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CONVEX COMPACT SURFACES WITH NO BOUND ON THEIR SYNTHETIC RICCI CURVATURE

by Constantin Vernicos

Abstract. — Using refraction in the setting of normed vector spaces allows us to present an example of a convex compact surface which admits no lower bound on its Ricci curvature as defined by Lott–Villani and Sturm.

Résumé (*Surfaces convexes et compactes n'admettant pas de borne de leur courbure de Ricci synthétiques*). — L'utilisation de la notion de réfraction dans le cadre des espaces vectoriels normés permet de construite un exemple de surface convexe et compacte qui n'est pas de courbure de Ricci minorée telle que défini par Lott-Villani et Sturm.

Introduction and statement of results

Many notions of curvature bounds adapted to a metric measure space have been defined to extend the ones existing in Riemannian geometry. Most of them heavily rely on comparison to the Euclidean space and that is why they are quite restrictive. For instance, a normed vector space is CAT(0) if and only if it is an Euclidean space; as a consequence, the only Finsler spaces which can be CAT(0) are Riemannian (see also [2]). The same thing happens with the

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Alexandroff spaces. It is even more general in that case, for an Alexandroff metric space happens to be almost Riemannian manifold (see [1] for a precise statement).

Some older notions, such as the Busemann convexity, are less restrictive. However, they might not pass to the Gromov–Hausdorff limit of a sequence of metric measured spaces; for instance, this happens when one approximates a non-strictly convex norm by strictly convex ones. The family of strictly convex normed spaces obtained are Busemann convex and converge to the non-strictly convex ones which are not. In the light of the current interest in understanding the limit spaces arising as limits of Riemannian metric space, with Ricci curvature bounded from below, for instance, this is a huge flaw.

Following the work of Lott & Villani [6] and Sturm [15, 16], a new family of notions of curvature bounded spaces arose. They involve the convexity of an operator on the L^2-Wasserstein space, which is a metrization of the space of probability measures with finite 2-th moment. Among them one finds the spaces satisfying the curvature dimension condition $CD(K, N)$ or the measure contraction property $MCP(K, N)$. The latter may be seen as a measure analog to the Busemann convexity, the former as a generalization of having Ricci curvature bounded from below by K and being of dimension less than N. We will refer to this last notion as synthetic Ricci curvature and describe such spaces as admitting a lower bound on their synthetic Ricci curvature. An example is given by a normed vector space of dimension n which satisfies the curvature dimension condition $CD(0, n)$ (see [17] in the Appendix).

Another point of view on curvature in metric spaces is based on analytical inequalities. For instance, Cordero-Erausquin, McCann, and Schmuckenschläger [3] looked at the Brascamp–Lieb inequality which is a generalization of the Prekopa–Leindler inequality that can be used to prove the Brunn–Minkoswki inequality in the Euclidean space.

The interesting aspect on which this paper is based is that most notions of curvature deriving from the work of Lott–Villani and Sturm imply a Brunn–Minkowski inequality, hence our focus on this inequality (see also [8, 9] for a recent study on the relation between the Brunn–Minkowski and the CD condition).

Our main result is the following:

THEOREM 1. — *There exists a compact $C^{1,1}$ convex surface in R^3 with the norm $\|(x, y, z)\| = \sqrt{x^2 + y^2} + |z|$ which admits no lower bounds on its synthetic Ricci curvature.*

The idea of that example came from the study of reflections and refraction in normed (not necessarily reflexive) vector spaces. Section 2 focuses on a specific example which allows us to obtain our convex set in Section 3.

It is worth mentioning here that the specific example of Section 2 also shows that the CD property is not preserved by gluing two CD spaces along their isometric boundaries. This behavior distinguishes the CD property from other properties, such as Alexandroff spaces (see [13, 5]).

The main reason why the example in Section 2 is not a CD space is due to the particular structure of geodesics which branch along a hyperplane. It is known that such branching does not go along with the CD property unless one has a particular measure and metric structure (see [7], pointed out to us by an anonymous referee as this paper was not available when the present work was done).

Section 3 is a perturbation of Section 2's example which smooths the space a bit and probably gets rid of the branching, but without allowing a synthetic curvature lower bound. One must also emphasize here that if the smoothing were C^2 then a lower bound would exist. Hence the nonexistence is not an immediate thing.

1. Definitions and notations

A metric measured space (X, d, μ) is a space X endowed with a distance d and a measure μ, usually a Borel one. Let us fix a metric measured space. For any pairs of point m_0, $m_1 \in X$, we call $m_s \in X$ an s-intermediate point from m_0 to m_1 if and only if

$$d(m_0, m_s) = sd(m_0, m_1) \quad \text{and} \quad d(m_s, m_1) = (1-s)d(m_0, m_1).$$

Let K_0 and K_1 be two compact sets in X, the set of s-intermediate points from points of K_0 to points of K_1 will be denoted by

$$M_s(K_0, K_1).$$

If $M_s(K_0, K_1)$ is not measurable, we will still denote its outer measure by

$$\mu\big(M_s(K_0, K_1)\big).$$

Let us first start with the classical Brunn–Minkowski inequality:

Definition 1.1 (Classical Brunn–Minkowski inequality). — Let N be greater than 1. We say that the Brunn–Minkowski inequality $BM(0, N)$ holds in the metric measured space (X, d, μ) if for every pair of compact sets K_0 and K_1, the following inequality is satisfied:

$$\mu^{1/N}\big(M_s(K_0, K_1)\big) \geq (1-s)\mu^{1/N}(K_0) + s\mu^{1/N}(K_1). \tag{1}$$

We also say that $BM(0, +\infty)$ holds if and only if

$$\mu\big(M_s(K_0, K_1)\big) \geq \mu^{1-s}(K_0)\mu^s(K_1). \tag{2}$$

Remark 1.2. — Notice that if for some $n \in R^*$, and t,a and $b \in \mathbb{R}$, the inequality $t \geq (sa^{1/n} + (1-s)b^{1/n})^n$ holds, then from the concavity of the logarithm we have

$$\begin{aligned}\ln t &\geq n \ln\bigl(sa^{1/n} + (1-s)b^{1/n}\bigr) \\ &\geq s \ln a + (1-s) \ln b.\end{aligned}$$

Hence, any $BM(0,N)$ implies $BM(0,\infty)$.

Now the general Brunn–Minkowski inequality $BM(K,N)$ requires the introduction of a family of functions depending on K, N, and $s \in [0,1]$ denoted by $\tau_{K,N}^{(s)} \colon \mathbb{R}^+ \to \mathbb{R}^+$. For a fixed $s \in [0,1]$ and $\theta \in \mathbb{R}^+$, $\tau_{K,N}^{(s)}(\theta)$ is continuous, nonincreasing in N, and nondecreasing in K. Its exact definition is not important for our applications, refer to [16].

Definition 1.3 (Generalized Brunn–Minkowski inequality). — Let N be greater than 1 and $K \in \mathbb{R}$. We say that the Brunn–Minkowski inequality $BM(K,N)$ holds in the metric measured space (X,d,μ) if for every pair of compact set K_0 and K_1, the following inequality is satisfied:

$$\mu^{1/N}\bigl(M_s(K_0,K_1)\bigr) \geq \tau_{K,N}^{(1-s)}(\vartheta)\mu^{1/N}(K_0) + \tau_{K,N}^{(s)}(\vartheta)\mu^{1/N}(K_1). \tag{3}$$

where ϑ is the minimal (respectively maximal) length of a geodesic between a point in K_0 and a point in K_1 if $K \geq 0$ (respectively $K < 0$).

We can also define the $BM(K,+\infty)$ as follows:

$$\mu\bigl(M_s(K_0,K_1)\bigr) \geq \mu^{1-s}(K_0)\mu^s(K_1)e^{Ks(1-s)\vartheta^2/2}. \tag{4}$$

The curvature dimension property, denoted by $CD(K,N)$, is a generalization of the following sentence on metric measures spaces:

> The space has dimension less than N and the Ricci curvature is bigger than K.

It is defined in terms of a convexity property of the entropy along geodesics in the space of probability of the metric space (see [16] for more precise statements).

For our purpose we only need to know the following properties of a space satisfying a curvature dimension property (see K.T. Sturm [16]).

Property 1.4. — *Let (X,d,μ) be a metric measured space, $K \in \mathbb{R}$. The following implications are valid:*

1. *Suppose $CD(K,N)$ holds. If $K' \leq K$, then $CD(K',N)$ holds as well. If $N' > N$, then $CD(K,N')$ holds as well.*
2. *Suppose $CD(K,N)$ holds. Then for any $\alpha,\beta > 0$, the metric measured space $(X,\alpha d,\beta\mu)$ satisfies the $CD(K/\alpha^2,N)$ condition.*

3. *$CD(0,N)$ implies $BM(0,N)$ and, more generally, $CD(K,N)$ implies $BM(K,N)$.*
4. *$CD(K,N)$ implies the Bishop–Gromov volume growth inequality with the Riemannian space of constant curvature K and dimension N.*

2. Brunn–Minkowski inequality is not preserved in a two-layer Banach space

In this section we are going to consider the vector space $\mathbb{R}^2$ and the hyperplane $\mathcal{H} = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$. We are going to put the classical Euclidean ℓ^2 norm $\|(x,y)\|_2 = \sqrt{x^2+y^2}$ on the half-space $y > 0$ and the ℓ^1 norm $\|(x,y)\|_1 = |x| + |y|$ on the half-space $y < 0$. Given $P = (x,y)$ and $Q = (x',y')$ in $\mathbb{R}^2$ we define the distance $d_{2,1}$ by

$$\begin{cases} \text{if } y > 0, y' > 0 & d_{2,1}(P,Q) = \|P-Q\|_2 \\ \text{if } y < 0, y' < 0 & d_{2,1}(P,Q) = \|P-Q\|_1 \\ \text{if } y > 0 \text{ and } y' < 0 & d_{2,1}(P,Q) = \inf_{Z\in\mathcal{H}} \|Z-P\|_2 + \|Q-Z\|_1 \end{cases}$$

This is actually the length distance when curves on the upper half-plane are measured thanks to their Euclidean length, and on the lower half-plane thanks to their ℓ^1-length. It is important here that the restriction of both norms coincides on the hyperplane $\mathcal{H}$.

Now let us specify the measures $\mathfrak{m}$ we will use here. In Finsler geometry there is no canonical measure as in Riemannian geometry. One has to choose a consistent normalization of the Lebesgue measure on each tangent space (see [1]). One possibility is to fix the volume of each tangent ball equal to π, this gives the so-called Busemann volume. In our case, if we denote by λ the standard Lebesgue measure, that is such that π is the measure of the standard Euclidean disk, then on the lower half-space our measure would be $\alpha\lambda$ with $\alpha = \pi/2$. Other normalization exists (see again [1]).

We shall denote by $(\mathbb{R}^2, d_{2,1}, \mathfrak{m})$ the metric measured space obtained this way.

Properties 2.1. — *Let $\alpha \in \mathbb{R}$ and $X_0 = (\rho,\theta)$ be in the upper half-plane in polar coordinates centered at the point $O_\alpha = (\alpha, 0)$. Consider $X_1 = (\alpha, y)$ be in the lower half-plane in Cartesian coordinates ($y < 0$), then*

- *the geodesic joining X_0 to X_1 is composed of the line segment from X_0 to the point O_α and from the point O_α to X_1. It is unique;*
- *the distance between X_0 to X_1 is equal to $\rho - y$;*
- *let X_s be the s-intermediate point between X_0 to X_1,*
    1. *if $s(\rho - y) < \rho$, then X_s belongs to the upper half-plane and lies on the affine segment from X_0 to the point O_α, and $X_s = \big((1-s)\rho + sy, \theta\big)$ in polar coordinates;*

2. *if* $s(\rho - y) > \rho$*, then* X_s *belongs to the lower half-plane and lies in the line* $x = \alpha$*, and* $X_s = \big(\alpha, (1-s)\rho + sy\big)$ *in Cartesian coordinates.*

Proof. — Without loss of generality we take $\alpha = 0$, hence O_α is the origin. First notice that even if geodesic segments are not necessarily straight line segments in ℓ^1, straight line segments are always geodesics. Hence, consider a path comprised of two line segments from $X_0 = (a, b)$ to X_1 which passes through $(t, 0)$ with $t \neq 0$. Then the length we get is $l(t) = \sqrt{(a-t)^2 + b^2} + |t| + |y|$. If we take the derivative with respect to t outside 0, one gets

$$l'(t) = \frac{-(a-t)}{\sqrt{(a-t)^2 + b^2}} + \frac{t}{|t|}.$$

For $t < 0$ we get as the numerator of $l'(t)$

$$(t-a) - \sqrt{(a-t)^2 + b^2} < 0$$

and for $t > 0$ we get

$$(t-a) + \sqrt{(a-t)^2 + b^2} > 0,$$

which proves that those paths are not geodesics, and that we need to pass through the origin.

We now need to prove uniqueness. As seen above, any geodesic between these points has to pass through the origin. Hence, on the upper half-plane, as there is only one geodesic between any two points, we do not have any choice.

Now on the lower half-plane, let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a path between the origin and the point $(0, y)$, Let us parametrize this path by $\gamma(t) = \big(x(t), y(t)\big)$ and suppose that for some $0 < t < 1$ we have $x(t) \neq 0$. Then we have

$$\big\|\big(x(t), y(t)\big)\big\|_1 + \big\|\big(x(t), y(t) - y\big)\big\|_1 = 2|x(t)| + |y(t)| + |y(t) - y| > |y(t)| + |y(t) - y|$$

that is to say that it is shorter to pass through $(0, y(t))$ than $(x(t), y(t))$, hence it is not a geodesic.

This implies that the only geodesic between $(0,0)$ and $(0, y)$ is the segment between these two points. The other properties are easy to check. □

Proposition 2.2. — *In the metric space* $(\mathbb{R}^2, d_{2,1}, \mathfrak{m})$ *no Brunn–Minkowski inequality holds, i.e., for any* $K \in \mathbb{R}$ *and* $N \in (1, +\infty]$*, the Brunn–Minkowski inequality* $BM(K, N)$ *does not hold.*

Proof. — First one can notice that for $N < +\infty$ the space $(\mathbb{R}^2, d_{2,1}, \mathfrak{m})$ is invariant under linear dilations. This implies that if it is $BM(K, N)$ then it is $BM(0, N)$.

Let (ρ, θ) be the polar coordinates in $\mathbb{R}^2$. Consider the annulus

$$K_0 = \{(\rho, \theta) \mid 6 \leq \rho \leq 8, \pi/3 \leq \theta \leq 2\pi/3\},$$

and the affine segment

$$I = \{(x, y) \in \mathbb{R}^2 \mid -101 \leq y \leq -100, x = 0\}.$$

Now let $X_0 = (\rho_0, \theta)$ be in K_0, and $X_I = (0, -100 - t)$ in I. Following the previous section, there is a unique geodesic from X_0 to X_I, and is composed of the affine segment joining X_0 to the origin $O = (0, 0)$ and the affine segment joining the origin to X_I. We therefore have $\|X_0\|_2 = \rho_0$ and $\|X_I\|_1 = 100 + t$, from which we deduce that the distance between these two points is $\rho_0 + 100 + t$. Now following Properties 2.1, as

$$(\rho_0 + 100 + t)/2 > 106/2 = 53 > 8 \geq \rho_0,$$

for $s \geq 1/2$, the point $X_s = \big(0, (1 - s)\rho_0 + s(-100 - t)\big)$ is the s-intermediate point on the geodesic from X_0 to X_I, From this we easily deduce that the 1/2-intermediate set from X_0 to I is

$$\frac{1}{2}K_0 + \frac{1}{2}I = \big\{(x, y) \mid x = 0, -47,5 \leq y \leq -46\big\}.$$

This suffices to prove that $BM(0, N)$ is not satisfied as

$$(\mathfrak{m})^{\frac{1}{N}}\left(\frac{1}{2}K_0 + \frac{1}{2}I\right) = 0 < \frac{1}{2}\mathfrak{m}^{\frac{1}{N}}(K_0). \tag{5}$$

Now let us prove that $BM(K, +\infty)$ is never satisfied. For $s > 1/2$, the s-intermediate set from X_0 to I is easily seen to be

$$(1 - s)K_0 + sI = \big\{(0, y) \in \mathbb{R}^2 \mid -101s + 6(1 - s) \leq y \leq -100s + 8(1 - s)\big\}.$$

We start by considering some $0 < \varepsilon < 1$, whose value will be chosen at the end, and replace I with

$$K_1 = \big\{(x, y) \in \mathbb{R}^2 \mid |x| \leq \varepsilon, |y + 100, 5| \leq 0, 5\big\}.$$

The next step is to introduce the positive slices of K_1 for $0 < \alpha \leq \varepsilon < \rho$,

$$I_\alpha = \{(\alpha, y) \in \mathbb{R}^2 \mid -101 \leq y \leq -100\}$$

and to identify their intermediate sets $M_s(K_0, I_\alpha)$. In order to do this we compute the distance between X_0 and $(\alpha, 0)$, which gives

$$\rho_\alpha = \sqrt{\rho_0^2 - 2\alpha\rho_0 \cos\theta + \alpha^2},$$

and it is now easy to check that for X_0 in K_0 we have

$$\begin{aligned}(6 - \alpha) \leq (\rho_0 - \alpha) < \sqrt{\rho_0^2 - 2\alpha\rho_0 + \alpha^2} \\ \rho_\alpha \leq \sqrt{\rho_0^2 + 2\alpha\rho_0 + \alpha^2} < (\rho_0 + \alpha) \leq (8 + \alpha).\end{aligned} \tag{6}$$

The description we were seeking is therefore (recall that $s > 1/2$)

$$M_s(K_0, I_\alpha) = \Big\{(\alpha, y) \in \mathbb{R}^2 \mid -101s + (1-s)\sqrt{6^2 - 6\alpha + \alpha^2} \le y \le -100s + \sqrt{8^2 + 8\alpha + \alpha^2}(1-s)\Big\}. \tag{7}$$

To obtain an upper bound on its area we notice that it can be seen as a subset as follows:

$$M_s(K_0, I_\alpha) \subset \Big\{(\alpha, y) \in \mathbb{R}^2 \mid -101s + (1-s)(6-\alpha) \le y \le -100s + (1-s)(8+\alpha)\Big\}. \tag{8}$$

Therefore, the area of the intermediate set $K_s = M_s(K_0, K_1)$ is less than

$$\varepsilon \cdot \big(16 - 15s\big)$$

up to some multiplicative constant C, depending on the normalization chosen for the Lebesgue measure.

This also tells us (depending on the sign of K) that (see definition 3 for the definition of ϑ)

$$105 \le 106 - \varepsilon \le \vartheta(\varepsilon) \le 108 + \varepsilon \le 109. \tag{9}$$

The area of K_1 is exactly 2ε. Hence, for some fixed constant C, we have

$$\frac{\mathfrak{m}\big(M_s(K_0, K_1)\big)}{\mathfrak{m}^s(K_1)} \le \varepsilon^{1-s} \cdot C^{1-s} \cdot \Big(8 - \frac{15}{2}s\Big) \tag{10}$$

We need now to compare, as $s \to 1$, the right-hand part of (10) with

$$\mathfrak{m}^{1-s}(K_0)e^{Ks(1-s)\vartheta(\varepsilon)^2/2},$$

which is the same as comparing $\varepsilon \cdot C \cdot \big(8 - \frac{15}{2}s\big)^{1/(1-s)}$ with

$$\mathfrak{m}(K_0)e^{Ks\vartheta(\varepsilon)^2/2}.$$

This last term converges towards $\mathfrak{m}(K_0)e^{K\vartheta(\varepsilon)^2/2}$, while the first to $\varepsilon \cdot C \cdot e^{15/2}$.

To conclude, as $\vartheta(\varepsilon)$ stays bounded, we can find and fix an ε small enough such that

$$\varepsilon \cdot C \cdot e^{15/2} < \frac{1}{2}\mathfrak{m}(K_0)e^{K\theta(\varepsilon)^2/2}.$$

Then, for values of s close enough to 1, we will obtain

$$\mathfrak{m}^{1-s}(K_0)e^{Ks(1-s)\vartheta(\varepsilon)^2/2} > \frac{\mathfrak{m}\big(M_s(K_0, K_1)\big)}{\mathfrak{m}^s(K_1)}, \tag{11}$$

which contradicts $BM(K, +\infty)$. □

Remark 2.3. — Actually, our proof, and notably the inequality (5), implies that the space is not $MCP(0, N)$.

Let f be any norm, we can define a new distance as above. That is, given $P=(x,y)$ and $Q=(x',y')$ in $\mathbb{R}^2$ we define the distance $d_{2,f}$ by

$$\begin{cases}\text{if } y>0, y'>0 & d_{2,f}(P,Q)=\|P-Q\|_2\\ \text{if } y<0, y'<0 & d_{2,f}(P,Q)=f(P-Q)\\ \text{if } y>0 \text{ and } y'<0 & d_{2,f}(P,Q)=\inf_{Z\in\mathcal{H}}\|Z-P\|_2+f(Q-Z).\end{cases}$$

Let us denote by $(\mathbb{R}^2, d_{2,f}, \mathfrak{m}_f)$ this metric measure space, where $\mathfrak{m}_f$ is the Busemann measure.

Proposition 2.4. — *There exists a Minkowski norm f on $\mathbb{R}^2$ such that that $BM(-1,+\infty)$ does not hold in the metric space $(\mathbb{R}^2, d_{2,f}, \mathfrak{m}_f)$.*

Proof. — Recall that a Minkowski norm f is twice differentiable on $\mathbb{R}^2\setminus\{0\}$, with a definite positive Hessian.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Minkowski norms, converging towards the ℓ^1 norm. Up to a rescaling we can suppose that the intersection of their unit ball with $\mathcal{H}$ coincides with the intersection of the unit ball of the ℓ^1 norm. In any case, we will consider the measures $\mathfrak{m}_n$ such that $\mathfrak{m}_n=\lambda$ is the Lebesgue measure on the upper half-plane, and $\mathfrak{m}_n=\alpha_n\lambda$ on the lower half-plane, where

$$\alpha_n=\frac{\pi}{\lambda(\{f_n\le 1\})}.$$

(Observe also that we can choose the norms f_n such that their tangents at their point of intersection with H are orthogonal to H. This will be useful in the last section of this paper.)

Given $P=(x,y)$ and $Q=(x',y')$ in $\mathbb{R}^2$ we define the distance $d_{2,n}$ by

$$\begin{cases}\text{if } y>0, y'>0 & d_{2,n}(P,Q)=\|P-Q\|_2\\ \text{if } y<0, y'<0 & d_{2,n}(P,Q)=f_n(P-Q)\\ \text{if} y>0 \text{ and } y'<0 & d_{2,n}(P,Q)=\inf_{Z\in\mathcal{H}}\|Z-P\|_2+f_n(Q-Z).\end{cases}$$

Then the sequence of metric measure spaces $(\mathbb{R}^2, d_{2,n}, \mathfrak{m}_\mathfrak{n})$ converges in the Gromov–Hausdorff measured topology towards $(\mathbb{R}^2, d_{2,1}, \mathfrak{m})$.

Consider again the sets K_0 and K_1 and the associated intermediate set $M_s(K_0,K_1)$ as in the proof of Proposition 2.2. Then for any n we would get another intermediate set $M_{s,n}(K_0,K_1)$, and another function $\theta(n)$. which is the maximum (respectively minimum) between two points from K_0 to K_1 or from K_1 to K_0 with respect to $d_{2,n}$. Following our assumption, we have that $\theta(n)$ converges towards the θ of the limit, $\mathfrak{m}_\mathfrak{n}(K_0)=\mathfrak{m}(K_0)=$ does not change, while $\mathfrak{m}_n(K_1)$ converges towards $\mathfrak{m}(K_1)$ thanks to the Gromov–Hausdorff measured convergence.

We suppose that s is close enough to 1 to be on the lower half-plane.

We need to prove that

$$\liminf \mathfrak{m}_n\big(M_{s,n}(K_0,K_1)\big)\le \mathfrak{m}\big(M_s(K_0,K_1)\big)$$

as n goes to infinity. First notice that $M_s(K_0,K_1)$ is a compact closed set, and so are the sets $M_{s;n}(K_0,K_1)$. Secondly, the geodesics from a point on the upper half-space to the lower half-space are unique, because both norms are strictly convex. Hence, the geodesics are converging to the geodesics, thus $M_{s;n}(K_0,K_1)$ converges to a subset K'_s of $M_s(K_0,K_1)$.

Therefore we get

$$\liminf \mathfrak{m}_n(K_s(n)) \leq \mathfrak{m}(K'_s) \leq \mathfrak{m}\big(M_s(K_0,K_1)\big).$$

Now let us take K_0, K_1 as in the proof of Proposition 2.2 and s close enough to 1 such that

$$\mathfrak{m}^{1-s}(K_0)e^{-s(1-s)\vartheta(\varepsilon)^2/2} > \frac{\mathfrak{m}\big(M_s(K_0,K_1)\big)}{\mathfrak{m}^s(K_1)},$$

then for any n large enough we would also get

$$\mathfrak{m_n}^{1-s}(K_0)e^{-s(1-s)\vartheta(n)^2/2} > \frac{\mathfrak{m_n}\big(M_{s,n}(K_0,K_1)\big)}{\mathfrak{m}_n{}^s(K_1)},$$

which concludes our proof, because any f_n for n large enough can be chosen. □

3. A compact Finsler surface with no lower Ricci bound embedded in a Minkowski space

3.1. First example. — Let us consider in the three-dimensional Euclidean space, the two-dimensional disk

$$\mathcal{S} = \big\{(x,y,z) \mid z = 0, x^2 + y^2 \leq 1\big\},$$

and let $\mathcal{B}$ be the convex hull of

$$\mathcal{S} \cup \big\{(0,0,1),(0,0,-1)\big\}.$$

We now endow $\mathbb{R}^3$ with the norm $\|\cdot\|_{\mathcal{B}}$ whose unit ball is $\mathcal{B}$. In other words, for any $(x,y,z) \in \mathbb{R}^3$,

$$\|(x,y,z)\|_{\mathcal{B}} = \sqrt{x^2+y^2} + |z|.$$

The affine planes normal to the vector $(0,0,1)$ endowed with the norm induced by $\|\cdot\|_{\mathcal{B}}$ are all isometric to the two-dimensional Euclidean plane. In the same way, the affine planes containing the direction $(0,0,1)$ are isometric to the ℓ^1-plane (i.e., that is the Manhattan distance).

In this normed vector space, we will consider $\mathcal{C}_\rho$ the boundary of the cube obtained as the convex hull H_ρ of the eight points

$$\big\{(\pm\rho,\pm\rho,\pm\rho)\big\}.$$

The cube $\mathcal{C}_\rho$ admits two faces which are Euclidean, and four faces which are ℓ^1. The measures considered are the induced Hausdorff measures on each face. In other words, $\lambda(\mathcal{B}) = \frac{4}{3}\pi$ and for any linear subspace L of dimension 2, the measure is the Lebesgue measure λ_L normalized such that

$$\lambda_L(\mathcal{B} \cap L) = \pi.$$

Proposition 3.1. — *Let $\mathbb{R}^3$ be endowed with the norm $\|\cdot\|_{\mathcal{B}}$. Then the cube $\mathcal{C}_1$ with the metric induced by $\|\cdot\|_{\mathcal{B}}$ does not satisfy any curvature dimension.*

Proof. — Let us denote by d_ρ the distance induced on $\mathcal{C}_\rho$ by $\|\cdot\|_{\mathcal{B}}$ and $\mathfrak{m}_\rho$ the induced Hausdorff measure. Focus on two adjacent faces of $\mathcal{C}_\rho$, one Euclidean and the second one ℓ^1. Then we are locally exactly as in section 2, and therefore the same computations as in section 2 show that for any $\rho \in \mathbb{R}^*$, the Brunn–Minkowski $BM(K,N)$ inequality does not hold for any $N \in \mathbb{N} \cup \{+\infty\}$ and any $K \in \mathbb{R}$.

Therefore, in $(\mathcal{C}_\rho, d_\rho, \mathfrak{m}_\rho)$ the curvature dimension $CD(K,N)$ does not hold for any K and any N. □

Corollary 3.2. — *There exists a $C^{1,1}$ compact and convex surface in $(\mathbb{R}^3, \|\cdot\|_{\mathcal{B}})$ such that for any $N \in \mathbb{N} \cup \{+\infty\}$ and any $K \in \mathbb{R}$, $CD(K,N)$ does not hold.*

Remark here that in our example, both the $C^{1,1}$ assumption and the fact that the norm restricted to the surface is not smooth are important. If the objects are too smooth, there is always some K and N for which it is $CD(K,N)$.

Proof. — Let $B(\varepsilon)$ be the Euclidean ball of radius ε. Let $H(\varepsilon)$ be the Minkowski sum of the cube H_1 and this ball, that is,

$$H(\varepsilon) = B(\varepsilon) + H_1 = \{x + y \mid x \in B(\varepsilon),\ y \in H_1\},$$

and let $C(\varepsilon)$ be its boundary with induced metric by $\|\cdot\|_{\mathcal{B}}$ and the induced Hausdorff two-dimensional measure. Then $C(\varepsilon)$ is $C^{1,1}$, and as ε goes to zero, it converges in the Gromov–Hausdorff measured topology towards $\mathcal{C}_1$.

Actually, $C(\varepsilon)$ is obtained by translating the faces of the cube $\mathcal{C}_1$ outward at a Euclidean distance ε and then closing by rolling the Euclidean ball of radius ε along the edges, from the inside.

Hence the difference is on the surface obtained along these curved edges. On the flat section we have the same distance as in $\mathcal{C}_1$.

Fix some $K = -1$ and $N = +\infty$. We can use the annulus K_0 and the rectangle K_1 from the proof of Proposition 2.2, translated to be on the faces of $C(\varepsilon)$. We shall still denote by K_0 and K_1 these translated domains whose induced measures remain unchanged by invariance of the measure by translation. The only thing that will change is the s-intermediate set from K_0 to K_1, denoted by $M_{s,\varepsilon}(K_0, K_1)$.

Fix an s such that we get the inequality (11) as in proof of Proposition 2.2 for K_0, K_1, and $M_s(K_0, K_1)$. This set will be on the plane containing K_1 for s close enough to 1.

Then as ε goes to zero, the corresponding sequence of s-intermediate sets $M_{s,\varepsilon}(K_0,K_1)$ converges towards a subset of $M_s(K_0,K_1)$ and, thus,

$$\liminf_{\varepsilon\to 0} \mathfrak{m}_\varepsilon\big(M_{s,\varepsilon}(K_0,K_1)\big) \leq \mathfrak{m}\big(M_s(K_0,K_1)\big),$$

where $M_s(K_0,K_1)$ is the same as in the proof of Proposition 2.2. Hence, for some ε small enough, we would get the same contradiction.

Let us now fix such an ε for $K=-1$.

Let h_ρ be the dilation of ratio ρ of center the origin. Consider the images of K_0, K_1, and $M_{s,\epsilon}(K_0,K_1)$ by h_ρ, they all lie on the boundary of $H_\rho + B(\rho\cdot\varepsilon)$. Furthermore the image of $M_{s,\epsilon}(K_0,K_1)$ by h_ρ is the s-intermediate set from $h_\rho(K_0)$ to $h_\rho(K_1)$ on the boundary of $H_\rho+B(\rho\cdot\varepsilon)$. Therefore, we still get the inequality 11 which is invariant by dilations, which proves that the boundary of $H_\rho+B(\rho\cdot\varepsilon)$ is not $BM(-1,+\infty)$ as well.

Hence, for any $\rho>0$, the boundary of $H_\rho+B(\rho\cdot\varepsilon)$ does not satisfy the Brunn–Minkowski inequality $BM(-1,+\infty)$ and is not $CD(-1,+\infty)$.

Now let us suppose that $\partial\big(H_1+B(\varepsilon)\big)$ is $CD(K,N)$ for some $K<-1$. Then $\big(\partial\big(H_1+B(\varepsilon)\big),\rho d,\rho^2\lambda\big)$ is $CD(K/\rho^2,N)$. Observe now that h_ρ is an isometry between $\big(\partial\big(H_1+B(\varepsilon)\big),\rho d,\rho^2\lambda\big)$ and $\partial\big(H_\rho+B(\rho\cdot\varepsilon)\big)$, because

$$d\big(h_\rho(x),h_\rho(y)\big)=\rho\cdot d(x,y),$$

but then for $\rho^2>-K$, we get that $\partial\big(H_\rho+B(\rho\cdot\varepsilon)\big)$ is $C(-1,N)$, which contradicts the choice of ε. □

The question I am often asked with this example is why $C(\varepsilon)$ does not satisfy some $CD(K,N)$ with $K\to-\infty$ as $\varepsilon\to 0$? In the above proof, one can see that this is due to the very nature of all the objects defined here, which behave nicely with respect to dilation on one side, and the translation on the other side. That is to say that the very specificity of the Lebesgue measure, that its homogeneity by dilation and invariance by translation are important here.

Another point of view should be from the point of view of optics, as was explained to me a long time ago. The laws of refraction are an approximation, that is to say that in reality, there is no discontinuity of the differential of a ray of light, but to our eyes it looks like that. In other words, the intersection between two media behaves as $C(\varepsilon)$ for ε small, but our $CD(K,N)$ eyes see $C(0)=\mathcal{C}_1$.

3.2. Second example. — This second example is to justify that one can get an example with a smoother norm.

Let $\mathcal{H}$ in $\mathbb{R}^3$ be the x-axis (that is the line $z=0$ and $y=0$). Consider f a norm in the plane $y=0$ such that $(\mathbb{R}^2,d_{2,f},\mathfrak{m}_f)$ does not satisfy $CD(-1,+\infty)$ as in Proposition 2.4. Then consider $\mathcal{B}_f$ the convex obtained by rotating the norm f around the z-axis.

Then let us denote by $\|\cdot\|_f$ the norm whose unit ball coincides with $\mathcal{B}_f$.

Proposition 3.3. — *There exists a $C^{1,1}$ compact and convex surface in $(\mathbb{R}^3, \|\cdot\|_f)$ such that for any $N \in \mathbb{N} \cup \{+\infty\}$ and any $K \in \mathbb{R}$, $CD(K,N)$ does not hold.*

Proof. — Again, let us consider the family of cubes C_ρ with our two translated sets K_0 and K_1. Then for any ρ, C_ρ with the induced metric is not $BM(-1,+\infty)$.

Then let us once again consider the set $C(\varepsilon)$, then for some ε small enough it will not be $BM(-1,+\infty)$ as in the previous example. And again, by homotating the sets contradicting $BM(-1,+\infty)$, we obtain that for any $\rho > 0$, the boundary of $H_\rho + B(\varepsilon \cdot \rho)$ is not $BM(-1,+\infty)$.

Again, the same reasoning by contradiction as in the proof of Corollary 3.2 shows that $C_1 + B(\varepsilon)$ cannot satisfy any $CD(K,N)$ for any K and any N. □

4. Concluding remarks

The current work has been the subject of various talks and discussions with many colleagues having their own idea about what is a good notion of curvature in metric measured spaces.

The first main problem which forbids the notion of synthetic Ricci curvature to apply in our first example is the branching occuring when one passes from one media to another. It is also related to the Finslerian nature of our spaces.

Both these problems exclude all the notions of curvatures that have been presented to us by our various colleagues. For instance, one could decide to work with spaces admitting a Gromov–Bishop comparison theorem, as some nice theorems and results in Riemannian geometry are actually based on the fact that manifolds with Ricci curvature bounded from below admit such a comparison. An easy computation shows that the metric space $(\mathbb{R}^2, d_{2,1}, \mathfrak{m})$ does not satisfy such a comparison with the standard hyperbolic plane.

Notice that by smoothing our norm, we still get a surface without synthetic Ricci curvature bounded from below, but without branching. This illustrates the fact that by being close to a branching space is also problematic.

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