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CHARACTERIZATION OF THE TWO-DIMENSIONAL FIVEFOLD TRANSLATIVE TILES

by Qi Yang & Chuanming Zong

Abstract. — In 1885, Fedorov discovered that a convex domain can form a lattice tiling of the Euclidean plane, if and only if it is a parallelogram or a centrally symmetric hexagon. This paper proves the following results. Besides parallelograms and centrally symmetric hexagons, there is no other convex domain that can form any two, three or fourfold translative tiling in the Euclidean plane. In particular, it characterizes all two-dimensional fivefold translative tiles, which are parallelograms, centrally symmetric hexagons, two classes of octagons and one class of decagons.

Résumé (*Caractérisation des pavages translatifs quintuples à deux dimensions*). — En 1885, Fedorov découvrait qu'un domaine convexe peut former un réseau-pavage de la plane euclidienne si et seulement s'il est un parallélogramme ou un hexagone symmétrique centralement. Cet article démontre les résultats suivants: outre les parallélogrammes et les hexagones symmétriques centralement, il n'y aucun autre domaine convexe qui peut former dans la plane eucldienne un pavage translatif double ou triple ou quadruple. En particulier, il caractérise tous les pavages translatifs quintuples en deux dimensions, qui sont parallélogrammes, hexagones symmétriques centralement, deux classes d'octogones, et une classe de décagons.

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1. Introduction

In 1885, Fedorov [6] proved that *a convex domain can form a lattice tiling in the plane if and only if it is a parallelogram or a centrally symmetric hexagon; a convex body can form a lattice tiling in the space if and only if it is a parallelotope, a hexagonal prism, a rhombic dodecahedron, an elongated dodecahedron, or a truncated octahedron.* As a generalized inverse problem of Fedorov's discovery, in 1900 Hilbert [13] listed the following question in the second part of his 18th problem: *Whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible.* To verify Hilbert's problem in the plane, in 1917 Bieberbach suggested to Reinhardt (see [19]) to determine all the two-dimensional convex tiles. However, to complete the list turns out to be challenging and dramatic. Over the years, the list has been successively extended by Reinhardt, Kershner, James, Rice, Stein, Mann, McLoud-Mann and Von Derau (see [15, 27]); its completeness has been mistakenly announced several times! In 2017, M. Rao [18] announced a completeness proof based on computer checks.

Let K be a convex body with (relative) interior $\mathrm{int}(K)$ and (relative) boundary $\partial(K)$, and let X be a discrete set, both in $\mathbb{E}^n$. We call $K+X$ a *translative tiling* of $\mathbb{E}^n$ and call K a *translative tile*, if $K+X=\mathbb{E}^n$ and the translates $\mathrm{int}(K)+\mathbf{x}_i$ are pairwise disjoint. In other words, if $K+X$ is both a packing and a covering in $\mathbb{E}^n$. In particular, we call $K+\Lambda$ a *lattice tiling* of $\mathbb{E}^n$ and call K a *lattice tile*, if Λ is an n-dimensional lattice. It is apparent that a translative tile must be a convex polytope. Usually, a lattice tile is called a *parallelohedron.*

As one can predict, to determine the parallelohedra in higher dimensions is complicated. According to Fedorov [6], there are exact five types of parallelohedra in $\mathbb{E}^3$. Through the works of Delone [3], Štogrin [23] and Engel [5], we know that there are exact 52 combinatorially different types of parallelohedra in $\mathbb{E}^4$. A computer classification for the five-dimensional parallelohedra was announced by Dutour Sikirić, Garber, Schürmann and Waldmann [4] only in 2015.

Let Λ be an n-dimensional lattice. The *Dirichlet–Voronoi cell* of Λ is defined by

$$C=\left\{\mathbf{x}:\mathbf{x}\in\mathbb{E}^n,\ |\mathbf{x},\mathbf{o}|\leq|\mathbf{x},\Lambda|\right\},$$

where $|X,Y|$ denotes the Euclidean distance between X and Y. Clearly, $C+\Lambda$ is a lattice tiling, and the Dirichlet–Voronoi cell C is a parallelohedron. In 1908, Voronoi [22] made a conjecture that *every parallelohedron is a linear transformation image of the Dirichlet–Voronoi cell of a suitable lattice.* In $\mathbb{E}^2$, $\mathbb{E}^3$ and $\mathbb{E}^4$, this conjecture was confirmed by Delone [3] in 1929. In higher dimensions, it is still open.

To characterize the translative tiles is another fascinating problem. At the first glance, translative tilings should be more complicated than lattice tilings. However, the dramatic story had a happy ending! It was shown by Minkowski [17] in 1897 that *every translative tile must be centrally symmetric*. In 1954, Venkov [21] proved that *every translative tile must be a lattice tile* (*parallelohedron*) (see [1] for generalizations). Later, a new proof for this beautiful result was independently discovered by McMullen [16].

Let X be a discrete multiset in $\mathbb{E}^n$ and let k be a positive integer. We call $K+X$ a *k-fold translative tiling* of $\mathbb{E}^n$ and call K a *translative k-tile*, if every point $\mathbf{x}\in\mathbb{E}^n$ belongs to at least k translates of K in $K+X$, and every point $\mathbf{x}\in\mathbb{E}^n$ belongs to at most k translates of $\mathrm{int}(K)$ in $\mathrm{int}(K)+X$. In other words, $K+X$ is both a k-fold packing and a k-fold covering in $\mathbb{E}^n$ (see [7, 27]). In particular, we call $K+\Lambda$ a k-fold lattice tiling of $\mathbb{E}^n$ and call K a *lattice k-tile*, if Λ is an n-dimensional lattice. Apparently, a translative k-tile must be a convex polytope. In fact, similarly to Minkowski's characterization, it was shown by Gravin, Robins and Shiryaev [10] that *a translative k-tile must be a centrally symmetric polytope with centrally symmetric facets.*

Multiple tilings were first investigated by Furtwängler [8] in 1936 as a generalization of Minkowski's conjecture on cube tilings. Let C denote the n-dimensional unit cube. Furtwängler made a conjecture that *every k-fold lattice tiling $C+\Lambda$ has twin cubes. In other words, every multiple lattice tiling $C+\Lambda$ has two cubes sharing a whole facet.* In the same paper, he proved the two and three-dimensional cases. Unfortunately, when $n\geq 4$, this beautiful conjecture was disproved by Hajós [12] in 1941. In 1979, Robinson [20] determined all the integer pairs $\{n,k\}$ for which Furtwängler's conjecture is false. We refer to Zong [25, 26] for detailed accounts on this fascinating problem and to pages 82–84 of Gruber and Lekkerkerker [11] for some generalizations.

Let P denote an n-dimensional centrally symmetric convex polytope, let $\tau(P)$ be the smallest integer k, such that P can form a k-fold translative tiling in $\mathbb{E}^n$, and let $\tau^*(P)$ be the smallest integer k, such that P can form a k-fold lattice tiling in $\mathbb{E}^n$. For convenience, we define $\tau(P)=\infty$, if P cannot form translative tiling of any multiplicity. Clearly, for every centrally symmetric convex polytope, we have

$$\tau(P)\leq\tau^*(P).$$

In 1994, Bolle [2] proved that *every centrally symmetric lattice polygon is a lattice multiple tile.* However, little is known about the multiplicity. Let Λ denote the two-dimensional integer lattice and let P_8 denote the octagon with vertices $(\frac{1}{2},\frac{3}{2})$, $(\frac{3}{2},\frac{1}{2})$, $(\frac{3}{2},-\frac{1}{2})$, $(\frac{1}{2},-\frac{3}{2})$, $(-\frac{1}{2},-\frac{3}{2})$, $(-\frac{3}{2},-\frac{1}{2})$, $(-\frac{3}{2},\frac{1}{2})$ and $(-\frac{1}{2},\frac{3}{2})$, as shown in Figure 1. As a particular example of Bolle's theorem, it was discovered by Gravin, Robins and Shiryaev [10] that $P_8+\Lambda$ *is a sevenfold lattice tiling of* $\mathbb{E}^2$.

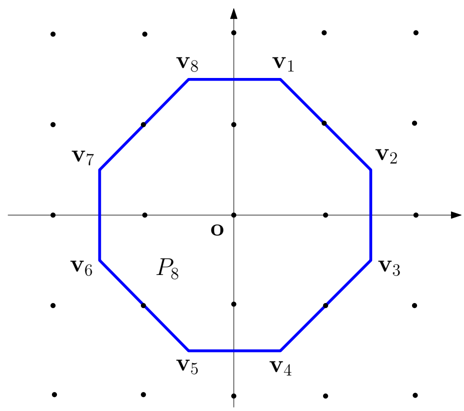

Figure 1

In 2000, Kolountzakis [14] proved that, if D is a two-dimensional convex domain, which is not a parallelogram, and $D + X$ is a multiple tiling in $\mathbb{E}^2$, then X must be a finite union of translated two-dimensional lattices. In 2013, a similar result in $\mathbb{E}^3$ was discovered by Gravin, Kolountzakis, Robins and Shiryaev [9].

In 2017, Yang and Zong [24] studied multiple lattice tilings by proving the following results. *Besides parallelograms and centrally symmetric hexagons, there is no other convex domain that can form any two, three or fourfold lattice tiling in the Euclidean plane. However, there are particular octagons and decagons that can form fivefold lattice tilings.* Afterwards, Zong [29] characterized all the two-dimensional fivefold lattice tiles. *A convex domain can form a fivefold lattice tiling of the Euclidean plane, if and only if it is a parallelogram, a centrally symmetric hexagon, under a suitable affine linear transformation, a centrally symmetric octagon with vertices* $\mathbf{v}_1 = (-\alpha, -\frac{3}{2})$, $\mathbf{v}_2 = (1-\alpha, -\frac{3}{2})$, $\mathbf{v}_3 = (1+\alpha, -\frac{1}{2})$, $\mathbf{v}_4 = (1-\alpha, \frac{1}{2})$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \alpha < \frac{1}{4}$, *or with vertices* $\mathbf{v}_1 = (\beta, -2)$, $\mathbf{v}_2 = (1+\beta, -2)$, $\mathbf{v}_3 = (1-\beta, 0)$, $\mathbf{v}_4 = (\beta, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$, $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $\frac{1}{4} < \beta < \frac{1}{3}$, *or a centrally symmetric decagon with* $\mathbf{u}_1 = (0,1)$, $\mathbf{u}_2 = (1,1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (\frac{3}{2}, 0)$, $\mathbf{u}_5 = (1, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges.*

This paper proves the following theorems.

Theorem 1.1. — *Besides parallelograms and centrally symmetric convex hexagons, there is no other convex domain that can form a two, three, or fourfold translative tiling of the Euclidean plane.*

THEOREM 1.2. — *A convex domain can form a fivefold translative tiling of the Euclidean plane, if and only if it is a parallelogram, a centrally symmetric hexagon, under a suitable affine linear transformation, a centrally symmetric octagon with vertices* $\mathbf{v}_1 = \left(\frac{3}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_2 = \left(-\frac{1}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_3 = \left(\frac{\alpha}{4} - \frac{3}{2}, 0\right)$, $\mathbf{v}_4 = \left(\frac{\alpha}{4} - \frac{3}{2}, 1\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \alpha < \frac{2}{3}$, *or with vertices* $\mathbf{v}_1 = (2 - \beta, -3)$, $\mathbf{v}_2 = (-\beta, -3)$, $\mathbf{v}_3 = (-2, -1)$, $\mathbf{v}_4 = (-2, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \beta \leq 1$, *or a centrally symmetric decagon with* $\mathbf{u}_1 = (0, 1)$, $\mathbf{u}_2 = (1, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (\frac{3}{2}, 0)$, $\mathbf{u}_5 = (1, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges.*

REMARK 1.3. — Comparing this with Zong's work [29], it is easy to show that all fivefold translative tiles are fivefold lattice tiles.

2. Basic preparation

Let P_{2m} denote a centrally symmetric convex $2m$-gon centered at the origin, let $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_{2m}$ be the $2m$ vertices of P_{2m} enumerated clock-wise, and let G_1, G_2, …, G_{2m} be the $2m$ edges, where G_i is ended by $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$. For convenience, we write

$$V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2m}\}$$

and

$$\Gamma = \{G_1, G_2, \ldots, G_{2m}\}.$$

Assume that $P_{2m} + X$ is a $\tau(P_{2m})$-fold translative tiling in $\mathbb{E}^2$, where $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$ is a discrete multiset with $\mathbf{x}_1 = \mathbf{o}$. Now, let us observe the local structures of $P_{2m} + X$ at the vertices $\mathbf{v} \in V + X$.

Let $X^{\mathbf{v}}$ denote the subset of X consisting of all points $\mathbf{x}_i$, such that

$$\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i.$$

Since $P_{2m} + X$ is a multiple tiling, the set $X^{\mathbf{v}}$ can be divided into disjoint subsets $X_1^{\mathbf{v}}$, $X_2^{\mathbf{v}}$, …, $X_t^{\mathbf{v}}$, such that the translates in $P_{2m} + X_j^{\mathbf{v}}$ can be re-enumerated as $P_{2m} + \mathbf{x}_1^j$, $P_{2m} + \mathbf{x}_2^j$, …, $P_{2m} + \mathbf{x}_{s_j}^j$ satisfying the following conditions (as shown by Figure 2 in two cases):

1. $\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i^j$ *holds for all* $i = 1, 2, \ldots, s_j$.
2. *Let* $\angle_i^j$ *denote the inner angle of* $P_{2m} + \mathbf{x}_i^j$ *at* $\mathbf{v}$ *with two half-line edges* $L_{i,1}^j$ *and* $L_{i,2}^j$, *where* $L_{i,1}^j$, $\mathbf{x}_i^j - \mathbf{v}$ *and* $L_{i,2}^j$ *are in clock order. Then, the inner angles join properly as*

$$L_{i,2}^j = L_{i+1,1}^j$$

holds for all $i = 1, 2, \ldots, s_j$, *where* $L_{s_j+1,1}^j = L_{1,1}^j$.

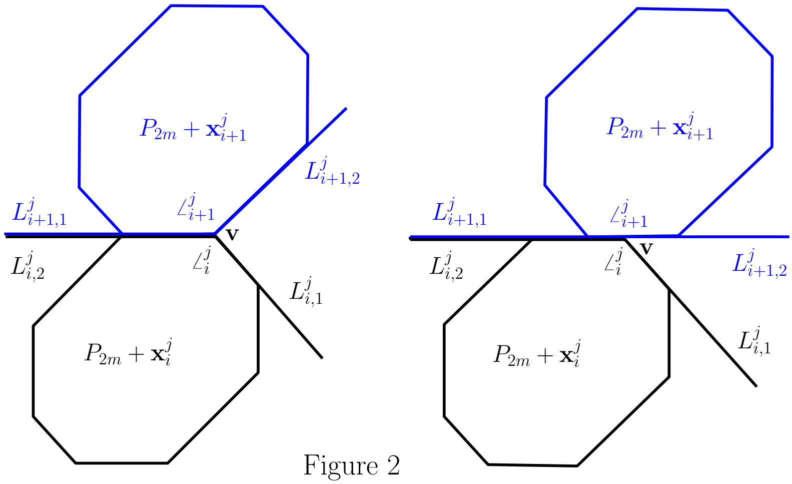

Figure 2

For convenience, we call such a sequence $P_{2m}+\mathbf{x}^j_1$, $P_{2m}+\mathbf{x}^j_2$, ..., $P_{2m}+\mathbf{x}^j_{s_j}$ an *adjacent wheel* at $\mathbf{v}$. In other words, if $\mathbf{v}$ belongs to the boundary of a tile, then we follow this tile around, moving from tile to tile, until it closes up again. It is easy to see that

$$\sum_{i=1}^{s_j}\angle^j_i = 2w_j\cdot\pi$$

hold for positive integers w_j. Then we define

$$\varpi(\mathbf{v})=\sum_{j=1}^{t}w_j=\frac{1}{2\pi}\sum_{j=1}^{t}\sum_{i=1}^{s_j}\angle^j_i$$

and

$$\varphi(\mathbf{v})=\sharp\left\{\mathbf{x}_i:\ \mathbf{x}_i\in X,\ \mathbf{v}\in \mathrm{int}(P_{2m})+\mathbf{x}_i\right\}.$$

In other words, $\varpi(\mathbf{v})$ is the tiling multiplicity produced by the boundary, and $\varphi(\mathbf{v})$ is the tiling multiplicity produced by the interior.

Clearly, if $P_{2m}+X$ is a $\tau(P_{2m})$-fold translative tiling of $\mathbb{E}^2$, then

$$\tau(P_{2m})=\varphi(\mathbf{v})+\varpi(\mathbf{v}) \tag{1}$$

holds for all $\mathbf{v}\in V+X$.

Now we introduce some basic results which will be useful in this paper.

Lemma 2.1. — *Assume that P_{2m} is a centrally symmetric convex $2m$-gon centered at the origin and $P_{2m}+X$ is a $\tau(P_{2m})$-fold translative tiling of the plane, where $m\geq 4$. If $\mathbf{v}\in V+X$ is a vertex and $G\in\Gamma+X$ is an edge with $\mathbf{v}$ as one* *of its two ends, then there are at least* $\lceil (m-3)/2 \rceil$ *different translates* $P_{2m}+\mathbf{x}_i$ *satisfying both*

$$\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i$$

and

$$G \setminus \{\mathbf{v}\} \subset \operatorname{int}(P_{2m}) + \mathbf{x}_i.$$

Proof. — Since adjacent wheels are circular, without loss of generality, let $P_{2m}+\mathbf{x}_1$, $P_{2m}+\mathbf{x}_2$, …, $P_{2m}+\mathbf{x}_s$ be an adjacent wheel at $\mathbf{v}$, such that G is the first edge appearing in the wheel and let $\angle_i$ denote the inner angle of $P_{2m}+\mathbf{x}_i$ at the vertex $\mathbf{v}$.

Let n denote the smallest index, such that

$$\sum_{i=1}^{n} \angle_i = \omega \cdot \pi \tag{2}$$

holds with some positive integer ω. Then the angle sequence $\angle_1$, $\angle_2$, …, $\angle_n$ has no pair $\angle_i$ and $\angle_j$ satisfying $\angle_i = \angle_j$. Otherwise, one can make the index n smaller. If $\angle_j$ and $\angle_{j+k}$ are two opposite angles of P_{2m} appearing in the angle sequence with $1 \le j < j+k \le n$, it is easy to see that

$$\sum_{i=0}^{k-1} \angle_{j+i} = \omega' \cdot \pi$$

holds with a positive integer ω' and $\omega \ge \omega'$. Therefore, to estimate ω we may assume that the angle sequence $\angle_1$, $\angle_2$, …, $\angle_n$ has no opposite angle pair of P_{2m}.

Clearly, $\angle_i = \pi$, if and only if $\mathbf{v}$ is a relative interior point of an edge of $P_{2m}+\mathbf{x}_i$ (such as $\angle_5$ in Figure 3) and, therefore,

$$\sum_{i=1}^{n} \angle_i < n \cdot \pi. \tag{3}$$

On the other hand, if ℓ of the n angles are π and $n-\ell < m$, then $m-n+\ell$ pairs of the opposite angles of P_{2m} do not appear in the angle sequence. Thus, we have

$$\sum_{i=1}^{n} \angle_i > \ell \cdot \pi + (m-1) \cdot \pi - (m-n+\ell) \cdot \pi = (n-1) \cdot \pi, \tag{4}$$

which together with (3) contradicts (2). Therefore, to avoid the contradiction, we must have

$$n - \ell = m,$$

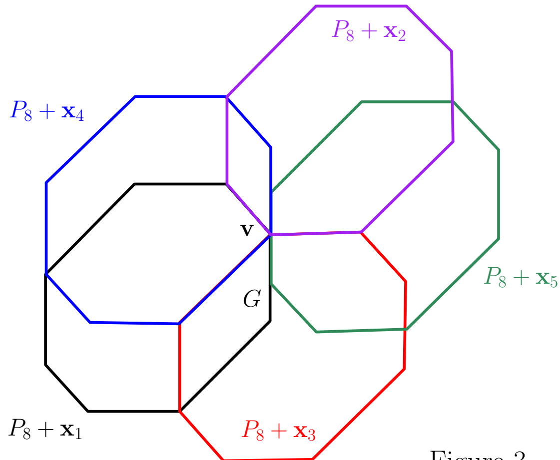

Figure 3

and each pair of the opposite angles of P_{2m} has a representative in the sequence $\angle_1$, $\angle_2$, …, $\angle_n$. Consequently, we have

$$\sum_{i=1}^{n} \angle_i \geq \frac{(2m-2)\cdot\pi}{2} = (m-1)\cdot\pi. \tag{5}$$

If $\mathbf{v} \in \partial(P_{2m}) + \mathbf{x}_i$, $G \subset P_{2m} + \mathbf{x}_i$, and G is not an edge of $P_{2m} + \mathbf{x}_i$, then by the convexity and symmetry of P_{2m} it follows that $G \setminus \{\mathbf{v}\} \subset \mathrm{int}(P_{2m}) + \mathbf{x}_i$. Therefore, it follows by (5) that $G \setminus \{\mathbf{v}\}$ is covered by at least

$$\left\lceil \frac{m-1}{2} \right\rceil - 1 = \left\lceil \frac{m-3}{2} \right\rceil$$

of the s translates $\mathrm{int}(P_{2m}) + \mathbf{x}_i$. Lemma 2.1 is proved. □

Lemma 2.2. — *Assume that P_{2m} is a centrally symmetric convex $2m$-gon centered at the origin, $P_{2m} + X$ is a translative multiple tiling of the plane, and $\mathbf{v} \in V + X$. Then we have*

$$\varpi(\mathbf{v}) = \kappa \cdot \frac{m-1}{2} + \ell \cdot \frac{1}{2},$$

where κ is a positive integer, and ℓ is the number of the edges in $\Gamma + X$, which take $\mathbf{v}$ as an interior point.

Proof. — Assume that $P_{2m} + \mathbf{x}_1$, $P_{2m} + \mathbf{x}_2$, …, $P_{2m} + \mathbf{x}_s$ is an adjacent wheel at $\mathbf{v}$ and let $\angle_i$ denote the inner angle of $P_{2m} + \mathbf{x}_i$ at $\mathbf{v}$. Of course, we have $\angle_i = \pi$, if $\mathbf{v}$ is not a vertex of $P_{2m} + \mathbf{x}_i$.

Assume that $\angle_1 < \pi$ and let n to be the smallest index, such that

$$\sum_{i=1}^{n} \angle_i = \omega\pi \tag{6}$$

holds with a positive integer ω. We proceed to show that each pair of the opposite angles of P_{2m} has one and only one representative in $\angle_1$, $\angle_2$, …, $\angle_n$.

If, on the contrary, $\angle_j$ and $\angle_{j+k}$ are two of these n angles, $\angle_j < \pi$, which are either identical or opposite. Then, it is easy to see that

$$\sum_{i=0}^{k-1} \angle_{j+i} = \omega'\pi \tag{7}$$

holds with a positive integer ω'. For convenience, we assume that $\angle_j$, $\angle_{j+1}$, …, $\angle_{j+k-1}$ have neither a identical nor an opposite pair. Then, by repeating the argument between (2) and (5) in the proof of Lemma 2.1, one can deduce that each pair of the opposite angles of P_{2m} has one and only one representative in $\angle_j$, $\angle_{j+1}$, …, $\angle_{j+k-1}$. Consequently, one of these k angles is either identical or opposite to $\angle_1$, which contradicts the minimum assumption on n and ω.

Then, applying the argument between (2) and (5) to $\angle_1$, $\angle_2$, …, $\angle_n$, it can be deduced that

$$\sum_{i=1}^{n} \angle_i = (m-1)\pi + \ell_1\pi, \tag{8}$$

where ℓ_1 is the number of the π angles in $\angle_1$, $\angle_2$, …, $\angle_n$. In fact, it is $n - m$.

By repeating this process to $\angle_{n+1}$, $\angle_{n+2}$, …, $\angle_s$ if necessary, it follows that

$$\sum_{i=1}^{s} \angle_i = \kappa'(m-1)\pi + \ell'\pi, \tag{9}$$

and, therefore,

$$\varpi(\mathbf{v}) = \frac{1}{2\pi}\sum\sum_{i=1}^{s} \angle_i = \kappa\cdot\frac{m-1}{2} + \ell\cdot\frac{1}{2}, \tag{10}$$

where the first sum is over all adjacent wheels at $\mathbf{v}$, κ' and κ are suitable positive integers, and ℓ' and ℓ are suitable nonnegative integers. In fact, ℓ is the number of the edges that take $\mathbf{v}$ as an interior point.

Lemma 2.2 is proved. □

LEMMA 2.3. — *If m is an odd positive integer, P_{2m} is a centrally symmetric convex $2m$-gon centered at the origin $\mathbf{o}$, and $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_{2m}$ are the middle points of its edges enumerated clockwise, then we have*

$$\sum_{i=1}^{m} (-1)^i \mathbf{u}_i = \mathbf{o}.$$

Proof. — Since $\mathbf{u}_i$ is the middle point of G_i, we have

$$\begin{cases} \mathbf{v}_2 = 2\mathbf{u}_1 - \mathbf{v}_1, \\ \mathbf{v}_3 = 2\mathbf{u}_2 - \mathbf{v}_2, \\ \quad \cdots \\ \mathbf{v}_{m+1} = 2\mathbf{u}_m - \mathbf{v}_m, \end{cases}$$

which implies

$$-\mathbf{v}_1 = \mathbf{v}_{m+1} = -\mathbf{v}_1 - 2\sum_{i=1}^{m}(-1)^i\mathbf{u}_i \tag{11}$$

and, therefore,

$$\sum_{i=1}^{m}(-1)^i\mathbf{u}_i = \mathbf{o}.$$

The lemma is proved. □

The following lemma will be useful in the proofs of Lemma 3.5 and Lemma 3.8.

Lemma 2.4 (Bolle [2]). — *A convex polygon is a k-fold lattice tile for a lattice Λ and some positive integer k, if and only if the following conditions are satisfied:*

1. *It is centrally symmetric.*
2. *When it is centred at the origin, in the relative interior of each edge G there is a point of $\frac{1}{2}\Lambda$.*
3. *If the midpoint of G is not in $\frac{1}{2}\Lambda$, then G is a lattice vector of Λ.*

3. Proofs of the theorems

Lemma 3.1. — *Let P_{2m} be a centrally symmetric convex $2m$-gon, then*

$$\tau(P_{2m}) \geq \begin{cases} m-1, & \text{if } m \text{ is even,} \\ m-2, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. — Assume that $P_{2m}+X$ is a $\tau(P_{2m})$-fold translative tiling in the Euclidean plane and assume that $\mathbf{v} \in V+X$. Then it follows by Lemma 2.1 that

$$\varphi(\mathbf{v}) \geq \left\lceil \frac{m-3}{2} \right\rceil. \tag{12}$$

Let $P_{2m}+\mathbf{x}_1$, $P_{2m}+\mathbf{x}_2$, …, $P_{2m}+\mathbf{x}_s$ be an adjacent wheel at $\mathbf{v}$ and let $\angle_1$, $\angle_2$, …, $\angle_s$ be the corresponding angle sequence. By (5) we have

$$\varpi(\mathbf{v}) \geq \frac{1}{2\pi}\sum_{i=1}^{s}\angle_i \geq \left\lceil \frac{m-1}{2} \right\rceil. \tag{13}$$

Then, it follows by (1), (12) and (13) that

$$\tau(P_{2m}) \geq \left\lceil \frac{m-3}{2} \right\rceil + \left\lceil \frac{m-1}{2} \right\rceil = \begin{cases} m-1, & \text{if } m \text{ is even,} \\ m-2, & \text{if } m \text{ is odd.} \end{cases}$$

Lemma 3.1 is proved. □

LEMMA 3.2. — *Let P_{14} be a centrally symmetric convex tetradecagon, then*

$$\tau(P_{14}) \geq 6.$$

Proof. — Assume that $P_{14}+X$ is a $\tau(P_{14})$-fold translative tiling in $\mathbb{E}^2$ and $\mathbf{v} \in V+X$. By Lemma 2.1 and Lemma 2.2, we have

$$\varphi(\mathbf{v}) \geq \left\lceil \frac{7-3}{2} \right\rceil = 2 \tag{14}$$

and

$$\varpi(\mathbf{v}) = \kappa \cdot 3 + \ell \cdot \frac{1}{2} \geq 3, \tag{15}$$

where κ is a positive integer and ℓ is a nonnegative integer.

Now, to show the lemma it is sufficient to deal with the following two cases.

Case 1. — $\varpi(\mathbf{v}) \geq 4$ *holds for a vertex* $\mathbf{v} \in V+X$. Then, by (1) and (14) we get

$$\tau(P_{14}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{16}$$

Case 2. — $\varpi(\mathbf{v}) = 3$ *holds for every vertex* $\mathbf{v} \in V+X$. First, let us observe a simple fact. If $\varpi(\mathbf{v}) = 3$ holds at $\mathbf{v} \in V+X$ and $P_{14}+\mathbf{x}_1$, $P_{14}+\mathbf{x}_2$, …, $P_{14}+\mathbf{x}_s$ is an adjacent wheel at $\mathbf{v}$, then it follows from (15) that s must be seven and $\mathbf{v}$ is a common vertex of all these translates, as shown by Figure 4. Then, by Lemma 2.1, every vertex $\mathbf{v}_i^*$ connecting with $\mathbf{v}$ by an edge is an interior point of two of the seven translates in the wheel.

Then, we have

$$\varpi(\mathbf{v}_1^*) = \varpi(\mathbf{v}_2^*) = \varpi(\mathbf{v}_3^*) = \varpi(\mathbf{v}_4^*) = \varpi(\mathbf{v}_5^*) = \varpi(\mathbf{v}_6^*) = \varpi(\mathbf{v}_7^*) = 3. \tag{17}$$

Therefore, for each vertex $\mathbf{v}_i^*$, there are two different points $\mathbf{y}_{i,1}, \mathbf{y}_{i,2} \in X$, such that

$$\mathbf{v}_i^* \in \partial(P_{14}) + \mathbf{y}_{i,j}, \qquad j = 1,\ 2$$

and

$$\mathbf{v} \in \text{int}(P_{14}) + \mathbf{y}_{i,j}, \qquad j = 1,\ 2.$$

If $\mathbf{y}_{i,j} \notin \{\mathbf{y}_{1,1}, \mathbf{y}_{1,2}\}$ holds for one of these points, and then we have

$$\varphi(\mathbf{v}) \geq 3$$

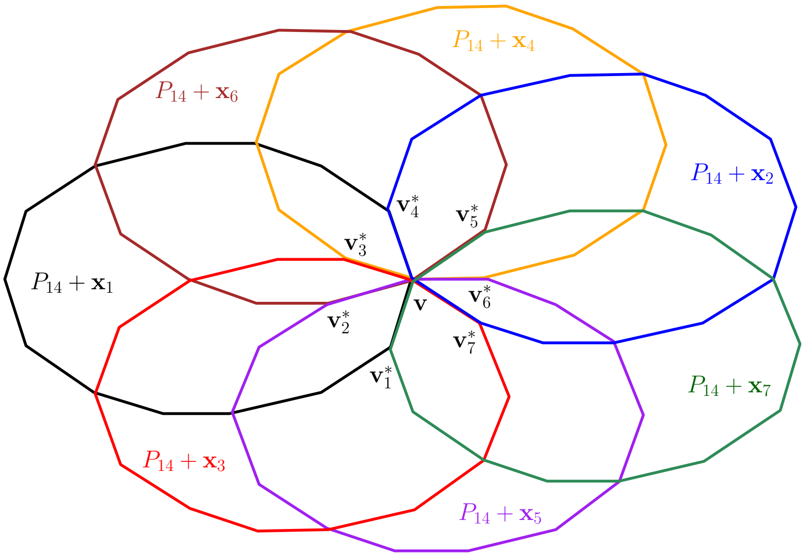

Figure 4

and, therefore,

$$\tau(P_{14}) \geq \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{18}$$

If $\mathbf{y}_{i,j} \in \{\mathbf{y}_{1,1}, \mathbf{y}_{1,2}\}$ holds for all of these points, then we must have

$$\{\mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*, \mathbf{v}_4^*, \mathbf{v}_5^*, \mathbf{v}_6^*, \mathbf{v}_7^*\} \subset \partial(P_{14}) + \mathbf{y}_{1,1}.$$

It is known that $(D+\mathbf{x}) \cap (D+\mathbf{y})$ is centrally symmetric for all $\mathbf{x}$ and $\mathbf{y}$ whenever D is centrally symmetric. Then, by Figure 4 it is easy to see that $(P_{14}+\mathbf{y}_{1,1}) \cap (P_{14}+\mathbf{x}_1)$ is a parallelogram with vertices $\mathbf{v}_1^*$, $\mathbf{v}$, $\mathbf{v}_4^*$ and $\mathbf{v}_1^* + (\mathbf{v}_4^* - \mathbf{v})$, and $(P_{14}+\mathbf{y}_{1,1}) \cap (P_{14}+\mathbf{x}_7)$ is a parallelogram with vertices $\mathbf{v}_1^*$, $\mathbf{v}$, $\mathbf{v}_5^*$ and $\mathbf{v}_1^* + (\mathbf{v}_5^* - \mathbf{v})$. Consequently, by symmetry one can deduce that $P_{14}+\mathbf{y}_{1,1}$ is an hexagon with vertices $\mathbf{v}_1^*$, $\mathbf{v}_1^* + (\mathbf{v}_4^* - \mathbf{v})$, $\mathbf{v}_4^*$, $\mathbf{v} + (\mathbf{v} - \mathbf{v}_1^*)$, $\mathbf{v}_5^*$ and $\mathbf{v}_1^* + (\mathbf{v}_5^* - \mathbf{v})$, which contradicts the assumption that P_{14} is a tetradecagon.

As a conclusion, for every centrally symmetric convex tetradecagon, we have

$$\tau(P_{14}) \geq 6. \tag{19}$$

The lemma is proved. □

Lemma 3.3. — *Let P_{12} be a centrally symmetric convex dodecagon, then we have*

$$\tau(P_{12}) \geq 6.$$

Proof. — First of all, it follows by Lemma 2.1 that

$$\varphi(\mathbf{v}) \geq \left\lceil \frac{6-3}{2} \right\rceil = 2 \tag{20}$$

holds for all $\mathbf{v} \in V + X$. On the other hand, by Lemma 2.2 we have

$$\varpi(\mathbf{v}) = \kappa \cdot \frac{6-1}{2} + \ell \cdot \frac{1}{2} \geq 3. \tag{21}$$

Thus, to show the lemma it is sufficient to deal with the following two cases.

Case 1. — $\varpi(\mathbf{v}) \geq 4$ *holds for a vertex* $\mathbf{v} \in V + X$. Then it follows by (1) and (20) that

$$\tau(P_{12}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{22}$$

Case 2. — $\varpi(\mathbf{v}) = 3$ *holds for a vertex* $\mathbf{v} \in V + X$. Assume that $P_{12} + \mathbf{x}_1$, $P_{12} + \mathbf{x}_2$, …, $P_{12} + \mathbf{x}_s$ is an adjacent wheel at $\mathbf{v}$. By (21) it can be deduced that there is a $G \in \Gamma + X$, such that

$$\mathbf{v} \in \mathrm{int}(G).$$

Let $\mathbf{v}'$ and $\mathbf{v}^*$ denote the two ends of G. By Lemma 2.1 and the convexity of P_{12} it follows that X has four different points $\mathbf{y}_1'$, $\mathbf{y}_2'$, $\mathbf{y}_1^*$ and $\mathbf{y}_2^*$ satisfying

$$\begin{aligned} \mathbf{v}' &\in \partial(P_{12}) + \mathbf{y}_i', & i &= 1,\ 2, \\ \mathbf{v}^* &\in \partial(P_{12}) + \mathbf{y}_i^*, & i &= 1,\ 2, \\ \mathbf{v} &\in \mathrm{int}(P_{12}) + \mathbf{y}_i', & i &= 1,\ 2, \qquad \text{and} \\ \mathbf{v} &\in \mathrm{int}(P_{12}) + \mathbf{y}_i^*, & i &= 1,\ 2. \end{aligned}$$

Consequently, we have

$$\varphi(\mathbf{v}) \geq 4,$$

and, therefore,

$$\tau(P_{12}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 7. \tag{23}$$

The conclusion of these two cases that

$$\tau(P_{12}) \geq 6 \tag{24}$$

holds for every centrally symmetric dodecagon. Lemma 3.3 is proved. □

Lemma 3.4 (Yang and Zong [24]). — *Let P_{10} be a centrally symmetric decagon centred at the origin, then we have*

$$\tau^*(P_{10}) \geq 5.$$

Lemma 3.5. — *Let P_{10} be a centrally symmetric decagon centred at the origin, then we have*

$$\tau(P_{10}) \geq 5,$$

where the equality holds, if and only if P_{10} is a fivefold lattice tile.

Proof. — Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_{10}$ denote the ten vertices of P_{10} enumerated clockwise, let G_i denote the edge with ends $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ and let $\mathbf{u}_i$ denote the middle point of G_i. Suppose that X is a discrete subset of $\mathbb{E}^2$, and $P_{10}+X$ is a $\tau(P_{10})$-fold translative tiling of the plane. First of all, it follows from Lemma 2.1 that

$$\varphi(\mathbf{v}) \geq \left\lceil \frac{5-3}{2} \right\rceil = 1 \tag{25}$$

holds for every $\mathbf{v} \in V+X$. On the other hand, by Lemma 2.2 we have

$$\varpi(\mathbf{v}) = \kappa \cdot 2 + \ell \cdot \frac{1}{2}, \tag{26}$$

where κ is a positive integer, and ℓ is the number of the edges that contain $\mathbf{v}$ as a relative interior point.

Now we prove the lemma by dealing with two cases.

Case 1. — $\ell \neq 0$ *holds at a vertex* $\mathbf{v} \in V+X$. In other words, there is an edge $G \in \Gamma + X$, such that $\mathbf{v} \in \mathrm{int}(G)$. Clearly, by (26) we have $\varpi(\mathbf{v}) \geq 3$.

Suppose that $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ are the two ends of G. By Lemma 2.1, there are two different points $\mathbf{y}_1 \in X^{\mathbf{v}_1^*}$ and $\mathbf{y}_2 \in X^{\mathbf{v}_2^*}$, such that

$$\mathbf{v} \in (\mathrm{int}(P_{10}) + \mathbf{y}_1) \cap (\mathrm{int}(P_{10}) + \mathbf{y}_2).$$

Then we have $\varphi(\mathbf{v}) \geq 2$. If $\varpi(\mathbf{v}) \geq 4$, one can deduce that

$$\tau(P_{10}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{27}$$

If $\varpi(\mathbf{v}) = 3$, by (26) one can deduce that $P_{10} + X^{\mathbf{v}}$ consists of seven translates $P_{10} + \mathbf{x}_1$, $P_{10} + \mathbf{x}_2$, ..., $P_{10} + \mathbf{x}_7$, and there is another $G' \in \Gamma + X$, which contains $\mathbf{v}$ as an interior point. Suppose that G is an edge of $P_{10} + \mathbf{x}_6$, and G' has two ends $\mathbf{v}_5^*$ and $\mathbf{v}_6^*$. We deal with three subcases.

Subcase 1.1. — $G' || G$ *and* $G' \neq G$. Without loss of generality, we assume that $\mathbf{v}_5^*$ is between $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$. Then, by Lemma 2.1 we have $\mathbf{y}_i \in X^{\mathbf{v}_i^*}$, such that

$$\mathbf{v} \in \mathrm{int}(P_{10}) + \mathbf{y}_i, \quad i = 1,\ 2,\ 5.$$

It is obvious that $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_5$ are pairwise distinct. Thus, we have $\varphi(\mathbf{v}) \geq 3$ and, therefore,

$$\tau(P_{10}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{28}$$

Subcase 1.2. — $G' = G$. Then $P_{10} + X^{\mathbf{v}}$ can be divided into two adjacent wheels, as shown by Figure 5.

Let $P_{10} + \mathbf{x}_6$ and $P_{10} + \mathbf{x}_7$ be the two translates that contain G as a common edge. Without loss of generality, suppose that $G = G_6 + \mathbf{x}_7$ and $\mathbf{v} = \mathbf{v}_7 + \mathbf{x}_1$, as shown in Figure 5. Let L be the straight line determined by $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, let G_1^* be the edge of $P_{10} + \mathbf{x}_1$ lying on L with ends $\mathbf{v}$ and $\mathbf{v}_3^*$ and let G_2^* be the edge of $P_{10} + \mathbf{x}_1$ with ends $\mathbf{v}$ and $\mathbf{v}_4^*$.

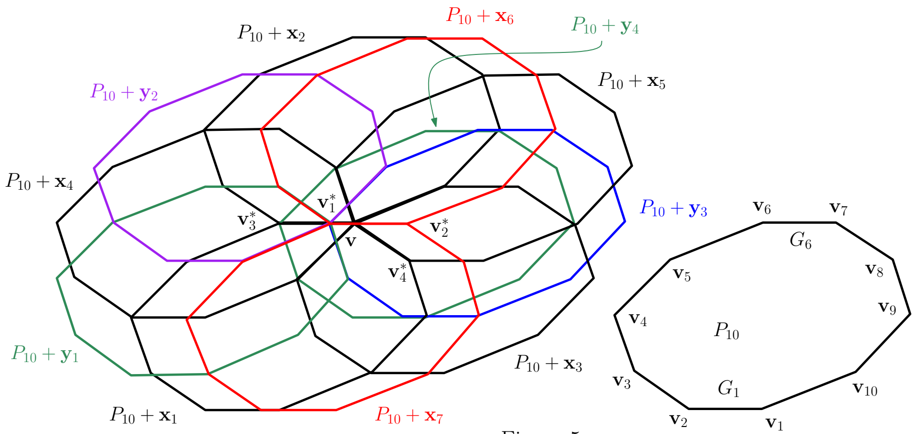

Figure 5

It is easy to see that $\varpi(\mathbf{v}_1^*) \geq 3$ and $\varphi(\mathbf{v}_1^*) \geq 2$, since $\mathbf{v}_1^*$ is an interior point of G_1^*. If $\varpi(\mathbf{v}_1^*) \geq 4$, then we have $\tau(P_{10}) \geq 6$. If $\varpi(\mathbf{v}_1^*) = 3$, the adjacent wheel at $\mathbf{v}_1^*$ can be divided into two adjacent wheels. Since $\mathbf{v}_1^* = \mathbf{v}_6 + \mathbf{x}_7$, by Lemma 2.1 and the structure of the adjacent wheel that consists of five translates, we have three points $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3 \in X^{\mathbf{v}_1^*}$, such that

$$\mathbf{v}_1^* = \mathbf{v}_8 + \mathbf{y}_1, \quad \mathbf{v}_3^* \in \operatorname{int}(P_{10}) + \mathbf{y}_1, \tag{29}$$

$$\mathbf{v}_1^* = \mathbf{v}_{10} + \mathbf{y}_2, \quad \mathbf{v}_3^* \in \operatorname{int}(P_{10}) + \mathbf{y}_2, \tag{30}$$

and

$$\mathbf{v}_1^* = \mathbf{v}_4 + \mathbf{y}_3, \quad \mathbf{v} \in \operatorname{int}(P_{10}) + \mathbf{y}_3. \tag{31}$$

Clearly, we also have $\mathbf{v}_3^* \in \operatorname{int}(P_{10}) + \mathbf{x}_4$. Since $\mathbf{v}_1^* \in \operatorname{int}(P_{10}) + \mathbf{x}_4$, we thus have $\mathbf{x}_4 \notin \{\mathbf{y}_1, \mathbf{y}_2\}$, $\varphi(\mathbf{v}_3^*) \geq 3$ and

$$\tau(P_{10}) = \varphi(\mathbf{v}_3^*) + \varpi(\mathbf{v}_3^*) \geq 5, \tag{32}$$

where the equality may hold only if $\varpi(\mathbf{v}_3^*) = 2$. When $\varpi(\mathbf{v}_3^*) = 2$, by Lemma 2.1 and the structure of the adjacent wheel with five translates, there is a point $\mathbf{y}_4 \in X^{\mathbf{v}_3^*}$, such that

$$\mathbf{v}_3^* = \mathbf{v}_4 + \mathbf{y}_4, \quad \mathbf{v} \in \operatorname{int}(P_{10}) + \mathbf{y}_4. \tag{33}$$

Furthermore, by Lemma 2.1 we have a point $\mathbf{y}_5 \in X^{\mathbf{v}_4^*}$, such that $\mathbf{v} \in \operatorname{int}(P_{10}) + \mathbf{y}_5$. By (31), (33) and convexity we have

$$\mathbf{v}_4^* \in (\operatorname{int}(P_{10}) + \mathbf{y}_3) \cap (\operatorname{int}(P_{10}) + \mathbf{y}_4),$$

$\mathbf{y}_5 \notin \{\mathbf{y}_3, \mathbf{y}_4\}$, $\varphi(\mathbf{v}) \geq 3$ and, thus,

$$\tau(P_{10}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{34}$$

Subcase 1.3. — $G' \nparallel G$. Suppose that G is an edge of $P_{10}+\mathbf{x}_6$ with ends $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, which contains $\mathbf{v}$ as an interior point. Since $G' \nparallel G$, there is a translate $P_{10}+\mathbf{x}'$ in $X^{\mathbf{v}}$ that meets $P_{10}+\mathbf{x}_6$ at a non-singleton part of G. Let L be the line determined by $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$. Let G_1^* be the edge of $P_{10}+\mathbf{x}'$ lying on L with ends $\mathbf{v}_3^*$ and $\mathbf{v}$, where $\mathbf{v}_1^* \in \mathrm{int}(G_1^*)$.

First, since $\ell \neq 0$ at $\mathbf{v}_1^*$, by (26) we have

$$\varpi(\mathbf{v}_1^*) \geq 3. \tag{35}$$

On the other hand, since $G' \nparallel G$, the local arrangement $P_{10}+X^{\mathbf{v}}$ cannot be divided into smaller adjacent wheels. Then, two of the seven translates in $P_{10}+X^{\mathbf{v}}$ contain both $\mathbf{v}_3^*$ and $\mathbf{v}_1^*$ as interior points. Furthermore, by Lemma 2.1, there is a translate $P_{10}+\mathbf{y}$ in $P_{10}+X^{\mathbf{v}_3^*}$ that contains $\mathbf{v}_1^*$ as an interior point and, therefore, $\varphi(\mathbf{v}_1^*) \geq 3$. Then, by (35) we get

$$\tau(P_{10}) = \varphi(\mathbf{v}_1^*) + \varpi(\mathbf{v}_1^*) \geq 6. \tag{36}$$

Case 2. — $\ell = 0$ *holds for all vertices* $\mathbf{v} \in V+X$. Then by (26) it is sufficient to assume that $\varpi(\mathbf{v})$ can take only two values, 2 or 4.

Subcase 2.1. — $\varpi(\mathbf{v}) = 4$ *holds at a vertex* $\mathbf{v} \in V+X$. Then the local arrangements $P_{10}+X^{\mathbf{v}}$ can be divided into two adjacent wheels, each containing five translates. Suppose that $P_{10}+\mathbf{x}_1$, $P_{10}+\mathbf{x}_2$, …, $P_{10}+\mathbf{x}_5$ is such a wheel at $\mathbf{v}$ and $\mathbf{v}=\mathbf{v}_k+\mathbf{x}_1$. Then, as shown in Figure 6, the wheel can be determined by $P_{10}+\mathbf{x}_1$ explicitly as follows:

$$\begin{aligned}
\mathbf{v} &= \mathbf{v}_{k+4}+\mathbf{x}_2, \quad G_{k+4}+\mathbf{x}_2 = G_{k-1}+\mathbf{x}_1,\\
\mathbf{v} &= \mathbf{v}_{k+8}+\mathbf{x}_3, \quad G_{k+8}+\mathbf{x}_3 = G_{k+3}+\mathbf{x}_2,\\
\mathbf{v} &= \mathbf{v}_{k+2}+\mathbf{x}_4, \quad G_{k+2}+\mathbf{x}_4 = G_{k+7}+\mathbf{x}_3,\\
\mathbf{v} &= \mathbf{v}_{k+6}+\mathbf{x}_5, \quad G_{k+6}+\mathbf{x}_5 = G_{k+1}+\mathbf{x}_4,
\end{aligned}$$

where $\mathbf{v}_{10+i}=\mathbf{v}_i$ and $G_{10+i}=G_i$.

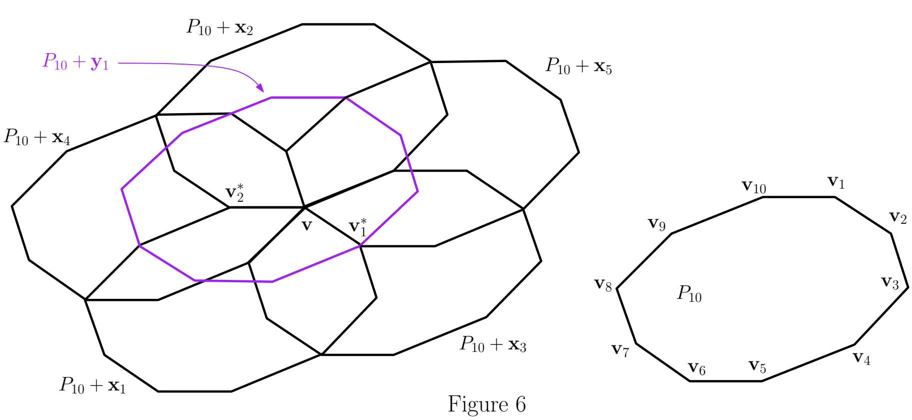

Figure 6

Without loss of generality, as shown by Figure 6, we take $\mathbf{v} = \mathbf{v}_1 + \mathbf{x}_1$, $\mathbf{v}_1^* = \mathbf{v}_2 + \mathbf{x}_1$ and $\mathbf{v}_2^* = \mathbf{v}_{10} + \mathbf{x}_1$. By Lemma 2.1, for each $\mathbf{v}_i^*$, there is a point $\mathbf{y}_i \in X^{\mathbf{v}_i^*}$, such that

$$\mathbf{v} \in \mathrm{int}(P_{10}) + \mathbf{y}_i.$$

In fact, by the previous analysis, we have $\mathbf{y}_1 = \mathbf{v}_1^* - \mathbf{v}_4$. Therefore, by convexity and symmetry,

$$\mathbf{v}_2^* \in \mathrm{int}(P_{10}) + \mathbf{y}_1.$$

Thus, the two points $\mathbf{y}_1$ and $\mathbf{y}_2$ are different. Then we have

$$\varphi(\mathbf{v}) \geq 2,$$

and

$$\tau(P_{10}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{37}$$

Subcase 2.2. — $\varpi(\mathbf{v}) = 2$ *hold for all vertices* $\mathbf{v} \in V + X$. Let P_{10} be a centrally symmetric convex decagon centred at the origin with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_{10}$ enumerated in anti-clock order. Let G_i denote the edge with ends $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ and let $\mathbf{u}_i$ denote the middle point of G_i. Then, we define

$$\begin{cases} \mathbf{a}_1 = \mathbf{u}_1 - \mathbf{u}_6, \\ \mathbf{a}_2 = \mathbf{u}_2 - \mathbf{u}_7, \\ \mathbf{a}_3 = \mathbf{u}_3 - \mathbf{u}_8, \\ \mathbf{a}_4 = \mathbf{u}_4 - \mathbf{u}_9, \\ \mathbf{a}_5 = \mathbf{u}_5 - \mathbf{u}_{10}. \end{cases}$$

By Lemma 2.3 we have

$$\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{o}. \tag{38}$$

Assume that $\mathbf{x}_1 = \mathbf{o} \in X$. Since $\varpi(\mathbf{v}) = 2$ holds for every vertex $\mathbf{v} \in V + X$, by studying the structure of the adjacent wheel at $\mathbf{v}$ we have

$$\sum z_i \mathbf{a}_i \in X, \quad z_i \in \mathbb{Z}.$$

For convenience, we define

$$\Lambda = \left\{ \sum z_i \mathbf{a}_i : \ z_i \in \mathbb{Z} \right\}. \tag{39}$$

Suppose that the adjacent wheel at $\mathbf{v}_1$ is $P_{10} + \mathbf{x}_i$, $i = 1, 2, \ldots, 5$. Let $\mathbf{v}_i^*$ be the common vertex of $P_{10} + \mathbf{x}_i$ and $P_{10} + \mathbf{x}_{i+1}$ other than $\mathbf{v}_1$, as shown in Figure 7, where $\mathbf{x}_6 = \mathbf{x}_1$ and $\mathbf{x}_1 = \mathbf{o}$. By Lemma 2.1, we have $\mathbf{y}_i \in X^{\mathbf{v}_i^*}$, such

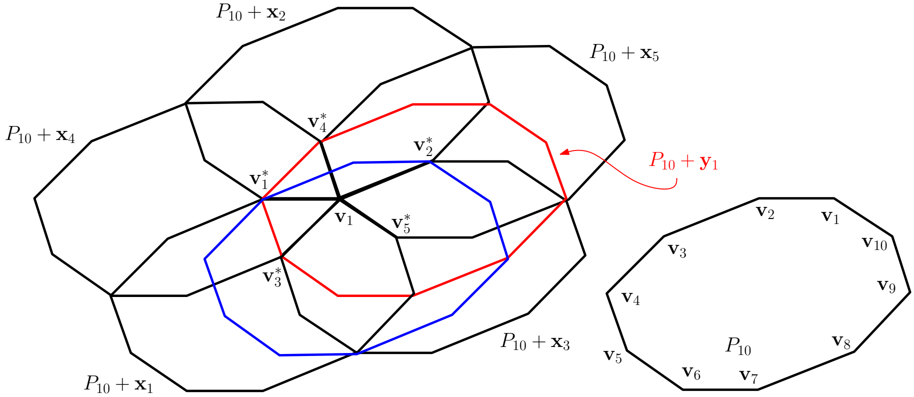

Figure 7

that $\mathbf{v}_1 \in \text{int}(P_{10}) + \mathbf{y}_i$. In fact, it can be explicitly deduced by the adjacent wheels at $\mathbf{v}_1$, $\mathbf{v}_1^*$, $\mathbf{v}_2^*$, $\mathbf{v}_3^*$, $\mathbf{v}_4^*$ and $\mathbf{v}_5^*$ that

$$\begin{cases} \mathbf{y}_1 = \mathbf{v}_1^* - \mathbf{v}_4 = \mathbf{a}_2 - \mathbf{a}_3, \\ \mathbf{y}_2 = \mathbf{v}_2^* - \mathbf{v}_{10} = \mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_4, \\ \mathbf{y}_3 = \mathbf{v}_3^* - \mathbf{v}_6 = \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_4 - \mathbf{a}_5, \\ \mathbf{y}_4 = \mathbf{v}_4^* - \mathbf{v}_2 = -\mathbf{a}_5 + \mathbf{a}_3 - \mathbf{a}_2, \\ \mathbf{y}_5 = \mathbf{v}_5^* - \mathbf{v}_8 = -\mathbf{a}_5 - \mathbf{a}_1 + \mathbf{a}_2. \end{cases} \tag{40}$$

For example, if $P_{10} + \mathbf{y}_2$ satisfying $\mathbf{v}_{10} + \mathbf{y}_2 = \mathbf{v}_2^*$, one can obtain $P_{10} + \mathbf{y}_2$ by moving P_{10} successively to $P_{10}+\mathbf{a}_1$, $P_{10}+\mathbf{a}_1-\mathbf{a}_3$, and then to $P_{10}+\mathbf{a}_1-\mathbf{a}_3+\mathbf{a}_4$.

By (40) and symmetry it can be shown that $\mathbf{y}_i \neq \mathbf{y}_{i+1}$, where $\mathbf{y}_6 = \mathbf{y}_1$. For example, if $\mathbf{y}_1 = \mathbf{y}_2$ (as shown in Figure 7), then by symmetry we will get that $(P_{10}+\mathbf{x}_2)\cap(P_{10}+\mathbf{y}_1)$ is a parallelogram and $\mathbf{y}_1 = \mathbf{v}_1^*-\mathbf{v}_3$, which contradicts the first equation of (40). Thus, any triple of $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_5\}$ cannot be identical and, therefore, $\varphi(\mathbf{v}) \geq 3$. Consequently, we get

$$\tau(P_{10}) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 5, \tag{41}$$

where the equality may hold only if $\varphi(\mathbf{v}) = 3$.

When $\varphi(\mathbf{v}) = 3$, the five points $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_5$ have to satisfy one of the following five groups of conditions:

(i) $\mathbf{y}_1 = \mathbf{y}_3$ and $\mathbf{y}_2 = \mathbf{y}_4$;
(ii) $\mathbf{y}_1 = \mathbf{y}_3$ and $\mathbf{y}_2 = \mathbf{y}_5$;
(iii) $\mathbf{y}_1 = \mathbf{y}_4$ and $\mathbf{y}_2 = \mathbf{y}_5$;
(iv) $\mathbf{y}_1 = \mathbf{y}_4$ and $\mathbf{y}_3 = \mathbf{y}_5$ and
(v) $\mathbf{y}_2 = \mathbf{y}_4$ and $\mathbf{y}_3 = \mathbf{y}_5$.

Case (i). — $\mathbf{y}_1 = \mathbf{y}_3$ *and* $\mathbf{y}_2 = \mathbf{y}_4$. Then, by (40) and (38) we get

$$\begin{cases} \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_4 - \mathbf{a}_5, \\ \mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_4 = -\mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_5, \\ \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{o}, \end{cases}$$

$$\begin{cases} 2\mathbf{a}_2 - 2\mathbf{a}_4 + \mathbf{a}_5 = \mathbf{a}_1 + (\mathbf{a}_3 - \mathbf{a}_4), \\ \mathbf{a}_4 - 2\mathbf{a}_5 = 2\mathbf{a}_1 - (\mathbf{a}_3 - \mathbf{a}_4), \\ \mathbf{a}_2 - \mathbf{a}_5 = \mathbf{a}_1 + (\mathbf{a}_3 - \mathbf{a}_4) \end{cases}$$

and, therefore,

$$\begin{cases} \mathbf{a}_1 = \mathbf{a}_1, \\ \mathbf{a}_2 = -2\mathbf{a}_1 + 4(\mathbf{a}_3 - \mathbf{a}_4), \\ \mathbf{a}_3 = -4\mathbf{a}_1 + 6(\mathbf{a}_3 - \mathbf{a}_4), \\ \mathbf{a}_4 = -4\mathbf{a}_1 + 5(\mathbf{a}_3 - \mathbf{a}_4), \\ \mathbf{a}_5 = -3\mathbf{a}_1 + 3(\mathbf{a}_3 - \mathbf{a}_4), \end{cases}$$

which means that Λ is a lattice with a basis $\{\mathbf{a}_1,\ \mathbf{a}_3 - \mathbf{a}_4\}$. Furthermore, since $\mathbf{u}_i = \frac{1}{2}\mathbf{a}_i \in \frac{1}{2}\Lambda$, it follows by Lemma 2.4 that $P_{10} + \Lambda$ is, indeed, a multiple lattice tiling. Thus, for this particular P_{10} by (39) and Lemma 3.4 we have

$$\tau(P_{10}) \geq \tau^*(P_{10}) \geq 5, \tag{42}$$

where the equalities hold only if $P_{10} + X$ is a fivefold lattice tiling.

Case (ii). — $\mathbf{y}_1 = \mathbf{y}_3$ *and* $\mathbf{y}_2 = \mathbf{y}_5$. Then, by (40) and (38) we have

$$\begin{cases} \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_4 - \mathbf{a}_5, \\ \mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_4 = -\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_5, \\ \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{o}, \end{cases}$$

$$\begin{cases} \mathbf{a}_1 + \mathbf{a}_4 + \mathbf{a}_5 = -\mathbf{a}_3 + 2(\mathbf{a}_2 + \mathbf{a}_5), \\ 3\mathbf{a}_1 + 4\mathbf{a}_5 = 2(\mathbf{a}_2 + \mathbf{a}_5), \\ \mathbf{a}_1 - \mathbf{a}_4 + 2\mathbf{a}_5 = -\mathbf{a}_3 + (\mathbf{a}_2 + \mathbf{a}_5) \end{cases}$$

and, therefore,

$$\begin{cases} \mathbf{a}_1 = 8\mathbf{a}_3 - 6(\mathbf{a}_2 + \mathbf{a}_5), \\ \mathbf{a}_2 = 6\mathbf{a}_3 - 4(\mathbf{a}_2 + \mathbf{a}_5), \\ \mathbf{a}_3 = \mathbf{a}_3, \\ \mathbf{a}_4 = -3\mathbf{a}_3 + 3(\mathbf{a}_2 + \mathbf{a}_5), \\ \mathbf{a}_5 = -6\mathbf{a}_3 + 5(\mathbf{a}_2 + \mathbf{a}_5), \end{cases}$$

which means that Λ is a lattice with a basis $\{\mathbf{a}_3,\ \mathbf{a}_2 + \mathbf{a}_5\}$. Furthermore, since $\mathbf{u}_i = \frac{1}{2}\mathbf{a}_i \in \frac{1}{2}\Lambda$, it follows by Lemma 2.4 that $P_{10} + \Lambda$ is, indeed, a multiple lattice tiling. Thus, for this particular P_{10} by (39) and Lemma 3.4 we have

$$\tau(P_{10}) \geq \tau^*(P_{10}) \geq 5, \tag{43}$$

where the equalities hold only if $P_{10}+X$ is a fivefold lattice tiling.

Case (iii). — $\mathbf{y}_1=\mathbf{y}_4$ *and* $\mathbf{y}_2=\mathbf{y}_5$. Then, by (40) and (38) we get

$$\begin{cases}\mathbf{a}_2-\mathbf{a}_3=-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_5,\\ \mathbf{a}_1-\mathbf{a}_3+\mathbf{a}_4=-\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_5,\\ \mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_4+\mathbf{a}_5=\mathbf{o},\end{cases}$$
$$\begin{cases}2\mathbf{a}_2-\mathbf{a}_3+2\mathbf{a}_5=-(\mathbf{a}_1-\mathbf{a}_2)+\mathbf{a}_4,\\ \mathbf{a}_2+2\mathbf{a}_5=-3(\mathbf{a}_1-\mathbf{a}_2),\\ \mathbf{a}_3+\mathbf{a}_5=-(\mathbf{a}_1-\mathbf{a}_2)+\mathbf{a}_4\end{cases}$$

and, therefore,

$$\begin{cases}\mathbf{a}_1=4\mathbf{a}_4+6(\mathbf{a}_1-\mathbf{a}_2),\\ \mathbf{a}_2=4\mathbf{a}_4+5(\mathbf{a}_1-\mathbf{a}_2),\\ \mathbf{a}_3=3\mathbf{a}_4+3(\mathbf{a}_1-\mathbf{a}_2),\\ \mathbf{a}_4=\mathbf{a}_4,\\ \mathbf{a}_5=-2\mathbf{a}_4-4(\mathbf{a}_1-\mathbf{a}_2),\end{cases}$$

which means that Λ is a lattice with a basis $\{\mathbf{a}_4,\ \mathbf{a}_1-\mathbf{a}_2\}$. Furthermore, since $\mathbf{u}_i=\frac{1}{2}\mathbf{a}_i\in\frac{1}{2}\Lambda$, it follows by Lemma 2.4 that $P_{10}+\Lambda$ is, indeed, a multiple lattice tiling. Thus, for this particular P_{10} by (39) and Lemma 3.4 we have

$$\tau(P_{10}) \geq \tau^*(P_{10}) \geq 5, \tag{44}$$

where the equalities hold only if $P_{10}+X$ is a fivefold lattice tiling.

Case (iv). — $\mathbf{y}_1=\mathbf{y}_4$ *and* $\mathbf{y}_3=\mathbf{y}_5$. Then, by (40) and (38) we have

$$\begin{cases}\mathbf{a}_2-\mathbf{a}_3=-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_5,\\ \mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_4-\mathbf{a}_5=-\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_5,\\ \mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_4+\mathbf{a}_5=\mathbf{o},\end{cases}$$
$$\begin{cases}\mathbf{a}_2-\mathbf{a}_4+\mathbf{a}_5=\mathbf{a}_3-(\mathbf{a}_1+\mathbf{a}_5),\\ 3\mathbf{a}_2+2\mathbf{a}_5=\mathbf{a}_3+3(\mathbf{a}_1+\mathbf{a}_5),\\ \mathbf{a}_2+\mathbf{a}_4=\mathbf{a}_3+(\mathbf{a}_1+\mathbf{a}_5)\end{cases}$$

and therefore

$$\begin{cases} \mathbf{a}_1 = 4\mathbf{a}_3 - 5(\mathbf{a}_1 + \mathbf{a}_5), \\ \mathbf{a}_2 = 3\mathbf{a}_3 - 3(\mathbf{a}_1 + \mathbf{a}_5), \\ \mathbf{a}_3 = \mathbf{a}_3, \\ \mathbf{a}_4 = -2\mathbf{a}_3 + 4(\mathbf{a}_1 + \mathbf{a}_5), \\ \mathbf{a}_5 = -4\mathbf{a}_3 + 6(\mathbf{a}_1 + \mathbf{a}_5), \end{cases}$$

which means that Λ is a lattice with a basis $\{\mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_5\}$. Furthermore, since $\mathbf{u}_i = \frac{1}{2}\mathbf{a}_i \in \frac{1}{2}\Lambda$, it follows by Lemma 2.4 that $P_{10} + \Lambda$ is indeed a multiple lattice tiling. Thus, for this particular P_{10} by (39) and Lemma 3.4 we have

$$\tau(P_{10}) \geq \tau^*(P_{10}) \geq 5, \tag{45}$$

where the equalities hold only if $P_{10} + X$ is a fivefold lattice tiling.

Case (v). — $\mathbf{y}_2 = \mathbf{y}_4$ *and* $\mathbf{y}_3 = \mathbf{y}_5$. Then, by (40) and (38) we have

$$\begin{cases} \mathbf{a}_1 - \mathbf{a}_3 + \mathbf{a}_4 = -\mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_5, \\ \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_4 - \mathbf{a}_5 = -\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_5, \\ \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 - \mathbf{a}_4 + \mathbf{a}_5 = \mathbf{o}, \end{cases}$$

$$\begin{cases} 2\mathbf{a}_1 - \mathbf{a}_3 = -2\mathbf{a}_5, \\ 3\mathbf{a}_1 + \mathbf{a}_3 - 3\mathbf{a}_4 = 3(\mathbf{a}_2 - \mathbf{a}_4) - \mathbf{a}_5, \\ \mathbf{a}_1 + \mathbf{a}_3 - 2\mathbf{a}_4 = (\mathbf{a}_2 - \mathbf{a}_4) - \mathbf{a}_5 \end{cases}$$

and, therefore,

$$\begin{cases} \mathbf{a}_1 = 3\mathbf{a}_5 + 3(\mathbf{a}_2 - \mathbf{a}_4), \\ \mathbf{a}_2 = 6\mathbf{a}_5 + 5(\mathbf{a}_2 - \mathbf{a}_4), \\ \mathbf{a}_3 = 8\mathbf{a}_5 + 6(\mathbf{a}_2 - \mathbf{a}_4), \\ \mathbf{a}_4 = 6\mathbf{a}_5 + 4(\mathbf{a}_2 - \mathbf{a}_4), \\ \mathbf{a}_5 = \mathbf{a}_5, \end{cases}$$

which means that Λ is a lattice with a basis $\{\mathbf{a}_5,\ \mathbf{a}_2 - \mathbf{a}_4\}$. Furthermore, since $\mathbf{u}_i = \frac{1}{2}\mathbf{a}_i \in \frac{1}{2}\Lambda$, it follows by Lemma 2.4 that $P_{10} + \Lambda$ is indeed a multiple lattice tiling. Thus, for this particular P_{10} by (39) and Lemma 3.4, we have

$$\tau(P_{10}) \geq \tau^*(P_{10}) \geq 5, \tag{46}$$

where the equalities hold only if $P_{10} + X$ is a fivefold lattice tiling.

As a conclusion of these cases, Lemma 3.5 is proved. □

Lemma 3.6 (Zong [28, 29]). — *A centrally symmetric convex decagon can form a fivefold lattice tiling in* $\mathbb{E}^2$*, if and only if, under a suitable affine linear transformation, it takes* $\mathbf{u}_1 = (0,1)$, $\mathbf{u}_2 = (1,1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (\frac{3}{2}, 0)$, $\mathbf{u}_5 = (1, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ *and* $\mathbf{u}_{10} = -\mathbf{u}_5$ *as the middle points of its edges.*

Remark 3.7 (Zong [28, 29]). — Let W denote the quadrilateral with vertices $\mathbf{w}_1 = (-\frac{1}{2}, 1)$, $\mathbf{w}_2 = (-\frac{1}{2}, \frac{3}{4})$, $\mathbf{w}_3 = (-\frac{2}{3}, \frac{2}{3})$ and $\mathbf{w}_4 = (-\frac{3}{4}, \frac{3}{4})$. A centrally symmetric convex decagon can take $\mathbf{u}_1 = (0, 1)$, $\mathbf{u}_2 = (1, 1)$, $\mathbf{u}_3 = (\frac{3}{2}, \frac{1}{2})$, $\mathbf{u}_4 = (\frac{3}{2}, 0)$, $\mathbf{u}_5 = (1, -\frac{1}{2})$, $\mathbf{u}_6 = -\mathbf{u}_1$, $\mathbf{u}_7 = -\mathbf{u}_2$, $\mathbf{u}_8 = -\mathbf{u}_3$, $\mathbf{u}_9 = -\mathbf{u}_4$ and $\mathbf{u}_{10} = -\mathbf{u}_5$ as the middle points of its edges, if and only if one of its vertices is an interior point of W.

Lemma 3.8. — *For every centrally symmetric convex octagon P_8 we have*

$$\tau(P_8) \geq 5,$$

where the equality holds, if and only if, under a suitable affine linear transformation, it is one with vertices $\mathbf{v}_1 = \left(\frac{3}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_2 = \left(-\frac{1}{2} - \frac{5\alpha}{4}, -2\right)$, $\mathbf{v}_3 = \left(\frac{\alpha}{4} - \frac{3}{2}, 0\right)$, $\mathbf{v}_4 = \left(\frac{\alpha}{4} - \frac{3}{2}, 1\right)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \alpha < \frac{2}{3}$, *or with vertices* $\mathbf{v}_1 = (2 - \beta, -3)$, $\mathbf{v}_2 = (-\beta, -3)$, $\mathbf{v}_3 = (-2, -1)$, $\mathbf{v}_4 = (-2, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ *and* $\mathbf{v}_8 = -\mathbf{v}_4$, *where* $0 < \beta \leq 1$.

Proof. — Suppose that X is a discrete subset of $\mathbb{E}^2$, and $P_8 + X$ is a $\tau(P_8)$-fold translative tiling of the plane. First of all, it follows from Lemma 2.1 that

$$\varphi(\mathbf{v}) \geq \left\lceil \frac{4-3}{2} \right\rceil = 1 \tag{47}$$

holds for all $\mathbf{v} \in V + X$. On the other hand, by Lemma 2.2 we have

$$\varpi(\mathbf{v}) = \kappa \cdot \frac{3}{2} + \ell \cdot \frac{1}{2}, \tag{48}$$

where κ is a positive integer and ℓ is a nonnegative integer. In fact, ℓ is the number of the edges that take $\mathbf{v}$ as an interior point. Thus, to prove the lemma, it is sufficient to deal with the following four cases:

Case 1. — $\varpi(\mathbf{v}) \geq 5$ *holds for a vertex* $\mathbf{v} \in V + X$. It follows by (1) and (47) that

$$\tau(P_8) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{49}$$

Case 2. — $\varpi(\mathbf{v}) = 4$ *holds for a vertex* $\mathbf{v} \in V + X$. It follows by (48) that $\ell \neq 0$ and therefore $\mathbf{v} \in \text{int}(G)$ holds for some $G \in \Gamma + X$. Assume that $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ are the two ends of G. Applying Lemma 2.1 to $\{\mathbf{v}_1^*, G\}$ and $\{\mathbf{v}_2^*, G\}$, respectively, one can deduce that

$$\varphi(\mathbf{v}) \geq 2$$

and, therefore,

$$\tau(P_8) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{50}$$

Case 3. — $\varpi(\mathbf{v}) = 3$ *holds for a vertex* $\mathbf{v} \in V + X$. Then (48) has and only has two groups of solutions $\{\kappa, \ell\} = \{1, 3\}$ or $\{2, 0\}$.

Subcase 3.1. — $\{\kappa, \ell\} = \{1, 3\}$. Then, there are three edges G'_1, G'_2 and G'_3 in $\Gamma + X$ satisfying

$$\mathbf{v} \in \text{int}(G'_i), \quad i = 1,\ 2,\ 3.$$

Next, we study the multiplicity by considering the relative positions of these edges.

Subcase 3.1.1. — $G'_1 = G'_2 = G'_3$. Assume that $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ are the two ends of G'_1. Then $X^{\mathbf{v}_1^*}$ has two identical points. By computing the angle sum of all the adjacent wheels at $\mathbf{v}_1^*$ it can be deduced that

$$\varpi(\mathbf{v}_1^*) \geq 4.$$

Then, by Case 1 and Case 2 we get

$$\tau(P_8) = \varpi(\mathbf{v}_1^*) + \varphi(\mathbf{v}_1^*) \geq 6. \tag{51}$$

Subcase 3.1.2. — $G'_2 = G'_3$ *and* $G'_1 \nparallel G'_2$. Then there are two adjacent wheels at $\mathbf{v}$, one has five translates $P_8 + \mathbf{x}_1$, $P_8 + \mathbf{x}_2$, ..., $P_8 + \mathbf{x}_5$, and the other has two translates $P_8 + \mathbf{x}'_1$ and $P_8 + \mathbf{x}'_2$, as shown by Figure 8.

By re-enumeration we may assume that $\angle_1$, $\angle_2$, $\angle_3$ and $\angle_4$ are inner angles of P_8 and $\angle_5 = \pi$, as shown by Figure 8. Guaranteed by linear transformation, we assume that the two edges G_1 and G_3 of P_8 are horizontal and vertical, respectively. Suppose that $G'_1 = G_1 + \mathbf{x}_5$. Let $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ be the two ends of G'_1, let L denote the straight line determined by $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, let G_3^* denote the

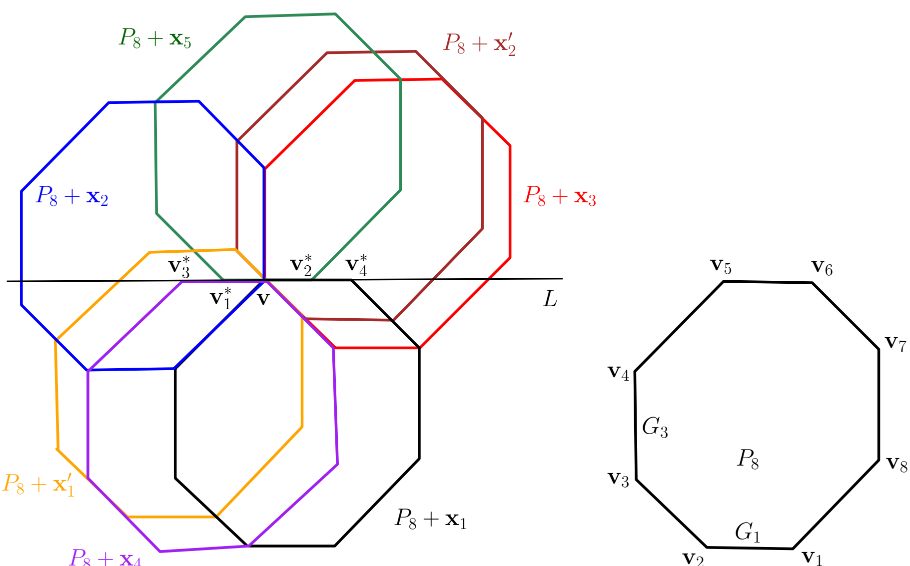

Figure 8

edge of $P_8+\mathbf{x}_4$ lying on L with two ends $\mathbf{v}$ and $\mathbf{v}_3^*$, and let G_4^* denote the edge of $P_8+\mathbf{x}_1$ lying on L with two ends $\mathbf{v}$ and $\mathbf{v}_4^*$.

By Lemma 2.1, there is a point $\mathbf{y}_1\in X^{\mathbf{v}_1^*}$, such that $\mathbf{v}_3^*\in \mathrm{int}(P_8)+\mathbf{y}_1$. Clearly, by the convexity of P_8, both $\mathbf{v}_3^*$ and $\mathbf{v}_1^*$ belong to $\mathrm{int}(P_8)+\mathbf{x}_1'$. Thus, we have $\mathbf{y}_1\neq\mathbf{x}_1'$. Meanwhile, since both $\mathbf{v}_3^*$ and $\mathbf{v}_1^*$ belong to $\mathrm{int}(P_8)+\mathbf{x}_2$, we have $\mathbf{x}_2\neq\mathbf{y}_1$ and, therefore,

$$\varphi(\mathbf{v}_3^*)\geq 3.$$

Similarly, we have $\varphi(\mathbf{v}_1^*)\geq 3$, $\varphi(\mathbf{v}_2^*)\geq 3$ and $\varphi(\mathbf{v}_4^*)\geq 3$. Then, by (48) we get

$$\tau(P_8)=\varphi(\mathbf{v}_i^*)+\varpi(\mathbf{v}_i^*)\geq 5, \tag{52}$$

where the equality may hold only if

$$\varpi(\mathbf{v}_1^*)=\varpi(\mathbf{v}_2^*)=\varpi(\mathbf{v}_3^*)=\varpi(\mathbf{v}_4^*)=2. \tag{53}$$

By (48) it is easy to see that the local configuration of $P_8+X^{\mathbf{v}}$ is essentially unique when $\varpi(\mathbf{v})=2$. In other words, it is determined by the one that $\mathbf{v}$ is not its vertex. Consequently, the set X has four points $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ and $\mathbf{y}_4$ satisfying

$$\mathbf{v}_1^*=\mathbf{v}_4+\mathbf{y}_1,\mathbf{v}\in \mathrm{int}(P_8)+\mathbf{y}_1, \tag{54}$$

$$\mathbf{v}_2^*=\mathbf{v}_7+\mathbf{y}_2,\mathbf{v}\in \mathrm{int}(P_8)+\mathbf{y}_2, \tag{55}$$

$$\mathbf{v}_3^*=\mathbf{v}_3+\mathbf{y}_3,\mathbf{v}\in \mathrm{int}(P_8)+\mathbf{y}_3, \qquad \text{and} \tag{56}$$

$$\mathbf{v}_4^*=\mathbf{v}_8+\mathbf{y}_4,\mathbf{v}\in \mathrm{int}(P_8)+\mathbf{y}_4. \tag{57}$$

Clearly, by the convexity of P_8 we have $\mathbf{y}_1\neq\mathbf{y}_2$, $\mathbf{y}_1\neq\mathbf{y}_3$ and $\mathbf{y}_2\neq\mathbf{y}_4$. For convenience, we write $\mathbf{v}_i=(x_i,y_i)$. If $\mathbf{y}_2=\mathbf{y}_3$, then by (55) and (56) we have

$$y_3=y_7. \tag{58}$$

If $\mathbf{y}_1=\mathbf{y}_4$, then by (54) and (57) we get

$$y_4=y_8. \tag{59}$$

However, it is obvious that (58) and (59) cannot hold simultaneously. Therefore, we still get

$$\varphi(\mathbf{v})\geq 3$$

and, therefore,

$$\tau(P_8)=\varphi(\mathbf{v})+\varpi(\mathbf{v})\geq 6. \tag{60}$$

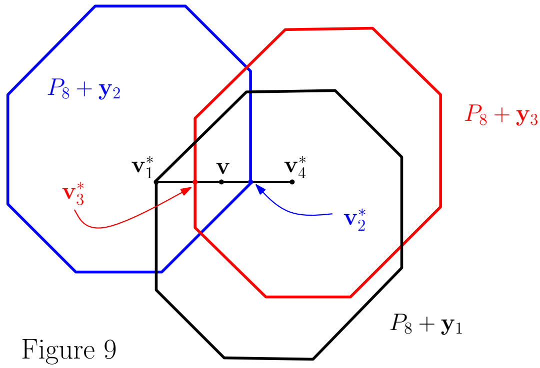

Figure 9

Subcase 3.1.3. — *$G_1' \neq G_2'$ and $G_1' \parallel G_2'$.* Let $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ be the two ends of G_1', and let $\mathbf{v}_3^*$ and $\mathbf{v}_4^*$ be the two ends of G_2'. Without loss of generality, we suppose that $\mathbf{v}_3^*$ is between $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, as shown by Figure 9. By Lemma 2.1, X has three points $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ satisfying both

$$\mathbf{v}_i^* \in \partial(P_8)+\mathbf{y}_i, \quad i=1,\ 2,\ 3$$

and

$$\mathbf{v} \in \mathrm{int}(P_8)+\mathbf{y}_i, \quad i=1,\ 2,\ 3.$$

By the convexity of P_8 it is easy to see that these three points are pairwise distinct. Then, we get

$$\varphi(\mathbf{v}) \geq 3$$

and, therefore,

$$\tau(P_8)=\varpi(\mathbf{v})+\varphi(\mathbf{v}) \geq 6. \tag{61}$$

Subcase 3.1.4. — *$G_1' \nparallel G_2'$, $G_1' \nparallel G_3'$ and $G_2' \nparallel G_3'$.* By studying the angle sum at $\mathbf{v}$ it can be deduced that $P_8+X^{\mathbf{v}}$ is an adjacent wheel of seven translates. Suppose that $\mathbf{x}_2 \in X^{\mathbf{v}}$ and G_1' is an edge of $P_8+\mathbf{x}_2$. Since G_1', G_2' and G_3' are mutually non-collinear, $X^{\mathbf{v}}$ has two points $\mathbf{x}_1$ and $\mathbf{x}_3$, such that $\mathbf{v}$ is a common vertex of both $P_8+\mathbf{x}_1$ and $P_8+\mathbf{x}_3$, and $P_8+\mathbf{x}_2$ joins both $P_8+\mathbf{x}_1$ and $P_8+\mathbf{x}_3$ at non-singleton parts of G_1', respectively. Let $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ be the two ends of G_1', let L denote the straight line determined by $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, let G_1^* denote the edge of $P_8+\mathbf{x}_1$ lying on L with ends $\mathbf{v}$ and $\mathbf{v}_3^*$, and let G_2^* denote the edge of $P_8+\mathbf{x}_3$ lying on L with ends $\mathbf{v}$ and $\mathbf{v}_4^*$, as shown in Figure 10.

By studying the corresponding angles of the adjacent wheel at $\mathbf{v}$, it is easy to see that $P_8+X^{\mathbf{v}}$ has exact two translates which contain both $\mathbf{v}_1^*$ and $\mathbf{v}_3^*$ as interior points. On the other hand, by Lemma 2.1, $P_8+X^{\mathbf{v}_3^*}$ has at least one more translate that contains $\mathbf{v}_1^*$ as an interior point. Thus, we have

$$\varphi(\mathbf{v}_1^*) \geq 3.$$

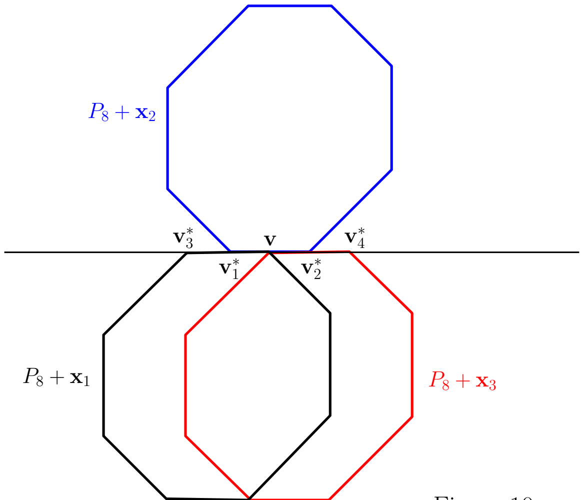

Figure 10

Similarly, we have $\varphi(\mathbf{v}_2^*)\geq 3$, $\varphi(\mathbf{v}_3^*)\geq 3$ and $\varphi(\mathbf{v}_4^*)\geq 3$. Then, by (48) we get

$$\tau(P_8)=\varphi(\mathbf{v}_i^*)+\varpi(\mathbf{v}_i^*)\geq 5, \tag{62}$$

where the equality may hold only if

$$\varpi(\mathbf{v}_1^*)=\varpi(\mathbf{v}_2^*)=\varpi(\mathbf{v}_3^*)=\varpi(\mathbf{v}_4^*)=2. \tag{63}$$

By repeating the argument between (53) and (60), it can be deduced that

$$\tau(P_8)=\varphi(\mathbf{v})+\varpi(\mathbf{v})\geq 6. \tag{64}$$

Subcase 3.2. — $\{\kappa,\ell\}=\{2,0\}$ *holds at every vertex* $\mathbf{v}\in V+X$. Then $P_8+X^{\mathbf{v}}$ is an adjacent wheel of eight translates $P_8+\mathbf{x}_1$, $P_8+\mathbf{x}_2$, …, $P_8+\mathbf{x}_8$, as shown in Figure 11. Let $\mathbf{v}_i^*$ be the second vertex of $P_8+\mathbf{x}_i$ connecting to $\mathbf{v}$ by an edge. Since $\varpi(\mathbf{v})=3$, every $\mathbf{v}_i^*$ is an interior point of exactly two of these eight translates. Consequently, for every $\mathbf{v}_i^*$, there are two different translates $P_8+\mathbf{y}_i$ and $P_8+\mathbf{y}_i'$ in $P_8+X^{\mathbf{v}_i^*}$ that both contain $\mathbf{v}$ as an interior point.

On the other hand, it can be easily deduced that there is only one point $\mathbf{x}\in X$, such that both $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$ belong to $\partial(P_8)+\mathbf{x}$ and $\mathbf{v}\in \mathrm{int}(P_8)+\mathbf{x}$. It is $\mathbf{v}_2^*-\mathbf{v}+\mathbf{x}_1$. Therefore, at least one of the two points $\mathbf{y}_2$ and $\mathbf{y}_2'$ is different from both $\mathbf{y}_1$ and $\mathbf{y}_1'$. Then, we get

$$\varphi(\mathbf{v})\geq 3$$

and

$$\tau(P_8)=\varphi(\mathbf{v})+\varpi(\mathbf{v})\geq 6. \tag{65}$$

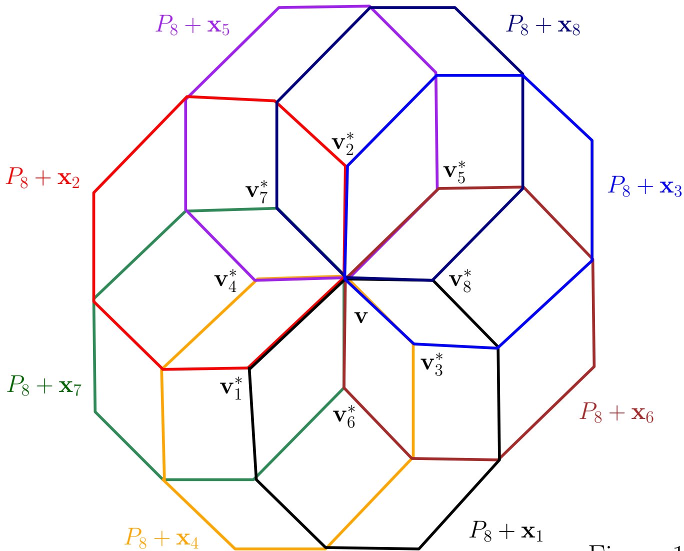

Figure 11

Case 4. — $\varpi(\mathbf{v}) = 2$ holds for a vertex $\mathbf{v} \in V + X$. It follows by (48) that $\varpi(\mathbf{v}) = 2$ holds, if and only if $\kappa = 1$ and $\ell = 1$. In other words, $P_8 + X^{\mathbf{v}}$ is an adjacent wheel of five translates. By re-enumeration we may assume that $\angle_1$, $\angle_2$, $\angle_3$ and $\angle_4$ are inner angles of P_8 and $\angle_5 = \pi$, as shown by Figure 12. Guaranteed by linear transformation, we assume that the edges G_1 and G_3 of P_8 are horizontal and vertical, respectively.

Let G_1^* denote the edge of $P_8 + \mathbf{x}_5$, such that $\mathbf{v} \in \mathrm{int}(G_1^*)$ with two ends $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, let L denote the straight line determined by $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$, let G_3^* denote the edge of $P_8 + \mathbf{x}_4$ lying on L with ends $\mathbf{v}$ and $\mathbf{v}_3^*$, and let G_4^* denote the edge of $P_8 + \mathbf{x}_1$ lying on L with ends $\mathbf{v}$ and $\mathbf{v}_4^*$. If $\varpi(\mathbf{v}_1^*) \geq 3$ or $\varpi(\mathbf{v}_2^*) \geq 3$, by Case 1, Case 2 and Subcase 3.1 we have $\tau(P_8) \geq 6$. Therefore, by considering the adjacent wheels at $\mathbf{v}_1^*$, $\mathbf{v}_2^*$, $\mathbf{v}_3^*$ and $\mathbf{v}_4^*$ successively, $\tau(P_8) \leq 5$ may happen only if

$$\varpi(\mathbf{v}_1^*) = \varpi(\mathbf{v}_2^*) = \varpi(\mathbf{v}_3^*) = \varpi(\mathbf{v}_4^*) = 2. \tag{66}$$

Since the configuration of $P_8 + X^{\mathbf{v}}$ is essentially unique if $\varpi(\mathbf{v}) = 2$, by considering the wheel structures at $\mathbf{v}$, $\mathbf{v}_1^*$, $\mathbf{v}_2^*$, $\mathbf{v}_3^*$ and $\mathbf{v}_4^*$, there are four points $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ and $\mathbf{y}_4$ in X satisfying

$$\mathbf{v}_1^* = \mathbf{v}_4 + \mathbf{y}_1, \mathbf{v} \in \mathrm{int}(P_8) + \mathbf{y}_1, \tag{67}$$

$$\mathbf{v}_2^* = \mathbf{v}_7 + \mathbf{y}_2, \mathbf{v} \in \mathrm{int}(P_8) + \mathbf{y}_2, \tag{68}$$

$$\mathbf{v}_3^* = \mathbf{v}_3 + \mathbf{y}_3, \mathbf{v} \in \mathrm{int}(P_8) + \mathbf{y}_3, \qquad \text{and} \tag{69}$$

$$\mathbf{v}_4^* = \mathbf{v}_8 + \mathbf{y}_4, \mathbf{v} \in \mathrm{int}(P_8) + \mathbf{y}_4. \tag{70}$$

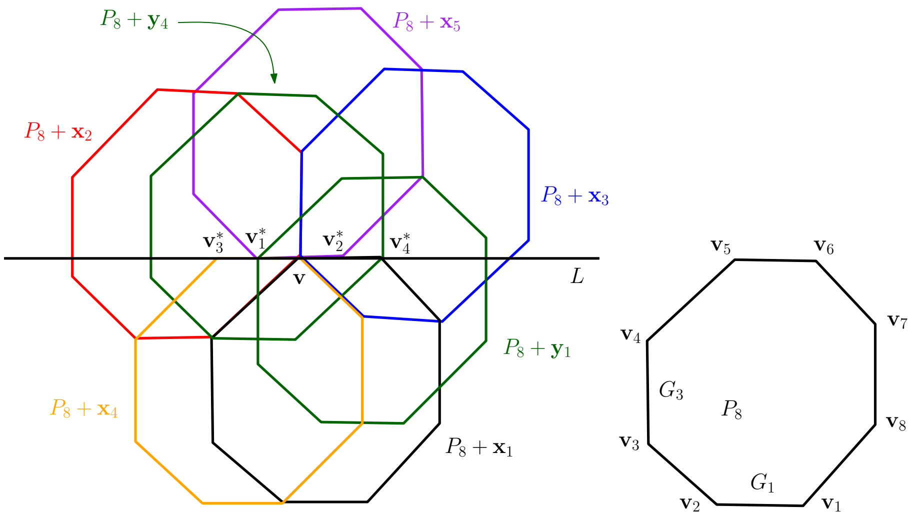

Figure 12

By the convexity of P_8 it follows that $\mathbf{y}_1 \neq \mathbf{y}_2$, $\mathbf{y}_1 \neq \mathbf{y}_3$ and $\mathbf{y}_2 \neq \mathbf{y_4}$. For convenience, we write $\mathbf{v}_i = (x_i, y_i)$. If $\mathbf{y}_1 = \mathbf{y}_4$, and then by (67) and (70) we have

$$y_4 = y_8. \tag{71}$$

If $\mathbf{y}_2 = \mathbf{y}_3$, then by (68) and (69) we have

$$y_3 = y_7. \tag{72}$$

It is obvious that (71) and (72) cannot hold simultaneously. Therefore, we have either $\mathbf{y}_1 \neq \mathbf{y}_4$ or $\mathbf{y}_2 \neq \mathbf{y}_3$.

On the other hand, since $\varpi(\mathbf{v}) = 2$, the three inequalities $\mathbf{y}_3 \neq \mathbf{y}_4$, $\mathbf{y}_2 \neq \mathbf{y}_3$ and $\mathbf{y}_1 \neq \mathbf{y_4}$ cannot hold simultaneously. Otherwise, it can be deduced that

$$\varphi(\mathbf{v}) \geq 4$$

and, therefore,

$$\tau(P_8) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{73}$$

Since $\mathbf{y}_1 = \mathbf{y}_4$ and $\mathbf{y}_2 = \mathbf{y}_3$ are symmetric, with respect to $\mathbf{v}$ in Figure 12, it is sufficient to deal with two subcases.

Subcase 4.1. — $\mathbf{y}_2 = \mathbf{y}_3$. Let $\mathbf{v}_1'$ and $\mathbf{v}_2'$ be the two vertices of $P_8 + \mathbf{x}_2$ that are adjacent to $\mathbf{v}$, as shown in Figure 13. First, we have $\mathbf{v}_1' \in \mathrm{int}(P_8) + \mathbf{x}_5$. Second, by convexity it is easy to see that $\mathbf{v}_1' \in \mathrm{int}(P_8) + \mathbf{y_4}$. Since $\mathbf{y}_2 = \mathbf{y}_3$, we have $y_3 = y_7$. Then $\mathbf{v}_1'$ is an interior point of $P_8 + \mathbf{y_2}$ as well. Thus we get

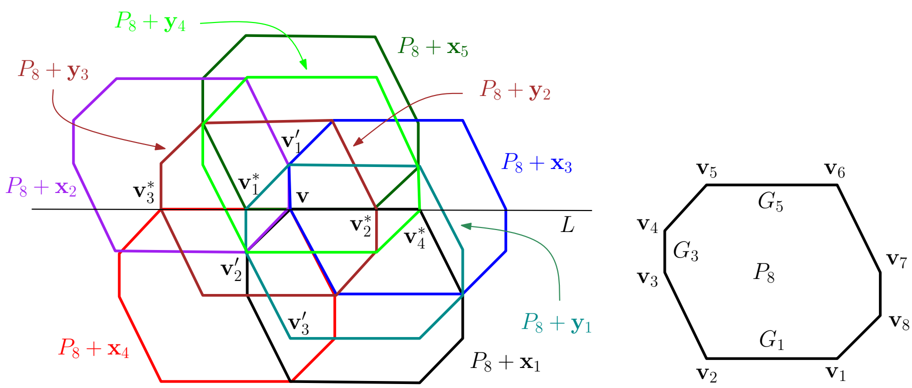

Figure 13

$\varphi(\mathbf{v}'_1) \geq 3$ and

$$\tau(P_8) = \varpi(\mathbf{v}'_1) + \varphi(\mathbf{v}'_1) \geq 5, \tag{74}$$

where the equality may hold only if

$$\varpi(\mathbf{v}'_1) = 2. \tag{75}$$

By Lemma 2.1, there is a point $\mathbf{y_5} \in X^{\mathbf{v}'_1}$, such that $\mathbf{v} \in \mathrm{int}(P_8) + \mathbf{y_5}$.

Subcase 4.1.1. — $\mathbf{v}'_1$ *is a vertex of* $P_8 + \mathbf{y_5}$. If $\mathbf{v}'_1$ is a vertex of $P_8 + \mathbf{y_1}$, as shown by Figure 13, then by symmetry one can deduce that $\mathbf{v}'_2$ is a vertex of $P_8 + \mathbf{y_1}$. Similarly, it follows by (70) that $\mathbf{v}'_2$ is a vertex of $P_8 + \mathbf{y}_4$ as well. Then some points near to $\mathbf{v}'_2$ belong to all $\mathrm{int}(P_8) + \mathbf{y}_1$, $\mathrm{int}(P_8) + \mathbf{y}_4$ and $\mathrm{int}(P_8) + \mathbf{x}_1$. Thus, we have

$$\varpi(\mathbf{v}'_2) \geq 3. \tag{76}$$

Let $\mathbf{v}'_3$ denote the vertex $\mathbf{v}_2 + \mathbf{y}_1$ of $P_8 + \mathbf{y}_1$, as shown in Figure 13. By Lemma 2.1, there is a point $\mathbf{z} \in X^{\mathbf{v}'_3}$, such that $\mathbf{v}'_2 \in \mathrm{int}(P_8) + \mathbf{z}$. Then it can be deduced that $\mathbf{v}'_3 \in \mathrm{int}(P_8) + \mathbf{x_4}$ and, thus, $\mathbf{z} \neq \mathbf{x}_4$. Since $y_3 = y_7$, it can be shown that $\mathbf{v}'_3 \notin P_8 + \mathbf{y_2}$ and, therefore, $\mathbf{z} \neq \mathbf{y}_2$. In addition, we have

$$\mathbf{v}'_2 \in (\mathrm{int}(P_8) + \mathbf{x_4}) \cap (\mathrm{int}(P_8) + \mathbf{y_2})\,.$$

Thus, we have

$$\varphi(\mathbf{v}'_2) \geq 3$$

and, consequently,

$$\tau(P_8) = \varphi(\mathbf{v}'_2) + \varpi(\mathbf{v}'_2) \geq 6. \tag{77}$$

If $\mathbf{v}'_1$ is not a vertex of $P_8 + \mathbf{y_1}$, remembering the subcase assumption, then we have $\mathbf{y_1} \neq \mathbf{y_5}$. In fact, in this case all $\mathbf{y}_1$, $\mathbf{y}_3$, $\mathbf{y}_4$ and $\mathbf{y}_5$ are pairwise distinct.

Thus, we have

$$\varphi(\mathbf{v}) \geq 4$$

and

$$\tau(P_8) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{78}$$

Subcase 4.1.2. — $\mathbf{v}_1'$ *is an interior point of an edge of* $P_8 + \mathbf{y_5}$. It follows from the convexity of P_8 that $\mathbf{v}_1'$ is an interior point of both $P_8 + \mathbf{y}_4$ and $P_8 + \mathbf{y}_3$. Therefore, we have $\mathbf{y_5} \notin \{\mathbf{y}_3, \mathbf{y}_4\}$. If $\mathbf{y_5} \neq \mathbf{y}_1$, then all $\mathbf{y}_1$, $\mathbf{y}_3$, $\mathbf{y}_4$ and $\mathbf{y}_5$ are pairwise distinct. Thus, we have

$$\varphi(\mathbf{v}) \geq 4$$

and

$$\tau(P_8) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{79}$$

If $\mathbf{y_5} = \mathbf{y}_1$, then all $\mathbf{y}_1$, $\mathbf{y}_3$ and $\mathbf{y}_4$ are pairwise distinct and, therefore,

$$\tau(P_8) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 5. \tag{80}$$

Now, we try to figure out the equality cases in (80).

Notice that $\mathbf{v}_1'$ is an interior point of $P_8 + \mathbf{x}_5$, and $P_8 + \mathbf{y_1}$ has only two edges $G_4 + \mathbf{y}_1$ and $G_5 + \mathbf{y}_1$, which contain interior points of $P_8 + \mathbf{x}_5$. Since $\varpi(\mathbf{v}_1') = 2$ (see (75)), by studying the structure of the adjacent wheel at $\mathbf{v}_1'$, one can deduce that $\mathbf{v}_1'$ must be an interior point of $G_5 + \mathbf{y}_1$. Then we have

$$y_5 - y_4 = y_4 - y_3 \tag{81}$$

and

$$y_3 - y_2 = 2(y_4 - y_3). \tag{82}$$

Let $\mathbf{v}_5^*$ and $\mathbf{v}_6^*$ be the two ends of $G_5 + \mathbf{y}_1$. Suppose that $\mathbf{v}_5^*$ is on the left-hand side of $\mathbf{v}_1'$. By Lemma 2.1, there is a point $\mathbf{y}_6 \in X^{\mathbf{v}_5^*}$, such that $\mathbf{v}_1' \in \mathrm{int}(P_8) + \mathbf{y}_6$.

It is obvious that $\mathbf{v}_5^*$ is an interior point of both $P_8 + \mathbf{x_5}$ and $P_8 + \mathbf{y_2}$. Thus, we have $\mathbf{y}_6 \notin \{\mathbf{y_2}, \mathbf{x_5}\}$. If $\mathbf{v}_5^*$ is not lying on the boundary of $P_8 + \mathbf{y_4}$, then we have $\mathbf{y_4} \neq \mathbf{y_6}$. Consequently, all $\mathbf{y}_2$, $\mathbf{y}_4$, $\mathbf{y}_6$ and $\mathbf{x}_5$ are pairwise distinct. Then we have

$$\varphi(\mathbf{v}_1') \geq 4$$

and

$$\tau(P_8) = \varphi(\mathbf{v}_1') + \varpi(\mathbf{v}_1') \geq 6. \tag{83}$$

Thus, to save $\tau(P_8) = 5$, the point $\mathbf{v}_5^*$ must belong to the boundary of $P_8 + \mathbf{y_4}$. Furthermore, since the y-coordinate of $\mathbf{v}_5^*$ is equal to the y-coordinates of both $\mathbf{v}_1'$ and $\mathbf{v}_3 + \mathbf{y}_4$, the point $\mathbf{v}_5^*$ must be the vertex $\mathbf{v}_3 + \mathbf{y}_4$ of $P_8 + \mathbf{y_4}$.

Let v denote the x-coordinate of $\mathbf{v}$ and let w_1, w_2 and w_3 denote the x-coordinates of $\mathbf{v}_3+\mathbf{y}_4$, $\mathbf{v}_1^*$ and $\mathbf{v}_5^*$, respectively. First, by computing the x-coordinate of $\mathbf{v}_4^*$ in two ways we get

$$w_1+(x_7-x_6)+(x_6-x_5)+(x_5-x_4)=v+(x_6-x_5)$$

and, thus,

$$w_1=v-(x_7-x_6)-(x_5-x_4). \tag{84}$$

On the other hand, since $\mathbf{y}_2=\mathbf{y}_3$, by computing the distance between $\mathbf{v}_3^*$ and $\mathbf{v}_4^*$ in two ways we get

$$(x_7-x_6)+(x_6-x_5)+(x_5-x_4)+v-w_2=2(x_6-x_5)$$

and, thus,

$$w_2=v+(x_7-x_6)-(x_6-x_5)+(x_5-x_4).$$

Since $\mathbf{v}_5^*$ is the left vertex of $G_5+\mathbf{y}_1$, we get

$$w_3=w_2+(x_5-x_4)=v+(x_7-x_6)-(x_6-x_5)+2(x_5-x_4). \tag{85}$$

Then, $\mathbf{v}_5^*=\mathbf{v}_3+\mathbf{y}_4$ implies $w_1=w_3$ and

$$2(x_7-x_6)+3(x_5-x_4)=x_6-x_5. \tag{86}$$

In conclusion, recalling (81) and (82), a centrally symmetric octagon P_8 with G_1 horizontal, G_3 vertical and $\mathbf{y}_2=\mathbf{y}_3$ is a fivefold translative tile only if

$$\begin{cases} y_5-y_4=y_4-y_3,\\ y_3-y_2=2(y_4-y_3),\\ x_6-x_5=2(x_7-x_6)+3(x_5-x_4). \end{cases} \tag{87}$$

Guaranteed by linear transformations, by choosing $y_4-y_3=1$, $x_6-x_5=2$ and $x_5-x_4=\alpha$ and keeping the symmetry in mind, one can deduce that the candidates are the octagons $D_8(\alpha)$ with vertices $\mathbf{v}_1=\left(\frac{3}{2}-\frac{5\alpha}{4},-2\right)$, $\mathbf{v}_2=\left(-\frac{1}{2}-\frac{5\alpha}{4},-2\right)$, $\mathbf{v}_3=\left(\frac{\alpha}{4}-\frac{3}{2},0\right)$, $\mathbf{v}_4=\left(\frac{\alpha}{4}-\frac{3}{2},1\right)$, $\mathbf{v}_5=-\mathbf{v}_1$, $\mathbf{v}_6=-\mathbf{v}_2$, $\mathbf{v}_7=-\mathbf{v}_3$ and $\mathbf{v}_8=-\mathbf{v}_4$, where $0<\alpha<\frac{2}{3}$.

Let $\Lambda(\alpha)$ denote the lattice generated by $\mathbf{u}_1=(2,0)$ and $\mathbf{u}_2=(1+\frac{\alpha}{2},1)$. By Lemma 2.4 it can be shown that $D_8(\alpha)+\Lambda(\alpha)$ is, indeed, a fivefold tiling of $\mathbb{E}^2$.

Subcase 4.2. — $\mathbf{y}_3=\mathbf{y}_4$. First of all, we have $\varphi(\mathbf{v})\geq 3$ and, therefore,

$$\tau(P_8)=\varphi(\mathbf{v})+\varpi(\mathbf{v})\geq 5. \tag{88}$$

Next, we try to figure out the equality cases in (88).

As shown by Figure 14, by (69) and (70) we get

$$y_3=y_8 \tag{89}$$

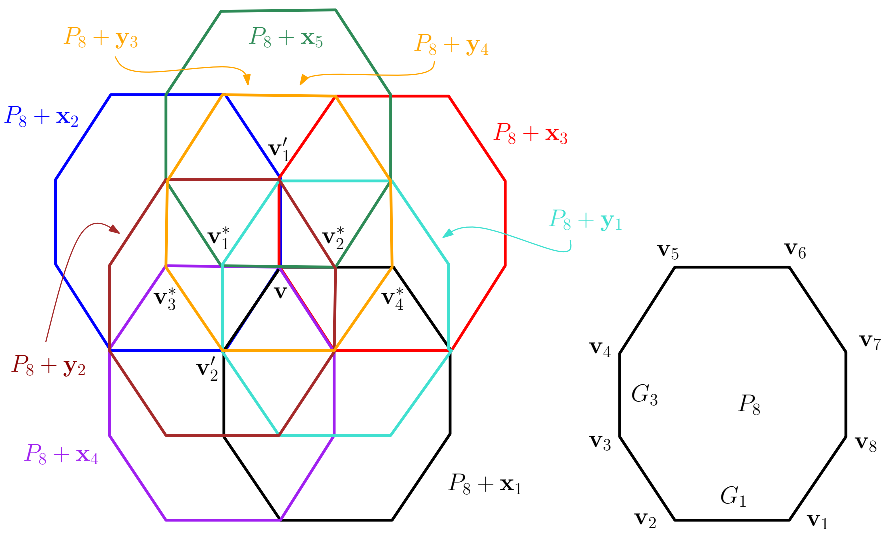

Figure 14

and

$$x_8 - x_3 = 2(x_1 - x_2). \tag{90}$$

By Lemma 2.1, there is $\mathbf{y}_5 \in X^{\mathbf{v}'_1}$, such that $\mathbf{v} \in \mathrm{int}(P_8) + \mathbf{y}_5$. Since $y_3 = y_8$, by convexity we have $\mathbf{v}'_1 \in \mathrm{int}(P_8) + \mathbf{y}_3$, and then $\mathbf{y}_5 \neq \mathbf{y}_3$. If both $\mathbf{y}_5 \neq \mathbf{y}_1$ and $\mathbf{y}_5 \neq \mathbf{y}_2$ hold simultaneously, then we have

$$\varphi(\mathbf{v}) \geq 4,$$

and, therefore,

$$\tau(P_8) = \varphi(\mathbf{v}) + \varpi(\mathbf{v}) \geq 6. \tag{91}$$

Thus, the equality in (88) holds only if $\mathbf{y}_5 = \mathbf{y}_1$ or $\mathbf{y}_5 = \mathbf{y}_2$.

Suppose that $\mathbf{y}_5 = \mathbf{y}_1$. If $\mathbf{v}'_1$ is a vertex of $P_8 + \mathbf{y}_1$, then we have

$$y_5 - y_4 = y_4 - y_3. \tag{92}$$

If $\mathbf{v}'_1$ is an interior point of an edge of $P_8 + \mathbf{y}_1$, eliminated by Case 1, Case 2 and Subcase 3.1 at $\mathbf{v}'_1$, it may be assumed that $\varpi(\mathbf{v}'_1) = 2$. Then, by studying the structure of the adjacent wheel at $\mathbf{v}'_1$, one can deduce that $\mathbf{v}'_1$ must be an interior point of $G_5 + \mathbf{y}_1$. Since $G_5 + \mathbf{y}_1$ is horizontal, we also obtain (92).

In conclusion, recalling (89) and (90), a centrally symmetric octagon P_8 with G_1 horizontal, G_3 vertical and $\mathbf{y}_3 = \mathbf{y}_4$ is a fivefold translative tile only if

$$\begin{cases} y_3 = y_8, \\ y_5 - y_4 = y_4 - y_3, \\ x_8 - x_3 = 2(x_1 - x_2). \end{cases} \tag{93}$$

Guaranteed by linear transformation, by choosing $y_4 - y_3 = 2$, $x_1 - x_2 = 2$ and $x_6 = \beta$ and keeping symmetry in mind, one can deduce that the candidates are the octagons $D_8(\beta)$ with vertices $\mathbf{v}_1 = (2 - \beta, -3)$, $\mathbf{v}_2 = (-\beta, -3)$, $\mathbf{v}_3 = (-2, -1)$, $\mathbf{v}_4 = (-2, 1)$, $\mathbf{v}_5 = -\mathbf{v}_1$, $\mathbf{v}_6 = -\mathbf{v}_2$, $\mathbf{v}_7 = -\mathbf{v}_3$ and $\mathbf{v}_8 = -\mathbf{v}_4$, where $0 < \beta \leq 1$.

Let $\Lambda(\beta)$ denote the lattice generated by $\mathbf{u}_1 = (2, 0)$ and $\mathbf{u}_2 = (1 + \frac{\beta}{2}, 2)$. By Lemma 2.4, it can be proved that $D_8(\beta) + \Lambda(\beta)$ is, indeed, a fivefold tiling of $\mathbb{E}^2$.

Lemma 3.8 is proved. □

Proof of Theorem 1.1. — It has been shown by Gravin, Robins and Shiryaev [10] that a convex body can form a multiple translative tiling in $\mathbb{E}^n$ only if it is a centrally symmetric polytope with centrally symmetric facets. Then, Theorem 1.1 follows from Lemma 3.1, Lemma 3.5 and Lemma 3.8. □

Proof of Theorem 1.2. — Assume that P_{2m} is a centrally symmetric $2m$-gon satisfying $\tau(P_{2m}) = 5$. First, by Fedorov's theorem and Lemma 3.1 we have $4 \leq m \leq 7$. Second, by Lemma 3.3 and Lemma 3.2 we get $m \neq 6$ and 7, respectively. When $m = 5$, the theorem follows by Lemma 3.5 and Lemma 3.6. Finally, when $m = 4$, the theorem follows from Lemma 3.8. □

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