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*Towards geometric Satake correspondence for Kac-Moody algebras —
Cherkis bow varieties and affine Lie algebras of type A*

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TOWARDS GEOMETRIC SATAKE CORRESPONDENCE FOR KAC-MOODY ALGEBRAS, CHERKIS BOW VARIETIES AND AFFINE LIE ALGEBRAS OF TYPE A

BY HIRAKU NAKAJIMA

ABSTRACT. – We give a provisional construction of the Kac-Moody Lie algebra module structure on the hyperbolic restriction of the intersection cohomology complex of the Coulomb branch of a framed quiver gauge theory, as a refinement of the conjectural geometric Satake correspondence for Kac-Moody algebras proposed in an earlier paper with Braverman, Finkelberg in 2019. This construction assumes several geometric properties of the Coulomb branch under the torus action. These properties are checked in affine type A , via the identification of the Coulomb branch with a Cherkis bow variety established in a joint work with Takayama.

RÉSUMÉ. – Nous donnons une construction provisoire de la structure de module d’algèbre de Lie de Kac-Moody sur la restriction hyperbolique du complexe de cohomologie d’intersection de la branche de Coulomb d’une théorie de jauge avec carquois, comme un raffinement de la correspondance géométrique conjecturale de Satake pour les algèbres de Kac-Moody proposé dans un article antérieur avec Braverman, Finkelberg en 2019. Cette construction suppose plusieurs propriétés géométriques de la branche de Coulomb sous l’action du tore. Ces propriétés sont vérifiées dans le type affine A , via l’identification de la branche Coulomb avec une variété d’arc Cherkis établie dans un travail commun avec Takayama.

Introduction

Let $Q = (Q_0, Q_1)$ be a quiver without edge loops and \mathfrak{g}_{KM} be the corresponding symmetric Kac-Moody Lie algebra. Let $\mathcal{M}(\lambda, \mu)$ be the Coulomb branch of the framed quiver gauge theory associated with dimension vectors specified by a dominant weight λ and a weight μ with $\mu \leq \lambda$, defined as an affine algebraic variety by Braverman, Finkelberg, and the author [6]. (See also the earlier paper [29] for motivation and references to physics literature.) In the subsequent paper [7, §3(viii)] it was conjectured that there is a geometric construction of an integrable highest weight \mathfrak{g}_{KM} -module structure on the direct sum (over μ) via $\mathcal{M}(\lambda, \mu)$: Recall $\mathcal{M}(\lambda, \mu)$ is equipped with an action of the torus $T = (\mathbb{C}^\times)^{Q_0}$, the Pontryagin dual of the fundamental group of the gauge group, which is \mathbb{Z}^{Q_0} in this case.

The conjecture in [7] refines and generalizes the earlier conjecture by Braverman-Finkelberg [4], which uses instanton moduli spaces on $\mathbb{R}^4/(\mathbb{Z}/\ell\mathbb{Z})$ in the affine case. The Coulomb branch $\mathcal{M}(\lambda, \mu)$ for an affine type with dominant μ is conjectured to be instanton moduli spaces. (This is proved for affine type A .)

Let Φ denote the hyperbolic restriction functor ([3, 10]) with respect to a generic one parameter subgroup in T . Let us apply it to the intersection cohomology complexes IC of $\mathcal{M}(\lambda, \mu)$ with coefficients in \mathbb{Q} . It is conjectured that Φ is hyperbolic semismall in the sense of [5, 3.5.1], and the fixed point set is either empty or a single point. Hence $\mathcal{V}_\mu(\lambda) \stackrel{\text{def.}}{=} \Phi(\mathrm{IC}(\mathcal{M}(\lambda, \mu)))$ is a vector space. The main part of the conjecture states that $\mathcal{V}(\lambda) = \bigoplus_\mu \mathcal{V}_\mu(\lambda)$ has the structure of an integrable highest weight $\mathfrak{g}_{\mathrm{KM}}$ -module $V(\lambda)$ with the highest weight λ so that $\mathcal{V}_\mu(\lambda)$ is a weight space with weight μ . It is regarded as the geometric Satake correspondence for the Kac-Moody Lie algebra $\mathfrak{g}_{\mathrm{KM}}$, as a generalization of the usual geometric Satake for a finite dimensional complex reductive group due to Lusztig, Ginzburg, Beilinson-Drinfeld and Mirković-Vilonen [18, 12, 1, 19]. (See also [11] for a review of the conjecture.) In particular, the hyperbolic restriction functor was used in Mirković-Vilonen's realization of representations. Ginzburg's realization of $V(\lambda)$ by the cohomology over the affine Grassmannian has no analog in general $\mathfrak{g}_{\mathrm{KM}}$ setting at the moment when this paper is written.

In this paper, we give a provisional construction of the $\mathfrak{g}_{\mathrm{KM}}$ -module structure, assuming several geometric properties of $\mathcal{M}(\lambda, \mu)$. This is a refinement of the conjecture in [7], as well as its supporting evidence since these geometric properties are technical in nature, and not mysterious unlike the $\mathfrak{g}_{\mathrm{KM}}$ -module structure. We then check the properties when $\mathfrak{g}_{\mathrm{KM}}$ is of affine type A , using the identification of relevant Coulomb branches with Cherkis bow varieties proved by Takayama and the author [31]. Bow varieties are symplectic reduction, and easier to handle than Coulomb branches. We also use Hanany-Witten transition (see §3(iv)) at various points. It is an isomorphism between two bow varieties. This technique is useful for the following reason. When a bow variety satisfies a balanced condition (see §3(ii)), it is isomorphic to Coulomb branch $\mathcal{M}(\lambda, \mu)$. A fixed point component of a balanced bow variety is another bow variety, but it does not necessarily satisfy the balanced condition. We then show that it is isomorphic to one with the balanced condition by Hanany-Witten transition.

The idea of the construction is simple. The $\mathfrak{g}_{\mathrm{KM}}$ -structure should be compatible with restriction to a Levi subalgebra \mathfrak{l} , and realized by the hyperbolic restriction functor with respect to a one parameter subgroup $\chi_{\mathfrak{l}}$ corresponding to the Levi subalgebra. When the one parameter subgroup is generic, the Levi subalgebra is Cartan, and we recover the above construction. This compatibility is well-known for the usual geometric Satake correspondence and is a key ingredient of the construction. Therefore we define operators e_i, f_i, h_i corresponding to $i \in Q_0$ by using the hyperbolic restriction with respect to χ_i for the Levi subalgebra \mathfrak{l}_i and the reduction to the case A_1 . It is easy to prove the conjecture in the A_1 case. The check of the defining relations on e_i, f_i, h_i , say $[e_i, f_j] = 0$ for $i \neq j$, is reduced to rank 2 cases. By considering tensor products as explained below, it is enough to check them when λ is a fundamental weight. For $\mathfrak{sl}(3)$ relevant bow varieties are affine spaces, and we check them by direct computation. We also realize the embedding $\widehat{\mathfrak{sl}}(n) \rightarrow \widehat{\mathfrak{gl}}(\infty)$ by a

variant of a family $\mathcal{M}(\lambda, \mu)$ below. This argument covers the case $\widehat{\mathfrak{sl}}(2)$. Since we consider only affine types, these are enough.

Unlike in [19] we take \mathbb{Q} as field of coefficients. We believe that some of the arguments survive even in positive characteristic, but we leave the study for the future.

Suppose that Q is of finite type, and hence \mathfrak{g}_{KM} is a finite dimensional complex simple Lie algebra \mathfrak{g} of type ADE . Then $\mathcal{M}(\lambda, \mu)$ is isomorphic to a transversal slice to an orbit in the closure of another orbit in the affine Grassmannian when μ is dominant [7]. The group G^\vee for the affine Grassmannian is Langlands dual to the simply-connected G with Lie algebra \mathfrak{g} . (Hence representations of G are nothing but representations of \mathfrak{g} .) This is one of the reasons why we expect the geometric Satake correspondence for \mathfrak{g}_{KM} via $\mathcal{M}(\lambda, \mu)$. Moreover the hyperbolic restriction $\Phi(\text{IC}(\mathcal{M}(\lambda, \mu)))$ is naturally identified with one in the affine Grassmannian even for non-dominant μ by a recent result by Krylov [16]. Therefore the \mathfrak{g} -module structure is induced from the usual geometric Satake correspondence.

From this point of view, our construction above resembles the definition of Kashiwara crystal structure on the set of irreducible components of Mirković-Vilonen cycles by Braverman-Gaitsgory [9]. It is also similar to Vasserot's construction [35] of a \mathfrak{g} -module structure. The main difference between these constructions and ours is a construction of an isomorphism between multiplicity spaces appearing in the hyperbolic restriction with respect to χ_i , which we will explain in §1(ii). In our case, the isomorphism is given by the factorization property of Coulomb branches. This isomorphism comes for free or is unnecessary in the usual setting [9, 35]. See §5(viii) for a comparison between our construction and the usual one.

After the author gave a talk on this work at Sydney, B. Webster explained to him an approach to a construction of a \mathfrak{g}_{KM} -module structure via symplectic duality. It is not clear to the author how much can be said in this approach at the time this paper is written. The construction in this paper is nothing to do with the symplectic dual side, which is a quiver variety, where a geometric construction of \mathfrak{g}_{KM} -modules was given in [21, 22]. See [7, §3(viii)] for parallel explanation of two constructions.

After the first version of this paper was written, the author and Weekes generalize the definition of the Coulomb branch of a framed quiver gauge theory to a quiver with *symmetrizer* [32]. Our geometric Satake correspondence is naturally generalized to cover the case when \mathfrak{g}_{KM} is *symmetrizable* but not necessarily symmetric. The type of the quiver with symmetrizer is Langlands dual $\mathfrak{g}_{\text{KM}}^\vee$ as in the usual geometric Satake correspondence.

The paper is organized as follows. In §1 we formulate conjectures on geometric properties of Coulomb branches under the torus action. In §2 we fix notation for weights of affine Lie algebras. In §3 we review the quiver description and important properties of bow varieties studied in [31]. §4 is the heart of this paper and is devoted to the study of torus action on bow varieties. In §5 we use results in §4 to define a \mathfrak{g}_{KM} structure on the hyperbolic restriction for affine type A . In the appendix we parametrize torus fixed points in bow varieties when they are smooth. Fixed points are in bijection to Maya diagrams which appear in the infinite wedge space.

Notation

The symmetric group of n letters is denoted by \mathfrak{S}_n .

Let J_k denote the regular nilpotent Jordan matrix of size k :

$$J_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & \end{bmatrix}.$$

For an irreducible algebraic variety X we denote by $\mathrm{IC}(X)$ its intersection cohomology complex associated with the trivial rank 1 local system on its regular locus with rational coefficients.

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1. Conjectures

1(i). Earlier conjectures

We first recall conjectures from [7].

Let $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ and $\mathfrak{g}_{\mathrm{KM}}$ be as in the Introduction. Let $\mathcal{M}(\lambda, \mu)$ be the Coulomb branch associated with a dominant weight λ and a weight μ with $\mu \leq \lambda$. It has a structure of a Poisson variety such that its regular locus is symplectic (see [6, §3(iv) and Prop. 6.15]). It is conjectured that there are only finitely many symplectic leaves. This conjecture is known when $\mathfrak{g}_{\mathrm{KM}}$ is affine type A . See Theorem 3.13 below. We assume it from now, and we call symplectic leaves *strata* of $\mathcal{M}(\lambda, \mu)$.

Let T be the torus acting on $\mathcal{M}(\lambda, \mu)$ as in the Introduction. Let us take a generic 1-parameter family $\chi: \mathbb{C}^\times \rightarrow T$ and consider a diagram

$$\mathrm{pt} = \mathcal{M}(\lambda, \mu)^T \xleftarrow{p} \mathfrak{A}_\chi(\lambda, \mu) \xrightarrow{j} \mathcal{M}(\lambda, \mu),$$

where $\mathfrak{A}_\chi(\lambda, \mu)$ is the attracting set with respect to χ . When there is no fear of confusion, we denote it simply by \mathfrak{A} . Here j is the inclusion, and p is the map given by taking the limit $\rho(t)$ for $t \rightarrow 0$. The hyperbolic restriction functor $\Phi = p_* j^!$ is defined on the derived category of equivariant constructible sheaves.

CONJECTURE 1.1 ([7, Conj. 3.25]). – (1) $\mathcal{M}(\lambda, \mu)^T = \mathcal{M}(\lambda, \mu)^{\chi(\mathbb{C}^\times)}$ is either a single point or empty.

(2) Intersections of $\mathfrak{A}_\chi(\lambda, \mu)$ with symplectic leaves of $\mathcal{M}(\lambda, \mu)$ are lagrangian. Hence Φ is hyperbolic semi-small for the intersection cohomology complex $\mathrm{IC}(\mathcal{M}(\lambda, \mu))$. In particular, $\mathcal{V}_\mu(\lambda) \stackrel{\mathrm{def.}}{=} \Phi(\mathrm{IC}(\mathcal{M}(\lambda, \mu)))$ is a vector space.

(3) The direct sum $\mathcal{V}(\lambda) = \bigoplus \mathcal{V}_\mu(\lambda)$ has the structure of an integrable highest weight $\mathfrak{g}_{\mathrm{KM}}$ -module $V(\lambda)$.

1(ii). New conjectures

Let us introduce several notations in order to state conjectural geometric properties and the construction of the $\mathfrak{g}_{\mathrm{KM}}$ -module structure in more detail.

Set $|\mathbf{v}| = \sum_{i \in Q_0} \mathbf{v}_i$ and $\mathbb{A}^{\mathbf{v}} = \mathbb{A}^{|\mathbf{v}|} / \prod_{i \in Q_0} \mathfrak{S}_{\mathbf{v}_i}$, where $\lambda - \mu = \sum_i \mathbf{v}_i \alpha_i$ with simple roots α_i . We consider $\mathbb{A}^{\mathbf{v}}$ as the configuration space of Q_0 -colored points in \mathbb{A} . We have the factorization morphism $\varpi: \mathcal{M}(\lambda, \mu) \rightarrow \mathbb{A}^{\mathbf{v}}$ ([6, (3.17)]). This was denoted by Ψ in the context of bow varieties [31], and played fundamental roles in the analysis of Coulomb branches and their identification with bow varieties. In particular, it enjoys the factorization property that says $\mathcal{M}(\lambda, \mu)$ factorizes over an open subset of disjoint configurations. (See §3(iii) for a brief review.)

Let $w_{i,r}, y_{i,r}$ ($i \in Q_0, r = 1, \dots, \mathbf{v}_i$) be regular functions on $\mathcal{M}(\lambda, \mu) \times_{\mathbb{A}^{\mathbf{v}}} \mathbb{A}^{|\mathbf{v}|}$ introduced in [7, §3(iii)] ($y_{i,r}$ was denoted by $\bar{y}_{i,r}$ there) and [31, §6.8.1]. They induce a birational isomorphism $\mathcal{M}(\lambda, \mu) \times_{\mathbb{A}^{\mathbf{v}}} \mathbb{A}^{|\mathbf{v}|} \dashrightarrow (\mathbb{A} \times \mathbb{G}_m)^{|\mathbf{v}|}$. The factorization morphism ϖ corresponds to the projection to the first factor \mathbb{A} . In other words, it is given by $((w_{i,r})_r \bmod \mathfrak{S}_{\mathbf{v}_i})_i$. See §3(viii) for the definition of $w_{i,r}, y_{i,r}$ in terms of bow varieties.

We take the one parameter subgroup χ of T from the ‘negative’ Weyl chamber, i.e., $\chi(t) = (t^{m_j})_{j \in Q_0}$ with $m_j < 0$ for all $j \in Q_0$. We consider the corresponding hyperbolic restriction functor Φ with respect to χ . Choosing $i \in Q_0$, we take another one parameter subgroup χ_i so that $\chi_i(t) = (t^{m_j})$ with $m_i = 0, m_j < 0$ for $j \neq i$. This χ_i lives at the boundary of the chamber containing χ . We then consider the fixed point set $\mathcal{M}(\lambda, \mu)^{\chi_i}$ with respect to χ_i .

CONJECTURE 1.2. – (1) The fixed point set $\mathcal{M}(\lambda, \mu)^{\chi_i}$ is either empty or isomorphic to a Coulomb branch $\mathcal{M}_{A_1}(\lambda', \mu')$ of an A_1 type framed quiver gauge theory with weights λ', μ' , where $\mu' = \langle \mu, h_i \rangle$.

(2) The intersection of $\mathcal{M}(\lambda, \mu)^{\chi_i}$ with a stratum is either empty or a stratum $\mathcal{M}_{A_1}^s(\kappa', \mu')$ of $\mathcal{M}_{A_1}(\lambda', \mu')$. (Here $\mathcal{M}_{A_1}^s(\kappa', \mu')$ is the smooth locus of $\mathcal{M}_{A_1}(\kappa', \mu')$. The stratification of $\mathcal{M}_{A_1}(\lambda', \mu')$ was determined in [31, §7.5].)

(3) The restrictions $w_{j,r}, y_{j,r}|_{\mathcal{M}(\lambda, \mu)^{\chi_i}}$ are zero for $j \neq i$, and are equal to the corresponding functions on $\mathcal{M}_{A_1}(\lambda', \mu')$ or zero for $j = i$. In particular, the restriction of the i -th component of the factorization morphism ϖ of $\mathcal{M}(\lambda, \mu)$ to $\mathcal{M}_{A_1}(\lambda', \mu')$ is equal to the factorization morphism of $\mathcal{M}_{A_1}(\lambda', \mu')$ up to adding 0.

This conjecture will be shown for affine type A in Theorem 4.14.

The condition (3) uniquely fixes the isomorphism $\mathcal{M}(\lambda, \mu)^{\chi_i} \cong \mathcal{M}_{A_1}(\lambda', \mu')$ since the restrictions of $w_{i,r}, y_{i,r}$ give the birational coordinates on $\mathcal{M}_{A_1}(\lambda', \mu')$.

Once we prove that $\mathcal{V}(\lambda) \cong V(\lambda)$, λ' is determined as follows. Let \mathfrak{l}_i be the i -th Levi subalgebra of \mathfrak{g}_{KM} . We consider \mathfrak{l}_i -modules with highest weight $\mu + \mathbf{v}\alpha_i$ ($\mathbf{v} \in \mathbb{Z}_{\geq 0}$) which appear in the restriction of the integrable highest weight module $V(\lambda)$. Then $(\lambda' - \mu')/2$ is the maximum among such \mathbf{v} . When Conjecture 1.2 will be discussed, this would not be clear. See the proof of Theorem 4.14.

REMARK 1.3. – After the first version of this paper was written, we find that the above $\lambda' = \mu' + 2\mathbf{v}$ is characterized as

$$(1.4) \quad \mathcal{M}(\lambda, \mu + \mathbf{v}\alpha_i)^T \neq \emptyset, \quad \mathcal{M}(\lambda, \mu + (\mathbf{v} + 1)\alpha_i)^T = \emptyset$$

during a discussion with D. Muthiah. This condition is phrased in terms of Coulomb branches, and is equivalent to the above representation theoretic one by Conjecture 1.1(1). The condition (1.4) is checked for affine type A . See Remark 4.18.

Since χ_i lives in the boundary of a chamber containing χ , the hyperbolic restriction Φ factors as $\Phi = \Phi^i \circ \Phi_i$. Here Φ_i is the hyperbolic restriction with respect to χ_i , and Φ^i is the hyperbolic restriction with respect to χ , restricted to the fixed point set $\mathcal{M}(\lambda, \mu)^{\chi_i}$. Assuming Conjecture 1.1(2) (and also that for type A_1 , which was already proved in [31, Prop. 7.33]), we see that Φ_i is hyperbolic semismall in the sense of [5, 3.5.1], hence it sends $\text{IC}(\mathcal{M}(\lambda, \mu))$ to a semisimple perverse sheaf. See the argument in [5, App. A].

We further conjecture

CONJECTURE 1.5. – $\Phi_i(\text{IC}(\mathcal{M}(\lambda, \mu)))$ is a direct sum of $\text{IC}(\mathcal{M}_{A_1}(\kappa', \mu'))$ with various κ' with $\mu' \leq \kappa' \leq \lambda'$.

This conjecture means that we do not have *nontrivial* local systems on the regular locus of $\mathcal{M}_{A_1}(\kappa', \mu')$. For affine type A , it follows from the smoothness of deformation of $\mathcal{M}(\lambda, \mu)$, and compatibility between Conjecture 1.2 and the deformation of $\mathcal{M}(\lambda, \mu)$ explained later. See the proof of Proposition 5.6. By Conjecture 1.5

$$(1.6) \quad \Phi_i(\text{IC}(\mathcal{M}(\lambda, \mu))) \cong \bigoplus_{\kappa'} M_{\kappa', \mu'}^{\lambda, \mu} \otimes \text{IC}(\mathcal{M}_{A_1}(\kappa', \mu'))$$

for vector spaces $M_{\kappa', \mu'}^{\lambda, \mu}$, called *multiplicity spaces*. Hence we also deduce

$$\mathcal{V}_\mu(\lambda) = \Phi(\text{IC}(\mathcal{M}(\lambda, \mu))) \cong \bigoplus_{\kappa'} M_{\kappa', \mu'}^{\lambda, \mu} \otimes \Phi^i(\text{IC}(\mathcal{M}_{A_1}(\kappa', \mu')))$$

as $\Phi = \Phi^i \circ \Phi_i$.

The factor $\Phi^i(\text{IC}(\mathcal{M}_{A_1}(\kappa', \mu')))$ is $\mathcal{V}_{\mu'}(\kappa')$ for the finite A_1 case. Therefore it should be the weight space of a finite dimensional irreducible $\mathfrak{sl}(2)$ -module. We indeed construct operators $\Phi^i(\text{IC}(\mathcal{M}_{A_1}(\kappa', \mu'))) \xrightleftharpoons[e]{f} \Phi^i(\text{IC}(\mathcal{M}_{A_1}(\kappa', \mu' - 2)))$ in Theorem 5.4.

The factorization property of the Coulomb branch says that there is a χ_i -equivariant isomorphism between open subsets of $\mathcal{M}(\lambda, \mu) \times (\mathbb{A} \times \mathbb{G}_m)$ and $\mathcal{M}(\lambda, \mu - \alpha_i)$ after a base change. See Theorem 3.6. Here χ_i acts trivially on $\mathbb{A} \times \mathbb{G}_m$. The multiplicity spaces $M_{\kappa', \mu'}^{\lambda, \mu}$,

$M_{\kappa', \mu' - 2}^{\lambda, \mu - \alpha_i}$ are determined by restriction to open subsets. Therefore we have an isomorphism $M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu' - 2}^{\lambda, \mu - \alpha_i}$. (See Proposition 5.10.) We define operators e_i, f_i

$$\mathcal{V}_\mu(\lambda) \xrightleftharpoons[e_i]{f_i} \mathcal{V}_{\mu - \alpha_i}(\lambda),$$

as (the above isomorphism) $\otimes (e, f$ for A_1 case).

The multiplicity space $M_{\kappa', \mu'}^{\lambda, \mu}$ is $\text{Hom}_{\mathfrak{l}_i}(V_{\mathfrak{l}_i}(\tilde{\kappa}'), \mathcal{V}(\lambda))$ once $\mathcal{V}(\lambda)$ is equipped with an \mathfrak{l}_i -module as above, where $\tilde{\kappa}' = \mu + (\kappa' - \mu')\alpha_i/2$. Here $V_{\mathfrak{l}_i}(\tilde{\kappa}')$ is a finite dimensional irreducible representation of \mathfrak{l}_i with highest weight $\tilde{\kappa}'$. Note $\tilde{\kappa}'$ and hence $\text{Hom}_{\mathfrak{l}_i}(V_{\mathfrak{l}_i}(\tilde{\kappa}'), \mathcal{V}(\lambda))$ are unchanged under simultaneous shifts $\mu \rightarrow \mu - \alpha_i, \mu' \rightarrow \mu' - 2$. Thus the construction of the isomorphism $M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu' - 2}^{\lambda, \mu - \alpha_i}$ is crucial in the definition of a \mathfrak{g}_{KM} -module structure on $\mathcal{V}(\lambda)$. This isomorphism is obscure in the usual geometric Satake correspondence, as we mentioned in the Introduction.

1(iii). Tensor products

Next, we consider the realization of tensor products in this framework. We recall [7, Conj. 3.27] and give its refinement as in the previous subsection.

Take a decomposition $\lambda = \lambda^1 + \lambda^2$ into a sum of two dominant weights λ^1, λ^2 . Then it gives a one parameter subgroup in the flavor symmetry group of the quiver gauge theory. (See [6, §3(viii)].) This gives rise to a family $\underline{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{A}^1$ parametrized by the affine line \mathbb{A}^1 together with $\pi: \widetilde{\mathcal{M}}(\lambda, \mu) \rightarrow \underline{\mathcal{M}}(\lambda, \mu)$ which is expected to be a small birational morphism, and the second family $\widetilde{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{A}^1$ is topologically trivial over \mathbb{A}^1 . We further conjecture that the fixed point set $\underline{\mathcal{M}}(\lambda, \mu)^\chi$ is a union of finitely many copies of \mathbb{A}^1 , glued at the origin such that components correspond, in bijection, to a decomposition $\mu = \mu^1 + \mu^2$ with $\mathcal{V}_{\mu^1}(\lambda^1), \mathcal{V}_{\mu^2}(\lambda^2) \neq 0$. Finally, we conjecture that there are isomorphisms

$$(1.7) \quad \Phi \circ \pi_*(\text{IC}(\widetilde{\mathcal{M}}(\lambda, \mu))) \cong \psi \circ \Phi(\text{IC}(\underline{\mathcal{M}}(\lambda, \mu))) \cong \bigoplus_{\mu = \mu^1 + \mu^2} \mathcal{V}_{\mu^1}(\lambda^1) \otimes \mathcal{V}_{\mu^2}(\lambda^2),$$

where $\widetilde{\mathcal{M}}(\lambda, \mu)$ is the fiber of $\widetilde{\mathcal{M}}(\lambda, \mu)$ over 0, and ψ is the nearby cycle functor. The first isomorphism is a consequence of the triviality of $\widetilde{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{A}^1$ and the commutativity of the nearby cycle and hyperbolic restriction functors. (See Remark 1.11 below for a comment on the commutativity.)

Since the second isomorphism in (1.7) was stated in [7, Conj. 3.27(2)] without reason, let us give an explanation. Take a fiber $\mathcal{M}^{\nu^{\bullet, \mathbb{C}}}(\lambda, \mu)$ of $\underline{\mathcal{M}}(\lambda, \mu)$ over $1 \in \mathbb{A}^1$. The parameter $\nu^{\bullet, \mathbb{C}}$ is determined by a nonzero complex number ν which corresponds to the second weight λ^2 , while 0 corresponds to λ^1 . See §4(ii) for detail. Then

CONJECTURE 1.8. – (1) The fixed point set $\mathcal{M}^{\nu^{\bullet, \mathbb{C}}}(\lambda, \mu)^\chi$ is bijective to

$$\bigsqcup_{\mu^1 + \mu^2 = \mu} \mathcal{M}(\lambda^1, \mu^1)^\chi \times \mathcal{M}(\lambda^2, \mu^2)^\chi.$$

(The bijection realizes the above bijection between irreducible components of $\underline{\mathcal{M}}(\lambda, \mu)^\chi$ and decomposition.)

(2) The image of a fixed point in $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)^X$ under the factorization morphism ϖ is supported at 0 and \dot{v} . The factorization gives an isomorphism $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)$ and $\mathcal{M}(\lambda^1, \mu^1) \times \mathcal{M}(\lambda^2, \mu^2)$ in neighborhoods of fixed points.

The factorization isomorphism in (2) gives the second isomorphism in (1.7). We will prove this conjecture in Proposition 4.7 for affine type A .

We thus have

$$(1.9) \quad \mathcal{V}(\lambda) = \bigoplus_{\mu} \Phi(\mathrm{IC}(\mathcal{M}(\lambda, \mu))) \rightarrow \bigoplus_{\mu} \Phi \circ \pi_*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu))) \xrightarrow[(1.7)]{\cong} \mathcal{V}(\lambda^1) \otimes \mathcal{V}(\lambda^2),$$

where the middle arrow is induced from the inclusion of the direct summand $\mathrm{IC}(\mathcal{M}(\lambda, \mu)) \hookrightarrow \pi_*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))$. The composite is conjectured ([7, Conj. 3.27(3)]) to be a homomorphism of $\mathfrak{g}_{\mathrm{KM}}$ -modules.

We refine this conjecture as follows. We apply the above construction of the $\mathfrak{g}_{\mathrm{KM}}$ -module structure via χ_i to $\pi_*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))$. The construction is compatible with the inclusion of the direct summand, hence the middle arrow in (1.9) is a $\mathfrak{g}_{\mathrm{KM}}$ -homomorphism.

We then prove that $\xrightarrow[(1.7)]{\cong}$ in (1.9) is a $\mathfrak{g}_{\mathrm{KM}}$ -homomorphism by a reduction to the A_1 case, which is easy to show by direct computation. The A_1 reduction is guaranteed by a geometric property of the fixed point set in the deformed case, as will be shown in Proposition 5.13:

CONJECTURE 1.10. – Let us take χ_i as above. The fixed point set $\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)^{\chi_i}$ is a union of A_1 type Coulomb branches where the parameter is induced from the original parameter $v^{\bullet, \mathbb{R}}$.

We do not have a good understanding of what we mean by the induced parameter. At this moment, we just say that the entries of new parameters, once we forget multiplicities, are entries $v_h^{\bullet, \mathbb{R}}$ of the original parameter.

We check this for affine type A in Lemma 5.8.

REMARK 1.11. – The reference for the commutativity was [30, Prop. 5.4.1] in [7, Conj. 3.27(2)]. The last part of the proof of [30, Prop. 5.4.1(2)] works only the case when $\widetilde{\mathcal{M}}(\lambda, \mu)$ is smooth. This is true for affine type A , but not in general. We need to use $\tilde{c}_X^* = \tilde{c}_X^!$. The commutativity, as well as the commutativity between the nearby cycle functor and the isomorphism $p_* j^! \cong p_1^-(j^-)^*$ are proved in [33]. Here p^- , j^- is the projection and the inclusion for the attracting set with respect to the opposite 1-parameter subgroup χ^{-1} . The isomorphism is a main theorem of [3], and used here as semisimplicity of $\Phi_i(\mathrm{IC}(\mathcal{M}(\lambda, \mu)))$.

2. Weights of affine Lie algebras

We fix our convention on weights of affine Lie algebras of type A in this section.

We denote the central extension of the loop Lie algebra of $\mathfrak{sl}(n)$ by $\widehat{\mathfrak{sl}}(n)$ while the affine Lie algebra containing the degree operator d is denoted by $\mathfrak{sl}(n)_{\mathrm{aff}}$. We also use versions for $\mathfrak{gl}(n)$, which are denoted by $\widehat{\mathfrak{gl}}(n)$, $\mathfrak{gl}(n)_{\mathrm{aff}}$ respectively.

Let us take the Cartan subalgebra $\mathfrak{h}_{\mathfrak{gl}(n)}$ of $\mathfrak{gl}(n)$ as the space of diagonal matrices. The weight lattice $P_{\mathfrak{gl}(n)}$ of $\mathfrak{gl}(n)$ is \mathbb{Z}^n , where the i -th coordinate vector e_i is $\mathfrak{h}_{\mathfrak{gl}(n)} \rightarrow \mathbb{C}$ given by taking the i -th diagonal entry of $h \in \mathfrak{h}_{\mathfrak{gl}(n)}$. The weight lattice $P_{\mathfrak{sl}(n)}$ of $\mathfrak{sl}(n)$ is its

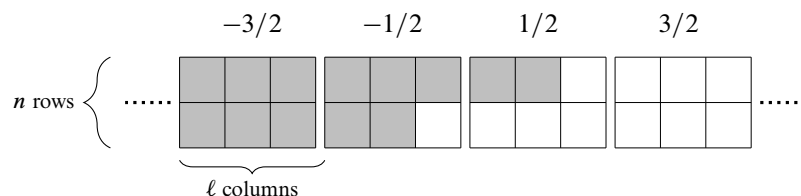
quotient $\mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$, considered as n -tuples of integers $[\lambda_1, \dots, \lambda_n]$ up to simultaneous shifts. We let $\alpha_i \stackrel{\text{def.}}{=} e_i - e_{i+1} \bmod \mathbb{Z}[1, \dots, 1]$, the i -th simple root of $\mathfrak{sl}(n)$. The i -th fundamental weight Λ_i of $\mathfrak{sl}(n)$ is $(\underbrace{1, \dots, 1}_{i \text{ times}}, \underbrace{0, \dots, 0}_{(n-i) \text{ times}}) \bmod \mathbb{Z}[1, \dots, 1]$.

We denote simple roots of $\mathfrak{sl}(n)_{\text{aff}}$ by $\alpha_0, \dots, \alpha_{n-1}$. Here the primitive positive imaginary root is $\delta = \alpha_0 + \dots + \alpha_{n-1}$, hence $\alpha_0 = \delta - (\alpha_1 + \dots + \alpha_n)$. We denote fundamental weights by $\Lambda_0, \dots, \Lambda_{n-1}$. Our convention is $\langle d, \Lambda_i \rangle = 0$, $\langle d, \alpha_i \rangle = \delta_{0i}$. The weight ‘lattice’ $P_{\mathfrak{sl}(n)_{\text{aff}}}$ of $\mathfrak{sl}(n)_{\text{aff}}$ is $\bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i \oplus \mathbb{C}\delta$. (This is not a lattice, but we keep this terminology.) The weight lattice $P_{\widehat{\mathfrak{sl}(n)}}$ of $\widehat{\mathfrak{sl}(n)}$ is identified with $\bigoplus_{i=0}^{n-1} \mathbb{Z}\Lambda_i$. The level of a weight λ in $P_{\mathfrak{sl}(n)_{\text{aff}}}$ (or $P_{\widehat{\mathfrak{sl}(n)}}$) is $\langle c, \lambda \rangle$. If $\lambda = \sum_{i=0}^{n-1} \mathbf{w}_i \Lambda_i (+a\delta)$, the level is equal to $\sum_{i=0}^{n-1} \mathbf{w}_i$. We often fix a level ℓ , then the set of level ℓ weights is identified with $\mathbb{Z}^n / \mathbb{Z}[1, \dots, 1] \times \mathbb{C}$ (or $\mathbb{Z}^n / \mathbb{Z}[1, \dots, 1]$), where \mathbf{w}_i with $1 \leq i \leq n$ is read off from $\mathbb{Z}^n / \mathbb{Z}[1, \dots, 1]$, and \mathbf{w}_0 is given by $\ell - \sum_{i=1}^{n-1} \mathbf{w}_i$. Namely $[\lambda_1, \dots, \lambda_n]$ defines $\mathbf{w}_1 = \lambda_1 - \lambda_2, \dots, \mathbf{w}_{n-1} = \lambda_{n-1} - \lambda_n$. In the same way a level ℓ weight of $\mathfrak{gl}(n)_{\text{aff}}$ (resp. $\widehat{\mathfrak{gl}(n)}$) is an element in $\mathbb{Z}^n \times \mathbb{C}$ (resp. \mathbb{Z}^n).

Let W_{aff} be the Weyl group of $\mathfrak{sl}(n)_{\text{aff}}$. It is the semi-direct product $W \ltimes \mathbb{Z}^{n-1}$ of a finite Weyl group (= the symmetric group of n letters) and the root lattice $\mathbb{Z}^{n-1} \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}\alpha_i$. It acts on the set of level ℓ weights of $P_{\widehat{\mathfrak{sl}(n)}}$, identified with $\mathbb{Z}^n / \mathbb{Z}[1, \dots, 1]$ by permutation for the W part, and translation by $\ell \mathbb{Z}^{n-1}$ for the root lattice part. The fundamental alcove is $\{[\lambda_1, \dots, \lambda_n] \mid \lambda_1 \geq \dots \geq \lambda_n \geq \lambda_1 - \ell\}$. A level ℓ weight λ is dominant if and only if the corresponding $[\lambda_1, \dots, \lambda_n]$ is contained in the fundamental alcove.

It is convenient to extend λ_i to all $i \in \mathbb{Z}$ so that $\lambda_{i+n} = \lambda_i - \ell$. For example, the fundamental alcove consists of $\dots \geq \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_n \geq \lambda_{n+1} \geq \dots$. The j -th permutation s_j ($j = 0, \dots, n-1$) of W_{aff} acts on $(\lambda_i)_{i \in \mathbb{Z}}$ by exchanging λ_i and λ_{i+1} if $i \equiv j \bmod n$.

A level ℓ dominant weight of $\widehat{\mathfrak{gl}(n)}$ corresponds to a *generalized Young diagram with the level ℓ constraint* (see [27, App. A], [31, §7.6])



where a box is indexed by (i, σ, N) with $1 \leq i \leq n$, $1 \leq \sigma \leq \ell$, $N \in \mathbb{Z} + 1/2$ and we put a gray box \blacksquare if $\ell(N - 1/2) + \sigma \leq \lambda_i$. The above figure is $[\lambda_1, \lambda_2] = [2, -1]$ for $n = 2$, $\ell = 3$. We define the transpose of a generalized Young diagram by the transposition of each rectangle.

Then we get a sequence $[\lambda_1, \dots, \lambda_\ell]$ which satisfies $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq \lambda_1 - n$, i.e., a generalized Young diagram with the level n constraint. The above example gives $[\lambda_1, \lambda_2, \lambda_3] = [1, 1, -1]$.

Recall that a simultaneous shift $[\lambda_1, \dots, \lambda_n] \mapsto [\lambda_1 - 1, \dots, \lambda_n - 1]$ does not change the weight of $P_{\widehat{\mathfrak{sl}(n)}}$. It corresponds to $[\lambda_1, \dots, \lambda_\ell] \mapsto [\lambda_2, \dots, \lambda_\ell, \lambda_1 - n]$. If we consider $[\lambda_1, \dots, \lambda_\ell]$ as a level n weight of $P_{\widehat{\mathfrak{sl}(\ell)}}$, it corresponds to a diagram automorphism $0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow (\ell - 1) \rightarrow 0$.

3. Bow varieties

3(i). Definition

Let us recall the quiver description of bow varieties in [31]. It is associated with a bow diagram such as Figure 1. A bow diagram consists of \times , \circ on a circle, and nonnegative

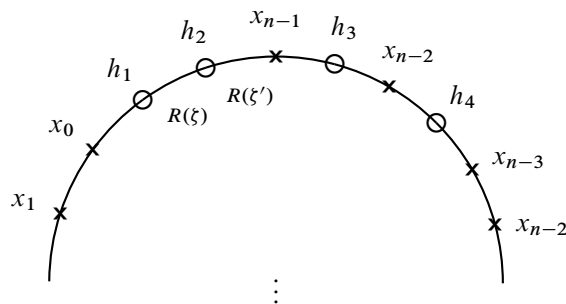


FIGURE 1. Bow diagram

integers $R(\zeta)$ for segments ζ cut by either \times or \circ . We index \times as x_0, x_1, \dots, x_{n-1} , \circ as h_1, h_2, \dots, h_ℓ in anticlockwise and clockwise orientation respectively. The number n of \times will be the rank of the affine Lie algebra $\mathfrak{sl}(n)_{\text{aff}}$. The number ℓ of \circ will be the level of an integrable highest weight representation. In Figure 1 only R for segments ζ, ζ' between h_1 and h_2, h_2 and x_{n-1} are drawn for simplicity.

We assign a vector space V_ζ for each segment ζ with $\dim V_\zeta = R(\zeta)$. We also assign a 1-dimensional vector \mathbb{C}_{x_i} for each x_i .

We assume $n > 1$ throughout this paper except Remarks 4.21 and 5.15.

We have a complex parameter $v_h^{\mathbb{C}}$ and a real parameter $v_h^{\mathbb{R}}$ for each h (i.e., h_σ for $1 \leq \sigma \leq \ell$), and also one additional pair $v_*^{\mathbb{C}}, v_*^{\mathbb{R}}$. We use the convention in [31, (6.3)], rather than in [31, §2.2].

A quiver description consists of the following:

1. A linear endomorphism $B_\zeta: V_\zeta \rightarrow V_\zeta$.
2. Let x be \times . Let $\text{o}(x), \text{i}(x)$ be the adjacent segments so that $\xrightarrow{\text{o}(x)} \underset{\times}{x} \xrightarrow{\text{i}(x)}$ in the anticlockwise orientation. We assign triple of linear maps

$$\begin{aligned} A_x: V_{\text{o}(x)} &\rightarrow V_{\text{i}(x)}, \\ a_x: \mathbb{C}_x &\rightarrow V_{\text{i}(x)}, \quad b_x: V_{\text{o}(x)} \rightarrow \mathbb{C}_x. \end{aligned}$$

3. Let h be \circ . Let $\text{o}(h), \text{i}(h)$ be the adjacent segments so that $\xrightarrow{\text{o}(h)} \underset{\circ}{h} \xrightarrow{\text{i}(h)}$ in the anticlockwise orientation. We assign a pair of linear maps

$$C_h: V_{\text{o}(h)} \rightarrow V_{\text{i}(h)}, \quad D_h: V_{\text{i}(h)} \rightarrow V_{\text{o}(h)}.$$

See [31, Fig. 1].

We denote the direct sum $\bigoplus B_\zeta \in \text{End}(\bigoplus V_\zeta)$ by B , and similarly for a, b, C, D . However we also denote B_ζ by B when ζ is clear from the context. The same applies for A_x , etc.

We require the following conditions:

- (a) Let $x, o(x), i(x)$ as in (2) above. As a linear map $V_{o(x)} \rightarrow V_{i(x)}$ we have

$$(B_{i(x)} + \delta_{x,x_0} v_*^{\mathbb{C}})A_x - A_x B_{o(x)} + a_x b_x = 0.$$

They satisfy the two conditions (S1),(S2):

(S1): There is no nonzero subspace $0 \neq S \subset V_{o(x)}$ with $B_{o(x)}(S) \subset S, A(S) = 0 = b(S)$.

(S2): There is no proper subspace $T \subsetneq V_{i(x)}$ with $B_{i(x)}(T) \subset T, \text{Im } A + \text{Im } a \subset T$.

- (b) Let $h, o(h), i(h)$ as in (3). As endomorphisms on $V_{i(h)}$ and $V_{o(h)}$, we have

$$C_h D_h + B_{i(h)} = v_h^{\mathbb{C}}, \quad D_h C_h + B_{o(h)} = v_h^{\mathbb{C}}$$

respectively.

Because of these defining equations, we often omit B_{ζ} when ζ has \bigcirc on the boundary.

- (c) We say (A, B, C, D, a, b) is $v^{\mathbb{R}}$ -semistable if the following conditions are satisfied:

(v1): Suppose a graded subspace $S = \bigoplus S_{\zeta} \subset \bigoplus V_{\zeta}$ invariant under A, B, C, D with $b(S) = 0$ is given. We further assume that A_x is an isomorphism $S_{o(x)} \xrightarrow{A_x} S_{i(x)} \cong$ for all $\frac{o(x)}{x} \xrightarrow{\bullet} \frac{i(x)}{x}$. Then

$$v_*^{\mathbb{R}} \dim S_{i(x_0)} + \sum_h v_h^{\mathbb{R}} (\dim S_{i(h)} - \dim S_{o(h)}) \leq 0.$$

Here $i(h), o(h)$ are determined by h by the rule in (3).

(v2): Suppose a graded subspace $T = \bigoplus T_{\zeta} \subset \bigoplus V_{\zeta}$ invariant under A, B, C, D with $\text{Im } a \subset T$ is given. We further assume that the restriction of A induces an isomorphism $V_{o(x)}/T_{o(x)} \xrightarrow{A_x} V_{i(x)}/T_{i(x)} \cong$ for all $\frac{o(x)}{x} \xrightarrow{\bullet} \frac{i(x)}{x}$. Then

$$v_*^{\mathbb{R}} \text{codim } T_{i(x_0)} + \sum_h v_h^{\mathbb{R}} (\text{codim } T_{i(h)} - \text{codim } T_{o(h)}) \geq 0.$$

We say (A, B, C, D, a, b) is $v^{\mathbb{R}}$ -stable if we have strict inequalities in (v1), (v2) unless $S_{\zeta} = 0, T_{\zeta} = V_{\zeta}$ for all ζ .

We have a natural group action of $\mathcal{G} := \prod \text{GL}(V_{\zeta})$ by conjugation, which preserves the above conditions. Let $\widetilde{\mathcal{M}}^{v\text{-ss}}$ (resp. $\widetilde{\mathcal{M}}^{v\text{-s}}$) denote the set of $v^{\mathbb{R}}$ -semistable (resp. $v^{\mathbb{R}}$ -stable) points satisfying other conditions (a),(b). (We understand that the parameter v is a pair $(v^{\mathbb{C}}, v^{\mathbb{R}})$.) We introduce the S -equivalence relation \sim on the $\widetilde{\mathcal{M}}^{v\text{-ss}}$ by defining $m \sim m'$ if and only if orbit closures $\overline{\mathcal{G}m}$ and $\overline{\mathcal{G}m'}$ intersect in the $\widetilde{\mathcal{M}}^{v\text{-ss}}$. Let \mathcal{M}^v denote $\widetilde{\mathcal{M}}^{v\text{-ss}}/\sim$. It is a geometric invariant theory quotient, hence in particular has a structure of a quasi-projective variety. Let $\mathcal{M}^{v\text{-s}}$ denote the open subvariety of \mathcal{M} given by $\widetilde{\mathcal{M}}^{v\text{-s}}/\mathcal{G}$.

When $v^{\mathbb{C}} = 0 = v^{\mathbb{R}}$, we denote $\mathcal{M}^v, \mathcal{M}^{v\text{-s}}$ by $\mathcal{M}, \mathcal{M}^s$ for brevity. We also understand that $\mathcal{M}^{v^{\mathbb{C}}}$ is a bow variety with vanishing real parameters, i.e., $v^{\mathbb{C}}$ is understood as an abbreviation of $(0, v^{\mathbb{C}})$. We also use $\mathcal{M}^{v^{\mathbb{R}}}$ for a bow variety with vanishing complex parameters, i.e., $v^{\mathbb{R}}$ means $(v^{\mathbb{R}}, 0)$.

We have a projective morphism

$$(3.1) \quad \pi: \mathcal{M}^v \rightarrow \mathcal{M}^{v^{\mathbb{C}}},$$

where $v^{\mathbb{C}}$ is the complex part of v , i.e., $v = (v^{\mathbb{R}}, v^{\mathbb{C}})$, and $\mathcal{M}^{v^{\mathbb{C}}}$ is a bow variety with vanishing real parameters as above.

REMARK 3.2. – Note that the inequality in $(\mathbf{v}\mathbf{1})$ can be rewritten as

$$v_{*}^{\mathbb{R}} \dim S_{i(x_0)} + \sum_h (v_h^{\mathbb{R}} - v_{h+1}^{\mathbb{R}}) \dim S_{i(h)} \leq 0.$$

The first term can be absorbed in the second term with h such that $i(h)$ is connected to $i(x_0)$ through triangle parts. For example, we have

$$(v_{*}^{\mathbb{R}} + v_1^{\mathbb{R}} - v_2^{\mathbb{R}}) \dim S_{i(h_1)} + \sum_{h \neq 1} (v_h^{\mathbb{R}} - v_{h+1}^{\mathbb{R}}) \dim S_{i(h)}$$

in Figure 1.

From this reformulation, it is clear that an overall shift of $v_h^{\mathbb{R}}$ is irrelevant. The same is true for $v_h^{\mathbb{C}}$ as we can simultaneously subtract a scalar from all B_{ξ} 's. Therefore the total number of real or complex parameters is ℓ .

3(ii). Coulomb branch

We say that the *balanced condition* is satisfied if $R(o(h)) = R(i(h))$ for any $\frac{o(h)}{h} \rightarrow \frac{i(h)}{h}$. Then $R(\zeta)$ depends only on the arc $x_{i-1} \rightarrow x_i$ which contains ζ . In particular R is determined by an n -tuple of integers $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ corresponding to $x_0 \rightarrow x_1, x_1 \rightarrow x_2, \dots, x_{n-1} \rightarrow x_0$. Let $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_{n-1}$ be the numbers of \bigcirc on the corresponding arcs. Let $\mathbf{v} = (\mathbf{v}_0, \dots, \mathbf{v}_{n-1})$, $\mathbf{w} = (\mathbf{w}_0, \dots, \mathbf{w}_{n-1})$.

THEOREM 3.3 ([31, §6]). – Suppose the balanced condition is satisfied, and determine \mathbf{v}, \mathbf{w} as above. Then the corresponding bow variety with parameters $v^{\mathbb{R}} = 0, v^{\mathbb{C}} = 0$ is isomorphic to the Coulomb branch of the framed affine quiver gauge theory with dimensions \mathbf{v}, \mathbf{w} .

Here the Coulomb branch is one defined in [6]. We take

$$\mathbf{N} = \bigoplus_{i=0}^{n-1} \mathrm{Hom}(\mathbb{C}^{\mathbf{v}_{i-1}}, \mathbb{C}^{\mathbf{v}_i}) \oplus \mathrm{Hom}(\mathbb{C}^{\mathbf{w}_i}, \mathbb{C}^{\mathbf{v}_i}),$$

as a representation of $\mathbf{G} \stackrel{\text{def.}}{=} \prod_{i=0}^{n-1} \mathrm{GL}(\mathbf{v}_i)$, and consider the variety

$$\mathcal{R} = \{([g], s) \in \mathbf{G}_{\mathcal{K}}/\mathbf{G}_{\mathcal{O}} \times \mathbf{N}_{\mathcal{O}} \mid g^{-1}s \in \mathbf{N}_{\mathcal{O}}\},$$

where $\mathcal{O} = \mathbb{C}[[z]] \subset \mathcal{K} = \mathbb{C}((z))$. Then we consider its equivariant Borel-Moore homology group $H_{*}^{\mathbf{G}_{\mathcal{O}}}(\mathcal{R})$ equipped with the convolution product. The Coulomb branch $\mathcal{M}_{\mathcal{C}}$ of the gauge theory associated with \mathbf{G}, \mathbf{N} is defined as the spectrum of the ring $H_{*}^{\mathbf{G}_{\mathcal{O}}}(\mathcal{R})$.

This \mathbf{N} is equipped with an action of a larger group $\tilde{\mathbf{G}} = (\mathbf{G} \times T(\mathbf{w}))/\mathbb{C}^{\times} \times \mathbb{C}_{\mathrm{dil}}^{\times}$ ($T(\mathbf{w}) = \prod_i T^{\mathbf{w}_i}$), where \mathbb{C}^{\times} is the diagonal scalar in $\mathbf{G} \times T(\mathbf{w})$ and $\mathbb{C}_{\mathrm{dil}}^{\times}$ acts on \mathbf{N} by scaling $\mathrm{Hom}(\mathbb{C}^{\mathbf{v}_{n-1}}, \mathbb{C}^{\mathbf{v}_0})$. We can consider the spectrum of $H_{*}^{\tilde{\mathbf{G}}_{\mathcal{O}}}(\mathcal{R})$, which is a deformation of $\mathcal{M}_{\mathcal{C}}$ parametrized by $\mathrm{Spec} H_{\mathbf{G}_F}^{*}(\mathrm{pt}) = \mathrm{Lie}(\mathbf{G}_F)$, where $\mathbf{G}_F = \tilde{\mathbf{G}}/\mathbf{G}$. Theorem 3.3 is generalized to this deformation (see [31, §6.8.2]). Therefore it is identified with $\mathcal{M}^{v^{\mathbb{C}}}$ (with vanishing real parameters) where $v^{\mathbb{C}} \in \mathrm{Lie}(\mathbf{G}_F)$ is identified with a tuple of complex numbers above under the standard coordinates. (The above definition of bow varieties is slightly modified from one in [31, §2] so that we have the identification of parameters. See [31, §6.2.2]. We also change the

action of $\mathbb{C}_{\text{dil}}^\times$ from the scaling on $\bigoplus_{i=0}^{n-1} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_{i+1}})$ to the scaling only on the factor $i = n - 1$.)

Similarly the variety $\mathcal{R}_{\tilde{\mathbf{G}}, \mathbf{N}}$ defined for the larger group $\tilde{\mathbf{G}}$ (and \mathbf{N}) defines a quasi-projective variety equipped with a projective morphism to $\mathcal{M}^{v^\mathbb{C}}$ depending on a \mathbb{Q} -coweight κ of \mathbf{G}_F . This is identified with \mathcal{M}^v so that this projective morphism coincides with $\pi: \mathcal{M}^v \rightarrow \mathcal{M}^{v^\mathbb{C}}$ in (3.1), when the real parameters $v^\mathbb{R}$. (See [8, §4].)

In what follows, we will not use the original definition of Coulomb branches and discuss only bow varieties. Hence we do not explain the further detail.

REMARK 3.4. – If the balanced condition is satisfied, the ordering of parameters v_h in the arc $x_i \rightarrow x_{i+1}$ is irrelevant. Consider the relevant part

$$\begin{array}{ccccccc}
 & B_o(x_i) & & B_i(x_i) & & B_o(x_{i+1}) & & B_i(x_{i+1}) \\
 & \curvearrowright & & \curvearrowright & & \curvearrowright & & \curvearrowright \\
 V_o(x_i) & \xrightarrow{A_{x_i}} & V_i^0 & \rightleftharpoons V_i^1 & \rightleftharpoons \cdots & \rightleftharpoons V_i^{w_i} & \xrightarrow{A_{x_{i+1}}} & V_{i(x_{i+1})} \\
 & \searrow b_{x_i} & \nearrow a_{x_i} & & & \searrow b_{x_{i+1}} & \nearrow a_{x_{i+1}} & \\
 & & \mathbb{C}_{x_i} & & & & \mathbb{C}_{x_{i+1}} &
 \end{array}$$

We can apply reflection functors [26] at $V_i^1, \dots, V_i^{w_i-1}$ to change the ordering of parameters v_h . Since the balanced condition means $\dim V_i^0 = \dots = \dim V_i^{w_i}$, dimensions of vector spaces are preserved under reflection functors.

3(iii). Factorization

Let $\mathbb{A}^{\mathbf{v}} = \prod_{i=0}^{n-1} \mathbb{A}^{v_i} / \mathfrak{S}_{v_i} = \text{Spec } H_{\mathbf{G}}^*(\text{pt})$. In the proof of Theorem 3.3, the factorization morphism $\varpi: \mathcal{M}^v \rightarrow \mathbb{A}^{\mathbf{v}}$ played an important role. Since we will use it later, let us recall its definition and properties. Suppose that \mathcal{M}^v is a bow variety with the balanced condition. For each x_i we consider $i(x_i)$ and the associated linear map $B_{i(x_i)}$. We count its eigenvalues with multiplicities and let it as the i -th component of ϖ . Since $B_{i(x_i)}$ and $B_{o(x_{i+1})}$ have the same eigenvalues by the defining equations thanks to the following lemma, we can also use $B_{o(x_{i+1})}$.

LEMMA 3.5. – Let $C: V \rightarrow V', D: V' \rightarrow V$ be a pair of linear maps. We have

$$\text{tr}_{V'}(t + CD)^N = \text{tr}_V(t + DC)^N + t^N (\dim V' - \dim V)$$

for any $N \in \mathbb{Z}_{\geq 0}$.

Let $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ be a decomposition of the dimension vector \mathbf{v} . Let $(\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''})_{\text{disj}}$ be the open subset of $\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''}$ consisting of $(w'_{i,1} + \dots + w'_{i,v'_i}, w''_{i,1} + \dots + w''_{i,v''_i})_{i=0}^{n-1}$ such that $w'_{i,j} \neq w''_{i,k}, w'_{i,j} \neq w''_{i\pm 1,k}, w'_{n-1,\sigma} \neq w''_{0,\tau} + v_{*,\mathbb{C}}, w'_{0,\sigma} \neq w''_{n-1,\tau} + v_{*,\mathbb{C}}$, where σ, τ run over the set of indices for data at $i, i \pm 1, n - 1$ or 0 appropriately. Note that the last two conditions correspond to the additional term $\delta_{x,x_0} v_{*,\mathbb{C}}$ in the defining equation.

THEOREM 3.6 ([31, Th. 6.9]). – (1) \mathcal{M}^v satisfies the factorization property:

$$\mathcal{M}^v \times_{\mathbb{A}^{\mathbf{v}}} (\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''})_{\text{disj}} \cong (\mathcal{M}^v(\mathbf{v}', \mathbf{w}) \times \mathcal{M}^v(\mathbf{v}'', \mathbf{w})) \times_{\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''}} (\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''})_{\text{disj}},$$

where $\mathcal{M}^v(\mathbf{v}', \mathbf{w})$, $\mathcal{M}^v(\mathbf{v}'', \mathbf{w})$ are bow varieties associated with dimension vectors \mathbf{v}', \mathbf{w} and \mathbf{v}'', \mathbf{w} respectively.

(2) \mathcal{M}^v is normal, and all fibers of ϖ have the same dimension.

In fact, the balanced condition is *not* essential in (1) once we note that eigenvalues of $B_{i(x_i)}$ and $B_{o(x_{i+1})}$ may differ, but differences are determined by $v_h^{\mathbb{C}}$ and differences of dimensions of $V_{i(h)}$ and $V_{o(h)}$ thanks to Lemma 3.5.

When there is no fear of confusion and the open subset $(\mathbb{A}^{\mathbf{v}'} \times \mathbb{A}^{\mathbf{v}''})_{\text{disj}}$ is clear from the context, we simply write the above isomorphism after the base change as $\mathcal{M}^v \approx \mathcal{M}^v(\mathbf{v}', \mathbf{w}) \times \mathcal{M}^v(\mathbf{v}'', \mathbf{w})$ for brevity.

In the context of Coulomb branches the factorization morphism ϖ corresponds to $H_{\mathbf{G}}^*(\text{pt}) \rightarrow H^{\mathbf{G}^\circ}(\mathcal{R})$. The factorization property above follows from the localization theorem in the equivariant cohomology and was essentially proved in [6, §5]. The factorization of bow varieties is an essential ingredient for the identification of bow varieties and Coulomb branches in Theorem 3.3.

3(iv). Hanany-Witten transition

Let us recall the Hanany-Witten transition of bow varieties, which is formulated as isomorphisms between bow varieties with adjacent \times and \circ swapped [31, §7].

Consider the following part of bow data:

$$\begin{array}{ccccc} \begin{array}{c} B_1 \\ \curvearrowright \\ V_1 \end{array} & \begin{array}{c} \xrightarrow{C} \\ \xleftarrow{D} \end{array} & \begin{array}{c} B_2 \\ \curvearrowright \\ V_2 \end{array} & \xrightarrow{A} & \begin{array}{c} B_3 \\ \curvearrowright \\ V_3 \end{array} \\ & & \searrow b & & \nearrow a \\ & & \mathbb{C} & & \end{array} \quad \begin{array}{l} CD + B_2 = v^{\mathbb{C}}, \quad DC + B_1 = v^{\mathbb{C}}, \\ B_3 A - AB_2 + ab = 0. \end{array}$$

We assume the triangle part is not x_0 for a moment. We replace this part by

$$\begin{array}{ccccc} \begin{array}{c} B_1 \\ \curvearrowright \\ V_1 \end{array} & \xrightarrow{A^{\text{new}}} & \begin{array}{c} B_2^{\text{new}} \\ \curvearrowright \\ V_2^{\text{new}} \end{array} & \begin{array}{c} \xrightarrow{C^{\text{new}}} \\ \xleftarrow{D^{\text{new}}} \end{array} & \begin{array}{c} B_3 \\ \curvearrowright \\ V_3 \end{array} \\ & \searrow b^{\text{new}} & & \nearrow a^{\text{new}} & \\ & & \mathbb{C} & & \end{array} \quad \begin{array}{l} C^{\text{new}} D^{\text{new}} + B_3 = v^{\mathbb{C}}, \quad D^{\text{new}} C^{\text{new}} + B_1 = v^{\mathbb{C}}, \\ B_2^{\text{new}} A^{\text{new}} - A^{\text{new}} B_1 + a^{\text{new}} b^{\text{new}} = 0. \end{array}$$

so that we have a commutative square with the exact middle row:

$$\begin{array}{ccccccc}
 & & \alpha = \begin{bmatrix} D \\ A \\ b \end{bmatrix} & & \beta = [AC \ (B_3 - v^{\mathbb{C}}) \ a] & & \\
 & V_2 & \xrightarrow{\quad} & V_1 \oplus V_3 \oplus \mathbb{C} & \xrightarrow{\quad} & V_3 & \\
 & \parallel & & \parallel & & \uparrow C^{\text{new}} & \\
 0 \longrightarrow & V_2 & \xrightarrow{\alpha} & V_1 \oplus V_3 \oplus \mathbb{C} & \xrightarrow{\beta^{\text{new}}} & V_2^{\text{new}} \longrightarrow 0 & \\
 & \uparrow C & & \parallel & & \parallel & \\
 & V_1 & \xrightarrow{\alpha^{\text{new}} = \begin{bmatrix} v^{\mathbb{C}} - B_1 \\ C^{\text{new}} A^{\text{new}} \\ b^{\text{new}} \end{bmatrix}} & V_1 \oplus V_3 \oplus \mathbb{C} & \xrightarrow{\beta^{\text{new}} = [A^{\text{new}} \ -D^{\text{new}} \ a^{\text{new}}]} & V_2^{\text{new}} & .
 \end{array}$$

This gives an isomorphism between bow varieties where adjacent \bigcirc and \times are swapped, and the dimensions of vector spaces are changed by the rule

$$\dim V_2 + \dim V_2^{\text{new}} = \dim V_1 + \dim V_3 + 1.$$

When the triangle part is x_0 , the defining equations become $(B_3 + v_*^{\mathbb{C}})A - AB_2 + ab = 0$, $(B_2^{\text{new}} + v_*^{\mathbb{C}})A^{\text{new}} - A^{\text{new}}B_1 + a^{\text{new}}b^{\text{new}} = 0$. Thus B_3 and B_2^{new} must be shifted, hence other defining equations must be changed to

$$C^{\text{new}}D^{\text{new}} + B_3 = v^{\mathbb{C}} - v_*^{\mathbb{C}}, \quad D^{\text{new}}C^{\text{new}} + B_1 = v^{\mathbb{C}} - v_*^{\mathbb{C}}.$$

We consider two \mathbb{C}^\times -actions on the relevant part. The first one is the action induced from the weight 1 action on \mathbb{C} , hence $a \mapsto t^{-1}a$, $b \mapsto tb$, $a^{\text{new}} \mapsto t^{-1}a^{\text{new}}$, $b^{\text{new}} \mapsto tb^{\text{new}}$, and other data are unchanged. The second one is $A, b \mapsto tA, tb$, $A^{\text{new}}, b^{\text{new}} \mapsto tA^{\text{new}}, tb^{\text{new}}$, and others are unchanged. See [31, §6.9.2].

The following was not stated in [31], but clear from the definition.

LEMMA 3.7. – *The Hanany-Witten transition respects the $(\mathbb{C}^\times)^2$ -action.*

Note also that the factorization morphism does not essentially change under Hanany-Witten transition by Lemma 3.5. It is because we can use spectra of B_1 , B_3 for the definition of the factorization morphism, which are unchanged under Hanany-Witten transition.

3(v). Invariants

Let h be \bigcirc in a bow diagram. Let N_h be $R(i(h)) - R(o(h))$ where $i(h)$, $o(h)$ are as in (3) of §3(i). Let $h_\sigma, h_{\sigma+1}$ be consecutive \bigcirc . We define

$$N(h_\sigma, h_{\sigma+1}) \stackrel{\text{def.}}{=} N_{h_\sigma} - N_{h_{\sigma+1}} + (\text{the number of } \times \text{ between } h_{\sigma+1} \rightarrow h_\sigma),$$

where $h_{\sigma+1} \rightarrow h_\sigma$ means that on the arc starting from $h_{\sigma+1}$ towards h_σ in the anticlockwise direction.

Similarly we define N_x and $N(x_i, x_{i+1})$ in the same way by replacing \bigcirc by \times , and the anticlockwise direction by clockwise one.

Then $N(h_\sigma, h_{\sigma+1})$, $N(x_i, x_{i+1})$ are invariant under Hanany-Witten transition ([31, Lem. 7.6]).

We have two other invariants

$$(3.8) \quad - \sum_{\sigma=1}^{\ell} N_{h_{\sigma}}^2 + \sum_{i=0}^{n-1} (R(o(x_i)) + R(i(x_i))), \quad - \sum_{i=0}^{n-1} N_{x_i}^2 + \sum_{\sigma=1}^{\ell} (R(o(h_{\sigma})) + R(i(h_{\sigma}))),$$

where $\frac{o(x_i) \quad x_i \quad i(x_i)}{\mathbf{x}}$ and $\frac{o(h_{\sigma}) \quad h_{\sigma} \quad i(h_{\sigma})}{\bigcirc}$ are invariant under Hanany-Witten transition ([31, Lem. 7.6]).

The following is stated in [31, Prop. 7.19], but it is based on a wrong statement.

PROPOSITION 3.9. – *There is at most one bow diagram satisfying the balanced condition among those obtained by successive applications of Hanany-Witten transitions.*

Proof. – Let us suppose a bow diagram satisfying the balanced condition is given. Then $N(h_{\sigma}, h_{\sigma+1})$ is the number of \mathbf{x} between $h_{\sigma+1} \rightarrow h_{\sigma}$. Hence the collection $\{N(h_{\sigma}, h_{\sigma+1})\}_{\sigma=1}^{\ell}$ determines the distribution of \bigcirc and \mathbf{x} . On the other hand the vector \mathbf{w} in §3(ii) is given by the number of \bigcirc on the arc $x_i \rightarrow x_{i+1}$. Therefore \mathbf{w} is determined up to a cyclic permutation. This is because the numbering of \mathbf{x} by x_i is not fixed by $N(h_{\sigma}, h_{\sigma+1})$, but the only ambiguity is given by a shift $x_i \mapsto x_{i+i_0}$ (modulo n) for some i_0 . (This ambiguity was overlooked in [31, Prop. 7.19].)

But this shift cannot be achieved by Hanany-Witten transitions. By Hanany-Witten transitions, $N_{h_{\sigma}}$ is changed by the number of \mathbf{x} crossing $\bigcirc^{h_{\sigma}}$ in the anticlockwise direction minus the number of \mathbf{x} crossing in the clockwise direction. Therefore in order to keep $N_{h_{\sigma}}$ vanishing, those two numbers must be equal. Therefore the numbering for the first \mathbf{x} after $\bigcirc^{h_{\sigma}}$ (in either direction) remains the same. Thus the shift is not possible. Thus the numbering of \mathbf{x} by x_i is unique, hence \mathbf{w} is determined.

Next note that $N(x_i, x_{i+1})$ is the i -th entry of $\mathbf{u} = \mathbf{w} - C\mathbf{v} = (\mathbf{w}_i + \mathbf{v}_{i-1} + \mathbf{v}_{i+1} - 2\mathbf{v}_i)_{i=0}^{n-1}$ [31, Lem. 7.18]. Therefore the collection $\{N(x_i, x_{i+1})\}$ and \mathbf{w} determine \mathbf{v} up to an addition of a multiple of ${}^t(1, 1, \dots, 1)$. But an addition of ${}^t(1, 1, \dots, 1)$ increases two invariants in (3.8) by $2n$ and 2ℓ respectively. Hence \mathbf{v} , i.e., numbers $R(\zeta)$ on segments are determined. \square

3(vi). Another form

Let us take a bow diagram satisfying the balanced condition, and define dimension vectors $\mathbf{v} = (\mathbf{v}_0, \dots, \mathbf{v}_{n-1})$, $\mathbf{w} = (\mathbf{w}_0, \dots, \mathbf{w}_{n-1})$ as in §3(ii). We apply Hanany-Witten transitions successively so that we separate \mathbf{x} and \bigcirc as follows.

$$(3.10) \quad \mathbf{v}_0 + \sum_i i \mathbf{w}_i$$

See the proof of [31, Cor. 7.21]. We do not move \bigcirc across x_0 , hence the dimension \mathbf{v}_0 next to x_0 is unchanged. Numbers ${}^t\lambda_{\sigma}$, μ_i above \bigcirc , \mathbf{x} indicate the values of $N_{h_{\sigma}}$, N_{x_i} respectively.

Two numbers \mathbf{v}_0 and $\mathbf{v}_0 + \sum_i i \mathbf{w}_i$ are dimensions of vector spaces on two segments, between x_0 and h_ℓ , h_1 and x_1 respectively.

The numbers ${}^t\lambda_\sigma$, μ_i are N_{h_σ} and N_{x_i} respectively. In order to explain how ${}^t\lambda_\sigma$, μ_i are given in terms of \mathbf{v} , \mathbf{w} , we introduce weights of $P_{\mathfrak{sl}(n)_{\text{aff}}}$, $P_{\widehat{\mathfrak{gl}}(n)}$. We first define two weights λ , μ of $P_{\mathfrak{sl}(n)_{\text{aff}}}$ by

$$(3.11) \quad \lambda = \sum_{i=0}^{n-1} \mathbf{w}_i \Lambda_i, \quad \mu = \sum_{i=0}^{n-1} (\mathbf{w}_i \Lambda_i - \mathbf{v}_i \alpha_i).$$

We have $\langle d, \lambda \rangle = 0$, $\langle d, \mu \rangle = -\mathbf{v}_0$. (Note that this is different from the convention in [31, §7.6] by $-\mathbf{v}_0 \delta$. Since we change \mathbf{v}_0 , the current convention is more natural.)

Let ℓ be the level of λ , which is equal also to the level of μ . It is $\langle c, \lambda \rangle = \langle c, \mu \rangle$, where c is the central element in $\widehat{\mathfrak{gl}}(n)$. Concretely it is equal to $\sum_{i=0}^{n-1} \mathbf{w}_i$, hence the number of \bigcirc . Therefore we can number \bigcirc as h_1, \dots, h_ℓ as in (3.10).

We define two integer vectors $[\lambda_1, \dots, \lambda_n]$, $[\mu_1, \dots, \mu_n]$ by

$$\lambda_i = \sum_{j=i}^{n-1} \mathbf{w}_j, \quad \mu_i = \mathbf{v}_{n-1} - \mathbf{v}_0 + \sum_{j=i}^{n-1} \mathbf{u}_j,$$

where \mathbf{u}_i is the i -th entry of $\mathbf{u} = \mathbf{w} - C\mathbf{v}$ as in the proof of Proposition 3.9. We consider them as level ℓ weights of $\widehat{\mathfrak{gl}}(n)$. Note that $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i$. It means that the pairings with the central element $\text{diag}(1, \dots, 1)$ in $\mathfrak{gl}(n)$ (charges) are the same for λ and μ .

Note that λ is dominant by its definition. Hence it is contained in the fundamental alcove, i.e.,

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq \lambda_1 - \ell.$$

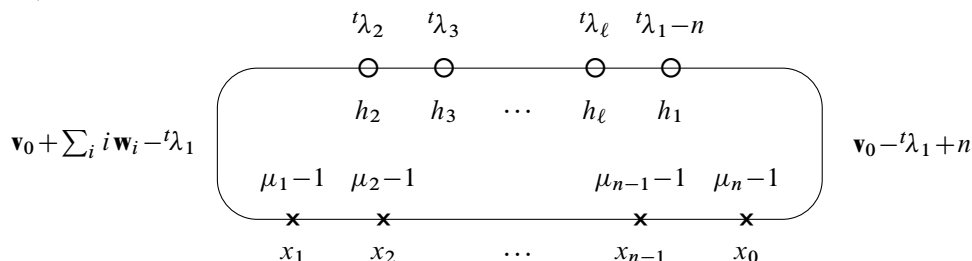
Let us consider the corresponding generalized Young diagram $[\lambda_1, \dots, \lambda_n]$ and its transpose $[{}^t\lambda_1, \dots, {}^t\lambda_\ell]$. The latter is a generalized Young diagram with the level n constraint, i.e.,

$${}^t\lambda_1 \geq {}^t\lambda_2 \geq \dots \geq {}^t\lambda_\ell \geq {}^t\lambda_1 - n.$$

See §2. Now numbers in (3.10) are given by these rules.

Recall that we did not move \bigcirc over x_0 in this procedure. Let us move \bigcirc anticlockwise overall x_i including x_0 to return back to the same picture as (3.10). But this process changes numbers on \bigcirc , \mathbf{x} , and also dimensions of vector spaces on two segments. The result is as follows.

(3.12)



Note that $[{}^t\lambda_1, \dots, {}^t\lambda_\ell] \mapsto [{}^t\lambda_2, \dots, {}^t\lambda_\ell, {}^t\lambda_1 - n]$ corresponds to a simultaneous shift $[\lambda_1, \dots, \lambda_n] \mapsto [\lambda_1 - 1, \dots, \lambda_n - 1]$ (see §2). Therefore this process shifts both λ and μ simultaneously.

By [31, Prop. 7.20] (more precisely its dual version), bow diagrams (3.10) can be transformed to a balanced one by successive applications of Hanany-Witten transition, as $[{}^t\lambda_1, \dots, {}^t\lambda_\ell]$ is in the fundamental alcove. And it is unique by Proposition 3.9. Dimension vectors are read off from numbers in (3.10) as

$$\begin{aligned} \mathbf{w}_i &= \lambda_i - \lambda_{i+1} \quad (1 \leq i \leq n-1), & \mathbf{w}_0 &= \lambda_n - (\lambda_1 - \ell), \\ \mathbf{u}_i &= \mu_i - \mu_{i+1} \quad (1 \leq i \leq n-1), & \mathbf{u}_0 &= \mu_n - (\mu_1 - \ell), \end{aligned}$$

and ambiguity $(\mathbf{v}_0, \dots, \mathbf{v}_{n-1}) + \mathbb{Z}(1, \dots, 1)$ is fixed by one of the invariants in (3.8) by

$$\mathbf{v}_0 + \dots + \mathbf{v}_{n-1} = -\frac{1}{2} \sum_{\sigma=1}^{\ell} {}^t\lambda_{\sigma}^2 + \frac{1}{2} \sum_{i=0}^{n-1} (R(\mathbf{o}(x_i)) + R(\mathbf{i}(x_i))).$$

Let us denote the bow variety with the balanced condition with dimension vectors \mathbf{v}, \mathbf{w} by $\mathcal{M}(\lambda, \mu)$ hereafter, where λ, μ are given by (3.11). (Note that it was denoted by $\mathcal{M}(\mu, \lambda)$ in [31].)

3(vii). Stratification

By [31, Prop. 4.5, Th. 7.26].

THEOREM 3.13. – (1) Suppose $\ell \neq 1$. We have a stratification

$$\mathcal{M}(\lambda, \mu) = \bigsqcup_{\kappa, \underline{k}} \mathcal{M}^s(\kappa, \mu) \times S^{\underline{k}}(\mathbb{C}^2 \setminus \{0\}/(\mathbb{Z}/\ell\mathbb{Z})),$$

where $\underline{k} = [k_1, k_2, \dots]$ is a partition and κ is a dominant weight with $\mu \leq \kappa \leq \lambda - |\underline{k}|\delta$. The same is true if we replace $\mathbb{C}^2 \setminus \{0\}/(\mathbb{Z}/\ell\mathbb{Z})$ by \mathbb{C}^2 and we allow only $\kappa = \lambda - |\underline{k}|\delta$ when $\ell = 1$.

(2) Take a generic real parameter $v^{\mathbb{R}}$. Then $\pi: \mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)$ in (3.1) is a semismall resolution with respect to the above stratification, and all strata are relevant.

Here we understand $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)$ has vanishing complex parameters.

3(viii). Birational coordinate system

We consider a bow variety with the balanced condition. Suppose that we have vector spaces and linear maps between $\mathbf{i}(x_i)$ and $\mathbf{o}(x_{i+1})$ as in Remark 3.4. Recall that the factorization morphism ϖ is given by eigenvalues with multiplicities of $B_{\mathbf{i}(x_i)}$ or $B_{\mathbf{o}(x_{i+1})}$ by §3(iii). Let $w_{i,r}$ ($r = 1, \dots, \mathbf{v}_i$) be eigenvalues. Let $C_{\mathbf{w}_i \dots \mathbf{1}, i}: V_i^0 \rightarrow V_i^{\mathbf{w}_i}$ be the composite of linear maps $V_i^0 \rightarrow V_i^1 \rightarrow \dots \rightarrow V_i^{\mathbf{w}_i}$. We define

$$y_{i,r} \stackrel{\text{def.}}{=} b_{x_{i+1}} \prod_{\substack{1 \leq s \leq \mathbf{v}_i \\ s \neq r}} (B_{\mathbf{o}(x_{i+1})} - w_{i,s} \text{id}) C_{\mathbf{w}_i \dots \mathbf{1}, i} a_{x_i}.$$

Then $w_{i,r}, y_{i,r}$ are regular functions on $\mathcal{M}(\lambda, \mu) \times_{\mathbb{A}^1} \mathbb{A}^{|\mathbf{v}|}$. The isomorphism between the Coulomb branch and the bow variety is constructed so that $w_{i,r}, y_{i,r}$ coincide with one defined in [7, §3(iii)] ($y_{i,r}$ was denoted by $\bar{y}_{i,r}$ there) and [31, §6.8.1], as certain homology classes.

Under Hanany-Witten transitions, eigenvalues of $B_{o(x_i)}$ remain unchanged except 0 by Lemma 3.5. It is also obvious that $y_{i,r}$, as a map from \mathbb{C} at x_{i-1} to \mathbb{C} at x_i , is unchanged. Therefore $w_{i,r}$, $y_{i,r}$ are given by the same formula for bow varieties, not necessarily with balanced condition.

4. Torus action

We consider the $T = (\mathbb{C}^\times)^{\mathcal{Q}_0}$ -action given by $\pi_1(\mathbf{G}) \cong \mathbb{Z}^n$ [6, §3(v)]. (We formally add the factor \mathbb{C}^\times even when $\mathbf{v}_i = 0$ so that T depends only on \mathcal{Q} .) Let (s_0, \dots, s_{n-1}) denote the standard coordinates of T , where s_i corresponds to $\pi_1(\mathrm{GL}(\mathbf{v}_i))$ at the vertex i . By [31, §6.9.2], the action is given by one induced by $s_0 \cdots s_{i-1}$ on \mathbb{C}_{x_i} ($1 \leq i \leq n-1$) and A, b at the vertex x_0 are multiplied by $s_0 \cdots s_{n-1}$. By Lemma 3.7 Hanany-Witten transitions are equivariant under the torus action.

4(i). Torus fixed points

Recall that the T -fixed point in $\mathcal{M}(\lambda, \mu)$ is either a single point or empty [31, Prop. 7.30]. (Recall also $\mathcal{M}(\lambda, \mu)$ is $\mathcal{M}^v(\lambda, \mu)$ for $v^{\mathbb{C}} = v^{\mathbb{R}} = 0$.) Let us review the proof as we will study fixed points with respect to smaller tori by using the same argument. Let us give a slight improvement simultaneously.

We have a stratification $\mathcal{M}(\lambda, \mu) = \bigsqcup \mathcal{M}^s(\kappa, \mu) \times S^{\underline{k}}(\mathbb{C}^2 \setminus \{0\}/(\mathbb{Z}/\ell\mathbb{Z}))$ ($\ell > 1$), $\bigsqcup \mathcal{M}^s(\kappa, \mu) \times S^{\underline{k}}(\mathbb{C}^2)$ ($\ell = 1$), where \underline{k} is a partition, and κ is a dominant weight between μ and $\lambda - |\underline{k}|\delta$. (See Theorem 3.13.) This stratification is compatible with the T -action. On the factor $S^{\underline{k}}(\mathbb{C}^2 \setminus \{0\}/(\mathbb{Z}/\ell\mathbb{Z}))$ or $S^{\underline{k}}(\mathbb{C}^2)$, the action is induced from the \mathbb{C}^\times -action on \mathbb{C}^2 given by $t \cdot (x, y) = (tx, t^{-1}y)$ where $t = s_0 \cdots s_{n-1}$. In particular, the T -fixed point set is empty for $S^{\underline{k}}(\mathbb{C}^2 \setminus \{0\}/(\mathbb{Z}/\ell\mathbb{Z}))$ unless $\underline{k} = \emptyset$ when $\ell > 1$. When $\ell = 1$, it is the single point $k[0]$ of $S^{(k)}(\mathbb{C}^2)$ when \underline{k} only has a single entry k and is empty otherwise. Thus it is enough to determine $\mathcal{M}^s(\kappa, \mu)^T$.

PROPOSITION 4.1. – *The T -fixed point set $\mathcal{M}^s(\kappa, \mu)^T$ is a single point if $\kappa = \mu^+$, the dominant weight in the Weyl group orbit of μ . It is empty otherwise.*

Proof. – Let us apply Hanany-Witten transitions successively to separate \bigcirc and \mathbf{x} . Moreover we move \bigcirc all over \mathbf{x} many times so that $\mu_i \stackrel{\text{def.}}{=} N_{x_i} \geq 0$ for all i . See §3(vi).

Let us take a representative (A, B, C, D, a, b) of a point in $\mathcal{M}^s(\kappa, \mu)^T$. We have a homomorphism $\rho = (\rho_\xi): T \rightarrow \mathcal{G} = \prod_\xi \mathrm{GL}(V_\xi)$ such that

$$\begin{aligned} A_{x_i} &= \rho_{i(x_i)}(s)^{-1} A_{x_i} \rho_{o(x_i)}(s) \quad (i \neq 0), & s_0 \cdots s_{n-1} A_{x_0} &= \rho_{i(x_0)}(s)^{-1} A_{x_0} \rho_{o(x_0)}(s), \\ B_\xi &= \rho_\xi(s)^{-1} B_\xi \rho_\xi(s), & (s_0 \cdots s_{i-1})^{-1} a_{x_i} &= \rho_{i(x_i)}(s)^{-1} a_{x_i}, \\ s_0 \cdots s_{i-1} b_{x_i} &= b_{x_i} \rho_{o(x_i)}(s) \quad (i \neq 0), & s_0 \cdots s_{n-1} b_{x_0} &= b_{x_0} \rho_{o(x_0)}(s), \\ C_{h_\sigma} &= \rho_{i(h_\sigma)}(s)^{-1} C_{h_\sigma} \rho_{o(h_\sigma)}(s), & D_{h_\sigma} &= \rho_{o(h_\sigma)}(s)^{-1} D_{h_\sigma} \rho_{i(h_\sigma)}(s), \end{aligned}$$

where $\xrightarrow{o(x_i) \quad x_i \quad i(x_i)} \mathbf{x} \xrightarrow{o(h_\sigma) \quad h_\sigma \quad i(h_\sigma)} \bigcirc$. We consider the weight space decomposition of V with respect to ρ . Then A_{x_i} ($i \neq 0$), B_ξ , C_{h_σ} and D_{h_σ} preserve weight spaces, while A_{x_0} shifts weights by $(s_0 \cdots s_{n-1})^{-1}$. And a_{x_i} sends \mathbb{C}_{x_i} to the $s_0 \cdots s_{i-1}$ weight space, b_{x_i} is 0 on weight

spaces other than $s_0 \cdots s_{i-1}$ ($i \neq 0$), $s_0 \cdots s_{n-1}$ ($i = 0$). In particular, we classify weights to n classes $(s_0 \cdots s_{i-1})(s_0 \cdots s_{n-1})^{\mathbb{Z}}$ ($i = 0, \dots, n-1$) so that \mathbb{C}_{x_i} can be ‘communicated’ with only weight spaces in the i -th class. Thus the data is a direct sum of n pieces.

Let us consider the direct summand for \mathbb{C}_{x_i} and the corresponding bow diagram. Since a_{x_j}, b_{x_j} vanish for $j \neq i$, A_{x_j} is an isomorphism thanks to the condition (S1,2). Then we can identify $V_{o(x_j)}$ with $V_{i(x_j)}$ so that we may assume that the bow diagram has only one \mathbf{x} . Moreover we can unwind the circle to a line as A_{x_0} shifts weight by $(s_0 \cdots s_{n-1})^{-1}$. Thus the bow diagram is

$$(4.2) \quad \begin{array}{ccccccccccccccc} & & & & & & \mu_i & & & & & & & & \\ & & & & & & \times & & & & & & & & \\ \cdots & \bigcirc & \cdots & h_m & \cdots & h_2 & h_1 & x_i & h_0 & h_{-1} & \cdots & h_{-n} & \cdots \end{array}$$

Note that μ_i remains the same as one for the original bow diagram, as A_{x_i} is an isomorphism in other summands. In particular, the above μ_i is ≥ 0 as we have assumed so in the original bow diagram.

By the necessary condition for $\mathfrak{M}_0^{\text{reg}} \neq \emptyset$ in [21, Lem. 8.1], [22, Lem. 4.7] we have $N(h_\sigma, h_{\sigma+1}) \geq 0$ for any σ . (To show $N(h_0, h_1) \geq 0$, we use the Hanany-Witten transition. See the proof of [31, Th. 7.26] for detail.) On the other hand, $\sum N(h_\sigma, h_{\sigma+1}) = 1$ by definition. Therefore $N(h_\sigma, h_{\sigma+1}) \neq 0$ at most one σ . Since $N_{h_\sigma} = 0$ if $|\sigma|$ is sufficiently large, we have $N_{h_1} = \cdots = N_{h_{\mu_i}} = 1$, other $N_{h_\sigma} = 0$. Thus the data looks like

$$(4.3) \quad \begin{array}{ccccccc} & & & & B_{o(x_i)} & & \\ & & & & \curvearrowright & & \\ \mathbb{C} & \rightleftarrows & \mathbb{C}^2 & \rightleftarrows & \cdots & \rightleftarrows & \mathbb{C}^{\mu_i} \\ & & & & & \searrow^{b_{x_i}} & \\ & & & & & & \mathbb{C}_{x_i} \end{array}$$

Note that $B_{o(x_i)}$ is nilpotent by the defining equation. The condition (S1,2) says b_{x_i} is cocyclic vector for $B_{o(x_i)}$. Hence $b_{x_i}, B_{o(x_i)}$ can be moved to ${}^t e_{\mu_i}, {}^t J_{\mu_i}$ by conjugation. Once the action of $\text{GL}(\mu_i)$ is killed, the remaining data C_i, D_i are regarded as a point of a quiver variety of type A_{μ_i-1} , which is the nilpotent cone of $\mathfrak{sl}(\mu_i - 1)$. See [21, §7]. Therefore the fixed point set is a single point. Alternatively, we apply Hanany-Witten transitions μ_i times to move x_i to the left. Then we arrive at the bow variety with all vector spaces V_ζ vanishing. It is a single point.

Since dimensions of vector spaces are determined by μ , the weight κ with $\mathcal{M}^s(\kappa, \mu)^T \neq \emptyset$ is determined uniquely by μ . Let us show that $\kappa = \mu^+$. Recall that the projection of κ to $P_{\widehat{\mathfrak{sl}(n)}}$ can be read off from N_{h_σ} ($\sigma = 1, \dots, \ell$) as in (3.10). Namely κ is the transpose of the generalized Young diagram associated with $N_{h_1} \geq N_{h_2} \geq \cdots \geq N_{h_\ell} \geq N_{h_1} - n$. The coefficient of δ is fixed by either of two invariants (3.8).

First, note that κ is unchanged under permutations of μ_i . It is because vector spaces are direct sums of vector spaces for \mathbb{C}_{x_i} , hence the ordering is not relevant. Thus we may assume $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Next we change μ to $(\mu_n + \ell, \mu_2, \dots, \mu_{n-1}, \mu_1 - \ell)$. Then all vector spaces in the upper semicircle of (3.10) between x_1 and x_0 change their dimensions by $\mu_n + \ell - \mu_1$, which is $\mathbf{u}_0 = \mathbf{w}_0 + \mathbf{v}_1 + \mathbf{v}_{n-1} - 2\mathbf{v}_0$. Thus this process does not change κ ,

and replaces μ by $s_0\mu$. Here s_0 is the simple reflection for the 0-th simple root. If we extend μ_i to $i \in \mathbb{Z}$ by $\mu_{i+n} = \mu_i - \ell$ as in §2, s_0 exchanges μ_j and μ_{j+1} for $j \equiv 0 \pmod n$. It is also clear that s_0 does not change κ in this description. These two operations generate the Weyl group of $\mathfrak{sl}(n)_{\text{aff}}$. Hence we may assume that μ is in the fundamental alcove, i.e., $\mu_1 \geq \dots \geq \mu_n \geq \mu_1 - \ell$. Moreover we can shift μ_i simultaneously by the process explained in §3(vi). So we make $\mu_n = 0$. Then μ determines a Young diagram with at most $(n-1)$ rows and ℓ columns. Then N_{h_σ} is the number of rows which have length more than σ . Namely $[N_{h_1}, \dots, N_{h_\ell}]$ is the transpose of the Young diagram. Moreover the vector space between x_0 and h_ℓ is 0 as $\ell \geq \mu_i$ for any i . It means that $\kappa = \mu$. \square

COROLLARY 4.4. – *The followings are equivalent:*

- (1) $\mathcal{M}(\lambda, \mu)$ has a T -fixed point.
- (2) $\lambda \geq \mu^+$ in the dominance order.
- (3) μ is a weight of the integrable highest weight module with the highest weight λ .

The equivalence between (2) and (3) is a consequence of [14, Prop. 12.5]. The equivalence between (1) and (2) follows from an observation that $\mathcal{M}(\lambda, \mu)$ contains $\mathcal{M}^s(\mu^+, \mu)$ as a stratum if and only if $\lambda \geq \mu^+$ in the dominance order. This confirms Conjecture 1.1(1) and a part of (3), that is $V_\mu(\lambda) = 0 \Leftrightarrow \mathcal{M}(\lambda, \mu)^T = \emptyset$, for affine type A .

Note also that the above will follow without the combinatorial argument in Proposition 4.1, once we will endow $\mathcal{V}(\lambda)$ with a \mathfrak{g}_{KM} -module structure and identify it with the integrable highest weight module.

Let us take a generic 1-parameter family $\chi: \mathbb{C}^\times \rightarrow T$ and consider a diagram

$$\text{pt} = \mathcal{M}(\lambda, \mu)^T \xleftarrow{p} \mathfrak{A}_\chi(\lambda, \mu) \xrightarrow{j} \mathcal{M}(\lambda, \mu),$$

where $\mathfrak{A}_\chi(\lambda, \mu)$ is the attracting set with respect to ρ . When there is no fear of confusion, we denote it simply by \mathfrak{A} . Here j is the inclusion, and p is the map given by taking the limit $\rho(t)$ for $t \rightarrow 0$. Then the following confirms [7, Conjecture 3.25(2)] for affine type A .

THEOREM 4.5 ([31, Prop. 7.33]). – *The intersection of \mathfrak{A} with strata in Theorem 3.13 is lagrangian. In particular, the hyperbolic restriction functor $\Phi = p_* j^!$ is hyperbolic semismall.*

Let

$$\mathcal{V}_\mu(\lambda) \stackrel{\text{def.}}{=} \Phi(\text{IC}(\mathcal{M}(\lambda, \mu))).$$

Thanks to the above theorem, this is a vector space. We also set $\mathcal{V}(\lambda) = \bigoplus \mathcal{V}_\mu(\lambda)$.

REMARK 4.6. – Since Φ is hyperbolic semismall, we have

$$\mathcal{V}_\mu(\lambda) = \Phi(\text{IC}(\mathcal{M}(\lambda, \mu))) \cong H_{2\dim}(\mathfrak{A}_\chi(\lambda, \mu)),$$

and hence $\mathcal{V}_\mu(\lambda)$ possesses a basis parametrized by irreducible components of $\mathfrak{A}_\chi(\lambda, \mu)$ of $\dim = \dim \mathcal{M}(\lambda, \mu)/2$ [19, Prop. 3.10]. After identifying $\mathcal{V}_\mu(\lambda)$ with a weight space of an integrable highest weight representation of an affine Lie algebra, irreducible components are regarded as Mirković-Vilonen cycles for affine Lie algebras, hence for double affine Grassmannian. This generalizes the construction in [27, §6] for dominant μ .

4(ii). Deformed case

Let us next choose a parameter v^\bullet such that $v_{\star}^{\bullet, \mathbb{C}} = 0 = v_{\star}^{\bullet, \mathbb{R}}$ and $v_h^{\bullet, \mathbb{C}} = v_h^{\bullet, \mathbb{R}}$ is either \dot{v} or 0, where \dot{v} is a nonzero real number. As before we denote by $v^{\bullet, \mathbb{C}}$, the complex part of v^\bullet and understand it as one with vanishing real parameters. This gives us a decomposition $\lambda = \lambda^1 + \lambda^2$: recall $\lambda = \sum \mathbf{w}_i \Lambda_i$, and \mathbf{w}_i is the number of \bigcirc 's between x_i and x_{i+1} . Let \mathbf{w}_i^1 , \mathbf{w}_i^2 be numbers of \bigcirc 's with $v_h^{\mathbb{C}} = 0$ and $= \dot{v}$ respectively. Then we have $\mathbf{w}_i = \mathbf{w}_i^1 + \mathbf{w}_i^2$ and define $\lambda^1 = \sum \mathbf{w}_i^1 \Lambda_i$, $\lambda^2 = \sum \mathbf{w}_i^2 \Lambda_i$. Thanks to Remark 3.4, the results in this subsection do not depend on how \dot{v} is distributed to v_h^\bullet . They depend only on λ^1, λ^2 .

The following confirms [7, Conjecture 3.27(1) and the first half of (2)] for affine type A.

PROPOSITION 4.7. – *The T -fixed points $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)^T$ are finite, and correspond to decomposition $\mu = \mu^1 + \mu^2$ such that $\mathcal{M}(\lambda^1, \mu^1)^T$, $\mathcal{M}(\lambda^2, \mu^2)^T$ are nonempty. Moreover the image of $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)^T$ under the factorization morphism ϖ is supported at 0 and \dot{v} , and the multiplicities are determined by the decomposition $\mu = \mu^1 + \mu^2$. Therefore around the point corresponding to $\mu = \mu^1 + \mu^2$, the bow variety $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)$ is isomorphic to a neighborhood of the unique T -fixed point in $\mathcal{M}(\lambda^1, \mu^1) \times \mathcal{M}(\lambda^2, \mu^2)$ by the factorization.*

More explicitly, if $p \in \mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)^T$ corresponds to a decomposition $\mu = \mu^1 + \mu^2$, we have $\varpi(p) = (\mathbf{v}_i^1[0] + \mathbf{v}_i^2[\dot{v}])_{i \in \mathcal{Q}_0}$ with $\lambda^1 - \mu^1 = \sum \mathbf{v}_i^1 \alpha_i$, $\lambda^2 - \mu^2 = \sum \mathbf{v}_i^2 \alpha_i$.

Proof. – We argue as in the proof of Proposition 4.1 to decompose a fixed point as sum of the data associated with (4.2) over i . Each summand looks like

$$(4.8) \quad \begin{array}{ccccccc} V_m & \xrightleftharpoons[D_{m-1}]{C_{m-1}} & V_{m-1} & \xrightleftharpoons[D_1]{C_1} & V_1 & \xrightarrow[A]{B_1} & V_0 & \xrightleftharpoons[D_{-1}]{C_{-1}} & \cdots & \xrightleftharpoons[D_{-n}]{C_{-n}} & V_{1-n} & \xrightarrow{B_0} & V_{-n} \\ & & & & & \searrow b & \nearrow a & & & & & & \\ & & & & & & \mathbb{C} & & & & & & \end{array}$$

instead of (4.3). By the defining equation we see that eigenvalues of B_0, B_1 are either 0 or \dot{v} . Let us decompose $V_0 = V'_0 \oplus V''_0$, $V_1 = V'_1 \oplus V''_1$ by eigenvalues, the prime for 0 and the double prime for \dot{v} . We have inherited decomposition $V_h = V'_h \oplus V''_h$ for other h 's. Then we have factorization $\mathcal{M}^{v^{\bullet, \mathbb{C}}} \approx {}'\mathcal{M}^{v^{\bullet, \mathbb{C}}} \times {}''\mathcal{M}^{v^{\bullet, \mathbb{C}}}$ around the fixed point. (See the paragraph after Theorem 3.6.)

Take a two way part h and consider the corresponding C_h, D_h . If $v_h^{\mathbb{C}} = 0$ (resp. \dot{v}), then C_h, D_h are isomorphisms on the factor V'' (resp. V'). Therefore we can absorb the action of $\mathrm{GL}(V''_{i(h)})$ to that of $\mathrm{GL}(V''_{o(h)})$ by normalizing $C_h|_{V''_{o(h)}}$ to the identity homomorphism. The remaining $D_h|_{V''_{i(h)}}$ is fixed by the defining equation. Thus we can eliminate $V''_{i(h)}$. The same applies for V' . After this normalization each factor gives a fixed point in a bow variety with parameter $v = 0$, one classified in Proposition 4.1. Therefore it is a form in (4.3).

We return back to the balanced bow variety $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)$ by successive applications of Hanany-Witten transitions. Eigenvalues of B 's are preserved (Lemma 3.5), hence we have the factorization $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu) \approx \mathcal{M}(\lambda^1, \mu^1) \times \mathcal{M}(\lambda^2, \mu^2)$ corresponding to the above factorization. Here we eliminate several summands of V', V'' as above. Since the fixed point corresponds to a fixed point in $\mathcal{M}(\lambda^1, \mu^1) \times \mathcal{M}(\lambda^2, \mu^2)$, it is the one described in Proposition 4.1. In particular we must have $\mathcal{M}(\lambda^1, \mu^1)^T, \mathcal{M}(\lambda^2, \mu^2)^T \neq \emptyset$. Conversely

if $\mathcal{M}(\lambda^1, \mu^1)^T, \mathcal{M}(\lambda^2, \mu^2)^T \neq \emptyset$, we get a fixed point in $\mathcal{M}(\lambda, \mu)$ after adding removed summands of V', V'' . \square

We allow scaling of $v^{\bullet, \mathbb{C}}$ in the defining equation to get families

$$\underline{\mathcal{M}}(\lambda, \mu) = \bigsqcup_{v' \in \mathbb{C} v^{\bullet, \mathbb{C}}} \mathcal{M}^{v'}(\lambda, \mu), \quad \widetilde{\mathcal{M}}(\lambda, \mu) = \bigsqcup_{v' \in \mathbb{C} v^{\bullet, \mathbb{C}}} \mathcal{M}^{v^{\bullet, \mathbb{R}}, v'}(\lambda, \mu)$$

parametrized by \mathbb{C} , where the fiber at 0 (resp. 1) is $\mathcal{M}(\lambda, \mu)$ (resp. $\mathcal{M}^{v^{\bullet, \mathbb{C}}}(\lambda, \mu)$) for the first family. The fiber at 0 of the second family is $\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)$. We have $\pi: \widetilde{\mathcal{M}}(\lambda, \mu) \rightarrow \underline{\mathcal{M}}(\lambda, \mu)$ as in (3.1). Let us denote its fiber over 0 as $\pi: \mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)$. The latter is a stratified semismall birational morphism ([31, Prop. 4.5]), hence the former is a stratified small birational morphism. Moreover $\widetilde{\mathcal{M}}(\lambda, \mu)$ is a topologically trivial family. In fact, by hyperKähler rotation, we can consider it as a family of bow varieties with the same defining equation (the parameter is $v^{\bullet, \mathbb{R}}$) with varying stability conditions with parameters in $\mathbb{R} \operatorname{Re} v^{\bullet, \mathbb{C}}$. But we only consider submodules whose dimension vectors are perpendicular to $v^{\bullet, \mathbb{R}}$, hence slopes appearing in inequalities in $(v\mathbf{1}, \mathbf{2})$ in §3(i) are automatically vanishing. Therefore $\operatorname{Re} v^{\bullet, \mathbb{C}}$ -(semi)stability and 0-(semi)stability are equivalent. Therefore the nearby cycle functor ψ for the family $\underline{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{C}$ sends $\operatorname{IC}(\underline{\mathcal{M}}(\lambda, \mu))$ to $\pi_*(\operatorname{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$. Now the remaining half of [7, Conjecture 3.27(2)] follows as

COROLLARY 4.9. – *We have*

$$\begin{aligned} \Phi \circ \pi_*(\operatorname{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu))) &\cong \psi \circ \Phi(\operatorname{IC}(\underline{\mathcal{M}}(\lambda, \mu))) \\ &\cong \bigoplus_{\substack{\mu = \mu^1 + \mu^2 \\ \mathcal{M}(\lambda^1, \mu^1)^T, \mathcal{M}(\lambda^2, \mu^2)^T \neq \emptyset}} \Phi(\operatorname{IC}(\mathcal{M}(\lambda^1, \mu^1))) \otimes \Phi(\operatorname{IC}(\mathcal{M}(\lambda^2, \mu^2))). \end{aligned}$$

The first isomorphism is given by the triviality of the family $\widetilde{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{C}$ and the commutativity of the nearby cycle and hyperbolic restriction functors by [33]. The second isomorphism is given by the factorization in Proposition 4.7.

Recall we denote $\Phi(\operatorname{IC}(\mathcal{M}(\lambda, \mu)))$ by $\mathcal{V}_\mu(\lambda)$. Since $\pi: \mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)$ is an isomorphism over the open locus $\mathcal{M}^s(\lambda, \mu)$, the direct image $\pi_*(\operatorname{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$ contains $\operatorname{IC}(\mathcal{M}(\lambda, \mu))$ as a direct summand with multiplicity one. Therefore we have natural homomorphisms, inclusion and projection $\operatorname{IC}(\mathcal{M}(\lambda, \mu)) \hookrightarrow \pi_*(\operatorname{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$. They induce

$$(4.10) \quad \bigoplus_{\mu} \Phi(\operatorname{IC}(\mathcal{M}(\lambda, \mu))) \hookrightarrow \bigoplus_{\mu} \Phi \circ \pi_*(\operatorname{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu))).$$

4(iii). Weyl group action

Let us consider a real parameter with $v_*^{\circ, \mathbb{R}} = 0$ but other $v_h^{\circ, \mathbb{R}}$ are generic. We consider $\mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)$. As in Corollary 4.9 we have

$$(4.11) \quad \Phi \circ \pi_*(\operatorname{IC}(\mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu))) \cong \bigoplus_{\sum \mu_i^{a_i} = \mu} \bigotimes_{i=0}^{n-1} \bigotimes_{a_i=1}^{\mathbf{w}_i} \Phi(\operatorname{IC}(\mathcal{M}(\Lambda_i, \mu_i^{a_i}))).$$

We divide the hamiltonian reduction in the definition of a balanced bow variety in two steps: the first by $\operatorname{GL}(V_\zeta)$'s when both ends of ζ are \bigcirc , and the second by the remaining $\operatorname{GL}(V_\zeta)$'s. In the first step, we obtain products of triangle parts and quiver varieties of

type $A_{\mathbf{w}_i-1}$ with dimension vectors $(\mathbf{v}_i, \mathbf{v}_i, \dots, \mathbf{v}_i, \mathbf{v}_i)$, $(\mathbf{v}_i, 0, \dots, 0, \mathbf{v}_i)$. For $\zeta^{v^{\circ, \mathbb{R}}}$ this first step gives the inverse image of Slodowy slice $\mathcal{S}((\mathbf{w}_i - 1)^{\mathbf{v}_i}, 1^{\mathbf{v}_i})$ under the projection $\pi: T^*\mathcal{F} \rightarrow \overline{\mathcal{N}}(\mathbf{w}_i^{\mathbf{v}_i})$. Here $((\mathbf{w}_i - 1)^{\mathbf{v}_i}, 1^{\mathbf{v}_i})$, $(\mathbf{w}_i^{\mathbf{v}_i})$ denote Jordan type of nilpotent orbits, and $T^*\mathcal{F}$ is the cotangent bundle of the partial flag variety of flags of subspaces of $\mathbb{C}^{\mathbf{w}_i \mathbf{v}_i}$ of dimensions $\mathbf{v}_i, 2\mathbf{v}_i, \dots, (\mathbf{w}_i - 1)\mathbf{v}_i$. For $\mathcal{M}(\lambda, \mu)$ the first step gives the intersection of Slodowy slice and $\overline{\mathcal{N}}(\mathbf{w}_i^{\mathbf{v}_i})$. (See e.g., [31, §7.4].)

The projection $\pi: T^*\mathcal{F} \rightarrow \overline{\mathcal{N}}(\mathbf{w}_i^{\mathbf{v}_i})$ is very similar to the Springer resolution. (In fact, it is exactly the Springer resolution when $\mathbf{v}_i = 1$.) Recall that the Springer representation is a $\mathfrak{S}_{\mathbf{w}_i}$ -action on $\pi_*(\mathbb{Q}_{T^*\mathcal{F}}[\dim T^*\mathcal{F}])$ for $\mathbf{v}_i = 1$ in Lusztig's construction [17] (see also [2, §2.6]).

LEMMA 4.12. – (1) *A perverse sheaf $\pi_*(\mathrm{IC}(\mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu)))$ is equipped with a $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ -action.*
 (2) *The induced $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ -action on the hyperbolic restriction $\Phi \circ \pi_*(\mathrm{IC}(\mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu)))$ coincides with the permutation among factors $a_i = 1, \dots, \mathbf{w}_i$ with common i in (4.11).*

Taking the sum over μ , we rewrite the right hand side of (4.11) as

$$\bigotimes_{i=0}^{n-1} \mathcal{V}(\Lambda_i)^{\otimes \mathbf{w}_i},$$

as $\mathcal{V}(\Lambda_i) = \bigoplus_{\mu_i^{a_i}} \Phi(\mathrm{IC}(\mathcal{M}(\Lambda_i, \mu_i^{a_i})))$. Thus this analog of Springer representation permutes tensor factors.

Proof. – (1) We follow Lusztig's construction. We consider the family $\widetilde{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{A}^{\ell-1}$ where the real parameter is the one we already chose $v^{\circ, \mathbb{R}}$. Complex parameters $v^{\mathbb{C}}$ have $v_*^{\mathbb{C}} = 0$, but other parts $v_h^{\mathbb{C}}$ are arbitrary. The number of parameters is $\sum_i \mathbf{w}_i - 1 = \ell - 1$. (Recall that an overall shift of $v_h^{\mathbb{C}}$ is irrelevant.) For $\underline{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{A}^{\ell-1}$, we choose complex parameter as above while the real parameter $v^{\mathbb{R}}$ is 0. The product of symmetric groups $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ acts on $\mathbb{A}^{\ell-1}$, where $\mathfrak{S}_{\mathbf{w}_i}$ permutes $v_h^{\mathbb{C}}$'s in the arc $x_i \rightarrow x_{i+1}$. This action can be lifted to $\underline{\mathcal{M}}(\lambda, \mu)$ by reflection functors [26] as we have already mentioned in Remark 3.4. Alternatively we use the identification of $\underline{\mathcal{M}}(\lambda, \mu)$ with the spectrum of $H_*^{\mathbf{G}^{\circ}}(\mathcal{R})$, and observe that it has a quotient by $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ given by $\mathrm{Spec} H_*^{\overline{\mathbf{G}}^{\circ}}(\mathcal{R})$ with $\overline{\mathbf{G}} = (\mathbf{G} \times \prod_i \mathrm{GL}(\mathbf{w}_i))/\mathbb{C}^\times$. We now have a commutative diagram

$$\begin{array}{ccccc} \mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu) & \xrightarrow{\tilde{\iota}} & \widetilde{\mathcal{M}}(\lambda, \mu) & \longrightarrow & \mathbb{A}^{\ell-1} \\ \downarrow \pi & & \downarrow \underline{\pi} & & \downarrow \\ \mathcal{M}(\lambda, \mu) & \xrightarrow{\iota} & \underline{\mathcal{M}}(\lambda, \mu) / \prod_i \mathfrak{S}_{\mathbf{w}_i} & \longrightarrow & \mathbb{A}^{\ell-1} / \prod_i \mathfrak{S}_{\mathbf{w}_i}, \end{array}$$

where the leftmost column consists of fibers over 0.

Since the left square is Cartesian, we have $\iota^* \pi_*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu))) \cong \pi_* \tilde{\iota}^*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))$ by base change. Since $\widetilde{\mathcal{M}}(\lambda, \mu)$ is a topologically trivial family as explained in §4(ii), we have $\tilde{\iota}^*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))[1 - \ell] \cong \mathrm{IC}(\mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu))$. On the other hand, $\underline{\pi}$ is a stratified small birational morphism again as in §4(ii). Hence $\pi_*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))$ is the IC associated with the local system given by the $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ -covering defined by the restriction of $\underline{\pi}$ to the open subset of $\mathbb{A}^{\ell-1} / \prod_i \mathfrak{S}_{\mathbf{w}_i}$ consisting of disjoint configurations. In particular, $\pi_*(\mathrm{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))$,

and hence also $\iota^* \pi_* (\text{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))[1 - \ell] = \pi_* (\text{IC}(\mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu)))$ is equipped with a $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ -action.

(2) Since Φ is a functor, we have a $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ -action on $\Phi \circ \pi_* (\text{IC}(\mathcal{M}^{v^{\circ, \mathbb{R}}}(\lambda, \mu))) = \Phi \circ \tilde{\iota}^* (\text{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))[1 - \ell]$. We exchange Φ and $\tilde{\iota}^*$ after we replace $\Phi = p_* j^!$ by $p_i^-(j^-)^*$ by [3] and use the base change. As in Proposition 4.7 the fixed point components in $\widetilde{\mathcal{M}}(\lambda, \mu)^T$ correspond to decomposition $\mu = \sum \mu_i^{a_i}$. Moreover each factor corresponds to h , as $v_h^{\mathbb{C}}$'s take different values on the open subset of disjoint configurations. Therefore $\prod_i \mathfrak{S}_{\mathbf{w}_i}$ -action permutes fixed point components according to permutations of $\mu_i^{a_i}$'s. Note also that $\tilde{\iota}^* (\text{IC}(\widetilde{\mathcal{M}}(\lambda, \mu)))[1 - \ell]$ and $\psi(\text{IC}(\widetilde{\mathcal{M}}(\lambda, \mu))|_{\mathbb{C}^{\times}, \bullet, \mathbb{C}})$ are isomorphic as $\widetilde{\mathcal{M}}(\lambda, \mu)$ is a topologically trivial family. The same is true for those applied with Φ . Therefore (4.11) is given also by $\tilde{\iota}^*[1 - \ell]$. \square

4(iv). Weyl chambers on the space of one parameter subgroups

Let us take a one-parameter subgroup $\chi(t) = (t^{m_0}, \dots, t^{m_{n-1}}) \in T$ ($m_i \in \mathbb{Z}$). We consider the corresponding fixed point $\mathcal{M}(\lambda, \mu)^\chi$. Since we can replace t by t^N for $N \in \mathbb{Z} \setminus \{0\}$, we can also consider $m_i \in \mathbb{Q}$. We regard $\underline{m} = (m_0, \dots, m_{n-1}) \in \mathbb{Q}^n$ as an element of the Cartan subalgebra of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$ (without the degree operator). Recall that the roots of $\widehat{\mathfrak{sl}}(n)$ are $(k, k, \dots, k) = k\delta$ ($k \neq 0$) and $\pm(0, \dots, 0, \frac{1}{j}, \dots, 1, 0, \dots, 0) + k\delta$ ($1 \leq j < i \leq n$, $k \in \mathbb{Z}$).

LEMMA 4.13. – Suppose that $\langle \underline{m}, \alpha \rangle \neq 0$ for any root α of $\widehat{\mathfrak{sl}}(n)$.

Then $\mathcal{M}(\lambda, \mu)^\chi = \mathcal{M}(\lambda, \mu)^T$.

Proof. – Recall a stratification $\mathcal{M}(\lambda, \mu) = \bigsqcup \mathcal{M}^s(\kappa, \mu) \times S^k(\mathbb{C}^2 \setminus \{0\}/(\mathbb{Z}/\ell\mathbb{Z}))$ ($\ell > 1$), $\bigsqcup \mathcal{M}^s(\kappa, \mu) \times S^k(\mathbb{C}^2)$ ($\ell = 1$). The action on symmetric products is induced from the \mathbb{C}^\times -action on \mathbb{C}^2 given by $t \cdot (x, y) = (tx, t^{-1}y)$ where $t = s_0 \cdots s_{n-1}$. We have

$$(\mathbb{C}^2)^T = (\mathbb{C}^2)^\chi \iff 0 \neq \sum_{i=0}^{n-1} m_i = \langle \underline{m}, \delta \rangle.$$

Thus T -fixed points and $\chi(\mathbb{C}^\times)$ -fixed points in symmetric products are the same if \underline{m} is not in the imaginary root hyperplane $\delta = 0$.

Next we consider $\mathcal{M}^s(\kappa, \mu)^\chi$ as in §4(i). If

$$(s_0 \cdots s_{i-1})(s_0 \cdots s_{j-1})^{-1} \notin (s_0 \cdots s_{n-1})^{\mathbb{Z}}$$

on the image of χ , data for \mathbb{C}_{x_i} and \mathbb{C}_{x_j} live on different weight spaces, hence we have a direct sum decomposition as before. The above condition is

$$m_{i-1} + m_{i-2} + \cdots + m_j \notin \mathbb{Z} \sum_{i=0}^{n-1} m_i,$$

if $i > j$ and similar for $i < j$. This means that \underline{m} is not in root hyperplanes

$$(0, \dots, 0, \frac{1}{j}, \dots, 1, 0, \dots, 0) + k\delta$$

for any k . \square

Thus we get a geometric interpretation of Weyl chambers in terms of one parameter subgroups.

4(v). Fixed points with respect to smaller tori

Let us consider the ‘negative’ chamber $m_i < 0$ for $i = 0, \dots, n-1$ on the space of one parameter subgroups. It does not intersect with root hyperplanes, hence the fixed point set with respect to $\chi(t) = (t^{m_0}, \dots, t^{m_{n-1}})$ coincides with the T -fixed point set. Let us consider a one parameter subgroup $\chi_i(t)$, which lies on the boundary of the negative chamber.

THEOREM 4.14. – *Take a one parameter subgroup $\chi_i(t)$ with $m_i = 0$ for some i and $m_j < 0$ otherwise.*

(1) *The χ_i -fixed point set $\mathcal{M}(\lambda, \mu)^{\chi_i}$ is either empty or isomorphic to a Coulomb branch $\mathcal{M}_{A_1}(\lambda', \mu')$ of an A_1 type framed quiver gauge theory with weights λ', μ' , where $\mu' = \langle \mu, h_i \rangle = \mu_i - \mu_{i+1}$ ($i \neq 0$), $\ell + \mu_n - \mu_1$ ($i = 0$). Moreover the intersection of $\mathcal{M}(\lambda, \mu)^{\chi_i}$ with a stratum is either empty or a stratum of $\mathcal{M}_{A_1}(\lambda', \mu')$.*

(2) *The restrictions of $w_{j,r}, y_{j,r}$ are zero for $j \neq i$, and are equal to one for $\mathcal{M}_{A_1}(\lambda', \mu')$ or zero for $j = i$.*

The weight λ' is determined from λ, μ in a combinatorial way as we will see during the proof. On the other hand, it is the largest highest weight with $\geq \mu'$ among those corresponding $\mathfrak{sl}(2)_i$ -modules appearing in the restriction of the integrable highest weight module $V(\lambda)$, once we establish $\mathcal{V}(\lambda) = V(\lambda)$. But it is not clear to the author how to show that the combinatorial expression of λ' coincides with the representation theoretic characterization directly.⁽¹⁾

Proof. – Let us first suppose $i \neq 0, n-1$. The same argument as in §4(i) shows that a point $\mathcal{M}^s(\kappa, \mu)^{\chi_i}$ is represented by (A, B, C, D, a, b) which decomposes to data for \mathbb{C}_{x_j} ($j \neq i, i+1$) and data for $\mathbb{C}_{x_i} \oplus \mathbb{C}_{x_{i+1}}$. We already know that the former data gives a single point by §4(i). We untwist the circle to the line as before, and the bow diagram for the latter is

$$(4.15) \quad \begin{array}{ccccccccccccccc} & & & & & & \mu_i & \mu_{i+1} & & & & & & & \\ & & & & & & \times & \times & & & & & & & \\ \circ & \circ & \circ & \circ & \circ & \times & \times & \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \cdots & h_m & \cdots & h_2 & h_1 & x_i & x_{i+1} & h_0 & h_{-1} & \cdots & h_{-n} & \cdots & & \end{array}$$

When $i = n-1$, we have $s_0 \cdots s_{n-2} = s_0 \cdots s_{n-1}$. Then the action on $\mathbb{C}_{x_{n-1}}$ and that on A, b at x_0 have the same weight. The argument above yields the same diagram above, if we understand $(i, i+1) = (n-1, 0)$ and $\mu_0 = \mu_n$.

When $i = 0$, we have ℓ \circ 's between x_0 and x_1 . (See (3.10).) So we get

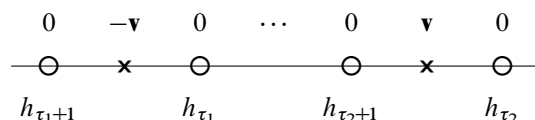
$$\begin{array}{ccccccccccc} & & & & \mu_n & & & \mu_1 & & & \\ & & & & \times & & \times & & & & \\ \cdots & \circ & \times & \circ & \cdots & \circ & \times & \circ & \cdots & & \\ & h_{\ell+1} & x_0 & h_\ell & & h_1 & x_1 & h_0 & & & \end{array}$$

⁽¹⁾ This sentence was written in the first version of the manuscript. See Remark 4.18 for the update.

Then we move ℓ \circ 's to the left of x_0 by Hanany-Witten transition to get the same diagram in (4.15) with μ_n replaced by $\mu_n + \ell$. The argument below remains if we understand $\mu_{i=0}$ is $\mu_n + \ell$.

We assume $\mu_i, \mu_{i+1} \geq 0$ as in the proof of Proposition 4.1. We have $N(h_\sigma, h_{\sigma+1}) \geq 0$ for any σ again as before. On the other hand, $\sum N(h_\sigma, h_{\sigma+1}) = 2$ by definition. Therefore $N(h_\sigma, h_{\sigma+1}) \neq 0$ at most two σ 's.

Let us suppose $N(h_\sigma, h_{\sigma+1}) \neq 0$ for $\sigma = \tau_1, \tau_2$ and $N(h_\sigma, h_{\sigma+1}) = 0$ otherwise. We assume $\tau_1 \geq \tau_2$. We have $N(h_{\tau_1}, h_{\tau_1+1}) = 1 = N(h_{\tau_2}, h_{\tau_2+1})$ if $\tau_1 > \tau_2$, $N(h_{\tau_1}, h_{\tau_1+1}) = 2$ if $\tau_1 = \tau_2$. We move x_i (resp. x_{i+1}) between h_{τ_1} and h_{τ_1+1} (resp. h_{τ_2} and h_{τ_2+1}) by successive applications of Hanany-Witten transition. Then N_{h_σ} becomes 0 for any σ . Thus vector spaces appear between x_i and x_{i+1} with the same dimension (let it be \mathbf{v}), and all others are 0. See



The balanced condition is satisfied, hence it gives a Coulomb branch. The gauge theory is of type A_1 with dimensions \mathbf{v}, \mathbf{w} , where $\mathbf{w} = \tau_1 - \tau_2$. It is nonempty if and only if $\mathbf{v} \geq 0$. More precisely, we consider the fixed point set in a stratum $\mathcal{M}^s(\kappa, \mu)$, hence the data above must satisfy the 0-stability condition. Therefore the fixed point set is the open stratum of the Coulomb branch.

Note that τ_1, τ_2 are determined by μ_i, μ_{i+1} and \mathbf{v} as

$$\tau_1 = \begin{cases} \mu_i + \mathbf{v} & (i \neq 0), \\ \mu_n + \ell + \mathbf{v} & (i = 0), \end{cases} \quad \tau_2 = \mu_{i+1} - \mathbf{v}.$$

In particular, we have $\mathbf{w} - 2\mathbf{v} = \mu_i - \mu_{i+1}$ ($i \neq 0$), $\ell + \mu_n - \mu_1$ ($i = 0$). This is determined only by μ_i, μ_{i+1} , hence only by the bow data for the original bow variety $\mathcal{M}(\lambda, \mu)$.

Let us return back to the data (4.15), and then to (3.10) in order to see how strata of $\mathcal{M}(\lambda, \mu)$ and the above are related.

If $\tau_1 > \tau_2 > 0$, the data looks like

(4.16)

$$\begin{array}{ccccccc} \mathbb{C} & \rightleftarrows & \mathbb{C}^2 & \rightleftarrows & \dots & \rightleftarrows & \mathbb{C}^{\tau_1 - \tau_2} & \rightleftarrows & \mathbb{C}^{\tau_1 - \tau_2 + 2} & \rightleftarrows & \dots & \rightleftarrows & \mathbb{C}^{\tau_1 + \tau_2} & \longrightarrow & \mathbb{C}^{\tau_2 + \mathbf{v}} & \xrightarrow{b_{x_{i+1}}} & \mathbb{C}_{x_{i+1}} \\ & & & & & & & & & & & & & \searrow^{b_{x_i}} & & \nearrow^{a_{x_i}} & & \searrow^{b_{x_{i+1}}} \\ & & & & & & & & & & & & & \mathbb{C}_{x_i} & & & & \mathbb{C}_{x_{i+1}} \end{array}$$

From left to right, vector spaces increase dimension by 1 until $\tau_1 - \tau_2$. Then increase by 2 until $\tau_1 + \tau_2$. If $\tau_1 = \tau_2$, the left edge starts as $\mathbb{C}^2 \rightleftarrows \mathbb{C}^4 \rightleftarrows \dots$.

Note $\tau_1 + \tau_2 = \mu_i + \mu_{i+1}$, $\tau_2 + \mathbf{v} = \mu_{i+1}$, $\tau_1 - \tau_2 = 2\mathbf{v} + \mu_i - \mu_{i+1}$. In particular, $\tau_1 + \tau_2$ and $\tau_2 + \mathbf{v}$ are determined by μ . Going to (3.10), we see that $\tau_1 - \tau_2$ (and hence \mathbf{v}) is determined from κ, μ . In fact, the sum of dimension of vector spaces in two way parts is $\frac{1}{4}(\tau_1 + \tau_2)^2 + \frac{1}{2}(\tau_1 + \tau_2) + \frac{1}{4}(\tau_1 - \tau_2)^2$. Hence $\tau_1 - \tau_2$ is determined by dimension vectors in the form (3.10).

If we add 2 to \mathbf{w} keeping $\mathbf{w} - 2\mathbf{v}$ unchanged, we go to a larger stratum in \mathcal{M}_{A_1} . It corresponds to adding $\mathbb{C} \rightleftharpoons \cdots \rightleftharpoons \mathbb{C}$ ($\tau_1 - \tau_2 + 1$ copies of \mathbb{C}). It goes to a larger stratum in $\mathcal{M}(\lambda, \mu)$ also.

If $\tau_1 \geq 0 \geq \tau_2$, the data looks like

$$\begin{array}{ccccccc} \mathbb{C} & \rightleftharpoons & \mathbb{C}^2 & \rightleftharpoons & \cdots & \rightleftharpoons & \mathbb{C}^{\tau_1} & \xrightarrow{\quad} & \mathbb{C}^{\mathbf{v}} & \xrightarrow{\quad} & \mathbb{C}^{-\tau_2} & \rightleftharpoons & \cdots & \rightleftharpoons & \mathbb{C}^2 & \rightleftharpoons & \mathbb{C} \\ & & & & & & \searrow & & \nearrow & & \searrow & & \nearrow & & & & & \\ & & & & & & \mathbb{C} & & & & \mathbb{C} & & & & & & & \end{array}$$

The sum of dimension of vector spaces in two way parts is

$$\frac{1}{4}(\tau_1 + \tau_2)^2 + \frac{1}{2}(\tau_1 + \tau_2) + \frac{1}{4}(\tau_1 - \tau_2)^2.$$

Hence $\tau_1 - \tau_2$ and \mathbf{v} are determined from κ, μ as in the first case. If we add 2 to \mathbf{w} keeping $\mathbf{w} - 2\mathbf{v}$ unchanged, we add \mathbb{C} to each entry of the top row, including changing 0 to \mathbb{C} at the leftmost and rightmost entries. This process also changes $\mathcal{M}^s(\kappa, \mu)$ to a larger stratum.

For type A_1 Coulomb branches, the closure relation on strata is a total order. We take $\mathcal{M}_{A_1}(\lambda', \mu')$ as the closure of the largest stratum. The intersection of $\mathcal{M}(\lambda, \mu)^{x_i}$ with $\mathcal{M}^s(\kappa, \mu)$ is either empty or $\mathcal{M}_{A_1}^s(\tau_1 - \tau_2, \tau_1 + \tau_2)$ above, which is a stratum of $\mathcal{M}_{A_1}(\lambda', \mu')$.

Let us consider the assertion (2). The data for \mathbb{C}_{x_j} with $j \neq i, i+1$ is of the form (4.2). Note that we have other triangle parts x_k ($k \neq j$) which are suppressed as $a_{x_k} = 0 = b_{x_k}$ and $A_{x_k} = \text{id}$. The data contribute to $w_{k,r}, y_{k,r}$ by zero including the case $k = j$ since B_{x_j} is nilpotent and $a_{x_j} = 0$ holds.

Next consider the data for $\mathbb{C}_{x_i} \oplus \mathbb{C}_{x_{i+1}}$ as in (4.16). The data contribute to $w_{i,r}, y_{i,r}$ by eigenvalues of $B_{i(x_i)}$ and $b_{x_{i+1}} \prod_{s \neq r} (B_{i(x_i)} - w_{i,s})a_{x_i}$. They are the same as $w_{i,r}, y_{i,r}$ for $\mathcal{M}_{A_1}(\lambda', \mu')$. As for $w_{k,r}, y_{k,r}$ with $k \neq i$, the data contribute by zero since $B_{o(x_i)}$ is nilpotent by the defining equation, and $a_{x_k} = 0$. \square

REMARK 4.17. – The corresponding result for a finite type quiver Q was proved by Krylov [16, Lem. 5.5]. In fact, he considered more general one parameter subgroups corresponding to any Levi subalgebra. The fixed point set is, in general, not a Coulomb branch of a framed quiver gauge theory for the semisimple part of the Levi subalgebra. The above argument does not work since the dominance order, restricted to weights that differ from μ' by a linear combination of roots, is not a total order.

REMARK 4.18. – During the proof of Theorem 4.14 we prove that a stratum $\mathcal{M}_{A_1}^s(\mu' + 2\mathbf{v}, \mu')$ of $\mathcal{M}_{A_1}(\lambda', \mu')$ is the intersection $\mathcal{M}^s(\kappa, \mu) \cap \mathcal{M}(\lambda, \mu)^{x_i}$.

Let us compute κ in terms of \mathbf{v}, μ . Note that κ is determined by two way parts. Looking at (4.16), we find that dimensions of vector spaces in two way parts are the same as those for the torus fixed point appearing in the proof of Proposition 4.1 after we replaced μ by $\tilde{\mu} = \mu + \mathbf{v}\alpha_i$. The same is true for the case $\tau_1 \geq 0 \geq \tau_2$, once we understand that vector spaces and linear maps go to the right of x_{i+1} when $\tilde{\mu}_{i+1} < 0$. Therefore by the proof of Proposition 4.1, κ is the dominant weight $\tilde{\mu}^+$ in the Weyl group orbit of $\tilde{\mu}$. Namely we have

$$\mathcal{M}_{A_1}^s(\mu' + 2\mathbf{v}, \mu') = \mathcal{M}^s(\tilde{\mu}^+, \mu) \cap \mathcal{M}(\lambda, \mu)^{x_i}, \quad \tilde{\mu} = \mu + \mathbf{v}\alpha_i.$$

We further observe that

$$\mathcal{M}^s(\tilde{\mu}^+, \mu) \subset \mathcal{M}(\lambda, \mu) \iff \lambda \geq \tilde{\mu}^+ \iff \mathcal{M}(\lambda, \tilde{\mu})^T \neq \emptyset,$$

where the second \iff is a consequence of Corollary 4.4. Since we take $\kappa = \tilde{\mu}^+$ with the largest \mathbf{v} , we verify the characterization of λ' in Remark 1.3.

4(vi). Another choice of the stability parameter

Let us consider a parameter v^\square with $v_*^{\square, \mathbb{C}}, v_*^{\square, \mathbb{R}}$ nonzero, but all other v_h^\square are zero. We have $\underline{\mathcal{M}}(\lambda, \mu), \mathcal{M}^{v^{\square, \mathbb{C}}}(\lambda, \mu), \widetilde{\mathcal{M}}(\lambda, \mu), \mathcal{M}^{v^{\square, \mathbb{R}}}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)$ as in §4(ii). Recall that we choose $v_*^\bullet = 0$ in §4(ii). Therefore this parameter v^\square is complementary to the choice v^\bullet there.

Let us consider the case $n = 2, \ell = 1, \lambda = \Lambda_0$ for notational simplicity. The data and equations are

$$\begin{array}{ccccc} & & \overset{C}{\curvearrowright} & & \\ & & B_1 & & \\ & & \curvearrowright & & \\ V_2 & \xrightarrow{A_1} & V_1 & \xrightarrow{A_0} & V_0 \\ & \searrow b_1 & \nearrow a_1 & \searrow b_0 & \nearrow a_0 \\ & & \mathbb{C} & & \mathbb{C} \end{array} \quad \begin{cases} [m](-DC + v_*^{\square, \mathbb{C}})A_0 - A_0B_1 + a_0b_0 = 0, \\ B_1A_1 + A_1CD + a_1b_1 = 0. \end{cases}$$

We assume the balanced condition, i.e., $\dim V_0 = \dim V_2$.

Let us analyze the fixed point set $\mathcal{M}^{v^{\square, \mathbb{C}}}(\lambda, \mu)^T$ as in Propositions 4.1 and 4.7. The data for a fixed point decompose as sum of (4.2) for \mathbb{C}_{x_i} ($i = 0, 1$), and each summand looks like (4.8). Recall that we shrink triangle parts, which do not communicate with \mathbb{C}_{x_i} to get (4.8). In particular, at a vector space V_i , the defining equation is $C_i D_i - D_{i-1} C_{i-1} + v_*^{\square, \mathbb{C}} = 0$. Therefore we see that eigenvalues of $-DC, B_1, -CD$ are in $\mathbb{Z}v_*^{\square, \mathbb{C}}$. We decompose V_0, V_1, V_2 into generalized eigenspaces, and apply the factorization. Because of the shift $v_*^{\square, \mathbb{C}}$, the resulted data look as

$$\begin{array}{ccccccc} & & & \overset{B_1}{\curvearrowright} & & & \\ \cdots & \xrightarrow{A_0} & V_0(m) & \xrightleftharpoons[D]{C} & V_2(m) & \xrightarrow{A_1} & V_1(m) & \xrightarrow{A_0} & V_0(m-1) & \xrightleftharpoons[D]{C} & V_2(m-1) & \xrightarrow{A_1} & \cdots \\ & \nearrow a_0 & & & \searrow b_1 & \nearrow a_1 & \searrow b_0 & \nearrow a_0 & & & \searrow b_1 & & \\ & & \mathbb{C} & & & \mathbb{C} & & \mathbb{C} & & & & \mathbb{C} \end{array}$$

where $V_i(m)$ corresponds to the eigenvalue $mv_*^{\square, \mathbb{C}}$ for the fixed point. (We are thinking of data in a neighborhood of the fixed point. So the eigenvalue is not precisely $mv_*^{\square, \mathbb{C}}$. But it is ‘close’.) Since we may assume CD, DC on $V_0(m), V_2(m)$ have eigenvalues different from 0 if $m \neq 0$, we see that C, D are isomorphisms. Therefore we can identify $V_0(m)$ and $V_2(m)$ by C , and then determine D by the equation. Therefore we can collapse all two way parts except one between $V_0(0)$ and $V_2(0)$. Thus bow data is for type A_∞ , type A diagram going to infinity in both directions, with single \bigcirc corresponding to the remaining two way part. We can absorb $v_*^{\square, \mathbb{C}}$ to B_i ’s once we do not have two way parts except one. (See [31, §3.1.4].) Note also that the balanced condition $\dim V_0(0) = \dim V_2(0)$ remains to be true as we observed $\dim V_0(m) = \dim V_2(m)$ for $m \neq 0$. Therefore the A_∞ type bow variety is $\mathcal{M}_{A_\infty}(\lambda', \mu')$ with $\lambda' = \Lambda_0, \mu' = \Lambda_0 - \sum_{m,i} \mathbf{v}_i(m) \alpha_{mn+i}$.

Moreover the factorization isomorphism $\mathcal{M}^{\square, \mathbb{C}}(\lambda, \mu) \approx \mathcal{M}_{A_\infty}(\lambda', \mu')$ is T -equivariant, if we let T act on the right hand side as follows: for \mathbb{C} in a triangle between $V_2(m)$ and $V_1(m)$, it acts by $s_0(s_0s_1)^{-m}$. For \mathbb{C} in a triangle between $V_1(m+1)$ and $V_0(m)$, it acts by $(s_0s_1)^{-m}$. The action of s_0s_1 on the triangle between V_1 and V_0 for the left hand side is absorbed into the shift $(s_0s_1)^{-m}$ in the right hand side. By the same argument as in the proof of Lemma 4.13, the T -fixed point set in $\mathcal{M}_{A_\infty}(\lambda', \mu')$ is *not* larger than the fixed point set with respect to the torus T_∞ acting by linearly independent weights on \mathbb{C} for triangles.

PROPOSITION 4.19. – *Let $\lambda = \Lambda_0$ as above and write $\lambda - \mu = \sum_{i=0}^{n-1} \mathbf{v}_i \alpha_i$. The T -fixed points $\mathcal{M}^{\square, \mathbb{C}}(\lambda, \mu)^T$ are finite, and correspond to a decomposition $\mathbf{v}_i = \sum_{m \in \mathbb{Z}} \mathbf{v}_i(m)$ such that the corresponding A_∞ bow variety $\mathcal{M}_{A_\infty}(\lambda', \mu')$ as above has a T -fixed point, where $\lambda' = \Lambda_0$, $\mu' = \Lambda_0 - \sum_{m,i} \mathbf{v}_i(m) \alpha_{mn+i}$. The image of $\mathcal{M}^{\square, \mathbb{C}}(\lambda, \mu)^T$ under the factorization morphism ϖ is supported in $\mathbb{Z}v_*^{\square, \mathbb{C}}$, and the multiplicities of $mv^{\square, \mathbb{C}}$ are given by $\mathbf{v}_0(m)$, $\mathbf{v}_1(m)$. Around the fixed point, the bow variety $\mathcal{M}^{\square, \mathbb{C}}(\lambda, \mu)$ is isomorphic to a neighborhood of the unique T -fixed point in $\mathcal{M}_{A_\infty}(\lambda', \mu')$.*

It is also easy to describe $\mathcal{M}_{A_\infty}(\lambda', \mu')^T$. See the appendix.

Next consider the morphism $\pi: \mathcal{M}^{\square, \mathbb{R}}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)$. Since $\ell = 1$, the stratification (see Theorem 3.13) is $\mathcal{M}(\lambda, \mu) = \bigsqcup \mathcal{M}^s(\lambda - |\underline{k}| \delta, \mu) \times S^{\underline{k}}(\mathbb{C}^2)$. The fiber of π over the stratum $\mathcal{M}^s(\lambda - |\underline{k}| \delta, \mu) \times S^{\underline{k}}(\mathbb{C}^2)$ is isomorphic to a product of the punctual Hilbert scheme of k_1 points, k_2 points, ... on \mathbb{C}^2 by [31, §4.3]. In particular, it is irreducible. Hence

$$\pi_*(\mathbb{C}_{\mathcal{M}^{\square, \mathbb{R}}(\lambda, \mu)}[\dim]) \cong \bigoplus \mathrm{IC}(\mathcal{M}(\lambda - |\underline{k}| \delta, \mu)) \boxtimes \mathbb{C}_{\overline{S^{\underline{k}}(\mathbb{C}^2)}}[\dim],$$

where $\mathbb{C}_X[\dim]$ denotes the shift of the constant sheaf on X by $\dim X$ and $\overline{S^{\underline{k}}(\mathbb{C}^2)}$ is the closure of $S^{\underline{k}}(\mathbb{C}^2)$ in $S^{|\underline{k}|}(\mathbb{C}^2)$. Therefore

$$(4.20) \quad \bigoplus_{\mu} \Phi \circ \pi_*(\mathbb{C}_{\mathcal{M}^{\square, \mathbb{R}}(\lambda, \mu)}[\dim]) \cong \left(\bigoplus_{\mu} \Phi(\mathrm{IC}(\mathcal{M}(\lambda, \mu))) \right) \otimes \left(\bigoplus_{\underline{k}} \mathbb{C}[\overline{S^{\underline{k}}(\mathbb{C}^2)}] \right),$$

where $S^{\underline{k}}(\mathbb{C})$ is the stratum of the symmetric product of the line \mathbb{C} corresponding to a partition \underline{k} , and its closure is the attracting set in $\overline{S^{\underline{k}}(\mathbb{C}^2)}$.

REMARK 4.21. – Suppose $n = 1$, $\ell = 1$. Let us denote the corresponding balanced bow variety by $\mathcal{M}(\mathbf{v}, 1)$ by using dimension vectors as in §3(ii). ($\mathbf{w} = 1$ as $\ell = 1$.) We have also $\mathcal{M}^{\square, \mathbb{R}}(\mathbf{v}, 1)$, etc. In this case $\mathcal{M}(\mathbf{v}, 1)$, $\mathcal{M}^{\square, \mathbb{R}}(\mathbf{v}, 1)$ are isomorphic to quiver varieties of Jordan type as the cobalanced condition is satisfied. In particular, they are isomorphic to the symmetric product $S^{\mathbf{v}}(\mathbb{C}^2)$ and Hilbert scheme $\mathrm{Hilb}^{\mathbf{v}}(\mathbb{C}^2)$ of \mathbf{v} points on \mathbb{C}^2 . The proof of Proposition 4.19 works in this case, and we recover a well-known fact that a fixed point in $\mathrm{Hilb}^{\mathbf{v}}(\mathbb{C}^2)$ corresponds to a partition of \mathbf{v} . (See e.g., [23, Ch. 5].)

Since $\mathcal{M}^s(\lambda - |\underline{k}| \delta, \mu) = \emptyset$ unless $\mu = \lambda - |\underline{k}| \delta$ in this case, we have

$$\bigoplus_{\mu} \Phi \circ \pi_*(\mathbb{C}_{\mathcal{M}^{\square, \mathbb{R}}(\lambda, \mu)}[\dim]) = \bigoplus_{\underline{k}} \mathbb{C}[\overline{S^{\underline{k}}(\mathbb{C})}]$$

instead of (4.20). This result is also known, see e.g., [23, Ch. 7].

4(vii). Hyperbolic restriction in two steps

Let us take one parameter subgroup as in Theorem 4.14. Since it depends on i , let us denote it by χ_i . We also take a one parameter subgroup $\chi(t) = (t^{m_0}, \dots, t^{m_{n-1}})$ with $m_j < 0$ for all j . We have a chamber structure on the space of one parameter subgroups, and χ_i lives in the boundary of the chamber containing χ . The result in this subsection remains true only under this assumption, but we keep notation for brevity. Let us denote the attracting sets with respect to χ and χ_i by \mathfrak{A} and \mathfrak{A}_i respectively. We have $\mathcal{M}(\lambda, \mu)^{\chi_i} \xleftarrow{p_i} \mathfrak{A}_i \xrightarrow{j_i} \mathcal{M}(\lambda, \mu)$. Note also that χ acts nontrivially on the fixed point set $\mathcal{M}(\lambda, \mu)^{\chi_i}$, and we have the corresponding attracting set, which will be studied in §5(i). Let us denote it by \mathfrak{A}^i . We have $\text{pt} = \mathcal{M}(\lambda, \mu)^\chi \xleftarrow{p^i} \mathfrak{A}^i \xrightarrow{j^i} \mathcal{M}(\lambda, \mu)^{\chi_i}$, as above. Then we form the diagram

$$\begin{array}{ccccc}
 \mathfrak{A} & \xrightarrow{j''} & \mathfrak{A}_i & \xrightarrow{j_i} & \mathcal{M}(\lambda, \mu) \\
 p'' \downarrow & & \downarrow p_i & & \\
 \mathfrak{A}^i & \xrightarrow{j^i} & \mathcal{M}(\lambda, \mu)^{\chi_i} & & \\
 p^i \downarrow & & & & \\
 \mathcal{M}(\lambda, \mu)^\chi & & & &
 \end{array}$$

where \mathfrak{A} is the fiber product of \mathfrak{A}_i and \mathfrak{A}^i over $\mathcal{M}(\lambda, \mu)^{\chi_i}$. Then as in [5, §4.5] \mathfrak{A} is the attracting set in $\mathcal{M}(\lambda, \mu)$ with respect to χ such that compositions $j_i \circ j''$, $p^i \circ p''$ give the diagram $\text{pt} = \mathcal{M}(\lambda, \mu)^\chi \xleftarrow{p^i \circ p''} \mathfrak{A} \xrightarrow{j_i \circ j''} \mathcal{M}(\lambda, \mu)$ which is the one given by χ .

PROPOSITION 4.22. — *There is a natural transformation from the hyperbolic restriction functor $\Phi = (p^i \circ p'')_*(j_i \circ j'')^!$ to $\Phi^i \circ \Phi_i$, the composition of two hyperbolic restriction functors by base change. Here $\Phi^i = p_*^i(j^i)^!$, $\Phi_i = (p_i)_*j_i^!$.*

5. Construction

After preparation in the previous sections, we are ready to define the action of generators e_i, f_i, h_i on $\mathcal{V}(\lambda)$. The operator h_i is defined so that $\mathcal{V}_\mu(\lambda)$ is the weight space with weight μ .

5(i). Type A_1

Let us consider the bow variety of type A_1 with the balanced condition. We suppose dimension vectors are $\mathbf{v}, \mathbf{w} \in \mathbb{Z}_{\geq 0}$. By Corollary 4.4 we assume $\mathbf{v} \leq \mathbf{w}$, as there is no fixed point otherwise. We take a one parameter subgroup $\chi(t) = t^m$ ($m < 0$) as in the previous subsection.

The following was observed in [16, several paragraphs after Th. 3.1], but let us give a proof in terms of bow varieties for completeness.

THEOREM 5.1. — *The attracting set \mathfrak{A} for the type A_1 balanced bow variety associated with \mathbf{v}, \mathbf{w} is isomorphic to $\mathbb{C}^{\mathbf{v}}$.*

Proof. – We start with the balanced diagram, two triangles at the left and right ends. We have \mathbf{w} two way parts in between. The leftmost and rightmost vector spaces are 0 and others are \mathbf{v} -dimensional. By Hanany-Witten transition, we move the leftmost triangle next to the rightmost one. The result is

$$\begin{array}{ccccccc} \mathbb{C} & \xrightleftharpoons[D_{\mathbf{w}}]{C_{\mathbf{w}}} & \mathbb{C}^2 & \xrightleftharpoons[D_{\mathbf{w}-1}]{C_{\mathbf{w}-1}} & \cdots & \xrightleftharpoons[D_3]{C_3} & \mathbb{C}^{\mathbf{w}-1} \\ & & & & & & \xrightleftharpoons[D_2]{C_2} & \mathbb{C}^{\mathbf{w}} & \xrightarrow{A} & \mathbb{C}^{\mathbf{v}} & & \\ & & & & & & \searrow^{b_+} & \swarrow_{a_+} & & \searrow_{b_-} & & \\ & & & & & & \mathbb{C}_+ & & & \mathbb{C}_- & & \end{array}$$

(Note: The diagram shows a sequence of vector spaces connected by two-way arrows. The first part is a chain of $\mathbb{C} \xrightleftharpoons[D_{\mathbf{w}}]{C_{\mathbf{w}}} \mathbb{C}^2 \xrightleftharpoons[D_{\mathbf{w}-1}]{C_{\mathbf{w}-1}} \cdots \xrightleftharpoons[D_3]{C_3} \mathbb{C}^{\mathbf{w}-1} \xrightleftharpoons[D_2]{C_2} \mathbb{C}^{\mathbf{w}}$. Then, $\mathbb{C}^{\mathbf{w}}$ is connected to $\mathbb{C}^{\mathbf{v}}$ by a single arrow A . Above $\mathbb{C}^{\mathbf{w}}$ and $\mathbb{C}^{\mathbf{v}}$ are loops labeled B_+ and B_- respectively. Below $\mathbb{C}^{\mathbf{w}}$ is \mathbb{C}_+ with an arrow b_+ from $\mathbb{C}^{\mathbf{w}}$ to \mathbb{C}_+ . Below $\mathbb{C}^{\mathbf{v}}$ is \mathbb{C}_- with an arrow b_- from $\mathbb{C}^{\mathbf{v}}$ to \mathbb{C}_- . There is also an arrow a_+ from \mathbb{C}_+ to $\mathbb{C}^{\mathbf{v}}$.

Let us study the attracting set. The action is induced from one given by $b_- \mapsto t^m b_-$ and other maps unchanged.

We have $b_- B_-^k a_+ = 0$ for any k as it has weight $m < 0$. But $b_- B_-^k$ ($k = 0, 1, \dots, \mathbf{v} - 1$) is a base of the dual of $\mathbb{C}^{\mathbf{v}}$ by the condition (S1). Hence $a_+ = 0$. The defining equation now becomes $B_- A = A B_+$. Therefore A is surjective by (S2). (This gives a direct proof that there is no fixed point unless $\mathbf{v} \leq \mathbf{w}$.)

By the equation $C_{i+1} D_{i+1} - C_i D_i = 0$, we see that $B_+ = -C_1 D_1$ is nilpotent by induction. Hence B_- is also nilpotent. By (S1) $b_- B_-^k$ ($k = 0, 1, \dots, \mathbf{v} - 1$) is a base of the dual of $\mathbb{C}^{\mathbf{v}}$, hence we write B_- , b_- as

$$B_- = {}^t J_{\mathbf{v}} \quad b_- = {}^t e_{\mathbf{v}}.$$

This normalization kills the action of $\mathrm{GL}(\mathbf{v})$. Here ${}^t e_{\mathbf{v}} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$.

Note that $\mathrm{Ker} A$ is B_+ -invariant, and $b_+|_{\mathrm{Ker} A} \in (\mathrm{Ker} A)^*$ is cocyclic with respect to $B_+|_{\mathrm{Ker} A}$ by (S1). Therefore

$$\begin{aligned} b_- B_-^{\mathbf{v}-1} A &= b_- A B_+^{\mathbf{v}-1}, & b_- B_-^{\mathbf{v}-2} A &= b_- A B_+^{\mathbf{v}-2}, & \dots, & & b_- A, \\ b_+ B_+^{\mathbf{w}-\mathbf{v}-1} &, & b_+ B_+^{\mathbf{w}-\mathbf{v}-2} &, & \dots, & & b_+ B_+, & b_+ \end{aligned}$$

is a base of the dual of $\mathbb{C}^{\mathbf{w}}$. If $\mathbf{w} > \mathbf{v}$, we have $A = [\mathrm{id}_{\mathbf{v}} \ 0]$, $b_+ = {}^t e_{\mathbf{w}}$ and

$$B_+ = {}^t \begin{bmatrix} J_{\mathbf{v}} & \vec{c} & 0 \\ 0 & J_{\mathbf{w}-\mathbf{v}} \end{bmatrix}, \quad \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{\mathbf{v}} \end{bmatrix} \text{ with } b_+ B_+^{\mathbf{w}-\mathbf{v}} = c_1 b_- A B_+^{\mathbf{v}-1} + \cdots + c_{\mathbf{v}} b_- A.$$

If $\mathbf{w} = \mathbf{v}$, we have $A = \mathrm{id}$, $B_+ = {}^t J_{\mathbf{v}}$, and b_+ is arbitrary. Once the action of $\mathrm{GL}(\mathbf{w})$ is killed, the remaining data C_i , D_i are regarded as a point of a quiver variety of type $A_{\mathbf{w}-1}$, which is the nilpotent cone of $\mathfrak{sl}(\mathbf{w})$. See [21, §7]. Therefore they are normalized by the remaining action of $\mathrm{GL}(\mathbf{w} - 1) \times \cdots \times \mathrm{GL}(1)$. Hence the attracting set is $\mathbb{C}^{\mathbf{v}}$ parametrizing $c_1, \dots, c_{\mathbf{v}}$ ($\mathbf{w} > \mathbf{v}$), or b_+ ($\mathbf{w} = \mathbf{v}$). \square

REMARK 5.2. – Suppose $\mathbf{w} > \mathbf{v}$. Then \mathcal{M} is isomorphic to the intersection of the nilpotent cone of $\mathfrak{sl}(\mathbf{w})$ and the space of $\mathbf{w} \times \mathbf{w}$ matrices of the following form:

$$\left(\begin{array}{c|c} \mathbf{v} \text{ columns} & \mathbf{w} - \mathbf{v} \text{ columns} \\ \hline \mathbf{v} \text{ rows} & \left[\begin{array}{cc} 0 \cdots 0 & * \\ 1 & 0 \\ 0 \cdots 0 & * \\ 0 \cdots 1 & * \end{array} \right] \\ \hline \mathbf{w} - \mathbf{v} \text{ rows} & \left[\begin{array}{cc} * \cdots * & 0 \cdots 0 \\ 0 \cdots 0 & 1 \quad 0 \\ 0 \cdots 0 & 0 \cdots 0 \\ 0 \cdots 0 & 0 \cdots 1 \end{array} \right] \end{array} \right),$$

where $*$ indicates an arbitrary complex number. We use [31, Prop. 3.2] twice, first to determine the upper left block, and second to give the remaining part. The space of matrices of the above form is a slice to the nilpotent matrix $(\mathbf{v}, \mathbf{w} - \mathbf{v})$ considered in [20] when $\mathbf{v} \leq \mathbf{w} - \mathbf{v}$. When $\mathbf{v} > \mathbf{w} - \mathbf{v}$, the space has larger dimension than the slice.

If $\mathbf{w} = \mathbf{v}$, the triangle parts (quotient-ed by $\mathrm{GL}(\mathbf{v})$) give the product of $\mathrm{GL}(\mathbf{w}) \times \mathbb{C}^{2\mathbf{w}}$ and the space of the matrix of the above form (with $\mathbf{w} - \mathbf{v} = 0$). Then \mathcal{M} is the hamiltonian reduction of the product of the nilpotent cone for $\mathfrak{sl}(\mathbf{w})$ and this space by the diagonal action of $\mathrm{GL}(\mathbf{w})$.

Since the hyperbolic restriction functor Φ is hyperbolic semismall, $\Phi(\mathrm{IC}(\mathcal{M}_{A_1}(\lambda, \mu)))$ ($\lambda = \mathbf{w}$, $\mu = \mathbf{w} - 2\mathbf{v}$) has a base parametrized by irreducible components of the attracting set $\mathfrak{A} = \mathfrak{A}_{A_1}(\lambda, \mu)$. Therefore

COROLLARY 5.3. – *The attracting set $\mathfrak{A}_{A_1}(\lambda, \mu)$ is irreducible.*

Hence $\Phi(\mathrm{IC}(\mathcal{M}_{A_1}(\lambda, \mu))) \cong \mathbb{C}[\mathfrak{A}_{A_1}(\lambda, \mu)]$.

THEOREM 5.4. – (1) *The direct sum $\bigoplus_{\mu} \mathcal{V}_{\mu}(\lambda) = \bigoplus_{\mu} \Phi(\mathrm{IC}(\mathcal{M}_{A_1}(\lambda, \mu)))$ has an $\mathfrak{sl}(2)$ -module structure, which is irreducible with dimension $\lambda + 1 = \mathbf{w} + 1$. Moreover homomorphisms in (4.10) intertwine $\mathfrak{sl}(2)$ -module structures when we endow an $\mathfrak{sl}(2)$ -module structure on the right hand side as the tensor product via Corollary 4.9.*

(2) $[\mathfrak{A}_{A_1}(\lambda, \mu)] = \frac{f^n}{n!} [\mathfrak{A}_{A_1}(\lambda, \lambda)]$ for $\lambda - \mu = 2n$.

The construction is explicit and will be given during the proof.

Proof. – The operator h is given by μ id on the summand $\mathcal{V}_{\mu}(\lambda)$.

If $\lambda = 0$, it is the trivial representation. We have nothing to do. Next consider the case $\lambda = 1$. We need to study $\mathcal{M}_{A_1}(1, \pm 1)$. In the $+$ case, we get a special case of the bow diagram studied in the proof of Proposition 4.1. In particular, it is a point. We have $\Phi(\mathrm{IC}(\mathcal{M}_{A_1}(1, 1))) \cong \mathbb{C}[\text{point}]$. In the $-$ case we normalize A, b_- to 1 to kill the action, and determine B_- from the equation $B_- + a_+ b_+ = 0$. The remaining data are a_+, b_+ , hence we have $\mathcal{M}_{A_1}(1, -1) \cong \mathbb{C}^2$. The attracting set \mathfrak{A} is given by $a_+ = 0$ as above, therefore $\mathfrak{A} \cong \mathbb{C}$. We have $\Phi(\mathrm{IC}(\mathcal{M}_{A_1}(1, -1))) \cong \mathbb{C}[\mathfrak{A}]$.

We then define e, f on $\Phi(\mathrm{IC}(\mathcal{M}_{A_1}(1, 1))) \oplus \Phi(\mathrm{IC}(\mathcal{M}_{A_1}(1, -1)))$ by

$$e[\text{point}] = 0, \quad f[\text{point}] = [\mathfrak{A}], \quad f[\mathfrak{A}] = 0, \quad e[\mathfrak{A}] = [\text{point}].$$

This gives the two dimensional standard representation of $\mathfrak{sl}(2)$. The formula in (2) holds by definition.

Let us consider $\lambda > 1$. We take a real parameter $v^{\mathbb{R}}$ so that $v_1^{\mathbb{R}} < v_2^{\mathbb{R}} < \cdots < v_{\lambda-1}^{\mathbb{R}}$. There is no $v_*^{\mathbb{R}}$ as we are considering type A_1 bow varieties. The condition **(v2)** is automatically satisfied, and **(v1)** says that a graded subspace S as in **(v1)** must be zero. In particular, $v^{\mathbb{R}}$ -semistability and $v^{\mathbb{R}}$ -stability are equivalent, and \mathcal{M}^v is smooth.

Recall that we constructed an isomorphism between the hyperbolic restriction Φ of $\pi_*(\mathrm{IC}(\mathcal{M}_{A_1}^{v^{\mathbb{R}}}(\lambda, \mu)))$ and

$$\bigoplus_{\substack{\mu_i = \pm 1 \\ \mu_1 + \cdots + \mu_{\mathbf{w}} = \mu}} \bigotimes_{i=1}^{\lambda} \mathcal{V}_{\mu_i}(1)$$

in Corollary 4.9. Therefore for the direct sum

$$(5.5) \quad \bigoplus_{\mu} \Phi(\pi_*(\mathrm{IC}(\mathcal{M}_{A_1}^{v^{\mathbb{R}}}(\lambda, \mu)))) \cong \bigotimes_{i=1}^{\lambda} (\mathcal{V}_1(1) \oplus \mathcal{V}_{-1}(1)) = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{\lambda \text{ times}}.$$

We endow an $\mathfrak{sl}(2)$ -module structure as the tensor product of the above construction for $\lambda = 1$.

Now we consider the hamiltonian reduction in the definition of the bow variety in two steps as in §4(iii). In this case, the first reduction gives the product of a quiver variety of type $A_{\lambda-1}$ with dimension vectors $(1, 2, \dots, \lambda-1)$, $(0, \dots, 0, \lambda)$ and another variety given by triangles. The first quiver variety is the cotangent bundle of the flag variety for $\mathrm{SL}(\lambda)$. Thus we have \mathfrak{S}_{λ} action on $\bigoplus_{\mu} \pi_*(\mathrm{IC}(\mathcal{M}_{A_1}^{v^{\mathbb{R}}}(\lambda, \mu)))$ and its hyperbolic restriction (5.5). The action on the latter is given by permutation of factors.

Now $\bigoplus_{\mu} \mathcal{V}_{\mu}(\lambda) = \bigoplus_{\mu} \Phi(\mathrm{IC}(\mathcal{M}_{A_1}(\lambda, \mu)))$ is the direct summand of (5.5) consisting of \mathfrak{S}_{λ} -fixed vectors. Therefore it is isomorphic to the symmetric power $S^{\lambda}(\mathbb{C}^2)$. In particular, it inherits an $\mathfrak{sl}(2)$ -module structure, which is irreducible with dimension $\lambda + 1$, as we promised. The assertion on tensor products is clear from the construction.

To check the formula (2) we need to compute $[\mathfrak{A}_{A_1}(\lambda, \mu)]$ in the tensor product (5.5). The isomorphism in (5.5) came from the factorization, and we use the base change from $\mathbb{A}^n/\mathfrak{S}_n$ to \mathbb{A}^n . Therefore we get a factor $n!$. \square

5(ii). Definition of operators e_i, f_i

Let us take χ, χ_i as in §4(vii).

PROPOSITION 5.6. – *Suppose $\mathcal{M}(\lambda, \mu)^{\chi_i}$ is $\mathcal{M}_{A_1}(\lambda', \mu')$ as in Theorem 4.14. The hyperbolic restriction Φ_i sends $\mathrm{IC}(\mathcal{M}(\lambda, \mu))$ to a direct sum of $\mathrm{IC}(\mathcal{M}_{A_1}(\kappa', \mu'))$ with various κ' with $\mu' \leq \kappa' \leq \lambda'$.*

Proof. – Since $\Phi_i = (p_i)_* j_i^!$ is hyperbolic semismall with respect to the natural stratification of $\mathcal{M}(\lambda, \mu)$, $\mathcal{M}_{A_1}(\lambda', \mu')$, the assertion means that there is no direct summand for an IC complex for a *non trivial* local system. Using a deformation by $v^{\mathbb{C}}$ as in §4(ii), it is enough to show that $\psi \circ \Phi_i(\mathrm{IC}(\mathcal{M}(\lambda, \mu)))$ does not contain such a summand. We take $v^{\mathbb{C}}$ generic so that fibers of $\mathcal{M}(\lambda, \mu) \rightarrow \mathbb{C}$ are smooth outside 0, unlike a degenerate case used in §4(ii).

Again as in §4(ii), we introduce the corresponding real parameter to construct a topologically trivial family $\underline{\mathcal{M}}(\lambda, \mu)$. Then we have

$$\psi \circ \Phi_i(\mathrm{IC}(\underline{\mathcal{M}}(\lambda, \mu))) = \pi_*(\mathbb{C}_{\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)^{x_i}}[\dim]),$$

where the right hand side is the direct sum of constant sheaves on connected components of the fixed point set $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)^{x_i}$ shifted by their dimensions. Note that $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)$ and $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)^{x_i}$ are smooth.

We analyze the fixed point set as in the proof of Theorem 4.14. After Hanany-Witten transitions, we arrive at

$$(5.7) \quad \begin{array}{ccccccc} V_m & \xrightleftharpoons[D_m]{C_m} & V_{m-1} & \xrightleftharpoons{\quad} \cdots \xrightleftharpoons{\quad} & V_2 & \xrightleftharpoons[D_2]{C_2} & V_+ \xrightarrow{A} V_- \\ & & & & & & \begin{array}{c} \text{\scriptsize B_+} \\ \curvearrowright \\ \text{\scriptsize B_-} \end{array} \\ & & & & & & \begin{array}{c} \searrow b \\ \mathbb{C} \end{array} \quad \begin{array}{c} \nearrow a \\ \mathbb{C} \end{array} \quad \begin{array}{c} \searrow b \\ \mathbb{C} \end{array} \end{array}$$

where dimensions of V_m, \dots, V_+, V_- may differ depending on components. As in the proof of Theorem 5.4, this is a hamiltonian reduction of the product of a quiver variety of type A , which is the cotangent bundle of a partial flag variety, and a variety given by the triangles by the action of $\mathrm{GL}(V_+)$. Since the hamiltonian reduction is compatible with the decomposition of the pushforward $\pi_*(\mathbb{C}_{\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)^{x_i}}[\dim])$ as before, the assertion follows from the corresponding result for nilpotent orbits of type A (the connectedness of stabilizers), or quiver varieties [24, Prop. 15.3.2]. \square

For a later purpose, we study the fixed point set $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)^{x_i}$. Here the parameter $v^{\mathbb{R}}$ is arbitrary, not necessarily generic in the proof of Proposition 5.6.

LEMMA 5.8. – *The fixed point set $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)^{x_i}$ is a union of A_1 type balanced bow varieties with real parameters induced from $v^{\mathbb{R}}$.*

Here the induced parameters mean the following: recall $v_h^{\mathbb{R}}$ is assigned for each $h = h_\sigma$ ($1 \leq \sigma \leq \ell$). We take the universal covering of the bow diagram so that σ runs over \mathbb{Z} . Then $\{h_\sigma\}$ for A_1 is a subset of $\{h_\sigma \mid \sigma \in \mathbb{Z}\}$, and the parameters $v_h^{\mathbb{R}}$ are given by the restriction.

Proof. – Let us change the fixed point component to the form of (5.7) as in the proof of Proposition 5.6.

We will show that the balanced condition is achieved by first applying reflection functors in [26], then next applying Hanany-Witten transitions. We consider the deformation $\underline{\mathcal{M}}$ as in the proof of Proposition 5.6. Since reflection functors are hyperKähler isometry, $\mathcal{M}^{v^{\mathbb{R}}}$ and $\mathcal{M}^{v^{\mathbb{C}}}$ are changed in the same way. Also Hanany-Witten transitions respect complex and real parameters. Therefore it is enough to show the statement for $\mathcal{M}^{v^{\mathbb{C}}}$.

Let $N_{h_2} = \dim V_+ - \dim V_2, \dots, N_{h_{m+1}} = \dim V_m$ as before. By applying reflection functors in [26] at the cost of change of the parameter v , we achieve the dominance condition $N_{h_2} \geq N_{h_3} \geq \dots \geq N_{h_{m+1}}$. By an argument in [31, Prop. 7.5] we can transform the bow diagram to a balanced one by Hanany-Witten transition, if $N_{h_2} \leq 2$, the number of triangles in (5.7).

As in the proof of Proposition 4.7, the data (5.7) factorizes according to eigenvalues of B_+ , which are entries of $v^{\mathbb{C}}$. Moreover we can normalize C_h to the identity on a component for an eigenvalue $\neq v_h^{\mathbb{C}}$. Then each factor is an A_1 type bow variety with the parameter 0, which was studied during the proof of Theorem 4.14. In particular, we have $N_{h\sigma} = 0, 1$ or 2 in each factor. Since other factors do not contribute to $N_{h\sigma}$, we have $N_{h\sigma} \leq 2$, in particular, $N_{h_2} \leq 2$.

Once we achieve the balanced condition, the ordering on parameters v_h is irrelevant by Remark 3.4. \square

We can apply [8, §4] after changing the stability parameter in the decreasing order so that [8, (4.2)] is satisfied. Hence $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)^{X_i}$ is isomorphic to a union of Coulomb branches of the quiver gauge theory of type A_1 with parameter induced from the original $v^{\mathbb{R}}$.

REMARK 5.9. – Let us give an alternative argument for Lemma 5.8 for finite type A . It is informed by Finkelberg.

Recall that $\mathcal{M}^{v^{\mathbb{R}}}(\lambda, \mu)$ is realized as an iterated convolution diagram $\widetilde{\mathcal{W}}_{\mu}^{\lambda}$ in [8, §5]. For type A_n , it is the moduli space of flags of lattices $L_0 \supset L_1 \supset \cdots \supset L_N$ such that $L_{s-1} \supset L_s \supset zL_{s-1}$, and $\dim L_{s-1}/L_s = i_s$, $1 \leq i_s \leq n$, where $L_0 = V[[z]]$, $V = \mathbb{C}v_0 \oplus \cdots \oplus \mathbb{C}v_n$. Here the sequence i_1, i_2, \dots, i_N is as follows. We consider coordinates of cocharacter of G_F corresponding to $v^{\mathbb{R}}$. We choose i_1 so that the maximum k_1 of coordinates is achieved at the vertex i_1 of the Dynkin diagram. Then we choose i_2 so that the next one k_2 is achieved at i_2 , and so on. The determinant bundle is $\mathcal{D}_s = \det(L_{s-1}/L_s)$ (or its dual?), and the ample line bundle for the stability parameter $v^{\mathbb{R}}$ is $\bigotimes \mathcal{D}_s^{\otimes k_s}$. Let our Levi subgroup be $\mathrm{GL}(V' := \mathbb{C}v_0 \oplus \mathbb{C}v_1)$ times the remaining torus T'' . Then the corresponding torus fixed points in the above convolution diagram will have many connected components. A typical connected component will be the moduli space of flags $(L'_0 \supset L'_1 \supset \cdots \supset L'_N) \oplus (L''_0 \supset L''_1 \supset \cdots \supset L''_N)$ where the first summand is a flag in $L'_0 = V'[[z]]$, while the second summand is a T'' -invariant flag in $L''_0 = V''[[z]]$, where $V'' = \mathbb{C}v_2 \oplus \cdots \oplus \mathbb{C}v_n$. There are finitely many possibilities for choices of L'_s . Clearly, $\det(L_{s-1}/L_s) = \det(L'_{s-1}/L'_s) \otimes \det(L''_{s-1}/L''_s)$, and the second factor is “constant” from the point of view of $\mathfrak{sl}(2)$ -Grassmannian. Thus the ample line bundle for the fixed point component is $\bigotimes \det(L'_{s-1}/L'_s)^{\otimes k_s}$.

Thanks to Proposition 5.6, we write the hyperbolic restriction $\Phi_i(\mathrm{IC}(\mathcal{M}(\lambda, \mu)))$ as in (1.6). The multiplicity space $M_{\kappa', \mu'}^{\lambda, \mu}$ is top degree Borel-Moore homology group of a subvariety $p_i^{-1}(x)$, where x is a point in the smooth locus of a stratum $\mathcal{M}_{A_1}^s(\kappa', \mu')$ of $\mathcal{M}_{A_1}(\lambda', \mu') = \mathcal{M}(\lambda, \mu)^{X_i}$. See §4(vii). By Proposition 5.6 the top degree Borel-Moore homology group form a trivial local system over $\mathcal{M}_{A_1}^s(\kappa', \mu')$. Therefore its fibers are canonically identified, i.e., independent of the choice of x .

PROPOSITION 5.10. – *The factorization gives us an isomorphism*

$$M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu'-2}^{\lambda, \mu-\alpha_i}.$$

The construction will be explained during the proof.

Proof. – Recall that the i -th component of the factorization morphism Φ of $\mathcal{M}(\lambda, \mu)$ is restricted to the factorization morphism of $\mathcal{M}_{A_1}(\lambda', \mu')$ (for which $\mathcal{M}_{A_1}(\kappa', \mu')$ is a stratum).

We consider the Coulomb branch of another quiver gauge theory obtained by increasing the i -th entry of the vector $\underline{\mathbf{v}}$ by 1, i.e., $\mathcal{M}(\lambda, \mu - \alpha_i)$. Let $\underline{\delta_i}$ be the dimension vector whose i -entry is 1 and other entries are 0. We take the open subset $(\mathbb{A}^{\underline{\mathbf{v}}} \times \mathbb{G}_m^{\underline{\delta_i}})_{\text{disj}}$ of $\mathbb{A}^{\underline{\mathbf{v}}} \times \mathbb{A}^{\underline{\delta_i}}$ consisting of pairs of colored configurations whose supports are disjoint as in §3(iii), and also the support of the second configuration is disjoint from 0. Then we have the factorization isomorphism

$$(5.11) \quad \mathcal{M}(\lambda, \mu - \alpha_i) \times_{\mathbb{A}^{\underline{\mathbf{v}}} + \underline{\delta_i}} (\mathbb{A}^{\underline{\mathbf{v}}} \times \mathbb{G}_m^{\underline{\delta_i}})_{\text{disj}} \cong (\mathcal{M}(\lambda, \mu) \times \mathbb{A} \times \mathbb{G}_m) \times_{\mathbb{A}^{\underline{\mathbf{v}}} \times \mathbb{A}^{\underline{\delta_i}}} (\mathbb{A}^{\underline{\mathbf{v}}} \times \mathbb{G}_m^{\underline{\delta_i}})_{\text{disj}}.$$

The second factor is $\mathcal{M}(\lambda, \lambda - \alpha_i)$ in the statement of Theorem 3.6. It can be replaced by the Coulomb branch of type A_1 theory with $\mathbf{w} = \mathbf{w}_i$, $\mathbf{v} = 1$ as entries of $\underline{\delta_i}$ are 0 except at i . Moreover we restrict it to the open subset where the factorization morphism is nonzero, hence we can further replace it by the Coulomb branch of the quiver gauge theory of type A_1 with $\mathbf{w} = 0$, $\mathbf{v} = 1$, which is just $\mathbb{A} \times \mathbb{G}_m$ as above.

This factorization is compatible with the one parameter subgroup χ_i , as χ_i does not change the i -th component of Φ . Note χ_i acts trivially on the factor $\mathbb{A} \times \mathbb{G}_m$. We have the factorization $\mathcal{M}_{A_1}^s(\kappa', \mu' - 2) \approx \mathcal{M}_{A_1}^s(\kappa', \mu') \times (\mathbb{A} \times \mathbb{G}_m)$ for strata of χ_i -fixed point sets, compatible with $\mathcal{M}(\lambda, \mu - \alpha_i) \approx \mathcal{M}(\lambda, \mu) \times (\mathbb{A} \times \mathbb{G}_m)$. Thus the desired isomorphism is given by the factorization. \square

By Proposition 4.22 and (1.6) we define operators e_i, f_i on $\bigoplus_{\mu} \mathcal{V}_{\mu}(\lambda)$ induced from e, f given by Theorem 5.4 for the A_1 case, as explained in Introduction.

5(iii). Tensor product

Let us slightly generalize the above construction of e_i, f_i . We replace $\text{IC}(\mathcal{M}(\lambda, \mu))$ by $\pi_*(\text{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$. Here $v^{\bullet, \mathbb{R}}$ is as in §4(ii). Then $\Phi_i \circ \pi_*(\text{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$ decomposes as $\bigoplus_{\kappa'} \widetilde{\mathcal{M}}_{\kappa', \mu'}^{\lambda, \mu} \otimes \text{IC}(\mathcal{M}_{A_1}(\kappa', \mu'))$ as in (1.6). Then we construct an isomorphism $\widetilde{\mathcal{M}}_{\kappa', \mu'}^{\lambda, \mu} \cong \widetilde{\mathcal{M}}_{\kappa', \mu'-2}^{\lambda, \mu-\alpha_i}$ in the same way as in the proof of Proposition 5.10 by the factorization. Then we define e_i, f_i as (this isomorphism) $\otimes (e, f \text{ for } A_1 \text{ case})$.

This construction is compatible with the original one: Since $\text{IC}(\mathcal{M}(\lambda, \mu))$ is a direct summand of $\pi_*(\text{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$, we have the induced inclusion and projection $M_{\kappa', \mu'}^{\lambda, \mu} \hookrightarrow \widetilde{\mathcal{M}}_{\kappa', \mu'}^{\lambda, \mu}$. They commute with isomorphisms $M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu'-2}^{\lambda, \mu-\alpha_i}$, $\widetilde{\mathcal{M}}_{\kappa', \mu'}^{\lambda, \mu} \cong \widetilde{\mathcal{M}}_{\kappa', \mu'-2}^{\lambda, \mu-\alpha_i}$ by the construction. Therefore maps in (4.10) intertwine e_i, f_i .

Now recall $\Phi \circ \pi_*(\text{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$ decomposes into a sum of tensor product by Corollary 4.9. Hence we have

$$(5.12) \quad \bigoplus_{\mu} \Phi \circ \pi_*(\text{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu))) \cong \mathcal{V}(\lambda^1) \otimes \mathcal{V}(\lambda^2).$$

PROPOSITION 5.13. – (1) *Homomorphisms in (4.10) intertwine operators e_i, f_i .*

(2) *The operators e_i, f_i defined on the left hand side of (5.12) just above are equal to the tensor product in the right hand side, i.e., e_i is given by $1 \otimes e_i + e_i \otimes 1$, etc.*

Proof. – The statement (1) is already proved above.

In order to understand $\Phi \circ \pi_*(\mathrm{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu)))$ in (5.12), we consider $\psi \circ \Phi(\mathrm{IC}(\underline{\mathcal{M}}(\lambda, \mu)))$ as in Corollary 4.9 and change it to $\Phi^i \circ \psi \circ \Phi_i(\mathrm{IC}(\underline{\mathcal{M}}(\lambda, \mu)))$. Then the triviality of the family $\underline{\mathcal{M}}(\lambda, \mu) \rightarrow \mathbb{C}$ and the commutativity of the nearby cycle and hyperbolic restriction functors give an isomorphism

$$\Phi_i \circ \pi_*(\mathrm{IC}(\mathcal{M}^{v^{\bullet, \mathbb{R}}}(\lambda, \mu))) \cong \psi \circ \Phi_i(\mathrm{IC}(\underline{\mathcal{M}}(\lambda, \mu))).$$

By Lemma 5.8 (or more precisely its version for the complex parameter $v^{\mathbb{C}}$), $\Phi_i(\mathrm{IC}(\underline{\mathcal{M}}(\lambda, \mu)))$ is a direct sum $\bigoplus \overline{M}_{\kappa', \mu'}^{\lambda, \mu} \otimes \mathrm{IC}(\underline{\mathcal{M}}_{A_1}(\kappa', \mu'))$ over the inverse image of $\mathbb{C} \setminus \{0\}$ under $\underline{\mathcal{M}}_{A_1}(\kappa', \mu') \rightarrow \mathbb{C}$. Moreover $\underline{\mathcal{M}}(\lambda, \mu)$ factors as

$$\underline{\mathcal{M}}(\lambda^1, \mu^1) \times \underline{\mathcal{M}}(\lambda^2, \mu^2)$$

around a T -fixed point by Proposition 4.7.

Therefore $\overline{M}_{\kappa', \mu'}^{\lambda, \mu}$ is the tensor product $M_{\kappa^{1'}, \mu^{1'}}^{\lambda^1, \mu^1} \otimes M_{\kappa^{2'}, \mu^{2'}}^{\lambda^2, \mu^2}$.

The factorization gives an isomorphism $\overline{M}_{\kappa', \mu'}^{\lambda, \mu} \cong \overline{M}_{\kappa', \mu' - 2}^{\lambda, \mu - \alpha_i}$ as in Proposition 5.10. We can apply two factorizations simultaneously. Therefore the isomorphism is induced from isomorphisms for factors $M_{\kappa^{1'}, \mu^{1'}}^{\lambda^1, \mu^1}$, $M_{\kappa^{2'}, \mu^{2'}}^{\lambda^2, \mu^2}$, and it is also independent of the choice of the decomposition $\mu = \mu^1 + \mu^2$.

Thus it is enough to check the assertion $e_i = e_i \otimes 1 + 1 \otimes e_i$ for

$$\psi \circ \Phi^i(\mathrm{IC}(\underline{\mathcal{M}}_{A_1}(\kappa', \mu'))) \cong \bigoplus_{\mu^{1'} + \mu^{2'} = \mu'} \Phi^i(\mathrm{IC}(\underline{\mathcal{M}}_{A_1}(\kappa^{1'}, \mu^{1'}))) \otimes \Phi^i(\mathrm{IC}(\underline{\mathcal{M}}_{A_1}(\kappa^{2'}, \mu^{2'}))),$$

the isomorphism given by the factorization as a special case of Corollary 4.9 for type A_1 . But this is clear from the definition as explained in the proof of Theorem 5.4. \square

5(iv). Type A_2

We next show the relation $[e_i, f_j] = 0$ if $i \neq j$ and the Serre relation. This is reduced to the rank 2 case. If i and j are not connected in the Dynkin diagram, the bow variety decomposes into a product. The assertion is trivial. Next we study the A_2 case. Thanks to Proposition 5.13, we may assume λ is a fundamental weight. We may further assume $\lambda = \Lambda_1$, the first fundamental weight, by a diagram automorphism. The bow variety $\mathcal{M}_{A_2}(\lambda, \mu)$ has a fixed point if and only if $\mu = \Lambda_1, \Lambda_1 - \alpha_1, \Lambda_1 - \alpha_1 - \alpha_2$. We apply Hanany-Witten transition to go to a bow diagram

$$\begin{array}{ccccccc} & 0 & 1 & \mathbf{v}_1 & \mathbf{v}_2 & 0 & \\ & \bigcirc & \times & \times & \times & & \end{array}$$

with $(\mathbf{v}_1, \mathbf{v}_2) = (0, 0), (1, 0), (1, 1)$.

In the case $\mu = \Lambda_1$, the bow variety is a single point. Let \mathfrak{A}_1 denote the corresponding attracting set, which is also a single point. We have $e_1[\mathfrak{A}_1] = 0 = e_2[\mathfrak{A}_1]$ as the corresponding bow varieties are empty. We also have $f_2[\mathfrak{A}_1] = 0$ since the corresponding bow variety does not have torus fixed points.

Next consider the case $\mu = \Lambda - \alpha_1$. We have

$$\begin{array}{ccccc}
 \overset{B_0=0}{\curvearrowright} & & \overset{B_1}{\curvearrowright} & & 0 \\
 \mathbb{C} & \xrightarrow{A_1} & \mathbb{C} & & \\
 \searrow b_1 & & \nearrow a_1 & \searrow b_2 & \\
 & \mathbb{C} & & \mathbb{C} & \mathbb{C}.
 \end{array}$$

We normalize $A_1 = b_2 = 1$, and determine B_1 from the equation $B_1 + a_1 b_1 = 0$. Therefore $\mathcal{M}_{A_2}(\lambda, \mu) \cong \mathbb{C}^2$ by the remaining variables (a_1, b_1) . The action is

$$(a_1, b_1) \mapsto (t^{m_1} a_1, t^{-m_1} b_1).$$

The attracting set is $\{a_1 = 0\}$, which is \mathbb{C} . Let us denote it by \mathfrak{A}_2 . We have $e_2[\mathfrak{A}_2] = 0 = f_1[\mathfrak{A}_2]$ as the corresponding bow varieties are empty.

For the case $\mu = \Lambda_1 - \alpha_1 - \alpha_2$, we have

$$\begin{array}{ccccccc}
 \overset{B_0=0}{\curvearrowright} & & \overset{B_1}{\curvearrowright} & & \overset{B_2}{\curvearrowright} & & \\
 \mathbb{C} & \xrightarrow{A_1} & \mathbb{C} & \xrightarrow{A_2} & \mathbb{C} & & \\
 \searrow b_1 & & \nearrow a_1 & \searrow b_2 & \nearrow a_2 & \searrow b_3 & \\
 & \mathbb{C} & & \mathbb{C} & & \mathbb{C} & \mathbb{C}.
 \end{array}$$

We normalize $A_1 = A_2 = b_3 = 1$, and determine B_1, B_2 from the equations $B_1 + a_1 b_1 = 0$, $B_2 - B_1 + a_2 b_2 = 0$. Therefore $\mathcal{M}_{A_2}(\lambda, \mu) \cong \mathbb{C}^4$ by the remaining variables (a_1, b_1, a_2, b_2) . The action is

$$(a_1, b_1, a_2, b_2) \mapsto (t^{m_1+m_2} a_1, t^{-m_1-m_2} b_1, t^{m_2} a_2, t^{-m_2} b_2).$$

The attracting set is $\{a_1 = a_2 = 0\}$, which is \mathbb{C}^2 . Let us denote it by \mathfrak{A}_3 . We have $f_1[\mathfrak{A}_3] = 0 = f_2[\mathfrak{A}_3]$, $e_1[\mathfrak{A}_3] = 0$ as above.

In order to calculate remaining actions of operators e_1, e_2, f_1, f_2 , we take one parameter subgroups with $m_1 = 0, m_2 < 0$ and $m_1 < 0, m_2 = 0$ respectively.

When $m_1 = 0$, the action is trivial in the case $\mu = \Lambda_1 - \alpha_1 - \alpha_2$. Therefore the hyperbolic restriction does nothing. Hence we have

$$f_1[\mathfrak{A}_1] = [\mathfrak{A}_2], \quad e_1[\mathfrak{A}_2] = [\mathfrak{A}_1].$$

When $m_2 = 0$, the attracting set remains \mathfrak{A}_2 , and the fixed point is a single point $(a_1, b_1) = 0$ for $\mu = \Lambda_1 - \alpha_1$, while the attracting set is $\{a_1 = 0\} \cong \mathbb{C}^3$ and the fixed point set is $\{a_1 = b_1 = 0\} \cong \mathbb{C}^2$ for $\mu = \Lambda_1 - \alpha_1 - \alpha_2$. Therefore we have

$$f_2[\mathfrak{A}_2] = [\mathfrak{A}_3], \quad e_2[\mathfrak{A}_3] = [\mathfrak{A}_2].$$

This finishes the calculation, and we see that this gives the 3-dimensional standard representation of $\mathfrak{sl}(3)$.

5(v). Reduction to A_∞ case

We are left to check the case affine A_1 . As in §5(iv) we may assume λ is a fundamental weight, and $\lambda = \Lambda_0$ by the diagram automorphism. The following argument works for general $n \geq 2$.

We apply the method used in §5(iii) to $\mathcal{M}^{v^{\square, \mathbb{R}}}(\lambda, \mu)$, $\mathcal{M}^{v^{\square, \mathbb{C}}}(\lambda, \mu)$ studied in §4(vi). We have

$$(5.14) \quad \bigoplus_{\mu} \Phi \circ \pi_*((\mathrm{IC}(\mathcal{M}^{v^{\square, \mathbb{R}}}(\lambda, \mu)))) \cong \bigoplus_{\mu'} \Phi(\mathrm{IC}(\mathcal{M}_{A_\infty}(\lambda', \mu'))),$$

where λ' is the 0-th fundamental weight for A_∞ . As in Proposition 5.13, we define operators e_i, f_i on the left hand side, and ask what they are in the right hand side. In $\mathcal{M}_{A_\infty}(\lambda', \mu')$ it is straightforward to check that the fixed point set $\mathcal{M}^{v^{\square, \mathbb{C}}}(\lambda, \mu)^{\chi_i}$ with respect to the degenerate one-parameter subgroup χ_i is mapped to the product of A_1 -type bow varieties for χ_{i+mn} ($m \in \mathbb{Z}$) under the isomorphism in Proposition 4.19. Therefore e_i, f_i are given by $\sum_{m \in \mathbb{Z}} e_{i+mn}, \sum_{m \in \mathbb{Z}} f_{i+mn}$ in the right hand side. This is nothing but an embedding of $\widehat{\mathfrak{sl}}(n)$ into $\widehat{\mathfrak{gl}}(\infty)$, see e.g., [15, Lecture 9]. In particular the relation $[e_i, f_j] = 0$ for $i \neq j$ and the Serre relation are satisfied also for $\widehat{\mathfrak{sl}}(2)$.

Let us look at (4.20). It is an isomorphism of $\widehat{\mathfrak{sl}}(n)$ -modules. The left hand side is the restriction of the Fock space for $\widehat{\mathfrak{gl}}(\infty)$ to $\widehat{\mathfrak{sl}}(n)$. In the right hand side the first factor is $V(\Lambda_0)$, while the second factor corresponds to the Fock space for the Heisenberg subalgebra in $\widehat{\mathfrak{gl}}(\infty)$.

REMARK 5.15. – Consider the case $n = 1$. (5.14) remains to be true. As we mentioned in Remark 4.21 the left hand side is $\bigoplus_{\underline{k}} \mathbb{C}[S^{\underline{k}}(\mathbb{C})]$. This space is equipped with the structure of the Fock space of the Heisenberg algebra so that $[S^{\underline{k}}(\mathbb{C})]$ corresponds to the monomial symmetric function for the partition \underline{k} [28, §2].

On the other hand the summand $\Phi(\mathrm{IC}(\mathcal{M}_{A_\infty}(\lambda', \mu')))$ in (5.14) corresponds to a fixed point in $\mathcal{M}^{v^{\square, \mathbb{C}}}(\lambda, \mu)$ corresponding to a partition given by $\lambda' - \mu' = \sum_m \mathbf{v}(m) \alpha_m$. (See in the appendix how $\mathbf{v}(m)$ corresponds to a Maya diagram. Then we use the standard bijection between a partition and a Maya diagram.) This fixed point was studied by Vasserot [34]: Its class is the Schur function for \underline{k} .

Since the correspondence Maya diagrams and Schur functions appear in boson-fermion correspondence (see e.g., [15, Lectures 5,6]), we see that (5.14) respects the Heisenberg algebra action where we consider the Heisenberg subalgebra in $\widehat{\mathfrak{gl}}(\infty)$ in the right hand side.

5(vi). Kashiwara crystal

Recall $\mathcal{V}_\mu(\lambda)$ has a base parametrized by irreducible components of the attracting set $\mathfrak{A}_\chi(\lambda, \mu)$ of $\dim = \dim \mathcal{M}(\lambda, \mu)/2$. (Remark 4.6) In [7, Remark 3.26(2)] it was conjectured that the union of irreducible components $\bigsqcup \mathrm{Irr} \mathfrak{A}_\chi(\lambda, \mu)$ has a structure of Kashiwara crystal, isomorphic to $B(\lambda)$, the crystal of the integrable highest weight module of the quantized enveloping algebra.

As we mentioned in the Introduction, our construction resembles the construction of Kashiwara crystal structure in [9], it is straightforward to apply the construction in [9] to our setting. Let us briefly sketch. We use the standard notation for crystal, e.g., as in [25].

1. We define Kashiwara operators $\widetilde{e}, \widetilde{f}$ for $\mathfrak{sl}(2)$ from the analysis in §5(i). Namely they send $[\mathfrak{A}_\chi(\lambda, \mu)]$ to $[\mathfrak{A}_\chi(\lambda, \mu \pm 2)]$ or 0.
2. We define $\widetilde{e}_i, \widetilde{f}_i$ in general by reduction to $\mathfrak{sl}(2)$ by the hyperbolic restriction with respect to χ_i as in §5(ii). In particular, we use the factorization isomorphism appeared in the proof of Proposition 5.10.
3. We consider irreducible components of attracting sets in $\mathcal{M}^{v^\bullet, \mathbb{C}}(\lambda, \mu)$ and $\mathcal{M}^{v^\bullet, \mathbb{R}}(\lambda, \mu)$ as in §4(ii). They are naturally identified via the topologically trivial family $\widetilde{\mathcal{M}}(\lambda, \mu)$. Let us denote it by $\text{Irr } \mathfrak{A}_\chi^{v^\bullet}(\lambda, \mu)$. Then $\bigsqcup_\mu \text{Irr } \mathfrak{A}_\chi^{v^\bullet}(\lambda, \mu)$ has a Kashiwara crystal structure. Moreover

- (a) The projection $\pi: \mathcal{M}^{v^\bullet, \mathbb{R}}(\lambda, \mu) \rightarrow \mathcal{M}(\lambda, \mu)$ induces an inclusion

$$\bigsqcup \text{Irr } \mathfrak{A}_\chi(\lambda, \mu) \subset \bigsqcup_\mu \text{Irr } \mathfrak{A}_\chi^{v^\bullet}(\lambda, \mu),$$

which is an embedding of crystals.

- (b) the factorization in Proposition 4.7 induces an isomorphism

$$\bigsqcup_\mu \text{Irr } \mathfrak{A}_\chi^{v^\bullet}(\lambda, \mu) \cong \bigsqcup_{\mu^1} \text{Irr } \mathfrak{A}_\chi(\lambda^1, \mu^1) \otimes \bigsqcup_{\mu^2} \text{Irr } \mathfrak{A}_\chi(\lambda^1, \mu^1)$$

of crystals.

4. If we view $[Y] \in \text{Irr } \mathfrak{A}_\chi(\lambda, \mu)$ as an element of $H_{2 \dim}(\mathfrak{A}_\chi(\lambda, \mu)) \cong V_\mu(\lambda)$, Kashiwara operator \widetilde{f}_i and f_i in the Lie algebra are related as

$$f_i[Y] = (\varepsilon_i(Y) + 1) \widetilde{f}_i[Y] + \sum_{Y': \varepsilon_i(Y') > \varepsilon_i(Y) + 1} c_{Y'}[Y']$$

for some constants $c_{Y'}$. (This property was explained in a different way in [9, Prop. 4.1].)

As in [9] the only remaining property we need to check is the highest weight property: for any $[Y] \in \text{Irr } \mathfrak{A}_\chi(\lambda, \mu)$ which is not $[\mathfrak{A}_\chi(\lambda, \lambda)]$, there exists i such that $\widetilde{e}_i[Y] \neq 0$. This will be discussed in the next subsection. At this moment, if we replace $\bigsqcup \text{Irr } \mathfrak{A}_\chi(\lambda, \mu)$ by the connected component containing $[\mathfrak{A}_\chi(\lambda, \lambda)]$, it is isomorphic to the crystal $B(\lambda)$.

5(vii). Irreducibility

So far we have constructed a \mathfrak{g}_{KM} -module structure on $\mathcal{V}(\lambda)$. It is integrable and has a vector v_λ correspond to the fundamental class of $\mathfrak{A}_\chi(\lambda, \lambda)$ which is killed by all e_i by definition. It remains to show that $\mathcal{V}(\lambda)$ is generated by v_λ . By the construction in §5(vi) it follows once we show that $\bigsqcup \text{Irr}(\mathfrak{A}_\chi(\lambda, \mu))$ has the highest weight property. Conversely if we show that $\mathcal{V}(\lambda)$ is generated by v_λ , there are no other irreducible components, hence $\bigsqcup \text{Irr}(\mathfrak{A}_\chi(\lambda, \mu)) \cong B(\lambda)$.

We expect that there is a direct argument showing the highest weight property of crystal, but we give two indirect arguments.

Let us show that the number of irreducible components in $\mathfrak{A}_\chi(\lambda, \mu)$ is equal to the weight multiplicities. Since we have constructed a \mathfrak{g}_{KM} -module structure, we can assume μ is dominant. Then the bow variety $\mathcal{M}(\lambda, \mu)$ is isomorphic to a quiver variety of affine type A , where the level ℓ and rank n are swapped. See [31, Prop. 7.20]. Moreover the attracting set $\mathfrak{A}_\chi(\lambda, \mu)$ is the tensor product variety studied in [25]. More precisely its intersection

with $\mathcal{M}^s(\lambda, \mu)$ is the modified version of the tensor product variety $\mathfrak{Z}_0^s(\mathbf{v}, \mathbf{w})$ introduced in [27, §6]. It was proved in [27, §6] that the number of $\text{Irr } \mathfrak{Z}_0^s(\mathbf{v}, \mathbf{w})$ is equal to the tensor product multiplicity of $V_{\mathfrak{gl}(\ell)_{\text{aff}}}(t\mu)$ in the tensor product of fundamental representations. By level-rank duality, this is equal to the weight multiplicity of $V_\mu(\lambda)$ for $\mathfrak{sl}(n)_{\text{aff}(n)}$.

The second argument uses the computation of the stalk of $\text{IC}(\mathcal{M}(\lambda, \mu))$ when μ is dominant in [4]. As far as dimension is concerned, the stalk and hyperbolic restriction give the same answer. Hence the result in [4, §7] can be used. Note that [4, §7] used a geometric construction of affine Lie algebra modules via quiver varieties and level rank duality. In this sense, the second argument is not far away from the first one.

5(viii). Finite dimensional cases

Suppose that Q is a quiver with symmetrizer as [32], mentioned in the Introduction. We further assume that it is of finite type. We denote the corresponding Lie algebra by \mathfrak{g}^\vee , and the Langlands dual Lie algebra by \mathfrak{g} . Then $\mathcal{M}(\lambda, \mu)$ is isomorphic to a generalized affine Grassmannian slice $\overline{\mathcal{W}}_\mu^\lambda$ by [7, 32]. Here the complex reductive group G^\vee for the affine Grassmannian has the Lie algebra \mathfrak{g}^\vee and of adjoint type. Let us fix a Borel subgroup B^\vee and an opposite Borel B_-^\vee . Let $\text{Gr}_{G^\vee} = G^\vee((z))/G^\vee[[z]]$ be the affine Grassmannian for G^\vee . Let $\text{Gr}_{G^\vee}^\lambda$ be the $G^\vee[[z]]$ -orbit through z^λ , $\overline{\text{Gr}}_{G^\vee}^\lambda$ its closure. When μ is dominant, $\overline{\mathcal{W}}_\mu^\lambda$ is a transversal slice to $\text{Gr}_{G^\vee}^\mu$ in $\overline{\text{Gr}}_{G^\vee}^\lambda$. For general μ , $\overline{\mathcal{W}}_\mu^\lambda$ is the moduli space parametrizing

- (a) a G^\vee -bundle \mathcal{P} on \mathbb{P}^1 .
- (b) A trivialization of \mathcal{P} over $\mathbb{P}^1 \setminus \{0\}$ having a pole of degree $\leq \lambda$ at $0 \in \mathbb{P}^1$.
- (c) A B^\vee -structure on \mathcal{P} of degree $w_0\mu$ having fiber B_-^\vee at $\infty \in \mathbb{P}^1$ with respect to the trivialization in (b).

See [7, §2(ii)] for more detail. Here we mention that it is equipped with $\mathbf{p}: \overline{\mathcal{W}}_\mu^\lambda \rightarrow \overline{\text{Gr}}_{G^\vee}^\lambda$ by (a), (b). It is a closed embedding when μ is dominant, and gives a slice to $\text{Gr}_{G^\vee}^\mu$ in $\overline{\text{Gr}}_{G^\vee}^\lambda$.

We identify $T = (\mathbb{C}^\times)^{Q_0}$ with the maximal torus of G^\vee , which is the intersection $B^\vee \cap B_-^\vee$. (We will not use G^\vee except in this subsection. Therefore we do not use the notation T^\vee .) The T -action is identified with the standard one on $\overline{\mathcal{W}}_\mu^\lambda$ under the isomorphism $\mathcal{M}(\lambda, \mu) \cong \overline{\mathcal{W}}_\mu^\lambda$. Let us use $\mathcal{M}(\lambda, \mu)$ instead of $\overline{\mathcal{W}}_\mu^\lambda$ hereafter.

We take an anti-dominant cocharacter $\chi: \mathbb{C}^\times \rightarrow T$ as above. By a result of Krylov [16] mentioned in Introduction, $\mathcal{V}_\mu(\lambda) = \Phi(\text{IC}(\mathcal{M}(\lambda, \mu)))$ is isomorphic to the hyperbolic restriction considered by Mirković-Vilonen [19] via a homomorphism induced by \mathbf{p} . Therefore it is equipped with a \mathfrak{g} -module structure, same as a G -module structure, as G is simply-connected, by the usual geometric Satake correspondence. On the other hand, Conjectures 1.2 and 1.5 were proved in [16], as we will explain below. Therefore the construction in §1(ii) is applicable.

THEOREM 5.16 ⁽²⁾. — *Operators e_i, f_i, h_i on $\bigoplus \mathcal{V}_\mu(\lambda)$ given by the construction in §1(ii) are equal to ones given by the usual geometric Satake correspondence.*

The proof occupies the rest of this subsection.

For type A_1 , both constructions are explicit. We can directly check that they are the same.

We consider the one parameter subgroup χ_i as in §1(ii). We then have a diagram $\mathrm{Gr}_{L_i} \xleftarrow{\tilde{p}_i} \mathrm{Gr}_{P_i^-} \xrightarrow{\tilde{j}_i} \mathrm{Gr}_{G^\vee}$, where L_i (resp. P_i^-) is the Levi (resp. parabolic) subgroup for χ_i . We further choose connected components $\mathrm{Gr}_{L_i, \bar{\mu}}, \mathrm{Gr}_{P_i^-, \bar{\mu}}$ for $\bar{\mu} \in \pi_1(L_i)$ corresponding to μ . This is an analog of a diagram in §4(vii), and two diagrams sit in a commutative diagram

$$(5.17) \quad \begin{array}{ccccc} \mathcal{M}(\lambda, \mu)^{\chi_i} & \xleftarrow{p_i} & \mathfrak{A}_i & \xrightarrow{j_i} & \mathcal{M}(\lambda, \mu) \\ \mathbf{p}_{L_i} \downarrow & & \downarrow & & \downarrow \mathbf{p} \\ \mathrm{Gr}_{L_i, \bar{\mu}} & \xleftarrow{\tilde{p}_i} & \mathrm{Gr}_{P_i^-, \bar{\mu}} & \xrightarrow{\tilde{j}_i} & \mathrm{Gr}_{G^\vee}. \end{array}$$

See [16, (5.1)]. Here we denote the composite of $\mathbf{p}: \mathcal{M}(\lambda, \mu) \rightarrow \overline{\mathrm{Gr}}_{G^\vee}^\lambda$ with the inclusion $\overline{\mathrm{Gr}}_{G^\vee}^\lambda \rightarrow \mathrm{Gr}_{G^\vee}$ also by \mathbf{p} for brevity. The leftmost vertical morphism \mathbf{p}_{L_i} is the corresponding morphism for L_i . Furthermore $\mathcal{M}(\lambda, \mu)^{\chi_i}$ is isomorphic to $\mathcal{M}_{A_1}(\lambda', \mu')$ by [16, Lemma 5.5]. (Note that we have the maximal λ' as we are considering the case when the semisimple part of L_i is of type A_1 .) Since the fixed point set is naturally regarded a generalized slice for L_i , let us change the notation from $\mathcal{M}_{A_1}(\lambda', \mu')$ to $\mathcal{M}_{L_i}(\lambda', \mu')$. We also understand λ', μ' as coweights of L_i , instead of integers.

The multiplicity vector space $M_{\kappa', \mu'}^{\lambda, \mu}$ is the top degree Borel-Moore homology group of a subvariety $p_i^{-1}(x)$, where x is a point in the smooth locus of $\mathcal{M}_{L_i}(\kappa', \mu')$. Here κ' is understood as a coweight of L_i . The left square of (5.17) is Cartesian by [16, Lemma 5.11]. Hence $p_i^{-1}(x)$ is isomorphic to $\tilde{p}_i^{-1}(\mathbf{p}_{L_i}(x)) \cap \overline{\mathrm{Gr}}_{G^\vee}^\lambda$. When we move a point y in the $L_i[[z]]$ -orbit $\mathrm{Gr}_{L_i}^{\kappa'}$, the top degree Borel-Moore homology group of $\tilde{p}_i^{-1}(y) \cap \overline{\mathrm{Gr}}_{G^\vee}^\lambda$ forms a local system, which is trivial as the $\mathrm{Gr}_{L_i}^{\kappa'}$ is simply-connected. (This triviality has been used in the usual geometric Satake correspondence. See e.g., [9, §3.1].) As a consequence, we have

LEMMA 5.18. – *Let X be an irreducible subvariety in $\mathrm{Gr}_{L_i}^{\kappa'}$. There is a natural bijection between irreducible components of $\tilde{p}_i^{-1}(X) \cap \overline{\mathrm{Gr}}_{G^\vee}^\lambda$ and those of $\tilde{p}_i^{-1}(\mathrm{Gr}_{L_i}^{\kappa'}) \cap \overline{\mathrm{Gr}}_{G^\vee}^\lambda$.*

As an application, we obtain a bijection between irreducible components of $p_i^{-1}(x)$ and those of $p_i^{-1}(x^-)$ for $x \in \mathcal{M}_{L_i}^s(\kappa', \mu')$, $x^- \in \mathcal{M}_{L_i}^s(\kappa', \mu' - \alpha_i)$, as $\bar{\mu} - \bar{\alpha}_i = \bar{\mu}$ and κ' is common. Hence we get an isomorphism $M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu' - \alpha_i}^{\lambda, \mu - \alpha_i}$. It also shows Conjecture 1.5, as the local system given by the top degree Borel-Moore homology of $p_i^{-1}(x)$ is trivial over $\mathcal{M}_{L_i}^s(\kappa', \mu')$. In fact, Krylov also compared $p_{i*} j_i^! \mathrm{IC}(\mathcal{M}(\lambda, \mu))$ with $\tilde{p}_{i*} \tilde{j}_i^! \mathrm{IC}(\overline{\mathrm{Gr}}_{G^\vee}^\lambda)$ by homomorphisms associated with vertical morphisms in (5.17). See [16, Proof of Th. 3.4].

Since the usual geometric Satake correspondence is constructed so that it is compatible with the restriction to Levi subgroups (see e.g., [1, Prop. 5.3.29]), operators e_i, f_i are given by the same method as in §1(ii). Thus we only need to check that the above isomorphism $M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu' - \alpha_i}^{\lambda, \mu - \alpha_i}$ is the same as one given by the factorization.

Let us factorize $\mathcal{M}(\lambda, \mu - \alpha_i)$ over $(\mathbb{A}^1 \times \mathbb{G}_m^{\delta_i})_{\mathrm{disj}}$ as in (5.11). The factorization induces an isomorphism between $p_i^{-1}(x)$ and $p_i^{-1}(x^-)$, where x^- corresponds to (x, p) where p is

a generic point in $\mathcal{M}(\lambda, \lambda - \alpha_i)$. In particular, we have a bijection between irreducible components of $p_i^{-1}(x)$ and $p_i^{-1}(x^-)$. We need to check that this bijection is the same as one given by Lemma 5.18. Let X be an irreducible component of $p_i^{-1}(x)$ and X^- the corresponding one of $p_i^{-1}(x^-)$ under the factorization. We show the following.

CLAIM. – $\mathbf{p}(X)$ and $\mathbf{p}(X^-)$ are contained in the same irreducible component of $\tilde{p}_i^{-1}(\mathrm{Gr}_{L_i}^{\kappa'}) \cap \overline{\mathrm{Gr}}_{G^\vee}^\lambda$.

Proof of the claim. – By the definition of generalized slices in [7, §2(ii)], $\mathcal{M}(\lambda, \mu)$ is an open subvariety of a larger subvariety $\mathcal{G}\mathcal{Z}_\lambda^{-\mu}$, which is the moduli space of (a),(b),(c) above, but the B -structure in (c) could have defects in $\mathbb{P}^1 \setminus \{\infty\}$. It is equipped with $\mathbf{p}: \mathcal{G}\mathcal{Z}_\lambda^{-\mu} \rightarrow \mathrm{Gr}_{G^\vee}$ such that the above \mathbf{p} is its restriction. The diagram (5.17) extends to $\mathcal{G}\mathcal{Z}_\lambda^{-\mu}$. We regard X, X^- as irreducible components of $p_i^{-1}((\mathcal{G}\mathcal{Z}_\lambda^{-\mu})^{\chi_i})$ and $p_i^{-1}((\mathcal{G}\mathcal{Z}_\lambda^{-\mu+\alpha_i})^{\chi_i})$ respectively.

We have a locally closed embedding $\mathcal{M}(\lambda, \mu) \times \mathbb{A} \hookrightarrow \mathcal{G}\mathcal{Z}_\lambda^{-\mu+\alpha_i}$ by regarding the factor \mathbb{A} as a defect of the B -structure. The open subset $\mathcal{M}(\lambda, \mu) \times \mathbb{G}_m$ is contained in the image of the factorization isomorphism $\mathcal{G}\mathcal{Z}_\lambda^{-\mu+\alpha_i} \approx \mathcal{G}\mathcal{Z}_\lambda^{-\mu} \times (\mathbb{A} \times \mathbb{A})$ via $\mathbb{G}_m = \mathbb{G}_m \times \{0\} \subset \mathbb{A} \times \mathbb{A}$. In fact, the second factor $\mathbb{A} \times \mathbb{G}_m$ in (5.17) is the Coulomb branch for type A_1 quiver with $\mathbf{w} = 0, \mathbf{v} = 1$. It is the space of based maps $\mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree 1. By allowing defects as above, we get the corresponding zastava space, isomorphic to $\mathbb{A} \times \mathbb{A}$. Since we are restricting to the inverse image of $\mathbb{G}_m^{\delta_i}$ under the factorization morphism, the first \mathbb{A} factor is restricted to \mathbb{G}_m . Note also that the factorization for $\mathcal{G}\mathcal{Z}_\lambda^{-\mu+\alpha_i}$ is an extension of that for $\mathcal{M}(\lambda, \mu - \alpha_i)$ by its construction.

The restriction of $\mathbf{p}: \mathcal{G}\mathcal{Z}_\lambda^{-\mu+\alpha_i} \rightarrow \mathrm{Gr}_{G^\vee}$ to $\mathcal{M}(\lambda, \mu) \times \mathbb{G}_m$ is equal to $\mathbf{p}: \mathcal{M}(\lambda, \mu) \rightarrow \mathrm{Gr}_{G^\vee}$, composed with the first projection, as the defect has nothing to do with the data (a),(b). Therefore X^- contains a subvariety that is mapped to $\mathbf{p}(X)$ under \mathbf{p} . Hence the claim is proved. \square

Therefore the isomorphism $M_{\kappa', \mu'}^{\lambda, \mu} \cong M_{\kappa', \mu'-2}^{\lambda, \mu-\alpha_i}$ is the same as one given by the factorization. This completes the proof of Theorem 5.16.

Let us also mention that Kashiwara crystal structure on the set of irreducible components of the attracting set in $\mathcal{M}(\lambda, \mu)$ given in §5(vi) is the same as one constructed in [16] for the same reason as above.

Appendix

Fixed points and Maya diagrams

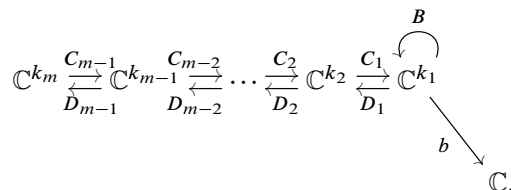
Let us choose a real parameter $v^\mathbb{R}$ so that $v_*^\mathbb{R} + v_h^\mathbb{R} < v_{h+1}^\mathbb{R}$ or $v_h^\mathbb{R} < v_{h+1}^\mathbb{R}$ according to $i(h)$ is connected to $i(x_0)$ through triangle parts or not. (See Remark 3.2.) A complex parameter $v^\mathbb{C}$ is arbitrary. It could be 0.

Because of this choice of $v^\mathbb{R}$, the condition **(v1)** is automatically satisfied, and **(v2)** says that a graded subspace T as in **(v2)** must be the whole V . In particular, $v^\mathbb{R}$ -semistability and $v^\mathbb{R}$ -stability are equivalent, and \mathcal{M}^v is smooth.

Let us study the torus fixed point set $(\mathcal{M}^v)^T$ as in Proposition 4.1. Let us also transform so that the bow diagram is of form (3.10). The data (A, B, C, D, a, b) decomposes into a direct

sum corresponding to \mathbb{C}_{x_i} ($0 \leq i \leq n-1$). And a summand corresponding to a bow diagram of a form (4.2).

Let us move x_i to the right by Hanany-Witten transitions until the vector space at the right of x_i becomes 0. Then we get data as



Let us first take the reduction by $\mathrm{GL}(k_2) \times \cdots \times \mathrm{GL}(k_m)$ and next take the reduction by $\mathrm{GL}(k_1)$. By the first step we get the product of (a) a quiver variety of type A_{m-1} with dimension vectors $\underline{v} = (k_m, \dots, k_2)$, $\underline{w} = (0, \dots, 0, k_1)$, and (b) a pair (B, b) of a $k_1 \times k_1$ -matrix and a co-vector in \mathbb{C}^{k_1} such that b is co-cyclic with respect to B . In the second step we set $B = -C_1 D_1$ and take the quotient by $\mathrm{GL}(k_1)$.

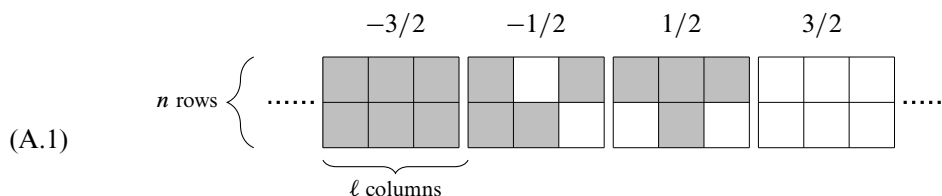
By the standard argument (cf. [21, Th. 7.3]) D_i ($1 \leq i \leq m-1$) is surjective. In particular, $\dim S_i$ is decreasing. However $C_1 D_1 = -B$ has a cocyclic vector, we must have $k_s - k_{s+1} = 0$ or 1. We fill \circ for two way parts with $k_s - k_{s+1} = 1$ as \bullet .

The defining equation determines the characteristic polynomial of $C_1 D_1$ (e.g., it is z^{k_1} if the complex parameter $v^\mathbb{C}$ is 0). Then $B = -C_1 D_1$ together with b forms a single free $\mathrm{GL}(k_1)$ -orbit. Note also C_1, D_1, \dots , are determined automatically once k_1, k_2, \dots , are specified, if B is fixed. Namely the corresponding bow variety is a single point.

Returning back to (4.2) by Hanany-Witten transitions, we find that \circ for h is filled as \bullet if and only if

$$\begin{cases} N_h = 1 & \text{if } h = h_j \text{ with } j > 0, \\ N_h = 0 & \text{if } h = h_j \text{ with } j \leq 0. \end{cases}$$

Though we move x_i only finite amount, we extend the above rule to any \circ . Hence we have a sequence of \circ 's going to infinite in both left and right with some filled as \bullet such that they are filled for h_j with sufficiently negative j , not filled sufficiently positive j . We have an infinite sequence for each x_i ($0 \leq i \leq n-1$), and we arrange them as follows:



where h_1 is the first column in the block $1/2$, h_2 is the second one, and h_0 is the last column in the block $-1/2$, and so on. This is a variant of a Maya diagram.

Conversely a diagram above gives a torus fixed point: Reading $(i+1)$ -th row, we determine the bow diagram corresponding to x_i ($0 \leq i \leq n-1$) including dimensions $R(\zeta)$. Then we take the sum over i .

Note that $R(o(x_i))$ (resp. $R(i(x_i))$) is equal to the number of \blacksquare (resp. \square) in blocks $1/2, 3/2, \dots$ (resp. $-1/2, -3/2, \dots$) of x_i . In particular, N_{x_i} is the difference between these numbers.

Since a summand corresponding to x_j ($j \neq i$) has isomorphic A_{x_i} , N_{x_i} is the same for this summand and the original bow diagram. In other words, if a diagram corresponds to a T -fixed point in \mathcal{M}^v of a bow diagram of form (3.10), we have a constraint

$$(A.2) \quad N_{x_i} = \begin{array}{l} \text{the number of } \blacksquare \text{ in the } (i+1)\text{-th row in blocks } 1/2, 3/2, \dots \\ - \text{the number of } \square \text{ in the } (i+1)\text{-th row in blocks } -1/2, -3/2, \dots \end{array}$$

Let us consider numbers of \blacksquare , \square in each column on the other hand. In blocks $1/2, 3/2, \dots$, \blacksquare contributes 1 to N_h . In blocks $-1/2, -3/2, \dots$, \square contributes -1 . Therefore

$$(A.3) \quad N_{h_\sigma} = \begin{array}{l} \text{the number of } \blacksquare \text{ in the } \sigma\text{-th column in blocks } 1/2, 3/2, \dots \\ - \text{the number of } \square \text{ in the } \sigma\text{-th row in blocks } -1/2, -3/2, \dots \end{array}$$

Though N_{x_i}, N_{h_σ} determine $R(\zeta)$ only up to overall shifts, the number \mathbf{v}_0 in (3.10) is given by

$$(A.4) \quad \mathbf{v}_0 = \begin{array}{l} \text{(the number of } \square \text{ in blocks } -1/2) \\ + 2(\text{the number of } \square \text{ in blocks } -3/2) \\ + 3(\text{the number of } \square \text{ in blocks } -5/2) + \dots \end{array}$$

Therefore

THEOREM A.5. – *The fixed point set $(\mathcal{M}^v)^T$ is in bijection to the set of diagrams (A.1) with constraint (A.2), (A.3) and (A.4).*

In particular, we have a natural bijection between fixed point sets of the bow variety and another bow variety given by the bow diagram with \times , \circ swapped. It includes the case of Higgs and Coulomb branches of the same quiver gauge theory of affine type A . This should be equal to the natural bijection given as a consequence of Hikita conjecture [13, 29].

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