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of conformal blocks from vertex algebras*

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ON FACTORIZATION AND VECTOR BUNDLES
OF CONFORMAL BLOCKS
FROM VERTEX ALGEBRAS

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AND Nicola TARASCA

ABSTRACT. – Representations of vertex operator algebras define sheaves of coinvariants and conformal blocks on moduli of stable pointed curves. Assuming certain finiteness and semisimplicity conditions, we prove that such sheaves satisfy the factorization conjecture and consequently are vector bundles. Factorization is essential to a recursive formulation of invariants, like ranks and Chern classes, and to produce new constructions of rational conformal field theories and cohomological field theories.

RÉSUMÉ. – Les représentations des algèbres d'opèrateurs vertex définissent des faisceaux de coinvariants et de blocs conformes sur des modules de courbes pointées stables. En supposant certaines conditions de finitude et de semi-simplicité, nous prouvons que de tels faisceaux satisfont la conjecture de factorisation et sont par conséquent des fibrés vectoriels. La factorisation est essentielle à une formulation récursive des invariants, comme les rangs et les classes de Chern, et à produire de nouvelles constructions de théories conformes rationnelles des champs et de théories cohomologiques des champs.

By assigning a module over a vertex operator algebra to each marked point on a stable pointed curve, one can construct dual vector spaces of coinvariants and conformal blocks, giving rise to sheaves on moduli spaces of stable pointed curves. The main result of this paper is that these sheaves satisfy the factorization property (Theorem 7.0.1), as conjectured in [76, 38]. Namely, if certain finiteness and semisimplicity conditions hold, vector spaces of coinvariants and conformal blocks at a nodal curve decompose as products of analogous spaces at each component of its normalization.

We also show that sheaves of coinvariants satisfy the sewing property (Theorem 8.5.1), a refined version of factorization at infinitesimal smoothings of nodal curves. From this and their projectively flat connection on families of smooth curves [21], we deduce they are vector bundles (VB corollary).

Our findings generalize a number of results known in special cases and for low genus. A historical account with references is given in §§0.2 and 0.3.

Factorization leads to recursive formulas for ranks and is used to show that the Chern characters define semisimple cohomological field theories, hence the Chern classes lie in the

tautological ring ([22], building on results for affine Lie algebras from [62, 63]). A study of these tautological classes may lead to progress on open questions: As proposed by Pandharipande [69], a computation of such Chern classes independently of the projective flatness of the connection [21] would yield relations in the tautological ring, and could be used to test Pixton's conjecture [70, 52].

For sheaves defined by integrable modules over affine Lie algebras, vector spaces of conformal blocks are canonically isomorphic to generalized theta functions (see [55] and references therein). When in addition the genus is zero, vector bundles of such coinvariants are globally generated [36], hence their Chern classes have positivity properties. For instance, first Chern classes are base-point-free and thus give rise to morphisms, some with images having modular interpretations [45, 46]. It is natural to expect that the more general vector bundles of coinvariants and conformal blocks studied here can be shown to have analogous properties under appropriate assumptions. This has been supported by a preliminary investigation [20].

To outline our results, we set some notation. We refer to a stable pointed coordinatized curve as a triple $(C, P_\bullet, t_\bullet)$ where $(C, P_\bullet)$ is a stable n-pointed curve, $P_\bullet = (P_1, \ldots, P_n)$, and $t_\bullet = (t_1, \ldots, t_n)$ with t_i a formal coordinate at the point P_i. Let $M^\bullet = (M^1, \ldots, M^n)$ be an n-tuple of finitely generated admissible modules over a vertex operator algebra V (see §§1.1-1.2). When $C \setminus P_\bullet$ is affine, the vector space of coinvariants $\mathbb{V}(V; M^\bullet)_{(C, P_\bullet, t_\bullet)}$ is defined as the largest quotient of the tensor product $\bigotimes_{i=1}^n M^i$ by the action of a Lie algebra determined by V and $(C, P_\bullet, t_\bullet)$, see §4.2. In general, by adding more marked points, one can reduce to the case when $C \setminus P_\bullet$ is affine, see (30). Before now, two Lie algebras have been used to define coinvariants: Zhu's Lie algebra $\mathfrak{g}_{C \setminus P_\bullet}(V)$ and the (former) chiral Lie algebra $\mathscr{L}_{C \setminus P_\bullet}(V)$. Here, we introduce a new chiral Lie algebra $\mathcal{L}_{C \setminus P_\bullet}(V)$.

Zhu's Lie algebra (§A.1) is defined when the vertex algebra V is quasi-primary generated and $\mathbb{Z}_{\geq 0}$-graded with lowest degree space of dimension one, for either *fixed* smooth curves [76, 3], or for *rational* stable pointed curves *with coordinates* [67]. To show that $\mathfrak{g}_{C \setminus P_\bullet}(V)$ is a Lie algebra, Zhu uses that any *fixed* smooth algebraic curve admits an atlas such that all transition functions are Möbius transformations. Transition functions between charts on families of curves of arbitrary genus are more complicated, and for an arbitrary vertex operator algebra, $\mathfrak{g}_{C \setminus P_\bullet}(V)$ is not well-defined.

The chiral Lie algebra $\mathscr{L}_{C \setminus P_\bullet}(V)$ is available for curves of arbitrary genus and for more general vertex operator algebras V, not necessarily quasi-primary generated. Defined for smooth pointed curves by Frenkel and Ben-Zvi [38, §19.4.14] and shown in [38] to coincide with that studied by Beilinson and Drinfeld [14], the chiral Lie algebra was extended to nodal curves in [21], enabling the construction of coinvariants on such curves.

In §3, we introduce and study a new chiral Lie algebra $\mathcal{L}_{C \setminus P_\bullet}(V)$, whose coinvariants are projectively flat over $\mathcal{M}_{g,n}$, and additionally satisfy factorization under some natural hypotheses. See §3.3 for a description of $\mathcal{L}_{C \setminus P_\bullet}(V)$ on nodal curves.

In [67], Nagatomo and Tsuchiya remark that the coinvariants on rational curves with coordinates using Zhu's Lie algebra are equivalent to those considered by Beilinson and Drinfeld. In Proposition A.2.1, we verify their genus zero statement, and further show that coinvariants from the chiral Lie algebra are isomorphic to those given by $\mathfrak{g}_{C \setminus P_\bullet}(V)$ whenever Zhu's Lie algebra is defined. For instance, when V is quasi-primary generated and one

works over a family of curves admitting an atlas where all transition functions are Möbius transformations, both perspectives are equivalent.

To state our main result, let $(C, P_\bullet, t_\bullet)$ be a stable pointed coordinatized curve, as above, with one node Q. Let $\widetilde{C} \to C$ be the normalization, $Q_\bullet := (Q_+, Q_-)$ the pair of preimages of Q in $\widetilde{C}$, and $s_\bullet := (s_+, s_-)$ with $s_\pm$ a formal coordinate at $Q_\pm$. Let $\mathscr{W}$ be a set of representatives of isomorphism classes of simple V-modules, and for $W \in \mathscr{W}$, let W' be its contragredient module (§1.8).

FACTORIZATION THEOREM (Theorem 7.0.1). – *Let V be a rational, C_2-cofinite vertex operator algebra with one-dimensional weight zero space, and let $M^\bullet$ be an n-tuple of finitely-generated admissible V-modules. One has:*

$$(1) \qquad \mathbb{V}\left(V; M^\bullet\right)_{(C, P_\bullet, t_\bullet)} \cong \bigoplus_{W \in \mathscr{W}} \mathbb{V}\left(V; M^\bullet \otimes W \otimes W'\right)_{(\widetilde{C}, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)} .$$

If $\widetilde{C} = C_+ \sqcup C_-$ is disconnected, with $Q_\pm \in C_\pm$, then (1) is isomorphic to

$$\bigoplus_{W \in \mathscr{W}} \mathbb{V}\left(V; M_+^\bullet \otimes W\right)_{X_+} \otimes \mathbb{V}\left(V; M_-^\bullet \otimes W'\right)_{X_-} ,$$

where $X_\pm := (C_\pm, P_\bullet|_{C_\pm} \sqcup Q_\pm, t_\bullet|_{C_\pm} \sqcup s_\pm)$, and $M_\pm^\bullet$ are the modules at those points $P_\bullet$ that are in $C_\pm$.

The isomorphism giving the factorization theorem is constructed in §7.

Propositions 3.3.1, 5.1.1, and 6.2.1 are at the heart of this work, and arise from the construction in §2 of the sheaf $\mathcal{V}_C$ globalizing a vertex algebra V over a nodal curve C, integral to the definition of the chiral Lie algebra. The sheaf $\mathcal{V}_C$ is a slight variant on the sheaf $\mathscr{V}_C$ that we defined in [21]. The two sheaves coincide for smooth curves, and as we explain here, we recover the results of [21] using $\mathcal{V}_C$ in place of $\mathscr{V}_C$. So while the projectively flat connection can be obtained in both constructions (see Proposition 2.7.3 for $\mathcal{V}_C$), to have factorization, we have used $\mathcal{V}_C$ (see §2.5, and §2.6). In Proposition 3.3.1, we explicitly describe the chiral Lie algebra on a nodal curve in terms of elements of the chiral Lie algebra on its normalization. This involves stable k-differentials that satisfy an infinite number of identities.

In Proposition 5.1.1, we show that vector spaces of coinvariants defined from the chiral Lie algebra and smooth curves of arbitrary genus are finite-dimensional. Known to be true in special cases, this result is a natural generalization of work of Abe and Nagatomo [3] for coinvariants defined from Zhu's Lie algebra and smooth curves with formal coordinates (see §0.3 for the history of the problem). As in [3], we assume here that V is C_2-cofinite, which implies C_k-cofiniteness for $k \geq 1$ [17, 54].

Proposition 6.2.1 reinterprets coinvariants at a nodal curve as coinvariants on the normalization by the action of a Lie subalgebra of the chiral Lie algebra. The proof of this result, for which we assume V is C_1-cofinite, uses that lowest weight V-modules admit spanning sets of PBW-type, following [54, Cor. 3.12].

In §8.7, following Tsuchimoto [72] in the case of simple affine vertex algebras, we show the sheaf of coinvariants $\mathbb{V}(V; M^\bullet)$ is coordinate-free and descends to the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g. As an application of the factorization theorem, we show:

VB COROLLARY. – *Let V be a rational, C_2-cofinite vertex operator algebra with one-dimensional weight zero space, and $M^\bullet$ be an n-tuple of finitely-generated admissible V-modules. Then $\mathbb{V}(V; M^\bullet)$ is a vector bundle of finite rank on $\overline{\mathcal{M}}_{g,n}$.*

The proof of the VB corollary is presented in §8.8. The result on the interior of $\overline{\mathcal{M}}_{g,n}$, the moduli space $\mathcal{M}_{g,n}$ of smooth pointed curves, follows from finite-dimensionality of coinvariants (Proposition 5.1.1) and the existence of a projectively flat connection [21]. Two ingredients are needed to give VB corollary on the whole space $\overline{\mathcal{M}}_{g,n}$: Theorem 8.4.2, a more general result on finite-dimensionality of coinvariants, and the sewing property (Theorem 8.5.1). Each of these involve evaluation of the sheaf of coinvariants on a family formed by infinitesimally smoothing a nodal curve (§8.1). The proof of the sewing theorem (§8) relies on the factorization theorem and a sewing procedure originally found in [74, §6.2].

0.1. Results on sheaves of coinvariants from [21]

As stated, the vertex algebra sheaf $\mathcal{V}_C$ defined in §2 is a slight variant on the sheaf $\mathscr{V}_C$ defined in [21]. These give rise to sheaves of chiral Lie algebras $\mathcal{L}_{C \backslash P_\bullet}(V)$ and $\mathscr{L}_{C \backslash P_\bullet}(V)$, respectively, which in turn define two sheaves of coinvariants on the moduli space of curves, potentially different on the boundary. As these sheaves agree on the interior $\mathcal{M}_{g,n}$, i.e., the moduli space of smooth pointed curves, the results of [21] on $\mathcal{M}_{g,n}$ hold for the sheaves of coinvariants defined here. In particular, the identification of the Atiyah algebra acting on sheaves of coinvariants on $\mathcal{M}_{g,n}$ [21, Theorem], and consequently, the computation of their Chern character on $\mathcal{M}_{g,n}$ [21, Corollary] hold here. Combining these results with the factorization theorem, we show in [22] that the sheaves of coinvariants defined here in the case of self-contragredient simple vertex algebras give rise to semisimple cohomological field theories, thus allowing one to determine their Chern character on $\overline{\mathcal{M}}_{g,n}$ in terms of the fusion rules.

A natural question is whether the two sheaves of coinvariants coincide on the whole of $\overline{\mathcal{M}}_{g,n}$. In particular, it is natural to ask whether sheaves of coinvariants defined by $\mathscr{L}_{C \backslash P_\bullet}(V)$ satisfy factorization.

0.2. History of factorization and sewing

Tsuchiya and Kanie used integrable modules at a fixed level over affine Lie algebras to form spaces of coinvariants on smooth pointed rational curves with coordinates [73]. Generalized by Tsuchiya, Ueno, and Yamada to moduli of stable pointed coordinatized curves of arbitrary genus, these sheaves were shown to satisfy a number of good properties including factorization and sewing [74]. Tsuchimoto [72] proved the bundles are independent of coordinates and descend to $\overline{\mathcal{M}}_{g,n}$. Beilinson, Feigin, and Mazur [15] showed that factorization holds for coinvariants defined by modules over Virasoro algebras. Our arguments follow [67] in the genus zero case after our study of the chiral Lie algebra allows one to replace Zhu's Lie algebra in the general case.

Sewing was proved for $g \in \{0, 1\}$ by Huang [47, 48, 50, 51]. His approach was to prove the operator product expansion and the modular invariance of intertwining operators, both conjectured by Moore and Seiberg [66]. Huang assumes that (i) $V = \bigoplus_{i \geq 0} V_i$ with $V_0 \cong \mathbb{C}$; (ii) Every $\mathbb{N}$-gradable weak V-module is completely reducible; and (iii) V is C_2-cofinite.

Our assumptions are (i); (ii′) V is rational; and (iii). Conditions (ii) and (iii) of Huang are equivalent to our conditions (ii′) and (iii) (see §1.7 for more details). Huang shows that if one assumes in addition (iv) $V \cong V'$, then the modular tensor categories he constructs for $g \in \{0, 1\}$ are *rigid* and *nondegenerate*. In case (i)-(iv) hold, V is sometimes called *strongly rational*.

Codogni in [19] proves factorization in case our hypotheses hold, with the additional assumptions (iv′) $V \cong V'$ and (v) V has no nontrivial modules. Like Nagatomo and Tsuchiya, Codogni works with coinvariants defined on the moduli space of curves with formal coordinates. We note that the additional assumption (iv′) gives that V is quasi-primary generated and in particular, coinvariants defined from Zhu's Lie algebra and the chiral Lie algebra are both well-defined and agree (see §).

Here we do not assume condition (iv) nor condition (v). There are examples satisfying (1–3) but not (v), see §9. Furthermore, through private communications with Sven Möller, we have learned the existence of a vertex operator algebra that satisfies our conditions (1–3) but not condition (iv). Therefore, while establishing factorization for all $g \geq 0$, our work also covers new examples of factorization for $g \in \{0, 1\}$.

0.3. History of coherence

In [38, page 3 and §5.5.4] the authors single out rational vertex algebras as good candidates for defining finite-dimensional coinvariants (and hence leading to finite-rank vector bundles). At that time, rationality and C_2-cofiniteness were conjectured to be equivalent conditions [27, 2]. This has been disproved, as Abe has given a C_2-cofinite, non-rational vertex operator algebra [1].

Coherence of sheaves of coinvariants was shown previously in special cases: for (1) integrable modules at positive integral levels over affine Lie algebras [74]; (2) modules over C_2-cofinite Virasoro vertex algebras [15]; (3) curves of genus zero and modules over C_2-cofinite vertex operator algebras $V = \bigoplus_{i \in \mathbb{N}} V_i$ such that $V_0 \cong \mathbb{C}$ [67]; (4) fixed smooth curves of positive genus and modules over a quasi-primary generated, C_2-cofinite vertex operator algebras $V = \bigoplus_{i \in \mathbb{N}} V_i$ such that $V_0 \cong \mathbb{C}$ [3]. After the first draft of this paper, [35] showed finite-dimensionality of spaces of coinvariants associated with trivial modules over elliptic curves under a weaker assumption than C_2-cofiniteness. In [20] it is shown that if V is generated in degree 1 and $g = 0$, then coinvariants are finite-dimensional.

1. Background

1.1. Vertex operator algebras

We work with a non-negatively graded *vertex operator algebra*, that is, a 4-tuple $(V, 1^V, \omega, Y(\cdot, z))$, throughout simply denoted V for short, such that:

1. $V = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} V_i$ is a vector space over $\mathbb{C}$ with dim $V_i < \infty$;

2. $1^V \in V_0$ (the *vacuum vector*), and $\omega \in V_2$ (the *conformal vector*);

3. $Y(\cdot, z) \colon V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$ is a linear function assigning to every element $A \in V$ the *vertex operator* $Y(A, z) := \sum_{i \in \mathbb{Z}} A_{(i)} z^{-i-1}$.

The datum $(V, 1^V, \omega, Y(\cdot, z))$ must satisfy the following axioms:

1. *(vertex operators are fields)* for all $A, B \in V$, $A_{(i)}B = 0$, for $i \gg 0$;

2. *(vertex operators of the vacuum)* $Y(1^V, z) = \mathrm{id}_V$:

$$1^V_{(-1)} = \mathrm{id}_V \qquad \text{and} \qquad 1^V_{(i)} = 0, \quad \text{for } i \neq -1,$$

and for all $A \in V$, $Y(A, z)1^V \in A + zV[\![z]\!]$:

$$A_{(-1)}1^V = A \qquad \text{and} \qquad A_{(i)}1^V = 0, \qquad \text{for } i \geq 0;$$

3. *(weak commutativity)* for all $A, B \in V$, there exists an $N \in \mathbb{Z}_{\geq 0}$ such that

$$(z - w)^N [Y(A, z), Y(B, w)] = 0 \quad \text{in } \mathrm{End}(V)[\![z^{\pm 1}, w^{\pm 1}]\!];$$

4. *(conformal structure)* for $Y(\omega, z) = \sum_{i \in \mathbb{Z}} \omega_{(i)} z^{-i-1}$,

$$\left[\omega_{(p+1)}, \omega_{(q+1)}\right] = (p - q)\, \omega_{(p+q+1)} + \frac{c}{12} \delta_{p+q, 0}\, (p^3 - p)\, \mathrm{id}_V.$$

Here $c \in \mathbb{C}$ is the *central charge* of V. Moreover:

$$\omega_{(1)}\big|_{V_i} = i \cdot \mathrm{id}_V, \quad \text{for all } i, \qquad \text{and} \qquad Y\left(\omega_{(0)}A, z\right) = \partial_z Y(A, z).$$

1.1.1. *Action of Virasoro.* – As we next explain, the conformal structure encodes an action of the Virasoro (Lie) algebra Vir on V. The *Witt (Lie) algebra* $\mathrm{Der}\,\mathcal{K}$ represents the functor assigning to a $\mathbb{C}$-algebra R the Lie algebra $\mathrm{Der}\,\mathcal{K}(R) := R(\!(z)\!)\partial_z$ generated over R by the derivations $L_p := -z^{p+1}\partial_z$, for $p \in \mathbb{Z}$, with relations $[L_p, L_q] = (p-q)L_{p+q}$. The *Virasoro (Lie) algebra* Vir represents the functor assigning to R the Lie algebra generated over R by a formal vector K and the elements L_p, for $p \in \mathbb{Z}$, with Lie bracket given by

$$[K, L_p] = 0, \qquad [L_p, L_q] = (p - q)L_{p+q} + \frac{K}{12}(p^3 - p)\delta_{p+q, 0}.$$

Setting $L_p = \omega_{(p+1)} \in \mathrm{End}(V)$, Axiom (4) gives an action of Vir on V with *central charge* $c \in \mathbb{C}$, that is, $K \in \mathrm{Vir}$ acts as $c \cdot \mathrm{id}_V$.

1.1.2. *Degree of $A_{(i)}$.* – As a consequence of the axioms, one has $A_{(i)}V_k \subseteq V_{k+d-i-1}$ for homogeneous $A \in V_d$ (see e.g., [77]). Thus, we have

(2) $$\deg A_{(i)} := \deg(A) - i - 1, \qquad \text{for homogeneous } A \text{ in } V.$$

Axiom (4) implies that L_0 acts as a *degree* operator on V, and L_{-1}, called the *translation* operator, is determined by $L_{-1}A = A_{(-2)}1^V$, for $A \in V$.

1.2. Modules of vertex operator algebras

Let $(V, 1^V, \omega, Y(\cdot, z))$ be a vertex operator algebra. A *weak V-module* is a pair $(M, Y^M(\cdot, z))$, where:

1. M is a vector space over $\mathbb{C}$;

2. $Y^M(\cdot, z): V \to \mathrm{End}(M)[\![z, z^{-1}]\!]$ is a linear function that assigns to $A \in V$ an $\mathrm{End}(M)$-valued *vertex operator* $Y^M(A, z) := \sum_{i \in \mathbb{Z}} A^M_{(i)} z^{-i-1}$.

The pair $(M, Y^M(\cdot, z))$ must satisfy the following axioms:

1. for all $A \in V$ and $v \in M$, one has $A^M_{(i)}v = 0$, for $i \gg 0$;

2. $Y^M\left(1^V, z\right) = \mathrm{id}_M$;

3. for all $A, B \in V$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that for all $v \in M$ one has

$$(z - w)^N \left[Y^M(A, z), Y^M(B, w)\right] v = 0;$$

4. for all $A \in V$ and $v \in M$, there exists $N \in \mathbb{Z}_{\geq 0}$, such that for all $B \in V$ one has

$$(w + z)^N \left(Y^M(Y(A, w)B, z) - Y^M(A, w + z)Y^M(B, z)\right) v = 0;$$

5. for $Y^M(\omega, z) = \sum_{i \in \mathbb{Z}} \omega_{(i)}^M z^{-i-1}$, one has

$$\left[\omega_{(p+1)}^M, \omega_{(q+1)}^M\right] = (p - q)\,\omega_{(p+q+1)}^M + \frac{c}{12}\,\delta_{p+q,0}\,(p^3 - p)\,\mathrm{id}_M,$$

where $c \in \mathbb{C}$ is the central charge of V. We identify $\omega_{(p+1)} \in \mathrm{End}(M)$ with an action of $L_p \in \mathrm{Vir}$ on M. Moreover $Y^M(L_{-1}A, z) = \partial_z Y^M(a, z)$.

In the literature, axiom (3) is referred to as *weak commutativity*, and axiom (4) as *weak associativity*. Weak associativity and weak commutativity are known to be equivalent to the *Jacobi identity* (see e.g., [25, 41, 56, 57]). Moreover, by [27, Lemma 2.2], axiom (5) is redundant.

Throughout, by V-modules we mean admissible V-modules. These are weak V-modules such that

1. $M = \bigoplus_{i \in \mathbb{N}} M_i$ is $\mathbb{N}$-graded and $A_{(i)}^M M_k \subseteq M_{k+\deg(A)-i-1}$ for every homogeneous $A \in V$.

Note that V is a module over itself (see [56, Thm 3.5.4] or [38, §3.2.1]). We give a description of V-modules in §1.5.

1.3. The Lie algebra ancillary to V

Given a formal variable t, we call

$$\mathfrak{L}_t(V) := \left(V \otimes \mathbb{C}((t))\right)/\mathrm{Im}\,\partial$$

the *Lie algebra* $\mathfrak{L}(V) = \mathfrak{L}_t(V)$ *ancillary to* V. Here

$$(3) \qquad\qquad \partial := L_{-1} \otimes \mathrm{id}_{\mathbb{C}((t))} + \mathrm{id}_V \otimes \partial_t.$$

The space $\mathfrak{L}(V)$ is spanned by series of type $\sum_{i \geq i_0} c_i A_{[i]}$, for $A \in V$, $c_i \in \mathbb{C}$, and $i_0 \in \mathbb{Z}$, where $A_{[i]}$ denotes the projection in $\mathfrak{L}(V)$ of $A \otimes t^i \in V \otimes \mathbb{C}((t))$. The Lie bracket of $\mathfrak{L}(V)$ is induced by

$$(4) \qquad\qquad \left[A_{[i]}, B_{[j]}\right] := \sum_{k \geq 0} \binom{i}{k} \left(A_{(k)} \cdot B\right)_{[i+j-k]}.$$

The axiom on the vacuum vector 1^V implies that $1_{[-1]}^V$ is central. The Lie algebra $\mathfrak{L}(V)$ is isomorphic to the current Lie algebra in [67]. For t we will use a formal coordinate at a point P on a curve, and we will denote $\mathfrak{L}_P(V) = \mathfrak{L}_t(V)$. A coordinate-free description of $\mathfrak{L}_t(V)$ is discussed in §2.8.

1.4. The universal enveloping algebra

For a vertex algebra V, there is a complete topological associative algebra $\mathscr{U}(V)$, defined originally by Frenkel and Zhu [43]. We review it here following the presentation in [38]. Consider the universal enveloping algebra $U(\mathfrak{L}(V))$ of $\mathfrak{L}(V)$, and its completion

$$\widetilde{U}(\mathfrak{L}(V)) := \varprojlim_N U(\mathfrak{L}(V))/I_N,$$

where I_N is the left ideal generated by $A_{[i]}$, for homogeneous $A \in V$ and $i \geq N + \deg(A)$. The *universal enveloping algebra* $\mathscr{U}(V)$ of V is defined as the quotient of $\widetilde{U}(\mathfrak{L}(V))$ by the two-sided ideal generated by the Fourier coefficients of the series

$$Y\left[A_{(-1)}B, z\right] - \; :Y[A,z]\,Y[B,z]:, \quad \text{for all } A, \; B \in V,$$

where $Y[A,z] = \sum_{i \in \mathbb{Z}} A_{[i]} z^{-i-1}$ and ": :" is the normal ordering (see [40]).

For an affine vertex algebra $V = V_\ell(\mathfrak{g})$, one has $\mathscr{U}(V)$ is isomorphic to a completion $\widetilde{U}_\ell(\widehat{\mathfrak{g}})$ of $U_\ell(\widehat{\mathfrak{g}})$, for all $\ell \in \mathbb{C}$. Here, $U_\ell(\widehat{\mathfrak{g}})$ is the quotient of $U(\widehat{\mathfrak{g}})$ by the two-sided ideal generated by $K - \ell$, where $K \in \widehat{\mathfrak{g}}/\mathfrak{g} \otimes \mathbb{C}((t))$, and

$$\widetilde{U}_\ell(\widehat{\mathfrak{g}}) := \varprojlim_N U_\ell(\widehat{\mathfrak{g}})/U_\ell(\widehat{\mathfrak{g}}) \cdot \mathfrak{g} \otimes t^N \mathbb{C}[\![t]\!].$$

For more detail and other examples see [38, §4.3.2, §5.1.8].

1.5. Action on V-modules

Both $\mathfrak{L}(V)$ and $\mathscr{U}(V)$ act on any V-module M via the Lie algebra homomorphism $\mathfrak{L}(V) \to \operatorname{End}(M)$ obtained by mapping $A_{[i]}$ to the Fourier coefficient $A_{(i)}$ of the vertex operator $Y^M(A,z) = \sum_i A_{(i)} z^{-i-1}$. Thus, the series $\sum_{i \geq i_0} c_i A_{[i]}$ acts on a V-module M via

$$\operatorname{Res}_{z=0} Y^M(A,z) \sum_{i \geq i_0} c_i z^i \, dz.$$

An $\mathfrak{L}(V)$-module need not be a V-module. On the other hand, there is an equivalence between the categories of weak V-modules and smooth $\mathscr{U}(V)$-modules (see [38, §5.1.6]). A $\mathscr{U}(V)$-module M is *smooth* if for any $w \in M$ and $A \in V$, one has $A_{[i]}w = 0$ for $i \gg 0$.

A V-module is *finitely generated* if it is finitely generated as a $\mathscr{U}(V)$-module. The modules in [67] are also finitely generated and admissible.

1.6. Correspondence between V-modules and $A(V)$-modules

A V-module W is *irreducible*, or *simple*, if it is non zero and it has no sub-representation other than the trivial representation 0 and W itself. We review here Zhu's associative algebra $A(V)$, and the one-to-one correspondence between isomorphism classes of finite-dimensional simple $A(V)$-modules and isomorphism classes of simple V-modules [77].

Zhu's algebra is the quotient $A(V) := V/O(V)$, where $O(V)$ is the subspace of V linearly spanned by elements of the form

$$\operatorname{Res}_{z=0} \frac{(1+z)^{\deg A}}{z^2} Y(A,z)B,$$

where A is homogeneous in V. The image of an element $A \in V$ in $A(V)$ is denoted by $o(A)$. The product in $A(V)$ is defined by

$$o(A) * o(B) = \text{Res}_{z=0} \frac{(1+z)^{\deg A}}{z} Y(A, z) B,$$

for homogeneous A in V. Nagatomo and Tsuchiya [67] consider an isomorphic copy of $A(V)$, which they refer to as the *zero-mode algebra*.

Given a V-module $W = \bigoplus_{i \geq 0} W_i$, one has that W_0 is an $A(V)$-module [77, Thm 2.2.2]. The action of $A(V)$ on W_0 is defined as follows: an element $o(A) \in A(V)$, image of a homogeneous element $A \in V$, acts on W_0 as the endomorphism $A_{(\deg A - 1)}$, a Fourier coefficient of $Y^W(A, z)$.

For the other direction, recall the triangular decomposition:

(5) $$\mathfrak{L}(V) = \mathfrak{L}(V)_{<0} \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_{>0},$$

determined by the degree $\deg(A_{[i]}) := \deg(A) - i - 1$, for homogeneous $A \in V$. From the Lie bracket (4) of $\mathfrak{L}(V)$, one checks that each summand above is a Lie subalgebra of $\mathfrak{L}(V)$. This induces a subalgebra $\mathscr{U}(V)_{\leq 0}$ of $\mathscr{U}(V)$.

Given a finite-dimensional $A(V)$-module E, the *generalized Verma $\mathscr{U}(V)$-module* is

$$M(E) := \mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E.$$

To make E into an $\mathscr{U}(V)_{\leq 0}$-module, one lets $\mathfrak{L}(V)_{<0}$ act trivially on E, and $\mathfrak{L}(V)_0$ act by the homomorphism of Lie algebras $\mathfrak{L}(V)_0 \twoheadrightarrow A(V)_{\text{Lie}}$ induced by the identity endomorphism of V [58, Lemma 3.2.1]. For homogeneous $A \in V_k$, the image of the element $A_{[k-1]} \in \mathfrak{L}(V)_0$ in $A(V)$ is $o(A)$. By construction, $M(E)$ is automatically a V-module.

Given an *irreducible* V-module $W = \bigoplus_{i \geq 0} W_i$, the space W_0 is a finite-dimensional *irreducible* $A(V)$-module; conversely, given a finite-dimensional *irreducible* $A(V)$-module E, there is a unique maximal proper V-submodule $N(E)$ of the V-module $M(E)$ with $N(E) \cap E = 0$ such that $L(E) = M(E)/N(E)$ is an *irreducible* V-module [77].

1.7. Rational vertex algebras

A vertex algebra V is *rational* if every finitely generated V-module is a direct sum of irreducible V-modules. A rational vertex algebra has only finitely many isomorphism classes of irreducible modules, and an irreducible module $M = \bigoplus_{i \geq 0} M_i$ over a rational vertex algebra satisfies $\dim M_i < \infty$ [28].

An *ordinary* V-module is a weak V-module which carries a $\mathbb{C}$-grading $M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$ such that: $L_0|_{M_\lambda} = \lambda \, \text{id}_{M_\lambda}$; $\dim M_\lambda < \infty$; and for fixed λ, one has $M_{\lambda+n} = 0$, for integers $n \ll 0$. Ordinary V-modules are admissible, and when V is rational, every simple admissible V-module is ordinary [28], [27, Rmk 2.4]. It follows that for rational V, finitely generated V-modules are direct sums of simple ordinary V-modules. In particular, a finitely generated admissible module $M = \bigoplus_{i \geq 0} M_i$ over a rational vertex algebra satisfies $\dim M_i < \infty$, for all i, and L_0 acts semisimply on such M.

When V is rational, the associative algebra $A(V)$ is semisimple [77]. For a rational vertex algebra V, given a simple module E over Zhu's algebra $A(V)$, the Verma module $M(E)$ remains simple. In general, Verma modules are not necessarily simple, but they are indecomposable. Hence complete reducibility implies that simple and indecomposable coincide.

1.8. Dual modules

Isomorphism classes of simple V-modules and simple $A(V)$-modules are in correspondence (§1.5). Here we describe the V-module (and corresponding $A(V)$-module) structures on their duals.

1.8.1. *Duality for V-modules.* – Let V be a vertex algebra. Following [41, §5.2], given a V-module $\left(M = \bigoplus_{i \geq 0} M_i, Y^M(-, z)\right)$, its *contragredient module* is $\left(M', Y^{M'}(-, z)\right)$, where M' is the graded dual of M, that is, $M' := \bigoplus_{i \geq 0} M_i^{\vee}$, with $M_i^{\vee} := \mathrm{Hom}_{\mathbb{C}}(M_i, \mathbb{C})$, and

$$Y^{M'}(-, z) \colon V \to \mathrm{End}\left(M'\right) [\![z, z^{-1}]\!]$$

is the unique linear map determined by

$$(6) \qquad \left\langle Y^{M'}(A, z)\psi, m \right\rangle = \left\langle \psi, Y^M\left(e^{zL_1}(-z^{-2})^{L_0} A, z^{-1}\right) m \right\rangle$$

for $A \in V$, $\psi \in M'$, and $m \in M$. Here and throughout, $\langle \cdot, \cdot \rangle$ is the natural pairing between a vector space and its graded dual.

1.8.2. *Duality for $A(V)$-modules.* – For a V-module M, the $A(V)$-module structure on $M_0^{\vee}$ requires the involution $\vartheta \colon \mathfrak{L}(V)^f \to \mathfrak{L}(V)^f$ induced from

$$(7) \qquad \vartheta\left(A_{[j]}\right) := (-1)^{a-1} \sum_{i \geq 0} \frac{1}{i!} \left(L_1^i A\right)_{[2a-j-i-2]}$$

for a homogeneous element $A \in V$ of degree a. Here $\mathfrak{L}(V)^f \subset \mathfrak{L}(V)$ denotes the quotient of $V \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ by $\mathrm{Im}\, \partial$. Observe that since the operator L_1 has negative degree, the above sum is finite, and that ϑ is a Lie algebra homomorphism [67, Prop 4.1.1] (note the sign difference between ϑ used here and the one in [67]). This involution appeared in [16], and it naturally arises from the action of the vertex operators on the contragredient module V'. Since ϑ restricts to an involution of $\mathfrak{L}(V)_0$ leaving $O(V)$ invariant, it induces an involution on Zhu's algebra $A(V)$. The following statement is a direct consequence of the definition of contragredient modules.

LEMMA 1.8.3. –   (i) *The image of $\psi \in M_0^{\vee}$ under the action of $\sigma \in \mathfrak{L}(V)_0$ is the linear functional*

$$\sigma \cdot \psi = -\psi \circ \vartheta(\sigma).$$

(ii) *The image of $\psi \in M_0^{\vee}$ under the action of $o(A) \in A(V)$ is*

$$o(A) \cdot \psi = -\psi \circ \vartheta(o(A)).$$

For homogeneous $A \in V$ of degree a and for $m \in M_0$, this is

$$\langle o(A) \cdot \psi, m \rangle = (-1)^a \left\langle \psi, \sum_{i \geq 0} \frac{1}{i!} \left(L_1^i A\right)_{(a-i-1)} m \right\rangle.$$

See [38, §10.4.8] for a geometric realization of the involution ϑ. In §3, we use ϑ to describe chiral Lie algebras on nodal curves.

1.9. Stable k-differentials

Let $(C, P_\bullet)$ be a stable n-pointed curve with at least one node, and let ω_C be the dualizing sheaf on C. We review here *(stable) k-differentials* on C, that is, sections of $\omega_C^{\otimes k}$, for an integer k. When $k \geq 1$, by $\omega_C^{\otimes -k}$ we mean $(\omega_C^\vee)^{\otimes k}$.

Let $\widetilde{C} \to C$ be the partial normalization of C at a node Q, let $Q_+, Q_- \in \widetilde{C}$ be the two preimages of Q, and set $Q_\bullet = (Q_+, Q_-)$. Note that the curve $\widetilde{C}$ may not be connected. Let s_+ and s_- be formal coordinates at the points Q_+ and Q_-, respectively. We write $Q_\pm$ to denote either point, and similarly use $s_\pm$ to denote either formal coordinate. For a section

$$\mu \in \mathrm{H}^0\left(\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet, \omega_{\widetilde{C}}^{\otimes k}\right) =: \widetilde{\Gamma},$$

let $\mu_{Q_\pm} \in \mathbb{C}((s_\pm))(ds_\pm)^k$ be the Laurent series expansion of μ at $Q_\pm$, that is, the image of μ under the restriction morphism

$$\mathrm{H}^0\left(\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet, \omega_{\widetilde{C}}^{\otimes k}\right) \to \mathrm{H}^0\left(D_{Q_\pm}^\times, \omega_{\widetilde{C}}^{\otimes k}\right) \simeq_{s_\pm} \mathbb{C}((s_\pm))(ds_\pm)^k.$$

Here $D_{Q_\pm}^\times$ is the punctured formal disk about $Q_\pm$, that is, the spectrum of the field of fractions of the completed local ring $\widehat{\mathscr{O}}_{Q_\pm}$, and $\simeq_{s_\pm}$ denotes the isomorphism given by fixing the formal coordinate $s_\pm$ at $Q_\pm$.

For a k-differential μ, define the *order* $\mathrm{ord}_{Q_\pm}(\mu)$ of μ at the point $Q_\pm$ as the highest integer m such that $\mu_{Q_\pm} \in s_\pm^m \mathbb{C}[\![s_\pm]\!](ds_\pm)^k$. For a k-differential μ with $\mathrm{ord}_{Q_\pm}(\mu) \geq -k$, the *$k$-residue* $\mathrm{Res}_{Q_\pm}^k(\mu)$ of μ at the point $Q_\pm$ is defined as the coefficient of $s_\pm^{-k}(ds_\pm)^k$ in $\mu_{Q_\pm}$.

The order and the k-residue at $Q_\pm$ are independent of the formal coordinate $s_\pm$ at $Q_\pm$. Here we only require the case $\mathrm{ord}_{Q_\pm}(\mu) \geq -k$. For the definition of the k-residue without the assumption $\mathrm{ord}_{Q_\pm}(\mu) \geq -k$, see e.g., [12].

LEMMA 1.9.1. – *Assume that $C \setminus P_\bullet$ is affine. For all integers k, one has*

$$\eta^* \mathrm{H}^0\left(C \setminus P_\bullet, \omega_C^{\otimes k}\right) = \left\{ \mu \in \widetilde{\Gamma} \ \middle| \ \begin{array}{l} \mathrm{ord}_{Q_\pm}(\mu) \geq -k, \\ \mathrm{Res}_{Q_+}^k(\mu) = (-1)^k \, \mathrm{Res}_{Q_-}^k(\mu) \end{array} \right\}.$$

Proof. – It is enough to prove the statement for $k \in \{-1, 0, 1\}$: indeed, for negative integers k, sections of $\omega_C^{\otimes k}$ on the affine $C \setminus P_\bullet$ are tensor products of sections of ω_C^{-1}, and the Laurent series expansions of sections of $\omega_C^{\otimes k}$ at Q_+ and Q_- are obtained as tensors of the Laurent series expansions of sections of $\omega_{\widetilde{C}}^{-1}$ at Q_+ and Q_-, respectively. An analogous argument may be made in case k is a positive integer.

When $k = 1$, the statement is about sections of ω_C, and by definition sections of ω_C are sections of $\omega_{\widetilde{C}}$ with at most simple poles at Q_+ and Q_- such that $\mathrm{Res}_{Q_+}(\mu) + \mathrm{Res}_{Q_-}(\mu) = 0$. When $k = 0$, the statement is about sections of $\mathscr{O}_C$, and indeed a regular function on C is a regular function μ on $\widetilde{C}$ such that $\mu(Q_+) = \mu(Q_-)$. When $k = -1$, by definition we have $\omega_C^{-1} = \mathscr{H}\mathrm{om}_{\mathscr{O}_C}(\omega_C, \mathscr{O}_C)$, and the statement follows from a direct computation using the cases $k \in \{0, 1\}$. $\qquad\square$

We have used the notation η^* to denote that the identification of elements of $\mathrm{H}^0(C \setminus P_\bullet, \omega_C^{\otimes k})$ with elements of $\widetilde{\Gamma}$ is naturally induced by the map η. We will use the same notation in the statement of Proposition 3.3.1.

1.10. A consequence of Riemann-Roch

We will frequently use the following corollary of the Riemann-Roch theorem. Let C be a smooth curve, possibly disconnected, with two non-empty sets of distinct marked points $P_\bullet = (P_1, \ldots, P_n)$ and $Q_\bullet = (Q_1, \ldots, Q_m)$. Assume that each irreducible component of C contains at least one of the marked points $P_\bullet$, so that $C \setminus P_\bullet$ is affine. Let s_i be a formal coordinate at the point Q_i, for each i. Fix an integer k. For all integers d and N, there exists $\mu \in \mathrm{H}^0\left(C \setminus P_\bullet \sqcup Q_\bullet, \omega_C^{\otimes k}\right)$ such that its Laurent series expansions at the points $Q_\bullet$ satisfy:

$$\mu_{Q_i} \equiv s_i^d (ds_i)^k \in \mathbb{C}((s_i))(ds_i)^k / s_i^N \mathbb{C}[\![s_i]\!](ds_i)^k, \qquad \text{for a fixed } i,$$

$$\mu_{Q_j} \equiv 0 \in \mathbb{C}((s_j))(ds_j)^k / s_j^N \mathbb{C}[\![s_j]\!](ds_j)^k, \qquad \text{for all } j \neq i.$$

This statement has appeared for instance in [76].

2. Sheaves of vertex algebras on curves

We describe the sheaf of vertex algebras $\mathcal{V}_C$ on a curve C with (at worst) simple nodal singularities in §2.5, and its flat connection in §2.7. This yields a coordinate-free description of the Lie algebra ancillary to V in §2.8.

2.1. The group scheme $\mathrm{Aut}\,\mathcal{O}$

Consider the functor which assigns to a $\mathbb{C}$-algebra R the group:

$$\mathrm{Aut}\,\mathcal{O}(R) = \left\{ z \mapsto \rho(z) = a_1 z + a_2 z^2 + \cdots \mid a_i \in R, a_1 \text{ a unit} \right\}$$

of continuous automorphisms of the algebra $R[\![z]\!]$ preserving the ideal $zR[\![z]\!]$. The group law is given by composition of series: $\rho_1 \cdot \rho_2 := \rho_2 \circ \rho_1$. This functor is represented by a group scheme, denoted $\mathrm{Aut}\,\mathcal{O}$.

To construct the sheaf of vertex algebras $\mathcal{V}_C$ on a stable curve C, we describe below the principal $(\mathrm{Aut}\,\mathcal{O})$-bundle $\mathscr{A}ut_C \to C$, and actions of $\mathrm{Aut}\,\mathcal{O}$ on the vertex operator algebra V and on $\mathscr{A}ut_C \times V$.

2.2. Coordinatized curves

Given a flat family $C \to S$ of stable curves of genus g, we construct an $(\mathrm{Aut}\,\mathcal{O})$-torsor $\mathscr{A}ut_{C/S} \to C/S$ in §2.2.2. This torsor is pulled back from a universal $(\mathrm{Aut}\,\mathcal{O})$-torsor $\mathcal{M}_{g,1} \to \overline{\mathcal{M}}_{g,1}$. To define these objects, we begin in §2.2.1 with $S = \mathrm{Spec}(\mathbb{C})$.

2.2.1. *The principal* $(\text{Aut}\,\mathcal{O})$*-bundle* $\mathscr{A}\!ut_C$ *at a fixed curve* C. – Assume first that C is a *smooth* curve. Let $\mathscr{A}\!ut_C$ be the infinite-dimensional smooth variety whose points consist of pairs (P, t), where P is a point in C, and t is a formal coordinate at P (see [11]). A formal coordinate t at P is an element of the completed local ring $\widehat{\mathcal{O}}_P$ such that $t \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2$, where $\mathfrak{m}_P$ is the maximal ideal of $\widehat{\mathcal{O}}_P$. There is a natural forgetful map $\mathscr{A}\!ut_C \to C$, with fiber at a point $P \in C$ equal to the set of formal coordinates at P:

$$\mathscr{A}\!ut_P = \left\{ t \in \widehat{\mathcal{O}}_P \mid t \in \mathfrak{m}_P \setminus \mathfrak{m}_P^2 \right\}.$$

The group scheme $\text{Aut}\,\mathcal{O}$ acts on fibers of $\mathscr{A}\!ut_C \to C$ by change of coordinates:

$$\mathscr{A}\!ut_C \times \text{Aut}\,\mathcal{O} \to \mathscr{A}\!ut_C, \qquad ((P, t), \rho) \mapsto (P, t \cdot \rho := \rho(t)).$$

This action is simply transitive, thus $\mathscr{A}\!ut_C$ is a principal $(\text{Aut}\,\mathcal{O})$-bundle on C. The choice of a formal coordinate t at P gives rise to the trivialization

$$\text{Aut}\,\mathcal{O} \xrightarrow{\simeq_t} \mathscr{A}\!ut_P, \qquad \rho \mapsto \rho(t).$$

For a *nodal* curve C, let $\widetilde{C} \to C$ denote the normalization of C. Assume for simplicity that C has only one node Q, and let Q_+ and Q_- be the two preimages in $\widetilde{C}$ of Q. A choice of formal coordinates s_+ and s_- at Q_+ and Q_-, respectively, determines a smoothing of the nodal curve C over $\text{Spec}(\mathbb{C}[\![q]\!])$ such that, locally around the point Q in C, the family is defined by $s_+ s_- = q$ (see §8.1 for more details). A principal $(\text{Aut}\,\mathcal{O})$-bundle on a nodal curve is equivalent to the datum of a principal $(\text{Aut}\,\mathcal{O})$-bundle on its normalization together with a gluing isomorphism between the fibers over the two preimages of each node. In particular, one constructs the principal $(\text{Aut}\,\mathcal{O})$-bundle $\mathscr{A}\!ut_C$ on C from the principal $(\text{Aut}\,\mathcal{O})$-bundle $\mathscr{A}\!ut_{\widetilde{C}}$ on $\widetilde{C}$ by identifying the fibers at the two preimages Q_+ and Q_- of the node Q in C by the following gluing isomorphism:

$$(8) \qquad \mathscr{A}\!ut_{Q_+} \simeq_{s_+} \text{Aut}\,\mathcal{O} \xrightarrow{\cong} \text{Aut}\,\mathcal{O} \simeq_{s_-} \mathscr{A}\!ut_{Q_-}, \qquad \rho(s_+) \mapsto \rho \circ \gamma(s_-),$$

where $\gamma \in \text{Aut}\,\mathcal{O}$ is defined as

$$(9) \qquad \gamma(z) := \frac{1}{1+z} - 1 = -z + z^2 - z^3 + \cdots.$$

Note that $(\gamma \circ \gamma)(z) = z$, hence (8) determines an involution of $\text{Aut}\,\mathcal{O}$. The isomorphism (8) is induced from the identification $s_+ = \gamma(s_-)$.

2.2.2. $\mathscr{A}\!ut_{C/S}$ *on a family* $C \to S$ *and the moduli space* $\mathcal{M}_{g,1}$. – The above construction of $\mathscr{A}\!ut_C$ can be carried out in families, and for a flat family $C \to S$ of stable curves, one thus obtains a principal $(\text{Aut}\,\mathcal{O})$-bundle $\mathscr{A}\!ut_{C/S} \to C/S$. This determines a principal $(\text{Aut}\,\mathcal{O})$-bundle $\mathcal{C}_g \to \overline{\mathcal{C}}_g$, where $\overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ is the universal curve over the moduli space of

stable curves of genus g. One has the following diagram made of fiber product squares:

$$
\begin{array}{ccc}
\mathscr{A}\!ut_{C/S} & \longrightarrow & \mathcal{C}_g \\
\downarrow & & \downarrow{\scriptstyle\mathrm{Aut}\,\mathcal{O}} \\
C & \longrightarrow & \overline{\mathcal{C}}_g \\
\downarrow & & \downarrow \\
S & \longrightarrow & \overline{\mathcal{M}}_g.
\end{array}
$$

It can be convenient to identify the universal curve $\overline{\mathcal{C}}_g$ with the moduli space of stable pointed curves $\overline{\mathcal{M}}_{g,1}$. This leads us to consider a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle $\mathcal{M}_{g,1} \to \overline{\mathcal{M}}_{g,1}$ identified with $\mathcal{C}_g \to \overline{\mathcal{C}}_g$. Namely:

DEFINITION 2.2.3. – Let $\mathcal{M}_{g,1}$ be the moduli space of *coordinatized stable pointed curves*. An object over a scheme S consists of a semistable curve $C \to S$ with a section $P \colon S \to C$ mapping to the smooth locus of C, together with a formally unramified thickening $S \times \mathrm{Spf}(\mathbb{C}[\![t]\!]) \to C$ of the section P, such that every genus one component has at least one special point and every rational component has at least three special points.

Here a special point is either a marked point or a node, counted with multiplicity. Moreover, $\mathrm{Spf}(\mathbb{C}[\![t]\!])$ is the formal spectrum of the complete topological ring $\mathbb{C}[\![t]\!]$ [38, §A.1.1].

The action of $\mathrm{Aut}\,\mathcal{O}$ via change of coordinates identifies $\mathcal{M}_{g,1}$ as a principal $(\mathrm{Aut}\,\mathcal{O})$-bundle over $\overline{\mathcal{M}}_{g,1}$. One has the diagram of Cartesian squares:

$$
\begin{array}{ccccc}
\mathscr{A}\!ut_{P/S} & \longrightarrow & \mathscr{A}\!ut_{C/S} & \longrightarrow & \mathcal{M}_{g,1} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle\mathrm{Aut}\,\mathcal{O}} \\
S & \xrightarrow{\ P\ } & C & \longrightarrow & \overline{\mathcal{M}}_{g,1} \\
& & \downarrow & & \downarrow \\
& & S & \longrightarrow & \overline{\mathcal{M}}_g.
\end{array}
$$

In particular, for a flat family $C \to S$ of stable curves of genus g, the space $\mathscr{A}\!ut_{C/S}$ is the pull-back of $\mathcal{M}_{g,1}$ via the moduli map $S \to \overline{\mathcal{M}}_g$.

For a nodal curve C, the fiber of $\mathscr{A}\!ut_C$ over a node can be described as follows. Assume for simplicity that C has only one node Q. Over the singular point $Q \in C$, the fiber $\mathscr{A}\!ut_Q$ can be identified with the space of formal coordinates at the point Q' in C', where (C', Q') is formed by stabilization of the unstable pointed curve (C, Q). Stabilization is carried out by blowing-up. As a result, C' consists of the partial normalization $\widetilde{C}$ of C at Q together with a rational exceptional component meeting $\widetilde{C}$ transversally at the two preimages Q_+ and $Q_- \in \widetilde{C}$ of the node Q. Such a rational component contains three special points: the two attaching points and a point labeled Q'. Up to isomorphism, one can identify this rational component with $\mathbb{P}^1$, with the points attached to Q_+ and Q_- identified with 0 and $\infty \in \mathbb{P}^1$, respectively, and the point Q' identified with $1 \in \mathbb{P}^1$. The fiber $\mathscr{A}\!ut_Q$ of $\mathscr{A}\!ut_C$ over the node Q in C is identified with the space of formal coordinates at the point 1 in $\mathbb{P}^1 \subset C'$.

Here are more details: Choose formal coordinates s_+ and s_- at Q_+ and Q_-, respectively. As mentioned in the previous section, a choice of such coordinates determines a smoothing of C over $\mathrm{Spec}(\mathbb{C}[\![q]\!])$ such that, locally around the point Q in C, the family is defined by $s_+ s_- = q$. After blowing-up such a family at the point Q, locally around the two resulting nodes incident to the exceptional component $\mathbb{P}^1$, the curve C' at $q = 0$ is given by equations $s_+ s_0 = 0$ and $s_\infty s_- = 0$, where s_0, s_∞ are formal coordinates at $0, \infty \in \mathbb{P}^1$, respectively. The formal coordinates s_0 and s_∞ satisfy $s_0 s_\infty = 1$. A pair of such coordinates s_0, s_∞ induce formal coordinates $s_0 - 1$ and $s_\infty - 1$ at $1 \in \mathbb{P}^1$, with change of coordinates given precisely by $s_0 - 1 = \gamma(s_\infty - 1)$, where γ is as in (9). It follows that for the nodal curve C', one has the identifications $s_+ = \gamma(s_0)$, $s_- = \gamma(s_\infty)$, and $s_0 - 1 = \gamma(s_\infty - 1)$. These identifications determine the identification of the space $\mathscr{A}ut_C$ with the fiber of the projection $\mathcal{M}_{g,1} \to \overline{\mathcal{M}}_g$ over the moduli point $[C] \in \overline{\mathcal{M}}_g$.

More generally, one defines $\mathcal{M}_{g,n}$ as the moduli space of objects of type $(C, P_\bullet, t_\bullet)$, where $(C, P_\bullet = (P_1, \ldots, P_n))$ is a stable n-pointed genus g curve, and $t_\bullet = (t_1, \ldots, t_n)$ with t_i a formal coordinate at P_i, for each i. The action of $(\mathrm{Aut}\,\mathcal{O})^{\oplus n}$ by change of coordinates endows $\mathcal{M}_{g,n}$ with the structure of an $(\mathrm{Aut}\,\mathcal{O})^{\oplus n}$-torsor over $\overline{\mathcal{M}}_{g,n}$. For further information about $\mathcal{M}_{g,n}$ over the locus of smooth curves, see [11] and [38, §6.5]; over stable curves with singularities, see [21, §3.2].

2.3. Action of $\mathrm{Aut}\,\mathcal{O}$ on V

Let $\mathrm{Der}_0\,\mathcal{O}$ be the Lie algebra functor attached to the group functor $\mathrm{Aut}\,\mathcal{O}$ in the sense of [23, Exp. II §3,4]. This is

$$\mathrm{Der}_0\,\mathcal{O}(R) = R[\![z]\!] z \partial_z.$$

The Lie algebra $\mathrm{Der}_0\,\mathcal{O}(R)$ is generated over R by the Virasoro elements L_p, for $p \geq 0$, hence $\mathrm{Der}_0\,\mathcal{O}$ is a Lie subalgebra of the Virasoro Lie algebra. The action of the Virasoro Lie algebra on a vertex operator algebra V restricts to an action of Der_0 on V. One can integrate this to obtain a left action of $\mathrm{Aut}\,\mathcal{O}$ on V defined as the inductive limit of the finite-dimensional submodules $V_{\leq k} := \bigoplus_{i \leq k} V_i$. This follows from the fact that L_0 acts semi-simply with integral eigenvalues, and L_p acts locally nilpotently for $p > 0$ [38, §6.3].

Explicitly, to compute the action on V of an element $\rho \in \mathrm{Aut}\,\mathcal{O}$, one proceeds as follows. The element $\rho(z)$ can be expressed as

$$\rho(z) = \exp\left(\sum_{i \geq 0} a_i z^{i+1} \partial_z\right)(z)$$

for some $a_i \in \mathbb{C}$ (see e.g., [38, §6.3.1]). Assuming $0 \leq \mathrm{Im}(a_0) < 2\pi$, the coefficients a_i are uniquely determined. Hence, ρ acts on V as $\exp\left(\sum_{i \geq 0} -a_i L_i\right)$.

As an example and for later use, the element $\gamma \in \mathrm{Aut}\,\mathcal{O}$ from (9) can be expressed as

$$(10) \qquad\qquad \gamma(z) = e^{-z^2 \partial_z}(-1)^{-z\partial_z}(z).$$

Thus, γ acts on V as $e^{L_1}(-1)^{L_0}$. This is a special case of the computation in [38, (10.4.3)] and is essential to the gluing isomorphism for $\mathcal{V}_C$ below.

The sheaf $\mathcal{V}_C$ is realized as a subsheaf of $\eta_*(\widetilde{\mathcal{V}})$ which coincides with $\eta_*(\widetilde{\mathcal{V}})$ on $C \setminus Q$. To describe its fiber over Q, we use the involution $\gamma \in \mathrm{Aut}\,\mathcal{O}$ from (9) and (10), hence for homogeneous $A \in V$ of degree a, one has

$$\gamma(A) = e^{L_1}(-1)^{L_0}A = (-1)^a \sum_{i \geq 0} \frac{1}{i!} L_1^i(A).$$

The fiber of $\mathcal{V}_C$ over Q is obtained by identifying the fibers of $\widetilde{\mathcal{V}}$ at Q_+ and Q_- via the gluing isomorphism induced by γ as in the diagram below:

$$
\begin{array}{ccc}
V & \xrightarrow{\hspace{2cm}\gamma\hspace{2cm}} & V \\[2pt]
{\scriptstyle\cong}\big\uparrow & & \big\downarrow{\scriptstyle\cong} \\[4pt]
\displaystyle\bigoplus_{k \geq 0} V_k \otimes_{\mathbb{C}} s_+^k & & \displaystyle\bigoplus_{k \geq 0} V_k \otimes_{\mathbb{C}} s_-^k \\[8pt]
{\scriptstyle\simeq s_+}\big\uparrow & & \big\downarrow{\scriptstyle\simeq s_-} \\[4pt]
\mathscr{A}\!ut_{Q_+} \times_{\mathrm{Aut}\,\mathcal{O}} V & \dashrightarrow{\hspace{0.3cm}\cong\hspace{0.3cm}} & \mathscr{A}\!ut_{Q_-} \times_{\mathrm{Aut}\,\mathcal{O}} V.
\end{array}
$$

Equivalently, after trivializing with respect to $s_{\pm}$, the gluing isomorphism is

$$\bigoplus_{k \geq 0} V_k \otimes_{\mathbb{C}} s_+^k \to \bigoplus_{k \geq 0} V_k \otimes_{\mathbb{C}} s_-^k, \qquad A \otimes_{\mathbb{C}} s_+^k \mapsto (-1)^k \sum_{i \geq 0} \frac{1}{i!} L_1^i(A) \otimes_{\mathbb{C}} s_-^{k-i}.$$

We next describe spaces of sections of $\mathcal{V}_C$ for the nodal curve C. Over an open set $U \subset C$ with $Q \notin U$, one has $\mathcal{V}_C|_U := \eta_*\big(\mathcal{V}_{\widetilde{C}}|_{\eta^{-1}(U)}\big)$. Since η is an isomorphism away from Q, sections of $\mathcal{V}_C$ over U are defined as in the case of smooth curves. To define spaces of sections of $\mathcal{V}_C$ over an open set containing Q, it is sufficient to do so on the formal neighborhood

$$(13) \qquad\qquad D_Q = \mathrm{Spec}\,\widehat{\mathscr{O}}_Q = \mathrm{Spec}(\mathbb{C}[\![s_+, s_-]\!]/(s_+s_-)).$$

The space of sections $\mathcal{V}_C(D_Q)$ is defined as the subspace of

$$\eta_*\widetilde{\mathcal{V}}(D_Q) = \bigoplus_{k \geq 0} V_k \otimes \big(s_+^k \mathbb{C}[\![s_+]\!] \oplus s_-^k \mathbb{C}[\![s_-]\!]\big)$$

consisting of elements in the kernel of the map

$$\bigoplus_{k \geq 0} V_k \otimes \big(s_+^k \mathbb{C}[\![s_+]\!] \oplus s_-^k \mathbb{C}[\![s_-]\!]\big) \longrightarrow V$$

induced by

$$\big(A \otimes s_+^a f(s_+), B \otimes s_-^b g(s_-)\big) \mapsto f(0)\,\gamma(A) - g(0)\,B$$

for homogeneous $A, B \in V$ with $\deg(A) = a$ and $\deg(B) = b$, and for all $f(s_+) \in \mathbb{C}[\![s_+]\!]$ and $g(s_-) \in \mathbb{C}[\![s_-]\!]$. Hence, $\mathcal{V}_C(D_Q)$ is spanned by elements

$$\Big(A \otimes s_+^a, (-1)^a \sum_{i \geq 0} \frac{1}{i!} L_1^i(A) \otimes s_-^{a-i}\Big) \quad \text{and} \quad \Big(B \otimes s_+^{b+1} f(s_+), D \otimes s_-^{d+1} g(s_-)\Big)$$

for homogeneous $A \in V_a$, $B \in V_b$, and $D \in V_d$, with $f(s_+) \in \mathbb{C}[\![s_+]\!]$ and $g(s_-) \in \mathbb{C}[\![s_-]\!]$. The sheaf $\mathcal{V}_C|_{D_Q}$ is naturally an $\widehat{\mathscr{O}}_Q$-module.

To summarize, the sheaf $\mathcal{V}_C$ on the nodal curve C can be described as follows. Consider the exact sequence of $\mathcal{O}_{\widetilde{C}}$-modules

$$0 \longrightarrow \mathcal{O}_{\widetilde{C}}(-Q_+ - Q_-) \otimes \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}} \longrightarrow V_{Q_+} \oplus V_{Q_-} \longrightarrow 0,$$

where $V_{Q_\pm}$ is the skyscraper sheaf supported at $Q_\pm$ with space of sections isomorphic to V via the choice of the coordinate $s_\pm$. Pushing forward this sequence along η, we obtain an exact sequence which fits in the diagram

(14)
$$0 \longrightarrow \eta_*\left(\mathcal{O}_{\widetilde{C}}(-Q_+ - Q_-) \otimes \widetilde{\mathcal{V}}\right) \longrightarrow \eta_*\widetilde{\mathcal{V}} \overset{q}{\longrightarrow} V_Q^{\oplus 2} \longrightarrow 0$$
$$\downarrow{\gamma \circ \pi_1 - \pi_2}$$
$$V_Q.$$

Here, $\pi_i \colon V_Q^{\oplus 2} \to V_Q$ is the natural projection, for $i = 1, 2$. The sheaf $\mathcal{V}_C$ is then defined as $\ker\left((\gamma \circ \pi_1 - \pi_2) \circ q\right)$.

2.6. The structure of the sheaf $\mathcal{V}_C$

We describe here some properties of $\mathcal{V}_C$. For a smooth curve C, the sheaf $\mathcal{V}_C$ is filtered by the sheaves $\mathcal{V}_{\leq k}$ defined as the sheaves of sections of the vector bundles of finite rank $\mathcal{A}ut_C \times_{\mathrm{Aut}\,\mathcal{O}} V_{\leq k}$. While the action of $\mathrm{Aut}\,\mathcal{O}$ on $V_{\leq k}$ is well-defined, the action of $\mathrm{Aut}\,\mathcal{O}$ on V_k is so only modulo $V_{\leq k-1}$, for each k. In particular, $\mathcal{V}_{\leq k}$ is well-defined, but $\mathcal{A}ut_C \times_{\mathrm{Aut}\,\mathcal{O}} V_k$ only makes sense as a quotient of $\mathcal{A}ut_C \times_{\mathrm{Aut}\,\mathcal{O}} V_{\leq k}$ modulo $\mathcal{A}ut_C \times_{\mathrm{Aut}\,\mathcal{O}} V_{\leq k-1}$.

Assuming for simplicity that the curve C has only one node Q, the sheaves

$$\left(\bigoplus_{i=0}^{k} V_i \otimes \mathcal{O}_{\widetilde{C}}\left(-i\,Q_+ - i\,Q_-\right) \otimes \widetilde{\pi}_* \mathcal{O}_{\mathcal{A}ut_{\widetilde{C}}}\right)^{\mathrm{Aut}\,\mathcal{O}}$$

provide an increasing filtration of the sheaf $\widetilde{\mathcal{V}}$ on the normalization $\widetilde{C}$ from (12). The restriction of the gluing isomorphism in §2.5 gives a gluing isomorphism between the fibers of such sheaves at the two preimages Q_+ and Q_- of the node Q, and hence induces an increasing filtration of $\mathcal{V}_C$. We denote the subsheaves of such a filtration as $\mathcal{V}_{\leq k}$, as in the smooth case.

Consider the associated graded sheaf

$$\mathrm{gr}_\bullet \mathcal{V}_C := \bigoplus_{k \geq 0} \mathrm{gr}_k \mathcal{V}_C, \qquad \text{where} \qquad \mathrm{gr}_k \mathcal{V}_C := \mathcal{V}_{\leq k}/\mathcal{V}_{\leq k-1}.$$

LEMMA 2.6.1. – *One has*

$$\mathrm{gr}_\bullet \mathcal{V}_C \cong \bigoplus_{k \geq 0} \left(\omega_C^{\otimes -k}\right)^{\oplus \dim V_k}.$$

This was proved in [38, §6.5.9] for smooth curves. The argument made there extends to stable curves if one replaces the sheaf of differentials Ω_C^1 with the dualizing sheaf ω_C. We sketch the proof for the reader's convenience.

Proof. – Consider V_k as the quotient $(\text{Aut}\,\mathcal{O})$-representation $V_{\leq k}/V_{\leq k-1}$, and let $A \in V_k$ be nonzero. One has $L_0 \cdot A = kA$ and $L_p \cdot A = 0$ in $V_k = V_{\leq k}/V_{\leq k-1}$ for $p > 0$. Assume first that C is smooth. Then it follows that

$$\mathscr{A} := \left(\mathbb{C}A \otimes \pi_* \mathcal{O}_{\mathscr{A}ut_C} \right)^{\text{Aut}\,\mathcal{O}}$$

is a line sub-bundle of $\text{gr}_k \mathcal{V}_C$. From [38, §6.5.9], the transition functions of $\mathscr{A}$ and $\omega_C^{\otimes -k}$ match, hence one concludes that $\mathscr{A} \cong \omega_C^{\otimes -k}$, and this implies the statement.

Next, we consider the case when C is nodal. Assume for simplicity that C has only one node Q. Consider the line bundle $\mathscr{A}$ constructed from the line bundle

$$\mathscr{A}' := \left(\mathbb{C}A \otimes \widetilde{\pi}_* \mathcal{O}_{\mathscr{A}ut_{\widetilde{C}}} \right)^{\text{Aut}\,\mathcal{O}} \otimes \mathcal{O}_{\widetilde{C}}(-kQ_+ - kQ_-)$$

on the normalization $\widetilde{C}$ of C by identifying the fibers at the preimages Q_+ and Q_- of the node. From the discussion of the smooth case, one has

$$\mathscr{A}' \cong \omega_{\widetilde{C}}^{\otimes -k} \left(-kQ_+ - kQ_- \right).$$

It remains to determine the isomorphisms used to identify the fibers at Q_+ and Q_-. The gluing isomorphism $A \mapsto e^{L_1}(-1)^{L_0}A$ of $\mathcal{V}_{\leq k}$ from §2.5 induces the gluing $A \mapsto (-1)^{L_0}A$ of $\text{gr}_k \mathcal{V}_C$. In particular, this is the gluing isomorphism of $\mathscr{A}$, which coincides with the gluing of sections for $\omega_C^{\otimes -k}$ given by the condition on the residues in Lemma 1.9.1. Hence one has $\mathscr{A} \cong \omega_C^{\otimes -k}$ in this case as well. The assertion therefore follows. $\square$

REMARK 2.6.2. – The reader will notice how both sheaves $(\omega_C^{\otimes -k})^{\oplus \dim V_k}$ and $\mathcal{V}_{\leq k}/\mathcal{V}_{\leq k-1}$ are defined using diagrams similar to the one in (14). From this point of view, the above proof can be seen as comparing the gluing data encoded by the vertical maps in the corresponding diagrams.

As a consequence of [38, §6.5.9], which is Lemma 2.6.1 for smooth curves, one has the following statement, which will be used throughout.

LEMMA 2.6.3. – *Let $(C, P_\bullet)$ be a smooth n-pointed curve. One has:*

$$\text{H}^0(C \setminus P_\bullet, \mathcal{V}_C) \cong \text{H}^0(C \setminus P_\bullet, \text{gr}_\bullet \mathcal{V}_C).$$

Proof. – We claim that on the affine open set $C \setminus P_\bullet$, one has

$$(15) \qquad \mathcal{V}_{\leq k} \cong \bigoplus_{i \leq k} \mathcal{V}_{\leq i}/\mathcal{V}_{\leq i-1} = \text{gr}_{\leq k} \mathcal{V}_C.$$

Assuming (15), then one has, for every $k \in \mathbb{Z}_{\geq 0}$, an injection

$$\phi_k \colon \text{gr}_{\leq k} \mathcal{V}_C \hookrightarrow \mathcal{V}_C,$$

altogether defining a map $\phi \colon \text{gr}_\bullet \mathcal{V}_C \longrightarrow \mathcal{V}_C$. The map ϕ gives the isomorphism we seek. To see that ϕ is injective, note that any element x in $\text{gr}_\bullet \mathcal{V}_C$ is in fact a finite sum, and hence x is in $\text{gr}_{\leq k} \mathcal{V}$ for some k. So if x is mapped to zero by ϕ, then x is mapped to zero by ϕ_k for some k. Since all maps ϕ_k are injective, x is zero. Surjectivity of ϕ follows from the fact that $\mathcal{V}_C$ is filtered by the $\mathcal{V}_{\leq k}$.

We prove (15) by induction on k with base case $k = 0$. Lemma 2.6.1 implies that $\mathrm{gr}_k \mathcal{V}_C$ is locally free. On affines, locally free sheaves are projective, and hence on the affine open set $C \setminus P_\bullet$, the following sequence splits:

$$0 \to \mathcal{V}_{\leq k-1} \to \mathcal{V}_{\leq k} \to \mathrm{gr}_k \mathcal{V}_C \to 0.$$

In particular, on $C \setminus P_\bullet$

$$\mathcal{V}_{\leq k} \cong \mathcal{V}_{\leq k-1} \oplus \mathrm{gr}_k \mathcal{V}_C \cong \mathrm{gr}_{\leq k-1} \mathcal{V}_C \oplus \mathrm{gr}_k \mathcal{V}_C,$$

and (15) holds.   □

2.7. The logarithmic connection

In this section, we describe the logarithmic connection on $\mathcal{V}_C$. For smooth curves, this was defined in [38, §6]. The adjective *logarithmic* is used here to emphasize that we work with possibly nodal curves. For this purpose, we replace the sheaf Ω_C^1, used in [38, §6], with the dualizing sheaf ω_C which arises from logarithmic differentials on the normalization of C. Hence a *logarithmic connection* on $\mathcal{V}_C$ is a $\mathbb{C}$-linear map $\nabla \colon \mathcal{V}_C \to \mathcal{V}_C \otimes \omega_C$ satisfying $\nabla(fs) = s \otimes df + f\nabla(s)$ for all local sections f of $\mathscr{O}_C$ and s of $\mathcal{V}_C$. We will use that on open sets $U \subset C$ where ω_C is trivial, one can describe a connection as an endomorphism of $\mathcal{V}_C|_U$.

2.7.1. *The connection on smooth curves.* – Let C be a smooth curve. On an open subset U of C admitting a global coordinate t (for instance, on an open U admitting an étale map $U \to \mathbb{A}^1$), one has a trivialization $\mathcal{V}_C|_U \simeq_t V \otimes \mathscr{O}_U$. On $\mathcal{V}_C|_U$, the connection ∇ is given by the endomorphism $L_{-1} \otimes \mathrm{id}_U + \mathrm{id}_V \otimes \partial_t$ (compare with (3)). This canonically defines a connection $\nabla \colon \mathcal{V}_C \to \mathcal{V}_C \otimes \omega_C$ independent of the choice of the coordinate t (as in [38, Thm 6.6.3]).

2.7.2. *The connection on nodal curves.* – Let C be a nodal curve, and let $\eta \colon \widetilde{C} \to C$ be its normalization.

PROPOSITION 2.7.3. – *The connection on $\mathcal{V}_{\widetilde{C}}$ described in §2.7.1 naturally induces a logarithmic connection $\nabla \colon \mathcal{V}_C \to \mathcal{V}_C \otimes \omega_C$.*

Proof. – For simplicity, we assume that C has only one node Q. Recall that the sheaf $\mathcal{V}_C$ is defined as a subsheaf of $\eta_*(\widetilde{\mathcal{V}})$, where $\widetilde{\mathcal{V}}$ is the sheaf in (12), and similarly, ω_C is a subsheaf of $\eta_* \omega_{\widetilde{C}}(Q_+ + Q_-)$. Since $\widetilde{C}$ is a smooth curve, restricting the prescription given in §2.7.1 to $\widetilde{\mathcal{V}} \subset \mathcal{V}_{\widetilde{C}}$ defines

$$\widetilde{\nabla} \colon \widetilde{\mathcal{V}} \longrightarrow \widetilde{\mathcal{V}} \otimes \mathscr{O}_{\widetilde{C}}(Q_+ + Q_-) \otimes \omega_{\widetilde{C}}.$$

Pushing forward to C along η and restricting to $\mathcal{V}_C \subset \eta_*(\widetilde{\mathcal{V}})$, we obtain

$$(16) \qquad\qquad \nabla \colon \mathcal{V}_C \longrightarrow \eta_* \left(\widetilde{\mathcal{V}} \otimes \omega_{\widetilde{C}}(Q_+ + Q_-) \right).$$

We claim that this factors through $\mathcal{V}_C \otimes \omega_C$, defining in this way the desired logarithmic connection. That this holds away from Q is clear. Therefore, it remains to check the assertion over the formal neighborhood D_Q from (13).

The sheaf ω_C is locally free over C, and by an argument analogous to the proof of Lemma 1.9.1, we obtain an explicit trivialization over D_Q as

$$\omega_C(D_Q) \cong \frac{\mathbb{C}[\![s_+, s_-]\!]}{(s_+ s_-)}\left(\frac{ds_+}{s_+}, -\frac{ds_-}{s_-}\right) \subset \mathbb{C}[\![s_+]\!]\frac{ds_+}{s_+} \oplus \mathbb{C}[\![s_-]\!]\frac{ds_-}{s_-}.$$

Using this, the restriction of (16) to D_Q can be explicitly written as

$$(17) \qquad \nabla : \mathcal{V}_C(D_Q) \longrightarrow \bigoplus_{k \geq 0} V_k \otimes \left(s_+^{k-1}\mathbb{C}[\![s_+]\!]\,ds_+ \oplus s_-^{k-1}\mathbb{C}[\![s_-]\!]\,ds_-\right).$$

To conclude, we show that this factors through $\mathcal{V}_C(D_Q) \otimes \left(\frac{ds_+}{s_+}, -\frac{ds_-}{s_-}\right)$. By definition, the image via ∇ of an element $(A \otimes f(s_+), B \otimes g(s_-)) \in \mathcal{V}_C(D_Q)$, with $A, B \in V$, is

$$(L_{-1}(A) \otimes f(s_+)ds_+ + A \otimes f'(s_+)ds_+, L_{-1}(B) \otimes g(s_-)ds_- + B \otimes g'(s_-)ds_-).$$

The coefficient of $\left(\frac{ds_+}{s_+}, -\frac{ds_-}{s_-}\right)$ is

$$(18) \qquad (L_{-1}(A) \otimes s_+ f(s_+) + A \otimes s_+ f'(s_+), -L_{-1}(B) \otimes s_- g(s_-) - B \otimes s_- g'(s_-)).$$

We are then left to check that if $(A \otimes f(s_+), B \otimes g(s_-)) \in \mathcal{V}_C(D_Q)$, then (18) is also an element of $\mathcal{V}_C(D_Q)$. To do so, it is enough to check that this is true for the elements of type

$$\left(A \otimes s_+^a, (-1)^a \sum_{i \geq 0} \frac{1}{i!}L_1^i(A) \otimes s_-^{a-i}\right) \in \mathcal{V}_C(D_Q)$$

for homogeneous $A \in V$ of degree a. Checking this amounts to showing that

$$(-1)^{a+1} \sum_{j=0}^{a+1} \frac{1}{j!}L_1^j(L_{-1}(A)) \otimes s_-^{a+1-j} + a\,(-1)^a \sum_{i=0}^{a} \frac{1}{i!}L_1^i(A) \otimes s_-^{a-i}$$

is equal to

$$(-1)^{a+1} \sum_{i=0}^{a} \frac{1}{i!}L_{-1}(L_1^i(A)) \otimes s_-^{a-i+1} + (-1)^{a+1} \sum_{i=0}^{a} \frac{a-i}{i!}L_1^i(A) \otimes s_-^{a-i}.$$

Here the sums are truncated, since the terms with larger values of i and j vanish for degree reasons. Comparing powers of s_-, we need to verify for every $k \in \{0, \ldots, a\}$ that

$$(19) \qquad \frac{(-1)^{a+1}}{(k+1)!}L_1^{k+1}L_{-1}(A) + \frac{a(-1)^a}{k!}L_1^k(A)$$

is equal to

$$(20) \qquad \frac{(-1)^{a+1}}{(k+1)!}L_{-1}L_1^{k+1}(A) + \frac{(a-k)(-1)^{a+1}}{k!}L_1^k(A)$$

in V. Repeatedly applying the commutator $[L_1, L_{-1}] = 2L_0$, one has

$$L_1^{k+1}L_{-1}(A) = L_{-1}L_1^{k+1}(A) + (2a-k)(k+1)\,L_1^k(A),$$

which indeed implies that (19) and (20) are equal in V. It follows that (17) factors through $\mathcal{V}_C(D_Q) \otimes \left(\frac{ds_+}{s_+}, -\frac{ds_-}{s_-}\right)$, hence the statement. $\square$

2.7.4. *The connection: summary*. – In conclusion, the connection is given by:

DEFINITION 2.7.5. – Let C be a curve which has at worst simple nodal singularities. We define the logarithmic connection $\nabla \colon \mathcal{V}_C \to \mathcal{V}_C \otimes \omega_C$ as the map induced by the following endomorphisms on formal neighborhoods of points of C: for a smooth point $P \in C$ with formal coordinate t at P, the endomorphism of $\mathcal{V}_C(D_P)$ given by $L_{-1} \otimes \mathrm{id}_{\mathbb{C}[\![t]\!]} + \mathrm{id}_V \otimes \partial_t$, and for a node $Q \in C$ locally given by $s_+ s_- = 0$, the endomorphism of $\mathcal{V}_C(D_Q)$ given by

$$(21) \qquad \left(L_{-1} \otimes s_+ \mathrm{id}_{\mathbb{C}[\![s_+]\!]} + \mathrm{id}_V \otimes s_+ \partial_{s_+}, -L_{-1} \otimes s_- \mathrm{id}_{\mathbb{C}[\![s_-]\!]} - \mathrm{id}_V \otimes s_- \partial_{s_-} \right).$$

2.8. The coordinate-free Lie algebra ancillary to V

As a first application, one obtains a coordinate-free version of the Lie algebra ancillary to V. For a punctured disk $D_P^\times$ about a smooth point P on C and a formal coordinate t at P, one has

$$(22) \qquad\qquad \mathrm{H}^0\left(D_P^\times, \mathcal{V}_C \otimes \omega_C / \mathrm{Im}\nabla \right) \xrightarrow{\simeq_t} \mathfrak{L}_t(V).$$

A section of $\mathcal{V}_C \otimes \omega_C$ on $D_P^\times$ with respect to the t-trivialization

$$B \otimes \sum_{i \geq i_0} a_i t^i dt \quad \in \quad V \otimes_{\mathbb{C}} \mathbb{C}(\!(t)\!) \otimes_{\mathbb{C}(\!(t)\!)} \mathbb{C}(\!(t)\!) dt \quad \simeq_t \mathrm{H}^0\left(D_P^\times, \mathcal{V}_C \otimes \omega_C \right)$$

maps to

$$\mathrm{Res}_{t=0} Y[B,t] \sum_{i \geq i_0} a_i t^i dt \quad \in \quad \mathfrak{L}_t(V),$$

where $Y[B,t] := \sum_{i \in \mathbb{Z}} B_{[i]} t^{-i-1}$. Sections in $\mathrm{Im}\nabla \subset \mathcal{V}_C \otimes \omega_C$ map to zero, and this defines a linear map from sections of $\mathcal{V}_C \otimes \omega_C / \mathrm{Im}\nabla$ on $D_P^\times$ to $\mathfrak{L}_t(V)$. The vector space $\mathrm{H}^0\left(D_P^\times, \mathcal{V}_C \otimes \omega_C / \mathrm{Im}\nabla \right)$ has the structure of a Lie algebra such that (22) is an isomorphism of Lie algebras [38, §§19.4.14, 6.6.9].

3. The new chiral Lie algebra $\mathcal{L}_{C \setminus P_\bullet}(V)$

3.1. Definition of the chiral Lie algebra

For $(C, P_\bullet)$ a stable n-pointed curve and V a vertex operator algebra, set

$$\mathcal{L}_{C \setminus P_\bullet}(V) := \mathrm{H}^0\left(C \setminus P_\bullet, \frac{\mathcal{V}_C \otimes \omega_C}{\mathrm{Im}\nabla} \right).$$

Here $\mathcal{V}_C$ and its logarithmic connection ∇ are as in §2.

3.2. The chiral Lie algebra maps to the Lie algebra ancillary to V

For each i, let t_i be a formal coordinate at P_i, let $D_{P_i}^\times$ be the punctured formal disk about P_i on C, and $\mathfrak{L}_{t_i}(V)$ be the Lie algebra ancillary to V (§1.3). Consider the linear map obtained as the composition

$$\text{(23)} \qquad \mathcal{L}_{C \setminus P_\bullet}(V) \to \mathrm{H}^0\left(D_{P_i}^\times, \mathcal{V}_C \otimes \omega_C / \mathrm{Im}\nabla\right) \xrightarrow{\cong} \mathfrak{L}_{t_i}(V).$$

The first map is canonical and obtained by restricting sections. The second map is the isomorphism of Lie algebras (22) and depends on the formal coordinates t_i. From (23), we construct the linear map

$$\text{(24)} \qquad \varphi_{\mathcal{L}} \colon \mathcal{L}_{C \setminus P_\bullet}(V) \to \bigoplus_{i=1}^n \mathrm{H}^0\left(D_{P_i}^\times, \mathcal{V}_C \otimes \omega_C / \mathrm{Im}\nabla\right) \xrightarrow{\cong} \bigoplus_{i=1}^n \mathfrak{L}_{t_i}(V).$$

After [38, §19.4.14], when C is smooth, the first map of (23) is a homomorphism of Lie algebras, hence so is $\varphi_{\mathcal{L}}$, next denoted simply φ. The map φ thus induces an action of $\mathcal{L}_{C \setminus P_\bullet}(V)$ on $\mathfrak{L}(V)^{\oplus n}$-modules. This will be used in §4. In Proposition 3.3.2 we show an analogous result for nodal curves.

3.3. A close look at the chiral Lie algebra for nodal curves

Let $(C, P_\bullet)$ be a stable n-pointed curve such that $C \setminus P_\bullet$ is affine. Assume for simplicity that C has exactly one simple node, which we denote by Q. Let $\eta \colon \widetilde{C} \to C$ be the normalization of C, let Q_+ and Q_- be the two preimages of Q, and set $Q_\bullet = (Q_+, Q_-)$. Let s_+ and s_- be formal coordinates at Q_+ and Q_-, respectively, such that locally around Q, the curve C is given by the equation $s_+ s_- = 0$. The chiral Lie algebra for $(\widetilde{C}, P_\bullet \sqcup Q_\bullet)$ is

$$\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V) = \mathrm{H}^0\left(\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet, \mathcal{V}_{\widetilde{C}} \otimes \omega_{\widetilde{C}} / \mathrm{Im}\,\nabla\right),$$

and consider the linear map given by restriction:

$$\text{(25)} \qquad \mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V) \to \mathrm{H}^0\left(D_{Q_+}^\times, \mathcal{V}_{\widetilde{C}} \otimes \omega_{\widetilde{C}} / \mathrm{Im}\,\nabla\right) \xrightarrow{\simeq_{s_+}} \mathfrak{L}_{Q_+}(V).$$

Recall the triangular decomposition of $\mathfrak{L}_{Q_\pm}(V)$ from (5):

$$\mathfrak{L}_{Q_\pm}(V) = \mathfrak{L}_{Q_\pm}(V)_{<0} \oplus \mathfrak{L}_{Q_\pm}(V)_0 \oplus \mathfrak{L}_{Q_\pm}(V)_{>0}.$$

Let $\sigma_{Q_\pm} \in \mathfrak{L}_{Q_\pm}(V)$ be the image of $\sigma \in \mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)$, and let $[\sigma_{Q_\pm}]_0$ be the image of $\sigma_{Q_\pm}$ under the projection $\mathfrak{L}_{Q_\pm}(V) \to \mathfrak{L}_{Q_\pm}(V)_0$.

Recall the involution ϑ of $\mathfrak{L}(V)^f$ in (7). This restricts to an involution on $\mathfrak{L}(V)_0$ given for homogeneous $A \in V$ of degree a by

$$\vartheta\left(A_{[a-1]}\right) = (-1)^{a-1} \sum_{i \geq 0} \frac{1}{i!} \left(L_1^i A\right)_{[a-i-1]}.$$

PROPOSITION 3.3.1. – *For $C \setminus P_\bullet$ affine, one has*

$$\eta^* \mathcal{L}_{C \setminus P_\bullet}(V) = \left\{ \sigma \in \mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V) \,\middle|\, \begin{array}{l} \text{(i) } \sigma_{Q_+}, \sigma_{Q_-} \in \mathfrak{L}(V)_{\leq 0} \\ \text{(ii) } [\sigma_{Q_-}]_0 = \vartheta\left([\sigma_{Q_+}]_0\right) \end{array} \right\}.$$

Proof. – Since $C \setminus P_\bullet$ is affine, one has

$$\mathcal{L}_{C \setminus P_\bullet}(V) = \mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C \right) / \nabla \mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \right)$$

and similarly, since $\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet$ is also affine, one has

$$\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V) = \mathrm{H}^0 \left(\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet, \mathcal{V}_{\widetilde{C}} \otimes \omega_{\widetilde{C}} \right) / \nabla \mathrm{H}^0 \left(\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet, \mathcal{V}_{\widetilde{C}} \right).$$

To characterize elements in $\mathcal{L}_{C \setminus P_\bullet}(V)$, we can first describe their lifts in the vector space $\mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C \right)$, and then show that the description descends to the quotient by the image of ∇.

By definition of $\mathcal{V}_C$ over nodal curves via the sheaf $\widetilde{\mathcal{V}}$ in (12) and by Lemma 1.9.1, we have the inclusion

$$(26) \qquad \eta^* \, \mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C \right) \subseteq \mathrm{H}^0 \left(\widetilde{C} \setminus P_\bullet, \widetilde{\mathcal{V}} \otimes \omega_{\widetilde{C}}(Q_+ + Q_-) \right).$$

To see that (26) implies (i), consider the composition of linear maps

$$(27) \qquad
\begin{array}{ccc}
\mathrm{H}^0 \left(\widetilde{C} \setminus P_\bullet, \widetilde{\mathcal{V}} \otimes \omega_{\widetilde{C}}(Q_+ + Q_-) \right) & \dashrightarrow & \bigoplus\limits_{k \geq 0} V_k \otimes_{\mathbb{C}} s_\pm^{k-1} \mathbb{C}[\![s_\pm]\!] ds_\pm \\
\downarrow & \nearrow_{\simeq s_\pm} & \\
\mathrm{H}^0 \left(D_{Q_\pm}, \widetilde{\mathcal{V}} \otimes \omega_{\widetilde{C}}(Q_+ + Q_-) \right), &&
\end{array}$$

where the left vertical map is the restriction, followed by the $s_\pm$-trivialization. By §2.8, the projection $V \otimes \mathbb{C}(\!(s_\pm)\!) ds_\pm \to \mathfrak{L}_{Q_\pm}(V)$ is given by

$$B \otimes \mu \mapsto \mathrm{Res}_{s_\pm = 0} \, Y[B, s_\pm] \mu \quad \in \quad \mathfrak{L}_{Q_\pm}(V).$$

It follows that the image of

$$(28) \qquad \bigoplus\limits_{k \geq 0} V_k \otimes_{\mathbb{C}} s_\pm^{k-1} \mathbb{C}[\![s_\pm]\!] ds_\pm \to \mathfrak{L}_{Q_\pm}(V)$$

lies in $\mathfrak{L}_{Q_\pm}(V)_{\leq 0}$. Composing (27) with (28), we deduce that for σ in (26), its image σ_{Q_+} in $\mathfrak{L}_{Q_\pm}(V)$ lies in $\mathfrak{L}_{Q_\pm}(V)_{\leq 0} \cong \mathfrak{L}(V)_{\leq 0}$, hence (i).

The assertion (ii) follows from the gluing isomorphisms that define $\mathcal{V}_C$ and ω_C as subsheaves of $\eta_* \widetilde{\mathcal{V}}$ and $\eta_* \omega_{\widetilde{C}}(Q_+ + Q_-)$, respectively, using diagrams as in (14). Given σ in (26), denote by $\left[\sigma_{Q_\pm} \right]_0$ its image via the composition of (27) with the projection

$$\bigoplus\limits_{k \geq 0} V_k \otimes_{\mathbb{C}} s_\pm^{k-1} \mathbb{C}[\![s_\pm]\!] ds_\pm \twoheadrightarrow \bigoplus\limits_{k \geq 0} V_k \otimes_{\mathbb{C}} s_\pm^{k-1} ds_\pm.$$

In view of a diagram as in (14), $\left[\sigma_{Q_\pm} \right]_0$ is the restriction of σ at the fibers at $Q_\pm$. For σ to correspond to a section of $\eta^* \mathrm{H}^0(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C)$, the elements $\left[\sigma_{Q_+} \right]_0$ and $\left[\sigma_{Q_-} \right]_0$ need to satisfy an identity coming from the gluing isomorphism between the fibers of $\widetilde{\mathcal{V}} \otimes \omega_{\widetilde{C}}(Q_+ + Q_-)$ at $Q_\pm$. For homogeneous $A \in V$ of degree a, the gluing isomorphism on fibers of $\eta_* \widetilde{\mathcal{V}}$ at $Q_\pm$ defining $\mathcal{V}_C$ is given by

$$A \otimes s_+^a \mapsto (-1)^a \sum\limits_{i \geq 0} \frac{1}{i!} L_1^i(A) \otimes s_-^{a-i}.$$

The gluing on fibers of $\eta_* \, \omega_{\widetilde{C}}(Q_+ + Q_-)$ at $Q_\pm$ defining ω_C is given by $s_+^{-1} ds_+ \mapsto -s_-^{-1} ds_+$. Combining the two, the induced gluing on fibers of $\widetilde{\mathcal{V}} \otimes \omega_{\widetilde{C}}(Q_+ + Q_-)$ at $Q_\pm$ is given by

$$(29) \qquad A \otimes s_+^{a-1} ds_+ \mapsto (-1)^{a-1} \sum_{i \geq 0} \frac{1}{i!} L_1^i(A) \otimes s_-^{a-i-1} ds_-.$$

After mapping to $\mathfrak{L}_{Q_\pm}(V)_0$ via the restriction of (28), the gluing (29) implies that the images of the elements $\left[\sigma_{Q_\pm}\right]_0$ in $\mathfrak{L}_{Q_\pm}(V)_0$, still denoted $\left[\sigma_{Q_\pm}\right]_0$, satisfy (ii) by definition of ϑ.

To show that the conditions are ∇-equivariant, it is enough to check on a neighborhood of Q. Since ∇ on $\mathcal{V}_C$ is constructed from ∇ on $\mathcal{V}_{\widetilde{C}}$, one verifies that

$$\eta^* \nabla \mathcal{V}_C(D_Q) \subset \nabla \mathcal{V}_{\widetilde{C}}(D_{Q_+}^\times \sqcup D_{Q_-}^\times),$$

hence the statement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We combine §3.2 with Proposition 3.3.1 to show the following:

PROPOSITION 3.3.2. – *The normalization map* $\eta \colon \widetilde{C} \to C$ *identifies* $\mathcal{L}_{C \setminus P_\bullet}(V)$ *with a Lie subalgebra of* $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)$. *Moreover, this induces an action of* $\mathcal{L}_{C \setminus P_\bullet}(V)$ *on* $\mathfrak{L}(V)^{\oplus n}$-*modules.*

Proof. – For simplicity, we may assume that C has a single node Q. By Proposition 3.3.1, $\eta^* \mathcal{L}_{C \setminus P_\bullet}(V)$ can be identified with the subspace of sections in $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)$ whose restriction to $Q_\pm$ is given by the subspace of $\mathfrak{L}_{Q_+}(V) \oplus \mathfrak{L}_{Q_-}(V) \simeq_{s_\pm} \mathfrak{L}(V)^{\oplus 2}$ generated by $\mathfrak{L}(V)_{<0}^{\oplus 2}$ and the elements of type $(A_{[a-1]}, \vartheta(A_{[a-1]})) \in \mathfrak{L}(V)_0^{\oplus 2}$ for homogeneous $A \in V$ of degree a.

Since $\mathfrak{L}(V)_{<0}$ and $\mathfrak{L}(V)_0$ are Lie subalgebras of $\mathfrak{L}(V)$ and ϑ is a Lie algebra morphism, it follows that $\eta^* \mathcal{L}_{C \setminus P_\bullet}(V)$ is a Lie subalgebra of $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)$.

To conclude, the analogue of the morphism (24) for the nodal $C \setminus P_\bullet$ is the composition of the Lie algebra morphisms:

$$
\begin{array}{c}
\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V) \\
{\scriptstyle \eta^*} \uparrow \qquad \searrow {\scriptstyle \widetilde{\varphi}} \\
\mathcal{L}_{C \setminus P_\bullet}(V) \dashrightarrow_{\varphi} \bigoplus_{i=1}^n \mathfrak{L}_{t_i}(V),
\end{array}
$$

where η^* is described in Proposition 3.3.1 and $\widetilde{\varphi}$ is as in (24). $\qquad\qquad\qquad\square$

3.4. A consequence of Riemann-Roch for chiral Lie algebras

We give here a statement parallel to §1.10 for chiral Lie algebras. Let C be a smooth curve, possibly disconnected, with two non-empty sets of distinct marked points $P_\bullet = (P_1, \ldots, P_n)$ and $Q_\bullet = (Q_1, \ldots, Q_m)$. For $i \in \{1, \ldots, m\}$, let s_i be a formal coordinate at Q_i. For $\sigma \in \mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)$, let σ_{Q_i} be the image of σ under the map given by restriction

$$\mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V) \to \mathrm{H}^0\left(D_{Q_i}^\times, \mathcal{V}_C \otimes \omega_C / \mathrm{Im}\,\nabla\right) \xrightarrow{\simeq_{s_i}} \mathfrak{L}_{Q_i}(V).$$

For an integer N, consider

$$\mathfrak{L}_{Q_i}(V, N Q_i) = V \otimes s_i^N \mathbb{C}[\![s_i]\!] / \mathrm{Im}\,\partial.$$

This is a Lie subalgebra of $\mathfrak{L}_{Q_i}(V)$.

PROPOSITION 3.4.1. – *Assume $C \setminus P_\bullet$ is affine, fix $E \in V$ homogeneous, and integers d and N. There exists $\sigma \in \mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)$ such that:*

$$\sigma_{Q_i} \equiv E_{[d]} \in \mathfrak{L}_{Q_i}(V)/\mathfrak{L}_{Q_i}(V, NQ_i), \qquad \text{for a fixed } i,$$

$$\sigma_{Q_j} \equiv 0 \in \mathfrak{L}_{Q_j}(V)/\mathfrak{L}_{Q_j}(V, NQ_j), \qquad \text{for all } j \neq i.$$

Proof. – Since $C \setminus P_\bullet$ is affine, so is $C \setminus P_\bullet \sqcup Q_\bullet$. As in Proof of Proposition 3.3.1, elements of $\mathcal{L}_{C \setminus P_\bullet \sqcup Q_\bullet}(V)$ can be lifted to sections of $\mathcal{V}_C \otimes \omega_C$ on $C \setminus P_\bullet \sqcup Q_\bullet$, and thus, via Lemma 2.6.3, described as sections of $\bigoplus_{k \geq 0}(\omega_C^{\otimes 1-k})^{\oplus \dim V_k}$ on $C \setminus P_\bullet \sqcup Q_\bullet$. The statement thus follows from the analogous property of sections of tensor products of ω_C, discussed in §1.10.   □

4. Spaces of coinvariants

Given a stable pointed curve $(C, P_\bullet)$ and a vertex operator algebra V, we define spaces of coinvariants using representations of the chiral Lie algebra.

4.1. Representations of the chiral Lie algebra

The chiral Lie algebra $\mathcal{L}_{C \setminus P_\bullet}(V)$ acts on the tensor product $M^\bullet := M^1 \otimes \cdots \otimes M^n$ of V-modules $M^1, \ldots, M^n$. For each i, let t_i be a formal coordinate at P_i, and $\mathfrak{L}_{t_i}(V)$ be the Lie algebra ancillary to V (§1.3). Each $\mathfrak{L}_{t_i}(V)$ acts on the V-module M^i as in §1.3, and the sum $\bigoplus_{i=1}^n \mathfrak{L}_{t_i}(V)$ acts diagonally on $M^\bullet$. The map (24) thus induces an action of $\mathcal{L}_{C \setminus P_\bullet}(V)$ on $M^\bullet$ as follows: for $\sigma \in \mathcal{L}_{C \setminus P_\bullet}(V)$ and $A^i \in M^i$, one has

$$\sigma \left(A^1 \otimes \cdots \otimes A^n \right) = \sum_{i=1}^n A^1 \otimes \cdots \otimes \sigma_{P_i} \left(A^i \right) \otimes \cdots \otimes A^n,$$

where σ_{P_i} is the restriction of the section σ to the punctured formal disk $D_{P_i}^\times$ about P_i on C.

4.2. Coinvariants

When $C \setminus P_\bullet$ is affine, the *space of coinvariants* at $(C, P_\bullet, t_\bullet)$ is

$$\mathbb{V}\left(V; M^\bullet\right)_{(C,P_\bullet,t_\bullet)} := M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V)} = M^\bullet / \mathcal{L}_{C \setminus P_\bullet}(V) \cdot M^\bullet.$$

This is the largest quotient of $M^\bullet$ on which $\mathcal{L}_{C \setminus P_\bullet}(V)$ acts trivially. In general, when $C \setminus P_\bullet$ is not necessarily affine, the *space of coinvariants* at $(C, P_\bullet, t_\bullet)$ is defined as the direct limit

$$(30) \qquad \mathbb{V}\left(V; M^\bullet\right)_{(C,P_\bullet,t_\bullet)} := \varinjlim_{(Q_\bullet,s_\bullet)} \mathbb{V}\left(V; M^\bullet \sqcup (V, \ldots, V)\right)_{(C,P_\bullet \sqcup Q_\bullet,t_\bullet \sqcup s_\bullet)},$$

where $Q_\bullet = (Q_1, \ldots, Q_m)$ ranges over the set of smooth points of C such that $P_\bullet \cap Q_\bullet = \emptyset$ and $C \setminus P_\bullet \sqcup Q_\bullet$ is affine, and $s_\bullet = (s_1, \ldots, s_m)$, with s_i a formal coordinate at Q_i, for each i. As in the case of affine Lie algebras [36, 61], the above direct limit is well defined thanks to the propagation of vacua theorem, which is discussed in §4.3.

The construction of the chiral Lie algebra in §3 extends to families of *smooth* pointed curves over an arbitrary smooth base, and one obtains sheaves of coinvariants as follows. Let $(C \to S, P_\bullet)$ be a family of smooth n-pointed curves. In this case, $C \setminus P_\bullet(S)$ is affine over S. Let t_i be formal coordinates at $P_i(S)$, for $i = 1, \ldots, n$. Equivalently, fix a formally

unramified thickening $S \times \mathrm{Spf}(\mathbb{C}[\![t_i]\!]) \to C$ of the section P_i, for each i. One then obtains sheaves of Lie algebras $\mathcal{L}_{C \setminus P_\bullet}(V)$ and of coinvariants

$$\mathbb{V}(V; M^\bullet)_{(C/S, P_\bullet, t_\bullet)} := \big(M^\bullet \otimes \mathscr{O}_S\big)_{\mathcal{L}_{C \setminus P_\bullet}(V)}$$

over S, for given V-modules $M^1, \ldots, M^n$.

We will extend sheaves of coinvariants over families of *stable* pointed curves over an arbitrary smooth base in §8.7.

4.3. Propagation of vacua

The propagation of vacua theorem, first proved by Tsuchiya, Ueno, and Yamada for spaces of coinvariants constructed from representations of affine Lie algebras [74, Prop 2.2.3, Cor 2.2.4], says that spaces of coinvariants associated to a stable n-pointed curve with coordinates remain invariant when adding a new marked point and the trivial module.

The result was established in the generality we need here by Codogni [19, Thm 3.6] (see also [21, Thm 6.2]). Other special cases were previously treated in the literature, including the case of coinvariants defined by quasi-primary generated vertex operator algebras V for which $V_0 \cong \mathbb{C}$ either at a fixed smooth pointed curve with coordinates [76, 3], or on stable pointed rational curves [67]. Moreover, propagation of vacua was proved for conformal blocks defined at a fixed smooth curve in [38, §10.3.1].

To state the theorem, we need the following setup. Let $(C \to S, P_\bullet, t_\bullet)$ be a family of stable n-pointed curves with coordinates. Let $Q \colon S \to C$ be a section such that $Q(S)$ is contained in the smooth locus of C and is disjoint from $P_i(S)$, for each $i \in \{1, \ldots, n\}$, and let r be a formal coordinate at $Q(S)$.

THEOREM 4.3.1 (Propagation of Vacua [19, Thm 3.6]). – *Let V be a vertex operator algebra with one-dimensional weight zero space. Assume that $C \setminus P_\bullet(S)$ is affine over S. The linear map*

$$M^\bullet \to M^\bullet \otimes V, \qquad u \mapsto u \otimes 1^V$$

induces a canonical $\mathscr{O}_S$-module isomorphism

$$\mathbb{V}\big(V; M^\bullet\big)_{(C/S, P_\bullet, t_\bullet)} \xrightarrow{\cong} \mathbb{V}\big(V; M^\bullet \otimes V\big)_{(C/S, P_\bullet \sqcup Q, t_\bullet \sqcup r)}.$$

Varying (Q, r), the induced isomorphisms are compatible. Moreover, as 1^V is fixed by the action of $\mathrm{Aut}\,\mathcal{O}$, the isomorphism is equivariant with respect to change of coordinates.

The proof requires two main ingredients: (1) the axiom on the vacuum vector; and (2) the existence of a PBW basis for V [44].

5. Finite-dimensionality of coinvariants

Using coinvariants by the action of Zhu's Lie algebra (§), Abe and Nagatomo show that spaces of coinvariants at smooth pointed curves of arbitrary genus are finite-dimensional [3].

We show here that the result of [3] extends to coinvariants by the action of the chiral Lie algebra. Moreover, we further extend the result in [3] by allowing the following twist of the chiral Lie algebra: given a smooth n-pointed curve $(C, P_\bullet)$, and an effective divisor $D = \sum_{i=1}^{m} n_i Q_i$ on C not supported at $P_\bullet$, consider

$$(31) \qquad \mathcal{L}_{C \setminus P_\bullet}(V, D) := \mathrm{H}^0\left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C(-D)/\mathrm{Im}\nabla\right),$$

where $\mathrm{Im}\nabla$ denotes the intersection of $\nabla(\mathcal{V}_C)$ and $\mathcal{V}_C \otimes \omega_C(-D)$. This is the space of sections in $\mathcal{L}_{C \setminus P_\bullet}(V)$ vanishing with order at least n_i at Q_i, for each i, and gives a Lie subalgebra of $\mathcal{L}_{C \setminus P_\bullet}(V)$.

5.1. C_2-cofiniteness

For $k \geq 2$ and a V-module M (e.g., $M = V$), set:

$$C_k(M) := \mathrm{span}_{\mathbb{C}}\left\{A_{(-k)}m \,:\, A \in V, m \in M\right\}.$$

One says that M is C_k-*cofinite* if $\dim_{\mathbb{C}} M/C_k(M) < \infty$. For a C_2-cofinite vertex operator algebra V with one-dimensional weight zero space, any finitely generated V-module is C_k-cofinite, for $k \geq 2$ [17]. As explained in [5], the C_2-cofiniteness has a natural geometric interpretation which generalizes the concept of lisse modules introduced in [15] for the Virasoro algebra.

PROPOSITION 5.1.1. – *Let V be a C_2-cofinite vertex operator algebra with one-dimensional weight zero space. Let C be a smooth curve with distinct points $P_1, \ldots, P_n$, and D an effective divisor on C not supported at $P_\bullet$. Fix formal coordinates t_i at P_i, for each i. For finitely generated V-modules $M^1, \ldots, M^n$, the coinvariants $M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V, D)}$ are finite-dimensional.*

Proof. – Recall the map from (23) obtained by fixing the formal coordinates t_i at P_i, for each i: $\mathcal{L}_{C \setminus P_\bullet}(V, D) \to \mathfrak{L}_{P_i}(V)$, $\sigma \mapsto \sigma_{P_i}$. For $k \in \mathbb{N}$, define

$$\mathcal{F}_k \mathcal{L}_{C \setminus P_\bullet}(V, D) := \left\{\sigma \in \mathcal{L}_{C \setminus P_\bullet}(V, D) \mid \deg \sigma_{P_i} \leq k, \text{ for all } i\right\},$$

which gives $\mathcal{L}_{C \setminus P_\bullet}(V, D)$ the structure of a filtered Lie algebra. Let

$$\mathcal{F}_k M^\bullet = \bigoplus_{0 \leq d \leq k} M^\bullet_d, \quad \text{where} \quad M^\bullet_d := \sum_{d_1 + \cdots + d_n = d} M^1_{d_1} \otimes \cdots \otimes M^n_{d_n}.$$

Since $\mathcal{F}_k \mathcal{L}_{C \setminus P_\bullet}(V, D) \cdot \mathcal{F}_l M^\bullet \subset \mathcal{F}_{k+l} M^\bullet$, the $\mathcal{L}_{C \setminus P_\bullet}(V, D)$-module $M^\bullet$ is a filtered $\mathcal{L}_{C \setminus P_\bullet}(V, D)$-module. One has an induced filtration on $M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V, D)}$:

$$\mathcal{F}_k\left(M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V, D)}\right) := \left(\mathcal{F}_k M^\bullet + \mathcal{L}_{C \setminus P_\bullet}(V, D) \cdot M^\bullet\right)/\mathcal{L}_{C \setminus P_\bullet}(V, D) \cdot M^\bullet.$$

Step 1. – Let U be a finite-dimensional subspace of V such that $V = U \oplus C_2(V)$. Contrary to [3], elements of U are not required to be quasi-primary here. Let d_U be the maximum of the degree of the homogeneous elements in U. Similar to [3, Lemma 4.1], by an application of the Riemann-Roch and the Weierstrass gap theorem, there exists an integer N such that

$$\mathrm{H}^0\left(C, \omega_C^{\otimes 1-k}(lP_i - D)\right) \neq 0, \quad \text{for all } k \leq d_U, l \geq N, i \in \{1, \ldots, n\}.$$

Step 2. – For a V-module M and with N as in Step 1, define the subset

$$C_N(U, M) = \operatorname{span}_{\mathbb{C}} \left\{ A_{(-l)}m \; : \; A \in U, m \in M, l \geq N \right\}.$$

We claim that for each i the set $M^1 \otimes \cdots \otimes C_N \left(U, M^i\right) \otimes \cdots \otimes M^n$ is in the kernel of the canonical surjective linear map

$$M^\bullet \xrightarrow{\pi} \operatorname{gr}_\bullet \left(M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V, D)} \right) := \underset{k \geq 0}{\oplus} \mathcal{F}_k \left(M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V, D)} \right) / \mathcal{F}_{k-1} \left(M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V, D)} \right).$$

For this, it is enough to show that $\pi \left(m_1 \otimes \cdots \otimes A_{(-l)}m_i \otimes \cdots \otimes m_n\right) = 0$, for homogeneous $A \in U$ of degree a, $m_i \in M^i_{d_i}$, and $l \geq N$. Note that $C \setminus P_i$ is affine for all i. As in the proof of Proposition 3.3.1, elements of $\mathcal{L}_{C \setminus P_i}(V, D) \subset \mathcal{L}_{C \setminus P_\bullet}(V)$ can be lifted to sections of $\mathcal{V}_C \otimes \omega_C(-D)$ on $C \setminus P_i$. By Lemmas 2.6.3 and 2.6.1, the vector space of such sections is isomorphic to the space of sections of

$$(32) \qquad \bigoplus_{k \geq 0} V_k \otimes \omega_C^{\otimes 1-k}(-D)$$

on $C \setminus P_i$. Following Step 1, there exists a section $\sigma = A \otimes \mu$ of (32) on $C \setminus P_i$ such that its image via the map $\mathcal{L}_{C \setminus P_i}(V, D) \to \mathfrak{L}_{P_i}(V)$ from (23) is

$$\sigma_{P_i} = A_{[-l]} + \sum_{j > -l} c_j A_{[j]}, \quad \text{for some } c_j \in \mathbb{C}.$$

One has $A_{[-l]} \cdot M^i_{d_i} \subset M^i_{d_i + a + l - 1}$ and $A_{[j]} \cdot M^i_{d_i} \subset M^i_{d_i + a + l - 2}$ for $j > -l$. Moreover, since μ is holomorphic at a point $P_r \neq P_i$, one has $\sigma_{P_r} = \sum_{s \geq 0} a_s A_{[s]}$, for some $a_s \in \mathbb{C}$. It follows that $\sigma_{P_r} \cdot M^r_{d_r} \subset M^r_{d_r + a - 1}$. From the identity

$$\sigma \left(m_1 \otimes \cdots \otimes m_n\right) = \sum_{r=1}^n m_1 \otimes \cdots \otimes \sigma_{P_r}(m_r) \otimes \cdots \otimes m_n,$$

one has

$$m_1 \otimes \cdots \otimes A_{(-l)}m_i \otimes \cdots \otimes m_n \in \mathcal{F}_{\sum_r d_r + a + l - 2}M^\bullet + \mathcal{L}_{C \setminus P_\bullet}(V, D) \cdot M^\bullet.$$

Since the element on the left-hand side is in $\mathcal{F}_{\sum_r d_r + a + l - 1}M^\bullet$, it follows that it maps to zero via π. The claim follows.

Step 3. – After Step 2, the map π factors through

$$(33) \qquad M^1 / C_N \left(U, M^1\right) \otimes \cdots \otimes M^n / C_N \left(U, M^n\right) \xrightarrow{\pi} \operatorname{gr}_\bullet \left(M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V, D)} \right).$$

By [3, Prop. 4.5], there is a positive integer k such that $C_k \left(M^i\right) \subset C_N \left(U, M^i\right)$ for all i. In particular, $\dim M^i / C_N \left(U, M^i\right) < \dim M^i / C_k \left(M^i\right)$. These are finite as the M^i are all C_k-cofinite by [17]. It follows that the source in (33) is finite-dimensional, hence so is the target. This implies that the coinvariants are finite-dimensional as well. □

The proof of Proposition 5.1.1 extends over families of smooth curves $C \to S$ with n disjoint sections $P_1, \ldots, P_n$, and for each i, a formal coordinate t_i at $P_i(S)$. Hence, we conclude:

COROLLARY 5.1.2. – *Let V be a C_2-cofinite vertex operator algebra with one-dimensional weight zero space. For any collection of finitely generated V-modules $M^1, \ldots, M^n$, the sheaf of coinvariants $\mathbb{V}(V; M^\bullet)_{(C/S, P_\bullet, t_\bullet)}$ is a coherent $\mathcal{O}_S$-module.*

6. The modules Z and $\overline{Z}$

In service of the proof of the factorization theorem, we consider the modules Z and $\overline{Z}$ in §6.1, and coinvariants constructed from them in §6.2.

6.1. Definitions and properties

Let V be a vertex operator algebra. Recall the associative algebra $\mathscr{U}(V)$ from §1.4. Consider the $\mathscr{U}(V)^{\otimes 2}$-module

$$Z := \left(\mathrm{Ind}_{\mathscr{U}(V)_{\leq 0}}^{\mathscr{U}(V)} A(V)\right)^{\otimes 2} = \left(\mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} A(V)\right)^{\otimes 2},$$

where $\mathscr{U}(V)_{<0}$ acts trivially on $A(V)$, and the action of $\mathscr{U}(V)_0$ on $A(V)$ is induced from the projection $\mathscr{U}(V)_0 \to A(V)$. With the notation from §1.6, one has that $Z = M(A(V))^{\otimes 2}$, where $M(A(V))$ is the generalized Verma $\mathscr{U}(V)$-module induced from the natural representation $A(V)$ of $A(V)$.

We will also consider a quotient $\overline{Z}$ of Z defined as follows. Let $\mathscr{P}$ be the subalgebra of $\mathscr{U}(V)^{\otimes 2}$ generated by $\mathscr{U}(V) \otimes_{\mathbb{C}} \mathscr{U}(V)_{<0}$, $\mathscr{U}(V)_{<0} \otimes_{\mathbb{C}} \mathscr{U}(V)$, and $\mathscr{U}(V)_0 \otimes_{\mathbb{C}} \mathscr{U}(V)_0$. Consider the $\mathscr{U}(V)^{\otimes 2}$-module

$$\overline{Z} := \mathrm{Ind}_{\mathscr{P}}^{\mathscr{U}(V)^{\otimes 2}} A(V) = \left(\mathscr{U}(V)^{\otimes 2}\right) \otimes_{\mathscr{P}} A(V),$$

where $\mathscr{U}(V) \otimes_{\mathbb{C}} \mathscr{U}(V)_{<0}$ and $\mathscr{U}(V)_{<0} \otimes_{\mathbb{C}} \mathscr{U}(V)$ act trivially on $A(V)$, and the action of $\mathscr{U}(V)_0 \otimes_{\mathbb{C}} \mathscr{U}(V)_0$ on $A(V)$ is induced via the natural surjection

$$\mathscr{U}(V)_0 \otimes_{\mathbb{C}} \mathscr{U}(V)_0 \to A(V) \otimes_{\mathbb{C}} A(V)$$

from the action of $A(V) \otimes_{\mathbb{C}} A(V)$ given by

$$(a \otimes b)(c) = a \cdot c \cdot (-\vartheta(b)), \quad \text{for } a \otimes b \in A(V) \otimes A(V), c \in A(V).$$

LEMMA 6.1.1. – *Let V be a rational vertex operator algebra. One has $\mathscr{U}(V)^{\otimes 2}$-module isomorphisms*

$$Z \cong \bigoplus_{W,Y \in \mathscr{W}} \left(W \otimes W_0^{\vee}\right) \otimes \left(Y \otimes Y_0^{\vee}\right) \text{ and } \overline{Z} \cong \bigoplus_{W \in \mathscr{W}} W \otimes W',$$

with $\mathscr{W}$ the set of representatives of isomorphism classes of simple V-modules.

Proof. – Since V is rational, the algebra $A(V)$ is semisimple [77]. From Wedderburn's theorem, one has $A(V) = \bigoplus_{E \in \mathscr{E}} E \otimes E^{\vee}$, where $\mathscr{E}$ is the finite set of representatives of isomorphism classes of simple $A(V)$-modules. Using the one-to-one correspondence between simple V-modules and simple $A(V)$-modules [77], and rationality of V which implies that the V-module induced from any simple $A(V)$-module is simple, it follows that each simple V-module is $W = \mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E$, for some $E \in \mathscr{E}$. Moreover, there exists a canonical V-module isomorphism $\mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E^{\vee} \cong (\mathscr{U}(V) \otimes_{\mathscr{U}(V)_{\leq 0}} E)'$, for $E \in \mathscr{E}$ [67, Prop. 7.2.1]. The statement follows by linearity. $\qquad\square$

6.2. Replacing coinvariants with Z

The main result of this section is Proposition 6.2.1, which generalizes [67, Prop. 7.2.2, Cor. 8.6.2] to curves of arbitrary genus. The statement describes coinvariants of the action of a Lie subalgebra $\mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\})$ of the chiral Lie algebra $\mathcal{L}_{C \backslash P_\bullet}(V)$.

We begin by defining $\mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\})$. For this, let C be a smooth curve, possibly disconnected, with two nonempty, disjoint sets of distinct marked points $P_\bullet = (P_1, \ldots, P_n)$ and $Q_\bullet = (Q_+, Q_-)$. Assume that $C \backslash P_\bullet$ is affine. After Lemmas 2.6.3 and 2.6.1, one has

$$\mathrm{H}^0\left(C \backslash P_\bullet, \mathcal{V}_C\right) \cong \bigoplus_{k \geq 0} \mathrm{H}^0\left(C \backslash P_\bullet, V_k \otimes_{\mathbb{C}} \omega_C^{\otimes -k}\right).$$

Fixing an isomorphism, consider the following Lie subalgebra of the chiral Lie algebra $\mathcal{L}_{C \backslash P_\bullet}(V)$:

$$(34) \qquad \mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\}) := \frac{\bigoplus_{k \geq 0} \mathrm{H}^0\left(C \backslash P_\bullet, V_k \otimes_{\mathbb{C}} \omega_C^{\otimes 1-k}(-kQ_+ - kQ_-)\right)}{\nabla \mathrm{H}^0(C \backslash P_\bullet, \mathcal{V}_C)}.$$

As in (31), $\nabla \mathrm{H}^0(C \backslash P_\bullet, \mathcal{V}_C)$ is the intersection of $\mathrm{Im} \nabla$ with the subspace

$$\bigoplus_{k \geq 0} \mathrm{H}^0\left(C \backslash P_\bullet, V_k \otimes_{\mathbb{C}} \omega_C^{\otimes 1-k}(-kQ_+ - kQ_-)\right)$$

of $\mathrm{H}^0(C \backslash P_\bullet, \mathcal{V}_C \otimes \omega_C)$.

Select formal coordinates t_i at P_i and s_i at Q_i. Let $\mathfrak{L}_{P_\bullet}(V) := \bigoplus_{i=1}^n \mathfrak{L}_{P_i}(V)$ and $\mathfrak{L}_{Q_\bullet}(V) := \mathfrak{L}_{Q_+}(V) \oplus \mathfrak{L}_{Q_-}(V)$. There are Lie algebra injections

$$(35) \qquad \mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\}) \to \mathfrak{L}_{P_\bullet}(V) \text{ and } \mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\}) \to \mathfrak{L}_{Q_\bullet}(V).$$

The image of an element of (34) in $\mathfrak{L}_{Q_\bullet}(V) \cong \mathfrak{L}(V)^{\oplus 2}$ via the restriction map in (35) is

$$\sum_{i \geq k} a_i A_{[i]} \oplus \sum_{j \geq k} b_j A_{[j]} \quad \in \quad \mathfrak{L}(V)_{<0}^{\oplus 2} \subset \mathfrak{L}_{Q_\bullet}(V)$$

for homogeneous $A \in V$ of degree $k \geq 0$ and coefficients $a_i, b_j \in \mathbb{C}$.

We use here the assumption that V is C_1-cofinite, i.e., $\dim_{\mathbb{C}} V/C_1(V) < \infty$, where $C_1(V)$ is the subspace of V linearly spanned by $A_{(-1)}B$ for $A, B \in V_{>0}$ and by $L_{-1}V$. If V is C_1-cofinite and $V_0 \cong \mathbb{C} 1^V$, then lowest weight V-modules admit spanning sets of PBW-type [54]. Also, C_2-cofinitess implies C_1-cofinitess [54], hence the assumption here is weaker.

PROPOSITION 6.2.1. – *Consider $(C, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)$, i.e., a smooth coordinatized $(n+2)$-pointed curve, possibly disconnected, such that $C \backslash P_\bullet$ is affine. Let V be a C_1-cofinite vertex operator algebra with one-dimensional weight zero space, and let $M^1, \ldots, M^n$ be V-modules. The map*

$$M^\bullet \to M^\bullet \otimes_{\mathbb{C}} Z, \qquad w \mapsto w \otimes 1^{A(V)} \otimes 1^{A(V)},$$

where $1^{A(V)} \in A(V)$ is the unit, induces an isomorphism of vector spaces

$$h: M^\bullet_{\mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\})} \xrightarrow{\cong} \left(M^\bullet \otimes_{\mathbb{C}} Z\right)_{\mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)}.$$

Proof. – We proceed in three steps.

Step 1. – We first show that the map h is well-defined. Observe that Z is naturally equipped with a left action of $\mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)$ induced by the Lie algebra homomorphisms $\mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V) \to \mathfrak{L}_{Q_\bullet}(V) \to \mathscr{U}(V)^{\otimes 2}$.

Given an element $\sigma \in \mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\}) \subset \mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)$, let $\sigma_{P_\bullet}$ be the image of σ in $\mathfrak{L}_{P_\bullet}(V)$ via the restriction map in (35), and similarly let σ_{Q_i} be its image in $\mathfrak{L}_{Q_i}(V)$, for $i = 1, 2$. Since $\sigma_{Q_i} \in \mathfrak{L}(V)_{<0}$, the elements $\sigma_{Q_+} \otimes 1$ and $1 \otimes \sigma_{Q_-}$ act trivially on $A(V) \otimes A(V) \subset Z$. This implies

(36) $\sigma_{P_\bullet}(w) \otimes 1^{A(V)} \otimes 1^{A(V)} = \sigma \left(w \otimes 1^{A(V)} \otimes 1^{A(V)} \right)$

$$- w \otimes \sigma_{Q_+} \left(1^{A(V)} \right) \otimes 1^{A(V)} - w \otimes 1^{A(V)} \otimes \sigma_{Q_-} \left(1^{A(V)} \right)$$

$$= \sigma \left(w \otimes 1^{A(V)} \otimes 1^{A(V)} \right)$$

for $w \in M^\bullet$. It follows that the image of any element is independent of the equivalence class representative in the quotient, hence the map h between the spaces of coinvariants is well-defined.

Step 2. – Next, we show that the map h is surjective: Given $w \otimes z_1 \otimes z_2$ in $M^\bullet \otimes Z$, there exists $w' \in M^\bullet$ such that

$$w \otimes z_1 \otimes z_2 \equiv w' \otimes 1^{A(V)} \otimes 1^{A(V)} \quad \mod \mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V) (M^\bullet \otimes Z).$$

By linearity, and reordering elements in $\mathscr{U}(V)$, we can reduce to the case

$$z_1 \otimes z_2 = D_l \cdots D_1 1^{A(V)} \otimes E_m \cdots E_1 1^{A(V)},$$

with each D_i and E_j in $\mathfrak{L}(V)_{\geq 0}$. The surjectivity is clear when $l = m = 0$. By induction on l (and similarly on m), it is then enough to show that

$$w \otimes z_1 \otimes z_2 \equiv w' \otimes z_1' \otimes z_2 \quad \mod \mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z)$$

for some w' in $M^\bullet$, when $z_1 = D_{[d]}\left(z_1' \right)$ for some homogeneous $D \in V$ and $D_{[d]}$ in $\mathfrak{L}(V)_{\geq 0}$. Each component of the curve C has at least one of the marked points in $P_\bullet$. By Proposition 3.4.1, there exists $\sigma \in \mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)$ such that

$$\sigma_{Q_+} \equiv D_{[d]} \in \mathfrak{L}_{Q_+}(V)/\mathfrak{L}_{Q_+}(V, NQ_+),$$
$$\sigma_{Q_-} \equiv 0 \in \mathfrak{L}_{Q_-}(V)/\mathfrak{L}_{Q_-}(V, NQ_-),$$

for $N \gg 0$. It is enough to take N such that

$$D_{[i]}(z_1') \otimes z_2 = z_1' \otimes D_{[i]}(z_2) = 0 \quad \text{in } Z \text{ for all } i \geq N.$$

Such N exists because $\mathscr{U}(V)$ acts smoothly on each factor. This implies

$$\sigma_{Q_+}(z_1') \otimes z_2 + z_1' \otimes \sigma_{Q_-}(z_2) = D_{[d]} \cdot z_1' \otimes z_2 = z_1 \otimes z_2.$$

It follows that

$$w \otimes z_1 \otimes z_2 = \sigma(w \otimes z_1' \otimes z_2) - \sigma_{P_\bullet}(w) \otimes z_1' \otimes z_2,$$

hence

$$w \otimes z_1 \otimes z_2 \equiv -\sigma_{P_\bullet}(w) \otimes z_1' \otimes z_2 \quad \mod \mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z).$$

Repeating the same argument for z_2, the surjectivity of h follows.

Step 3. – It remains to show that h is injective. Equivalently, we show the surjectivity of the dual map

$$h^\vee \colon \left(\left(M^\bullet \otimes_{\mathbb{C}} Z\right)_{\mathcal{L}_{C\backslash P_\bullet \sqcup Q_\bullet}(V)}\right)^\vee \to \left(M^\bullet_{\mathcal{L}_{C\backslash P_\bullet}(V,\{Q_+,Q_-\})}\right)^\vee$$

given by

$$h^\vee(\Phi)(w) = \Phi\left(w \otimes 1^{A(V)} \otimes 1^{A(V)}\right), \qquad \text{for } w \in M^\bullet.$$

For this, select an element $\overline{\Phi}$ in the target of $h^\vee$. We construct an element Φ in the source of $h^\vee$ such that $h^\vee(\Phi) = \overline{\Phi}$.

First, we define Φ as a linear functional on $M^\bullet \otimes_{\mathbb{C}} Z$. We use a spanning set for Z obtained as follows. Since V is C_1-cofinite and $V_0 \cong \mathbb{C}1^V$, lowest weight V-modules admit spanning sets of PBW-type [54, Cor. 3.12]. Since Z is the tensor product of two lowest weight V-modules, we conclude that Z admits a spanning set of PBW-type. Select a spanning set of PBW-type for Z, and consider a generating element

$$(37) \qquad z_1 \otimes z_2 = D^l_{[i_l]} \cdots D^1_{[i_1]} 1^{A(V)} \otimes E^m_{[j_m]} \cdots E^1_{[j_1]} 1^{A(V)} \quad \in Z$$

with $D^1, \ldots, D^l, E^1, \ldots, E^m \in V$, and $i_1, \ldots, i_l, j_1, \ldots, j_m \in \mathbb{Z}$ such that

$$D^l_{[i_l]}, \ldots, D^2_{[i_2]}, E^m_{[j_m]}, \ldots, E^2_{[j_2]} \in \mathfrak{L}(V)_{>0} \quad \text{and} \quad D^1_{[i_1]}, E^1_{[j_1]} \in \mathfrak{L}(V)_{\geq 0}.$$

Looking more closely at [54], one can assume that all $D^1, \ldots, D^l, E^1, \ldots, E^m$ are in the complement of $C_1(V)$ in V. Note that the elements $D^1_{[i_1]}, E^1_{[j_1]} \in \mathfrak{L}(V)_0$ generate the lowest weight space from the vector $1^{A(V)} \otimes 1^{A(V)}$. We proceed by induction on l and m. We start by defining

$$\Phi\left(w \otimes 1^{A(V)} \otimes 1^{A(V)}\right) := \overline{\Phi}(w).$$

Next, assume that Φ has been defined on elements $w \otimes z_1 \otimes z_2$ for $w \in M^\bullet$ and $z_1 \otimes z_2$ as in (37) for *fixed* l and m. Consider an element

$$w \otimes D^+_{[i_+]} z_1 \otimes z_2 \in M^\bullet \otimes_{\mathbb{C}} Z$$

for some $w \otimes z_1 \otimes z_2 \in M^\bullet \otimes_{\mathbb{C}} Z$, $D^+ \in V$, and $i_+ \in \mathbb{Z}$. Choose an integer $N \gg 0$ such that

$$D^+_{[i]} z_1 \otimes z_2 = z_1 \otimes D^+_{[i]} z_2 = 0 \quad \text{in } Z \text{ for all } i \geq N.$$

As in Step 2, such N exists because $\mathscr{U}(V)$ acts smoothly on each of the two factors of Z. By Proposition 3.4.1, there exists $\sigma \in \mathcal{L}_{C\backslash P_\bullet \sqcup Q_\bullet}(V)$ such that

$$(38) \qquad \begin{aligned} \sigma_{Q_+} &\equiv D^+_{[i_+]} \in \mathfrak{L}_{Q_+}(V)/\mathfrak{L}_{Q_+}(V, NQ_+), \\ \sigma_{Q_-} &\equiv 0 \in \mathfrak{L}_{Q_-}(V)/\mathfrak{L}_{Q_-}(V, NQ_-). \end{aligned}$$

Define

$$\Phi\left(w \otimes D^+_{[i_+]} z_1 \otimes z_2\right) := -\sum_{i=1}^{n} \Phi\left(w_1 \otimes \cdots \otimes \sigma_{P_i}(w_i) \otimes \cdots \otimes w_n \otimes z_1 \otimes z_2\right),$$

where $w = w_1 \otimes \cdots \otimes w_n \in M^\bullet$. Note that since the generating elements for Z are expressed in terms of $D^1, \ldots, D^l, E^1, \ldots, E^m$, which are not in $C_1(V)$, then Phi automatically respects the relations in Z, and hence defines a linear map on $M^\bullet \otimes_{\mathbb{C}} Z$. Since $\overline{\Phi}$ vanishes

on $\mathcal{L}_{C \backslash P_\bullet}(V, \{Q_+, Q_-\})$, the definition of Φ here is independent of the choice of σ satisfying (38). One proceeds in a similar way to define Φ on elements of type

$$w \otimes z_1 \otimes E^+_{[j_+]} z_2 \in M^\bullet \otimes_\mathbb{C} Z$$

for $w \otimes z_1 \otimes z_2 \in M^\bullet \otimes_\mathbb{C} Z$, $E^+ \in V$, and $j_+ \in \mathbb{Z}$. By induction, this defines Φ on a spanning set of PBW-type for $M^\bullet \otimes_\mathbb{C} Z$, hence by linearity, on $M^\bullet \otimes_\mathbb{C} Z$. A direct calculation shows that

$$\Phi\left(w \otimes D^a_{[i_a]} D^b_{[i_b]} z_1 \otimes z_2\right) - \Phi\left(w \otimes D^b_{[i_b]} D^a_{[i_a]} z_1 \otimes z_2\right) = \Phi\left(w \otimes \left[D^a_{[i_a]}, D^b_{[i_b]}\right] z_1 \otimes z_2\right)$$

and similarly

$$\Phi\left(w \otimes z_1 \otimes E^a_{[j_a]} E^b_{[j_b]} z_2\right) - \Phi\left(w \otimes z_1 \otimes E^b_{[j_b]} E^a_{[j_a]} z_2\right) = \Phi\left(w \otimes z_1 \otimes \left[E^a_{[j_a]}, E^b_{[j_b]}\right] z_2\right),$$

hence Φ is independent of the choice of the spanning set.

Finally, we check that Φ vanishes on $\mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes_\mathbb{C} Z)$. Fix τ in $\mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)$. By definition of Φ, one has

$$\Phi\left(w \otimes \tau_{Q_+}(z_1) \otimes z_2\right) + \Phi\left(w \otimes z_1 \otimes \tau_{Q_-}(z_2)\right) = -\Phi\left(\sigma_{P_\bullet}(w) \otimes z_1 \otimes z_2\right)$$

for any $\sigma \in \mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)$ satisfying conditions analogous to (38), hence we can choose $\sigma = \tau$. It follows that Φ vanishes on $\tau(M^\bullet \otimes_\mathbb{C} Z)$ for all $\tau \in \mathcal{L}_{C \backslash P_\bullet \sqcup Q_\bullet}(V)$, hence Φ is in the source of $h^\vee$. By construction, one has $h^\vee(\Phi) = \overline{\Phi}$, hence $h^\vee$ is surjective. This ends the proof. $\square$

7. Proof of the factorization theorem

Here we prove our main result, which we state in complete detail below. For this let us first set some notation. Let $(C, P_\bullet)$ be a stable n-pointed curve with exactly one simple node, denoted Q. Let $\widetilde{C} \to C$ be the normalization of C, let Q_+ and $Q_- \in \widetilde{C}$ be the two preimages of Q, and set $Q_\bullet = (Q_+, Q_-)$. The curve $\widetilde{C}$ may not be connected. Fix formal coordinates t_i at P_i, for each $i = 1, \ldots, n$, and $s_\pm$ at $Q_\pm$. Suppose $M^1, \ldots, M^n$ are V-modules, set $M^\bullet = \bigotimes_{i=1}^n M^i$, and let $\mathscr{W}$ be the set of representatives of isomorphism classes of simple V-modules. Consider the map

$$(39) \qquad M^\bullet \to \bigoplus_{W \in \mathscr{W}} M^\bullet \otimes_\mathbb{C} W \otimes_\mathbb{C} W', \qquad u \mapsto \bigoplus_{W \in \mathscr{W}} u \otimes 1^{W_0},$$

where $1^{W_0} = \mathrm{id}_{W_0} \in \mathrm{End}(W_0) \cong W_0 \otimes W_0^\vee$. Here W_0 is the degree zero space of the module $W = \bigoplus_{i \geq 0} W_i$. Recall that, by definition, the vector spaces W_0 and $W_0^\vee$ are finite-dimensional.

THEOREM 7.0.1 (Factorization theorem). – *Let V be a rational, C_2-cofinite vertex operator algebra with one-dimensional weight zero space. The map (39) gives rise to a canonical isomorphism of vector spaces*

$$\mathbb{V}(V; M^\bullet)_{(C, P_\bullet, t_\bullet)} \cong \bigoplus_{W \in \mathscr{W}} \mathbb{V}\left(V; M^\bullet \otimes_\mathbb{C} W \otimes_\mathbb{C} W'\right)_{(\widetilde{C}, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)}.$$

This isomorphism is equivariant with respect to change of the coordinates $t_\bullet$.

The proof we give here roughly follows the outline of the proof in [67, §8.6], with the generalizations to coinvariants defined using the chiral Lie algebra instead of Zhu's Lie algebra, and for curves of arbitrary genus, made possible by Propositions 3.3.1 and 6.2.1.

Proof. – By Definition (30), due to propagation of vacua (Theorem 4.3.1), we can reduce to the case $C \setminus P_\bullet$ affine, after possibly adding more marked points P_i and corresponding modules V. Using the formal coordinates t_i at P_i, for $i = 1, \ldots, n$, and $s_\pm$ at $Q_\pm$, one has Lie algebra homomorphisms

$$\mathcal{L}_{C \setminus P_\bullet}(V) \to \mathfrak{L}_{P_\bullet}(V) \quad \text{and} \quad \mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V) \to \mathfrak{L}_{P_\bullet}(V) \oplus \mathfrak{L}_{Q_\bullet}(V).$$

In the following, we show that (39) induces a canonical isomorphism

(40)
$$M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V)} \cong \bigoplus_{W \in \mathscr{W}} \left(M^\bullet \otimes_\mathbb{C} W \otimes_\mathbb{C} W' \right)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)}.$$

We will argue that there is a commutative diagram

$$
\begin{array}{ccc}
M^\bullet_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet}(V, \{Q_+, Q_-\})} & \xrightarrow[\cong]{h} & (M^\bullet \otimes_\mathbb{C} Z)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)} \\
\downarrow & & \downarrow \\
M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V)} & \xrightarrow[\cong]{f} & (M^\bullet \otimes_\mathbb{C} \overline{Z})_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)}.
\end{array}
$$

Then after Lemma 6.1.1, the isomorphism f gives (40).

Step 1. – The top horizontal isomorphism h is given by Proposition 6.2.1.

Step 2. – We argue that there is an inclusion

$$\iota \colon \mathcal{L}_{\widetilde{C} \setminus P_\bullet}(V, \{Q_+, Q_-\}) \hookrightarrow \mathcal{L}_{C \setminus P_\bullet}(V).$$

Indeed, by Proposition 3.3.1 an element of $\mathcal{L}_{C \setminus P_\bullet}(V)$ can be realized as σ in $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)$ such that $\sigma_{Q_\pm} \in \mathfrak{L}(V)_{\leq 0}$ and the restrictions $[\sigma_{Q_\pm}]_0$ of $\sigma_{Q_\pm}$ to $\mathfrak{L}(V)_0$ satisfy $[\sigma_{Q_-}]_0 = \vartheta [\sigma_{Q_+}]_0$. In particular, $\mathcal{L}_{\widetilde{C} \setminus P_\bullet}(V, \{Q_+, Q_-\})$ from (34) is identified as a Lie subalgebra of $\mathcal{L}_{C \setminus P_\bullet}(V)$ by pullback along the normalization, since its elements satisfy $\sigma_{Q_\pm} \in \mathfrak{L}(V)_{<0}$.

Step 3. – To show that the bottom horizontal map f is an isomorphism, it remains to verify that the kernel of the left vertical map is identified with the kernel of the right vertical map by the isomorphism h.

Step 3(a). – The kernel of the left vertical map is the space

$$K := \mathcal{L}_{C \setminus P_\bullet}(V) \left(M^\bullet_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet}(V, \{Q_+, Q_-\})} \right).$$

Note that for σ in $\mathcal{L}_{C \setminus P_\bullet}(V)$, the formula for the bracket (4) gives

$$\left[\varphi(\sigma), \varphi \left(\mathcal{L}_{\widetilde{C} \setminus P_\bullet}(V, \{Q_+, Q_-\}) \right) \right] \subset \varphi \left(\mathcal{L}_{\widetilde{C} \setminus P_\bullet}(V, \{Q_+, Q_-\}) \right),$$

where φ is as in (24). It follows that σ acts on the source of h. The left vertical map is thus the quotient by the action of $\mathcal{L}_{C \setminus P_\bullet}(V) / \mathcal{L}_{\widetilde{C} \setminus P_\bullet}(V, \{Q_+, Q_-\})$.

Step 3(b). – We conclude the argument by showing that the right vertical map is the quotient by $h(K)$. Recall that $h(w) = w \otimes 1^{A(V)} \otimes 1^{A(V)}$, for $w \in M^\bullet$. Hence, $h(K)$ is linearly spanned by

$$(41) \quad \sigma_{P_\bullet}(w) \otimes 1^{A(V)} \otimes 1^{A(V)} \equiv -w \otimes \sigma_{Q_+}\left(1^{A(V)}\right) \otimes 1^{A(V)} - w \otimes 1^{A(V)} \otimes \sigma_{Q_-}\left(1^{A(V)}\right)$$

modulo $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z)$, for $\sigma \in \mathcal{L}_{C \setminus P_\bullet}(V)$ and $w \in M^\bullet$. Since $\mathscr{U}(V)_{<0}$ acts trivially on $A(V)$, (41) is congruent to

$$-w \otimes \left[\sigma_{Q_+}\right]_0 \left(1^{A(V)}\right) \otimes 1^{A(V)} - w \otimes 1^{A(V)} \otimes \left[\sigma_{Q_-}\right]_0 \left(1^{A(V)}\right)$$

modulo $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z)$. From Proposition 3.3.1, one has $\left[\sigma_{Q_-}\right]_0 = \vartheta\left(\left[\sigma_{Q_+}\right]_0\right)$. By linearity, it follows that $h(K)$ is linearly spanned by

$$(42) \quad -w \otimes B_{[k-1]}\left(1^{A(V)}\right) \otimes 1^{A(V)} - w \otimes 1^{A(V)} \otimes \vartheta\left(B_{[k-1]}\right)\left(1^{A(V)}\right)$$

modulo $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z)$, for $w \in M^\bullet$ and homogeneous $B \in V$ of degree k. Here, $B_{[k-1]}$ and $\vartheta\left(B_{[k-1]}\right)$ are in $\mathfrak{L}(V)_0$, and recall that the action of $\mathfrak{L}(V)_0$ on $A(V)$ is induced from the projection $\mathfrak{L}(V)_0 \to A(V)$.

One has $b\left(1^{A(V)}\right) = \left(1^{A(V)}\right) b$ for all $b \in \mathfrak{L}(V)_0$, where $\left(1^{A(V)}\right) b$ denotes the right action of $b \in \mathfrak{L}(V)_0$ on $1^{A(V)}$. Using this in (42), we conclude that $h(K)$ is linearly spanned by

$$(43) \quad -w \otimes \left(1^{A(V)}\right) B_{[k-1]} \otimes 1^{A(V)} - w \otimes 1^{A(V)} \otimes \left(1^{A(V)}\right) \vartheta\left(B_{[k-1]}\right)$$

modulo $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)(M^\bullet \otimes Z)$, for $w \in M^\bullet$ and homogeneous $B \in V$ of degree k.

From Lemma 6.1.1, the target of h is isomorphic to

$$(44) \quad \begin{aligned} \left(M^\bullet \otimes Z\right)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)} &\cong \bigoplus_{W,Y \in \mathscr{W}} \left(M^\bullet \otimes W \otimes W_0^\vee \otimes Y \otimes Y_0^\vee\right)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)} \\ &\cong \bigoplus_{W,Y \in \mathscr{W}} \left(M^\bullet \otimes W \otimes Y\right)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)} \otimes W_0^\vee \otimes Y_0^\vee. \end{aligned}$$

The second isomorphism is due to the fact that $\mathfrak{L}_{Q_+}(V)$ acts on the left of $W \otimes W_0^\vee$, and similarly, $\mathfrak{L}_{Q_-}(V)$ acts on the left of $Y \otimes Y_0^\vee$, so that $\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)$ only acts on $M^\bullet \otimes W \otimes Y$. Expressing (43) via the isomorphisms in (44), $h(K)$ is linearly spanned by

$$-\left(M^\bullet \otimes W \otimes Y\right)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)} \otimes B_{[k-1]}\left(W_0^\vee\right) \otimes Y_0^\vee$$
$$-\left(M^\bullet \otimes W \otimes Y\right)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)} \otimes W_0^\vee \otimes \vartheta\left(B_{[k-1]}\right)\left(Y_0^\vee\right)$$

for $W, Y \in \mathscr{W}$ and homogeneous $B \in V$ of degree k. Here the right action of $B_{[k-1]}$ and $\vartheta\left(B_{[k-1]}\right)$ in $\mathfrak{L}(V)_0$ on $W_0^\vee$ and $Y_0^\vee$ is given by Lemma 1.8.3, and recall that ϑ is an involution. It follows that $h(K)$ in $(M^\bullet \otimes Z)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)}$ is isomorphic to

$$\bigoplus_{W,Y \in \mathscr{W}} \left(M^\bullet \otimes W \otimes Y\right)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)} \otimes \mathscr{I}\left(W_0^\vee, Y_0^\vee\right)$$

with $\mathscr{I}\left(W_0^\vee, Y_0^\vee\right) \subset W_0^\vee \otimes Y_0^\vee$ linearly spanned by

$$\psi_W \circ \vartheta\left(B_{[k-1]}\right) \otimes \psi_Y + \psi_W \otimes \psi_Y \circ B_{[k-1]},$$

where $\psi_W \in W_0^\vee$, $\psi_Y \in Y_0^\vee$ for $W, Y \in \mathscr{W}$, and $B \in V_k$, for $k \geq 0$. One has

$$W_0^\vee \otimes Y_0^\vee / \mathscr{I}\left(W_0^\vee, Y_0^\vee\right) = \mathrm{Hom}_{A(V)}\left(W_0, Y_0^\vee\right),$$
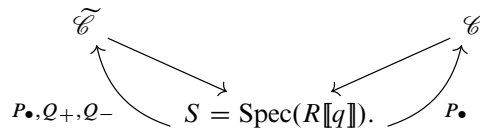
and by Schur's Lemma, this is isomorphic to $\mathbb{C}$ when $Y = W'$ and zero otherwise. This and the description of $\overline{Z}$ from Lemma 6.1.1 imply that, after taking the quotient of $(M^\bullet \otimes_{\mathbb{C}} Z)_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)}$ by $h(K)$, one obtains $(M^\bullet \otimes_{\mathbb{C}} \overline{Z})_{\mathcal{L}_{\widetilde{C} \setminus P_\bullet \sqcup Q_\bullet}(V)}$, hence the statement. $\qquad\square$

8. Sewing and local freeness

In this section we prove VB corollary. For this, we start with the sewing theorem (Theorem 8.5.1), a refined version of the factorization theorem. This requires the notion of formal smoothings, reviewed below.

8.1. Formal smoothings

For a $\mathbb{C}$-algebra R with smooth $\mathrm{Spec}(R)$, let $\mathscr{C}_0 \to S_0 := \mathrm{Spec}(R)$ be a flat family of nodal curves with a single simple node defined by a section Q and with n distinct smooth points given by sections $P_\bullet = (P_1, \ldots, P_n)$. Assume that $\mathscr{C}_0 \setminus P_\bullet(S_0)$ is affine over S_0. Up to an étale base change of S_0 of degree two, we can normalize $\mathscr{C}_0$ and obtain a smooth family of $(n+2)$-pointed curves $\widetilde{\mathscr{C}_0} \to S_0$ with sections $P_\bullet \sqcup (Q_+, Q_-)$, where $Q_\pm(S_0) \subset \widetilde{\mathscr{C}_0}$ are the preimages of the node in $\mathscr{C}_0/S_0$. Fix formal coordinates s_+ and s_- at $Q_+(S_0)$ and $Q_-(S_0)$, respectively. Such coordinates determine a *smoothing* of $(\mathscr{C}_0, P_\bullet)$ over $S := \mathrm{Spec}(R[\![q]\!])$. That is, a flat family $\mathscr{C} \to S = \mathrm{Spec}(R[\![q]\!])$ with sections $P_\bullet = (P_1, \ldots, P_n)$ such that the general fiber is smooth and the special fiber is identified with $\mathscr{C}_0 \to S_0$. The family $\widetilde{\mathscr{C}_0} \to S_0$ trivially extends to a family of smooth curves $\widetilde{\mathscr{C}} \to S$ with $n+2$ sections $P_\bullet$, Q_+, and Q_-:

$$
\begin{array}{ccc}
\widetilde{\mathscr{C}} & & \mathscr{C} \\
{\scriptstyle P_\bullet, Q_+, Q_-} \searrow & S = \mathrm{Spec}(R[\![q]\!]). & \swarrow {\scriptstyle P_\bullet}
\end{array}
$$

The formal coordinate at $Q_\pm(S_0)$ extends to a formal coordinate, still denoted $s_\pm$, at $Q_\pm(S)$ — that is, $s_\pm$ is a generator of the ideal of the completed local $R[\![q]\!]$-algebra of $\widetilde{\mathscr{C}}$ at $Q_\pm(R)$ — such that locally around the node, the family $\mathscr{C}$ is defined by $s_+ s_- = q$. For more details, see [61, p. 457] and [10, pp. 184-5]. We emphasize that the existence of such families holds over the formal base $S = \mathrm{Spec}(R[\![q]\!])$, or equivalently, over the complex open unit disk around S_0 in the analytic category, but fails over a more general base. Moreover, one still has that $\widetilde{\mathscr{C}} \setminus P_\bullet(S)$ and $\mathscr{C} \setminus P_\bullet(S)$ are affine over S.

8.2. The sheaf of vertex algebras over formal smoothings

We define here the sheaf of vertex algebras $\mathcal{V}_{\mathscr{C}}$ over a family $\mathscr{C} \to S$ as in §8.1. After §2.5, it is enough to describe $\mathcal{V}_{\mathscr{C}}$ on the formal neighborhood $\mathfrak{D}_Q$ of the node Q. The completed local ring $\widehat{\mathscr{O}}_Q$ consists of elements of the form $\sum_{i,j\geq 0} \alpha_{i,j}\, s_+^i s_-^j$ for $\alpha_{i,j} \in R$. After identifying $s_+ = (s_+, \frac{q}{s_-})$ and $s_- = (\frac{q}{s_+}, s_-)$, the ring $\widehat{\mathscr{O}}_Q$ is realized as the subring of $R(\!(s_+)\!)[\![q]\!] \oplus R(\!(s_-)\!)[\![q]\!]$ consisting of elements of type

$$
\sum_{i,j\geq 0} \alpha_{i,j} \left(s_+^{i-j} q^j, s_-^{j-i} q^i \right)
$$

with $\alpha_{i,j} \in R$. The space of sections of $\mathcal{V}_\mathscr{C}$ over $\mathfrak{D}_Q$ is generated by

$$
(45) \qquad \sum_{i,j \geq 0} \alpha_{i,j} \left(A \otimes s_+^{a+i-j} q^j, (-1)^a \sum_{k \geq 0} \frac{1}{k!} L_1^k(A) \otimes s_-^{a-k+j-i} q^i \right)
$$

for homogeneous $A \in V$ of degree a and $\alpha_{i,j} \in R$. For $q = 0$, one recovers the description given in §2.5 for nodal curves.

Similar to the case of a nodal curve, we can identify a logarithmic connection $\nabla \colon \mathcal{V}_\mathscr{C} \to \mathcal{V}_\mathscr{C} \otimes \omega_{\mathscr{C}/S}$. The description of ∇ on a smooth open set follows from the description of the connection ∇ over a smooth curve. Specifically, for every smooth point $P \in \mathscr{C} \setminus Q$, there is an open set $S' \subset S$ and an open set $U \subset \mathscr{C} \setminus Q$ over S' and containing P such that there exists an étale map $U \to \mathbb{A}^1_{S'}$. This implies that U has a global coordinate t, and thus $\mathcal{V}|_U \simeq_t V \otimes \mathscr{O}_U$. On the open set U, the connection ∇ is provided by the endomorphism of $\mathcal{V}|_U$ given by $L_{-1} \otimes \mathrm{id}_U + \mathrm{id}_V \otimes \partial_t$, as in §2.7.1.

We are left to describe ∇ over $\mathfrak{D}_Q$. Recall that $\left(\frac{ds_+}{s_+}, -\frac{ds_-}{s_-} \right)$ is a generator of $\omega_{\mathscr{C}/S}$ over $\widehat{\mathscr{O}}_Q$. Using this trivialization, the connection ∇ is defined by the endomorphism (21) extended over $R[\![q]\!]$ by linearity, that is:

$$
\left(L_{-1} \otimes s_+ \mathrm{id}_{R(\!(s_+)\!)[\![q]\!]} + \mathrm{id}_V \otimes s_+ \partial_{s_+}, -L_{-1} \otimes s_- \mathrm{id}_{R(\!(s_-)\!)[\![q]\!]} - \mathrm{id}_V \otimes s_- \partial_{s_-} \right).
$$

As in the proof of Proposition 2.7.3, one needs to show that the target of this map lies inside $\mathcal{V}_\mathscr{C}(\mathfrak{D}_Q)$, and not merely in $V \otimes \left(R(\!(s_+)\!)[\![q]\!] \oplus R(\!(s_-)\!)[\![q]\!] \right)$. This amounts to a verification as in the proof of Proposition 2.7.3.

8.3. The chiral Lie algebra over formal smoothings

Next, we globalize the Lie algebra $\mathfrak{L}(V)$ ancillary to V and the chiral Lie algebra.

Select a formal coordinate t_i at $P_i(S)$. On the formal neighborhood $\mathfrak{D}_{P_i} = \mathrm{Spec}\, R[\![t_i, q]\!]$, one has $\mathcal{V}_\mathscr{C}|_{\mathfrak{D}_{P_i}} \simeq_{t_i} (V \otimes R[\![t_i]\!])[\![q]\!]$. Since the restriction of ∇ over $\mathfrak{D}_{P_i}$ is given by $L_{-1} \otimes \mathrm{id}_{R[\![t_i, q]\!]} + \mathrm{id}_V \otimes \partial_{t_i}$, we have an identification as in (22):

$$
(46) \qquad \mathrm{H}^0 \left(\mathfrak{D}_{P_i}^\times, \mathcal{V}_\mathscr{C} \otimes \omega_{\mathscr{C}/S} / \mathrm{Im}\nabla \right) \xrightarrow{\simeq_{t_i}} \mathfrak{L}_{t_i}(V)[\![q]\!].
$$

We define the *chiral Lie algebra* assigned to V and $(\mathscr{C}, P_\bullet)$ as

$$
\mathcal{L}_{\mathscr{C} \setminus P_\bullet}(V) := \mathrm{H}^0 \left(\mathscr{C} \setminus P_\bullet(S), \frac{\mathcal{V}_\mathscr{C} \otimes \omega_{\mathscr{C}/S}}{\mathrm{Im}\nabla} \right).
$$

This extends §3.1. The map

$$
\mathcal{L}_{\mathscr{C} \setminus P_\bullet}(V) \to \bigoplus_{i=1}^n \mathrm{H}^0 \left(\mathfrak{D}_{P_i}^\times, \mathcal{V}_\mathscr{C} \otimes \omega_{\mathscr{C}/S} / \mathrm{Im}\nabla \right) \xrightarrow{\cong} \bigoplus_{i=1}^n \mathfrak{L}_{t_i}(V)[\![q]\!]
$$

induced by the restriction of sections and (46) is a Lie algebra morphism.

REMARK 8.3.1. – As in Proposition 3.3.1, $\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)$ can be described as the subspace of $\mathcal{L}_{\widetilde{\mathscr{C}}\setminus P_\bullet \sqcup Q_\bullet}(V)$ generated by those elements σ whose restrictions $\sigma_{Q_\pm}$ to $\mathfrak{L}_{Q_\pm}(V)\llbracket q \rrbracket$ satisfy

$$\sigma_{Q_+} = \sum_{i,j \geq 0} \alpha_{i,j} \, A_{[a+i-j-1]} \, q^j,$$

$$\sigma_{Q_-} = (-1)^{a-1} \sum_{i,j \geq 0} \alpha_{i,j} \sum_{k \geq 0} \frac{1}{k!} \left(L_1^k A \right)_{[a-k+j-i-1]} q^i$$

$$= \sum_{i,j \geq 0} \alpha_{i,j} \, \vartheta \left(A_{[a+i-j-1]} \right) q^i$$

for homogeneous $A \in V$ of degree a and integers $i, j \in \mathbb{Z}_{\geq 0}$. This uses that sections of $\mathcal{V}_{\mathscr{C}}$ over $\mathfrak{D}_Q$ are generated by (45), sections of $\omega_{\mathscr{C}/S}$ over $\mathfrak{D}_Q$ are generated by $\left(\frac{ds_+}{s_+}, -\frac{ds_-}{s_-} \right)$ over $\widehat{\mathscr{O}}_Q$, and the definition of ϑ from (7).

8.4. Sheaf of coinvariants over formal smoothings

Let $(\mathscr{C}/S, P_\bullet)$ be as in §8.1. Fix formal coordinates t_i at $P_i(S)$, for $i = 1, \ldots, n$. Given V-modules $M^1, \ldots, M^n$, let $M^\bullet := \bigotimes_{i=1}^n M^i$. One defines the sheaf of coinvariants $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)}$ as in §4, that is:

$$\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)} := \left(M^\bullet \otimes \mathscr{O}_S \right)_{\mathcal{L}_{\mathscr{C}\setminus P_\bullet}(V)},$$

The factorization theorem holds for the restriction of $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)}$ to the special fiber $\mathscr{C}_0 \to S_0$:

THEOREM 8.4.1. – *Let V be a rational, C_2-cofinite vertex operator algebra with one-dimensional weight zero space. The map* (39) *induces a canonical $\mathscr{O}_{S_0}$-module isomorphism*

$$\mathbb{V}(V; M^\bullet)_{(\mathscr{C}_0/S_0, P_\bullet, t_\bullet)} \cong \bigoplus_{W \in \mathscr{W}} \mathbb{V}\left(V; M^\bullet \otimes W \otimes W' \right)_{(\widetilde{\mathscr{C}}_0/S_0, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)}.$$

As a consequence of Corollary 5.1.2 and Theorem 8.4.1 we obtain the following result on coherence.

THEOREM 8.4.2. – *Let V be a rational, C_2-cofinite vertex operator algebra with one-dimensional weight zero space. For finitely generated V-modules $M^1, \ldots, M^n$, the sheaf $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)}$ is a coherent $\mathscr{O}_S$-module.*

Proof. – Corollary 5.1.2 implies that the restriction of $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)}$ to $S \setminus S_0$ is coherent. We are left to show that the restriction of $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)}$ to S_0, i.e., $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}_0/S_0, P_\bullet, t_\bullet)}$ is coherent. This follows from Theorem 8.4.1 and Corollary 5.1.2 applied to $\mathbb{V}\left(V; M^\bullet \otimes W \otimes W' \right)_{(\widetilde{\mathscr{C}}_0/S_0, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)}.$ $\square$

8.5. Sewing

Given a simple V-module $W = \bigoplus_{i \geq 0} W_i$, define

$$1^W := \sum_{i \geq 0} 1^{W_i} q^i \quad \in \quad (W \otimes W')[\![q]\!],$$

where $1^{W_i} := \mathrm{id}_{W_i} \in \mathrm{End}(W_i) \cong W_i \otimes W_i^\vee$. Consider the map

$$(47) \qquad M^\bullet \longrightarrow \bigoplus_{W \in \mathscr{W}} M^\bullet \otimes (W \otimes W')[\![q]\!], \qquad u \mapsto \bigoplus_{W \in \mathscr{W}} u \otimes 1^W.$$

The following result extends [67, Thm 8.4.6] to curves of arbitrary genus.

THEOREM 8.5.1 (Sewing Theorem). – *Let V be a rational, C_2-cofinite vertex operator algebra with one-dimensional weight zero space, and set $M^\bullet = \bigotimes_{i=1}^n M^i$ for V-modules M^i. The map (47) induces a canonical $\mathscr{O}_{S_0}[\![q]\!]$-module isomorphism Ψ such that the following diagram commutes*

$$
\begin{array}{ccc}
\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)} & \overset{\Psi}{\underset{\sim}{\to}} & \bigoplus_{W \in \mathscr{W}} \mathbb{V}(V; M^\bullet \otimes W \otimes W')_{(\widetilde{\mathscr{C}}_0/S_0, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)} \underset{\mathscr{O}_{S_0}}{\otimes} \mathscr{O}_{S_0}[\![q]\!] \\
\downarrow & & \downarrow \\
\mathbb{V}(V; M^\bullet)_{(\mathscr{C}_0/S_0, P_\bullet, t_\bullet)} & \overset{\cong}{\longrightarrow} & \bigoplus_{W \in \mathscr{W}} \mathbb{V}(V; M^\bullet \otimes W \otimes W')_{(\widetilde{\mathscr{C}}_0/S_0, P_\bullet \sqcup Q_\bullet, t_\bullet \sqcup s_\bullet)}.
\end{array}
$$

This isomorphism is equivariant with respect to change of the coordinates $t_\bullet$.

REMARK 8.5.2. – Theorems 8.4.1 and 8.5.1 give a canonical isomorphism

$$\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)} \cong \mathbb{V}\left(V; M^\bullet\right)_{(\mathscr{C}_0/S_0, P_\bullet, t_\bullet)} [\![q]\!].$$

In particular, this means that to the non-trivial deformation $\mathscr{C}$ of $\mathscr{C}_0$, there corresponds a trivial deformation of the space of conformal blocks.

Proof of Theorem 8.5.1. – As in the proof of the factorization theorem, we can reduce to the case $\mathscr{C} \setminus P_\bullet$ affine over S, and show that (47) induces a canonical $R[\![q]\!]$-module isomorphism, still denoted Ψ, such that the following diagram commutes

$$
\begin{array}{ccc}
(M^\bullet \otimes \mathscr{O}_S)_{\mathcal{L}_{\mathscr{C} \setminus P_\bullet}(V)} & \overset{\Psi}{\underset{\sim}{\to}} & \bigoplus_{W \in \mathscr{W}} \left(M^\bullet \otimes W \otimes W' \otimes \mathscr{O}_{S_0}\right)_{\mathcal{L}_{\widetilde{\mathscr{C}}_0 \setminus P_\bullet \sqcup Q_\bullet}(V)} \underset{\mathscr{O}_{S_0}}{\otimes} \mathscr{O}_{S_0}[\![q]\!] \\
\downarrow & & \downarrow \\
\left(M^\bullet \otimes \mathscr{O}_{S_0}\right)_{\mathcal{L}_{\mathscr{C}_0 \setminus P_\bullet}(V)} & \overset{\cong}{\longrightarrow} & \bigoplus_{W \in \mathscr{W}} \left(M^\bullet \otimes W \otimes W' \otimes \mathscr{O}_{S_0}\right)_{\mathcal{L}_{\widetilde{\mathscr{C}}_0 \setminus P_\bullet \sqcup Q_\bullet}(V)}.
\end{array}
$$

Here, the vertical maps are obtained by specializing at $q = 0$.

Step 1. – We show that (47) induces a well-defined map Ψ between spaces of coinvariants. For this, it is enough to show that for each $\sigma \in \mathcal{L}_{\mathscr{C}\backslash P_\bullet}(V)$ and $W \in \mathscr{W}$, one has $\sigma_{P_\bullet}(M^\bullet) \otimes 1^W = \sigma(M^\bullet \otimes 1^W)$, or equivalently

$$(48) \qquad \left(\sigma_{Q_+} \otimes 1 + 1 \otimes \sigma_{Q_-}\right)\left(1^W\right) = 0.$$

From Remark 8.3.1 and by linearity, we can reduce to the case when

$$\sigma_{Q_+} = A_{[a+i-j-1]}\, q^j \qquad \text{and} \qquad \sigma_{Q_-} = \vartheta\left(A_{[a+i-j-1]}\right) q^i$$

for homogeneous $A \in V$ of degree a and integers $i, j \geq 0$. The vanishing of (48) follows from the identity

$$(49) \qquad \left(A_{[a+i-j-1]} \otimes 1 + 1 \otimes \vartheta\left(A_{[a+i-j-1]}\right) q^{i-j}\right)\left(1^W\right) = 0$$

established in [67, Lemma 8.7.1] (there is a sign difference between the involution ϑ used here and the involution used in [67]). Thus we conclude that the map Ψ is well-defined and makes the diagram above commute.

Step 2. – Since (i) the target of Ψ is a free $\mathscr{O}_{S_0}[\![q]\!]$-module of finite rank, (ii) the source is finitely generated (Theorem 8.4.2), and (iii) Φ is an isomorphism modulo q (Theorem 8.4.1), Nakayama's lemma implies that Ψ is an isomorphism (this is as in [74, 61, 67]). $\qquad\square$

8.6. The sheaf of coinvariants on $\mathcal{M}_{g,n}$

Let $\widehat{\mathcal{M}}_{g,n}$ be the restriction of $\mathcal{M}_{g,n}$ over the locus $\mathcal{M}_{g,n}$ of smooth pointed curves (see §2.2.2 for definitions). The vertex algebra bundle and the chiral Lie algebra defined on smooth curves in §§2 and 3, respectively, give the vertex algebra bundle and the sheaf of chiral Lie algebras on $\widehat{\mathcal{M}}_{g,n}$. In §§8.2 and 8.3, the vertex algebra bundle and the sheaf of chiral Lie algebras are defined on formal smoothings of families of nodal curves. Gluing as in [13], one obtains the vertex algebra bundle and the sheaf of chiral Lie algebras on $\mathcal{M}_{g,n}$. Similarly, the sheaf of coinvariants defined on families of smooth curves in §4 gives the sheaf of coinvariants $\mathbb{V}(V; M^\bullet)$ on $\widehat{\mathcal{M}}_{g,n}$. In §8.4, the sheaf of coinvariants is defined on formal smoothings of families of nodal curves. By gluing as in [13], one obtains the *sheaf of coinvariants* $\mathbb{V}(V; M^\bullet)$ on $\mathcal{M}_{g,n}$.

8.7. The sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$

Throughout this section, we require every V-module M to further satisfy the following property: there exists a complex number c_M called the *conformal dimension* (or conformal weight) of M such that for every homogeneous $v \in M$ one has $L_0(v) = (\deg(v) + c_M)v$. This condition holds whenever $M = V$ or, for instance, when M is a simple V-module. Furthermore, if V is C_2-cofinite, then c_M is a rational number [64].

The sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$ is obtained from a two-step process [21, §6.3], reviewed next. Consider the group scheme $\mathrm{Aut}_+\mathcal{O}$ which represents the functor assigning to a $\mathbb{C}$-algebra R the group:

$$\mathrm{Aut}_+\mathcal{O}(R) = \left\{z \mapsto \rho(z) = z + a_2 z^2 + \cdots \mid a_i \in R\right\}.$$

This is a subgroup scheme of the group scheme $\mathrm{Aut}\,\mathcal{O}$. In particular, one has $\mathrm{Aut}\,\mathcal{O} = \mathbb{G}_m \ltimes \mathrm{Aut}_+\mathcal{O}$. The $(\mathrm{Aut}\,\mathcal{O})^{\oplus n}$-torsor $\mathcal{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ factors as the composition of an $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$-torsor and a $\mathbb{G}_m^{\oplus n}$-torsor:

(50)

$$
\begin{array}{ccc}
& & \mathcal{M}_{g,n} \\
& {\scriptstyle(\mathrm{Aut}_+\mathcal{O})^{\oplus n}}\nearrow & \downarrow {\scriptstyle(\mathrm{Aut}\,\mathcal{O})^{\oplus n}} \\
\overline{\mathcal{J}}^{\times}_{g,n} & & \\
& {\scriptstyle\mathbb{G}_m^{\oplus n}}\searrow & \downarrow \\
& & \overline{\mathcal{M}}_{g,n}.
\end{array}
$$

Here $\overline{\mathcal{J}}^{\times}_{g,n}$ is the space of tuples $(C, P_\bullet, \tau_\bullet)$, where $\tau_\bullet = (\tau_1, \dots, \tau_n)$ with τ_i a non-zero 1-jet of a formal coordinate at P_i for each i. As in [21, §6.3.1], $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$ acts on $\mathbb{V}(V; M^\bullet)$, and descending along the $(\mathrm{Aut}_+\mathcal{O})^{\oplus n}$-torsor in (50), one obtains a sheaf of coinvariants $\mathbb{V}^J(V; M^\bullet)$ on $\overline{\mathcal{J}}^{\times}_{g,n}$.

The idea for the descent of $\mathbb{V}^J(V; M^\bullet)$ to $\overline{\mathcal{M}}_{g,n}$ is inspired by Tsuchimoto [72] (and used to prove [21, Theorem 8.1]): first, one tensors $\mathbb{V}^J(V; M^\bullet)$ with an appropriate line bundle to obtain a new sheaf on which $\mathbb{G}_m^{\oplus n}$ acts; after descending the new sheaf, one then tensors back with the dual of the line bundle. Next, we detail this argument using root stacks.

The case $n > 1$ can be treated by iterating the procedure used for $n = 1$, hence we discuss only this latter case and set $M^1 =: M$. We will restrict to the case in which the conformal dimension c_M of M is a rational number, and we write $c_M = \frac{a}{d}$ for $a \in \mathbb{Z}$ and $d \in \mathbb{N}$. We consider line bundles $\mathcal{L}_{\mathcal{M}} = (\Psi^\vee)^{\otimes a}$ on $\mathcal{M} := \overline{\mathcal{M}}_{g,1}$, and $\mathcal{L}_{\mathcal{J}} = \pi^*\mathcal{L}_{\mathcal{M}}$ on $\mathcal{J} := \overline{\mathcal{J}}^{\times}_{g,1}$, where $\pi\colon \mathcal{J} \to \mathcal{M}$ is the map forgetting the 1-jets and Ψ is the cotangent line bundle corresponding to the marked point.

8.7.1. *Root stacks.* – We briefly review some properties of the root stacks $\sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$ and $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}}$. Our primary reference is [59], and more information can be found in [60, §§3.3 and 4], [68, §10.3], and [4, App. B].

The root stack $\sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$ is the stack parametrizing d-th roots of the line bundle $\mathcal{L}_{\mathcal{M}}$. In other words, $\sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$ represents the functor which associates to every scheme $\phi\colon Y \to \mathcal{M}$ the groupoid of pairs $(\mathcal{N}, f)$ where $\mathcal{N}$ is a line bundle on Y and $f\colon \mathcal{N}^{\otimes d} \to \phi^*\mathcal{L}_{\mathcal{M}}$ is an isomorphism. An isomorphism between $(\mathcal{N}_1, f_1)$ and $(\mathcal{N}_2, f_2)$ is an isomorphism of line bundles $g\colon \mathcal{N}_1 \to \mathcal{N}_2$ such that $f_2 \circ g^{\otimes d} = f_1$. One defines $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}}$ similarly.

Since $\mathcal{L}_{\mathcal{J}}$ is the pullback of $\mathcal{L}_{\mathcal{M}}$ along π, one has a Cartesian diagram

(51)

$$
\begin{array}{ccc}
\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}} & \xrightarrow{\sqrt[d]{\pi}} & \sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}} \\
{\scriptstyle p_{\mathcal{J}}}\downarrow & & \downarrow{\scriptstyle p_{\mathcal{M}}} \\
\mathcal{J} & \xrightarrow{\pi} & \mathcal{M}.
\end{array}
$$

In particular, $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}} \to \sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$ is a $\mathbb{G}_m$-torsor. The stacks $\sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$ and $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}}$ have universal line bundles $\mathcal{U}_{\mathcal{M}}$ and $\mathcal{U}_{\mathcal{J}} = \left(\sqrt[d]{\pi}\right)^* \mathcal{U}_{\mathcal{M}}$ such that $\mathcal{U}_{\mathcal{M}}^{\otimes d} = p_{\mathcal{M}}^*\mathcal{L}_{\mathcal{M}}$ and $\mathcal{U}_{\mathcal{J}}^{\otimes d} = p_{\mathcal{J}}^*\mathcal{L}_{\mathcal{J}}$.

The key property that we will use about $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}}$ (and $\sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$ as well) is that its category of quasi-coherent sheaves has an eigendecomposition with respect to the action of the inertial group μ_d of d-th roots of unity, and the degree zero component on $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}}$ (resp., $\sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$) consists of pullbacks of quasi-coherent sheaves on $\mathcal{J}$ (resp., $\mathcal{M}$). By [59, Lemma 3.1.1.7], the eigensheaves for the trivial character on the root stack are identified with sheaves on the base stack $\mathcal{M}$. This allows one to conclude that the pullback via $p_{\mathcal{J}}$ is fully faithful, i.e., two quasi-coherent sheaves on $\mathcal{J}$ are isomorphic if and only if they are isomorphic when pulled back to $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}}$.

8.7.2. *The final descent.* – Let $\mathbb{V}^J := \mathbb{V}^J(V; M)$. The quasi-coherent sheaf $p_{\mathcal{J}}^* \mathbb{V}^J \otimes \mathcal{U}_{\mathcal{J}}$ on $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}}$ has an action of $\mathbb{G}_m$ as in [21, §§4.2.1, 6.3.2][1]. Descending along the $\mathbb{G}_m$-torsor $\sqrt[d]{\mathcal{L}_{\mathcal{J}}/\mathcal{J}} \to \sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$, we obtain a sheaf $\mathcal{F}$ on $\sqrt[d]{\mathcal{L}_{\mathcal{M}}/\mathcal{M}}$ for which

$$(\sqrt[d]{\pi})^* \mathcal{F} \cong p_{\mathcal{J}}^* \mathbb{V}^J \otimes \mathcal{U}_{\mathcal{J}}.$$

Tensoring the above with $(\sqrt[d]{\pi})^* \mathcal{U}_{\mathcal{M}}^\vee = \mathcal{U}_{\mathcal{J}}^\vee$, we have

(52) $$(\sqrt[d]{\pi})^* \left(\mathcal{F} \otimes \mathcal{U}_{\mathcal{M}}^\vee \right) \cong p_{\mathcal{J}}^* \mathbb{V}^J.$$

It follows that $\mathcal{F} \otimes \mathcal{U}_{\mathcal{M}}^\vee$ lives in the degree zero component, hence it descends to a sheaf $\mathbb{V}(V; M)$ on $\mathcal{M}$ such that $p_{\mathcal{M}}^* \mathbb{V}(V; M) \cong \mathcal{F} \otimes \mathcal{U}_{\mathcal{M}}^\vee$. By its construction and the commutativity of (51), one has

$$(\pi \circ p_{\mathcal{J}})^* \mathbb{V}(V; M) = \left(p_{\mathcal{M}} \circ \sqrt[d]{\pi} \right)^* \mathbb{V}(V; M) = p_{\mathcal{J}}^* \mathbb{V}^J.$$

Since the pullback of sheaves to a root stack is fully faithful, we deduce that $\mathbb{V}^J = \pi^* \mathbb{V}(V; M)$. In particular, $\mathbb{V}^J$ descends to a sheaf $\mathbb{V}(V; M)$, which is therefore well-defined on $\overline{\mathcal{M}}_{g,1}$.

REMARK 8.7.3. – Here we list some observations.

1. When V is C_2-cofinite and rational, we can descend $\mathbb{V}^J(V; M)$ to a sheaf $\mathbb{V}(V; M)$ over $\overline{\mathcal{M}}_{g,n}$ for every finitely-generated V-module M. Indeed, since the category of V-modules is semisimple, we can decompose $M = \bigoplus_{\ell \in I} M^\ell$ with M^ℓ a simple V-module with rational conformal dimension and I a finite index set. This induces the decomposition $\mathbb{V}^J(V; M) = \bigoplus_{\ell \in I} \mathbb{V}^J(V; M^\ell)$, and we can apply the descent argument of §8.7.2 to each component $\mathbb{V}^J(V; M^\ell)$.

2. For simplicity, we have given the details of the argument when the conformal dimensions are rational. This is indeed the case for simple modules over a C_2-cofinite V, and allows for the use of a root stack defined by a line bundle. While not needed for this paper, an analogous argument can be made for complex, irrational conformal dimensions, using a gerbe that is a generalization of the root stack.

3. When $M^i = V$ for all i, the action of $\mathbb{G}_m^{\oplus n}$ on $\mathbb{V}^J(V; M^\bullet)$ from [21, §§4.2.1, 6.3.2] is compatible with the restriction of the action of $(\mathrm{Aut}\,\mathcal{O})^{\oplus n}$. In this case, the above construction simplifies, since the two descents along the $(\mathrm{Aut}_+ \mathcal{O})^{\oplus n}$-torsor and $\mathbb{G}_m^{\oplus n}$-torsor in (50) are equivalent to the descent along the $(\mathrm{Aut}\,\mathcal{O})^{\oplus n}$-torsor.

[1] For this action, one uses the filtration on $\mathbb{V}^J(V; M)$ induced by the $\mathbb{Z}_{\geq 0}$-grading of the module M.

4. The method used to construct the sheaf $\mathbb{V}(V; M^\bullet)$ on $\overline{\mathcal{M}}_{g,n}$ from $\mathbb{V}^J(V; M^\bullet)$ guarantees that, when $\mathbb{V}(V; M^\bullet)$ is of finite rank on $\mathcal{M}_{g,n}$, the Chern character of $\mathbb{V}(V; M^\bullet)$ on $\mathcal{M}_{g,n}$ is given by [21, Cor 9.1].

8.8. Proof of the VB corollary

By means of Theorems 8.4.2 and 8.5.1, one concludes that the sheaf of coinvariants $\mathbb{V}(V; M^\bullet)$ is a vector bundle of finite rank on $\mathcal{M}_{g,n}$, and this gives rise to a vector bundle of finite rank on $\overline{\mathcal{M}}_{g,n}$, as in [74], [71, §2.7], [61], [67]. We sketch the argument for completeness.

First, we argue that $\mathbb{V}(V; M^\bullet)$ is a vector bundle of finite rank on $\mathcal{M}_{g,n}$. The sheaf $\mathbb{V}(V; M^\bullet)$ on $\widehat{\mathcal{M}}_{g,n}$ is coherent (Corollary 5.1.2) and is equipped with a projectively flat connection [21]. As in [74], see also [71, §2.7], it follows that $\mathbb{V}(V; M^\bullet)$ is locally free of finite rank on $\widehat{\mathcal{M}}_{g,n}$. After Theorem 8.4.2 and gluing the sheaf as in [13], it follows that the sheaf $\mathbb{V}(V; M^\bullet)$ is also coherent on $\mathcal{M}_{g,n}$. It remains to show that $\mathbb{V}(V; M^\bullet)$ is locally free on $\mathcal{M}_{g,n}$. For this, consider a stable family of n-pointed nodal curves $(\mathscr{C}_0 \to \mathrm{Spec}(R), P_\bullet)$, and for simplicity, assume that it has only one simple node. Consider its formal smoothing $(\mathscr{C} \to \mathrm{Spec}(R[\![q]\!]), P_\bullet)$ as described in §8.1. For each i, fix a formal coordinate t_i at $P_i(S)$. The sewing theorem (Theorem 8.5.1) implies that $\mathbb{V}(V; M^\bullet)_{(\mathscr{C}/S, P_\bullet, t_\bullet)}$ is locally free of finite rank, hence we conclude the argument. For families of curves with more nodes, one proceeds similarly. It follows that $\mathbb{V}(V; M^\bullet)$ is a vector bundle of finite rank on $\mathcal{M}_{g,n}$.

Finally, since V is rational, by §8.7 and Remark 8.7.3(i), we can descend $\mathbb{V}(V; M^\bullet)$ to a sheaf of coinvariants on $\overline{\mathcal{M}}_{g,n}$. As the descent of a vector bundle is a vector bundle, this concludes the proof of the VB corollary. $\qquad\square$

9. Examples

Here we list examples of vertex operator algebras V satisfying the hypotheses of our theorems, namely: (1) $V = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} V_i$ with $V_0 \cong \mathbb{C}$; (2) V is rational; and (3) V is C_2-cofinite.

9.1. Virasoro VOAs

Given the Lie subalgebra $\mathrm{Vir}_{\geq 0} := \mathbb{C}K \oplus z\mathbb{C}[\![z]\!]\partial_z$ of the Virasoro Lie algebra Vir, and $c, h \in \mathbb{C}$, let $M_{c,h} := U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}_{\geq 0})} \mathbb{C}\mathbf{1}$, where $\mathbb{C}\mathbf{1}$ inherits the structure of a $\mathrm{Vir}_{\geq 0}$-module by setting $L_{p>0}\mathbf{1} = 0$, $L_0\mathbf{1} = h\mathbf{1}$, and $K\mathbf{1} = c\mathbf{1}$. There is a unique maximal proper submodule $J_{c,h} \subset M_{c,h}$. For $h = 0$, $J_{c,0}$ contains a submodule generated by the singular vector $L_{-1}\mathbf{1} \in M_{c,0}$ [37]. Set

$$L_{c,h} := M_{c,h}/J_{c,h}, \quad M_c := M_{c,0}/\langle L_{-1}\mathbf{1}\rangle, \quad \text{and} \quad \mathrm{Vir}_c := L_{c,0}.$$

If $c \neq c_{p,q} := 1 - \frac{6(p-q)^2}{pq}$, with relatively prime $p, q \in \mathbb{N}$ such that $1 < p < q$, then $M_c \cong \mathrm{Vir}_c$, that is, $J_{c,0} = \langle L_{-1}\mathbf{1}\rangle$, while for $c = c_{p,q}$, the submodule $J_{c,0}$ is generated by two singular vectors [37]. By [43, Thm 4.3], M_c and Vir_c are VOAs. Since $A(M_c) \cong \mathbb{C}[x]$ [43, Thm 4.6], M_c is not rational. However, Vir_c is rational if and only if $c = c_{p,q}$ [75, Thm 4.2] if and only if Vir_c is C_2-cofinite [29, Lemma 12.3] (see also [5, Prop. 3.4.1]). In this case, $A(\mathrm{Vir}_c)$ is a quotient of $\mathbb{C}[x]$, and simple Vir_c-modules are the $L_{c,h}$, for $h = \frac{(np-mq)^2-(p-q)^2}{4pq}$ with $0 < m < p$ and $0 < n < q$.

9.2. Simple affine VOAs

One may associate to a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ and $\ell \in \mathbb{Z}_{>0}$, a simple vertex operator algebra $L_\ell(\mathfrak{g})$, described in [43, §2], [56, §6.2] (see also [67, §A.1.1]). This is rational by [43, Thm 3.1.3] and C_2-cofinite by [77] (see also [29, Prop. 12.6], [5, Prop. 3.5.1]).

9.3. The moonshine module $V^\natural$

A rational vertex operator algebra V with no nontrivial simple V-modules is called *holomorphic*, and if (1) and (3) also hold, then V must have central charge divisible by 8 [31]. One example is given by the moonshine module $V^\natural$ of central charge 24, whose automorphism group is the monster group [42].

9.4. Even lattice vertex algebras

Vertex operator algebras V_L given by positive-definite even lattices L of finite rank [16] are rational [24] and C_2-cofinite [29]. Zhu's algebra is described in [26, Thm 3.4].

9.5. Exceptional W-algebras

Arakawa [7] has shown that a large class of simple W-algebras are C_2-cofinite, including the minimal series principal W-algebras [39] and the exceptional W-algebras of Kac-Wakimoto [53]. Moreover, the minimal series principal W-algebras and a large subclass of exceptional affine W-algebras are rational [6, 8, 9].

9.6. Orbifolds, commutants, and tensor products

More vertex operator algebras can be obtained from the examples discussed above through standard constructions resulting in orbifold algebras, commutants, and tensor products. These constructions often preserve our desired properties. For instance, for a finite subgroup G of the automorphism group $\mathrm{Aut}(V)$, the orbifold algebra V^G is conjecturally C_2-cofinite and rational when so is V. This holds for G solvable and V simple [65, 18]. For a subalgebra $\mathcal{A} \subset V$, the commutant $\mathrm{Com}(\mathcal{A}, V)$ [43] is conjecturally C_2-cofinite and rational if so are both $\mathcal{A}$ and V. This holds for parafermions [34]. The VOAs $V^1, \ldots, V^m$ are rational if and only if $V = \bigotimes_{i=1}^m V^i$ is rational [33]; in this case, if the V^i are C_2-cofinite, then so is V [27].

Appendix
Zhu's Lie algebra and isomorphic coinvariants

For a smooth curve C and a quasi-primary generated vertex algebra V with $V_0 \cong \mathbb{C}$, in addition to the chiral Lie algebra $\mathcal{L}_{C \setminus P_\bullet}(V)$ (§3.1), one also has Zhu's Lie algebra $\mathfrak{g}_{C \setminus P_\bullet}(V)$, reviewed in §A.1.

In Proposition A.2.1, we show that when defined, $\mathfrak{g}_{C \setminus P_\bullet}(V)$ is isomorphic to the image of $\mathcal{L}_{C \setminus P_\bullet}(V)$ under the Lie algebra homomorphism $\varphi_{\mathcal{L}}$ (Proposition A.2.1). Nagatomo and Tsuchiya extend the definition of $\mathfrak{g}_{C \setminus P_\bullet}(V)$ to stable pointed rational curves [67], and they indicate that their coinvariants are equivalent to those studied by Beilinson and Drinfeld in [14], suggesting they knew that Proposition A.2.1 holds in that case.

A quasi-primary vector is an element $A \in V$ such that $L_1 A = 0$, and V is quasi-primary generated if and only if $L_1 V_1 = 0$ [30]. A vertex algebra $V = \bigoplus_{i \geq 0} V_i$ with $V_0 \cong \mathbb{C}$ satisfies $L_1 V_1 = 0$ if and only if $V \cong V'$ (see [41, §5.3] and [32, §2]). In particular, in the results of Huang [49] and Codogni [19], the vertex algebras studied are quasi-primary generated.

A.1. The Lie algebra $\mathfrak{g}_{C \setminus P_\bullet}(V)$

In [77], given a smooth pointed curve $(C, P_\bullet)$ and a quasi-primary generated vertex operator algebra V for which $V_0 \cong \mathbb{C}$, Zhu defines a Lie algebra $\mathfrak{g}_{C \setminus P_\bullet}(V)$, generalizing the construction of Tsuchiya, Ueno, and Yamada for affine Lie algebras. Namely, consider

$$\mathfrak{g}_{C \setminus P_\bullet}(V) := \varphi_{\mathfrak{g}}\left(\bigoplus_{k \geq 0} V_k \otimes \mathrm{H}^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right)\right),$$

where

$$(53) \qquad \varphi_{\mathfrak{g}} : \bigoplus_{k \geq 0} V_k \otimes \mathrm{H}^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right) \to \bigoplus_{i=1}^{n} \mathfrak{L}_{t_i}(V)$$

is the map induced by

$$B \otimes \mu \mapsto \left(\mathrm{Res}_{t_i = 0} Y[B, t_i] \mu_{P_i} (dt_i)^k\right)_{i=1,\dots,n}.$$

Here t_i is a formal coordinate at the point P_i, $Y[B, t_i] := \sum_{k \in \mathbb{Z}} B_{[k]} t_i^{-k-1}$, and μ_{P_i} is the Laurent series expansion of μ at P_i, the image of μ via

$$\mathrm{H}^0\left(C \setminus P_\bullet, \omega_C^{\otimes 1-k}\right) \to \mathrm{H}^0\left(D_{P_i}^\times, \omega_C^{\otimes 1-k}\right) \simeq_{t_i} \mathbb{C}((t_i))(dt_i)^{1-k}.$$

When V is assumed to be quasi-primary generated with $V_0 \cong \mathbb{C}$, Zhu shows that $\mathfrak{g}_{C \setminus P_\bullet}(V)$ is a Lie subalgebra of $\mathfrak{L}(V)^{\oplus n}$. The argument uses that any *fixed* smooth algebraic curve admits an atlas such that all transition functions are Möbius transformations. Transition functions between charts on families of curves of arbitrary genus are more general, hence the need to consider the more involved construction for the chiral Lie algebra based on the (Aut $\mathcal{O}$)-twist of V in §2.

A.2. Isomorphism of coinvariants

When $\mathfrak{g}_{C \setminus P_\bullet}(V)$ is well-defined and $C \setminus P_\bullet$ is affine, one can define the space of coinvariants $M^\bullet_{\mathfrak{g}_{C \setminus P_\bullet}(V)}$ as the quotient of the $\mathfrak{L}(V)^{\oplus n}$-module $M^\bullet$ by the action of the Lie subalgebra $\mathfrak{g}_{C \setminus P_\bullet}(V)$ of $\mathfrak{L}(V)^{\oplus n}$. These spaces were introduced in [76], and studied also in [3, 67]. Recall the homomorphisms $\varphi_{\mathcal{L}}$ from (24) and $\varphi_{\mathfrak{g}}$ from (53).

PROPOSITION A.2.1. – *When $\mathfrak{g}_{C \setminus P_\bullet}(V)$ is well-defined (§A.1), one has*

$$\mathrm{Im}(\varphi_{\mathcal{L}}) \cong \mathrm{Im}(\varphi_{\mathfrak{g}}).$$

It follows that there exists an isomorphism of vector spaces

$$M^\bullet_{\mathfrak{g}_{C \setminus P_\bullet}(V)} \cong M^\bullet_{\mathcal{L}_{C \setminus P_\bullet}(V)}.$$

Proof. – One has

$$(54) \qquad \bigoplus_{k \geq 0} V_k \otimes \mathrm{H}^0 \left(C \setminus P_\bullet, \omega_C^{\otimes 1-k} \right) \cong \mathrm{H}^0 \left(C \setminus P_\bullet, \bigoplus_{k \geq 0} V_k \otimes \omega_C^{\otimes 1-k} \right)$$

$$\cong \mathrm{H}^0 \left(C \setminus P_\bullet, \bigoplus_{k \geq 0} \left(\omega_C^{\otimes 1-k} \right)^{\oplus \dim V_k} \right).$$

From Lemma 2.6.1, one has $\mathrm{gr}_\bullet \mathcal{V}_C \cong \bigoplus_{k \geq 0} \left(\omega_C^{\otimes -k} \right)^{\oplus \dim V_k}$. It follows that

$$\mathrm{H}^0 \left(C \setminus P_\bullet, \bigoplus_{k \geq 0} \left(\omega_C^{\otimes 1-k} \right)^{\oplus \dim V_k} \right) \cong \mathrm{H}^0 (C \setminus P_\bullet, \mathrm{gr}_\bullet \mathcal{V}_C \otimes \omega_C).$$

Now by Lemma 2.6.3,

$$(55) \qquad \mathrm{H}^0 \left(C \setminus P_\bullet, \mathrm{gr}_\bullet \mathcal{V}_C \otimes \omega_C \right) \cong \mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C \right).$$

On the other hand, as $C \setminus P_\bullet$ is assumed to be affine, one has

$$\mathcal{L}_{C \setminus P_\bullet}(V) \cong \mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C \right) / \nabla \mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \right).$$

The map $\varphi_\mathcal{L}$ is induced from the composition

$$(56) \qquad \mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C \right) \to \bigoplus_{i=1}^{n} \mathrm{H}^0 \left(D_{P_i}^\times, \mathcal{V}_C \otimes \omega_C \right) \to \bigoplus_{i=1}^{n} \mathfrak{L}_{t_i}(V).$$

The first map is canonical and obtained by restricting sections; the second is (22). By [38, §6.6.9], sections in $\nabla \mathrm{H}^0 \left(D_{P_i}^\times, \mathcal{V}_C \right)$ act trivially. Hence (56) induces a map from the Lie algebra $\mathcal{L}_{C \setminus P_\bullet}(V)$ to $\bigoplus_{i=1}^{n} \mathfrak{L}_{t_i}(V)$. It follows that the image of $\varphi_\mathcal{L}$ coincides with the image of $\mathrm{H}^0 \left(C \setminus P_\bullet, \mathcal{V}_C \otimes \omega_C \right)$ in $\bigoplus_{i=1}^{n} \mathfrak{L}_{t_i}(V)$ via (56). Composing (54) and (55), and by the definition of $\varphi_\mathfrak{g}$ in (53), the image of the map in (56) coincides with the image of $\varphi_\mathfrak{g}$. □

Acknowledgements

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