa_{-1} \cong \text{c-Ind}_{\tilde{J}}^{\hat{J}} \tilde{\sigma}$ by Lemma 8.3 and

$$\text{c-Ind}_{\tilde{J}}^{\hat{J}} \Lambda \cong (\text{c-Ind}_{\tilde{J}}^{\hat{J}} \tilde{\sigma}) \otimes \chi \otimes \hat{\kappa} \cong \kappa_{-1} \otimes \kappa_0 \otimes \cdots \otimes \kappa_d = \rho_d,$$

which finishes the proof. □

Corollary 11.9. — *The set of essentially tame supercuspidal representations of G is equal to the set of tame supercuspidal representations of G.*

Proof. — Let π be an irreducible supercuspidal representation of G. Since $\text{c-Ind}_{K^d(\Psi)}^{G} \rho^d(\Psi)$ is irreducible for any Yu's datum Ψ, π is tame supercuspidal if and only if $\pi \supset \rho^d(\Psi)$ for some Ψ. However, by Theorems 10.6 and 11.8 it holds if and only if π contains some compact induction of a maximal extension $(\tilde{J}, \Lambda)$ of a tame, maximal simple type, which is equivalent to π is essentially tame by 4.2. □

Remark 11.10. — In the condition of Theorems 10.6 or 11.8, suppose $G = \mathrm{GL}_N(F)$. Then we have $\tilde{J} = \tilde{J}(\lambda) = \hat{J}(\beta, \mathfrak{A})$ by Remark 2.19 (3). Therefore, $\tilde{J} = K^d(\Psi)$ and $\text{c-Ind}_{\tilde{J}}^{K^d(\Psi)} \Lambda = \Lambda$, which leads to Theorem 1.4.

12. Wild case

Let $[\mathfrak{A}, n, 0, \beta]$ be a Bushnell–Kutzko simple stratum. For the purpose of the paper, we assumed that $F[\beta]/F$ is tamely ramified.

Remark 12.1. — (cf. also [20])

1. If we remove the assumption that $F[\beta]/F$ is tame in our fixed simple stratum $[\mathfrak{A}, n, 0, \beta]$, then the sequence of fields $E_0, \dots, E_s$ attached to a defining sequence can not be chosen decreasing for $\subset$, in general. It always decreases for $[\bullet : F]$.
2. In a certain sense, we have explained that Bushnell–Kutzko and Sécherre's constructions are compatible with Yu's construction as they essentially are the same on their common domain of definition. Does there exist a construction generalizing both of them in a single formalism?
3. If one tries to obtain generalization of these approaches, one has to remove (among other things) the axiom of inclusions in the twisted Levi sequence by (1) of this remark and by definition of $\overrightarrow{G}$. This implies that one can not expect a factorable construction $\rho_d = \otimes \kappa^i$ as Yu's one.

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