RIEMANN–ROCH POLYNOMIALS OF THE KNOWN HYPERKÄHLER MANIFOLDS

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With an Appendix by Yalong Cao and Chen Jiang

Abstract. — We compute explicit formulas for the Euler characteristic of line bundles in the two exceptional examples of Hyperkähler Manifolds introduced by O'Grady. In an Appendix, Chen Jiang and Yalong Cao use our formulas to compute the Chern numbers of the example of O'Grady in dimension 10.

Résumé (*Polynômes de Riemann-Roch pour les variétés hyperkählériennes connues*). — Nous calculons des formules explicites pour la caractéristique d'Euler de fibrés en droites pour les deux exemples exceptionnels de variétés hyperkählériennes introduits par O'Grady. Dans un appendice, Chen Jiang et Yalong Cao utilisent nos formules pour calculer le nombre de Chern de l'exemple d'O'Grady en dimension 10.

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1. Introduction

A compact Kähler manifold is called *hyperkähler* (HK) if it is simply connected and carries a holomorphic symplectic form that spans $H^{2,0}$. HK manifolds can be thought of as the higher-dimensional analogues of $K3$ surfaces, and they constitute one of the three fundamental classes of varieties with vanishing first Chern class [1].

Although any two $K3$ surfaces are deformation equivalent, this fact no longer holds in higher dimensions. The first two series of examples of deformation types in each (necessarily even) dimension were described by Beauville [1]: the first series, denoted by $K3^{[n]}$, is given by the Hilbert scheme of n points in a K3 surface. The other one is a submanifold in the Hilbert scheme of n points in an abelian surface. Generalizing the construction of a Kummer surface, this ($2n$-dimensional) deformation type is denoted by Kum_n.

Later, O'Grady introduced two new deformation types in dimensions 6 and 10 ([24],[25]), now denoted by OG6 and OG10, respectively. The construction of both exceptional examples is done by resolving a singular moduli space of sheaves on a K3 surface for OG10 and an abelian surface for OG6. In view of these analogies, it is expected that the projective geometry of HK manifolds of $K3^{[5]}$-type (respectively Kum_3-type) should be related with that of OG10-type (respectively OG6-type).

The main result of this paper (cf. Theorem 2.6) gives, for the HK manifolds described by O'Grady, closed formulas that compute the Euler characteristic of any line bundle in terms of numerical polynomials that only depend on the Beauville–Bogomolov form—a canonical quadratic form in the second cohomology group of any HK. Surprisingly the formulas turn out to be exactly the same as those of the series described by Beauville.

In order to compute these polynomials we use two different methods. The first one exploits a recent description in [15] of OG10 as a compactification of a fibration associated with a cubic 4-fold. The second one is based on the explicit descriptions of some uniruled divisors in two different models of OG6 given in [17] and [21].

Observe that in [3], the authors give a closed formula for the Riemann–Roch polynomial of OG6 in terms of the so-called λ-invariant; in our work the closed formula is obtained directly.

Finally, we would like to point out the very recent paper [14], where Chen Jiang proves the positivity of the coefficients of the Riemann–Roch polynomial for HK manifolds in general.

2. Preliminaries

Let X be an HK manifold of dimension $2n$ and q_X its Beauville–Bogomolov form [1]. Recall that the Fujiki constant c_X is defined as the rational number

such that for all $\alpha \in H^2(X)$ we have the so-called Fujiki relation:

$$(1) \qquad \int_X \alpha^{2n} = c_X q_X(\alpha)^n$$

REMARK. — The polarized form of Fujiki's relation is

$$(2)$$
$$\int_X \alpha_1 \smile \cdots \smile \alpha_{2n} = \frac{c_X}{(2n)!} \sum_{\sigma \in S_{2n}} q_X(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \cdots q_X(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})$$

Huybrechts further generalized this relation to all polynomials in the Chern classes, more specifically, he proved the following:

THEOREM 2.1 ([9], Corollary 23.17). — *Assume* $\alpha \in H^{4j}(X, \mathbb{Q})$ *is of type* $(2k, 2k)$ *for all small deformations of* X. *Then there exists a constant* $C(\alpha) \in \mathbb{Q}$ *such that*

$$(3) \qquad \int_X \alpha \smile \beta^{2n-2k} = C(\alpha) \cdot q_X(\beta)^{n-k}$$

for all $\beta \in H^2(X, \mathbb{Q})$.

REMARK. — If we set $\alpha = 1$ in Theorem 2.1 we obtain the Fujiki relation (1) and also that $a_n = c_X$.

The odd Chern classes (hence the odd Todd classes) of X vanish since the symplectic form on X induces an isomorphism between T_X and its dual. The Todd classes are topological invariants of X, so for any line bundle L in X, we combine Theorem 2.1 with Hirzebruch–Riemann–Roch theorem to get

$$(4) \quad \chi(X, L) = \sum_{i=0}^{n} \frac{1}{(2i)!} \int_X Td_{2n-2i}(X) \smile c_1(L)^{2i} = \sum_{i=0}^{n} \frac{a_i}{(2i)!} \cdot q_X(L)^i$$

where $a_i := C(Td_{2n-2i}(X))$.

DEFINITION 2.2 (*Huybrechts, Nieper–Wißkirchen, Riess*). — The Riemann–Roch polynomial of X, denoted by $RR_X(t)$, is the polynomial

$$RR_X(t) = \sum_{i=0}^{n} \frac{a_i}{(2i)!} t^i.$$

Let us list a few well-known properties of this polynomial.

LEMMA 2.3. — *Let* X *be an HK variety of dimension* $2n$. *The following properties hold:*

1. RR_X *depends only on the deformation class of* X.
2. *The constant term is* $a_0 = n + 1$.
3. *The coefficient of the highest-order term is* $a_n = c_X$ *and is positive.*
4. *The coefficient* a_{n-1} *is positive.*

Proof. — We have already observed that the Todd classes are a deformation invariant of X. Hence, each a_i (and therefore RR_X) is also a deformation invariant of X. The constant term of RR_X is the holomorphic Euler characteristic of X, this was computed [1] to be $n + 1$. The constant $a_n = C(Td_0(X))$ is given by (1) so it is equal to c_X. Observe that c_X is positive because the left-hand side of the Fujiki relation (1) is a volume form.

By the first item we can assume X to be projective. Nieper [22] computed

$$\int_X c_2(X) \smile c_1(L)^{2n-2} = \binom{2n-2}{n-1} \left(\int_X c_2(X)(\sigma\overline{\sigma})^{n-1} \right) \cdot q_X(L)^{n-1}.$$

The second Todd class of X is a positive multiple of $c_2(X)$, and if L is an ample line bundle, then $q_X(L) > 0$. Therefore, a_{n-1} is positive if and only if $\int_X c_2(X)(\sigma\overline{\sigma})^{n-1}$ is positive. Fixing an HK metric compatible with the symplectic structure, the last quantity is a positive multiple of the L^2-norm of the Riemann curvature tensor (see [22]), hence positive.   □

In view of the previous Lemma, we can speak of the Riemann–Roch polynomial for a deformation *type*. This has been done for the two series of examples introduced by Beauville.

EXAMPLE 2.4 ([7], Lemma 5.1). — Let X be an HK of $K3^{[n]}$-type, then the Riemann–Roch polynomial is given by

$$RR_X(t) = \binom{t/2 + n + 1}{n}.$$

EXAMPLE 2.5 ([22], Lemma 5.2). — Let X be an HK of Kum_n-type, then the Hilbert polynomial takes the form

$$RR_X(t) = (n+1)\binom{t/2 + n}{n}.$$

We will say that the Riemann–Roch polynomial is of $K3^{[n]}$-type or Kum_n-type if it corresponds to one of the two examples above. Now we can state precisely the main result of this section.

THEOREM 2.6. — *The Riemann–Roch polynomials for the deformation class of OG6 and OG10 are of Kum_3-type and $K3^{[5]}$-type, respectively.*

The theorem will be proved in Propositions 4.2 and 5.5 below.

3. Abelian fibered CY varieties

Let $\pi : X \to B$ be a flat surjective morphism with connected fibers between projective normal complex varieties. Denote by X_b the schematic fiber of $b \in B$. For the rest of this section, we assume that

- X has rational singularities and ω_X is trivial.
- every smooth fiber X_b is an abelian variety.

Denote by $\mathcal{O}_B(1)$ an ample line bundle on B and let $F = \pi^*(\mathcal{O}_B(1))$ be the pullback. Let L be a π-ample line bundle on X. Whenever X_b is smooth, the restriction $L_b := L|_{X_b}$ defines a polarization of the abelian variety X_b.

Recall that to any polarization on an abelian variety, one can associate a tuple of positive integers $(d_1, \ldots, d_n)$ which is called the polarization type, see [12], in the following way: Since X_b is an abelian variety, we have an identification $H^2(X_b, \mathbb{Z}) \cong \bigwedge^2 H_1(X_b, \mathbb{Z})^\vee$, hence we can interpret L_b as an alternating integral form on the lattice $H_1(X_b, \mathbb{Z})$. Therefore, we can find a basis of $H_1(X_b, \mathbb{Z})$ for which L_b has the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where $D = \operatorname{diag}(d_1, \ldots, d_n)$ is an integral diagonal matrix with $d_i > 0$ and $d_i | d_{i+1}$. We will denote by $(d_1, \ldots, d_n)$ the type of L_b. Since the morphism is flat, the type remains constant on the smooth locus of π. The following is a generalization of [29, Claim 12]:

THEOREM 3.1. — *Let L be a π-ample line bundle on X and let $(d_1, \ldots, d_n)$ be the type of L_b for a smooth fiber X_b. Then for any $m \in \mathbb{Z}$, the sheaf $\pi_*(L \otimes F^{\otimes m})$ is locally free of rank $d_1 \cdots d_n$ and all higher direct images vanish. Moreover,*

$$h^p(X_b, (L \otimes F^{\otimes m})|_{X_b}) = \begin{cases} d_1 \cdots d_n & p = 0, \\ 0 & p > 0. \end{cases}$$

Proof. — Let $k > 0$ be an integer such that $M = L \otimes F^{\otimes k}$ is ample. Let X_b be a smooth fiber, and denote by M_b the restriction of M. Then

$$h^p(X_b, M_b) = h^p(X_b, L_b) = \begin{cases} d_1 \cdots d_n & p = 0, \\ 0 & p > 0. \end{cases}$$

Therefore, the higher direct image sheaves $R^p \pi_* M$ are torsion for $p > 0$. Let $\epsilon : \widetilde{X} \to X$ be a resolution of singularities of X. Since X has rational singularities, $\epsilon_*(\omega_{\widetilde{X}}) = \omega_X = \mathcal{O}_X$ and $R^q \epsilon_* \omega_{\widetilde{X}} = 0$ for every $q > 0$. Therefore, the Grothendieck spectral sequence

$$R^p \pi_* (R^q \epsilon_* \omega_{\widetilde{X}} \otimes M) \implies R^{p+q} (\epsilon \circ \pi)_* (\omega_{\widetilde{X}} \otimes \epsilon^* M)$$

degenerates and so $R^p(\epsilon \circ \pi)_*(\omega_{\widetilde{X}} \otimes \epsilon^* M) \cong R^p \pi_*(M)$. On the other hand, the divisor $\epsilon^*(M)$ is big and nef, so Theorem 2.2. in [10] states that $R^p(\epsilon \circ \pi)_*(\omega_{\widetilde{X}} \otimes \epsilon^* M)$ is torsion free for $p \geq 0$. We conclude that $R^p \pi_*(M)$ must vanish for $p > 0$.

Theorem 12.11 of [11] states that if $H^p(X_b, M_b)$ vanishes for all $b \in B$, then the natural map

$$R^{p-1} \pi_* M \otimes_{\mathcal{O}_b} k(b) \to H^{p-1}(X_b, M_b)$$

is an isomorphism for all $b \in B$. Since $H^{n+1}(X_b, M_b)$ vanishes for all $b \in B$ by dimension reasons, and by the previous reasoning also $R^n\pi_*(M) = 0$, then $H^n(X_b, M_b)$ vanishes for all $b \in B$. Continuing by reverse induction, we find that $H^p(X_b, M_b) = 0$ for all $p > 0$ and all $b \in B$.

Finally, $\pi : X \to B$ is a flat family and

$$h^0(X_b, M_b) = \chi(X_b, M_b)$$

is topological, so for all $b \in B$, we find that $h^0(X_b, L_b)$ agrees with the value $d_1 \cdots d_n$ for a smooth fiber. Theorem 12.11 of [11] implies that the sheaf $\pi_*(M)$ is locally free of rank $d_1 \cdots d_n$ as claimed. The projection formula for higher direct images implies

$$R^p\pi_*(L \otimes F^{\otimes m}) \cong R^p\pi_*(L \otimes F^{\otimes k}) \otimes \mathcal{O}_B(m - k) \cong R^p\pi_*M \otimes \mathcal{O}_B(m - k)$$

for any $m \in \mathbb{Z}$, the proposition follows. $\qquad\square$

Now that we have proved that the sheaf π_*L is locally free, we can study its positivity properties. The next theorem closely follows Mourougane's strategy in [18]. We will need the following well-known lemma. Recall that a vector bundle $\mathcal{E}$ on a projective variety is called *nef* if the canonical line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef.

LEMMA 3.2. — *Let $\mathcal{E}$ be a vector bundle over a projective variety Y. If there exists an ample line bundle M on Y such that for all s the vector bundle $\mathcal{E}^{\otimes s} \otimes M$ is globally generated, then $\mathcal{E}$ is nef.*

Proof. — See Example 6.2.13 in [16]. $\qquad\square$

THEOREM 3.3. — *Let L be a big and nef line bundle on X that is π-ample. Then the vector bundle π_*L is nef.*

Proof. — For any integer $s > 0$ define $X^{(s)} := X \times_B \cdots \times_B X$ and let $\pi^{(s)} : X^{(s)} \to B$ be the induced map. This is a flat map because flatness is preserved under base-change. Denote by $\mathrm{pr}_i : X^{(s)} \to X$ the i-th projection. Define the line bundle $L^{(s)} := \otimes_{i=1}^s \mathrm{pr}_i^*(L)$.

CLAIM. — $\pi_*(L)^{\otimes s} \cong \pi_*^{(s)}(L)$.

PROOF OF THE CLAIM. — Indeed, we will proceed by induction, the case $s = 1$ being trivial. Use the following diagram:

(5)
$$
\begin{array}{ccc}
X^{(s)} & \xrightarrow{\ \mathrm{pr}_s\ } & X \\
{\scriptstyle p}\downarrow & {\scriptstyle \pi^{(s)}}\searrow & \downarrow{\scriptstyle \pi} \\
X^{(s-1)} & \xrightarrow[\ \pi^{(s-1)}\ ]{} & B
\end{array}
$$

where p denotes the canonical map given by base-change. Apply projection formula twice and flat base change

$$
\begin{aligned}
\pi_*^{(s)}(L^{(s)}) &= \pi_*^{(s)}(p^*(L^{(s-1)} \otimes \mathrm{pr}_s^*(L))) \\
&= \pi_*^{(s-1)}(p_*(p^*(L^{(s-1)} \otimes \mathrm{pr}_s^*(L)))) \\
&= \pi_*^{(s-1)}(L^{(s-1)} \otimes p_*(\mathrm{pr}_s^*(L))) \\
&= \pi_*^{(s-1)}(L^{(s-1)} \otimes \pi^{(s-1)^*}(\pi_*(L)) \\
&= \pi_*(L^{(s-1)}) \otimes \pi_*(L)) = \pi_*(L)^{\otimes s}.
\end{aligned}
$$

Last equality follows by induction hypothesis. $\qquad\square$

We want to apply Lemma 3.2 to our case. By Theorem 3.1 we know that $R^p \pi_* L = 0$ for $p > 0$, so the Leray spectral sequence for L degenerates and we have that

(6) $$ H^i(B, \pi_*(L \otimes F^{\otimes m})) = H^i(X, L \otimes F^{\otimes m}) = 0 $$

for all $i > 0$ and $m > 0$. By replacing $\mathcal{O}_B(1)$ with a suitable multiple we can assume it to be very ample, hence $\pi_* L$ is n-regular with respect to $\mathcal{O}_B(1)$ in the sense of Castelnuovo–Mumford. Therefore the vector bundle $\pi_* L \otimes \mathcal{O}_B(n)$ is globally generated. This is the case $s = 1$ of Lemma 3.2, for $s > 1$ is just an application of the Künneth formula. Indeed, for all $m > 0$, we have

$$ H^p(X^{(s)}, L^{(s)} \otimes \pi^{(s)^*}(\mathcal{O}_B(m))) = \bigoplus_{i+j=p} H^i(X^{(s-1)}, L^{(s-1)}) \otimes H^j(X, L \otimes F^{\otimes m}) $$

The line bundle L is big and nef, so by Kawamata–Viehweg, all of its higher cohomology groups vanish. We apply the Künneth formula once again and use equation (6) to conclude that $H^p(X^{(s)}, L^{(s)} \otimes \pi^{(s)^*}(\mathcal{O}_B(m)))$ must vanish for all $p > 0$ and $m \geq 0$. This ensures that the sheaf $\pi_{(s)_*}(\mathcal{O}_X(L))$ is n-regular. Therefore,

$$ \pi_{(s)_*}(L) \otimes \mathcal{O}_B(n) \cong \pi_*(\mathcal{O}_X(L))^{\otimes s} \otimes \mathcal{O}_B(n) $$

is also globally generated. By Lemma 3.2 the vector bundle $\pi_*(L)$ is nef. $\qquad\square$

4. Riemann–Roch polynomial for OG10

We will use the realization of OG10 constructed by Laza–Saccà–Voisin in [15], denoted by J. This has a Lagrangian fibration $\pi : J \to \mathbb{P}^5$. Let Θ be the relative theta divisor. This is π-ample by [15], Section 5. Let J_t be a smooth fiber of π and let Θ_t be the restriction of Θ to the fiber. By Theorem 3.1 we have that $\pi_* \mathcal{O}_J(\Theta)$ is locally free of rank $h^0(J_t, \Theta_t)$. The relative theta divisor is a principal polarization when restricted to any smooth fiber, hence $h^0(J_t, \Theta_t) = 1$. We obtain

PROPOSITION 4.1. — $\pi_*(\mathcal{O}_J(\Theta)) \cong \mathcal{O}_{\mathbb{P}^5}(k)$ *for some* $k \geq 0$. *In particular*

(7) $$\chi(J, \Theta + mF) = h^0(J, \Theta + mF) = \binom{k + m + 5}{5}$$

for all $m \geq 0$.

The first equality follows by Theorem 3.1. The class F is isotropic with respect to q_J, we use the polarized version of the Fujiki formula to get

$$\int_J \Theta^5 \smile F^5 = \frac{c_J}{10!} \cdot (5!)^2 \cdot 2^5 \cdot q_J(F, \Theta)^5.$$

On the other hand, we can compute the left-hand side to be

$$\int_J \Theta^5 \smile F^5 = \int_{J_t} \Theta^5|_{J_t} = \int_{J_t} \Theta_t^5 = 5!$$

Hence $q_J(\Theta, F) = 1$. In particular $q_J(\Theta + mF) = q_J(\Theta) + 2m$.

PROPOSITION 4.2. — *The Riemann–Roch polynomial of OG10 is of* $K3^{[5]}$-*type.*

Proof. — In (7) we take $t = q(\Theta + mF) = q(\Theta) + 2m$ to get

$$RR_J(t) = \binom{k + \frac{t - q_J(\Theta)}{2} + 5}{5}.$$

Evaluating at zero we get the following equation:

$$6 = \binom{k - \frac{q_J(\Theta)}{2} + 5}{5}$$

whose only rational solution is $k - \frac{q_J(\Theta)}{2} = 1$, the result follows from Example 2.4.   □

We conclude this section by showing that the same strategy can be used to compute the Riemann–Roch polynomial for HK manifolds of $K3^{[n]}$-type. Let (S, L) be a K3 surface with $\mathrm{Pic}(S)$ generated by L and let $L^2 = 2d$. By Riemann–Roch we have that $|L| \cong \mathbb{P}^{d+1}$ and every smooth curve C in the linear system is of genus $d + 1$. Let $\mathcal{C}/\mathbb{P}^{d+1}$ be the universal family of all curves linearly equivalent to C. By the hypothesis on the Picard group of S, we have that every curve in the linear system is reduced and irreducible, and therefore its compactified Jacobian is well defined as the moduli space of rank-one torsion-free sheaves on the curve of degree k. Thus, we get a fibration $\overline{\mathrm{Jac}}^k(\mathcal{C}/\mathbb{P}^{d+1}) \to \mathbb{P}^{d+1}$ whose general fiber is a $d+1$-dimensional abelian variety.

On the other hand, the moduli space $M(0, L, k - d)$ of L-stable sheaves on S with Mukai vector $(0, L, k - d)$ is a HK variety of dimension $2d + 2$ by [20, Example 0.5]. The general element of this moduli space is again a degree-k line

bundle on a smooth curve in the linear system $|L|$, thought of as a torsion sheaf on S. We can also think of the fibration structure by considering the map

$$(8) \qquad\qquad \text{Supp} : M(0, L, k - d) \longrightarrow \mathbb{P}^{d+1}$$

taking a sheaf $\mathcal{F}$ to its support $\text{Supp}\mathcal{F} \in |L|$. With the hypothesis on S, these two spaces are isomorphic. The following theorem is well known. See, for example, Section 3 in [8] for details.

THEOREM 4.3. — *Let (S, L) be a K3 surface with* $\text{Pic}(S) \cong \mathbb{Z}L$, *then all the fibers of the morphism defined in* (8) *are irreducible. Moreover, the space* $M(0, L, 0)$ *admits a canonically defined theta divisor* Θ_L *given by*

$$\Theta_L = \{\mathcal{F} \in M(0, L, 0) \text{ such that } h^0(S, \mathcal{F}) = h^1(S, \mathcal{F}) \neq 0\}$$

which is ample on each fiber and with $q(\Theta_L) = -2$.

The same proof as in the OG10 deformation type computes the Riemann–Roch polynomial, and since the dimension of $M(0, L, 0)$ is $2d + 2$, we cover all the dimensions of manifolds of K3$^{[n]}$-type.

5. Riemann–Roch polynomial for OG6

Although there does exist an HK manifold of OG6-type with a Lagrangian fibration, we cannot use the same strategy as we did for OG10 because the abelian varieties appearing as smooth fibers are not principally polarized, i.e., there is not an ample divisor restricting to a principal polarization on every smooth fiber. Hence, we use an alternative method based on the explicit description of some divisors.

Let X be an HK manifold of OG6-type. The formula (4) for X is

$$(9) \quad \chi(X, L) = a_0 + \frac{a_1}{2!}q_X(L) + \frac{a_2}{4!}q_X(L)^2 + \frac{a_3}{6!}q_X(L)^3 = RR_X(q_X(L))$$

By Lemma 2.3 we have $a_0 = 4$ and also $a_3 = c_X = 60$ by [27]. We will find divisors whose invariants reduce equation 9 to a linear system of equations. In order to do this, we will introduce the divisors $\widetilde{\Sigma}$ and $\widetilde{B}$. They are both effective divisors for the variety of OG6-type considered in [27]. Their Beauville–Bogomolov forms were already computed in Theorems 3.3.1 and 3.5.1 of [27] and the Euler characteristic will be computed here.

TABLE 5.1. Invariants for $\widetilde{\Sigma}$ and $\widetilde{B}$.

	χ	q_X
$\widetilde{\Sigma}$	-4	-8
$\widetilde{B}$	0	-2

In the rest of this section we are going to compute the Euler characteristics in Table 5.1. For this we need explicit descriptions of the divisors above. Let A be an abelian surface, and denote by $A^\vee$ its dual. Multiplication by -1 is an involution on both abelian surfaces and their product $A \times A^\vee$.

THEOREM 5.1 ([17], Corollary 2.8). — *The divisor $\widetilde{\Sigma}$ is a $\mathbb{P}^1$-bundle over the nonsingular variety*

$$\overline{\Sigma} := Bl_{Sing((A \times A^\vee)/\pm 1)}(A \times A^\vee)/ \pm 1.$$

THEOREM 5.2 ([21]). — *$\widetilde{B}$ is a $\mathbb{P}^1$-bundle over $K(A) \times K(A^\vee)$, where $K(A)$ denotes the smooth Kummer surface associated to A.*

PROPOSITION 5.3. — *With the notation as above we have*

$$\chi(\overline{\Sigma}, \mathcal{O}_{\overline{\Sigma}}) = -8 \quad and \quad \chi(\widetilde{B}, \mathcal{O}_{\widetilde{B}}) = 4.$$

Proof. — The Euler characteristic is multiplicative for smooth fibrations, so $\chi(\mathcal{O}_{\widetilde{\Sigma}}) = \chi(\mathcal{O}_{\overline{\Sigma}})$. The Hodge numbers are given by

$$H^{4,q}(\overline{\Sigma}) = H^{4,q}(A \times \widehat{A})^{inv}.$$

Since the action in cohomology is given on differential forms by multiplication by $(-1)^{\deg}$, the cohomology groups are

$$H^{4,q}(A \times \widehat{A}) = \begin{cases} 0 & \text{if } q = 2, 4 \\ 6 & \text{if } q = 3 \\ 1 & \text{if } q = 1, 5. \end{cases}$$

It follows that $\chi(\mathcal{O}_{\overline{\Sigma}}) = -8$. For the other divisor $\widetilde{B}$ in Table 5.1 the Theorem 5.2 yields $\chi(\mathcal{O}_{\widetilde{B}}) = \chi(\mathcal{O}_{K(A) \times K(A^\vee)}) = 4$, since both $K(A)$ and $K(A^\vee)$ are K3 surfaces. $\qquad\square$

The computations of the Euler characteristic given above compute the Euler characteristics for these divisors on their respective HK varieties by the following:

LEMMA 5.4. — *Let X be an HK variety of OG6-type and let $E \subseteq X$ be an effective smooth divisor, then $\chi(X, \mathcal{O}(E)) = 4 - \chi(E, \mathcal{O}_E)$.*

Proof. — The exact sequence induced by a nonzero section of E gives a formula for its Euler characteristic:

$$(10) \qquad \chi(X, \mathcal{O}(E)) = \chi(X, \mathcal{O}_X) + \chi(E, \mathcal{O}_E(E)) = 4 + \chi(E, \mathcal{O}_E(E)).$$

By adjunction, $\mathcal{O}_E(E) \cong \omega_E$. If E is smooth, then by Serre duality $\chi(E, \omega_E) = -\chi(E, \mathcal{O}_E)$. $\qquad\square$

Hence, by Lemma 5.4 and Proposition 5.3, we have computed the Euler characteristics in Table 5.1. The following proposition finishes the proof of the main theorem.

PROPOSITION 5.5. — *The Riemann–Roch polynomial of OG6 is of Kum_3-type.*

Proof. — If we plug into Equation (9) the known invariants of $\widetilde{\Sigma}$ and $\widetilde{B}$ given in Table 5.1, then we have a complete system of *linear* equations because the Beauville–Bogomolov square of both divisors is different. Hence, the constants a_1 and a_2 are uniquely determined. The coefficients $a_0, \ldots, a_4$ are the same as those of the Riemann–Roch polynomial of Kum_3-type given in Example 2.5.

$\square$

Appendix A. Chern numbers of OG10, by Yalong Cao and Chen Jiang

An interesting question is to compute topological invariants (e.g., topological Euler characteristic and Chern numbers) of hyperkähler varieties.

For an HK of $K3^{[n]}$-type, Ellingsrud–Göttsche–Lehn [7] showed that the Chern numbers can be efficiently calculated, but in terms of the Chern numbers of the varieties $(\mathbb{P}^2)^{[k]}$ and $(\mathbb{P}^1 \times \mathbb{P}^1)^{[k]}$ (which can be calculated by Bott's residue formula via [5, 6]), no explicit formula is known.

For an HK of Kum_n-type, Nieper [23] showed that the Chern numbers can be efficiently calculated, in terms of the Chern numbers of the varieties $(\mathbb{P}^2)^{[k]}$ (see also [2, 28]).

For OG6, the Chern numbers are computed by Mongardi–Rapagnetta–Saccà [17, Corollary 6.8] (see also [26]).

For OG10, the topological Euler characteristic is computed by Mozgovyy [19] (see also [13]) and the Hodge numbers are computed by de Cataldo–Rapagnetta–Saccà [4]. But the Chern numbers of OG10 have not yet been computed. In fact, Hodge numbers could not provide enough linear equations to solve Chern numbers (cf. [28, Section 5.2]).

As an application to the Riemann–Roch polynomial of OG10, combining with the Hodge numbers, we can compute 7 Chern numbers of OG10.

A.1. Relations of q_X and λ via Riemann–Roch polynomials. — Let X be an HK variety. For a line bundle L on X, Nieper [22, Definition 17] defined the *characteristic value* of L,

$$\lambda(L) := \begin{cases} \dfrac{24n \int_X \text{ch}(L)}{\int_X c_2(X)\text{ch}(L)} & \text{if well defined;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\lambda(L)$ is a positive (topological constant) multiple of $q_X(c_1(L))$ (cf. [22, Proposition 10]), more precisely,

$$\lambda(L) = \frac{12c_X}{(2n-1)C(c_2(X))}q_X(c_1(L)).$$

Here we denote the ratio

$$M_{\lambda,q_X} := \lambda(L)/q_X(c_1(L)) = \frac{12c_X}{(2n-1)C(c_2(X))}.$$

So for the Riemann–Roch polynomial, we can write it in terms of either q_X (say $\mathrm{RR}_{X,q_X}(t)$) or λ (say $\mathrm{RR}_{X,\lambda}(t)$). They coincide up to a multiple of invariable, that is, $\mathrm{RR}_{X,q_X}(t) = \mathrm{RR}_{X,\lambda}(M_{\lambda,q_X}t)$.

An observation (maybe well known to experts) is that, even though we do not know the value of $M_{\lambda,q_X} = \frac{12c_X}{(2n-1)C(c_2(X))}$ priorly, we can see it once we know the expression of the Riemann–Roch polynomial, that is, $\mathrm{RR}_{X,q_X}(t)$ determines M_{λ,q_X}, $(\mathrm{RR}_{X,\lambda}(t), c_X)$ determines M_{λ,q_X}. Hence, as a consequence, $\mathrm{RR}_{X,q_X}(t)$ and $(\mathrm{RR}_{X,\lambda}(t), c_X)$ determine each other.

LEMMA A.1. — *The coefficients of the first 2 leading terms of* $\mathrm{RR}_{X,q_X}(t)$ *determine* M_{λ,q_X}. *The coefficient of the leading term* $\mathrm{RR}_{X,\lambda}(t)$ *and* c_X *determines* M_{λ,q_X}.

Proof. — Write $\mathrm{RR}_{X,q_X}(t) = at^n + bt^{n-1} + (\text{lower terms})$. Then we have

$$\int_X \mathrm{td}(X)\exp(L) = \mathrm{RR}_{X,q_X}(q_X(L)).$$

Comparing both sides, this implies that

$$\int_X \frac{1}{(2n)!}L^{2n} = aq_X(L)^n,$$

and

$$\int_X \mathrm{td}_2(X)\frac{1}{(2n-2)!}L^{2n-2} = bq_X(L)^{n-1}.$$

Using Fujiki's relation, these implies that

$$c_X = (2n)!a$$

and

$$C(c_2(X)) = 12(2n-2)!b.$$

Hence $M_{\lambda,q_X} = \frac{12c_X}{(2n-1)C(c_2(X))} = \frac{2na}{b}$.

Write $\mathrm{RR}_{X,\lambda}(t) = a't^n + (\text{lower terms})$. Then

$$\mathrm{RR}_{X,q_X}(t) = \mathrm{RR}_{X,\lambda}(M_{\lambda,q_X}t) = a'M_{\lambda,q_X}^n t^n + (\text{lower terms}).$$

Then $(2n)!a'M_{\lambda,q_X}^n = c_X$, which determines M_{λ,q_X} as M_{λ,q_X} is a positive real number. $\qquad\square$

A.2. Chern numbers of OG10. —

THEOREM A.2. — *Let X be an HK of OG10-type. Then the Chern numbers of X are the following:*

$$(c_2^5, c_2^3 c_4, c_2^2 c_6, c_2 c_8, c_2 c_4^2, c_4 c_6, c_{10})$$
$$= (127\,370\,880, 53\,071\,200, 12\,383\,280, 1\,791\,720, 22\,113\,000, 5\,159\,700, 176\,904)$$

Proof. — Let X be an HK of OG10-type. Then the Riemann–Roch polynomial is

$$\mathrm{RR}_{X,q_X}(t) = \binom{t/2 + 6}{5}.$$

By the discussion of the first section, we know that

$$(11) \qquad \mathrm{RR}_{X,\lambda}(t) = \mathrm{RR}_{X,q_X}(4t) = \binom{2t + 6}{5}.$$

That is, for a line bundle L on X,

$$\chi(X, L) = \binom{2\lambda(L) + 6}{5}.$$

On the other hand, Nieper [22, Theorem 5.2] proved that

$$\chi(X, L) = \int_X \exp\left(-2 \sum_{k=1}^{\infty} \frac{B_{2k}}{4k} \mathrm{ch}_{2k}(X) T_{2k}\left(\sqrt{\frac{\lambda(L)}{4} + 1} \right) \right)$$
$$(12) \qquad = \int_X \exp\left(-2 \sum_{k=1}^{\infty} b_{2k} s_{2k}(X) T_{2k}\left(\sqrt{\frac{\lambda(L)}{4} + 1} \right) \right)$$

where B_{2k} are the Bernoulli numbers, $b_{2k} = \frac{B_{2k}}{4k(2k)!}$ are modified Bernoulli numbers with $b_0 = 1$, $s_{2k} = (2k)! \mathrm{ch}_{2k}(X)$, and T_{2k} are even Chebyshev polynomials of the first kind. So by (11) and (12), we get an identity between polynomials in terms of $y = \sqrt{\lambda(L)/4 + 1}$:

$$(13) \qquad \binom{8y^2 - 2}{5} = \int_X \exp\left(-2 \sum_{k=1}^{\infty} b_{2k} s_{2k}(X) T_{2k}(y) \right).$$

Comparing coefficients of (13), we have 6 linear equations of 7 Chern numbers $c_2^5, c_2^3 c_4, c_2^2 c_6, c_2 c_8, c_2 c_4^2, c_4 c_6, c_{10}$. Of course this is not sufficient to get a unique solution. In fact, among 6 equations, there are only 3 linearly independent linear equations (from comparing coefficients of y^8, y^6, and y^2).

On the other hand, we have more equations for the Hodge numbers of X computed by de Cataldo–Rapagnetta–Saccà [4]. This gives us

$$\chi^1(X) = -111;$$
$$\chi^2(X) = 1\,062;$$
$$\chi^3(X) = -7\,151;$$
$$\chi^4(X) = 33\,534.$$

Here $\chi^p(X) = \sum_{q=0}^{\dim X}(-1)^q h^{p,q}(X) = \int_X \mathrm{ch}(\Omega_X^p)\mathrm{td}(X)$. Again, expressing the left-hand sides by Chern numbers, we get 4 linear equations of 7 Chern numbers. We can use Mathematica to solve the linear equations for 7 Chern numbers of OG10 as the following:

$$(c_2^5, c_2^3 c_4, c_2^2 c_6, c_2 c_8, c_2 c_4^2, c_4 c_6, c_{10})$$
$$= (127\,370\,880, 53\,071\,200, 12\,383\,280, 1\,791\,720, 22\,113\,000, 5\,159\,700, 176\,904).$$

$\square$

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