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VOLUME EN L'HONNEUR DE JEAN-MICHEL BONY

édité par

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***Résumé.* —** À l'occasion du soixantième anniversaire de Jean-Michel Bony, ses élèves, collaborateurs et amis ont tenu à lui dédier ce volume de la collection *Astérisque*. Il contient des articles de recherche en Analyse Microlocale linéaire, non-linéaire, algébrique..., illustrant la vivacité de ce domaine des mathématiques auquel J.-M. Bony a tant contribué.

***Abstract* (Around microlocal analysis. Volume in honor of Jean-Michel Bony)**

On the occasion of the sixtieth birthday of Jean-Michel Bony, his former students, collaborators and friends dedicate to him this *Astérisque* volume. It contains research articles on the linear, non-linear and algebraic microlocal analysis. . . , illustrating the vividness of this field of mathematics to which J.-M. Bony contributed so much.

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RÉSUMÉS DES ARTICLES

Nous considérons un exemple d'équation d'ondes quasi-linéaire qui se situe entre les exemples vraiment non-linéaires (pour lesquels l'explosion en temps fini est connue) et les exemples vérifiant la condition nulle (pour lesquels la solution existe globalement et est asymptotiquement libre). Nous montrons l'existence globale, bien que des arguments d'optique géométrique non-linéaire indiquent un comportement non libre de la solution à l'infini. La méthode de la preuve fait intervenir la commutation avec des champs dépendant de u, et utilise des idées proches de celles du calcul paradifférentiel.

Dans cet article, nous étudions l'existence et l'unicité locale de solutions pour une équation d'onde quasilinéaire cubique. Les classiques estimations de Strichartz ne sont pas adaptées dans ce cas. Nous démontrons des estimations bilinéaires pour des solutions d'équations d'ondes à coefficients variables. Les deux outils principaux sont le calcul paradifférentiel de Bony et la microlocalisation au sens du calcul pseudodifférentiel de Weyl-Hörmander.

Nous introduisons les notions de micro-support et régularité pour les ind-faisceaux et prouvons leur invariance par transformations de contact quantifiées. Nous appliquons ces résultats aux ind-faisceaux des solutions holomorphes tempérées des $\mathcal{D}$-modules. Nous prouvons que le micro-support d'un tel ind-faisceau est la variété caractéristique du $\mathcal{D}$-module correspondant et que le ind-faisceau est régulier si le $\mathcal{D}$-module est holonome régulier. Nous calculons enfin un exemple du ind-faisceau des solutions tempérées d'un $\mathcal{D}$-module irrégulier en dimension un.

Sur une algèbre de Lie réductive, les distributions invariantes qui sont vecteurs propres des opérateurs différentiels bi-invariants sont les solutions d'un système holonome. Il a été démontré par Kashiwara-Hotta que ce module est régulier. Nous résolvons ici une conjecture de Sekiguchi en montrant que ce résultat est encore vrai dans le cas plus général des paires symétriques.

Pour une classe d'opérateurs h-pseudodifférentiels non-autoadjoints, nous déterminons toutes les valeurs propres dans un domaine complexe indépendant de h et nous montrons que ces valeurs propres sont données par une condition de quantification de Bohr-Sommerfeld. Aucune condition d'integrabilité complète est supposée, et une étape géométrique de la démonstration est donnée par un théoreme du type KAM dans le complexe (sans petits dénominateurs).

Soit $X = (X_1, \ldots, X_m)$ un système de champs de vecteurs infiniment dégénérés. On montre d'abord l'inégalité de Sobolev logarithmique pour ce système de champs de vecteurs sur les espaces de fonctions associés, puis on étudie le problème de Dirichlet semi-linéaire pour des opérateurs somme de carrés de champs de vecteurs X.

En un point lisse d'une variété caractéristique définie par un polynôme homogène hyperbolique, le plan tangent détermine la vitesse de groupe. Dans cet article, on en déduit un algorithme algébrique de calcul de ce plan tangent en un point donné. Il n'est intéressant que là où la différentielle du polynôme s'annule.

Nous montrons que des arguments géométriques très simples, basés sur la réflexion de Schwarz, permettent souvent de décider si deux paires d'arcs analytiques tangents en $0 \in \mathbb{C}$ sont analytiquement équivalentes au voisinage de 0. Nous en déduisons la construction de familles nombreuses de germes, formellement mais non analytiquement conjugués, de difféomorphismes analytiques résonnants de $(\mathbb{C}, 0)$.

ABSTRACTS

We consider an example of a Quasilinear Wave Equation which lies between the genuinely nonlinear examples (for which finite time blowup is known) and the null condition examples (for which global existence and free asymptotic behavior is known). We show global existence, though geometrical optics techniques show that the solution does not behave like a free solution at infinity. The method of proof involves commuting with fields depending on u, and uses ideas close to that of the paradifferential calculus.

In this paper, we study the local wellposedness of a cubic quasilinear wave equation. The Strichartz estimate used for the solutions of linear variable coefficients wave equations are not relevant here. We prove bilinear estimates for solutions of linear wave equations with variable coefficients. The main tools are Bony's paradifferential calculus and the microlocalization in the sense of Weyl-Hörmander calculus.

We introduce the notions of micro-support and regularity for ind-sheaves, and prove their invariance by quantized contact transformations. We apply these results to the ind-sheaves of temperate holomorphic solutions of $\mathcal{D}$-modules. We prove that the micro-support of such an ind-sheaf is the characteristic variety of the corresponding $\mathcal{D}$-module and that the ind-sheaf is regular if the $\mathcal{D}$-module is regular holonomic. We finally calculate an example of the ind-sheaf of temperate solutions of an irregular $\mathcal{D}$-module in dimension one.

The invariant eigendistributions on a reductive Lie algebra are solutions of a holonomic $\mathcal{D}$-module which has been proved to be regular by Kashiwara-Hotta. We solve here a conjecture of Sekiguchi saying that in the more general case of symmetric pairs, the corresponding module is still regular.

For a class of non-selfadjoint h-pseudodifferential operators in dimension 2, we determine all eigenvalues in an h-independent domain in the complex plane and show that they are given by a Bohr–Sommerfeld quantization condition. No complete integrability is assumed, and as a geometrical step in our proof, we get a KAM–type theorem (without small divisors) in the complex domain.

Let $X = (X_1, \dots, X_m)$ be an infinitely degenerate system of vector fields, we prove firstly the logarithmic Sobolev inequality for this system on the associated Sobolev function spaces. Then we study the Dirichlet problem for the semilinear problem of the sum of square of vector fields X.

At a smooth point of the characteristic variety defined by a homogeneous hyperbolic polynomial, the tangent plane determines the group velocity. In this note an algebraic algorithm is derived for computing this tangent plane at a given point. This is interesting only where the differential of the polynomial vanishes.

We show that simple geometric arguments, based on the Schwarz reflection, allow in many cases to decide whether two pairs of tangent analytic arcs at $0 \in \mathbb{C}$ are conformally equivalent in a small neighborhood of 0. As an application, we exhibit big families of germs of analytic resonant diffeomorphisms of $(\mathbb{C}, 0)$, which are formally, but not analytically conjugate.

PRÉFACE

L'Analyse Microlocale s'est développée dans le dernier tiers du vingtième siècle à la suite des travaux fondateurs de M. Sato et de son école en Analyse Algébrique. Jean-Michel Bony est un des principaux artisans de l'extraordinaire fécondité de cette « analyse dans l'espace de phase », dont les concepts et méthodes vont tant faire progresser notre compréhension des équations aux dérivées partielles.

À l'occasion de son soixantième anniversaire, ses élèves, collaborateurs et amis ont donc souhaité offrir à Jean-Michel Bony ce recueil d'articles de recherche.

On y trouvera deux articles en Analyse Algébrique, un d'Analyse Complexe, trois en Analyse Microlocale linéaire et deux en Analyse Microlocale non-linéaire. M. Kashiwara et P. Schapira introduisent et développent leur théorie du micro-support des Ind-faisceaux. Y. Laurent prouve une conjecture de Sekiguchi sur la régularité des $\mathcal{D}$-modules associés à une paire symétrique. J.-M. Trépreau expose une approche géométrique de la classification holomorphe des paires d'arcs tangents. A. Melin et J. Sjöstrand élucident la théorie spectrale des opérateurs non autoadjoints en dimension 2. Y. Morimoto et C.-J. Xu étudient les sommes de carrés de champs de vecteurs inifiniment dégénérés. J. Rauch introduit un algorithme de calcul de la vitesse de groupe d'une équation hyperbolique. Enfin, l'article de S. Alinhac sur le comportement asymptotique des équations d'ondes non-linéaires et celui de H. Bahouri et J.-Y. Chemin sur le problème de Cauchy local pour les équations d'ondes quasi-linéaires illustrent l'efficacité des techniques d'analyse microlocale non-linéaire, dont Jean-Michel Bony, en inventant le paraproduit, fut le pionnier.

Gilles Lebeau

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AN EXAMPLE OF BLOWUP AT INFINITY FOR A QUASILINEAR WAVE EQUATION

by

Serge Alinhac

Dédié à J-M. Bony à l'occasion de son soixantième anniversaire

***Abstract.* —** We consider an example of a Quasilinear Wave Equation which lies between the genuinely nonlinear examples (for which finite time blowup is known) and the null condition examples (for which global existence and free asymptotic behavior is known). We show global existence, though geometrical optics techniques show that the solution does not behave like a free solution at infinity. The method of proof involves commuting with fields depending on u, and uses ideas close to that of the paradifferential calculus.

***Résumé* (Explosion à l'infini pour un exemple d'équation d'ondes quasi-linéaire)**
Nous considérons un exemple d'équation d'ondes quasi-linéaire qui se situe entre les exemples vraiment non-linéaires (pour lesquels l'explosion en temps fini est connue) et les exemples vérifiant la condition nulle (pour lesquels la solution existe globalement et est asymptotiquement libre). Nous montrons l'existence globale, bien que des arguments d'optique géométrique non-linéaire indiquent un comportement non libre de la solution à l'infini. La méthode de la preuve fait intervenir la commutation avec des champs dépendant de u, et utilise des idées proches de celles du calcul paradifférentiel.

In this text, Theorems, Propositions etc. are numbered according to the section where they appear, without any mention of the Chapter. When quoted in a different chapter, they appear with the additional mention of the Chapter. For instance, in Chapter III, section 2, there is Lemma 2. In Chapter IV, section 4, the same Lemma is quoted as Lemma III.2.

2000 *Mathematics Subject Classification.* — 35L40.
***Key words and phrases.* —** Quasilinear Wave Equation, Energy inequality, decay, blowup, geometrical optics, Poincaré inequality, paradifferential calculus, weighted norm.

Introduction

We prove in this paper the global existence (for ε small enough) of smooth solutions to the equation in $\mathbf{R}^3_x \times \mathbf{R}_t$

$$\partial_t^2 u - c^2(u)\Delta_x u = 0, c(u) = 1 + u,$$

with smooth and compactly supported initial data of size ε.

This result has been proved before only in the *radially symmetric case* by Lindblad [**13**], who also pointed out to some evidence that the nonradial solutions should have a very large lifespan. It turns out that the solutions do not behave at $t = +\infty$ like solutions of the free wave equation (that is, $u \sim \varepsilon/(1+t)$); most derivatives of u have, apart from the factor $\varepsilon/(1+t)$, an exponential growth $\exp C\tau$ at infinity, where $\tau = \varepsilon \log(1+t)$ is the slow time. This explains the title of this paper.

The method of proof is that of Klainerman [**11**], combining energy inequalities and commutations with appropriate "**Z**" fields. Because of the blowup at infinity, the fields we use have to be adapted to the geometry of the problem (as in Christodoulou-Klainerman [**7**]), and their coefficients smoothed out. This is very close to the paradifferential calculus of Bony [**6**], or, equivalently, to a Nash-Moser process.

I. Main result and ideas of the proof

We consider in $\mathbf{R}^3_x \times \mathbf{R}_t$ the equation

$$F(u) \equiv \partial_t^2 u - c^2(u)\Delta_x u = 0, \tag{1.1$_a$}$$

where we will take for simplicity $c = c(u) = 1 + u$, since higher powers of u produce only easily handled terms. The coordinates will be

$$x = (x_1, x_2, x_3), \quad t = x_0,$$

and

$$\partial u = (\partial_1 u, \partial_2 u, \partial_3 u, \partial_t u).$$

The initial data are

$$u(x,0) = \varepsilon u_1^0(x) + \varepsilon^2 u_2^0(x) + \cdots, \quad (\partial_t u)(x,0) = \varepsilon u_1^1(x) + \varepsilon^2 u_2^1(x) + \cdots, \tag{1.1$_b$}$$

for real C^∞ functions u_i^j, supported in the ball $|x| \leqslant M$.

We will use the usual polar coordinates $r = |x|$, $x = r\omega$, and define the rotation fields

$$R_1 = x_2\partial_3 - x_3\partial_2, \quad R_2 = x_3\partial_1 - x_1\partial_3, \quad R_3 = x_1\partial_2 - x_2\partial_1.$$

By Z_0 we will denote one of the standard Klainerman's fields

$$\partial_i, R_j, \quad S = t\partial_t + r\partial_r, \quad h_i = x_i\partial_t + t\partial_i. \tag{1.2}$$

For the Laplace operator, we have then

$$\Delta_x = \partial_r^2 + (2/r)\partial_r + (1/r^2)\Delta_\omega,$$

where the Laplace operator on the sphere Δ_ω is $\Delta_\omega = R_1^2 + R_2^2 + R_3^2$.

We define two linear operators

$$P \equiv c^{-1}\partial_t^2 - c\Delta, \quad P_1 \equiv c^{-1}\partial_t^2 - c(\partial_r^2 + r^{-2}\Delta_\omega), \tag{1.3}$$

such that, setting $u = \varepsilon/rU$, we have $Pu = 0$, $P_1 U = 0$. We also set

$$L \equiv c^{-1/2}\partial_t + c^{1/2}\partial_r, \quad L_1 \equiv c^{-1/2}\partial_t - c^{1/2}\partial_r,$$

for which we have

$$[L, L_1] = (L_1 u/2c)L_1 - (Lu/2c)L, \quad P_1 = LL_1 - cr^{-2}\Delta_\omega + (Lu/2c)L. \tag{1.4}$$

Remark that, since $c = c(u)$, iterated use of the fields $L, L_1, \partial_j, R_j, S$ will generate a considerable number of terms depending again on u. To master this phenomenon, we will have to construct an appropriate "Calculus". Finally, we set

$$\sigma_1 = M + 1 - r + t, \tag{1.5}$$

which is positive and roughly equivalent to the distance to the boundary of the light cone.

Our main result is the following Theorem.

Theorem. — *Let $s_0 \in \mathbf{N}$. For ε small enough, the Cauchy problem* (1.1) *has a global smooth solution u. Moreover, we have the estimates*

$$|Z_0^\alpha \partial u|_{L^2} \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad |\alpha| \leqslant s_0,$$

$$|\partial u| \leqslant C\varepsilon(1+t)^{-1}, \quad |Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0 - 2.$$

In the case of radially symmetric data, the solution u is a smooth function of (r^2, t). For this case, Lindblad [**13**] has proved global existence. We explain now the main ideas of the proof. In the whole paper, all constants will be denoted by C, unless otherwise specified.

I.1. A first insight using nonlinear geometrical optics

a. If w denotes the solution of the linearized problem on zero

$$(\partial_t^2 - \Delta)w = 0, \quad w(x,0) = u_1^0(x), \quad (w_t)(x,0) = u_1^1(x),$$

we know (see [**10**]) that, for some smooth F_0,

$$w \sim 1/rF_0(\omega, r-t), \quad r \to +\infty.$$

Taking εw as a rough approximation of u, we observe as in [**10**], [**1**] that the quadratic nonlinearity $u\Delta u$ produces a *slow time* effect, for the slow time $\tau \equiv \varepsilon \log(1+t)$. This means that, for large time, we expect formally u to be better approximated by

$$\varepsilon/rV(r-t, \omega, \tau),$$

for a smooth V satisfying $V(r-t,\omega,0) = F_0(\omega, r-t)$. Substituting the above expression of u in (1.1), we obtain

$$V_{\sigma\tau} + VV_{\sigma\sigma} = 0, \quad V(\sigma,\omega,0) = F_0(\omega,\sigma), \quad \sigma \equiv r-t. \tag{1.6}$$

As pointed out already in [**13**], this is in sharp contrast with what happens, for instance, for the equation $\partial_t^2 u - (1+u_t)\Delta u = 0$. In this case, a similar approach yields for V the equation $2V_{\sigma\tau} - V_\sigma V_{\sigma\sigma} = 0$, which is essentially Burgers'equation and blows up in finite time. Here, one easily sees that (1.6) has *global* solutions: this gives a hint that the lifespan of u could be very large (though not necessarily $+\infty$, see for instance the case of the null condition in two space dimensions [**1**]); the consequences of this fact are precisely stated in Theorem II.1.

b. Looking more closely, we see that the solution V of (1.6) satisfies

$$|V_\sigma| \leqslant C, \quad |\partial^\alpha_{\sigma,\omega,\tau} V| \leqslant Ce^{C\tau}.$$

Since we are willing to use Klainerman's method [**11**], we have to apply products Z_0^α to $(1.1)_a$, and use an energy inequality for P to control $|\partial Z_0^\alpha u|_{L^2}$. On the one hand, the boundedness of V_σ yields

$$|\partial u| \leqslant C\varepsilon/(1+t).$$

In the standard energy inequality for P (see [**10**] Prop. 6.3.2), this will cause an *amplification factor* of the initial energy of the form

$$\exp C\varepsilon \int_0^t ds/(1+s) = (1+t)^{C\varepsilon}.$$

Thus the best one can expect, using the energy method and Klainerman's inequality, is

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2},$$

which is the result we obtain. On the other hand, if we believe that u and its derivatives actually behave like ε/rV, we see that derivatives like $R_i u$ or $\partial_t^2 u$, etc. do behave like $\varepsilon/r(1+t)^{C\varepsilon}$, which matches with what we just obtained from the energy method. This is why we say that we have *blowup at infinity*: the solution u exists globally, but does not behave like a solution of the linear equation. This phenomenon has been observed already, for instance in the study by Delort [**8**] of the Klein-Gordon equation.

I.2. Commuting Klainerman's fields

a. If we apply for instance a rotation field R_i to $(1.1)_a$, we obtain

$$PR_i u - 2(R_i u)(\Delta u) = 0.$$

Writing the energy inequality for P, it is not possible to reasonably absorb the term $(R_i u)(\Delta u)$ using Gronwall's lemma since

$$\exp \int_0^t |R_i u|_{L^\infty} \sim \exp[C^{-1}(1+t)^{C\varepsilon}]$$

is far too big. On the other hand, applying Z_0^α to $(1.1)_a$ produces a term $(Z_0^\alpha u)(\Delta u)$ in the equation for $PZ_0^\alpha u$, which is a zero order term: to handle this term will require some type of Poincaré lemma, controlling $Z_0^\alpha u$ by $\partial Z_0^\alpha u$. Note that even in a finite strip $|r-t| \leqslant C$ close to the boundary of the light cone, such a term cannot be reasonably controlled since again Δu behaves exponentially in τ at infinity.

b. Hence we have to modify the standard fields Z_0 to get better commutation properties. Following the geometric approach of Christodoulou-Klainerman [**7**], we define an optic function (in fact, only an *approximate* optic function) $\psi = \psi(r, \omega, t)$ by

$$L\psi = 0, \quad \psi(0, \omega, t) = -M - 1 - t.$$

This is a substitute for the standard optic function $r - t + C$, whose level surfaces are the light cones $r = t + C$. To write down the modified fields Z_m, we first adapt Z_0 to the geometry of the operator by defining $H_0 = ct\partial_r + r/c\partial_t$. For some $a(R_i)$, $a(S)$, $a(H_0)$ to be defined, we set now

$$R_i^m = R_i + a(R_i)L_1, \quad S^m = S + a(S)L_1, \quad H_0^m = H_0 + a(H_0)L_1.$$

Let us pause to explain how this compares with the approach of [**7**]. In [**7**], the authors introduce an exact optic function, whose level surfaces give a foliation of outgoing cones. The rotation fields and L are defined to be tangent to these cones. This way of taking into account the exact geometry of the symbol has the advantage of producing in the computations relatively easily understandable geometric objects. On the other hand, it leads to rather tedious computations: may be, one is demanding too much. Here, since Lu and $(R_i/r)u$ are expected to behave much better than other derivatives of u, we consider that the effect of taking more complicated perturbations (of the standard fields) involving L or R_i/r would be negligible. The choice of the perturbation coefficients a is dictated only by *commutation* properties with L. Ideally, taking

$$(1.7)_a \qquad La(R_i) + a(R_i)(L_1u/(2c)) = -R_iu/(2c),$$

$$(1.7)_b \qquad La(S) + a(S)(L_1u/(2c)) = -Su/(2c), a(H_0) = -a(S),$$

would give

$$[R_i^m, L] = *L, \quad [S^m, L] = *L, \quad [H_0^m, L] = *L.$$

To avoid singularities at $r = 0$, we introduce in fact a cutoff $\overline{\chi} = \overline{\chi}(r/(1+t))$ in (1.7) (see III.1 and the commutation relations of Lemma III.3.1).

I.3. Induction on time. — The proof is by "induction on time" (see [**10**] for instance). We first make the induction hypothesis

$$\textbf{(IH)} \qquad |Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+\eta}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0, \quad \eta = 10^{-2}, \quad s_0 \geqslant 10.$$

This is a pointwise estimate, which is supposed to be valid up to some time T. The strategy of the proof is the following:

Step 1. — From (IH), we deduce (still for $t \leqslant T$) the better behavior in L^∞ norm of a small number of derivatives of u (see Proposition III.7)

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{\mu-1}, \quad |\alpha| \leqslant s_0 - 4.$$

Here, $\mu = 1/2 + 10^{-1}$.

Step 2. — Using the energy method of Klainerman, we bound in L^2 norm (still for $t \leqslant T$) a large number of derivatives of u (see VII.3)

$$|Z_0^\alpha \partial u|_{L^2} \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad |\alpha| \leqslant 2(s_0 - 4).$$

Step 3. — Using Klainerman's inequality, we obtain

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0 \leqslant 2s_0 - 10.$$

If $C\varepsilon \leqslant \eta/2$ and t is large, this is much better than (IH) and Theorem II.1 allows us to prove that for small enough ε, (IH) never stops being true and u exists globally.

To prove the L^∞ estimates of Step 1, we write the equation in the form

$$LL_1U = c/r^2\Delta_\omega U - (Lu/2c)LU,$$

and apply products Z_m^α to the left. In particular, we get $|L_1U| \leqslant C$, which eventually gives $|\partial u| \leqslant C\varepsilon/(1+t)$.

To prove L^2 estimates without loosing derivatives, we have to commute Z_m with P, which causes new problems we analyze now.

I.4. Smoothing

a. In the expression of $[Z_m, P]u$ necessarily appears the term $(Pa)L_1u$, containing $(r^{-2}\Delta_\omega a)L_1u$ and $(LL_1a)L_1u$. Since, from (1.7), we expect to control $R_i^k a$ in terms of $R_i^k Z_m u$ only, we see that we are missing *two* derivatives if we want to keep the full r^{-2} decay, or missing *one* if we rather write

$$r^{-2}\Delta_\omega = r^{-1}\sum (R_i/r)R_i.$$

In both cases, we have to put a *smoothing operator* S_θ in front of a. Here, θ is a big parameter, and $S_\theta v$ is roughly the smooth truncation of $\widehat{v}(\xi)$ for $|\xi| \leqslant \theta$. This is very close to the paradifferential approach introduced by Bony [**6**], where symbols say $a(x)\xi$ correspond to operators T_aD_x and not to aD_x. A typical application of these ideas is given in Alinhac [**4**], where instead of using true vectors fields $\sum a_i\partial_i$ tangent to some (non smooth) surface, we use $\sum T_{a_i}\partial_i$. In other words, we have to commute to the equation vector fields (here, the Z_m) tangent to characteristic surfaces of the operator (here, essentially the modified cones $\psi = const$), but these vector fields have to be smoothed first. Alternatively, one can say that we use a Nash-Moser procedure (see for instance [**5**]). As shown by Hörmander [**9**], the two approaches are essentially equivalent.

Since La is already known, we hope to neglect the term LL_1a and concentrate on $\Delta_\omega a$. If we take S_θ to be smoothing in the ω variables only, we have (with another S_θ on the right)

$$R_i S_\theta v \sim \theta S_\theta v.$$

Choosing $\theta = \theta(t)$, we hope for the decay factor $1/r$ to compensate for the growth $\theta(t)$ in the term such as $1/rR_iS_\theta v$. Unfortunately, since L and L_1 have variable coefficients, commutators arise in $LL_1S_\theta a$ which display second order derivatives of a with respect to ∂_r and ∂_t also. We are thus forced to introduce S_θ as a smoothing operator *both* in the variables ω and r, say

$$S_\theta = S^{(r)}_{\theta_1} S^{(\omega)}_{\theta_2},$$

where the two parameters $\theta_1(t)$ and $\theta_2(t)$ have to be determined.

b. According to the analysis of **a.**, we use now the smoothed modified fields

$$\widetilde{R}^m_i = R_i + \widetilde{a}(R_i)L_1, \quad \widetilde{S}^m = S + \widetilde{a}(S)L_1, \quad \widetilde{H}^m_i = H_i + \widetilde{a}(H_i)L_1,$$

where $H_i = ct\partial_i + x_i/c\partial_t$ and

$$\widetilde{a}(R_i) = S_\theta a(R_i), \quad \widetilde{a}(S) = S_\theta a(S), \quad \widetilde{a}(H_i) = -\omega_i \widetilde{a}(S) - (\omega \wedge \widetilde{a}(R))_i.$$

These fields are denoted by $\widetilde{Z}_m$. Of course, we have to develop a calculus for these fields and their commutators with S_θ, etc., which is very similar to the calculus of paradifferential operators. Needless to say, this part of the paper, corresponding to sections IV.3, IV.4, IV.5, is quite tedious, and should be skipped by the reader.

c. On the one hand, we have the formula (cf. Lemma IV.5.1)

$$[\partial_t, S_\theta] = \theta_1'/\theta_1 s_\theta + \theta_2'/\theta_2 s_\theta.$$

On the other hand, we need in our estimates to have $\theta_i'/\theta_i = O(\varepsilon(1+t)^{-1})$. Hence we are forced to take

$$\theta_i = \theta_i^0 (1+t)^{\varepsilon\beta_i}.$$

It turns out that the two speeds β_i will have to be chosen different: β_1 and $\beta_2 - \beta_1$ have to be big enough. This reflects the dissymmetry between the first order derivatives of u: $\varepsilon^{-1}(1+t)u_r$ is bounded while $\varepsilon^{-1}(1+t)R_iu$ may grow like $(1+t)^{C\varepsilon}$.

I.5. Structure of $[\widetilde{Z}_m, P]u$. — This is the heart of the matter. Since the Z_0 fields have been modified so as to improve the commutation with L (see **2.b**), we expect good formulas for $[\widetilde{Z}_m, LL_1]$ also. In contrast, computing the term $[\widetilde{Z}_m, \Delta_\omega]$ and taking the smoothing operator S_θ into account is rather tedious. The result is described in Proposition VI.1. It turns out that the most delicate terms to control are the ones containing a, especially

$$r^{-2}L_1\widetilde{a}\Delta_\omega u, \quad L\widetilde{a}L_1^2u, \quad L_1L\widetilde{a}\partial u, \quad (1+t)^{-1}L\widetilde{a}\partial u. \tag{1.8}$$

These terms are handled in part **C** of the proof of Proposition VII.1. Formulas for the higher order commutators $[\widetilde{Z}_m^\alpha, P]$ are also established, and require the full calculus for the fields $\widetilde{Z}_m$.

I.6. Energy inequalities. — Writing $P\widetilde{Z}_m u = -[\widetilde{Z}_m, P]u$, $v = \widetilde{Z}_m u$ and using an energy inequality for P, we have to check that the various terms of $[\widetilde{Z}_m, P]u$ can be absorbed from right to left in the inequality. To handle the first term in (1.8), we need an inequality displaying a better control of the special derivatives $(R_i/r)v$. Such inequalities have been already discussed and used in [**2**], [**3**]: the idea is to establish an energy inequality with a "ghost weight" $e^{b(r-t)}$, where b is bounded. Here, we use ψ instead of $r-t$, and take a weight

$$\exp(\tau+1)b(\psi), \quad b(s) = B(-s)^{-\nu}, \quad \nu > 0,\ B > 0,$$

where ν and B^{-1} have to be chosen small enough (see Proposition V.3.1). This weight does not disappear, but is bounded below and above by $C(1+t)^{C\varepsilon}$, which is allowed in our context.

I.7. Poincaré Lemma. — As explained in **2.a**, we need a Poincaré Lemma to control the zero order term $(\Delta u)v$ in the linearized operator acting on v. In the context of the weighted L^2 norms explained in § 6, we obtained roughly the formula (see Proposition V.2)

$$\int_{r\geqslant t/2} e^p(\Delta u)^2 v^2 dx \leqslant C\varepsilon^2(1+t)^{-2}\int_{r\geqslant t/2} e^p v_r^2 dx, \quad p = (\tau+1)b(\psi).$$

The *miracle* here is that we only know

$$|\Delta u| \leqslant C\varepsilon(1+t)^{-1+C_1\varepsilon}\sigma_1^{\mu-2}$$

and still get the estimate we would obtain if we had $C_1 = 0$. This is due to the special structure of $L_1^2 U$ displayed in Lemmas II.3.3 and II.3.5.1, which say roughly

$$L_1^2 U \sim \psi_r h(\psi), \quad |h(s)| \leqslant C(1+|s|)^{-3/2+4\eta}.$$

To prove the inequality, we make the change of variable $s = \psi(r,\omega,t)$ in the integrals, and proceed as usual in the s variables.

I.8. Calculus for systems of modified Z_0 fields. — In the course of this paper, we use in fact several systems of modified fields, each of which giving birth to a special calculus. For instance, besides the two main systems of the Z_m of Chapter III.1 and the $\widetilde{Z}_m$ of Chapter IV.1 mentioned above, we have

i) The enlarged calculus for Z_m and the system $\overline{Z}_0$ in the proof of Proposition III.7,

ii) The new system Z_m and the system $\overline{Z}_0$ in the proof of Proposition IV.1,

iii) The system $\overline{Z}_m$ in the proof of Proposition VII.2.

We deliberately made the following choice: rather than building before the proofs of these results a tight wall of Lemmas that no reader can cross, we chose to rather write *"Scheherazade type"* of proofs, where the needed Lemmas are displayed and proved exactly when one needs them. This allows the reader to view Proposition III.7, Proposition IV.1, Proposition VII.2 as black boxes which need not be opened in a first approach, and avoids confusion between the different systems of fields.

The plan of the paper is as follows: in part II, we prove the large time existence theorem (needed to start the induction) and discuss the first consequences of the induction hypothesis, in particular the boundedness of $\varepsilon^{-1}(1+t)\partial u$ and the special structure of $L_1^2 U$. Chapter III is devoted to obtain the improved L^∞ estimates on u. In part IV, the smooth modified fields $\widetilde{Z}_m$ are defined and many lemmas display the calculus for these fields. The weighted energy norms, the energy inequality and the Poincaré Lemma are proved in Chapter V. The structure of the commutators $[\widetilde{Z}_m, P]$ and $[\widetilde{Z}_m^\alpha, P]$ are discussed in VI. Finally, using V and VI, simultaneous weighted L^2 estimates of $\widetilde{Z}_m^{k+1}\partial u$ and $\widetilde{Z}_m^k \partial a$ are obtained in VII, allowing us to finish the proof of the main result in VII.3.

II. Large time existence, induction hypothesis and first consequences

II.1. Large time existence. — We consider the Cauchy problem I.1.1. Our first result displays a very large lifespan of the solution.

Theorem 1. — *Let $\overline{\tau} > 0$ and $s_0 \in \mathbf{N}$. Then, if ε is small enough, the solution u to the Cauchy problem (1.1) exists and is C^∞ for $\tau \equiv \varepsilon \log(1+t) \leqslant \overline{\tau}$. Moreover, we have for some C the estimates*

$$|Z_0^\alpha \partial u| \leqslant C\varepsilon(1+t)^{-1}\sigma_1^{-1/2}, \quad |\alpha| \leqslant s_0. \tag{1.1}$$

Proof. — We only sketch the proof, since it is very close to the proof of Theorem 6.5.3 in [**10**], using "induction on time". There are two main differences:

i) The approximate solution u_a can be constructed without time limitation.

ii) The structure of the equation on the difference $\dot{u} = u - u_a$ is slightly different.

Let us review this more closely.

i) Construction of an approximate solution

a. Let w satisfy

$$w_{tt} - \Delta w = 0, \quad w(x,0) = u_1^0(x), \quad w_t(x,0) = u_1^1(x).$$

Then w can be written

$$w = 1/rF(\omega, 1/r, r-t),$$

where F is defined in [**10**], (6.2.5). Note that here F is supported, like w, in $-M \leqslant r-t \leqslant M$. When $t \to +\infty$,

$$w \sim 1/rF_0(\omega, r-t), \quad F_0(\omega,\sigma) = F(\omega,0,\sigma).$$

We consider now, for $\omega \in S^2$, $\tau \geqslant 0$, $\sigma \equiv r-t \leqslant M$, the Cauchy problem

$$\partial^2_{\sigma\tau} V + V\partial^2_{\sigma\sigma} V = 0, \quad V(\sigma,\omega,0) = F_0(\omega,\sigma).$$

We claim that this problem has a smooth solution for $0 \leqslant \tau \leqslant \overline{\tau}$, supported for $\sigma \leqslant M$. In fact, set

$$\sigma = \phi(s,\omega,\tau), \quad W(s,\omega,\tau) = V(\phi,\omega,\tau).$$

We have

$$W_s = \phi_s V_\sigma, W_{s\tau} = \phi_{s\tau} V_\sigma + \phi_s(V_{\sigma\tau} + \phi_\tau V_{\sigma\sigma}),$$
$$\partial_\tau(W_s/\phi_s) = (V_{\sigma\tau} + \phi_\tau V_{\sigma\sigma}).$$

We choose now ϕ defined by

$$\phi_s = \exp(\tau\partial_\sigma F_0), \quad \phi(M,\omega,\tau) = M,$$

and set $W = \phi_\tau$. Note that $\phi(s,\omega,0) = s$, and W_s is zero for $|s| \geqslant M$. Since $W_s/\phi_s = \partial_\sigma F_0(\omega,\sigma)$, we have

$$0 = \partial_\tau(W_s/\phi_s) = (V_{\sigma\tau} + VV_{\sigma\sigma})(\phi,\omega,\tau).$$

Moreover, for $\tau = 0$, $W_s = \partial_s F_0$, $W(M,\omega,0) = F_0(\omega,M) = 0$, hence

$$W(s,\omega,0) = F_0(\omega,s), \quad V(\sigma,\omega,0) = F_0(\omega,\sigma).$$

Finally, for $\sigma \leqslant \phi(-M,\omega,\tau)$, V is a smooth function of (ω,τ). In particular, $|V| \leqslant C$.

b. We introduce now two smooth real cutoff functions

$$\chi_1 = \chi_1(\varepsilon t), \quad \chi_2 = \chi_2(r/(1+t)),$$

where $\chi_1(s)$ is zero for $s \geqslant 2$ and one for $s \leqslant 1$, while $\chi_2(s)$ is zero for $s \leqslant 1/2$ and one for $s \geqslant 2/3$. We define the approximate solution by

$$u_a = \varepsilon\chi_1 w + \varepsilon/r(1-\chi_1)\chi_2 V(r-t,\omega,\tau).$$

As in [**10**], we have for all α the estimates $|Z_0^\alpha u_a| \leqslant C\varepsilon/(1+t)$. We set also $J_a = \partial_t^2 u_a - (1+u_a)^2\Delta u_a$. To prove the analogue to Lemma 6.5.5 of [**10**], we have to note that

$$\partial_t^2(\chi_2 V) = \chi_2\partial_t^2 V + 2(\partial_t\chi_2)(-V_\sigma + \varepsilon/(1+t)V_\tau) + (\partial_t^2\chi_2)V,$$
$$\partial_r^2(\chi_2 V) = \chi_2\partial_r^2 V + 2(\partial_r\chi_2)(V_\sigma) + (\partial_r^2\chi_2)V.$$

In these expressions, note that

$$\partial\chi_2 = O(1/(1+t)), \quad \partial^2\chi_2 = O(1/(1+t)^2), \quad \chi_2' V_\sigma \equiv 0.$$

For $t \geqslant 2/\varepsilon$, we obtain

$$J_a = -2\varepsilon^2/r^2(V_{\sigma\tau} + VV_{\sigma\sigma}) + O(\varepsilon/(1+t)^3).$$

Thanks to the equation on V, we finally obtain in this region, for all α,

$$|Z_0^\alpha J_a| \leqslant C\varepsilon(1+t)^{-3}.$$

In the first period $\varepsilon t \leqslant 1$ or in the transition region $1 \leqslant \varepsilon t \leqslant 2$, the discussion is the same as in [**10**], and we obtain

$$|Z_0^\alpha J_a| \leqslant C\varepsilon^2 |\log\varepsilon|(1+t)^{-2}.$$

The main difference here with [**10**] is that V is no longer zero for $\sigma \leqslant -M$. Hence the support of J_a is only contained in the region $(1+t)/2 \leqslant r \leqslant M+t$, and

$$|Z_0^\alpha J_a|_{L^2} \leqslant C\varepsilon^2|\log\varepsilon|(1+t)^{-1/2}, \quad t \leqslant 2/\varepsilon,$$
$$|Z_0^\alpha J_a|_{L^2} \leqslant C\varepsilon(1+t)^{-3/2}, \quad t \geqslant 2/\varepsilon.$$

We obtain finally

$$\int_{\tau\leqslant\overline{\tau}} |Z_0^\alpha J_a|_{L^2} dt \leqslant C\varepsilon^{3/2}|\log\varepsilon|.$$

ii) The induction argument. — We write the equation on $u = u_a + \dot{u}$ in the form

$$(cP)\dot{u} = \partial_t^2 \dot{u} - (1+u_a+\dot{u})^2\Delta\dot{u} = -J_a + (\Delta u_a)(2(1+u_a)+\dot{u})\dot{u}.$$

We make the induction hypothesis

$$|Z_0^\alpha \partial\dot{u}| \leqslant \varepsilon\sigma_1^{-1/2}/(1+t), \quad |\alpha| \leqslant s_0.$$

This means that this pointwise estimate is supposed to hold for $t \leqslant T$, for some T. We will eventually prove that T satisfies $\varepsilon\log(1+T) \geqslant \overline{\tau}$. First, since $|\partial u_a + \partial\dot{u}| \leqslant C\varepsilon(1+t)^{-1}$, we can use the standard energy inequality for the operator cP to evaluate $|\partial\dot{u}|_{L^2}$. We wish to apply Z_0^α to the left to the equation on $\dot{u}$, with $|\alpha| \leqslant 2s_0$. Since we have, for constants $C_{\alpha\beta}$,

$$[Z_0^\alpha, (\partial_t^2 - \Delta)] = \sum_{|\beta|<|\alpha|} C_{\alpha\beta} Z_0^\beta (\partial_t^2 - \Delta),$$

we write the equation in the form

$$(\partial_t^2 - \Delta)\dot{u} = ((1+u)^2 - 1)\Delta\dot{u} - J_a + (\Delta u_a)(2(1+u_a)+\dot{u})\dot{u} \equiv G.$$

Applying Z_0^α, we obtain $(\partial_t^2 - \Delta)Z_0^\alpha\dot{u} = Z_0^\alpha G - \sum C_{\alpha\beta}Z_0^\beta G$. In $Z_0^\alpha G$, we distinguish the term $((1+u)^2-1)\Delta Z_0^\alpha\dot{u}$ which we take back to the left-hand side to get $(cP)(Z_0^\alpha\dot{u})$.

a. We ignore the factor $(2(1+u_a)+\dot{u})$ accompanying $(\Delta u_a)\dot{u}$ in G. For terms

$$(Z_0^\gamma \Delta u_a)(Z_0^\delta \dot{u}), \quad |\gamma|+|\delta| \leqslant |\alpha|,$$

we use the inequality $|\sigma_1^{-1}v|_{L^2} \leqslant C|\partial v|_{L^2}$. Since $\sigma_1|Z_0^\gamma\Delta u_a| \leqslant C\varepsilon/(1+t)$, such terms are absorbed using Gronwall's inequality.

b. We ignore the factor $2(1+(u_a+\dot{u})/2)$ accompanying $(u_a+\dot{u})\Delta\dot{u}$ in G. We have to deal with terms

1) $$(u_a+\dot{u})[Z_0^\alpha, \Delta]\dot{u},$$

2) $$(Z_0^\gamma(u_a+\dot u))(Z_0^\delta \Delta \dot u), \quad |\delta|<|\alpha|, \quad |\gamma|+|\delta|\leqslant|\alpha|.$$

We use (the stars denoting irrelevant coefficients)

$$[Z_0^\alpha,\Delta]=\sum_{|\beta|\leqslant|\alpha|-1} *\partial^2 Z_0^\beta, \partial=\sigma_1^{-1}\sum *Z_0.$$

Hence

$$[Z_0^\alpha,\Delta]\dot u=\sigma_1^{-1}\sum_{|\gamma|\leqslant|\alpha|} *\partial Z_0^\gamma \dot u.$$

On the other hand,

$$\sigma_1^{-1}|u_a+\dot u|\leqslant C\varepsilon/(1+t),$$

thus the term 1) will be controlled using Gronwall's inequality.

For 2), we remark first that the part $(Z_0^\gamma u_a)(Z_0^\delta\Delta\dot u)$ is easily handled. For the other part, we distinguish which factor we are going to evaluate in L^2 norm. If $|\gamma|\leqslant s_0$, we write as before

$$|Z_0^\gamma \dot u Z_0^\delta \Delta\dot u|_{L^2}\leqslant \sum_{|\beta|\leqslant|\alpha|} C|\sigma_1^{-1}Z_0^\gamma\dot u|_{L^\infty}|\partial Z_0^\beta \dot u|_{L^2}$$

and use Gronwall's inequality. If $|\gamma|\geqslant s_0+1$, we write

$$|(\sigma_1^{-1}Z_0^\gamma\dot u)(\sigma_1 Z_0^\delta\Delta\dot u)|_{L^2}\leqslant\sum_{|\beta|\leqslant s_0} C|\partial Z_0^\beta\dot u|_{L^\infty}|\partial Z_0^\gamma\dot u|_{L^2}$$

and use once again Gronwall's inequality.

Finally, we obtain

$$|Z_0^\alpha\partial\dot u|_{L^2}\leqslant C\varepsilon^{3/2}|\log\varepsilon|, \quad |\alpha|\leqslant 2s_0.$$

Using Klainerman's inequality, we obtain for $|\alpha|\leqslant 2s_0-2$

$$|Z_0^\alpha\partial\dot u|\leqslant C\varepsilon^{3/2}|\log\varepsilon|\sigma_1^{-1/2}(1+t)^{-1}.$$

If $2s_0-2\geqslant s_0$, that is $s_0\geqslant 2$ and ε is small enough, we obtain the statement by the usual induction argument. □

II.2. The optic function. — We assume in what follows that u is defined and C^∞ for $t\leqslant T'$; u is also defined in any finite strip $-C\leqslant t\leqslant 0$ for small enough ε. We extend the integral curve $r=t+M$ of L for negative time until it reaches the t-axis. All objects and estimates related to u will implicitly be considered as defined in the corresponding region. We define the optic function $\psi=\psi(r,\omega,t)$ by

$$L\psi=0, \quad \psi(0,\omega,t)=-M-1-t.$$

Then $\psi\leqslant C<0$ in the region of interest. As in [**12**], the function ψ is a substitute for the usual phase $r-t-M-1$. The cones $\psi=$ const will be considered as deformations of the standard cones $\sigma_1=$ const, and later on, the geometry of the fields Z_0 will be adapted to these new cones.

Lemma 2. — *For $\tau\leqslant\overline{\tau}$, we have for C big enough*

i) $|\psi| \leqslant C(\sigma_1 + \tau^2)$,
ii) *for* $\sigma_1 \geqslant C\tau^2$, $C|\psi| \geqslant \sigma_1$.

Proof. — From a given point $M_0 = (r_0, \omega_0, t_0)$ we draw backward, for some big enough C, the integral curve Γ of L, along with the curves Γ_1 and Γ_2 respectively defined by

$$\sigma_1' = -C\varepsilon(1+t)^{-1}\sigma_1^{1/2}, \quad \sigma_1' = C\varepsilon(1+t)^{-1}\sigma_1^{1/2}.$$

According to the bound of u deduced from (1.1), the first curve is above, the second below Γ. The three curves meet $r = 0$ at t_1, θ, t_2, with $t_1 \geqslant -M - 1 - \psi \geqslant t_2$. By integration,

$$2\sigma_1^{1/2}(t) + C\varepsilon \log(1+t_0) = 2\sigma_1^{1/2}(t_1) + C\varepsilon \log(1+t_1),$$

hence

$$|\psi| \leqslant \sigma_1(t_1) \leqslant C(\sigma_1 + (\varepsilon \log(1+t))^2).$$

From the second differential equation, we get, if $\sigma_1 \geqslant (C\varepsilon \log(1+t))^2$, $4|\psi| \geqslant \sigma_1$. □

II.3. Induction hypothesis and its consequences. — We already know that u exists as a C^∞ function for $t \leqslant T'$, $\varepsilon \log(1+T') \geqslant \overline{\tau}$. For some $s_0 \in \mathbf{N}$ and some small $\eta > 0$ to be fixed later independently of ε (we will take in fact $s_0 \geqslant 10$ and, say, $\eta = 10^{-2}$), we assume now

$$\textbf{(IH)} \qquad |Z_0^\alpha \partial u| \leqslant \overline{C}\varepsilon(1+t)^{-1+\eta}\sigma_1^{-1/2}, \quad t \leqslant T \leqslant T', \quad |\alpha| \leqslant s_0.$$

From now on, all estimates will take place for $t \leqslant T$, and will use the induction hypothesis (IH). We will eventually prove that $T = T'$, thus getting global existence.

II.3.1. Estimates on the optic function

Lemma 3.1. — *For C big enough, we have*

i) $|\psi| \leqslant C(\sigma_1 + \varepsilon^2(1+t)^{2\eta})$.
ii) *For* $\sigma_1 \geqslant C\varepsilon^2(1+t)^{2\eta}$, $C|\psi| \geqslant \sigma_1$.
iii) *Everywhere for* $\tau \geqslant \overline{\tau}$, *we have*

$$\sigma_1 \leqslant C\varepsilon^2(1+t)^{2\eta}|\psi|, \quad |\psi| \leqslant C\varepsilon^2(1+t)^{2\eta}\sigma_1.$$

The proof is exactly the same as the proof of Lemma 2.

II.3.2. Structure of L_1U. — Since, from (IH), ∂u is much smaller than $\varepsilon(1+t)^{-1}$ as soon as $\sigma_1 \geqslant \gamma(1+t)$ (for any $\gamma > 0$), most of the estimates we need will take place in the "exterior" region R_e defined by

$$r \geqslant M + t/2, \quad \overline{\tau} \leqslant \tau.$$

The part of the boundary of R_e which is the union of $r = M + t/2$ and $\tau = \overline{\tau}$ will be denoted by γ. First of all, to establish later an energy inequality, we need to prove that $|\partial U|$ is bounded.

Lemma 3.2. — *In R_e, we have*

$$L_1 U = V_T(\psi,\omega) + \rho_1,$$

with

$$|V_T(s,\omega)| \leqslant C(1+|s|)^{-1/2+2\eta}, \quad |\rho_1| \leqslant C\varepsilon|\psi|^{1/2}(1+t)^{-1+2\eta}.$$

Proof. — First, we obtain from (IH) the estimates

$$|Z_0^\alpha \partial U| \leqslant C(1+t)^\eta \sigma_1^{-1/2}.$$

Set now

$$f = cr^{-2}\Delta_\omega U - (Lu/2c)LU,$$

for which $LL_1U = f$. Noting that $(r+t)(\partial_t + \partial_r) = \sum \omega_i h_i + S$, we get

$$|Lu| \leqslant C|(\partial_t+\partial_r)u| + C|u||\partial u| \leqslant \varepsilon(1+t)^{-2+2\eta}\sigma_1^{1/2},$$

$$|LU| \leqslant C|(\partial_t+\partial_r)U| + C|u||\partial U| \leqslant (1+t)^{-1+2\eta}\sigma_1^{1/2},$$

hence

$$|f| \leqslant (1+t)^{-2+\eta}\sigma_1^{1/2} \leqslant C\varepsilon(1+t)^{-2+2\eta}|\psi|^{1/2}.$$

If we draw from a point M in R_e the integral curve Γ of L, meeting γ at M', we denote by Γ_- and Γ_+ respectively the backward and forward parts of Γ in R_e. We set then

$$V_T(\psi,\omega) = (L_1U)(M') + \int_\Gamma f, \rho_1 = -\int_{\Gamma_+} f.$$

Here, the integrals are taken along Γ. From the estimates on f, we get (uniformly in T)

$$|\rho_1| \leqslant C\varepsilon|\psi|^{1/2}\int_t^{+\infty}(1+s)^{-2+2\eta}ds \leqslant C\varepsilon|\psi|^{1/2}(1+t)^{-1+2\eta}.$$

To estimate V_T, we compare both sides on γ, using Lemma 2. □

In the rest of the paper, to simplify notations, we drop the dependence of V on (T,ω).

II.3.3. The quantities a_1, b_1. — Let us define and fix in the sequence a cutoff function $\overline{\chi}$ by

$$\overline{\chi} = \overline{\chi}(r/(C+t)),$$

where $0 \leqslant \overline{\chi} \leqslant 1$ is smooth, zero for $s \leqslant 1/2$ and one for $s \geqslant 2/3$ and $C = 2(M+1)$. We define now a_1, b_1 by

$$Lb_1 = -\overline{\chi}L_1u/2c, \quad b_1(0,t) = 0, \quad a_1 = \exp b_1.$$

The following Lemma indicates the precise structure of b_1.

Lemma 3.3. — *We have*

i) $b_1 = -(\tau/2)V(\psi) + \rho_2, |\rho_2| \leqslant C,$
ii) $a_1\psi_r = 1 + \rho_3$, $|\rho_3| \leqslant C\varepsilon$.

Proof

a. By definition, using Lemma 3.2,

$$g = L(b_1 + (\tau/2)V(\psi)) = -(\varepsilon/2rc)\rho_1 - (\varepsilon\overline{\chi}/2c^{1/2}r^2)U + (\varepsilon(1-\overline{\chi})/2rc)L_1U \\ + \frac{\varepsilon}{1+t}V(c^{-1/2}/2 - 1/2c) + \varepsilon(V/2c)((1+t)^{-1} - r^{-1}),$$

hence

$$|g| \leqslant C\varepsilon(1+t)^{-2+\eta}\sigma_1^{1/2} + C\varepsilon^2(1+t)^{-2+2\eta}|\psi|^{1/2} \\ + C\varepsilon^2(1+t)^{-2+\eta}\sigma_1^{1/2}|\psi|^{-1/2+2\eta} + C\varepsilon\sigma_1(1+t)^{-2}|\psi|^{-1/2+2\eta} \\ \leqslant C\varepsilon^2(1+t)^{-2+2\eta}|\psi|^{1/2+2\eta}.$$

Thus,

$$b_1 + \tau/2V(\psi) = \rho_2^1(\psi) + \rho_2^2,$$

with

$$|\rho_2^2| \leqslant C\varepsilon^2(1+t)^{-1+2\eta}|\psi|^{1/2+2\eta}.$$

Since $b_1 + \tau/2V$ is bounded on γ, ρ_2^1 is bounded, which proves i).

b. We have $L\psi_r + c^{-1/2}u_r\psi_r = 0$. Hence

$$L\log(a_1\psi_r) = -(L_1u/2c + c^{-1/2}u_r) + (1-\overline{\chi})L_1u/2c.$$

Now $L_1u + 2c^{1/2}u_r = Lu$,

$$|L_1u + 2c^{1/2}u_r| \leqslant C\varepsilon(1+t)^{-2+2\eta}\sigma_1^{1/2}, \quad |L\log(a_1\psi_r)| \leqslant C\varepsilon(1+t)^{-2+2\eta}\sigma_1^{1/2}.$$

Since $a_1\psi_r(0,t) = 1/c = 1 + O(\varepsilon)$, we obtain ii). □

II.3.4. Improved estimates on the optic function

Lemma 3.4. — *For C big enough,we have the estimates*

i) $|\psi| \leqslant C\sigma_1 + C(1+t)^{C\varepsilon}$,

ii) *If $\sigma_1 \geqslant C(1+t)^{C\varepsilon}$, then $C|\psi| \geqslant \sigma_1$. In all cases, we have*

$$\sigma_1 \leqslant C|\psi|(1+t)^{C\varepsilon}, \quad |\psi| \leqslant C\sigma_1(1+t)^{C\varepsilon}.$$

Proof

a. From Lemma 3.2, we obtain $|L_1U| \leqslant C|\psi|^{-1/2+2\eta}$, since $|\psi| \leqslant C(1+t)$. Hence, using (IH),

$$|\partial_rU| \leqslant C(1+t)^{-1+\eta}\sigma_1^{1/2} + C|\psi|^{-1/2+2\eta} \leqslant |\psi|^{-1/2+2\eta}.$$

Using the estimates $a_1 \leqslant C(1+t)^{C\varepsilon}$ and $a_1\psi_r \geqslant 1/2$ from Lemma 3.3, we have

$$|\partial_rU| \leqslant C(1+t)^{C\varepsilon}|\psi|^{-1/2+2\eta}\psi_r,$$

and by integration

$$|U| \leqslant C_0(1+t)^{C_0\varepsilon}|\psi|^{1/2+2\eta}.$$

b. Just as in Lemma 2, let us consider the integral curve Γ of L through a point M_0, and denote by Γ_1 and Γ_2 respectively the curves

$$\sigma_1' = - \pm C\varepsilon/(1+t)(1+t)^{C\varepsilon} A_0^{1/2+2\eta},$$

where $A_0 = |\psi(M_0)|$ and C is big enough. Let us call respectively σ_1^1, $\overline{\sigma}_1$, σ_1^2 the values of σ_1 at the points where Γ_1, Γ, Γ_2 intersect γ. Since σ_1 is decreasing along γ, and Γ is above Γ_2 and below Γ_1, we have

$$\sigma_1^2 \leqslant \overline{\sigma}_1 \leqslant \sigma_1^1.$$

Integrating the equation for Γ_1, we thus get

$$A_0 \leqslant C\overline{\sigma}_1 \leqslant C\sigma_1^1 \leqslant C\sigma_1(M_0) + CA_0^{1/2+2\eta}(1+t_0)^{C\varepsilon},$$

which gives i).

c. Using Γ_2, we get

$$\sigma_1(M_0) \leqslant \sigma_1^2 + C(1+t_0)^{C\varepsilon} A_0^{1/2+2\eta} \leqslant \sigma_1^2 + C(1+t_0)^{C\varepsilon}(\sigma_1(M_0))^{1/2+2\eta},$$

hence $\sigma_1(M_0) \leqslant \sigma_1^2 + C(1+t_0)^{C\varepsilon}$. If

$$\sigma_1(M_0) \geqslant 2C(1+t_0)^{C\varepsilon},$$

we obtain $\sigma_1(M_0) \leqslant 2\sigma_1^2$. Since $|\psi| \geqslant 1/2\sigma_1$ on γ, we have finally

$$A_0 \geqslant (1/2)\overline{\sigma}_1 \geqslant (1/2)\sigma_1^2 \geqslant (1/4)\sigma_1(M_0),$$

which is ii). □

We conclude from this Lemma that $|\psi|$ is not quite equivalent to σ_1: there exists a *blind zone*

$$\sigma_1 \leqslant C(1+t)^{C\varepsilon}$$

in which we cannot ensure that $|\psi|$ is big even is σ_1 is. This is due to a possible drift of the integral curves of L toward the cone $r = t + M$. Inside this blind zone, we can only prove $|L_1 u| \leqslant C\varepsilon(1+t)^{-1}$, while $|L_1 u| \leqslant C\varepsilon(1+t)^{-1}\sigma_1^{-1/2+0}$ outside.

II.3.5. Structure of $L_1^2 U$. — To prove later the Poincaré Lemma, we need to elucidate the special structure of $L_1^2 U$.

Lemma 3.5.1. — *In R_e, we have $a_1 L_1^2 U = h_T(\psi,\omega) + \rho_4$, with*

$$|h_T(s,\omega)| \leqslant C(1+|s|)^{-3/2+4\eta}, \quad |\rho_4| \leqslant C(1+t)^{-3/2+4\eta}.$$

Proof. — We have first

$$[L, a_1 L_1] = -(a_1/2c)LuL + (1-\chi)a_1(L_1 u/2c)L_1,$$

hence

$$g = L(a_1 L_1^2 U) = -(a_1/2c)LuLL_1 U + a_1 L_1(LL_1 U) + (1-\chi)a_1(L_1 u/2c)L_1^2 U.$$

But

$$L_1(LL_1U) = -L_1u/r^2\Delta_\omega U - 2c^{3/2}/r^3\Delta_\omega U - c/r^2(L_1\Delta_\omega U) + L_1Lu/2cLU$$
$$- LuL_1u/2c^2LU + (Lu)^2/4c^2LU - LuL_1u/4c^2L_1U + Lu/2cLL_1U,$$

thus $|g| \leqslant C(1+t)^{-5/2+4\eta}$. By integrating L, we get the structure of $a_1L_1^2U$ with the estimate on ρ_4; comparing then both sides on γ yields the estimate on h. □

Finally, we have to evaluate the smallness of ψ_{rr}.

Lemma 3.5.2. — *We have, for $r \geqslant M + t/2$, the estimate*

$$|\psi_{rr}|/\psi_r^2 \leqslant C\tau(1+|\psi|)^{-3/2+4\eta} + C\varepsilon(1+|\psi|)^{-3/2+4\eta}.$$

Proof. — First $\psi_{tt} = c^2\psi_{rr} + (cu_r - u_t)\psi_r, (\partial_t + c\partial_r)\psi_t + u_t\psi_r = 0,$

$$(\partial_t + c\partial_r)\psi_{tt} = -u_{tt}\psi_r - 2u_t\psi_{rt}.$$

Hence

$$(\partial_t + c\partial_r)(\psi_{tt}/\psi_t^2) = u_{tt}/(c\psi_t) - 2u_t^2/(c^2\psi_t).$$

For $r \leqslant M + t/2$, the right hand side is less than $C\varepsilon(1+t)^{-5/2+2\eta}$; since $\psi_{tt}/\psi_t^2(0,t) = 0$, we obtain by integration

$$|\psi_{tt}/\psi_t^2| \leqslant C\varepsilon|\psi|^{-3/2+2\eta}.$$

Now, for $r \geqslant M + t/2$,

$$u_{tt} = c\varepsilon/(4r)L_1^2U + O(\varepsilon|\psi|^{-1/2})(1+t)^{-2+4\eta},$$

hence

$$|u_{tt}/(c\psi_t) - 2u_t^2/(c^2\psi_t^2)| \leqslant C\varepsilon(1+t)^{-1}|h(\psi)| + O(\varepsilon|\psi|^{-1/2}(1+t)^{-2+4\eta}),$$

which gives by integration from $r = M + t/2$ the desired estimate. □

III. Improved L^∞ estimates on u

In this chapter, we will prove that the L^∞ estimates (IH) on u imply in fact the much better estimates of Proposition 7.

III.1. Modified vector fields. — In order to control u and its derivatives in the spirit of Klainerman [**11**], we will need modified vectors fields Z_m ("m" for modified), which are perturbations of the standard vectors fields Z_0 defined in (II.1.2). First, we set

$$H_0 = c(u)t\partial_r + \frac{r}{c(u)}\partial_t, \quad H_i = c(u)t\partial_i + \frac{x_i}{c(u)}\partial_t, \quad 1 \leqslant i \leqslant 3,$$

thus defining hyperbolic rotations adapted to the operator P. Note that

$$H_0 = \sum \omega_i H_i, \quad H_i = \omega_i H_0 + ct(\partial_i - \omega_i\partial_r).$$

For each of the fields R_i, S, H_0 we define now $a(R_i), a(S), a(H_0)$ by

$$(1.1)_a \qquad La(R_i) + \overline{\chi}a(R_i)(L_1u/(2c)) = -\overline{\chi}R_iu/(2c),$$

$$(1.1)_b \qquad La(S) + \overline{\chi}a(S)(L_1u/(2c)) = -\overline{\chi}Su/(2c), \quad a(H_0) = -a(S),$$

$$(1.1)_c \qquad a(R_i)(0,t) = 0, \quad a(R_i)(x,0) = 0, \quad a(S)(0,t) = 0, \quad a(S)(x,0) = 0.$$

Remember that $\overline{\chi}$ is a standard cutoff defined in II.3.3. Thus the coefficients a are smooth functions (as long as u exists), vanishing for $r \geqslant t + M$ or $r \leqslant t/2 + M + 1$. The set of the coefficients

$$a(R_i), a(S), a_1$$

will be denoted by (Coeff$'$). We then define the *modified fields* R_i^m, S^m, H_0^m and K by

$$(1.2) \quad R_i^m = R_i + a(R_i)L_1, \quad S^m = S + a(S)L_1, \quad H_0^m = H_0 + a(H_0)L_1, \quad K = a_1L_1.$$

We will write these equalities simply as $Z_m = Z + aL_1$, where Z will be one of the adapted vector fields R_i, S, H_0 or 0, and a will stand for the corresponding coefficient as in (1.2). Remark that

$$(r + ct)L = \sqrt{c}(H_0 + S) = \sqrt{c}(H_0^m + S^m).$$

We finally define the family Φ' as the collection of the fields $Z_m = R_i^m, S^m, H_0^m, K$. As usual, Z_m^k will simply denote a product of k fields taken among Φ'. It is always understood here that some of the fields in Φ' are singular at $r = 0$, and they will be considered only for $r \geqslant \gamma_0(1 + t)$ $(\gamma_0 > 0)$.

In what follows, we will simply write f to denote a real C^∞ function of the (finitely many) variables

$$\varepsilon, u, \omega, \sigma_1(1 + t)^{-1}, (1 + t)^{-\nu_i}, \sigma_1^{-\nu_i}, \nu_i > 0.$$

Remark that $\overline{\chi} = f$. Finally, we denote by N_k one of the quantities

$$\varepsilon^{-1}(1 + t)\sigma_1^{-1}Z_m^k u, \quad \varepsilon^{-1}(1 + t)Z_m^k Lu, \quad \varepsilon^{-1}(1 + t)Z_m^k L_1u,$$

$$\sigma_1^{-1}Z_m^{k-1}a, \quad Z_m^{k-1}La, \quad Z_m^{k-1}L_1a, \quad a \in (\text{Coeff}').$$

We add the convention that 1 is also a N_0. We need now develop a calculus for these modified fields. To simplify the notation, we dispense in general with writing sums of terms of the same kind. For instance, we will write N_k for a sum of various N_k, Z_m for a sum of Z_m, etc.

III.2. Some calculus Lemma

Lemma 2. — *We have the following identities:*

i) $Z_m^k f = \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,
ii) $Z_m^k N_p = \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k + p$, $k_i \geqslant p$ *for some* i,
iii) $Z_m^k t = t \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,
iv) $Z_m^k \sigma_1 = \sigma_1 \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots k_j \leqslant k$.

Proof. — In view of the structure of the formulas, it is enough to prove them for $k = 1$ and any p.

We have

$$R_i\omega = f, \quad R_i\sigma_1 = 0, \quad R_i t = 0,$$
$$S\omega = 0,\ S\sigma_1 = -M - 1 + \sigma_1,\ St = t, S(\sigma_1/1+t) = f,\ S(1+t)^{-\nu} = f,\ S\sigma_1^{-\nu} = f,$$
$$H_0\omega = 0, \quad H_0\sigma_1 = f\sigma_1 N_0, \quad H_0 t = ft, \quad H_0(\sigma_1/1+t) = fN_0.$$

On the other hand,

$$L_1\omega = 0, \quad L_1\sigma_1 = f, \quad L_1 t = f, \quad L_1(\sigma_1/1+t) = f/1+t,$$
$$L_1(1+t)^{-\nu} = f/1+t, \quad L_1(\sigma_1^{-\nu}) = f/\sigma_1.$$

Hence

$$Z_m u = fN_0 + fN_0N_1, Z_m(\sigma_1/1+t) = fN_0 + fN_1,$$
$$Z_m(1+t)^{-\nu} = fN_0 + fN_1, Z_m\sigma_1^{-\nu} = fN_0 + fN_1,$$

and $Z_m f = f + fN_0 + fN_1$. Thus, i), iii) and iv) are proved. Now

$$L_1((1+t)/\sigma_1) = \sigma_1^{-1} f((1+t)/\sigma_1), \quad Z((1+t)/\sigma_1) = ((1+t)/\sigma_1)fN_0,$$

hence

$$Z_m((1+t)/\sigma_1) = ((1+t)/\sigma_1)(fN_0 + fN_1).$$

Thus, with $A = L$ or $A = L_1$,

$$Z_m[\varepsilon^{-1}((1+t)/\sigma_1)Z_m^p u] = (fN_0 + fN_1)N_p + N_{p+1},$$
$$Z_m[\varepsilon^{-1}(1+t)Z_m^p Au] = (fN_0 + fN_1)N_p + N_{p+1},$$
$$Z_m[\sigma_1^{-1}Z_m^p a] = (fN_0 + fN_1)N_{p+1} + N_{p+2},$$

which proves ii). □

III.3. Commutation Lemmas. — For fields X_i, Y, we will note

$$(adX)Y = [X, Y], \quad (adX^k)Y = [X_1, [X_2, \dots Y] \dots].$$

The following Lemmas justify the introduction of the modified fields Z_m: they just commute better with L than the standard fields Z_0.

Lemma 3.1. — *We have*

i)
$$[Z_m, L] = f dL_1 + f N_0 N_1 L,$$

ii)
$$[Z_m, L_1] = f N_0 N_1 L + f N_0 N_1 L_1,$$

iii)
$$(adZ_m^k)L = \sum f N_{k_1} \cdots N_{k_j} L + \sum f (Z_m^q d) N_{l_1} \cdots N_{l_i} L_1, \quad k_1 + \cdots + k_j \leqslant k, \quad q + l_1 + \cdots + l_i \leqslant k-1,$$

iv)
$$(adZ_m^k)L_1 = \sum f N_{k_1} \cdots N_{k_j} L + \sum f N_{l_1} \cdots N_{l_i} L_1, \quad k_1 + \cdots + k_j \leqslant k, \quad l_1 + \cdots + l_i \leqslant k.$$

Here, d denotes one of the quantities $d = (1-\overline{\chi}) Z_m u = \varepsilon f N_1$.

Proof. — Since $d = \varepsilon f N_1$, thanks to Lemma 2, it is enough to prove the formulas for $k = 1$. We have

$$[R_i, L] = -R_i u/(2c) L_1, \qquad [R_i, L_1] = -R_i u/(2c) L,$$
$$[S, L] = -L - Su/(2c) L_1, \quad [S, L_1] = -L_1 - Su/(2c) L,$$
$$[H_0, L] = \Big(-1 + \frac{r-ct}{2c\sqrt{c}} Lu\Big) L + \Big(\frac{r+ct}{2c\sqrt{c}} Lu - H_0 u/(2c)\Big) L_1,$$
$$[H_0, L_1] = \Big(\frac{r-ct}{2c\sqrt{c}} L_1 u - H_0 u/(2c)\Big) L + \Big(1 + \frac{r+ct}{2c\sqrt{c}} L_1 u\Big) L_1.$$

Remark here that

$$(r-ct) L_1 = \sqrt{c}(H_0 - S), \quad (r+ct) L = \sqrt{c}(H_0 + S),$$

hence the above formulas simplify to

$$[H_0, L] = \Big(-1 + \frac{r-ct}{2c\sqrt{c}} Lu\Big) L + Su/(2c) L_1,$$
$$[H_0, L_1] = -Su/(2c) L + \Big(1 + \frac{r+ct}{2c\sqrt{c}} L_1 u\Big) L_1.$$

Since

$$[aL_1, L] = -(La) L_1 - a L_1 u/(2c) L_1 + a Lu/(2c) L, \quad [aL_1, L_1] = -(L_1 a) L_1,$$

we obtain, thanks to the choices of the a for each Z,

$$[R_i^m, L] = -(1-\overline{\chi}) R_i^m u/(2c) L_1 + a Lu/(2c) L,$$
$$[S^m, L] = -(1-\overline{\chi}) S^m u/(2c) L_1 + (a Lu/(2c) - 1) L,$$
$$[H_0^m, L] = (1-\overline{\chi}) S^m u/(2c) L_1 + \Big(\frac{r-ct}{2c\sqrt{c}} Lu + a Lu/(2c) - 1\Big) L,$$
$$[K, L] = a Lu/(2c) L - (1-\overline{\chi}) K u/(2c) L_1.$$

Similarly,

$$[R_i^m, L_1] = -1/(2c)(R_i^m u - aL_1 u)L - (L_1 a)L_1,$$
$$[S^m, L_1] = -1/(2c)(S^m u - aL_1 u)L - (1 + L_1 a)L_1,$$
$$[H_0^m, L_1] = -1/(2c)(S^m u - aL_1 u)L + \Big(1 + \frac{r+ct}{2c\sqrt{c}} L_1 u - L_1 a\Big)L_1.$$

If we remark that

$$aLu = \sigma_1^{-1} a\varepsilon\sigma_1/(1+t)\varepsilon^{-1}(1+t)Lu = fN_0N_1,$$
$$(r-ct)Lu = r-ct/(r+ct)\sqrt{c}(H_0^m u + S^m u) = fN_1,$$

we can write

$$[Z_m, L] = f(1-\overline{\chi})(Z_m u)L_1 + fN_0N_1L = fdL_1 + fN_0N_1L,$$
$$[Z_m, L_1] = fN_0N_1L + fN_0N_1L_1.$$

□

Lemma 3.2. — *We have*

i) $$[Z_m^k, L] = \sum fN_{k_1}\cdots N_{k_j} Z_m^p L + \sum fZ_m^q dN_{l_1}\cdots N_{l_i} Z_m^r L_1,$$
$$p \leqslant k-1, \quad p + k_1 + \cdots + k_j \leqslant k, \quad r \leqslant k-1, \quad q + r + l_1 + \cdots + l_i \leqslant k-1.$$

ii) $$[Z_m^k, L_1] = \sum fN_{k_1}\cdots N_{k_j} Z_m^p L + \sum fN_{l_1}\cdots N_{l_i} Z_m^r L_1,$$
$$p \leqslant k-1, \quad p + k_1 + \cdots + k_j \leqslant k, \quad r \leqslant k-1, \quad r + l_1 + \cdots + l_i \leqslant k.$$

iii) $$[Z_m^k, L_1] = \sum_{\substack{p\leqslant k-1\\ p+\sum k_i\leqslant k}} fN_{k_1}\cdots N_{k_j} Z_m^p L + \sum_{(\sum l_j \leqslant k-1)} fN_{l_1}\cdots N_{l_i} Z_m^{l_{i+1}} L_1 a Z_m^{l_{i+2}} L_1$$
$$+ \sum_{(p+\sum k_i \leqslant k-1)} fN_{k_1}\cdots N_{k_j} Z_m^p L_1 + \sum_{(\sum l_j + q + r \leqslant k-1)} fN_{l_1}\cdots N_{l_i} Z_m^q d Z_m^r L_1.$$

Proof. — For $k=1$, the formulas i) and ii) follow from Lemma 3.1. For iii) we write

$$[Z_m, L_1] = fN_0N_1L + (fN_0 - L_1 a)L_1.$$

Since

$$[Z_m^{k+1}, A] = Z_m^k[Z_m, A] + [Z_m^k, A]Z_m,$$

we obtain easily the Lemma by induction, using Lemma 2. □

For technical reasons, we will need the following variant of Lemma 3.2.

Lemma 3.3. — *If* $Lw = g$, *we have*

$$LZ_m^k w = \sum fN_{l_1}\cdots N_{l_i} Z_m^{l_{i+1}} g + \sum fZ_m^{q_1} d \cdots Z_m^{q_i} dN_{k_1}\cdots N_{k_j} L_1 Z_m^{k_{j+1}} w$$
$$+ \sum (1+t)^{-1} fZ_m^{q_1} d \cdots Z_m^{q_i} dN_{k_1}\cdots N_{k_j} Z_m^{k_{j+1}} w$$
$$= \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3$$

In $\sum_1$, $\sum l_j \leqslant k$. *In* $\sum_2$, $i \geqslant 1$, $i + \sum q_j + \sum k_i \leqslant k, k_{j+1} \leqslant k-1$. *In* $\sum_3$,

$$i \geqslant 1, \quad i + \sum q_j + \sum k_i \leqslant k+1, \quad 1 \leqslant k_{j+1} \leqslant k-1.$$

Proof. — For $k = 1$,

$$LZ_m w = Z_m g - [Z_m, L]w = Z_m g + fN_0N_1 g + fN_0 dL_1 w.$$

Hence the formula is correct, with $\sum_3 = 0$. Now

$$Z_m L Z_m^k w = fN_0N_1 LZ_m^k w + fN_0 dL_1 Z_m^k w + LZ_m^{k+1} w,$$

$$LZ_m^{k+1} w = Z_m \sum\nolimits_1 + Z_m \sum\nolimits_2 + Z_m \sum\nolimits_3 + fN_0N_1(\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3) + fN_0 dL_1 Z_m^k w.$$

The last term belongs to $\sum_2$ for $k+1$. The terms involving $\sum_1$ again belong to $\sum_1$ for $k+1$. The terms involving $\sum_3$ again belong to $\sum_3$ for $k+1$, and $fN_0N_1 \sum_2$ belongs to $\sum_2$ for $k+1$. In $Z_m \sum_2$, the only nontrivial term is the one containing

$$Z_m L_1 Z_m^{k_{j+1}} w = fN_0N_1 LZ_m^{k_{j+1}} w + fN_0N_1 L_1 Z_m^{k_{j+1}} w + L_1 Z_m^{k_{j+1}+1} w.$$

The last two terms give terms belonging to $\sum_2$ for $k+1$. For the first, we write

$$LZ_m^{k_{j+1}} w = f(1+t)^{-1} \sum Z_m^{k_{j+1}+1} w,$$

and the corresponding terms belong to $\sum_3$ for $k+1$. □

III.4. A computation of Z_m^k

Lemma 4. — *We have,* Z_0 *denoting the standard fields defined in I.1.2*

$$Z_m^k = \sum fN_0^{l_0} N_1^{l_1} N_{k_1} \cdots N_{k_j} Z_0^p, \quad 1 \leqslant p \leqslant k, k_i \geqslant 2, \quad p + \sum (k_i - 1) \leqslant k.$$

Proof. — We use the formula $\partial_i = \sigma_1^{-1} \sum f Z_0$. We get by inspection $Z_m = \sum fN_0N_1 Z_0$, which implies the Lemma for $k = 1$, and the Lemma follows in general by induction, using Lemma 2. □

III.5. Estimates of the N_k

Proposition 5. — *We have, for* $k \leqslant s_0 - 3$

$$|N_k|_{L^\infty} \leqslant C(1+t)^{C\varepsilon}.$$

Proof

a. We have $La = -\overline{\chi}/(2c) Z_m u \equiv F_0 = \varepsilon f N_1$. Hence

$$L(\sigma_1^{-1} a) = -\overline{\chi}/(2c)\sigma_1^{-1} Z_m u + f u \sigma_1^{-2} a = \varepsilon(1+t)^{-1} fN_0N_1 \equiv F_1,$$

$$LL_1 a = [L, L_1]a + L_1 La = L_1 u/(2c) L_1 a - Lu/(2c) F_0 + L_1 F_0 \equiv F_2.$$

Also $LL_1 U = c/r^2 \Delta_\omega U - Lu/(2c) LU \equiv G$.

b. From Lemma 2, we get

$$Z_m^l F_1 = \varepsilon(1+t)^{-1} \sum{}' fN_{k_1} \cdots N_{k_j}, \sum k_i \leqslant l+1,$$

where here and later $\sum'$ means that not all N_{k_i} are one. We now evaluate F_2:

$$F_2 = f\varepsilon(1+t)^{-1} N_0 N_1 + L_1 F_0,$$

$$\begin{aligned} L_1F_0 &= fL_1uZ_mu + f(1+t)^{-1}Z_mu + fN_0N_1Lu + fN_0N_1L_1u + fZ_mL_1u \\ &= f\varepsilon(1+t)^{-1}N_0^2N_1. \end{aligned}$$

We thus obtain

$$Z_m^kF_2 = \varepsilon(1+t)^{-1}\sum{}' fN_{k_1}\cdots N_{k_j}, \quad k_1+\cdots+k_j \leqslant k+1.$$

c. We have in fact

$$Z_mr = fr + fa, \varepsilon Z_mU = fru + fau + rZ_mu,$$

hence

$$\begin{aligned} \varepsilon Z_m^{k+1}U = rZ_m^{k+1}u + r\sum_{\substack{1\leqslant p\leqslant k\\ k_1+\cdots k_j+p\leqslant k+1}} fN_{k_1}\cdots N_{k_j}Z_m^pu + r\sum_{p+k_1+\cdots+k_j\leqslant k} fN_{k_1}\cdots N_{k_j}Z_m^pu \\ + \sum fN_{l_i}\cdots N_{l_i}Z_m^qaZ_m^pu, \\ p+q+l_i+\cdots+l_i\leqslant k. \end{aligned}$$

Thus

$$\begin{gathered} \varepsilon^{-1}\sigma_1^{-1}(1+t)Z_m^{k+1}u = f\sigma_1^{-1}Z_m^{k+1}U + fN_0\sigma_1^{-1}Z_m^ka + \sum{}' fN_{k_1}\cdots N_{k_j}, \\ k_1+\cdots+k_j\leqslant k+1, \quad k_i\leqslant k. \end{gathered}$$

Similarly, we obtain, with $A = L$ or $A = L_1$

$$\varepsilon AU = \pm c^{1/2}u + rAu, \quad \varepsilon Z_mAU = fN_0Z_mu + frAu + faAu + rZ_mAu.$$

The last three terms are handled as before. For the first term, we write

$$\begin{gathered} Z_m^k(fN_0Z_mu) = fN_0Z_m^{k+1}u + \sum fN_{k_1}\cdots N_{k_j}Z_m^pu, \\ p+k_1+\cdots+k_j\leqslant k+1, \quad 1\leqslant p\leqslant k. \end{gathered}$$

Thus

$$\begin{aligned} \varepsilon^{-1}(1+t)Z_m^{k+1}Au = fZ_m^{k+1}AU + fN_0\varepsilon^{-1}Z_m^{k+1}u + fN_0\sigma_1^{-1}Z_m^ka \\ + \sum{}' fN_{k_1}\cdots N_{k_j}, \\ k_1+\cdots+k_j\leqslant k+1, \quad k_i\leqslant k. \end{aligned} \tag{5.1}$$

d. Using Lemma 3.3 for $w = \sigma_1^{-1}a$ and $g = F_1$ or $w = L_1a$ et $g = F_2$, we obtain

$$LZ_m^k(\sigma_1^{-1}a) = F_1^k, \quad LZ_m^kL_1a = F_2^k, \quad F_i^0 = F_i..$$

To estimate the right hand sides, we need the following Lemma.

Lemma 5.1. — *In any region $r\leqslant\gamma(1+t), \gamma<1$, we have*

$$(1+t)|L_1w| \leqslant C\sum|Z_mw|.$$

Proof. — We have the identity

$$\frac{r-ct}{\sqrt{c}}L_1 = H_0^m - S^m + 2aL_1.$$

On the other hand, a rough estimate of a shows that

$$\sigma_1^{-1}|a| \leqslant C\varepsilon\sigma_1^{-1/2}(1+t)^{C\varepsilon+\eta}.$$

Hence, in the region we consider, $\sigma_1^{-1}|a|$ is as small as we want, and the Lemma follows. □

We have, with the notations of Lemma 3.3, applied for the index k with $w = a\sigma_1$, $g = F_1$,

$$F_1^k = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

From the structure of F_1, we get

$$\textstyle\sum_1 = \varepsilon(1+t)^{-1}\sum fN_{l_1}\cdots N_{l_i}, \quad \sum l_j \leqslant k+1.$$

Using Lemma 3.3 and the structure of $d = \varepsilon fN_1$, we have

$$|\textstyle\sum_2| \leqslant C|d|(1+t)^{-1}|N_{k+1}| + C\varepsilon(1+t)^{-1}\sum |N_{k_1}|\cdots|N_{k_{j+1}}|, \quad k_i \leqslant k.$$

Note that $|d| \leqslant C\varepsilon(1+t)^{-\eta}$. We have a similar estimate for $\sum_3$. The computations are completely similar for F_2^k.

e. We have now to control the values of $Z_m^k(\sigma_1^{-1}a)$, $Z_m^kL_1a$, $Z_m^{k+1}L_1U$ on the boundary $r = M + t/2$.

Lemma 5.2. — *On the boundary $r = t/2 + M$, for $k \leqslant s_0 - 1$,*

i) *$Z_m^kL_1a = 0$, $|Z_m^k(\sigma_1^{-1}a)| \leqslant C$,*

ii) *$|Z_m^{k+1}L_1U| \leqslant C$.*

Proof. — Close to this boundary, a is either identically one or zero: the value of $Z_m^kL_1a$ is zero. For the U term, we remark that we can replace $Z_m \neq K$ by the corresponding Z, K by L_1. For such fields Z (including L_1), we have

$$Z = Z_0 + ftu/\sigma_1 Z_0 = fN_0Z_0.$$

Denoting only here by N_k the terms

$$\varepsilon^{-1}\sigma_1^{-1}(1+t)Z^ku, \quad \varepsilon^{-1}(1+t)Z^kLu, \quad \varepsilon^{-1}(1+t)Z^kL_1u,$$

we get as before

$$Z^k = \textstyle\sum fN_{l_1}\cdots N_{l_i}Z_0^{l_{i+1}}, \quad \sum l_j \leqslant k, \quad l_{i+1} \geqslant 1.$$

Hence, with $A = 1, L, L_1$,

$$Z^kAu = \textstyle\sum fN_{l_1}\cdots N_{l_i}Z_0^{l_{i+1}}Au.$$

By induction, starting from $|N_0| \leqslant C$ by the induction hypothesis, we get $|N_k| \leqslant C$ for $k \leqslant s_0$. Finally

$$Z^{k+1}L_1U = \textstyle\sum fN_{l_1}\cdots N_{l_i}Z_0^{l_{i+1}}L_1U,$$

hence the conclusion. The proof is similar for $Z_m^k \sigma_1^{-1}$. □

f. We now set
$$\phi_l = \sum |N_l|_{L^\infty},$$
and assume by induction $\phi_l \leqslant C(1+t)^{C\varepsilon}, l \leqslant k$ (we have already shown and used that $|N_0| \leqslant C$). Because of the structure of Lemma 4, we need first control ϕ_1 without using Lemma 4. Let
$$G = c/r^2 \Delta_\omega U - Lu/(2c)LU,$$
$$G_1 = LZ_m L_1 U = Z_m G + f d L_1^2 U + f N_0 N_1 G = f N_0 N_1 G + f N_0 N_1 Z_0 G + f d L_1^2 U.$$
It is clear that
$$|Z_0^l(cr^{-2}\Delta_\omega U)| \leqslant C(1+t)^{-2+\eta}\sigma_1^{1/2}.$$
On the other hand,
$$L = c^{-1/2}(\partial_t + \partial_r) + (c^{1/2} - c^{-1/2})\partial_r = f(1+t)^{-1}\sum Z_0 + fu\partial,$$
and as usual
$$fu\partial = fu\sigma_1^{-1}\sum Z_0 = \varepsilon(1+t)^{-1} f N_0 \sum Z_0.$$
Finally $L = (1+t)^{-1}\sum f N_0 Z_0$. Hence
$$Lu/(2c)LU = (1+t)^{-2} f N_0^2 Z_0 u Z_0 U,$$
$$|Z_0^l(Lu/(2c)LU)| \leqslant C(1+t)^{-2+2\eta}.$$
Adding, we get
$$|Z_0^l G| \leqslant C(1+t)^{-3/2+\eta}.$$
We also have
$$|dL_1^2 U| \leqslant C(1+t)^{-2+2\eta}(1+\phi_1),$$
hence
$$|G_1| \leqslant C(1+t)^{-3/2+\eta}\phi_1 + C(1+t)^{-1-\eta}.$$
From this estimate, we get by integrating
$$|Z_m L_1 U| \leqslant C(1+t)^{C\varepsilon} + C\int_0^t \phi_1 ds/(1+s)^{1+\eta}.$$
Integrating the equations on $\sigma_1^{-1}a$ and $L_1 a$, we get, using the estimates on F_i^0 established in **d.**,
$$\sigma_1^{-1}|a| + |L_1 a| \leqslant C(1+t)^{C\varepsilon} + C\varepsilon\int_0^t \phi_1 ds/(1+s) + C\int_0^t \phi_1 ds/(1+s)^{1+\eta}.$$
Now,
$$\begin{aligned}|(1+t)(\varepsilon\sigma_1)^{-1} Z_m u| &\leqslant C(1+t)\varepsilon^{-1}|\partial_r Z_m u|_{L^\infty}\\ &\leqslant C(1+t)\varepsilon^{-1}|LZ_m u|_{L^\infty} + C(1+t)\varepsilon^{-1}|L_1 Z_m u|_{L^\infty}.\end{aligned}$$
At this point we need the refinement iii) in Lemma 3.2:
$$(1+t)\varepsilon^{-1}|[Z_m, L_1]u| \leqslant C(1+t)\varepsilon^{-1}(|Lu|\phi_1 + |L_1 u|(1+|L_1 a|)).$$

Since $(1+t)\varepsilon^{-1}|Lu| \leqslant C(1+t)^{-\eta}$, we have

$$|(1+t)(\sigma_1\varepsilon)^{-1}Z_m u| \leqslant C(1+t)\varepsilon^{-1}|LZ_m u| + C(1+t)\varepsilon^{-1}|Z_m L_1 u| + C(1+t)^{-\eta}\phi_1 + C(1+t)^{C\varepsilon} + C|L_1 a|.$$

We use Lemma 3.1 to evaluate the first term:

$$(1+t)\varepsilon^{-1}|LZ_m u| \leqslant (1+t)\varepsilon^{-1}|Z_m Lu| + C|d|(1+t)\varepsilon^{-1}|L_1 u| + C|N_1|(1+t)\varepsilon^{-1}|Lu|.$$

But

$$|Z_m Lu| \leqslant C\phi_1|Z_0 Lu| + C(1+t)^{C\varepsilon}|Z_0 Lu|.$$

Since

$$(1+t)\varepsilon^{-1}|Z_0^q Lu| \leqslant C(1+t)^{-\eta},$$

we get

$$(1+t)\varepsilon^{-1}|Z_m Lu| \leqslant C + C(1+t)^{-\eta}\phi_1,$$

and finally

$$(1+t)\varepsilon^{-1}|LZ_m u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_1 + C\varepsilon\phi_1.$$

From (5.1) we get now

$$(1+t)\varepsilon^{-1}|Z_m L_1 u| \leqslant C|Z_m L_1 U| + \varepsilon^{-1}|Z_m u| + C|\sigma_1^{-1}a| + C(1+t)^{C\varepsilon}.$$

Since

$$\varepsilon^{-1}|Z_m u| \leqslant C + C(1+t)^{-\eta}\phi_1,$$

and, from the very definition of a, $|La| \leqslant C\varepsilon\phi_1$, we get finally

$$\phi_1 \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_1 + C\varepsilon\phi_1 + C\int_0^t \phi_1\varepsilon ds/(1+s) + C\int_0^t \phi_1 ds/(1+s)^{1+\eta}.$$

The conclusion follows by Gronwall's Lemma, since $|\phi_1| \leqslant C$ for finite t.

g. To control ϕ_k, $k \geqslant 2$, we essentially have to repeat the argument of **f.**, using Lemma 4 when necessary. Setting $LZ_m^{k+1}L_1 U = G_{k+1}$, we estimate first G_{k+1} using Lemma 3.3, which requires controlling $Z_m^l G, l \leqslant k+1$. Thanks to Lemma 4,

$$Z_m^l G = \sum f N_0^{l_0} N_1^{l_1} N_{k_1} \cdots N_{k_j} Z_0^p G,$$

and we already know $|Z_0^l G| \leqslant C(1+t)^{-3/2+\eta}$. Hence

$$|Z_m^l G| \leqslant C(1+t)^{-1-\eta}, \quad l \leqslant k,$$
$$|Z_m^{k+1} G| \leqslant C(1+t)^{-1-\eta}\phi_{k+1} + C(1+t)^{-1-\eta}.$$

We obtain from Lemma 3.3, applied for the index $k+1$ with $w = L_1 U, g = G$, and the induction hypothesis on ϕ_l

$$|G_{k+1}| \leqslant C(1+t)^{-1-\eta} + C(1+t)^{-1-\eta}\phi_{k+1} + C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1-\eta}|Z_m^{k+1}L_1 U|.$$

From this estimate, we get by integration down to the boundary $r = t/2 + M$

$$|Z_m^{k+1} L_1 U| \leqslant C(1+t)^{C\varepsilon} + C\int_0^t |Z_m^{k+1} L_1 U| ds/(1+s)^{1+\eta} + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta},$$

hence

$$|Z_m^{k+1} L_1 U| \leqslant C(1+t)^{C\varepsilon} + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta}.$$

Integrating the equations on $Z_m^k(\sigma_1^{-1}a)$ and $Z_m^k L_1 a$ we get, using the estimates on F_i^k established in **d.**,

$$|Z_m^k(\sigma_1^{-1}a)| + |Z_m^k L_1 a| \leqslant C(1+t)^{C\varepsilon} + C\varepsilon\int_0^t \phi_{k+1} ds/(1+s) + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta}.$$

Now,

$$\begin{aligned} |(1+t)(\varepsilon\sigma_1)^{-1} Z_m^{k+1} u| &\leqslant C(1+t)\varepsilon^{-1} |\partial_r Z_m^{k+1} u|_{L^\infty} \\ &\leqslant C|(1+t)\varepsilon^{-1} L Z_m^{k+1} u|_{L^\infty} + C|(1+t)\varepsilon^{-1} L_1 Z_m^{k+1} u|_{L^\infty}. \end{aligned}$$

At this point we need the refinement iii) in Lemma 3.2:

$$\begin{aligned} |(1+t)\varepsilon^{-1}[Z_m^{k+1}, L_1]u| &\leqslant |(1+t)\varepsilon^{-1} L u|\phi_{k+1} + C(1+t)^{C\varepsilon} \\ &\quad + C|Z_m^k L_1 a||(1+t)\varepsilon^{-1} L_1 u| + C(1+t)^{C\varepsilon} + C\sum_{q\leqslant k}(1+t)^{C\varepsilon}|Z_m^q d|. \end{aligned}$$

We have $|(1+t)\varepsilon^{-1} L u| \leqslant C(1+t)^{-\eta}$. Using Lemma 2 and Lemma 4, we get

$$Z_m^q d = \sum_{\substack{q_1+q_2=q \\ \sum l_i \leqslant q_1}} f N_{l_1}\cdots N_{l_j} Z_m^{q_2+1} u,$$

$$Z_m^l u = \sum f N_0^{l_0} N_1^{l_1} N_{k_1}\cdots N_{k_j} Z_0^p u.$$

Since $|Z_0^p u| \leqslant C\varepsilon(1+t)^{-2\eta}$ we obtain

$$(1+t)^{C\varepsilon}|Z_m^q d| \leqslant C + C(1+t)^{-\eta}\phi_{k+1}, \quad q \leqslant k.$$

Finally

$$\begin{aligned} |(1+t)(\sigma_1\varepsilon)^{-1} Z_m^{k+1} u| &\leqslant C|(1+t)\varepsilon^{-1} L Z_m^{k+1} u| + C|(1+t)\varepsilon^{-1} Z_m^{k+1} L_1 u| \\ &\quad + C(1+t)^{-\eta}\phi_{k+1} + C(1+t)^{C\varepsilon} + C|Z_m^k L_1 a|. \end{aligned}$$

We use Lemma 3.3 to evaluate the first term:

$$|(1+t)\varepsilon^{-1} L Z_m^{k+1} u| \leqslant \textstyle\sum_1 + \sum_2 + \sum_3.$$

We obtain

$$\textstyle\sum_2 + \sum_3 \leqslant C(1+t)^{C\varepsilon} + C\varepsilon\phi_{k+1},$$

$$\textstyle\sum_1 \leqslant C|(1+t)\varepsilon^{-1} L u||N_{k+1}| + C(1+t)^{C\varepsilon} \displaystyle\sum_{p\leqslant k+1}(1+t)\varepsilon^{-1}|Z_m^p L u|.$$

Using again Lemma 4, we obtain for $p \leqslant k+1$,

$$|Z_m^p L u| \leqslant C|N_{k+1}||Z_0^q L u| + C(1+t)^{C\varepsilon}|Z_0^q L u|.$$

Since $(1+t)\varepsilon^{-1}|Z_0^q Lu| \leqslant C(1+t)^{-\eta}$ by the induction hypothesis, we get

$$\textstyle\sum_1 \leqslant C + C(1+t)^{-\eta}\phi_{k+1},$$

and finally

$$|(1+t)\varepsilon^{-1} L Z_m^{k+1} u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_{k+1} + C\varepsilon\phi_{k+1}.$$

From (5.1) we get now

$$|(1+t)\varepsilon^{-1} Z_m^{k+1} L_1 u| \leqslant C|Z_m^{k+1} L_1 U| + C\varepsilon^{-1}|Z_m^{k+1} u| + C|\sigma_1^{-1} Z_m^k a| + C(1+t)^{C\varepsilon}.$$

From Lemma 4 we have

$$|\varepsilon^{-1} Z_m^{k+1} u| \leqslant C + C(1+t)^{-\eta}\phi_{k+1}, |Z_m^k La| \leqslant C + C(1+t)^{-\eta}\phi_{k+1}.$$

From Lemma 2 we have

$$|\sigma_1^{-1} Z_m^k a| \leqslant C|Z_m^k(\sigma_1^{-1}a)| + C(1+t)^{C\varepsilon},$$

thus finally

$$\begin{aligned}
\phi_{k+1} &\leqslant |\sigma_1^{-1} Z_m^k a| + |Z_m^k La| + |Z_m^k L_1 a| + |(1+t)(\varepsilon\sigma_1)^{-1} Z_m^{k+1} u| \\
&\qquad + |(1+t)\varepsilon^{-1} Z_m^{k+1} Lu| + |(1+t)\varepsilon^{-1} Z_m^{k+1} L_1 u| \\
&\leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_{k+1} + C\varepsilon\phi_{k+1} \\
&\qquad + C|Z_m^k(\sigma_1^{-1}a)| + C|Z_m^k L_1 a| + C|Z_m^{k+1} L_1 U| \\
&\leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\phi_{k+1} C\varepsilon\phi_{k+1} \\
&\qquad + C\int_0^t \phi_{k+1}\varepsilon ds/(1+s) + C\int_0^t \phi_{k+1} ds/(1+s)^{1+\eta}.
\end{aligned}$$

The conclusion follows by Gronwall's Lemma. □

III.6. Improved estimates of the N_k. — We will need later to know that the N_k have a better behavior inside the light cone.

Proposition 6. — *Let $\mu > 1/2$. For $\eta > 0$ small enough, we have for $k \leqslant s_0 - 3$, with the exception of $N_0 = 1$, the estimates*

$$|N_k| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

Proof. — We follow here the proof of Proposition 5 and use the notations there.

a. We have

$$L(\sigma_1^{1-\mu} L_1 U) = \sigma_1^{1-\mu} G + f N_0 \varepsilon(1+t)^{-1}(\sigma_1^{1-\mu} L_1 U), \quad |\sigma_1^{1-\mu} G| \leqslant C(1+t)^{-1-\eta}.$$

Since $|\sigma_1^{1-\mu} L_1 U| \leqslant C$ on $r = t/2 + M$, we get by integrating the equation

$$|L_1 U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

On the other hand, we know $|LU| \leqslant C\sigma_1^{1/2}(1+t)^{-1+\eta} \leqslant C\sigma_1^{-1/2+\eta}$. Hence $|\partial_r U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}$, which implies

$$|U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu}, \quad (1+t)(\sigma_1\varepsilon)^{-1}|u| \leqslant C(+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

Finally, with $A = L$ or $A = L_1$,

$Au = \varepsilon/rAU - (Ar/r)u,\ (1+t)\varepsilon^{-1}|Au| \leqslant C|AU| + C(1+t)^{-1}|U| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$

This proves the estimate for N_0.

b. We assume now

$$|N_l| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}, \quad l \leqslant k,$$

and set $\psi_k = \sum |\sigma_1^{1-\mu} N_k|_{L^\infty}$. We follow the proof of Proposition 5, g, just looking more closely to the powers of σ_1. Set $V_{k+1} = \sigma_1^{1-\mu} Z_m^{k+1} L_1 U$. We have

$$LV_{k+1} = \sigma_1^{1-\mu} G_{k+1} + fN_0\varepsilon(1+t)^{-1}V_{k+1}.$$

We see that

$$|\sigma_1^{1-\mu} Z_0^l G| \leqslant C(1+t)^{-1-2\eta},$$

hence

$$\sigma_1^{1-\mu}|Z_m^l G| \leqslant C(1+t)^{-1-\eta}, \quad l \leqslant k,$$
$$\sigma_1^{1-\mu}|Z_m^{k+1} G| \leqslant C(1+t)^{-1-\eta}(1+|N_{k+1}|).$$

Using Lemma 3.3 with $w = L_1U, g = G$, we get

$$G_{k+1} = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

We have from the above estimates

$$\begin{aligned}\sigma_1^{1-\mu}|\textstyle\sum_1| &\leqslant C(1+t)^{-1-\eta}(1+|N_{k+1}|) + C(1+t)^{-1-\eta}|N_{k+1}| + C(1+t)^{-1-\eta}\\ &\leqslant C(1+t)^{-1-\eta/2}.\end{aligned}$$

Since we get easily

$$\sigma_1^{1-\mu}|Z_m^l L_1 U| \leqslant C(1+t)^{C\varepsilon}, \quad l \leqslant k,$$

we have, using $|d| \leqslant C\varepsilon(1+t)^{-\eta}$ and the estimate on $|Z_m^l d|$ already established,

$$\begin{aligned}\sigma_1^{1-\mu}|\textstyle\sum_2| &\leqslant C|d|(1+t)^{-1}|V_{k+1}| + C|Z_m^k d|(1+t)^{-1+C\varepsilon} + C(1+t)^{-1-\eta}\\ &\leqslant C(1+t)^{-1-\eta}(1+|V_{k+1}|)\end{aligned}$$

and a similar estimate for $\sum_3$. Finally

$$\sigma_1^{1-\mu}|G_{k+1}| \leqslant C(1+t)^{-1-\eta/2}(1+|V_{k+1}|).$$

We already know that $|V_{k+1}| \leqslant C$ on the boundary $r = t/2 + M$, hence by integration we obtain

$$|V_{k+1}| \leqslant C(1+t)^{C\varepsilon}.$$

c. We have, still with the notations of Proposition 5,

$$L(\sigma_1^{-\mu}a) = \sigma_1^{1-\mu}F_1 + fN_0\varepsilon(1+t)^{-1}(\sigma_1^{-\mu}a),$$
$$L(\sigma_1^{1-\mu}L_1 a) = \sigma_1^{1-\mu}F_2 + fN_0\varepsilon(1+t)^{-1}(\sigma_1^{1-\mu}L_1 a).$$

Set now

$$LZ_m^k(\sigma_1^{-\mu}a) = \overline{F}_1^k, \quad LZ_m^k(\sigma_1^{1-\mu}L_1 a) = \overline{F}_2^k.$$

To estimate $\overline{F}_1^k$, we use Lemma 3.3 with

$$w = \sigma_1^{-\mu} a, \quad g = \sigma_1^{1-\mu} F_1 + f N_0 \varepsilon (1+t)^{-1} (\sigma_1^{-\mu} a).$$

We find

$$\overline{F}_1^k = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

We have

$$Z_m^l g = \sigma_1^{1-\mu} \sum_{\substack{l_1+l_2=l \\ \sum k_i \leqslant l_1}} f N_{k_1} \cdots N_{k_j} Z_m^{l_2} F_1 + \varepsilon(1+t)^{-1} \sum_{\substack{l_1+l_2=l \\ \sum k_i \leqslant l_1}} f N_{k_1} \cdots N_{k_j} Z_m^{l_2} (\sigma_1^{-\mu} a).$$

Since

$$\sigma_1^{1-\mu} |Z_m^k F_1| \leqslant C\varepsilon (1+t)^{-1} |\sigma_1^{1-\mu} N_{k+1}| + C\varepsilon (1+t)^{-1+C\varepsilon},$$
$$\sigma_1^{1-\mu} |Z_m^{l_2} F_1| \leqslant C\varepsilon (1+t)^{-1+C\varepsilon}, \quad l_2 \leqslant k-1,$$

the first sum is less than

$$C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1} |\sigma_1^{1-\mu} N_{k+1}|.$$

The second sum is less than

$$C\varepsilon(1+t)^{-1} |Z_m^k(\sigma_1^{-\mu} a)| + C\varepsilon(1+t)^{-1+C\varepsilon},$$

and finally

$$|\textstyle\sum_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1} (|\sigma_1^{1-\mu} N_{k+1}| + |Z_m^k(\sigma_1^{-\mu} a)|).$$

Just as before, we also get

$$|\textstyle\sum_2| + |\sum_3| \leqslant C(1+t)^{-1-\eta} (1 + |Z_m^k(\sigma_1^{-\mu} a)|),$$

hence

$$|\overline{F}_1^k| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C(1+t)^{-1-\eta} + C\varepsilon(1+t)^{-1} \psi_{k+1}$$
$$+ (C\varepsilon(1+t)^{-1} + C(1+t)^{-1-\eta}) |Z_m^k(\sigma_1^{-\mu} a)|$$

and a similar estimate for $\overline{F}_2^k$. Integration along L, we get

$$|Z_m^k(\sigma_1^{-\mu} a)| + |Z_m^k(\sigma_1^{1-\mu} L_1 a)| \leqslant C(1+t)^{C\varepsilon} + C \int_0^t \psi_{k+1} \varepsilon ds/(1+s).$$

d. We have

$$|(1+t)\varepsilon^{-1} \sigma_1^{-\mu} Z_m^{k+1} u| \leqslant C(1+t)\varepsilon^{-1} |\sigma_1^{1-\mu} L Z_m^{k+1} u|_{L^\infty} + C(1+t)\varepsilon^{-1} |\sigma_1^{1-\mu} L_1 Z_m^{k+1} u|_{L^\infty}.$$

Just as before, using point iii) in Lemma 3.2, we obtain

$$(1+t)\varepsilon^{-1} |\sigma_1^{1-\mu} [Z_m^{k+1}, L_1] u| \leqslant (1+t)\varepsilon^{-1} |Lu| \psi_{k+1} + C(1+t)^{C\varepsilon}$$
$$+ C(1+t)\varepsilon^{-1} |L_1 u| |\sigma_1^{1-\mu} Z_m^k L_1 a| + C,$$

hence

$$(1+t)\varepsilon^{-1}|\sigma_1^{-\mu} Z_m^{k+1} u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu} L Z_m^{k+1} u| + C(1+t)^{-\eta}\psi_{k+1} + C|\sigma_1^{1-\mu} Z_m^k L_1 a| + C(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu} Z_m^{k+1} L_1 u|.$$

Exactly as before, we get

$$(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu} L Z_m^{k+1} u| \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\psi_{k+1}.$$

From (5.1) we get now

$$(1+t)\varepsilon^{-1}|\sigma_1^{1-\mu} Z_m^{k+1} L_1 u| \leqslant C|\sigma_1^{1-\mu} Z_m^{k+1} L_1 U| + C\varepsilon^{-1}|\sigma_1^{1-\mu} Z_m^{k+1} u| + C|\sigma_1^{-\mu} Z_m^k a| + C(1+t)^{C\varepsilon}.$$

Using Lemma 4, we obtain

$$|\sigma_1^{1-\mu} Z_m^{k+1} u| \leqslant C, \quad |\sigma_1^{1-\mu} Z_m^k L a| \leqslant C.$$

We also have from Lemma 2

$$|\sigma_1^{-\mu} Z_m^k a| \leqslant |Z_m^k(\sigma_1^{-\mu} a)| + C(1+t)^{C\varepsilon},$$
$$|\sigma_1^{1-\mu} Z_m^k L_1 a| \leqslant |Z_m^k(\sigma_1^{1-\mu} L_1 a)| + C(1+t)^{C\varepsilon}.$$

Finally,

$$\psi_{k+1} \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-\eta}\psi_{k+1} + C\int_0^t \psi_{k+1}\varepsilon ds/(1+s),$$

which yields the result by Gronwall Lemma. □

III.7. Back to the standard fields. — In this section, we will transform the estimates on u given in terms of the fields Z_m into estimates given in terms of the standard fields Z_0. Remember that we have fixed $\mu > 1/2$ (μ as close as we want to $1/2$).

Proposition 7. — *We have, for $k \leqslant s_0 - 4$, the estimates*

$$|Z_0^k u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{\mu},$$
$$|Z_0^k \partial u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sigma_1^{\mu-1}.$$

Proof

1. First, we need control b_1.

Lemma 7.1. — *We have, for $\alpha \leqslant s_0 - 3$,*

$$|Z_m^{\alpha} b_1| \leqslant C(1+t)^{C\varepsilon}.$$

Proof. — We use Lemma 3.3 with $w = b_1$, $g = -\overline{\chi}/2cL_1 u = f\varepsilon/(1+t)N_0$. We obtain

$$L Z_m^k b_1 = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

Since

$$\textstyle\sum_1 = \varepsilon/(1+t)\sum f N_{l_1}\cdots N_{l_i} N_{k_i}\cdots N_{k_j}, \quad \sum l_j + \sum k_l \leqslant k,$$

we obtain $|\sum_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}$.

Exactly as before, using Lemma 5.1, we have

$$|\textstyle\sum_2| \leqslant C(1+t)^{-1-\eta}|Z_m^k b_1| + C\varepsilon(1+t)^{-1+C\varepsilon}\displaystyle\sum_{l\leqslant k-1}|Z_m^l b_1|.$$

For $\sum_3$, we get simply $|\sum_3| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}\sum_{l\leqslant k-1}|Z_m^l b_1|$.

We already know that $|b_1| \leqslant C(1+t)^{C\varepsilon}$. By induction, assuming already

$$\sum_{l\leqslant k-1}|Z_m^l b_1| \leqslant C(1+t)^{C\varepsilon}$$

we obtain

$$|LZ_m^k b_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C(1+t)^{-1-\eta}|Z_m^k b_1|.$$

Integrating yields the desired estimate. □

2. We have $Z_m = Z + aL_1$, but we have only a good control of a/σ_1, not of a. This forces us to display the fact that L_1 is a better field than the Z_m. To motivate some technical definitions which will be given in 3., we present the following attempt to express $\sigma_1 L_1$ in terms of the Z_m. We first write

$$\frac{r-ct}{\sqrt{c}}L_1 = H_0 - S = H_0^m - S^m + 2a(S)L_1.$$

We introduce now a cutoff in the blind zone. For this, set $q = q_0\sigma_1^{-1}\exp C_0\tau$, and define $\chi_1 = \chi_1(q)$, where $\chi_1(s)$ is zero for $s \leqslant 1$ and one for $s \geqslant 2$. We write then

$$\begin{aligned}\frac{r-t}{\sqrt{c}}L_1 &= H_0^m - S^m + 2\chi_1 a(S)L_1 + \frac{tu}{\sqrt{c}}L_1 + 2a(S)(1-\chi_1)L_1,\\ \sigma_1 DL_1 &= (C_1 - 2a(S)\sqrt{c}\chi_1)L_1 - \sqrt{c}(H_0^m - S^m),\\ D &= (1 - \sigma_1^{-1}tu - 2\sqrt{c}(1-\chi_1)\sigma_1^{-1}a(S)).\end{aligned}$$

Since

$$|a/\sigma_1| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1},$$

we have

$$|(1-\chi_1)a(S)/\sigma_1| \leqslant Cq_0^{\mu-1}(1+t)^{C\varepsilon - C_0\varepsilon(1-\mu)}.$$

If we choose q_0 and C_0 large enough, we obtain

$$|(1-\chi_1)\sigma_1 a(S)| \leqslant 1/4.$$

Hence, for ε small enough, D^{-1} will be a smooth function of

$$u, \quad tu/\sigma_1, \quad (1-\chi_1)\sigma_1^{-1}a(S).$$

We fix now this choice of q_0, C_0.

3. We have now to develop a calculus analogous to that of Chapter III, and enlarged so as to contain the cutoff in q we have just introduced. We denote by N_0 as before one of the quantities

$$1, \quad \varepsilon^{-1}(1+t)\sigma_1^{-1}u, \quad \varepsilon^{-1}(1+t)Lu, \quad \varepsilon^{-1}(1+t)L_1 u.$$

When we want to emphasize the fact that N_0 is not 1 but actually involves u, we write N_0'. We denote now by N_k, for $k \geqslant 1$, one of the quantities

$$\varepsilon^{-1}(1+t)\sigma_1^{-1}Z_m^k u, \quad \varepsilon^{-1}(1+t)Z_m^k Lu, \quad \varepsilon^{-1}(1+t)Z_m^k L_1 u,$$
$$\sigma_1^{-1}Z_m^{k-1}a, \quad g_0(q)Z_m^{k-1}a, \quad Z_m^{k-1}La, \quad Z_m^{k-1}L_1 a, \quad a \in (\mathrm{Coeff}).$$

Here, g_0 is any smooth function, vanishing for $q \leqslant 1/2$, whose derivative belongs to C_0^∞. This is of course a slight abuse of notation, since the g_0 actually used in the whole computation are generated by χ_1 and finitely many derivatives of χ_1. Hence, for these enlarged N_l, we still have

$$|N_l| \leqslant C(1+t)^{C\varepsilon}, \quad l \leqslant s_0 - 3.$$

In fact,

$$|g_0(q)Z_m^{l-1}a| \leqslant C|q^{-1}g_0(q)|(1+t)^{C_0\varepsilon}|\sigma_1^{-1}a|$$

and $q \geqslant 1/2$ on the support of g_0.

In view of **2.**, we enlarge a little the definition of f. We will denote by f a smooth function of

$$\varepsilon,\ u,\ \omega,\ \sigma_1(1+t)^{-1},\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i},\ g(q),\ N_0.$$

Here, g is any smooth function whose derivative belongs to $C_0^\infty(\mathbf{R}_+^*)$. Finally, we need to introduce nonlinear analogues to N_1, denoted by ν_1. We define ν_1 as any smooth function of

$$\varepsilon, u, \omega, \sigma_1(1+t)^{-1}, (1+t)^{-\mu}, \sigma_1^{-\mu}, g(q), N_0, (1-\chi_1(q))\sigma_1^{-1}a.$$

In some sense, we see that ν_1 is a generalization of f to order one derivatives. Of course, the quantity D^{-1} from **2** is a ν_1.

4. Some calculus Lemmas

We have to prove that the analogue to Lemma III.2 for the enlarged quantities is correct.

Lemma 2'. — *We have the following identities:*

i) $Z_m^k f = \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,
ii) $Z_m^k N_p = \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k + p$, *and, for some* i, $k_i \geqslant p$,
iii) $Z_m^k t = t \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k$,
iv) $Z_m^k \sigma_1 = \sigma_1 \sum f N_{k_1} \cdots N_{k_j}$, $k_1 + \cdots + k_j \leqslant k$.
v) $Z_m^k \nu_1 = \sum \nu_1 N_1^l N_{l_1} \cdots N_{l_j}$, $l_i \geqslant 2$, $\sum(l_i - 1) \leqslant k$.

Proof

a. We try first i) for $k = 1$. With $q = q_0\sigma_1^{-1}\exp C_0\tau$,

$$R_i(q) = 0, \quad S(q) = fq, \quad H_0(q) = fq\widetilde{N}_0, \quad L_1(q) = fq\sigma_1^{-1}, \quad L(q) = fq\sigma_1^{-1},$$

hence

$$Z_m(q) = Z(q) + aL_1(q) = fqN_0 + fqN_1,$$
$$Z_m(g(q)) = qg'(q)(f + fN_1) = f + fN_1.$$

Finally, $Z_m N_0 = f + fN_1$.

b. From **a.**, we have $Z_m g_0(q) = g_0(q) f N_1$, hence

$$Z_m \widetilde{N}_p = fN_1 N_p + N_{p+1}$$

and ii) is proved for $k = 1$ and any p.

c. iii) and iv) are clear for $k = 1$. Thus, by induction, i)-iv) are proved.

d. To prove v) for $k = 1$, we just have to check the factor $(1 - \chi_1)\sigma_1^{-1} a$:

$$Z_m[(1 - \chi_1)\sigma_1^{-1} a] = fN_1^2 + fN_2.$$

Now, by induction,

$$Z_m^{k+1}\nu_1 = \sum \nu_1 (N_1^{l+2} + N_1^l N_2) N_{l_1} \cdots N_{l_j} + \sum l \nu_1 N_1^{l-1}(fN_2 + fN_1^2) N_{l_1} \cdots N_{l_j} \\ + \sum_{(\sum k_i \leqslant l_i + 1)} \nu_1 N_1^l N_{l_1} \cdots (\sum fN_{k_1} \cdots N_{k_r}) \cdots N_{l_j}.$$

For a term $N_2 N_{l_1} \cdots N_{l_j}$, the sum of indexes is less than or equal to $k + j + 2 = k + 1 + j + 1$, as desired. For a term in the last sum, we note that $Z_m N_p$ contains at least one factor N_q, $q \geqslant p$ if $p \geqslant 2$. Let r' be the number of k_i greater than or equal to two: $1 \leqslant r' \leqslant r$. The sum of indexes corresponding to these terms is less than the sum of all indexes, which is less than or equal to $(\sum l_i) + 1 \leqslant k + j + 1 \leqslant k + 1 + j - 1 + r'$ as desired. □

We define, for $k \geqslant 1$,

$$M_k = \nu_1 N_1^l N_{l_1} \cdots N_{l_j}, \quad l \geqslant 0, \quad l_i \geqslant 2, \quad \sum (l_i - 1) \leqslant k - 1.$$

This definition is justified by Lemma 2', v). Remark that

$$M_1 = \nu_1 N_1^l, \quad M_1 M_k = M_k, \quad M_k M_l = M_{k+l-1},$$

and

$$\sum_{(\sum k_i \leqslant k)} \nu_1 N_{k_1} \cdots N_{k_j} = M_k, \quad Z_m M_k = \sum M_{k+1}, \quad Z_m^p M_k = \sum M_{k+p}.$$

5. We are now ready to prove Proposition 7. Denote by $\overline{Z}_0$ the fields

$$R_i, \quad S, \quad h_0 = t\partial_r + r\partial_t, \quad \partial_t.$$

Lemma 7.5. — *We have*

$$\overline{Z}_0^k = \sum M_q a_1^{-l} (Z_m^{r_1} b_1) \cdots (Z_m^{r_i} b_1) Z_m^p,$$

with

$$p \geqslant 1, \quad 0 \leqslant l \leqslant k, \quad r_i \geqslant 1, \quad q - 1 + \sum r_j + p \leqslant k.$$

Proof

a. Consider first $k = 1$. We write, according to **2.**,

$$\sigma_1 L_1 = \nu_1((f + fN_1)L_1 + fZ_m) = M_1 Z_m + M_1 a_1^{-1} Z_m.$$

Then

$$R_i = R_i^m - \sigma_1^{-1} a(R_i)(M_1 Z_m + M_1 a_1^{-1} Z_m),$$

and similarly for S and H_0. Then

$$H_0 = ct\partial_r + r/c\partial_t = h_0 + ftuL + ftuL_1,$$
$$h_0 = H_0 + f(H_0^m + S^m) + M_1 Z_m + M_1 a_1^{-1} Z_m.$$

Finally,

$$\partial_t = 2\sqrt{c}(L + L_1) = fZ_m + fa_1^{-1} Z_m.$$

b. Now

$$\overline{Z}_0^{k+1} = (M_1 + M_1 a_1^{-1}) Z_m(\overline{Z}_0^k),$$

and the formula follows at once by induction, since

$$M_1 M_q = M_q, \quad Z_m M_l = M_{l+1}, \quad Z_m a_1^{-l} = -l a_1^{-l} Z_m b_1. \qquad \square$$

From this Lemma, we get, for $l \leqslant s_0 - 3$,

$$|\varepsilon^{-1}(1+t)\sigma_1^{-1}\overline{Z}_0^l u| + |\varepsilon^{-1}(1+t)\overline{Z}_0^l Lu| + |\varepsilon^{-1}(1+t)\overline{Z}_0^l L_1 u| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

For $l \leqslant s_0 - 4$, we can in fact enlarge this estimate to have also

$$|\varepsilon^{-1}(1+t)\overline{Z}_0^l \partial u| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}.$$

To prove this, we write

$$\partial_t = 2\sqrt{c}(L + L_1), \quad \partial_i = (\omega_i/2\sqrt{c})(L - L_1) - 1/r(\omega \wedge R)_i, \quad R_i = \overline{Z}_0.$$

From the weak control

$$|\overline{Z}_0^l u| \leqslant C(1+t)^{C\varepsilon}$$

already proved, we get

$$|\overline{Z}_0^l f(\omega, u)| + |r\overline{Z}_0^l(\omega/r)| \leqslant C(1+t)^{C\varepsilon}.$$

6. Finally, we want to replace, in the above formula, the fields $\overline{Z}_0$ by Z_0. But all fields Z_0 can be expressed in terms of $\overline{Z}_0$. In fact, R_i, S and ∂_t are already $\overline{Z}_0$, and

$$h_i = t\partial_i + x_i\partial t = \omega_i h_0 - t/r(\omega \wedge R)i,$$

$$\partial_i = \omega_i(-\partial_t + (r+t)^{-1}(h_0 + S)) - 1/r(\omega \wedge R)_i.$$

Thus

$$Z_0 = \sum f(\omega, (1+t)^{-1}, r(1+t)^{-1})\overline{Z}_0.$$

This implies that we have the desired estimates of Proposition 7. $\square$

IV. A calculus of modified Klainerman's vector fields

IV.1. Definitions and L^∞ estimates of the perturbation coefficients. — In the previous chapter III, we have already used modified fields

$$Z_m = Z + aL_1$$

where the a have been defined by III.1.1. Our final result in Chapter III was the estimates, for $k \leqslant s_0 - 4$,

$$|Z_0^k u| \leqslant C_1\varepsilon(1+t)^{-1+C_1\varepsilon}\sigma_1^{\mu},$$
$$|Z_0^k \partial u| \leqslant C_1\varepsilon(1+t)^{-1+C_1\varepsilon}\sigma_1^{\mu-1}.$$

For aesthetic as well as technical reasons, we will start again from scratch and define new, and better supported coefficients a, by the formula

$$\begin{aligned} La(R_i) + \chi(q)a(R_i)(L_1u/2c) &= -\chi(q)R_iu/2c, \\ La(S) + \chi(q)a(S)(L_1u/2c) &= -\chi(q)Su/2c, \\ a(H_0) = -a(S),\ a(R_i)(0,t) = 0,&\ a(R_i)(x,0) = 0, \\ a(S)(0,t) = 0,&\ a(S)(x,0) = 0. \end{aligned} \tag{1.1}$$

Here $q = q_0\sigma_1^{-1}\exp C_0\tau$, where q_0 is taken to be

$$q_0 = 1/2\exp(-C_0\varepsilon\log 2)$$

in such a way that the boundary of the support of $\chi(q)$ intersects $r = t+M$ at $t=1$. The big constant C_0 is still to be determined. The function $\chi(s)$ is a real C^∞ function being zero for $s \leqslant 1/2$ and one for $s \geqslant 1$. The aesthetic reason is to perturb as little as possible the standard (adapted) fields Z. It turns out that it is enough to take perturbation coefficients a supported in a logarithmic zone $\sigma_1 \leqslant C(1+t)^{C\varepsilon}$. The technical reason will appear in the proof of Proposition VII.1, where powers of σ_1 on support of a have to be bounded by factors $(1+t)^{\gamma_i\varepsilon}$ for appropriate γ_i.

Proposition 1. — *The coefficients $a(R_i)$ and $a(S)$ defined by (1.1) are zero for t small, for $r \geqslant M+t$ or $q \leqslant 1/2$. Moreover, we can choose C_0 such that, for $k \leqslant s_0 - 5$, we have*

$$|\sigma_1^{-1}Z_0^k a| + |Z_0^k\partial a| \leqslant C(1+t)^{C\varepsilon}.$$

Proof

a. To prove the claim about the supports, we have to check that the domain left to the curve

$$\sigma_1 - 2q_0\exp C_0\tau = 0$$

is an influence domain of the t-axis (where a is zero) for L. But, on this curve,

$$\begin{aligned} L(\sigma_1 - 2q_0\exp C_0\tau) &= L\sigma_1 - (q_0/\sqrt{c})(\exp C_0\tau)C_0\varepsilon/(1+t) \\ &= -\sigma_1\varepsilon/(\sqrt{c}(1+t))((1+t)u/(\varepsilon\sigma_1) + C_0). \end{aligned}$$

If C_0 is big enough, this is negative, proving the claim.

b. To estimate a and its derivatives, we will use the same method as in Chapter III, except that we already know estimates on u. Exactly as in III.1, we define

$$R_i^m = R_i + a(R_i)L_1, \quad S^m = S + a(S)L_1, \quad H_0^m = H_0 + a(H_0)L_1.$$

We forget about K now, and take the family Φ' of the fields Z_m as the collection of the fields

$$R_i^m, S^m, \quad H_0^m, L_1.$$

We will write f to denote a real C^∞ function of the variables

$$\varepsilon,\ u,\ \omega,\ \sigma_1(1+t)^{-1},\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i},\ g(q),\ N_0,$$

where

$$N_0 = 1,\ \varepsilon^{-1}(1+t)\sigma_1^{-1}u,\ \varepsilon^{-1}(1+t)Lu,\ \varepsilon^{-1}(1+t)L_1u,$$

and g is any smooth function whose derivative belongs to $C_0^\infty(\mathbf{R}_+^*)$. We denote by N_k, $k \geqslant 1$ one of the quantities

$$\varepsilon^{-1}(1+t)\sigma_1^{-1}Z_m^k u,\ \varepsilon^{-1}(1+t)Z_m^k Lu,\ \varepsilon^{-1}(1+t)Z_m^k L_1 u,$$
$$\sigma_1^{-1}Z_m^{k-1}a,\ Z_m^{k-1}La,\ Z_m^{k-1}L_1a.$$

This machinery is the same as in III, except that we have enlarged f with $g(q)$ and N_0. As we can see from the proof of Lemma 2' in section III.7, all the calculus and commutation Lemmas of III (that is, Lemma 2, Lemma 3.1, 3.2, 3.3 and Lemma 4) remain valid with these new definitions. We will refer to these calculus lemmas just as Lemma 2, Lemma 3.1, etc. The only difference in the commutation relations is that

$$[L_1, L] = -L_1u/(2c)L_1 + Lu/(2c)L,$$

which means that, in Lemma 3.1, 3.2 or 3.3, we have either $d = (1-\chi(q))Z_m u$ or $d = L_1 u$.

c. We will need the following correspondence between the fields Z_m and the standard fields Z_0.

Lemma 1.1. — *We have*

$$Z_m^k = \sum f N_{k_1}\cdots N_{k_j} Z_0^p + \sum f N_{l_1}\cdots N_{l_i} Z_m^{r_1}(a/\sigma_1)\cdots Z_m^{r_q}(a/\sigma_1)Z_0^p.$$

In the first sum, $p \geqslant 1$, $\sum k_i + p \leqslant k$. In the second sum, $p \geqslant 1$, $q \leqslant k$ and $\sum l_j + \sum r_i + p \leqslant k$.

Proof. — We have $\partial = f\sigma_1 Z_0$. For $k=1$, we write

$$L_1 = f\partial_i + f\partial_t = f\sigma_1 Z_0,$$
$$R_i^m = R_i + aL_1 = Z_0 + fa/\sigma_1 Z_0, \quad S^m = S + fa/\sigma_1 Z_0.$$

Then

$$H_0 = t\partial_r + r\partial_t + f(1+t)u\partial = \sum \omega_i(t\partial_i + x_i\partial_t) + ftu/\sigma_1 Z_0 = fZ_0,$$
$$H_0^m = fZ_0 + fa/\sigma_1 Z_0,$$

which proves the claim. For $k \geqslant 2$, we write $Z_m^{k+1} = Z_m Z_m^k$, and the Lemma follows by induction, since a term

$$Z_m Z_0^p = f Z_0^{p+1} + f a/\sigma_1 Z_0^{p+1}$$

adds one term in each sum. □

d. The following Lemma will be crucial in the whole construction.

Lemma 1.2. — *Assume $1/2 < \mu < 2/3$. Then we have*

$$|a_t| + |a_r| + |a/\sigma_1^\mu| \leqslant C_3(1+t)^{C_3\varepsilon},$$

where C_3 depends only on C_1 and u and not on C_0. Moreover, if C_0 is big enough, we have, on the support of $1-\chi$, the estimates

$$|a/\sigma_1| \leqslant C, \quad |Z_m u| \leqslant C|Z_0 u|, \quad |Z_m \partial u| \leqslant C|Z_0 \partial u|,$$

$$|Z_0 u/\sigma_1|(1 + |\partial a| + \varepsilon^{-1}(1+t)(|Z_0 \partial u| + |Z_0 u/\sigma_1|) \leqslant C\varepsilon(1+t)^{-1-\varepsilon}.$$

Proof

a. In fact, with $b = a/\sigma_1^\mu$,

$$Lb = -\mu b(L\sigma_1/\sigma_1) - \chi b L_1 u/(2c) - \chi/(2c)\sigma_1^{-\mu} Z_0 u, \quad L\sigma_1 = -u/\sqrt{c},$$

hence

$$|Lb| \leqslant C_2\varepsilon(1+t)^{-1} b + C_2\varepsilon(1+t)^{-1+C_1\varepsilon}.$$

By integration, we get

$$|b| \leqslant (C_2/C_1)(1+t)^{(C_1+C_2)\varepsilon}.$$

We have now

$$L_1 L a = -L_1(\chi/(2c))(aL_1 u + Z_0 u) - \chi/(2c)(L_1 a L_1 u + f\partial Z_0 u + f(a/\sigma_1) Z_0 L_1 u).$$

But, since $L_1 q = f q \sigma_1^{-1}$,

$$L_1(\chi/(2c)) = 1/(2c)\chi'(q) f q \sigma_1^{-1} - \chi/(2c^2) L_1 u = f/\sigma_1,$$

$$L_1 L a = f(a/\sigma_1) L_1 u + f Z_0 u/\sigma_1 + f\varepsilon(1+t)^{-1} L_1 a + f\partial Z_0 u + f(a/\sigma_1) Z_0 L_1 u.$$

$$LL_1 a = f\varepsilon(1+t)^{-1} L_1 a + f(Z_0 L_1 u)(a/\sigma_1) + f\varepsilon/(1+t)(a/\sigma_1) \\ + f Z_0 u/\sigma_1 + f\partial Z_0 u + f(Z_0 u/\sigma_1)(a/\sigma_1) = g_1.$$

We deduce that

$$|LL_1 a| \leqslant C\varepsilon/(1+t)|L_1 a| + C\varepsilon/(1+t)(1+t)^{C\varepsilon},$$

where again C does not depend on C_0. Since La is bounded independently of C_0, we get by integration the first part of the Lemma.

b. From **a.**, we get $|a/\sigma_1| \leqslant C$ as soon as $C_0(1-\mu) \geqslant C_3$. Then, for any v,

$$|Z_m v| \leqslant C|Z_0 v| + C|a/\sigma_1||Z_0 v| \leqslant C|Z_0 v|.$$

Since

$$|Z_0 u(a/\sigma_1)^2| \leqslant C\varepsilon(1+t)^{-1+C_1\varepsilon+2C_3\varepsilon}\sigma_1^{3\mu-2},$$

we obtain on the support of $1-\chi$

$$|Z_0 u/\sigma_1| + |Z_0 u(a/\sigma_1)^2| \leqslant C\varepsilon(1+t)^{-1+C_4\varepsilon}(1+t)^{-(2-3\mu)C_0\varepsilon},$$

where C_4 does not depend on C_0. This completes the proof. □

e. From now on we assume that C_0 has been fixed big enough for the estimates of Lemma 1.2 to hold. We now assume by induction

$$|N_l| \leqslant C(1+t)^{C\varepsilon}, \quad l \leqslant k,$$

which is true for $k=0$. In particular, in view of Lemma 2,

$$|Z_m^l(a/\sigma_1)| \leqslant C(1+t)^{C\varepsilon}, \quad l \leqslant k-1.$$

Using Lemma 1.1 for the index k, we obtain

$$|Z_m^k \partial u| + |\sigma_1^{-1} Z_m^k Z_0 u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

We write now

$$L(a/\sigma_1) = -\chi/(2c)(a/\sigma_1)L_1 u - \chi/(2c)(Z_0 u/\sigma_1) + f\varepsilon/(1+t)(a/\sigma_1) = g.$$

Applying Lemma 3.3 for the index k with $w = a/\sigma_1$, we get

$$LZ_m^k(a/\sigma_1) = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

Since

$$g = f\varepsilon(1+t)^{-1}(a/\sigma_1) + f(Z_0 u/\sigma_1),$$

we have

$$\begin{aligned}
&|Z_m^l g| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad l \leqslant k-1,\\
&|Z_m^k g| \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k(a/\sigma_1)|.
\end{aligned}$$

Hence

$$|\textstyle\sum_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k(a/\sigma_1)|.$$

In $\sum_3$, all terms are controlled by induction, and $|\sum_3| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}$. In $\sum_2$, if $k_{j+1} \leqslant k-2$, we just write $L_1 Z_m^{k_{j+1}} w = Z_m^{k_{j+1}+1} w$ and the term in controlled by induction. If $k_{j+1} = k-1$, the corresponding term is just $fdL_1 Z_m^{k-1} w$. If $d = L_1 u$, we remember $L_1 = Z_m$ and keep the term as it is. If $d = (1-\chi)Z_m u$, we need to use that L_1 is a better field than the Z_m. We write as in **2**, Proposition III.7,

$$\begin{aligned}
r - t/\sqrt{c}L_1 &= H_0^m - S^m + 2aL_1 + tu/\sqrt{c}L_1,\\
\sigma_1 L_1 &= fZ_m + faL_1 + fL_1.
\end{aligned}$$

Iterating this, we obtain

$$\sigma_1 L_1 = fZ_m + f(a/\sigma_1)Z_m + fa^2/\sigma_1 L_1.$$

Using the corresponding inequality to estimate the term at hand, we get

$$|fdL_1Z_m^{k-1}w| \leqslant C|d|/\sigma_1[(C+C|a/\sigma_1|)|Z_m^k w| + Ca^2/\sigma_1|L_1Z_m^{k-1}w|].$$

From Lemma 1.2, we obtain finally in all cases,

$$|\textstyle\sum_2| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k(a/\sigma_1|.$$

Integrating the equation on $Z_m^k(a/\sigma_1)$, we obtain

$$|Z_m^k(a/\sigma_1)| \leqslant C(1+t)^{C\varepsilon}, \quad \sigma_1^{-1}|Z_m^k a| \leqslant C(1+t)^{C\varepsilon}.$$

f. Since

$$|Z_0pLu| + |Z_0^p L_1 u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad p \leqslant k+1,$$

we obtain, using now Lemma 1.1 with the index $k+1$, applied to u, Lu or L_1u,

$$|\sigma_1^{-1}Z_m^{k+1}u| + |Z_m^{k+1}Lu| + |Z_m^{k+1}L_1u| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

Similarly, since $La = fZ_0u + f(a/\sigma_1)Z_0u$, we obtain directly $|Z_m^k La| \leqslant C$.

g. Remember that

$$LL_1a = f\varepsilon(1+t)^{-1}L_1a + f(Z_0L_1u)(a/\sigma_1) + f\varepsilon/(1+t)(a/\sigma_1) \\ + fZ_0u/\sigma_1 + f\partial Z_0u + f(Z_0u/\sigma_1)(a/\sigma_1) = g_1.$$

Applying Lemma 3.3 for the index k and $w = L_1a$, we obtain

$$LZ_m^k L_1 a = \textstyle\sum_1 + \sum_2 + \sum_3 .$$

As before, we get first

$$|Z_m^l g_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad l \leqslant k-1,$$
$$|Z_m^k g_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k L_1 a|,$$

which gives

$$|\textstyle\sum_1| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k L_1 a|.$$

The analysis of $\sum_2$ and $\sum_3$ are strictly identical to the ones we have done for controlling $Z_m^k(a/\sigma_1)$. Finally

$$|LZ_m^k L_1 a| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-1}|Z_m^k l_1 a|,$$

which gives by integration the desired estimate, and proves that $|N_{k+1}| \leqslant C(1+t)^{C\varepsilon}$.

h. It remains now to translate this result using the standard fields Z_0. As in **5**, Proposition III.7, we denote by $\overline{Z}_0$ the fields

$$R_i,\ S,\ h_0 = r\partial_t + t\partial_r,\ \partial_t.$$

Lemma 1.3. — *We have*

$$\overline{Z}_0^k = \textstyle\sum fN_{k_1}\cdots N_{k_i}(Z_m^{r_1}a)\cdots(Z_m^{r_j}a)Z_m^p,$$

with

$$p \geqslant 1, \quad j \leqslant k, \quad \textstyle\sum k_j + \sum r_i + p \leqslant k.$$

Proof. — The argument is not the same as in Lemma 7.5, since we have defined no ν_1 here. We have

$$R_i = R_i^m - aZ_m, \quad S = S^m - aZ_m, \quad H_0 = H_0^m + aZ_m,$$

$$H_0 = h_0 + fZ_m + f(1+t)uL_1.$$

Remembering that

$$\sigma_1 L_1 = fZ_m + fL_1 + faL_1,$$

we get

$$f(1+t)uL_1 = f((1+t)u/\sigma_1)(fZ_m + fL_1 + faL_1) = fZ_m + faZ_m.$$

This proves the Lemma for $k = 1$, and the general case follows by induction. □

Now, since support a is contained in $q \geqslant 1/2$,

$$|Z_m^r a| \leqslant |\sigma_1^{-1} Z_m^r a| \sigma_1 \leqslant C(1+t)^{C\varepsilon},$$

hence this Lemma, applied to a, La or $L_1 a$, yields

$$|\sigma_1^{-1} \overline{Z}_0^l a| + |\overline{Z}_0^l La| + |\overline{Z}_0^l L_1 a| \leqslant C(1+t)^{C\varepsilon}.$$

The transition from $\overline{Z}_0$ to Z_0 is now identical with 7.5 c, and this completes the proof. □

IV.2. Smoothing operators

IV.2.1. Smoothing operators on the sphere. — We will need, in the spirit of the paradifferential calculus of J.M. Bony [**6**], smoothing operators S_θ^2 acting on functions on the unit sphere S^2. To define these S_θ^2, we will fix

$$\phi_2 \in C_0^\infty(\mathbf{R}^2), \quad 0 \leqslant \phi_2 \leqslant 1, \quad \int \phi_2 = 1,$$

and a partition of unity on S^2

$$\chi_+ + \chi_- = 1,$$

where $\chi_\pm$ is one for $\pm x_3 \geqslant 0$ and vanishes near the pole $(0, 0, -\pm 1)$. For w defined on the sphere, we set

$$S_\theta^2 w = \sum_{(+,-)} (\phi_{2,\theta} * [(\chi_\pm w)(p_{-\pm}^{-1})])(p_{-\pm}),$$

where $p_\pm$ are the stereographic projections from the poles $(0, 0, \pm 1)$, and

$$\phi_{2,\theta}(y) = \theta^2 \phi_2(\theta y).$$

The operators S^2_θ enjoy the usual properties

$$(2.1.1)_a \qquad ||S^2_\theta w|| \leqslant C||w||,$$

$$(2.1.1)_b \qquad ||S^2_\theta w - w|| \leqslant \theta^{-k}\sum_{l\leqslant k}||R^l w||,$$

$$(2.1.1)_c \qquad ||R^k S^2_\theta w|| \leqslant C\theta^k||w||.$$

Here, $||.||$ stands for the L^2 or L^∞ norm on the sphere, and

$$R^k = R_{i_1}\cdots R_{i_k}.$$

When computing with the S_θ, we think of them as if they were only the convolution with $\phi_{2,\theta}$, omitting for simplicity the cutoff functions etc. Note that if we abandon the property $\int\phi_2 = 1$, properties $(2.1.1)_a$ and $(2.1.1)_c$ remain.

IV.2.2. Smoothing operators. — We choose now

$$\phi_1\in C^\infty_0(\mathbf{R}),\quad 0\leqslant\phi_1\leqslant 1,\quad \int\phi_1 = 1,\quad \operatorname{supp}\phi_1\subset\{r\leqslant 0\},$$

and set

$$S^1_\theta w(r,\omega,t) = \int\theta\phi_1(\theta(r-r'))w(r',\omega,t)dr'.$$

This is the standard smoothing operator in the r-variable. We will use it only in a fixed domain on the form

$$r\geqslant\gamma_1(1+t),\quad \gamma_1 > 0,$$

acting on functions supported in $r\leqslant M+t$. With to different (big) parameters θ_1 and θ_2 to be chosen later, and $\theta = (\theta_1,\theta_2)$, we define finally

$$S_\theta w(r,\omega,t) = S^1_{\theta_1}S^2_{\theta_2}w.$$

It is clear that, for some C (independent of t) we have the inequality

$$|S_\theta w(.,t)|_{L^2_x}\leqslant C|w(.,t)|_{L^2_x}.$$

This inequality holds also if the integrals of the ϕ_i are not normalized to be one, in which case, to avoid confusion, we denote the corresponding operators by s_θ.

Computing commutators of S_θ with various fields, we will also need operators

$$s^{k,l}_\theta[p;q]w\equiv s_\theta[p;q]w,\quad p=(p_1,\ldots,p_k),\ q=(q_1,\ldots,q_l)$$

defined by

$$\begin{aligned} s_\theta[p;q]w = \Big\{\int & \theta_1^{1+k}\phi_1(\theta_1(r-r'))\theta_2^{2+l}\phi_2(\theta_2(y-y')) \\ & [p_1(r,p_+^{-1}(y),t)-p_1(r',p_+^{-1}(y),t)]\cdots[p_k(r,p_+^{-1}(y),t)-p_k(r',p_+^{-1}(y),t)] \\ & [q_1(r',p_+^{-1}(y),t)-q_1(r',p_+^{-1}(y'),t)]\cdots[q_l(r',p_+^{-1}(y),t)-q_l(r',p_+^{-1}(y'),t)] \\ & \chi(y')w(r',y',t)dr'dy'\Big\}(p_+), \end{aligned}$$

or similar integral involving p_-. Here, ϕ_1 and ϕ_2 need not have integral one, and χ is an arbitrary function in $C_0^\infty(\mathbf{R}^2)$. Note that $s_\theta[p;q]$ is automatically normalized to take into account the effects of the factors

$$p_i(r) - p_i(r'), \quad q_j(y) - q_j(y').$$

The continuity of these operators is given in the following Lemma.

Lemma 2.2. — *We have (uniformly in t)*

i) $|s_\theta[p;q]w|_{L^\infty} \leqslant C|w|_{L^\infty}\Pi|\partial_r p_i|_{L^\infty}\Pi|Rq_i|_{L^\infty}$,
ii) $|s_\theta[p;q]w|_{L^2} \leqslant C|w|_{L^2}\Pi|\partial_r p_i|_{L^\infty}\Pi|Rq_i|_{L^\infty}$,
iii) $|s_\theta[p;q]w|_{L^2} \leqslant C|w|_{L^\infty}|\partial_r p_1|_{L^2}\Pi_{i\geqslant 2}|\partial_r p_i|_{L^\infty}\Pi|Rq_i|_{L^\infty}$,
iv) $|s_\theta[p;q]w|_{L^2} \leqslant C|w|_{L^\infty}|Rq_1|_{L^2}\Pi|\partial_r p_i|_{L^\infty}\Pi_{i\geqslant 2}|Rq_i|_{L^\infty}$.

Proof. — The first two points are obvious. To prove iii) or iv), it is enough to consider, for instance, an integral

$$\int \theta^3\phi_2(\theta(y-y'))|b_1(y)-b_1(y')|dy'.$$

Since $|b_1(y)-b_1(y')| \leqslant |y-y'|\int_0^1 |\partial b_1|(y'+s(y-y'))ds$,

$$\Big|\int \theta^3\phi_2(\theta(y-y'))(b_1(y)-b_1(y'))dy'\Big|_{L^2} \leqslant C\int_0^1 ds\int \theta^2\psi(\theta z)|\partial b_1|^2(y'+sz)dy'dz \leqslant C|\partial b_1|^2_{L^2},$$

which gives the result. □

IV.3. Modified Klainerman's fields

a. We define now fields $\widetilde{Z}_m$, analogous to the fields Z_m used in chapter III, but with two important differences:

i) $\widetilde{Z}_m$ has to have smooth coefficients everywhere and not only outside $r=0$.

ii) The perturbation coefficients a from $Z_m = Z + aL_1$ have to be smoothed by S_θ, so as to bear extra derivatives (as occurs typically in a Nash-Moser scheme, see [**5**] for instance).

From now on, we fix, for some

$$\beta_1 > 0,\ \beta_2 > 0,\ \beta_2 \geqslant \beta_1,\ \theta_2^0 \geqslant \theta_1^0 \geqslant 1,\ \varepsilon\beta_2 \leqslant 1$$

to be chosen later,

$$\theta_i = \theta_i(t) = \theta_i^0(1+t)^{\beta_i\varepsilon}.$$

The coefficients $a(R_i), a(S)$ have already been defined in IV. 1. We define $a = a(H_i)$ by

$$a(H_i) = -\omega_i a(S) - (\omega\wedge a(R))_i.$$

We set now (recalling $H_i = ct\partial_i + x_i/c\partial_t$)

$$(3.1)_a \qquad \widetilde{R}_i^m = R_i + \widetilde{a}(R_i)L_1,$$

$$(3.1)_b \qquad \widetilde{S}^m = S + \widetilde{a}(S)L_1,$$

$$(3.1)_c \qquad \widetilde{H}_i^m = H_i + \widetilde{a}(H_i)L_1,$$

$$(3.1)_d \qquad \widetilde{K} = L_1 + L = (2/\sqrt{c})\partial_t.$$

Here

$$\widetilde{a}(R_i) = S_\theta a(R_i), \widetilde{a}(S) = S_\theta a(S),$$

and, for technical reasons,

$$(3.2) \qquad \widetilde{a}(H_i) = -\omega_i \widetilde{a}(S) - (\omega \wedge \widetilde{a}(R))_i.$$

We do not use H_0 since it does not satisfy i). Remark also that

$$(3.3) \qquad \sum \omega_i a(H_i) = -a(S), \quad \sum \omega_i \widetilde{a}(H_i) = -\widetilde{a}(S).$$

Thanks to these choices, we get

$$\frac{r+ct}{\sqrt{c}} L = \sum \omega_i H_i + S = \sum \omega_i \widetilde{H}_i^m + \widetilde{S}^m.$$

The set of the coefficients

$$a(R_i),\ a(S),\ a(H_i)$$

will be denoted by (Coeff), while the set of

$$\widetilde{a}(R_i),\ \widetilde{a}(S),\ \widetilde{a}(H_i)$$

will be denoted by $\widetilde{(\mathrm{Coeff})}$. We will denote by $\widetilde{\Phi}$ the collection of the fields

$$\widetilde{R}_i^m,\ \widetilde{S}^m,\ \widetilde{H}_i^m,\ \widetilde{K},$$

and call $\widetilde{Z}_m$ any of them. Except for $\widetilde{K}$, we will write simply

$$\widetilde{Z}_m = Z + \widetilde{a}L_1,$$

where Z means one of R_i, S, H_i.

b. We denote by $\widetilde{N}_0$ one of the quantities

$$1,\ \varepsilon^{-1}(1+t)\sigma_1^{-1}u,\ \varepsilon^{-1}(1+t)\partial u.$$

Remark that $|\widetilde{N}_0| \leqslant C$. When we want to emphasize the fact that $\widetilde{N}_0$ is not 1 but actually involves u, we write $\widetilde{N}_0'$. We denote by $\widetilde{N}_k$, for $k \geqslant 1$, one of the quantities

$$\varepsilon^{-1}(1+t)\sigma_1^{-1}\widetilde{Z}_m^k u, \quad \varepsilon^{-1}(1+t)\widetilde{Z}_m^k \partial u,$$

$$\sigma_1^{-1}\widetilde{Z}_m^{k-1}\widetilde{a}, \quad \widetilde{Z}_m^{k-1}\widetilde{a}, \quad \widetilde{Z}_m^{k-1}\partial\widetilde{a}, \quad \widetilde{a} \in \widetilde{(\mathrm{Coeff})}.$$

As before, we enlarge a little the definition of f. We will denote by f a smooth function of

$$\varepsilon,\ u,\ \omega,\ \sigma_1(1+t)^{-1},\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i},\ g(q),\ \widetilde{N}_0,\ \nu_i > 0.$$

Here, g is any smooth function whose derivative belongs to $C_0^\infty(\mathbf{R}_+^*)$.

c. We can now express L_1 in terms of the $\widetilde{Z}_m$.

Lemma 3.1. — *We have the relations*

i) $L_1 = f\widetilde{Z}_m$, $L = f/(1+t)\widetilde{Z}_m$,

ii) $Z = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$,

iii) $\sigma_1 L_1 = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$,

iv) $\sigma_1\partial_t = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$, $\sigma_1\partial_i = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$.

Proof

a. From the definition of $\widetilde{K}$, $L_1 = \widetilde{K} - L$. But $L = f\widetilde{Z}_m$, hence i). Writing $Z = \widetilde{Z}_m - \widetilde{a}L_1$ and using i), we get ii).

b. Once again

$$\frac{r-ct}{\sqrt{c}}L_1 = H_0 - S = \sum \omega_i \widetilde{H}_i^m - \widetilde{S}^m + 2\widetilde{a}(S)L_1.$$

As before, we deduce from this $\sigma_1 L_1 = f\widetilde{Z}_m + f\widetilde{N}_1\widetilde{Z}_m$, which is iii).

Finally,

$$\partial_i = f/(1+t)R + fL + fL_1 = \sigma_1^{-1}f(\widetilde{Z}_m - \widetilde{a}L_1) + f/(1+t)\widetilde{Z}_m + f\sigma_1^{-1}(\sigma_1 L_1),$$

which gives iv). □

IV.4. Some calculus Lemmas for the modified fields. — We have to prove the analogue to Lemma III.2.

Lemma 4.1. — *We have the following identities:*

i) $\widetilde{Z}_m^k f = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1+\cdots+k_j \leqslant k$,

ii) $\widetilde{Z}_m^k \widetilde{N}_p = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1+\cdots+k_j \leqslant k+p$, *and, for some* i, $k_i \geqslant p$,

iii) $\widetilde{Z}_m^k t = t\sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1+\cdots+k_j \leqslant k$,

iv) $\widetilde{Z}_m^k \sigma_1 = \sigma_1\sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}$, $k_1+\cdots+k_j \leqslant k$.

Proof

a. We try first i) for $k=1$. For the variables

$$\varepsilon,\ u,\ \omega,\ \sigma_1/(1+t),\ (1+t)^{-\nu_i},\ \sigma_1^{-\nu_i}$$

in f, we only have to check the action of H_i and L. But, analogously to H_0,

$$H_i\omega = f, \quad H_i t = ft, \quad H_i\sigma_1 = \sigma_1 f\widetilde{N}_0, \quad H_i(\sigma_1/(1+t)) = f\widetilde{N}_0$$

and the action of L is at least as good as that of L_1.

Now, with $q = q_0\sigma_1^{-1}\exp C_0\tau$,

$$R_i(q) = 0,\ S(q) = fq,\ H_i(q) = fq\widetilde{N}_0,\ L_1(q) = fq\sigma_1^{-1},\ L(q) = fq\sigma_1^{-1},$$

hence

$$\widetilde{Z}_m(q) = Z(q) + \widetilde{a}L_1(q) = fq\widetilde{N}_0 + fq\widetilde{N}_1,$$

$$\widetilde{Z}_m(g(q)) = qg'(q)f\widetilde{N}_1 = f\widetilde{N}_1.$$

Finally, $\widetilde{Z}_m\widetilde{N}_0 = f + f\widetilde{N}_1$.

b. We have

$$\widetilde{Z}_m\widetilde{N}_p = f\widetilde{N}_1\widetilde{N}_p + \widetilde{N}_{p+1}$$

and ii) is proved for $k = 1$ and any p. **c.** iii) and iv) are clear for $k = 1$. Thus, by induction, i)-iv) are proved. □

We define, for $k \geqslant 1$,

$$M_k = f\widetilde{N}_1^l\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_j}, \quad l \geqslant 0, \quad l_i \geqslant 2, \quad \sum(l_i - 1) \leqslant k - 1.$$

Remark that

$$M_1 = f\widetilde{N}_1^l, \quad M_1M_k = M_k, \quad M_kM_l = M_{k+l-1},$$

and

$$\sum_{(\sum k_i \leqslant k)} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j} = M_k.$$

As in **4** of Proposition III.7, we get easily

$$\widetilde{Z}_mM_k = \sum M_{k+1}, \widetilde{Z}_m^pM_k = \sum M_{k+p}.$$

We will state here for further reference the following commutation Lemmas.

Lemma 4.2. — *We have the formula*

i) $$[\widetilde{Z}_m^k, \partial] = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^p\partial,$$

ii) $$[\widetilde{Z}_m^k, \partial] = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\partial\widetilde{Z}_m^p.$$

In both sums, we have $p \leqslant k-1$, $\sum k_j + p \leqslant k$.

iii) $$[\widetilde{Z}_m^k, \partial] = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^p\partial + \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}(\widetilde{Z}_m^{r_1}A)\cdots(\widetilde{Z}_m^{r_q}A)\widetilde{Z}_m^p\partial.$$

Here, $A = \partial\widetilde{a}$ *or* $A = \sigma_1^{-1}\widetilde{a}$. *In the first sum, we have* $\sum k_j + p \leqslant k-1$. *In the second sum, we have* $q \geqslant 1$, $\sum k_j + \sum(r_i + 1) + p \leqslant k$.

Proof

a. Consider first $k = 1$. While $[R_i, \partial]$, $[S, \partial]$ are just ∂ multiplied by constants, we have

$$[H_i, \partial_j] = -\partial_j(ct)\partial_i - \partial_j(x_i/c)\partial_t = f\partial, [\widetilde{K}, \partial] = f\partial.$$

Now

$$[L_1, \partial] = f\partial u\partial + f(1+t)^{-1}\partial = f(1+t)^{-1}\partial.$$

Hence

$$[\widetilde{a}L_1, \partial] = f(1+t)^{-1}\widetilde{a}\partial + f\partial\widetilde{a}\partial = fA\partial.$$

This proves the Lemma for $k = 1$.

b. We write now

$$[\widetilde{Z}_m^{k+1}, \partial] = \widetilde{Z}_m[\widetilde{Z}_m^k, \partial] + [\widetilde{Z}_m, \partial]\widetilde{Z}_m^k,$$

and consider formula i). We see that the first term yields obviously terms of the desired form, while the second is

$$(f + f\widetilde{N}_1)([\partial, \widetilde{Z}_m^k] + \widetilde{Z}_m^k \partial).$$

This proves i). To prove ii), we write instead

$$\widetilde{Z}_m \partial \widetilde{Z}_m^p = [\widetilde{Z}_m, \partial]\widetilde{Z}_m^p + \partial \widetilde{Z}_m^{p+1}.$$

To prove iii), we see that $\widetilde{Z}_m([\widetilde{Z}_m^k, \partial])$ yields automatically good terms. For $[\widetilde{Z}_m, \partial]\widetilde{Z}_m^k$, we write this term as

$$(f + fA)([\partial, \widetilde{Z}_m^k] + \widetilde{Z}_m^k \partial),$$

which yields only terms of the desired form. □

Lemma 4.3. — *We have*

i) $$[\widetilde{Z}_m, R_j/r] = M_1/(1+t)\widetilde{Z}_m + M_1\sigma_1/(1+t)\partial + f/(1+t)(R_j\widetilde{a})\partial,$$

ii) $$[\widetilde{Z}_m^k, R_j/r] = (1+t)^{-1}M_l\widetilde{Z}_m^{p_1+1} + \sigma_1(1+t)^{-1}M_l\partial\widetilde{Z}_m^{p_1} + \theta_2(1+t)^{-1}M_l(\widetilde{Z}_m^{p_1}s_\theta a)\partial\widetilde{Z}_m^{p_2}.$$

In all terms of formula ii), we have $l - 1 + \sum p_i \leqslant k - 1$.

Proof

a. We have

$$[R_i, R_j] = -\varepsilon_{ijk}R_k, [S, R_j] = 0, [h_i, R_j] = -\varepsilon_{ijk}h_k.$$

Now

$$H_i = h_i + tu\partial_i - x_i u/c\partial_t, \quad [H_i, R_j] = -\varepsilon_{ijk}h_k + ftu\partial + ftRu\partial.$$

But

$$h_k = H_k - tu\partial_k + x_k u/c\partial_t = fR + \omega_k H_0 + ftu\partial, H_0 = f\sigma_1\partial + f\widetilde{Z}_m,$$

hence

$$[H_i, R_j/r] = fR/r + M_1\sigma_1/(1+t)\partial + f/(1+t)\widetilde{Z}_m$$

and the same is true for the other Z as well. Finally,

$$[\widetilde{a}L_1, R_j/r] = -R_j\widetilde{a}/rL_1 + \widetilde{a}[L_1, R_j/r],$$
$$[L_1, R_j/r] = fR/r^2 + f\varepsilon\sigma_1/(1+t)^2\widetilde{N}_1\partial,$$

which gives i), which is also ii) for $k = 1$, since $R_j\widetilde{a} = f\theta_2 s_\theta a$.

b. Since $[\widetilde{Z}_m, \partial] = M_1\partial$, ii) follows by induction from the properties of the M_k. □

IV.5. Some commutation Lemmas for the modified fields

Lemma 5.1. — *We have the formula*

$$\text{i)}\qquad [\partial_t, S_\theta]w = \frac{\theta_1'}{\theta_1}s_\theta w + \frac{\theta_2'}{\theta_2}s_\theta w,$$

$$\text{ii)}\qquad [\partial_t, S_\theta]w = \frac{\theta_1'}{\theta_1^2}s_\theta\partial_r w + \frac{\theta_2'}{\theta_2^2}(s_\theta w + s_\theta fRw),$$

$$\text{iii)}\qquad [b, S_\theta]w = \theta_1^{-1}s_\theta[b;]w + \theta_2^{-1}s_\theta[;b]w,$$

$$\text{iv)}\quad \theta_1[\partial_i, S_\theta]w = fs_\theta f\partial w + fs_\theta[;h(\omega)]f\partial w + fs_\theta[;h(\omega)]fw(1+t)^{-1} + fs_\theta[;h(\omega)w(1+t)^{-1}]1,$$

$$\text{v)}\qquad [\widetilde{Z}_m, S_\theta]w = f\theta_1^{-1}s_\theta[f\widetilde{N}_1^l;]M_1\widetilde{Z}_m w + f\theta_2^{-1}s_\theta[;f\widetilde{N}_1^l]M_1\widetilde{Z}_m w + f\widetilde{N}_1\theta_1^{-1}s_\theta f\partial w + f\widetilde{N}_1\theta_2^{-1}s_\theta w + f\widetilde{N}_1\theta_2^{-1}s_\theta M_1\widetilde{Z}_m w + f\theta_2^{-1}s_\theta[;fw]1.$$

Proof

a. We have

$$\begin{aligned}[\partial_t, S_\theta^1]w &= \frac{\theta'}{\theta}\int \theta(r\phi_1)_r(\theta(r-r'))w(r')dr' \\ &= -\frac{\theta'}{\theta}\int \partial_{r'}[(r\phi_1)(\theta(r-r'))]w(r')dr' \\ &= \frac{\theta'}{\theta^2}\int \theta(r\phi_1)(\theta(r-r'))\partial_r w(r')dr'.\end{aligned}$$

Similarly,

$$\begin{aligned}[\partial_t, S_\theta^2]w &= \theta'/\theta\{\int \theta^2[2\phi_2 + y\partial\phi_2](\theta(y-y'))(\chi_+ w)(p_-^{-1}(y'))dy'\}(p_-) + \cdots \\ &= \theta'/\theta\{\int \theta\sum\partial_j[(y_j\phi_2)(\theta(y-y'))](\chi_+ w)(p_-^{-1}(y'))dy'\}(p_-) + \cdots \\ &= \theta'/\theta^2\{\sum\int \theta^2(y_j\phi_2)(\theta(y-y'))\partial_j[(\chi_+ w)(p_-^{-1}(y'))]dy'\}(p_-) + \cdots.\end{aligned}$$

This gives the formula i) and ii). **b.** Let $p'_-R_i = \sum\alpha_i^j\partial_{y_j}$. We have

$$\begin{aligned}\partial_j\int\phi_{2,\theta}h(y')dy' &= \int\phi_{2,\theta}(\partial_j h)(y')dy', \\ \alpha_i^j(y)\partial_j\int\phi_{2,\theta}h(y')dy' &= \int\phi_{2,\theta}(\alpha_i^j(y)-\alpha_i^j(y'))(\partial_j h)(y')dy' + \int\phi_{2,\theta}(\alpha_i^j\partial_j h)(y')dy',\end{aligned}$$

hence, with $h(y) = (\chi_+ w)(p_-^{-1}(y))$,

$$[R_i, S_\theta^2]w = \theta^{-1}\Big\{\int \theta\phi_{2,\theta}(\alpha_i^j(y) - \alpha_i^j(y'))\partial_j h(y')dy'\Big\}(p_-)$$
$$- \theta^{-1}\Big\{\int \theta\phi_{2,\theta}[(wR_i(\chi_+))(p_-^{-1}(y)) - (wR_i(\chi_+))(p_-^{-1}(y'))]dy'\Big\}(p_-) + \cdots$$
$$+ wR_i(\chi_+ + \chi_-),$$

the dots meaning a similar term with p_+ and the last term being zero since we have a partition of unity. Since R commutes with S_θ^1, we obtain

$$\theta_2[R_i, S_\theta]w = s_\theta[;h]w + s_\theta[;h]fRw + s_\theta[;hw]1,$$

where $h = h(\omega)$ stands for various smooth functions of ω.

c. Now remark that, for any function b,

$$b(r,y)\int \theta_1\phi_1(\theta_1(r-r'))\theta_2^2\phi_2(\theta_2(y-y'))w(r',y')dr'dy'$$
$$= \int \dots [(b(r,y) - b(r',y)) + (b(r',y) - b(r',y')) + b(r',y')]w(r',y')dr'dy',$$

which gives iii).

d. Since

$$\partial_i = \omega_i\partial_r - 1/r(\omega\wedge R)_i,$$

we have

$$[\partial_i, s_\theta]w = [\omega_i, s_\theta]w_r + f/(1+t)[R, s_\theta]w + f/(1+t)[\omega, s_\theta]Rw + [1/r, s_\theta]fRw.$$

Since

$$1/r - 1/r' = -(r-r')/rr', \quad [1/r, s_\theta]w = (\theta_1 r)^{-1}s_\theta(w/r),$$

we obtain iv). **e.** With the above formula, we also obtain, using Lemma 2,

$$[\widetilde{a}L_1, S_\theta]w = \widetilde{a}/\sqrt{c}[\partial_t, S_\theta]w + [\widetilde{a}/\sqrt{c}, S_\theta]\partial_t w - [\widetilde{a}\sqrt{c}, S_\theta]\partial_r w.$$

Now, since

$$(b(r,y) - b(r',y))/\sigma_1(r') = b(r,y)/\sigma_1(r) - b(r',y)/\sigma_1(r') + (r'-r)b(r,y)/\sigma_1(r)\sigma_1(r'),$$

$$s_\theta[p;q](w/\sigma_1) = s_\theta[p_1,\dots,p_{i-1},p_i/\sigma_1,p_{i+1},\dots,p_k;q]w$$
$$+ p_i/\sigma_1 s_\theta[p_1,\dots,\widehat{p_i},\dots,p_k;q](w/\sigma_1),$$
$$s_\theta[p;q](w/\sigma_1) = s_\theta[p;q_1,\dots,q_i/\sigma_1,\dots,q_l]w.$$

Here, $\widehat{p_i}$ means as usual that p_i is omitted. We may write sometimes

$$s_\theta[\dots,p_i/\sigma_1,\dots;q]$$

instead of the correct

$$s_\theta[p_1,\dots,p_i/\sigma_1,\dots,p_k;q],$$

the ... meaning that the non-written terms are unchanged. We thus obtain for instance

$$[b,S_\theta]\partial w = \theta_1^{-1}s_\theta[b;]\partial w + \theta_2^{-1}s_\theta[;b]\partial w$$
$$= \theta_1^{-1}s_\theta[b/\sigma_1;]\sigma_1\partial w + b/\sigma_1 s_\theta\partial w + \theta_2^{-1}s_\theta[;b/\sigma_1]\sigma_1\partial w.$$

To summarize, using ii), we get

$$[\widetilde{a}L_1,S_\theta]w = f/\theta_1 s_\theta[f\widetilde{N}_1;]M_1\widetilde{Z}_m w + f/\theta_2 s_\theta[;f\widetilde{N}_1]M_1\widetilde{Z}_m w$$
$$+ f\widetilde{N}_1/\theta_1 s_\theta f\partial w + f/\theta_2 s_\theta M_1\widetilde{Z}_m w.$$

f. We also have

$$[S,S_\theta]w = t[\partial_t,S_\theta]w + [r,S_\theta]\partial_r w.$$

Since $[r,S_\theta]w = \theta_1^{-1}s_\theta w$,

$$[S,S_\theta]w = f/\theta_1 s_\theta f\partial w + f/\theta_2(s_\theta w + s_\theta M_1\widetilde{Z}_m w).$$

Similarly

$$[H_0,S_\theta]w = [c,S_\theta]t\partial_r w + [c^{-1},S_\theta]r\partial_t w + r/c[\partial_t,S_\theta]w + 1/c[r,S_\theta]\partial_t w$$
$$= f/\theta_1 s_\theta[tu;]\sigma_1^{-1}f\sigma_1\partial w + f/\theta_2 s_\theta[;f]M_1\widetilde{Z}_m w + f/\theta_1 s_\theta f\partial w$$
$$+ f/\theta_2(s_\theta w + s_\theta fRw).$$

Since, using the formula of **d.**,

$$s_\theta[tu;]v/\sigma_1 = s_\theta[f;]v + fs_\theta(v/\sigma_1),$$

we obtain again

$$[H_0,S_\theta]w = f/\theta_1 s_\theta[f;]M_1\widetilde{Z}_m w + f/\theta_1 s_\theta f\partial w + f/\theta_2 s_\theta[;f]M_1\widetilde{Z}_m w$$
$$+ f/\theta_2(s_\theta w + s_\theta fRw).$$

Since $H_i = \omega_i H_0 - ct/r(\omega\wedge R)_i$, we get the same formula for $[H_i,S_\theta]$, with the additional term $f/\theta_2 s_\theta[;fw]1$. This completes the proof. □

Lemma 5.2. — *We have the formula*

i) $$[\widetilde{Z}_m, s_\theta[p;q]]w = M_1 s_\theta[p;q]w + M_1/\theta_1 s_\theta[p;q]f\partial w + f/\theta_1 s_\theta[p;q]M_1\widetilde{Z}_m w$$
$$+\sum M_1 s_\theta[p_1,\dots,M_1\widetilde{Z}_m p_j,\dots,p_k;q]w + \sum M_1 s_\theta[p;q_1,\dots,M_1\widetilde{Z}_m q_j,\dots,q_l]w$$
$$+f/\theta_1 s_\theta[p,M_1;q]M_1\widetilde{Z}_m w + f/\theta_2 s_\theta[p;q,M_1]M_1\widetilde{Z}_m w$$
$$+\sum M_1 s_\theta[p_1,\dots,\widehat{p}_j,\dots p_k;q]f(\partial p_j)w$$
$$+\sum M_1/\theta_1 s_\theta[p;q_1,\dots,f\partial q_j,\dots,q_l]w + \sum M_1/\theta_2 s_\theta[p_1,\dots,\widehat{p}_j,\dots,p_k;q,f\partial p_j]w.$$

ii) $$\widetilde{Z}_m s_\theta w = M_1 s_\theta M_1\widetilde{Z}_m^r w, r\leqslant 1,$$

iii) $$\widetilde{Z}_m s_\theta[p;]w = \sum_{r_1+r_2\leqslant 1} M_1 s_\theta[M_1\widetilde{Z}_m^{r_1}p;]M_1\widetilde{Z}_m^{r_2}w + M_1 s_\theta(f\partial p)w + M_1 s_\theta[;f\partial p]w,$$

$$\text{iv)}\qquad \widetilde{Z}_m s_\theta[;q]w = \sum_{r_1+r_2\leqslant 1} s_\theta[;M_1\widetilde{Z}_m^{r_1}q]M_1\widetilde{Z}_m^{r_2}w + M_1 s_\theta(f\partial q)w + M_1 s_\theta[;f\partial q]w.$$

$$\text{v)}\qquad \widetilde{Z}_m^k s_\theta[;q]w = s_\theta[;M_{l_0}\widetilde{Z}_m^{r_1+p_1}q]M_{l_1}\widetilde{Z}_m^{r_2+p_2}w + M_{l_0}s_\theta M_{l_1}\widetilde{Z}_m^{r_1+p_1}q\widetilde{Z}_m^{r_2+p_2}w.$$

In formula v), we have for both terms

$$r_1+r_2\leqslant 1,\quad \textstyle\sum(l_i-1)+p_1+p_2\leqslant k-1.$$

Proof. — We prove only the delicate formula i), and v), the other formula ii), iii) and iv) being proved more easily along the same lines. We need only to get terms involving $\widetilde{Z}_m p, \widetilde{Z}_m q$ or terms in $\widetilde{Z}_m w$ with a small factor in front. **a.** We have easily the formula

$$\begin{aligned}[\partial_t, s_\theta[p;q]]w &= f/(1+t)s_\theta[p;q]w + \textstyle\sum s_\theta[p_1,\dots,\partial_t p_i,\dots,p_k;q]w\\ &\qquad\qquad + \textstyle\sum s_\theta[p;q_1,\dots,\partial_t q_i,\dots,q_l]w,\\ [\partial_r, s_\theta[p;q]]w &= \textstyle\sum s_\theta[p_1,\dots,\partial_r p_i,\dots,p_k;q]w + \sum s_\theta[p;q_1,\dots,\partial_r q_i,\dots,q_l]w.\end{aligned}$$

Also

$$[b, s_\theta[p;q]]w = \theta_1^{-1}s_\theta[b,p;q]w + \theta_2^{-1}s_\theta[p;b,q]w.$$

b. Since

$$\begin{aligned}&\partial_{y_j}\int \theta^{2+l}\phi_2(\theta(y-y'))(q_1(y)-q_1(y'))\cdots(q_l(y)-q_l(y'))h(y')dy' =\\ &\textstyle\sum \int \theta^{2+l}\phi_2(\theta(y-y'))(q_1(y)-q_1(y'))\cdots(\partial_j q_i(y)-\partial_j q_i(y'))\cdots(q_l(y)-q_l(y'))h(y')dy'\\ &\qquad + \int \theta^{2+l}\phi_2(\theta(y-y'))(q_1(y)-q_1(y'))\cdots(q_l(y)-q_l(y'))\partial_j h(y')dy',\end{aligned}$$

we need only push coefficients α_i^j through the last integral. We thus obtain as in **b**, Lemma 5.1,

$$\begin{aligned}[R_i, s_\theta[p;q]]w &= fs_\theta[p;q]w + \textstyle\sum fs_\theta[p_1,\dots,fRp_i,\dots p_k;q]w\\ &\qquad + \textstyle\sum fs_\theta[p;q_1,\dots,fRq_i,\dots q_l]w + \theta_2^{-1}s_\theta[p;h(\omega),q]fRw.\end{aligned}$$

This is of the desired form.

c. Similarly, since

$$r(p(r,y)-p(r',y)) = rp(r,y) - r'p(r',y) + (r'-r)(p(r',y)-p(r',y')+p(r',y')),$$

we obtain

$$\begin{aligned}rs_\theta[p;q]w &= s_\theta[p_1,\dots,rp_i,\dots,p_k;q]w\\ &\qquad + \theta_2^{-1}s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q,p_i]w + s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q]p_i w,\end{aligned}$$

and trivially

$$rs_\theta[p;q]w = \theta_1^{-1}s_\theta[p;q]w + s_\theta[p;q_1,\dots,rq_i,\dots,q_l]w.$$

Hence

$$[S, s_\theta[p;q]]w = fs_\theta[p;q]w + \theta_1^{-1}s_\theta[p;q]w_r + \sum\Big\{s_\theta[p_1,\dots,Sp_i,\dots p_k;q]w + s_\theta[p;q_1,\dots,Sq_j,\dots,q_l]w + s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q](\partial_r p_i)w + \theta_2^{-1}s_\theta[p_1,\dots,\widehat{p}_i,\dots,p_k;q,\partial_r p_i]w\Big\}.$$

This is of the desired form.

d. We have now

$$H_0 s_\theta[p;q]w = cs_\theta[p;q]t\partial_r w + r/cs_\theta[p;q]\partial_t w + fs_\theta[p;q]w + \sum\Big\{cs_\theta[\dots t\partial_r p_j\dots;q]w + r/cs_\theta[\dots\partial_t p_j\dots;q]w + cs_\theta[p;\dots t\partial_r q_j\dots]w + r/cs_\theta[p;\dots\partial_t q_j\dots]w\Big\}.$$

Since it is technically awkward to commute $1/c$ with s_θ, we proceed slightly differently. We write for instance

$$(r/c)s_\theta[\dots\partial_t p_j\dots;q]w = rcs_\theta[\dots\partial_t p_j\dots;q]w + f(1+t)us_\theta[\dots\partial_t p_j\dots;q]w.$$

Using the formula

$$\sigma_1 s_\theta[p;q]v = s_\theta[\dots,\sigma_1 p_j,\dots;q]v + s_\theta[\dots,\widehat{p}_j,\dots;q]p_j v + 1/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,p_j]v,$$
$$\sigma_1 s_\theta[p;q]v = s_\theta[p;\dots,\sigma_1 q_j,\dots]v + 1/\theta_1 s_\theta[p;q]v,$$

we obtain for the same typical term, remembering that $(1+t)u/\sigma_1$ is an f,

$$f(1+t)us_\theta[\dots,\partial_t p_j,\dots;q]w = fs_\theta[\dots,\sigma_1\partial_t p_j,\dots;q]w + fs_\theta[\dots,\widehat{p}_j,\dots;q](\partial_t p_j)w + f/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,\partial_t p_j]w.$$

Similarly, we have

$$f(1+t)us_\theta[p;\dots,\partial_t q_j,\dots]w = fs_\theta[p;\dots,\sigma_1\partial_t q_j,\dots]w + f/\theta_1 s_\theta[p;\dots,\partial_t q_j,\dots]w.$$

Using the formula of **c.** to move r to one of the factors p or q, we get

$$H_0 s_\theta[p;q]w = s_\theta[p;q]H_0 w + \sum fs_\theta[\dots,M_1\widetilde{Z}_m p_j,\dots;q]w + \sum fs_\theta[p;\dots,M_1\widetilde{Z}_m q_j,\dots]w + fs_\theta[p;q]w + f/\theta_1 s_\theta[p;q]\partial_t w + \sum\Big\{fs_\theta[\dots,\widehat{p}_j,\dots;q](\partial_t p_j)w + f/\theta_1 s_\theta[p;\dots,\partial_t q_j,\dots]w + f/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,\partial_t p_j]w\Big\} + f/\theta_1 s_\theta[u,p;q]f(1+t)\partial w + f/\theta_2 s_\theta[p;q,u]f(1+t)\partial w.$$

To see $(1+t)u/\sigma_1$ instead of u in the last two terms, we use the formula

$$s_\theta[\dots,\sigma_1 p_j,\dots;q]v = p_j s_\theta[\dots,\widehat{p}_j,\dots;q]v + s_\theta[p;q]\sigma_1 v,$$
$$s_\theta[p;\dots,\sigma_1 q_j,\dots]v = s_\theta[p;q]\sigma_1 v.$$

We thus obtain the desired form for $[H_0, s_\theta[p;q]]$.

e. Finally, we write

$$\begin{aligned}\widetilde{a}L_1 s_\theta[p;q]w &= f\widetilde{N}_1 s_\theta[p;q]w + \widetilde{a}/\sqrt{c}s_\theta[p;q]\partial_t w - \widetilde{a}\sqrt{c}s_\theta[p;q]\partial_r w \\ &\quad + \sum\Big\{\widetilde{a}/\sqrt{c}s_\theta[\dots,\partial_t p_j,\dots;q]w - \widetilde{a}\sqrt{c}s_\theta[\dots,\partial_r p_j,\dots;q]w \\ &\qquad + \widetilde{a}/\sqrt{c}s_\theta[p;\dots,\partial_t q_j,\dots]w - \widetilde{a}\sqrt{c}s_\theta[p;\dots,\partial_r q_j,\dots]w\Big\}.\end{aligned}$$

To see the term $\widetilde{a}L_1 w$, we will consider for instance

$$[\widetilde{a}/\sqrt{c}, s_\theta[p;q]]\partial_t w = 1/\theta_1 s_\theta[\widetilde{a}/\sqrt{c},p;q]\partial_t w + 1/\theta_2 s_\theta[p;q,\widetilde{a}/\sqrt{c}]\partial_t w.$$

In order to see $\widetilde{a}/\sigma_1 = \widetilde{N}_1$ instead of $\widetilde{a}$, we have to move around σ_1 using the formula of **d.** We get

$$\widetilde{N}_1/\theta_1 s_\theta[p;q]\partial_t w + 1/\theta_1 s_\theta[M_1,p;q]M_1\widetilde{Z}_m w + 1/\theta_2 s_\theta[p;q,M_1]M_1\widetilde{Z}_m w.$$

The computation is analogous with the term containing $\partial_r w$. To handle a term like

$$\widetilde{a}/\sqrt{c}s_\theta[\dots,\partial_t p_j,\dots;q]w,$$

we again have to move around σ_1: this term is equal to

$$\begin{aligned}f\widetilde{N}_1\sigma_1 s_\theta[\dots,\partial_t p_j,\dots;q]w &= f\widetilde{N}_1 s_\theta[\dots,\sigma_1\partial_t p_j,\dots;q]w \\ &\quad + f\widetilde{N}_1 s_\theta[\dots,\widehat{p}_j,\dots;q](\partial_t p_j)w + f\widetilde{N}_1/\theta_2 s_\theta[\dots,\widehat{p}_j,\dots;q,\partial_r p_j]w,\end{aligned}$$

and a similar expression for the terms involving $\partial_r p_j, \partial_t q_j, \partial_r q_j$.

To complete the proof, we note that $H_i = \omega_i H_0 + fR$, hence

$$[H_i, s_\theta[p;q]] = \omega_i[H_0, s_\theta[p;q]] + [\omega_i, s_\theta[p;q]]H_0 + f[R, s_\theta[p;q]] + [f, s_\theta[p;q]]R,$$

which yields only terms of the desired form. Finally,

$$\widetilde{K}s_\theta[p;q]w = 2/\sqrt{c}\{s_\theta[p;q]\partial_t w + s_\theta[\dots,\partial_t p_j,\dots;q]w + s_\theta[p;\dots,\partial_t q_j,\dots]w\}.$$

We could write $\partial_t w = f\widetilde{K}w$ and this would be enough for what we have in mind, but since we want a commutator, we proceed differently. We write

$$2/\sqrt{c}s_\theta[p;q]\partial_t w = s_\theta[p;q]\widetilde{K}w + (2/\sqrt{c}-2)s_\theta[p;q]\partial_t w + s_\theta[p;q](2-2/\sqrt{c})\partial_t w,$$

and again move around σ_1 in the first term to see u/σ_1 and $\sigma_1\partial w$. We obtain terms of the desired form with a gain of $1/(1+t)$ instead of θ_i^{-1}, and this is enough to complete the proof of i).

f. To prove v), we note that it is true for $k=1$, since $f\partial = M_1\widetilde{Z}_m$. Applying $\widetilde{Z}_m$ to v) and using ii) and iv), we get the formula by induction. □

Lemma 5.3. — *For all k, we have the formula*

$$\begin{aligned}[\widetilde{Z}_m^k, S_\theta]w = \theta_1^{-1}\Big\{\sum M_{l_0}s_\theta[M_{l_1};]M_{l_2}\widetilde{Z}_m^{p+r}w + M_{l_0}s_\theta[;M_{l_1}]M_{l_2}\widetilde{Z}_m^{p+r}w \\ + M_{l_0}s_\theta M_{l_1}\widetilde{Z}_m^{p+r}w + M_{l_0}s_\theta[;M_{l_1}\widetilde{Z}_m^{p}w]M_{l_2}\Big\}.\end{aligned}$$

In all terms, we have

$$r \leqslant 1, \quad \sum(l_i - 1) + p \leqslant k - 1.$$

Proof

a. We will prove by induction that $\widetilde{Z}_m^{k-1}[\widetilde{Z}_m, S_\theta]w$ is equal to the right hand side with the same conditions

$$r \leqslant 1, \quad \sum(l_i - 1) + p \leqslant k - 1.$$

For $k = 1$, by inspection of the formula for $[\widetilde{Z}_m, S_\theta]$ in Lemma 5.1, we see that this is true, since $\theta_1 \leqslant \theta_2$.

Assuming that this is true for k, we write $\widetilde{Z}_m^k = \widetilde{Z}_m \widetilde{Z}_m^{k-1}$ and examine the various terms. We have, using formula iii) of Lemma 5.2,

$$\widetilde{Z}_m s_\theta[M_{l_1};]M_{l_2}\widetilde{Z}_m^{p+r}w = M_1 s_\theta[M_1\widetilde{Z}_m^{p_1}(M_{l_1});]M_1\widetilde{Z}_m^{p_2}(M_{l_2}\widetilde{Z}_m^{p+r}w)$$
$$+ M_1 s_\theta(M_1\widetilde{Z}_m(M_{l_1})M_{l_2}\widetilde{Z}_m^{p+r}w) + M_1 s_\theta[;M_1\widetilde{Z}_m(M_{l_1})]M_{l_2}\widetilde{Z}_m^{p+r}w.$$

Since $\widetilde{Z}_m^q M_p = M_{p+q}$, we get

$$= M_1 s_\theta[M_{l_1+p_1};](M_{l_2+p_2}\widetilde{Z}_m^{p+r}w + M_{l_2}\widetilde{Z}_m^{p+p_2+r}w)$$
$$+ M_1 s_\theta(M_{l_1+l_2}\widetilde{Z}_m^{p+r}w) + M_1 s_\theta[;M_{l_1+1}]M_{l_2}\widetilde{Z}_m^{p+r}w.$$

Taking into account that

$$\widetilde{Z}_m(M_l\theta_1^{-1}) = \theta_1^{-1}M_{l+1},$$

we see that the action of $\widetilde{Z}_m$ on the first term of the right-hand side yields terms of the desired form. The issue is completely similar with the second term, and easy for the third. For the last term, we write

$$\widetilde{Z}_m s_\theta[;M_{l_1}\widetilde{Z}_m^p w]M_{l_2} = M_1 s_\theta[;M_{l_1+p_1}\widetilde{Z}_m^p w + M_{l_1}\widetilde{Z}_m^{p+p_1}w]M_{l_2}$$
$$+ M_1 s_\theta(M_{l_1+l_2}\widetilde{Z}_m^p w + M_{l_1+l_2-1}\widetilde{Z}_m^{p+1}w) + M_1 s_\theta[;M_{l_1+1}\widetilde{Z}_m^p w + M_{l_1}\widetilde{Z}_m^{p+1}w]M_{l_2},$$

and see that all terms are of the desired form.

b. Since

$$[\widetilde{Z}_m^k, S_\theta]w = \sum_{1\leqslant l\leqslant k} \widetilde{Z}_m^{k-l}[\widetilde{Z}_m, S_\theta]\widetilde{Z}_m^{l-1}w,$$

we see that this term is a sum of terms of the desired form with

$$\sum(l_i - 1) + p + l_1 \leqslant k - l + l - 1 = k - 1.$$

This completes the proof. □

Later on, we will need the following pseudo-commutator formula.

Lemma 5.4. — *We have the formula*

$$\widetilde{Z}_m^k s_\theta w = f s_\theta f \widetilde{Z}_m^k w + \theta_1\sigma_1 M_{l_0} s_\theta M_{l_1}\widetilde{Z}_m^p w + \theta_2^{-1}M_{l_0}s_\theta[;M_{l_1}]M_{l_2}\widetilde{Z}_m^{p+r}w$$
$$+ \theta_2^{-1}M_{l_0}s_\theta M_{l_1}\widetilde{Z}_m^{p+r}w + \theta_2^{-1}M_{l_0}s_\theta[;M_{l_1}\widetilde{Z}_m^p w]M_{l_2}.$$

In these sums,

$$r \leqslant 1, \quad \sum(l_i - 1) + p \leqslant k - 1.$$

Proof

a. We consider first $k = 1$, and set $\partial_t w + c\partial_r w = g$. As in the proof of Lemma 5.1, we have

$$[S, s_\theta]w = t[\partial_t, s_\theta]w + \theta_1^{-1} s_\theta w_r = f s_\theta w + s_\theta w,$$

since, by integration by parts, $s_\theta w_r = \theta_1 s_\theta w$. Next,

$$[H_0, s_\theta]w = [c, s_\theta]tw_r + [c^{-1}, s_\theta]rw_t + r/(ct)s_\theta w + c^{-1}\theta_1^{-1} s_\theta w_t.$$

First,

$$us_\theta w_r = u\theta_1 s_\theta w, s_\theta u w_r = s_\theta(uw)_r - s_\theta u_r w = \theta_1 s_\theta uw - s_\theta u_r w,$$
$$[c, s_\theta]w_r = [u, s_\theta]w_r = \theta_1[u, s_\theta]w + s_\theta u_r w.$$

Second,

$$[1/c, s_\theta]h = -1/c[u, s_\theta](h/c),$$
$$[u, s_\theta]rw_t/c = [u, s_\theta](rg/c - (rw)_r + w) = f(1+t)[u, s_\theta]fg + \theta_1 f(1+t)[u, s_\theta]fw.$$

Finally,

$$\theta_1^{-1} s_\theta w_t = \theta_1^{-1} s_\theta(g - (cw)_r + u_r w) = \theta_1^{-1} s_\theta g + s_\theta f w + \theta_1^{-1} s_\theta u_r w.$$

Collecting the terms, we obtain

$$[H_0, s_\theta]w = f s_\theta f w + f s_\theta f g + f\theta_1(1+t)[u, s_\theta]fw + f(1+t)[u, s_\theta]fg.$$

On the other hand,

$$\begin{aligned}\widetilde{a}L_1 s_\theta w &= f\widetilde{a}[\partial_t, s_\theta]w + f\widetilde{a}s_\theta(g - (cw)_r + u_r w) + f\widetilde{a}\theta_1 s_\theta w\\ &= \theta_1 M_1 s_\theta f w + f\widetilde{N}_1 s_\theta g,\\ s_\theta(\widetilde{a}L_1 w) &= s_\theta(f\widetilde{N}_1 g) + f\theta_1 s_\theta(f\widetilde{N}_1 w).\end{aligned}$$

Finally,

$$H_i s_\theta w = f[H_0, s_\theta]w + f s_\theta H_0 w + f[R, s_\theta]w + f s_\theta R w.$$

Collecting the terms, we obtain

$$\begin{aligned}\widetilde{Z}_m s_\theta w = f s_\theta f \widetilde{Z}_m w &+ \theta_1 M_1 s_\theta M_1 w + \theta_1 M_1(1+t)[u, s_\theta]M_1 w\\ &+ f\widetilde{N}_1 s_\theta g + s_\theta(f\widetilde{N}_1^r g) + f(1+t)[u, s_\theta]fg\\ &\qquad + f\theta_2^{-1}(s_\theta[;f]M_1\widetilde{Z}_m^r w + s_\theta[;fw]1).\end{aligned}$$

Opening the commutator term on w, we find

$$\theta_1 M_1(1+t)[u, s_\theta]M_1 w = \theta_1\sigma_1 M_1 s_\theta M_1 w.$$

Remember now that

$$g = \sqrt{c}Lw = f(1+t)^{-1}\widetilde{Z}_m w.$$

Hence

$$f(1+t)[u,s_\theta]fg = fs_\theta f\widetilde{Z}_m w,$$

and we replace also g by this value in the two other terms containing g. We thus obtain

$$\widetilde{Z}_m s_\theta w = fs_\theta f\widetilde{Z}_m w + (1+t)^{-1} f\widetilde{N}_1^{r_1} s_\theta f\widetilde{N}_1^{r_2}\widetilde{Z}_m w + \theta_1\sigma_1 M_1 s_\theta M_1 w + f\theta_2^{-1}(s_\theta[;f]M_1\widetilde{Z}_m^r w + s_\theta[;fw]1).$$

Thus the result is true for $k=1$, the second term in the right-hand side being of the form

$$\theta_2^{-1} M_1 s_\theta M_1 \widetilde{Z}_m w$$

since $(1+t)^{-1} = f\theta_2^{-1}$.

b. Let us assume now the formula for $\widetilde{Z}_m^l s_\theta w$, $l \leqslant k$. We obtain

$$\begin{aligned}\widetilde{Z}_m^{k+1} s_\theta w = \widetilde{Z}_m(\widetilde{Z}_m^k s_\theta w) &= M_1 s_\theta f\widetilde{Z}_m^k w + f(fs_\theta f\widetilde{Z}_m(f\widetilde{Z}_m^k w) \\ &+ \theta_1\sigma_1 M_1 s_\theta M_1(f\widetilde{Z}_m^k w)) + f\theta_2^{-1}(M_1 s_\theta[;M_1]M_1\widetilde{Z}_m^r(f\widetilde{Z}_m^k w) \\ &+ M_1 s_\theta M_1 f\widetilde{Z}_m^k w + M_1 s_\theta[;M_1\widetilde{Z}_m^k w]M_1) + \theta_1\sigma_1(M_1M_{l_0} + M_{l_0+1})s_\theta M_{l_1}\widetilde{Z}_m^p w \\ &+ \theta_1\sigma_1 M_{l_0}(M_1 s_\theta M_1 \widetilde{Z}_m^r(M_{l_1}\widetilde{Z}_m^p w)) + \widetilde{Z}_m(\cdots + \cdots + \cdots).\end{aligned}$$

The last three terms are identical to the corresponding terms in the proof of Lemma 5.3, we need not redo the computation. All other terms are easily seen to be of the desired form. □

IV.6. L^∞ estimates of the quantities $\widetilde{N}_k$

Proposition 6. — *Fix $\mu > 1/2$. For η small enough, θ_1^0 and β_1 big enough, we have the estimates (except of course for $\widetilde{N}_0 = 1$)*

i) $$|\widetilde{N}_k| \leqslant C(1+t)^{C_1\varepsilon}\sigma_1^{\mu-1}, \quad k \leqslant s_0 - 4.$$

ii) $$|\widetilde{Z}_m^{k-1} a| + |\widetilde{Z}_m^{k-1}\partial a| \leqslant C(1+t)^{C_1\varepsilon}, \quad k \leqslant s_0 - 4.$$

Here, C_1 does not depend on the θ_i.

Proof. — From Proposition III.7 and Proposition IV.1, we know that, for $k \leqslant s_0 - 4$,

$$\varepsilon^{-1}(1+t)|\sigma_1^{-1} Z_0^k u|, \quad \varepsilon^{-1}(1+t)|Z_0^k \partial u|, \quad \sigma_1^{-1}|Z_0^{k-1} a|, |Z_0^{k-1}\partial a|$$

are bounded by $C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}$, for $a = a(R_i), a(S)$. These estimates extend easily also to $a = a(H_i) = -\omega_i a(S) - (\omega \wedge a(R))_i$. Remark that, since a are supported in

$$\sigma_1 \leqslant C(1+t)^{C_0\varepsilon},$$

we can ignore the powers of σ_1 in estimates involving a.

a. First we estimate $\widetilde{N}_1$. From the properties of S_θ and Lemma 5.1, we get

$$|\widetilde{a}| \leqslant C(1+t)^{C\varepsilon}, \quad |\partial\widetilde{a}| \leqslant C|a|_{L^\infty} + C|\partial a|_{L^\infty} \leqslant C(1+t)^{C\varepsilon}.$$

Now
$$\widetilde{Z}_m = fN_0Z_0 + f\sigma_1^{-1}\widetilde{a}Z_0$$
gives the control of the terms of $\widetilde{N}_1$ involving u. Remark also
$$|\widetilde{Z}_m a| \leqslant C(1+t)^{C\varepsilon}|Z_0 a|_{L^\infty} \leqslant C(1+t)^{C\varepsilon}.$$

b. We need to establish an analogue to Lemma 1.1, including some refinement using the fact that u and a do not play the same role in the process of estimating the $\widetilde{N}_k$.

Lemma 6.1. — *We have the formula*

i) $\widetilde{Z}_m^k = \sum f\widetilde{N}_1^l\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}Z_0^p,$

ii) $\widetilde{Z}_m^k = \displaystyle\sum_{\substack{p\geqslant 1\\ \sum k_i+p\leqslant k}} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}Z_0^p + \sum_{q\geqslant 1} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^{r_1}(\widetilde{a}/\sigma_1)\cdots\widetilde{Z}_m^{r_q}(\widetilde{a}/\sigma_1)Z_0^p.$

In the sum of i), we have $p\geqslant 1$, $k_i\geqslant 2$, $\sum(k_i-1)+p\leqslant k$. In the second sum of ii), we have $p\geqslant 1$, $q\leqslant k$, $\sum k_j+\sum r_i+p\leqslant k$.

c. We assume $1\leqslant k\leqslant s_0-5$ and
$$|\widetilde{N}_l| \leqslant C(1+t)^{C\varepsilon}\sigma_1^{\mu-1}, \quad l\leqslant k,$$
$$|\widetilde{Z}_m^l a| + |\widetilde{Z}_m^l\partial a| \leqslant C(1+t)^{C\varepsilon}, \quad l\leqslant k-1.$$
Using first Lemma 6.1 i) for the index k, applied to a or ∂a, we get
$$|\widetilde{Z}_m^k a| + |\widetilde{Z}_m^k\partial a| \leqslant C(1+t)^{C\varepsilon}.$$
Using Lemma 5.3 for the index k and $w=a$, we see that the only terms which are not already controlled (using the induction hypothesis), are the terms
$$M_1s_\theta[M_k;]M_1\widetilde{Z}_m^r a, M_1s_\theta[;M_k]M_1\widetilde{Z}_m^r a, M_1s_\theta[;M_k a]M_1.$$
It is important to check the way θ_i enters the constants (that is, f) in Lemma 5.3: β_i and θ_i^0 enter the computation only through formula i) or ii) of Lemma 4.1. In these formula, θ_i^0 do not appear, and β_i appear only through $\beta_i\varepsilon$; replacing θ_2^{-1} by $f\theta_1^{-1}$, or θ_1^{-1} by f, gives f containing $\theta_1^0/\theta_2^0\leqslant 1$ or $(\theta_1^0)^{-1}\leqslant 1$ as constants, and negative powers of $(1+t)$ expressed with $\beta_i\varepsilon$. Hence, thanks to the constraints $\varepsilon\beta_i\leqslant 1$, all f entering the computation are bounded independently of the choices of the quantities θ_i^0, β_i. We thus obtain
$$|\widetilde{Z}_m^k\widetilde{a}| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}\theta_1^{-1}|\widetilde{N}_{k+1}|,$$
where here and later numbered constants C_2 and C_3 do not depend on θ_i. We obtain similarly
$$\theta_1|[\widetilde{Z}_m^k, s_\theta]\partial a| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}|\widetilde{N}_{k+1}|.$$
We have
$$\widetilde{Z}_m^k\partial\widetilde{a} = \widetilde{Z}_m^k[\partial, S_\theta]a + [\widetilde{Z}_m^k, S_\theta]\partial a + S_\theta\widetilde{Z}_m^k\partial a.$$

To evaluate the first term in the right-hand side, we use Lemma 5.1 ii) or iv) and Lemma 5.2. With the same reasoning as before, we obtain

$$\theta_1|\widetilde{Z}_m^k[\partial, S_\theta]a| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}|\widetilde{N}_{k+1}|.$$

Finally,

$$|\widetilde{Z}_m^k \partial\widetilde{a}| \leqslant C(1+t)^{C\varepsilon} + C_2(1+t)^{C_3\varepsilon}\theta_1^{-1}|\widetilde{N}_{k+1}|.$$

Using now Lemma 6.1 ii) for the index $k+1$, applied to u or ∂u, we get

$$\sigma_1^{-\mu}(1+t)\varepsilon^{-1}|\widetilde{Z}_m^{k+1}u| \leqslant C(1+t)^{C\varepsilon} + C_4(1+t)^{C_5\varepsilon}|\widetilde{Z}_m^k\widetilde{a}|,$$

and a similar formula for ∂u. Finally,

$$|\sigma_1^{1-\mu}\widetilde{N}_{k+1}| \leqslant C(1+t)^{C\varepsilon} + C_6(1+t)^{C_7\varepsilon}\theta_1^{-1}|\sigma_1^{1-\mu}\widetilde{N}_{k+1}|,$$

where as before the constants C_6 and C_7 are independent of θ_i. We choose then

$$\beta_1 \geqslant C_7, \quad \theta_1^0 \geqslant 2C_6$$

to obtain the desired estimate. □

V. Weighted L^2 norms, Poincaré Lemma and Energy Inequalities

V.1. Weighted norms. — For small $\nu > 0$ and big $B > 0$ to be chosen later, we set

$$b(s) = B(-s)^{-\nu}, \quad s \leqslant C < 0, \quad p = (\tau+1)b(\psi).$$

Remark that $p_r > 0$, since $b' > 0$ and $\psi_r > 0$. For fixed t, we define the L^2 weighted norm by

$$|w|_0^2 = \int (\exp p)|w|^2 dx.$$

We first have to clarify the control of $\sigma_1^{-1}w$ by ∂w in this norm.

Lemma 1.1. — *We have, for any smooth w supported in $|x| \leqslant M+t$,*

$$|\sigma_1^{-1}w|_0 \leqslant C|w_r|_0.$$

Proof. — For fixed ω, t, omitting these variables for simplicity, we write

$$w(r) = -\int_r^{M+t} w_r(s)ds,$$

$$w(r)^2 \leqslant C(\sigma_1(r))^{1-\mu}\int_r^{M+t}(\sigma_1(s))^\mu w_r^2(s)ds, \quad 0<\mu<1.$$

Hence, since p is increasing,

$$\int_0^{M+t} e^{p(r)}(\sigma_1(r))^{-2}w(r)^2r^2dr \leqslant C\int_0^{M+t} e^{p(s)}(\sigma_1(s))^\mu w_r^2(s)ds\int_0^s r^2(\sigma_1(r))^{-1-\mu}dr.$$

We split the right-hand side integral in

$$\int_0^{(M+t)/2} + \int_{(M+t)/2}^{M+t}.$$

In the first integral,

$$\sigma_1(r) \geqslant 1 + (M+t)/2,$$

hence it is less than

$$C\int_0^{(M+t)/2} e^{p(s)}s^2w_r^2(s)(\sigma_1(s)(1+(M+t)/2)^{-1})^\mu(s(1+(M+t)/2)^{-1})ds \leqslant C\int_0^{(M+t)/2} e^{p(s)}w_r^2(s)s^2ds.$$

In the second integral, we write

$$\int_0^s r^2(\sigma_1(r))^{-1-\mu}dr \leqslant C(M+t)^2(\sigma_1(s))^{-\mu},$$

and obtain that it is less than

$$\int_{(M+t)/2}^{M+t} e^{p(s)}w_r^2(s)(M+t)^2ds \leqslant C\int_{(M+t)/2}^{M+t} e^{p(s)}w_r^2(s)s^2ds.$$

Collecting the two bounds and integrating in ω finishes the proof. □

We now have to make sure that the smoothing operators behave properly.

Lemma 1.2. — *If β_1 is big enough, we have the formula*

i) $$|s_\theta[p;q]b|_0 \leqslant C|b|_0\Pi|\partial_r p_i|_{L^\infty}\Pi|Rq_j|_{L^\infty},$$

ii) $$|s_\theta[p;q]b|_0 \leqslant C|\partial_r p_1|_0|b|_{L^\infty}\Pi_{i\geqslant 2}|\partial_r p_i|_{L^\infty}\Pi|Rq_j|_{L^\infty},$$

iii) $$|s[p;q]b|_0 \leqslant C|Rq_1|_0|b|_{L^\infty}\Pi|\partial_r p_i|_{L^\infty}\Pi_{j\leqslant 2}|Rq_j|_{L^\infty}.$$

Proof. — We prove only iii), which is the more difficult. With

$$q_1(r',y)-q_1(r',y') = \Big(\int_0^1(\partial_y q_1)(sy+(1-s)y')ds\Big)(y-y'),$$

we can rewrite $s_\theta[p;q]b$ (assuming k factors p_i and l factors q_j) as sums of

$$\int_0^1 ds\int \theta_1^{1+k}\theta_2^{2+l-1}\phi_1(\theta_1(r-r'))\phi_2(\theta_2(y-y'))(p_1(r,y)-p_1(r',y))\cdots(p_k(r,y)-p_k(r',y))$$
$$(\partial_{y_j}q_1)(r',sy+(1-s)y')(q_2(r',y)-q_2(r',y'))\cdots(q_l(r',y)-q_l(r',y'))(\chi b)(r',y')dr'dy'.$$

To introduce e^p into this integral, we write

$$e^{p(r,y)} = e^{p(r,y)} - e^{p(r',y)} + e^{p(r',y)} - e^{p(r',sy+(1-s)y')} + e^{p(r',sy+(1-s)y')}.$$

For the integral corresponding to the last term, the previously proved L^2 estimate works, yielding the quantity $|Rq_1|_0$. The first two terms are bounded by

$$|e^p|_{L^\infty}(|r-r'||p_r|_{L^\infty} + |y-y'||p_y|_{L^\infty}),$$

and

$$|e^p| \leqslant C(1+t)^{C\varepsilon}, \quad |p_r| \leqslant C(1+t)^{C\varepsilon}.$$

From the equation $\psi_t + c\psi_r = 0$, we get

$$(\partial_t + c\partial_r)(\sigma_1^{-\mu} R_i\psi) = -\sigma_1^{-\mu} R_i u\psi_r + \mu u/\sigma_1(\sigma_1^{-\mu} R_i\psi),$$

and we already know that

$$|\sigma_1^{-\mu} R_i u\psi_r| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

Hence, while $\partial\psi$ is bounded for $r \leqslant t/2$, we get for $r \geqslant t/2$ and thus everywhere

$$|\sigma_1^{-\mu} R_i\psi| \leqslant C(1+t)^{C\varepsilon}.$$

This shows

$$|Rp| \leqslant C(\tau+1)|\psi|^{-1-\nu}|R\psi| \leqslant C(1+t)^{C\varepsilon}.$$

Thus the integrals corresponding to the first two terms we have just bounded are bounded by

$$C\theta_1^{-1}(1+t)^{C\varepsilon}|\partial_y q_1|_{L^2}|b|_{L^\infty}\Pi|\partial_r p_i|_{L^\infty}\Pi_{j\geqslant 2}|\partial_y q_j|_{L^\infty}.$$

Putting the weight e^p inside $|\partial_y q_1|_{L^2}$ costs only an extra factor $C(1+t)^{C\varepsilon}$, hence if β_1 is big enough, the claim is proved. □

V.2. The Poincaré Lemma. — The Poincaré Lemma is what we need to control the zero order term $(\Delta u)v$ in the linearized operator on u acting on v.

Proposition 2. — *Fix ν, $0 < \nu \leqslant 1/4$, and let $b(s) = B(-s)^{-\nu}$, $B > 0$. Then we can choose B such that, for any smooth v supported for $r \leqslant M + t$, we have, with $p = (\tau+1)b(\psi)$,*

$$\int_{r\geqslant t/2}(\exp p)(L_1^2 u)^2 v^2 dx \leqslant C\varepsilon^2(1+t)^{-2}\int_{r\geqslant t/2}(\exp p)v_r^2 dx + C\varepsilon^2\int_{r\geqslant t/2}(\exp p)(1+t)^{-7/2}\sigma_1^{-1}v^2 dx.$$

The point of this Lemma is this: the factor $L_1^2 u$ is well localized near the boundary of the light cone, but behaves only like $C\varepsilon(1+t)^{-1+C_1\varepsilon}$ there. In this Lemma, we get the inequality we would easily get if C_1 were zero.

Proof. — Using Lemma II.3.5.1, we get first, with $I = \int_{r\geqslant t/2}(\exp p)(L_1^2 U)^2 v^2 dx$

$$I \leqslant C\int_{r\geqslant t/2}(\exp p)(1+t)^{-3/2}\sigma_1^{-1}v^2 dx + 2\int_{r\geqslant t/2}(\exp p)a_1^{-2}h(\psi)^2 v^2 dx.$$

We perform a change of variables in the last integral, setting

$$s = \psi(r,\omega,t), \quad r = \phi(s,\omega,t), \quad \psi(\phi,\omega,t) = s.$$

The domain $t/2 \leqslant r \leqslant t + M$ is sent on the domain

$$\psi(t/2,\omega,t) \leqslant s \leqslant \psi(t+M,\omega,t) \equiv C(\omega),$$

since ψ is constant along any ray $r = t + M$. Hence, with $w(s,\omega,t) = v(\phi,\omega,t)$,

$$\int_{r\geqslant t/2} (\exp p) a_1^{-2} h(\psi)^2 v^2 dr = \int_{\psi(t/2)\leqslant s\leqslant C(\omega)} e^{\tau b(s)} a_1^{-2}(\phi) h^2(s) w^2 \phi_s ds.$$

We also have, from Lemma II.3.3

$$\phi_s / a_1^2(\phi) \leqslant 2/\phi_s,$$

hence

$$a_1^{-2}(\phi) h^2(s)\phi_s \leqslant C h^2(s)/\phi_s.$$

Now, with $\widetilde{b}(s) = e^{(\tau+1)b(s)}(\phi_s)^{-1}$,

$$\widetilde{b}(s)'/\widetilde{b}(s) = (\tau+1)b'(s) - \phi_{ss}/\phi_s.$$

But

$$\phi_s \psi_{rr}(\phi) = -\phi_{ss}/(\phi_s)^2,$$

and Lemma II.3.5.2 implies

$$|\phi_{ss}/\phi_s| \leqslant [|\psi_{rr}|/(\psi_r)^2](\phi) \leqslant C\tau(1+|s|)^{-3/2+4\eta} + C\varepsilon(1+|s|)^{-3/2+4\eta}.$$

Since $0 < \nu \leqslant 1/2 - 4\eta$, we can choose B big enough to ensure $\widetilde{b}' \geqslant 0$.

Proceeding as usual we write now

$$w(s) = \int_{C(\omega)}^{s} w_s(s')ds',$$

$$|w(s)|^2 \leqslant \Big(\int_s^{C(\omega)} w_s^2(s')\widetilde{b}(s')ds'\Big)\Big(\int_s^{C(\omega)} \frac{ds'}{\widetilde{b}(s')}\Big),$$

and, since $\widetilde{b}$ is increasing, the last integral is less than $(C(\omega)-s)/\widetilde{b}(s)$. Hence

$$\int_{\psi(t/2)}^{C(\omega)} \widetilde{b}(s)h(s)^2 w^2(s)ds \leqslant \Big(\int_{\psi(t/2)}^{C(\omega)} \widetilde{b}(s')w_s^2(s')ds'\Big)\Big(\int_{\psi(t/2)}^{C(\omega)} h(s)^2(C(\omega)-s)ds\Big),$$

and the last integral is bounded by

$$\int_{-\infty}^{C(\omega)} (1+|s|)^{-2+8\eta} ds \leqslant C.$$

Noting that $w_s = \phi_s v_r(\phi)$ and $\phi_s \widetilde{b}(s) = e^{\tau b(s)}$, we obtain by changing back the variables

$$\int_{\psi(t/2)}^{C(\omega)} \widetilde{b}(s) w_s^2(s)ds \leqslant \int_{t/2}^{t+M} (\exp \tau b(\psi)) v_r^2 dr.$$

Finally, we obtain easily

$$|L_1^2 u - \varepsilon/r L_1^2 U| \leqslant C\varepsilon(1+t)^{-2+2\eta}\sigma_1^{-1/2},$$

which completes the proof. □

V.3. The energy inequalities. — We present here one of the many possible variations on the ideas of [**3**].

Proposition 3.1. — *Let* $P = c^{-1}\partial_t^2 - c\Delta_x$ *and* $p = (\tau+1)b(\psi)$ *as in Proposition 2,* $T_i v = \partial_i v + (\omega_i/c)\partial_t v$. *Assuming that* u *satisfies the induction hypothesis* (IH) *on* $[0,T]$*, we have, for* $t \leqslant T$,

$$|(\partial v)(.,t)|_0^2 + C\int_0^t \int_{r\geqslant t'/2} (\exp p)(\tau+1)b'(\psi)\sum (T_i v)^2 dx dt'$$
$$\leqslant C|(\partial v)(.,0)|_0^2 + C\int_0^t\int_{\mathbf{R}^3} (\exp p)|Pv||v_t| dx dt'$$
$$+ C\varepsilon \int_0^t dt'/(1+t')|(\partial v)(.,t')|_0^2.$$

Proof. — We have

$$(\exp p)Pvv_t = \partial_t(1/2(\exp p)(v_t^2/c + c|v_x|^2)) - \sum \partial_i((\exp p)cv_iv_t) + (\exp p)Q,$$

with

$$Q = 1/(2c)(u_t/c - p_t)v_t^2 + \sum (u_i + cp_i)v_iv_t - 1/2(u_t + cp_t)|v_x|^2.$$

Writing explicitly the derivatives of p we get

$$Q = (\tau+1)/(2c)b'(\psi)[-c^2\psi_t \sum (v_i - (\psi_i/\psi_t)v_t)^2 - v_t^2/\psi_t(\psi_t^2 - c^2|\psi_x|^2)]$$
$$- \varepsilon(1+t)^{-1}b(\psi)/2(c^{-1}v_t^2 + c|v_x|^2) + u_t/2c^2v_t^2 + \sum u_iv_iv_t - u_t/2|v_x|^2.$$

Integrating this identity in the strip $[0,t]\times\mathbf{R}^3$, we obtain as usual the control of the energy

$$E(t) = 1/2\int_{\mathbf{R}^3} (\exp p)(v_t^2/c + c|v_x|^2),$$

and the terms of the last line in Q are bounded by

$$C\varepsilon\int_0^t E(t')dt'/(1+t').$$

Now, ψ is not an exact phase function for P. For $r \leqslant t/2$, $\partial\psi$ is bounded, hence the terms of the first line of Q are bounded by

$$C(\tau+1)|b'(\psi)||\partial v|^2 \leqslant C(1+\tau)(1+t)^{-1-\nu}|\partial v|^2,$$

which are negligible terms. For $r \geqslant t/2$, we write

$$\psi_t^2 - c^2|\psi_x|^2 = -c^2/r^2\sum (R_i\psi)^2.$$

From the equation $\psi_t + c\psi_r = 0$, we get

$$(\partial_t + c\partial_r)(\sigma_1^{-\mu}R_i\psi) = -\sigma_1^{-\mu}R_iu\psi_r + \mu u/\sigma_1(\sigma_1^{-\mu}R_i\psi),$$

and we already know that

$$|\sigma_1^{-\mu}R_iu\psi_r| \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

Hence
$$|\sigma_1^{-\mu} R_i\psi| \leqslant C(1+t)^{C\varepsilon}.$$
The error term
$$(\tau+1)b' v_t^2/\psi_t (R_i\psi)^2/r^2$$
is then bounded by
$$Cv_t^2(1+\varepsilon\log(1+t))(1+|\psi|)^{-1-\nu}\sigma_1^{2\mu}(1+t)^{-2+C\varepsilon} \leqslant C(1+t)^{-1-\eta}v_t^2,$$
which is negligible. Finally,
$$v_i - (\psi_i/\psi_t)v_t = v_i + (\omega_i/c)v_t - (v_t/\psi_t)(\psi_i + (\omega_i/c)\psi_t),$$
$$\psi_i + (\omega_i/c)\psi_t = \psi_i - \omega_i\psi_r.$$
Replacing $v_i - \psi_i/\psi_t v_t$ by $T_i v$ in Q gives an error term bounded by
$$(\tau+1)b'(\psi)|\psi_t|v_t^2(R_i\psi)^2/r^2,$$
which we have already seen to be negligible. □

In contrast with what could seem obvious, the energy inequality for L is non trivial.

Proposition 3.2. — *Let $p = (\tau+1)b(\psi)$ as in Proposition 2, and $\gamma > 0$. Then, for smooth functions v supported in $\gamma(1+t) \leqslant r \leqslant M+t$, we have the inequalities*

i) $$(1+t)^{-1}|e^{p/2}v(.,t)|_{L^2} \leqslant C\int_0^t (1+t')^{-1}|e^{p/2}(Lv)(.,t')|_{L^2}dt' + C\varepsilon\int_0^t (1+t')^{-2}|v(.,t')|_0 dt',\ v(x,0)=0,$$

ii) $$(1+t)^{-1}|(\partial v)(.,t)|_0 \leqslant C\int_0^t (1+t')^{-1}|(\partial Lv)(.,t')|_0 dt' + C\varepsilon\int_0^t (1+t')^{-2}|(\partial v)(.,t')|_0 dt',\ v(x,0)=v_t(x,0)=0.$$

Proof

a. We write
$$e^p\sqrt{c}Lvv = 1/2\partial_t(e^p v^2) + 1/2\partial_r(ce^p v^2) - (1/2)e^p v^2(p_t + cp_r) - (1/2)e^p u_r v^2,$$
and remark that
$$p_t + cp_r = (\tau+1)b'(\psi)(\psi_t + c\psi_r) + \varepsilon(1+t)^{-1}b(\psi).$$
Hence, integrating in r and t on $[0,+\infty[\times[0,t]$, we get
$$\int e^p\sqrt{c}Lvv\,drdt' = 1/2\int e^p v^2(r,\omega,t)dr - (1/2)\int e^p v^2(1+t')^{-1}[(1+t')u_r + \varepsilon b(\psi)]drdt',$$
which gives the bound
$$(1/2)\int e^p v^2(r,\omega,t)dr \leqslant C\varepsilon\int e^p v^2(1+t')^{-1}drdt' + C\int e^p|Lv||v|drdt'.$$

Integrating now also in ω and using again the support condition on v, we obtain

$$g(t)^2 \equiv ((1+t)^{-1}|e^{p/2}v(.,t)|_{L^2})^2 \leqslant C\varepsilon \int_0^t (1+t')^{-1} g^2(t')dt' + \int_0^t (1+t')^{-1}|e^{p/2}(Lv)(.,t')|_{L^2} g(t')dt',$$

which gives i).

b. With $Lv = h$, we have

$$h_t = Lv_t - (u_t/2c)L_1 v, h_r = Lv_r - (u_r/2c)L_1 v, R_i h = LR_i v - (R_i u/2c)L_1 v.$$

Using the inequality of **a.** for v_t, v_r yields the desired terms. For $R_i v$, we obtain

$$\rho_i(t) \equiv (1+t)^{-1}|e^{p/2}R_i v(.,t)|_{L^2} \leqslant C\varepsilon \int_0^t (1+t')^{-1}\rho_i(t')dt' + C\int_0^t (1+t')^{-1}[|e^{p/2}R_i h|_{L^2} + |R_i u|_{L^\infty}|e^{p/2}(\partial v)|_{L^2}]dt'.$$

Dividing both sides by $(1+t)$, using the support condition and the fact that $t' \leqslant t$ in the integrals, we get

$$(1+t)^{-1}|e^{p/2}(R_i/r)v(.,t)|_{L^2} \leqslant C\int_0^t (1+t')^{-1}|e^{p/2}(R_i/r)h|_{L^2}dt' + C\varepsilon\int_0^t (1+t')^{-2}|e^{p/2}(\partial v)|_{L^2}dt'.$$

Since $\partial_i = \omega_i \partial_r - (\omega \wedge (R/r))_i$, this gives ii). □

VI. Commutations with the operator P

VI.1. Computation of $[\widetilde{Z}_m, P]$ and consequences.

— Recall that

$$P = c^{-1}\partial_t^2 - c\Delta.$$

To establish formula describing $[\widetilde{Z}_m, P]$, we compute separately the two terms $[Z, P]$, which involves only u, and $[\widetilde{a}L_1, P]$.

Lemma 1.1. — *We have the formula* $(1.2)_a$, $(1.2)_b$, $(1.2)_c$, $(1.2)_d$. *Away from* $r = 0$, *we also have the formula*

$$(1.1)_a \quad [\widetilde{K}, P] = \widetilde{K}u/cP - \frac{1}{8c}(Lu + 3L_1 u)L_1^2 - \frac{1}{8c}(3Lu + L_1 u)L^2 - c^{-3/2}u_t L_1 L - \frac{1}{\sqrt{c}r^2}R_j u R_j \partial_t - (1/2c)[3/2c^3 u_t^2 - 1/4c^2 u_t u_r + 1/4cu_r^2 + 3(u_t^2/c^2 - |u_x|^2)]L_1 - (1/2c)[L_1 u/2c(\sqrt{c}/2u_r - u_t/\sqrt{c}) + 3(u_t^2/c^2 - |u_x|^2)]L,$$

$$(1.1)_b \quad [R_i, P] = (R_i u/c)P - (R_i u/2c)L_1^2 - (R_i u/2c)L^2 - (R_i u/c)LL_1 - R_i uLu/4c^2 L_1 - R_i u/2c^2(Lu + L_1 u/2)L,$$

$$(1.1)_c \quad [S, P] = (Su/c - 2)P - (Su/2c)L_1^2 - (Su/2c)L^2 - (Su/c)LL_1 - SuLu/4c^2 L_1 - Su/2c^2(Lu + L_1 u/2)L,$$

$$\begin{aligned}(1.1)_d \quad [H_i, P] = {} & (H_i u/c)P + (-H_i u/2c + \frac{\omega_i(r+ct)}{2c\sqrt{c}}Lu)L_1^2 \\ & + (-H_i u/2c + \frac{\omega_i(r-ct)}{2c\sqrt{c}}L_1 u)L^2 + (-H_i u/c + x_i u_t/c^2 - t\omega_i u_r)LL_1 \\ & - (tL_1 uL + tLuL_1)(\partial_i - \omega_i\partial_r) + 2ct/r^2 R_j u R_j \partial_i - 2x_i/cr^2 R_j u R_j \partial_t - 2u_t/c\partial_i \\ & - (H_i uLu/4c^2 - \frac{\omega_i(r-ct)}{4c^2\sqrt{c}}LuL_1 u + x_i/c\sqrt{c}(u_t^2/c^2 - |u_x|^2) + \partial_i u/\sqrt{c})L_1 \\ & - (H_i uLu/2c^2 + L_1 u/4c^2(H_i u - \omega_i Su) + \partial_i u/\sqrt{c} + \omega_i/c\sqrt{c}(ctu_r^2 - r|u_x|^2))L.\end{aligned}$$

Proof

a. We have

$$(1.2)_a \qquad [\widetilde{K}, P] = \widetilde{K}u/cP - \widetilde{K}u/c^2\partial_t^2 - c^{-1/2}\sum u_j\partial_{jt}^2,$$

$$(1.2)_b \quad [R_i, P] = [R_i, c^{-1}\partial_t^2] - [R_i, c\Delta] = -R_i u/c^2\partial_t^2 - R_i u\Delta = R_i u/cP - 2R_i u/c^2\partial_t^2.$$

Similarly, since $[S, \partial_t^2] = -2\partial_t^2$, $[S, \Delta] = -2\Delta$,

$$\begin{aligned}(1.2)_c \qquad [S, P] = [S, c^{-1}\partial_t^2] - [S, c\Delta] &= -Su/c^2\partial_t^2 - Su\Delta - 2/c\partial_t^2 + 2c\Delta \\ &= (Su/c - 2)P - 2Su/c^2\partial_t^2.\end{aligned}$$

b. We have

$$[\partial_t^2, H_i] = 2((c + tu_t)\partial_{it}^2 - x_i u_t/c^2\partial_t^2) + (2u_t + tu_{tt})\partial_i + x_i/c^2(2u_t^2/c - u_{tt})\partial_t,$$

$$[\Delta, H_i] = 2(tu_j\partial_{ij}^2 - x_i/c^2 u_j\partial_{jt}^2 + 1/c\partial_{it}^2) + t\Delta u\partial_i - 2/c^2 u_i\partial_t + x_i/c^2(2|u_x|^2/c - \Delta u)\partial_t.$$

Hence

$$[H_i, P] = H_i u/cP - 2H_i u/c^2\partial_t^2 - 2u_t/c(t\partial_{it}^2 - x_i/c^2\partial_t^2) + 2u_j(ct\partial_{ij}^2 - x_i/c\partial_{jt}^2) - 2u_t/c\partial_i - 2/c^2(cu_i + x_i(u_t^2/c^2 - |u_x|^2)\partial_t.$$

We can write

$$
\begin{aligned}
&ct\partial^2_{it} = \partial_t H_i - x_i/c\partial^2_t - (c+tu_t)\partial_i + x_iu_t/c^2\partial_t,\\
&ct\partial^2_{ij} = \partial_j H_i - x_i/c\partial^2_{jt} - tu_j\partial_i + (x_iu_j/c^2 - \delta_{ij}/c)\partial_t,\\
&ct\partial^2_{ij} = \partial_j H_i - (x_i/c^2t)\partial_t H_j + (x_ix_j/c^3t)\partial^2_t\\
&\qquad + (x_i/c^2t)(c+tu_t)\partial_j - tu_j\partial_i - c^{-1}(\delta_{ij} + (x_ix_j/c^3t)u_t - x_iu_j/c)\partial_t.
\end{aligned}
$$

Using these identities to express all second order derivatives as ∂^2_t modulo ∂H_k, we get

$$
\begin{aligned}
(1.2)_d\quad [H_i,P] &= H_iu/cP - 2H_iu/c^2\partial^2_t - 2u_t/c^2\partial_t H_i - 4(x_iu_j/c^2t)\partial_t H_j\\
&+ 2u_j\partial_j H_i + 4(x_i/c^3t)Su\partial^2_t + 2t(u_t^2/c^2 - |u_x|^2)\partial_i + 4(x_iu_j/c^2t)(c+tu_t)\partial_j\\
&- 4x_i/c^2(u_t^2/c^2 - |u_x|^2)\partial_t - 4/c(u_i + x_ix_ju_ju_t/c^3t)\partial_t.
\end{aligned}
$$

If we are away from $r=0$, we can handle differently, using the identity

$$\sum v_j\partial_j = v_r\partial_r + 1/r^2\sum R_jvR_j.$$

We write then

$$
\begin{aligned}
[H_i,P] &= H_iu/cP - 2H_iu/c^2\partial^2_t + 2x_iu_t/c^3\partial^2_t\\
&- 2tu_t/c\partial_t(\partial_i - \omega_i\partial_r + \omega_i\partial_r) + 2ct(u_r\partial_r\partial_i + 1/r^2R_juR_j\partial_i)\\
&- 2x_i/c(u_r\partial^2_{rt} + 1/r^2R_juR_j\partial_t) - 2u_t/c\partial_i - 2x_i/c^2(u_t^2/c^2 - |u_x|^2)\partial_t - 2u_i/c\partial_t\\
&= H_iu/cP - 2H_iu/c^2\partial^2_t - 2(tu_t/c\partial_t - ctu_r\partial_r)(\partial_i - \omega_i\partial_r)\\
&+2/r^2(ctR_juR_j\partial_i - x_i/cR_juR_j\partial_t) - 2u_t/c\partial_i - 2x_i/c^2(u_t^2/c^2 - |u_x|^2)\partial_t - 2u_i/c\partial_t + 2\sum,
\end{aligned}
$$

where $\sum$ means here the sum of the following four terms

$$\sum = x_iu_t/c^3\partial^2_t - t\omega_iu_t/c\partial^2_{rt} - x_iu_r/c\partial^2_{rt} + ctu_r\omega_i\partial^2_r.$$

Using the identities

$$
\begin{aligned}
2/c\partial^2_t &= 1/2L^2 + 1/2L_1^2 + LL_1 + Lu/4cL_1 + (L_1u + 2Lu)/4cL,\\
2c\partial^2_r &= 1/2L^2 + 1/2L_1^2 - LL_1 + Lu/4cL_1 + (L_1u - 2Lu)/4cL,\\
4\partial^2_{rt} &= L^2 - L_1^2 + Lu/2cL_1 - L_1u/2cL,
\end{aligned}
$$

we obtain from **a.** the desired forms for $[\widetilde{K},P]$, $[R_i,P]$ and $[S,P]$. In the present computation of $[H_i,P]$, we get

$$
\begin{aligned}
2\sum &= \frac{\omega_i(r-ct)}{2c\sqrt{c}}L_1uL^2 + \frac{\omega_i(r+ct)}{2c\sqrt{c}}LuL_1^2 + (x_iu_t/c^2 - t\omega_iu_r)LL_1\\
&+ (x_iu_t^2/c^3\sqrt{c} - t\omega_iu_r^2/\sqrt{c} + \omega_iL_1uSu/4c^2)L + \frac{\omega_i(r-ct)}{4c^2\sqrt{c}}LuL_1uL.
\end{aligned}
$$

After some algebraic manipulations, we get the result for $[H_i,P]$. □

Lemma 1.2. — *We have*

$$\begin{aligned}[\widetilde{a}L_1, P] &= (-L_1\widetilde{a} + \widetilde{a}L_1u/c)P - (L\widetilde{a} + \widetilde{a}L_1u/2c)L_1^2 + \widetilde{a}/c(Lu/2 - L_1u)L_1L \\ &\quad + r^{-2}R_j\widetilde{a}R_jL_1 - \widetilde{a}r^{-2}R_juR_jL - cr^{-2}(L_1\widetilde{a} + 2\widetilde{a}\sqrt{c}/r)\Delta_\omega \\ &\quad + [-L_1L\widetilde{a} - L_1u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a} \\ &\quad - \widetilde{a}(-c/r^2 + (-L_1u)^2/2c^2 + LuL_1u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1 \\ &\quad + [L_1\widetilde{a}(Lu/2c - \sqrt{c}/r) - \widetilde{a}(c/r^2 + L_1uu_t/c^2\sqrt{c} + 1/2c(u_t^2/c^2 - |u_x|^2))]L.\end{aligned}$$

Proof. — We have

$$\begin{aligned}[L_1, \partial_t^2] &= u_t/c\partial_tL + u_{tt}/2cL - 1/4c^{-5/2}u_t^2\partial_t - 3/4c^{-3/2}u_t^2\partial_r, \\ [L_1, \Delta] &= u_j/c\partial_jL + 2\sqrt{c}/r^3\Delta_\omega - 1/4c^{-5/2}|u_x|^2\partial_t \\ &\qquad - 3/4c^{-3/2}|u_x|^2\partial_r + \Delta u/2cL + 2\sqrt{c}/r^2\partial_r,\end{aligned}$$

hence, writing here $b = \widetilde{a}$,

$$\begin{aligned}[bL_1, P] &= bL_1u/cP - 2b/c^2L_1u\partial_t^2 - (Pb)L_1 - 2b_t/c\partial_tL_1 + 2cb_r\partial_rL_1 \\ &\quad + 2c/r^2R_jbR_jL_1 + bu_t/c^2\partial_tL - bu_r\partial_rL - b/r^2R_juR_jL - 2bc\sqrt{c}/r^3\Delta_\omega \\ &\qquad - 2bc\sqrt{c}/r^2\partial_r - b/4c(u_t^2/c^2 - |u_x|^2)(2L - L_1).\end{aligned}$$

The strategy is the following: after some algebraic arrangements, we express LL_1 using P only in the term $(L_1b)LL_1$, and take a careful look at the first order terms. We have first

$$\begin{aligned}-2b_t/c\partial_tL_1 + 2cb_r\partial_rL_1 &= -(Lb)L_1^2 - (L_1b)LL_1, \\ u_t/c^2\partial_tL - u_r\partial_rL &= (Lu/2c)L_1L + (L_1u/2c)L^2.\end{aligned}$$

Next

$$-2b/c^2L_1u\partial_t^2 = -b/cL_1u(LL_1 + 1/2L^2 + 1/2L_1^2 + Lu/4cL_1 + u_t/c\sqrt{c}L).$$

Now we replace, in the term $(L_1b)LL_1$,

$$LL_1 = P + c/r^2\Delta_\omega + 2c/r\partial_r - Lu/2cL,$$

which gives

$$\begin{aligned}[bL_1, P] &= (bL_1u/c - L_1b)P - (Lb + bL_1u/2c)L_1^2 + b(Lu/2c - L_1u/c)L_1L \\ &\quad + 2c/r^2R_jbR_jL_1 - b/r^2R_juR_jL - cr^{-2}(2b\sqrt{c}/r + L_1b)\Delta_\omega + Q_1,\end{aligned}$$

where the first order terms Q_1 are

$$\begin{aligned}Q_1 &= 2c/r(b_rL_1 - (L_1b)\partial_r) + q_1L_1 + q_2L, \\ q_1 &= -b/2c^2(L_1u)^2 - b/4c^2LuL_1u + bc/r^2 - L_1u/2cL_1b \\ &\qquad - L_1Lb + c/r^2\Delta_\omega b - 1/4c(u_t^2/c^2 - |u_x|^2), \\ q_2 &= -b/c^2\sqrt{c}L_1uu_t - bc/r^2 + LuL_1b/2c + 1/2c(u_t^2/c^2 - |u_x|^2).\end{aligned}$$

It is important to remark that

$$\begin{aligned} b_r L_1 - L_1 b\partial_r &= b_r(L - 2\sqrt{c}\partial_r) - (Lb - 2\sqrt{c}b_r)\partial_r = b_r L - Lb\partial_r \\ &= b_r L - (Lb)/2\sqrt{c}(L - L_1) = 1/2\sqrt{c}(-L_1 bL + LbL_1). \end{aligned}$$

Collecting the terms gives the result. □

Putting together the two above Lemmas yields the desired expression.

Lemma 1.3. — *We have the formula*

i) $$[\widetilde{R}_i^m, P] = (R_i u/c - L_1\widetilde{a} + \widetilde{a}L_1 u/c)P - \widetilde{A}(R_i)L_1^2 - R_i u/2cL^2$$
$$+(-R_i u/c + \frac{\widetilde{a}}{c}(Lu/2 - L_1 u))L_1 L + r^{-2}R_j\widetilde{a}R_j L_1 - \widetilde{a}r^{-2}R_j u R_j L - cr^{-2}(L_1\widetilde{a} + 2\widetilde{a}\sqrt{c}/r)\Delta_\omega$$
$$+ [-R_i u/2c^2(L_1 u + Lu/2) - L_1 L\widetilde{a} - L_1 u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a}$$
$$- \widetilde{a}(-c/r^2 + (-L_1 u)^2/2c^2 + LuL_1 u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1$$
$$+[-R_i u/4c^2 L_1 u + L_1\widetilde{a}(Lu/2c - \sqrt{c}/r) - \widetilde{a}(c/r^2 + L_1 u u_t/c^2\sqrt{c} + 1/2c(u_t^2/c^2 - |u_x|^2))]L,$$

ii) $$[\widetilde{S}^m, P] = (Su/c - 2 - L_1\widetilde{a} + \widetilde{a}L_1 u/c)P - \widetilde{A}(S)L_1^2 - (Su/2c)L^2$$
$$+(-Su/c + \frac{\widetilde{a}}{c}(Lu/2 - L_1 u)))L_1 L + r^{-2}R_j\widetilde{a}R_j L_1 - \widetilde{a}r^{-2}R_j u R_j L - cr^{-2}(L_1\widetilde{a} + 2\widetilde{a}\sqrt{c}/r)\Delta_\omega$$
$$+ [-Su/2c^2(L_1 u + Lu/2) - L_1 L\widetilde{a} - L_1 u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a}$$
$$- \widetilde{a}(-c/r^2 + (-L_1 u)^2/2c^2 + LuL_1 u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1$$
$$+[-Su/4c^2 L_1 u + L_1\widetilde{a}(Lu/2c - \sqrt{c}/r) - \widetilde{a}(c/r^2 + L_1 u u_t/c^2\sqrt{c} + 1/2c(u_t^2/c^2 - |u_x|^2))]L,$$

iii) $$[\widetilde{H}_i^m, P] = (H_i u/c - L_1\widetilde{a} + \widetilde{a}L_1 u/c)P - \widetilde{A}(H_i)L_1^2 - \frac{r - ct}{2cr}(\omega \wedge Ru)_i L_1^2$$
$$+ (-H_i u/c + x_i u_t/c^2 - t\omega_i u_r + \frac{\widetilde{a}}{c}(Lu/2 - L_1 u))L_1 L + (-H_i u/2c + \frac{\omega_i(r - ct)}{2c\sqrt{c}}L_1 u)L^2$$
$$- (tL_1 uL + tLuL_1)(\partial_i - \omega_i\partial_r) + 2ct/r^2 R_j u R_j\partial_i - 2x_i/cr^2 R_j u R_j\partial_t$$
$$+ r^{-2}R_j\widetilde{a}R_j L_1 - \widetilde{a}r^{-2}R_j u R_j L - cr^{-2}(L_1\widetilde{a} + 2\widetilde{a}\sqrt{c}/r)\Delta_\omega - 2u_t/c\partial_i$$
$$+ [-H_i u/2c^2(L_1 u + Lu/2) + \frac{\omega_i}{4c^2\sqrt{c}}L_1 u(-2c\sqrt{c}tu_r + (r - ct)Lu) - \partial_i u/\sqrt{c}$$
$$+ x_i/2c^2(u_t L_1 u - 2\sqrt{c}(u_t^2/c^2 - |u_x|^2)) - L_1 L\widetilde{a} - L_1 u/2cL_1\widetilde{a} + cr^{-2}\Delta_\omega\widetilde{a} + \sqrt{c}/rL\widetilde{a}$$
$$- \widetilde{a}(-c/r^2 + (-L_1 u)^2/2c^2 + LuL_1 u/4c^2 - 1/4c(u_t^2/c^2 - |u_x|^2))]L_1$$
$$+ [+L_1 u/4c^2(\omega_i Su - H_i u) - \partial_i u/\sqrt{c} + \omega_i/c\sqrt{c}(r|u_x|^2 - ctu_r^2) + Lu/2c(t\omega_i u_r - x_i u_t/c^2)$$
$$+ L_1\widetilde{a}(Lu/2c - \sqrt{c}/r) - \widetilde{a}(c/r^2 + L_1 u u_t/c^2\sqrt{c} + 1/2c(u_t^2/c^2 - |u_x|^2))]L.$$

Here,

$$\widetilde{A}(R_i) = L\widetilde{a}(R_i) + L_1u/2c\widetilde{a}(R_i) + R_iu/2c = L\widetilde{a}(R_i) + \widetilde{R}_i^m u/(2c),$$
$$\widetilde{A}(S) = L\widetilde{a}(S) + L_1u/2c\widetilde{a}(S) + Su/2c = L\widetilde{a}(S) + \widetilde{S}^m u/(2c),$$
$$\widetilde{A}(H_i) = -\omega_i\widetilde{A}(S) - (\omega\wedge\widetilde{A}(R))_i.$$

Proof. — The formula are obtained by just adding the formula of Lemma 1.1 and 1.2, and using $[L, L_1] = L_1u/2cL_1 - Lu/2cL$ to replace LL_1 by L_1L. The expressions of $\widetilde{A}(R_i)$ and $\widetilde{A}(S)$ are clear. We get

$$\widetilde{A}(H_i) = H_iu/2c - \frac{\omega_i(r+ct)}{2c\sqrt{c}}Lu + L\widetilde{a}(H_i) + L_1u/2c\widetilde{a}(H_i) - \frac{r-ct}{2cr}(\omega\wedge Ru)_i.$$

Since

$$H_i = \omega_iH_0 - ct/r(\omega\wedge R)_i, \omega_iH_0 = \frac{\omega_i(r+ct)}{\sqrt{c}}L - \omega_iS,$$

we obtain

$$\widetilde{A}(H_i) = L\widetilde{a}(H_i) + L_1u/2c\widetilde{a}(H_i) - \omega_iSu/2c - 1/2c(\omega\wedge Ru_i).$$

Using the definition of $\widetilde{a}(H_i)$, we get the result. □

We will dispatch the terms in Lemma 1.3 into three categories:

i) A term which can be written in the form

$$M_1\alpha\partial\widetilde{Z}_m, \quad M_1\alpha\sigma_1^{-1}\widetilde{Z}_m, M_1\alpha\partial$$

will be called "standard"(st.); otherwise, it will be called "special" (sp.).

ii) A standard term for which, for some $\gamma > 0$,

$$|\alpha| \leqslant C(1+t)^{-1-\gamma}$$

will be called integrable (int.). Otherwise, it will be called "just".

Rewriting appropriately the terms in Lemma 1.3, we obtain the following Proposition.

Proposition 1. — *We have*

$$[\widetilde{Z}_m, P] = \delta P + \textstyle\sum_1 + \sum_2 + \sum_3,$$
$$\delta = f\widetilde{Z}_mu + f\partial\widetilde{a} + f\partial u\widetilde{a},$$

i) $\sum_1$ *is a sum of standard integrable terms with*

$$\alpha = \sigma_1/(1+t)^2(\varepsilon^{-1}(1+t)\partial u), \quad \alpha = \sigma_1/(1+t)^2\widetilde{N}_1.$$

ii) $\sum_2$ *is the sum of the just standard terms*

$$\textstyle\sum_2 = f\partial u\partial\widetilde{Z}_m + f\partial\widetilde{Z}_mu\partial + f\partial u\partial,$$

iii) $\sum_3$ *is the sum of the special terms*

$$\sum_3 = -\widetilde{A}L_1^2 + r^{-2}R_j\widetilde{a}R_jL_1 + fr^{-2}L_1\widetilde{a}\Delta_\omega + cr^{-2}\Delta_\omega\widetilde{a}L_1 + f(1+t)^{-1}\partial uR_j\widetilde{a}\partial + fL_1L\widetilde{a}\partial + f\partial uL_1\widetilde{a}\partial + f(1+t)^{-1}L\widetilde{a}\partial.$$

Proof. — We proceed by inspecting the terms in Lemma 1.3, after an appropriate rewriting. We discuss only the terms in $[\widetilde{H}_i, P]$, which are the most difficult, examining the terms in the order they appear in the Lemma. The terms of the other $[\widetilde{Z}_m, P]$ have the same forms. The special terms will be discussed in the next Proposition.

1. The term $-\widetilde{A}L_1^2$ is special.

2. We have

$$r - ct = f + \sigma_1 f = f\sigma_1,$$

hence

$$\frac{r-ct}{2cr}(\omega \wedge Ru)_i = f\sigma_1/(1+t)R_ju,$$

and using Lemma IV.3.1,

$$\sigma_1/(1+t)R_juL_1^2 = M_1/(1+t)\widetilde{Z}_mu([\widetilde{Z}_m, L_1] + f\partial\widetilde{Z}_m).$$

Since $[\widetilde{Z}_m, L_1] = M_1\partial$, the term is

$$M_1\varepsilon\sigma_1/(1+t)^2\widetilde{N}_1(\partial + \partial\widetilde{Z}_m).$$

It is st. int. with $\alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1$.

3. Recalling that

$$L = \frac{\sqrt{c}}{r+ct}(\sum \omega_i\widetilde{H}_i + \widetilde{S}),$$

we note first

$$L_1(\frac{\sqrt{c}}{r+ct}) = \frac{r-ct}{2\sqrt{c}(r+ct)^2}L_1u, \quad L(\frac{\sqrt{c}}{r+ct}) = \frac{1}{2\sqrt{c}(r+ct)^2}((r-ct)Lu - 4c\sqrt{c}).$$

Hence

$$L_1L = \frac{f\sigma_1}{(1+t)^2}\partial u\widetilde{Z}_m + f/(1+t)\partial\widetilde{Z}_m, \tag{1.3}$$

$$L^2 = f/(1+t)^2\widetilde{Z}_m + f/(1+t)\partial\widetilde{Z}_m. \tag{1.4}$$

We write

$$M_1\widetilde{Z}_muL_1L = M_1\varepsilon\sigma_1/(1+t)\widetilde{N}_1(\sigma_1/(1+t)^2(\partial u)\widetilde{Z}_m + 1/(1+t)\partial\widetilde{Z}_m,$$

hence both terms are st. int., with

$$\alpha = \varepsilon^2\sigma_1^3/(1+t)^4\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

We write

$$ft\partial uL_1L = f(\partial u)^2\sigma_1/(1+t)\widetilde{Z}_m + f\partial u\partial\widetilde{Z}_m,$$

the second term is (just) while the first is st. int. with

$$\alpha = \varepsilon^2\sigma_1^2/(1+t)^3((1+t)\partial u/\varepsilon).$$

We write

$$f\widetilde{a}\partial uL_1L = (\sigma_1^{-1}\widetilde{a})(f\sigma_1^2(\partial u)^2/(1+t)^2\widetilde{Z}_m + f\sigma_1/(1+t)\partial u\partial\widetilde{Z}_m),$$

showing that both terms are st. int. with

$$\alpha = \varepsilon^2\sigma_1^3(1+t)^4\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

We write

$$fH_iuL^2 = M_1\widetilde{Z}_mu(f/(1+t)^2\widetilde{Z}_m + f/(1+t)\partial\widetilde{Z}_m),$$

hence both terms are st. int., with

$$\alpha = \varepsilon\sigma_1^2/(1+t)^3\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

4. To handle the term

$$tL_1uL(\partial_i - \omega_i\partial_r) = fL_1u\widetilde{Z}_m(\omega_iR_j/r) = M_1(\partial u)/(1+t)R_j + f\partial u[\widetilde{Z}_m, R_j/r] + f\partial u\partial\widetilde{Z}_m,$$

we need Lemma IV.4.3. The term

$$M_1((1+t)\partial u\varepsilon)\varepsilon/(1+t)^2\widetilde{Z}_m$$

is st. int. with

$$\alpha = \varepsilon\sigma_1/(1+t)^2((1+t)\partial u/\varepsilon).$$

According to Lemma IV.4.3, he middle-term is equal to

$$M_1\partial u/(1+t)\widetilde{Z}_m + M_1\sigma_1/(1+t)\partial u\partial + f\partial u/(1+t)R_j\widetilde{a}\partial.$$

The last term is sp., the first two are st. int. with

$$\alpha = \varepsilon\sigma_1/(1+t)^2((1+t)\partial u/\varepsilon).$$

5. We write the term $tLuL_1(\partial_i - \omega_i\partial_r)$ as

$$f\widetilde{Z}_muL_1(R/r) = f/(1+t)^2\widetilde{Z}_muR + f/(1+t)\widetilde{Z}_mu([L_1,R] + RL_1).$$

Since

$$[L_1,R] = fRuL, RL_1 = f\widetilde{N}_1^l\widetilde{Z}_mL_1, [\widetilde{Z}_m, L_1] = f\widetilde{N}_1\partial,$$

the term is

$$M_1/(1+t)^2\widetilde{Z}_mu\widetilde{Z}_m + M_1/(1+t)\widetilde{Z}_mu(\widetilde{Z}_muL + \partial + \partial\widetilde{Z}_m).$$

All three terms are st. int. with

$$\alpha = \varepsilon\sigma_1^2/(1+t)^3\widetilde{N}_1, \quad \alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1.$$

6. We write

$$f/(1+t)R_juR_j\partial_i = M_1/(1+t)\widetilde{Z}_mu([\widetilde{Z}_m, \partial_i] + \partial\widetilde{Z}_m).$$

In view of Lemma IV.4.2, both terms are st. int. with $\alpha = \varepsilon\sigma_1/(1+t)^2\widetilde{N}_1$. 7. The term $r^{-2}R_j\widetilde{a}R_jL_1$ is sp.

We write

$$\widetilde{a}r^{-2}R_juR_jL = M_1\widetilde{a}/(1+t)^2\widetilde{Z}_mu([\widetilde{Z}_m, L] + L\widetilde{Z}_m),$$

showing that both terms are st. int. with $\alpha = \varepsilon\sigma_1^2/(1+t)^3\widetilde{N}_1$. The next term is sp., then we write

$$f\widetilde{a}r^{-3}\widetilde{R}_j^2 = f\widetilde{a}/r^2 R_j(R_j/r) = M_1\sigma_1/(1+t)^2\widetilde{N}_1([\widetilde{Z}_m,\partial] + \partial + \partial\widetilde{Z}_m),$$

which gives three st. int. terms with $\alpha = \sigma_1/(1+t)^2\widetilde{N}_1$.

8. We reach now the first order terms. While

$$f\partial u\partial, \quad f(1+t)(\partial u)^2\partial = f\varepsilon\frac{(1+t)\partial u}{\varepsilon}\partial u\partial = f\partial u\partial,$$

are just, we write

$$f(\partial u)Zu\partial = M_1\widetilde{Z}_m u\partial u\partial,$$

which is a st. int. term with $\alpha = \varepsilon^2\sigma_1/(1+t)^2\widetilde{N}_1$.

9. The next four terms containing $\widetilde{a}$ are special.

10. We write then

$$f/(1+t)^2\widetilde{a}\partial = f\sigma_1/(1+t)^2\widetilde{N}_1\partial, \quad f\widetilde{a}(\partial u)^2\partial = f\varepsilon^2\sigma_1/(1+t)^2((1+t)\partial u\varepsilon)\partial,$$

hence the two terms are st. int. with

$$\alpha = \sigma_1/(1+t)^2\widetilde{N}_1, \quad \alpha = \sigma_1/(1+t)^2((1+t)\partial u/\varepsilon).$$

Finally, the terms in L have exactly the same structure, with the exception of

$$f(1+t)^{-1}L_1\widetilde{a}L = fL_1\widetilde{a}(\sigma_1/(1+t)^2)(\sigma_1^{-1}\widetilde{Z}_m)$$

which is st. int. with $\alpha = \sigma_1/(1+t)^2\widetilde{N}_1$. □

The following Lemma displays the structure of the most delicate terms in $\sum_3$.

Lemma 1.4. — *We have*

$$\begin{aligned}
\sqrt{c}L\widetilde{a} &= -S_\theta(\chi/(2\sqrt{c})(\widetilde{Z}_m u + (a-\widetilde{a})L_1 u)) + \varepsilon(1+t)^{-1}s_\theta a + [u, S_\theta]a_r,\\
\partial_r(\sqrt{c}L\widetilde{a}) &= -\partial_r S_\theta(\chi/(2\sqrt{c})(\widetilde{Z}_m u + (a-\widetilde{a})L_1 u))\\
&\quad + \varepsilon(1+t)^{-1}s_\theta a_r + u_r S_\theta a_r + \theta_1[u, s_\theta]a_r,\\
\partial_t(\sqrt{c}L\widetilde{a}) &= -\partial_t S_\theta(\chi/(2\sqrt{c})(\widetilde{Z}_m u + (a-\widetilde{a})L_1 u))\\
&\quad + \varepsilon/(1+t)^2 s_\theta a + \varepsilon/(1+t)s_\theta a_t + u_t S_\theta a_r\\
&\quad + u/(1+t)s_\theta a_r + (1+t)^{-1}s_\theta u a_r - S_\theta u_t a_r + S_\theta u_r a_t + \theta_1[u, s_\theta]a_t,\\
r^{-2}R_i\widetilde{a}R_i L_1 &= M_1(1+t)^{-2}\theta_2(s_\theta a)\partial\widetilde{Z}_m + M_1(1+t)^{-2}\theta_2(s_\theta a)\partial,\\
fr^{-2}L_1\widetilde{a}\Delta_\omega &= M_1(1+t)^{-1}(\partial\widetilde{a})(R/r)\widetilde{Z}_m\\
&\quad + M_1(\partial\widetilde{a})(\sigma_1/(1+t)^2)(\sigma_1^{-1}\widetilde{Z}_m) + M_1(1+t)^{-2}(\partial\widetilde{a})(\sigma_1\partial + \theta_2(s_\theta a))\partial),\\
cr^{-2}\Delta_\omega\widetilde{a}\partial &= f(1+t)^{-2}\theta_2^2(s_\theta a)\partial, f(1+t)^{-1}\partial uR\widetilde{a}\partial = f(1+t)^{-2}\theta_2(s_\theta a)\partial.
\end{aligned}$$

Moreover, we have

$$|(1+|\partial_t\widetilde{a}|+|\partial_r\widetilde{a}|+\widetilde{a}^2/\sigma_1+\varepsilon^{-1}(1+t)(|Z_0\partial u|+|Z_0u/\sigma_1|))\sigma_1^{-1}\widetilde{A}|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1-\varepsilon}.$$

Proof

a. We have

$$\sqrt{c}L\widetilde{a}=(\partial_t+c\partial_r)\widetilde{a}=\varepsilon/(1+t)s_\theta a+[c,S_\theta]a_r-S_\theta(\sqrt{c}La),$$

which gives the first formula. The second formula follows, since

$$\partial_r s_\theta a=s_\theta a_r,\quad \partial_r S_\theta b=\theta_1 s_\theta b.$$

The third is also clear, using the formula of Lemma IV.5.1, since $S_\theta b_r=\theta_1 s_\theta b$.

b. From the definition of s_θ, we have

$$R_i s_\theta b=h(\omega)\theta_2 s_\theta b,\quad \Delta_\omega S_\theta b=f\theta_2^2 s_\theta b.$$

Hence

$$\begin{aligned}
r^{-2}R_j\widetilde{a}R_jL_1&=h(\omega)r^{-2}\theta_2 s_\theta aR_jL_1=M_1(1+t)^{-2}\theta_2 s_\theta a(M_1\partial+L_1\widetilde{Z}_m),\\
cr^{-2}\Delta_\omega\widetilde{a}L_1&=f(1+t)^{-2}\theta_2^2 s_\theta a\partial,\\
f(1+t)^{-1}\partial uR_j\widetilde{a}\partial&=f(1+t)^{-2}\theta_2 s_\theta a\partial.
\end{aligned}$$

c. We have

$$fr^{-2}L_1\widetilde{a}R_j^2=fr^{-1}L_1\widetilde{a}R_j(R_j/r)=M_1(1+t)^{-1}\partial\widetilde{a}([\widetilde{Z}_m,R_j/r]+R_j/r\widetilde{Z}_m).$$

Using Lemma IV.4.3, we get the result.

d. Using Lemma IV.5.1 to evaluate $[\partial_t,S_\theta]$ and Lemma IV.3.1 to express R, we can write

$$\begin{aligned}
\sqrt{c}\widetilde{A}&=\varepsilon(1+t)^{-1}\theta_1^{-1}s_\theta a_r+\varepsilon(1+t)^{-1}\theta_2^{-1}s_\theta M_1\widetilde{Z}_m^r a+[u,S_\theta]a_r\\
&\quad-S_\theta(\chi/(2\sqrt{c})L_1u(a-\widetilde{a}))+\frac{1-\chi}{2\sqrt{c}}\widetilde{Z}_m u+\chi/(2\sqrt{c})\widetilde{Z}_m u-S_\theta(\chi/(2\sqrt{c})\widetilde{Z}_m u).
\end{aligned}$$

Using the already established formula $s_\theta(\sigma_1 b)=\sigma_1 s_\theta b+\theta_1^{-1}s_\theta b$, we can bound the first three terms of $\sigma_1^{-1}\widetilde{A}$ by

$$C\varepsilon(1+t)^{-1}\theta_1^{-1}|\partial a|+C\varepsilon(1+t)^{-1}\theta_2^{-1}|M_1|(|\sigma_1^{-1}\widetilde{Z}_m^r a|+|a_r|).$$

Next, since

$$\begin{aligned}
|b-S_\theta b|&\leqslant C\theta_1^{-1}|b_r|+C\theta_2^{-1}(|b|+|Rb|),\\
|a-\widetilde{a}|&\leqslant C\theta_1^{-1}|\partial a|+C\theta_2^{-1}(|a|+|M_1||\widetilde{Z}_m a|),\\
|\chi/(2\sqrt{c})\widetilde{Z}_m u/\sigma_1-S_\theta(\chi/(2\sqrt{c})\widetilde{Z}_m u/\sigma_1)|&\leqslant C\varepsilon(1+t)^{-1}(\theta_1^{-1}|M_1|+\theta_2^{-1}|M_2|).
\end{aligned}$$

Note that the error term produced when introducing σ_1^{-1} in S_θ is bounded by

$$C\varepsilon(1+t)^{-1}\theta_1^{-1}|M_1|.$$

Finally, observing that

$$|\widetilde{Z}_m u| \leqslant C|Z_0 u|, |\partial_t \widetilde{a}| + |\partial_r \widetilde{a}| \leqslant C|a|/(1+t) + C|a_t| + C|a_r|,$$

we see that we can use Lemma IV.1.2 to control the terms containing $(1-\chi)\sigma_1^{-1}\widetilde{Z}_m u$. Taking β_1 big enough with respect to $|M_2|$ yields the desired estimate. □

VI.2. Higher order commutators. — Taking a standard cutoff $\overline{\chi} = \overline{\chi}(r/(1+t))$ (that is $\overline{\chi}(s)$ is zero for $s \leqslant 1/2$ and one for $s \geqslant 2/3$), we write

$$[\widetilde{Z}_m, P] = \overline{\chi}[\widetilde{Z}_m, P] + (1-\overline{\chi})[\widetilde{Z}_m, P],$$

and use the formula of Lemma 1.3 for the first term, the formula (1.2) of Lemma 1.1 for the second.

We need now a Lemma describing the structure of $[\widetilde{Z}_m^k, P]$.

Lemma 2.1. — *Writing in short*

$$[\widetilde{Z}_m, P] = \delta P + Q,$$

we have

i) $$[\widetilde{Z}_m^k, P] = \sum \widetilde{Z}_m^{l_1}\delta \cdots \widetilde{Z}_m^{l_i}\delta \widetilde{Z}_m^p P + \sum \widetilde{Z}_m^{k_1}\delta \cdots \widetilde{Z}_m^{k_j}\delta \widetilde{Z}_m^q Q \widetilde{Z}_m^p.$$

By an abuse of notation, we do not put indexes for the δ and Q, though there is one for each $\widetilde{Z}_m$. In the first sum,

$$i \geqslant 1, \quad p + \sum(l_j+1) \leqslant k.$$

In the second sum,

$$q + p + \sum(k_i+1) \leqslant k-1.$$

ii) $$\widetilde{Z}_m^q[\widetilde{Z}_m, P]\widetilde{Z}_m^p = \sum \widetilde{Z}_m^{l_1}\delta \cdots \widetilde{Z}_m^{l_i}\delta \widetilde{Z}_m^{p_1} P + \sum \widetilde{Z}_m^{k_1}\delta \cdots \widetilde{Z}_m^{k_j}\delta \widetilde{Z}_m^{p_1} Q \widetilde{Z}_m^{p_2}.$$

In the first sum,

$$i \geqslant 1, \quad p_1 + \sum(l_j+1) \leqslant p+q+1.$$

In the second sum,

$$p_1 + p_2 + \sum(k_i+1) \leqslant p+q.$$

Proof. — For $k=1$, i) is clear. We write now

$$[\widetilde{Z}_m^{k+1}, P] = \widetilde{Z}_m[\widetilde{Z}_m^k, P] + [\widetilde{Z}_m, P]\widetilde{Z}_m^k.$$

We see that $\widetilde{Z}_m$ acting on both sums yields only correct terms. On the other hand,

$$[\widetilde{Z}_m, P]\widetilde{Z}_m^k = \delta[P, \widetilde{Z}_m^k] + \delta \widetilde{Z}_m^k P + Q\widetilde{Z}_m^k,$$

and all three terms are of the desired form. This proves i). The proof of ii) is completely similar. □

VII. L^2 estimates of u and a

Using the structure of $P\widetilde{Z}_m^{k+1}u$ displayed in VI, and the energy inequality for P, we want to estimate now $|\partial\widetilde{Z}_m^{k+1}u|_0$. Similarly, we will estimate $|\partial\widetilde{Z}_m^k a|_0$. To this aim, we introduce some notations. We set, with $a = a(R_i)$ or $a = a(S)$,

$$A_k = (1+t)^{-1}(|\sigma_1^{-1}\widetilde{Z}_m^{k-1}a|_0 + |\widetilde{Z}_m^{k-1}\partial a|_0),\ k \geqslant 1, \quad \phi_k = \sum |\widetilde{N}_k|_0,\ k \geqslant 0,$$

$$\phi'_k = \varepsilon^{-1}(1+t)(|\widetilde{Z}_m^k\partial u|_0 + |\sigma_1^{-1}\widetilde{Z}_m^k u|_0) + |\widetilde{Z}_m^{k-1}\partial\widetilde{a}|_0 + |\sigma_1^{-1}\widetilde{Z}_m^{k-1}\widetilde{a}|_0,\ k \geqslant 1.$$

The point of these notations is that the "bad" $\widetilde{N}_k$ is $\widetilde{Z}_m^k\widetilde{a}$, that we were forced to introduce to have Lemma IV.3.1. The quantity ϕ'_k is just ϕ_k deprived of this bad term. Note that, since $\widetilde{a}$ is supported for $\sigma_1 \leqslant C(1+t)^{C_0\varepsilon}$, we have

$$\phi_k \leqslant C(1+t)^{C_0\varepsilon}\phi'_k,$$

but this "small" amplification factor is very important in all this paper. According to Lemma V.1.1, we have

$$\phi'_k \sim \varepsilon^{-1}(1+t)|\widetilde{Z}_m^k\partial u|_0 + |\widetilde{Z}_m^{k-1}\partial\widetilde{a}|_0.$$

Since the energy inequality will control $\partial\widetilde{Z}_m^{k+1}u$, we introduce also

$$\phi''_k = \varepsilon^{-1}(1+t)|\partial\widetilde{Z}_m^k u|_0 + |\widetilde{Z}_m^{k-1}\partial\widetilde{a}|_0.$$

Thanks to Lemma IV.4.2, we see that, assuming

$$\phi_l \leqslant C(1+t)^{1+C\varepsilon}, \quad l \leqslant k,$$

we obtain

$$|[\widetilde{Z}_m^{k+1},\partial]u|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}|\widetilde{Z}_m^k\partial\widetilde{a}|_0.$$

It follows that

$$\phi'_{k+1} \leqslant C(1+t)^{1+C\varepsilon} + C\phi''_{k+1}, \phi''_{k+1} \leqslant C(1+t)^{1+C\varepsilon} + C\phi'_{k+1}.$$

VII.1. L^2 estimates of u

Proposition 1. — *We can choose β_1 and $\beta_2 - \beta_1$ big enough to ensure the following implication: Assume that, for $0 \leqslant l \leqslant k \leqslant 2(s_0-4)-1$, we have*

$$|\widetilde{N}_l|_0 \leqslant C(1+t)^{1+C\varepsilon}, \quad A_l \leqslant C(1+t)^{C\varepsilon}.$$

Then, for some $\gamma > 0$,

$$\begin{aligned}|P\widetilde{Z}_m^{k+1}u|_0 \leqslant{}& C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon^2(1+t)^{-1+C\varepsilon} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}\\ &+ C\varepsilon^2(1+t)^{-2}\phi'_{k+1} + C\varepsilon(1+t)^{-1-\gamma}A_{k+1} + C\varepsilon^2(1+t)^{-1}A_{k+1}\\ &+ C(1+t)^{-1+C\varepsilon}\Big(\int_{\sigma_1 \leqslant C(1+t)^{C_0\varepsilon}} e^p(T_i\widetilde{Z}_m^{k+1}u)^2dx\Big)^{1/2}.\end{aligned}$$

Remark that the statement of the Proposition does not change if we replace ϕ'_{k+1} by ϕ''_{k+1}.

Proof. — Before proceeding, let us explain how we classify the various terms of $P\widetilde{Z}_m^{k+1}u$. We call SC (for subcritical) the terms which can be estimated by already known quantities, that is, using the induction hypothesis. We wish these SC terms to be bounded in weighted norm either by $\varepsilon(1+t)^{-1-\gamma}$, or by $\varepsilon^2(1+t)^{-1+C\varepsilon}$ (γ will denote here various strictly positive numbers). In both cases,

$$\int_0^t |\text{SC term}|_0 dt' \leqslant C\varepsilon(1+t)^{C\varepsilon}.$$

The other terms are called C (critical) terms, and are more delicate to handle, since we want them to be bounded by quantities we control directly through energy inequalities, in such a way that application of Gronwall's Lemma will be possible without damage. More precisely, the quantities we expect to control are

$$|\partial\widetilde{Z}_m^{k+1}u|_0, \quad |\partial\widetilde{Z}_m^k a|_0,$$

using the inequalities for P and for L respectively. The C terms for which we have an easy control will be bounded by $\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}$ or $\varepsilon(1+t)^{-1-\gamma}A_{k+1}$. The limiting case will be C terms bounded by $\varepsilon^2(1+t)^{-2}\phi'_{k+1}$ or $\varepsilon^2(1+t)^{-1}A_{k+1}$. Finally, one term involves the special derivatives T_i, and is expected to be handled using the control of these special derivatives given in Proposition V.3.1.

A. 1. According to Lemma 2.1, ignoring $\overline{\chi}$ here, we have

$$P\widetilde{Z}_m^{k+1}u = \sum \widetilde{Z}_m^{k_1}\delta\cdots\widetilde{Z}_m^{k_j}\delta\widetilde{Z}_m^q(\textstyle\sum_1+\sum_2+\sum_3)(\widetilde{Z}_m^r u),$$

with

$$q+r+\textstyle\sum(k_i+1)\leqslant k.$$

We are going to write down operators like $\widetilde{Z}_m^q Q$, and estimate the corresponding terms. With the notations of Proposition 1.1,

$$\widetilde{Z}_m^q Q = \widetilde{Z}_m^q\textstyle\sum_1 + \widetilde{Z}_m^q\sum_2 + \widetilde{Z}_m^q\sum_3.$$

A.2. We have

$$\widetilde{Z}_m^q\textstyle\sum_1 = \displaystyle\sum_{q_1+q_2+q_3=q}\widetilde{Z}_m^{q_1}M_1\widetilde{Z}_m^{q_2}\alpha[\widetilde{Z}_m^{q_3}\partial\widetilde{Z}_m + \widetilde{Z}_m^{q_3}(\sigma_1^{-1}\widetilde{Z}_m) + \widetilde{Z}_m^{q_3}\partial].$$

Now $\widetilde{Z}_m^{q_1}M_1 = M_{1+q_1}$,

$$\widetilde{Z}_m^{q_2}(\sigma_1/(1+t)^2(\varepsilon^{-1}(1+t)\partial u)) = \sigma_1/(1+t)^2\textstyle\sum' f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}, \quad \sum l_j\leqslant q_2,$$

$$\widetilde{Z}_m^{q_2}(\sigma_1/(1+t)^2\widetilde{N}_1) = \sigma_1/(1+t)^2\textstyle\sum' f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_j}, \quad \sum k_i\leqslant 1+q_2.$$

Using Lemma IV.4.2, we also have

$$\widetilde{Z}_m^{q_3}\partial = \partial\widetilde{Z}_m^{q_3} + \textstyle\sum f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}\partial\widetilde{Z}_m^p, \quad \sum l_i+p\leqslant q_3,\ p\leqslant q_3-1,$$

$$\widetilde{Z}_m^{q_3}(\sigma_1^{-1}\widetilde{Z}_m) = \sigma_1^{-1}\textstyle\sum f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}\widetilde{Z}_m^{p'}, \quad \sum l_j+p'\leqslant q_3+1.$$

We will often use the following standard remark: we have

$$M_r = f\widetilde{N}_1^l\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_j}, \quad l_i\geqslant 2, \quad \textstyle\sum(l_i-1)\leqslant r-1.$$

Either all l_i are $\leqslant s_0-4$, or one of them at least is $\geqslant s_0-3$; in the latter case, noting $\sum'(l_i-1)$ the sum corresponding to the other indexes, we have

$$\sum{}'(l_i-1)+s_0-4\leqslant r-1.$$

If $r\leqslant 2(s_0-4)$, this implies that for all other indexes, $l_i\leqslant s_0-4$.

Hence

$$|M_{k+1}|_0\leqslant C(1+t)^{C\varepsilon}|\widetilde{N}_{k+1}|_0+C(1+t)^{1+C\varepsilon},$$
$$|M_r|_0\leqslant C(1+t)^{1+C\varepsilon},\quad r\leqslant k.$$

If $q_1=q=k$, the corresponding term in $\widetilde{Z}_m^q\sum_1\widetilde{Z}_m^r u$ is bounded in weighted L^2 norm by

$$|M_{k+1}|_0|\alpha\varepsilon(1+t)^{-1}\sigma_1^{\mu-1}|_{L^\infty}\leqslant C\varepsilon(1+t)^{-2-\gamma}(1+t+|\widetilde{N}_{k+1}|_0).$$

If $q_2=q=k$, the corresponding term has the same bound, and also if $q_3=q=k$ or $r=k$. In all other cases, the term is bounded by $C\varepsilon(1+t)^{-1-\gamma}$. Since $\delta=f\widetilde{N}_1$, $\widetilde{Z}_m^{k_i}\delta=M_{1+k_i}, 1+k_i\leqslant k$, the term involving $\sum_1$ in $P\widetilde{Z}_m^{k+1}u$ is bounded in weighted L^2 norm by

$$C\varepsilon(1+t)^{-1-\gamma}+C\varepsilon(1+t)^{-2-\gamma}|\widetilde{N}_{k+1}|_0.$$

A.3. We turn now to the terms involving $\sum_2$. We have

$$\widetilde{Z}_m^q\sum{}_2\widetilde{Z}_m^r u=\sum_{q_1+q_2=q}\widetilde{Z}_m^{q_1}(f\partial u)\widetilde{Z}_m^{q_2}\partial\widetilde{Z}_m^r u+\widetilde{Z}_m^q(f\partial\widetilde{Z}_m u\partial\widetilde{Z}_m^r u).$$

For the first term, we have as before,

$$\widetilde{Z}_m^{q_1}(f\partial u)=\varepsilon(1+t)^{-1}M_{q_1},\quad q_1\geqslant 1,$$
$$\widetilde{Z}_m^{q_2}\partial\widetilde{Z}_m^r u=\partial\widetilde{Z}_m^{q_2+r}u+\sum f\widetilde{N}_{l_1}\cdots\widetilde{N}_{l_i}\partial\widetilde{Z}_m^{p+r}u,\quad \sum l_j+p\leqslant q_2,\ p\leqslant q_2-1.$$

If $q_2+r=k$, necessarily $q_1=0$ and no δ terms are present, hence the corresponding term is bounded by

$$C\varepsilon(1+t)^{-1}|\widetilde{Z}_m^{k+1}u|_0.$$

If $q_2+r\leqslant k-1$, the weighted L^2 norm is bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$. For the second term, either $q=k$ and all derivatives fall on the middle term to give $f\partial\widetilde{Z}_m^{k+1}u\partial u$, or the powers of $\widetilde{Z}_m$ acting on u are all at most k. In the first case, the L^2 norm is bounded by

$$C\varepsilon/(1+t)|e^{p/2}\partial\widetilde{Z}_m^{k+1}u|_{L^2}.$$

In the second case, it is bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$. To summarize, the term involving $\sum_2$ in $P\widetilde{Z}_m^{k+1}u$ is bounded in weighted L^2 norm by

$$C\varepsilon^2(1+t)^{-1+C\varepsilon}+C\varepsilon(1+t)^{-1}|e^{p/2}\partial\widetilde{Z}_m^{k+1}u|_{L^2}.$$

B. We turn now to the special terms $\widetilde{Z}_m^q\sum_3\widetilde{Z}_m^p u$. We claim that, if $p+q\leqslant k-1$, all these terms are SC. In particular, any term in $P\widetilde{Z}_m^{k+1}u$ containing at least one δ factor will be SC, since then

$$p+q\leqslant k-1,\quad k_i\leqslant k-1.$$

It is also important to remark that, for SC terms supported for $\sigma_1 \leqslant C(1+t)^{C_0\varepsilon}$, powers of σ_1 are not crucial, since extra factors $(1+t)^{C\varepsilon}$ are admitted in the estimate of the Proposition. In what follows, the index r is always $r=0$ or $r=1$.

B.1. We have

$$\widetilde{Z}_m^q(r^{-2}R_j\widetilde{a}R_jL_1\widetilde{Z}_m^p u) = M_{l_1}(1+t)^{-2}\theta_2(\widetilde{Z}_m^{p_1}s_\theta a)\partial\widetilde{Z}_m^{p_2+p+r}u,$$

with $l_1-1+p_1+p_2 \leqslant q$. If $p+q\leqslant k-1$, the term is SC, and also for $p+q=k$, except for the C terms

$$M_1(1+t)^{-2}\theta_2(s_\theta a)\partial\widetilde{Z}_m^{k+1}u,\ M_{k+1}(1+t)^{-2}\theta_2(s_\theta a)\partial\widetilde{Z}_m^{r}u,\ M_1(1+t)^{-2}\theta_2(\widetilde{Z}_m^{k}s_\theta a)\partial\widetilde{Z}_m^{r}u.$$

Considering the last term, using Lemma IV.5.3, we see that all terms in $[\widetilde{Z}_m^k, s_\theta]a$ are SC, except

$$\begin{aligned}\theta_1^{-1}(M_1s_\theta[M_1;]M_1\widetilde{Z}_m^k a + M_1s_\theta[M_k;]M_1\widetilde{Z}_m^r a + M_1s_\theta[;M_1]M_1\widetilde{Z}_m^k a\\ + M_1s_\theta[;M_k]M_1\widetilde{Z}_m^r a + M_1s_\theta M_1\widetilde{Z}_m^k a + M_1s_\theta[;M_ka + M_1\widetilde{Z}_m^{k-1}a]M_1).\end{aligned}$$

These terms are bounded in weighted L^2 norm by $K_1\times|\widetilde{Z}_m^k a|_0, K_2\times|M_{k+1}|_0$. Here,

$$K_1 = \theta_1^{-1}|M_2|_{L^\infty},\quad K_2 = \theta_1^{-1}(|\widetilde{Z}_m^r a|_{L^\infty} + |M_1|_{L^\infty}).$$

Let us explain here once for all the meaning of such expressions. The notation M_1, M_2 etc. is a commodity not to write explicitly the exact powers of $\widetilde{N}_1$ involved. The point is that these powers, in the finite computation we are doing here (once s_0 has been chosen), never exceed some number depending on s_0. The important fact is that, according to Proposition IV.6, $\widetilde{N}_l, l\leqslant s_0-4$ is bounded in L^∞ norm by $C(1+t)^{C_1\varepsilon}$, where C_1 does not depend on θ. Hence, here and in what follows, we can choose β_1 big enough to have

$$|K_i| \leqslant C(1+t)^{-C'\varepsilon},$$

with C' as big as we want.

Returning to our term, we see that its norm does not exceed

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1} + C\varepsilon(1+t)^{-1-\gamma}A_{k+1}.$$

The same analysis applies to the terms coming from

$$cr^{-2}\Delta_\omega\widetilde{a}\partial,\quad f(1+t)^{-1}\partial uR\widetilde{a}\partial,$$

with the same result.

B.2. We have

$$\begin{aligned}\widetilde{Z}_m^q(fr^{-2}L_1\widetilde{a}\Delta_\omega\widetilde{Z}_m^p u) = \widetilde{Z}_m^q(M_1(1+t)^{-1}(\partial\widetilde{a})(R/r)\widetilde{Z}_m^{p+1}u)\\ + \widetilde{Z}_m^q(M_1(\partial\widetilde{a})(\sigma_1/(1+t)^2)(\sigma_1^{-1}\widetilde{Z}_m^{p+1}u)) + \widetilde{Z}_m^q(M_1(\partial\widetilde{a})(\sigma_1/(1+t)^2)\partial\widetilde{Z}_m^p u)\\ + \widetilde{Z}_m^q(M_1(1+t)^{-2}\theta_2(\partial\widetilde{a})(s_\theta a)\partial\widetilde{Z}_m^p u) = (1)+(2)+(3)+(4).\end{aligned}$$

The term (4) is handled just as in **B.1.** The term (3) is analogous to (2), with one less derivative. We have

$$(2) = M_{l_1}(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\sigma_1^{-1}\widetilde{Z}_m^{p_2+1}u,$$

with $l_1 - 1 + p_1 + p_2 \leqslant p + q$. The only C terms here are

$$M_1(\sigma_1/(1+t)^2)(\partial\widetilde{a})\sigma_1^{-1}\widetilde{Z}_m^{k+1}u, \quad M_1(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{k}\partial\widetilde{a})(\sigma_1^{-1}\widetilde{Z}_m u),$$

$$M_{k+1}(\sigma_1/(1+t)^2)(\partial\widetilde{a})(\sigma_1^{-1}\widetilde{Z}_m u).$$

They are bounded by $C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}$. The SC terms are bounded by $C\varepsilon(1+t)^{-1-\gamma}$. Using Lemma IV.4.3, we can write

$$\begin{aligned}(1) = &\sum_{l_1-1+p_1+p_2\leqslant q} M_{l_1}(1+t)^{-1}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})(R/r)\widetilde{Z}_m^{p_2+p+1}u\\ &+\sum_{l_1-1+p_1+p_2\leqslant q} M_{l_1}(1+t)^{-2}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\widetilde{Z}_m^{p_2+p+1}u\\ &+\sum_{l_1-1+p_1+p_2\leqslant q-1} M_{l_1}(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\partial\widetilde{Z}_m^{p_2+p+1}u\\ &+\sum_{l_1-1+p_1+p_2+p_3\leqslant q-1} M_{l_1}(\theta_2/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})(\widetilde{Z}_m^{p_2}s_\theta a)\partial\widetilde{Z}_m^{p_3+p+1}u.\end{aligned}$$

The last three terms come from the commutator of R/r with some power of $\widetilde{Z}_m$, and the last two are SC and bounded by $C\varepsilon(1+t)^{-1-\gamma}$. The second term is easily handled and bounded by

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}.$$

If $p_2 + p + 1 \leqslant k$, the first term can be rewritten as

$$M_{l_1}(1+t)^{-2}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\widetilde{Z}_m^{p_2+p+2}u.$$

The C terms are then

$$M_{k+1}(1+t)^{-2}(\partial\widetilde{a})\widetilde{Z}_m^{2}u, \quad M_1(1+t)^{-2}(\widetilde{Z}_m^{k}\partial\widetilde{a})\widetilde{Z}_m^{2}u,$$

$$M_{l_1}(\sigma_1/(1+t)^2)(\widetilde{Z}_m^{p_1}\partial\widetilde{a})(\sigma_1^{-1}\widetilde{Z}_m^{k+1}u), \quad l_1 + p_1 \leqslant 2.$$

All others are SC terms. The sum of all terms is bounded by

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}.$$

If $p_2 + p + 1 = k + 1$, we keep the first term as

$$M_1(1+t)^{-1}(\partial\widetilde{a})R/r\widetilde{Z}_m^{k+1}u.$$

If we think of R/r as $f\partial$, we cannot control this term. We have to keep in mind that $R/r = fT_i$ and keep the term as such for later treatment.

B.3. We have

$$\begin{aligned}\widetilde{Z}_m^q(f\partial uL_1\widetilde{a}\partial\widetilde{Z}_m^p u) &= \widetilde{Z}_m^q(f\varepsilon(1+t)^{-1}\partial\widetilde{a}\partial\widetilde{Z}_m^p u)\\ &= \sum_{p_1+p_2\leqslant p+q} f\varepsilon(1+t)^{-1}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\partial\widetilde{Z}_m^{p_2}u\\ &\qquad + \sum_{l_1-1+p_1+p_2\leqslant p+q-1} M_{l_1}\varepsilon(1+t)^{-1}(\widetilde{Z}_m^{p_1}\partial\widetilde{a})\partial\widetilde{Z}_m^{p_2}u.\end{aligned}$$

All terms in the second sum are SC. The only C term in the first sum is

$$f\varepsilon(1+t)^{-1}\widetilde{Z}_m^k\partial\widetilde{a}\partial u,$$

which is bounded by $C\varepsilon^2(1+t)^{-2}\phi'_{k+1}$. All SC terms from both sums are bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$.

C. To understand the behavior of the last three special terms

$$-\widetilde{A}L_1^2,\quad fL_1L\widetilde{a}\partial,\quad f(1+t)^{-1}L\widetilde{a}\partial,$$

we cannot consider $L\widetilde{a}$ as an $\widetilde{N}_1$. We need make explicit its closeness to La, and, in particular, show that a factor ε is present in its estimates.

C.1 We prove the following estimates:

$$\begin{aligned}&|\widetilde{Z}_m^l(\sqrt{c}L\widetilde{a})|_{L^\infty}\leqslant C\varepsilon(1+t)^{-1+C\varepsilon},\quad l\leqslant s_0-5,\\ &|\widetilde{Z}_m^l(\sqrt{c}L\widetilde{a})|_0\leqslant C\varepsilon(1+t)^{C\varepsilon},\quad l\leqslant k-1,\\ &|\widetilde{Z}_m^k(\sqrt{c}L\widetilde{a})|_0\leqslant C\varepsilon(1+t)^{C\varepsilon}+C\varepsilon(1+t)^{-1+C\varepsilon}\phi_{k+1}+C\varepsilon(1+t)^{C\varepsilon}A_{k+1}.\end{aligned}$$

From Lemma 1.4, we have with a self-defined E

$$\sqrt{c}L\widetilde{a}=-S_\theta E+\varepsilon(1+t)^{-1}s_\theta a+[u,S_\theta]a_r.$$

a. We have

$$\begin{aligned}\widetilde{Z}_m^l E = (\chi/(2\sqrt{c})(\widetilde{Z}_m^{l+1}u+(\widetilde{Z}_m^l a-\widetilde{Z}_m^l\widetilde{a})L_1u+\sum_{l'\geqslant 1}(\widetilde{Z}_m^{l-l'}a-\widetilde{Z}_m^{l-l'}\widetilde{a})\widetilde{Z}_m^{l'}L_1u))\\ +\sum_{l'\leqslant l-1}M_{l-l'}(\widetilde{Z}_m^{l'+1}u+\textstyle\sum(\widetilde{Z}_m^{l''}a-\widetilde{Z}_m^{l''}\widetilde{a})\widetilde{Z}_m^{l'-l''}L_1u).\end{aligned}$$

If $l\leqslant k-1$, all terms in

$$[\widetilde{Z}_m^l,S_\theta]E+S_\theta\widetilde{Z}_m^l E$$

are SC, and are easily seen to be bounded as indicated. If $l=k$, the only C terms in $\widetilde{Z}_m^k E$ are

$$f(\widetilde{Z}_m^{k+1}u+(\widetilde{Z}_m^k a-\widetilde{Z}_m^k\widetilde{a})L_1u).$$

These terms, and also all other SC terms, are bounded as desired. Since we do not care about factors $(1+t)^{C\varepsilon}$ here, we see that the same bounds are true also for all terms in $[\widetilde{Z}_m^k,S_\theta]E$.

b. We have

$$\widetilde{Z}_m^l(\varepsilon(1+t)^{-1}s_\theta a) = \varepsilon(1+t)^{-1}([\widetilde{Z}_m^l, s_\theta]a + s_\theta(\widetilde{Z}_m^l a) + \sum_{l_1+l_2\leqslant l} M_{l_1}\widetilde{Z}_m^{l_2}s_\theta a).$$

The terms in the last sum are all SC, and bounded as desired. For $l \leqslant k-1$, all other terms are also SC and appropriately bounded. For $l = k$,

$$|\widetilde{Z}_m^k a|_0 \leqslant C(1+t)^{1+C_0\varepsilon}A_{k+1},$$
$$[\widetilde{Z}_m^k, s_\theta]a|_0 \leqslant C(1+t)^{1+C\varepsilon}A_{k+1} + C(1+t)^{C\varepsilon}\phi_{k+1}.$$

c. We write

$$\widetilde{Z}_m^l([u, S_\theta]a_r) = \sum_{l_1+l_2=l} \widetilde{Z}_m^{l_1}u\widetilde{Z}_m^{l_2}S_\theta a_r - [\widetilde{Z}_m^l, S_\theta]ua_r - S_\theta\widetilde{Z}_m^l(ua_r).$$

We do not use here the bracket structure, estimating each term separately.

C.2. We prove the following estimates, where $A = \partial_r$ or $A = \partial_t$:

$$|\widetilde{Z}_m^l AL\widetilde{a}|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}, \quad l \leqslant s_0 - 5,$$
$$|\widetilde{Z}_m^l AL\widetilde{a}|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad l \leqslant k-1,$$
$$|\widetilde{Z}_m^k AL\widetilde{a}|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}\phi'_{k+1} + C\varepsilon A_{k+1}.$$

We handle only $A = \partial_t$, the other case being similar and easier.

a. We have first

$$\partial_t S_\theta E = S_\theta E_t + \varepsilon s_\theta(1+t)^{-1}E,$$
$$E_t = \partial_t(\chi/(2\sqrt{c}))(\widetilde{Z}_m u + (a-\widetilde{a})L_1 u)$$
$$+ (\chi/(2\sqrt{c}))(\partial_t\widetilde{Z}_m u + (a_t - \partial_t\widetilde{a})L_1 u + (a-\widetilde{a})\partial_t L_1 u).$$

We note first that

$$\partial(\chi/(2\sqrt{c})) = f\sigma_1^{-1}.$$

We observe now, using Lemma 1.4, that all terms in $\partial_t(\sqrt{c}L\widetilde{a})$ are either

i) linear in $\partial_t^r\widetilde{Z}_m u$ $(r \leqslant 1)$,

ii) bilinear in $\partial^r u$ $(r \leqslant 1)$ and $\partial^{r'}a$ $(r' \leqslant 1)$ or $\partial^{r'}\widetilde{a}$ $(r' \leqslant 1)$, with the exception of $(a-\widetilde{a})\partial_t L_1 u$,

iii) linear in $\partial^r a$ $(r \leqslant 1)$ with a coefficient at least as good as $\varepsilon(1+t)^{-1}$.

Since we do not care about factors $(1+t)^{C\varepsilon}$ in the estimation of SC terms, we obtain that all SC terms in $\widetilde{Z}_m^l AL\widetilde{a}$ have the desired bound. We concentrate therefore on C terms, which can occur only for $l = k$. If we ignore at first the bracket terms in $\widetilde{Z}_m^k\partial_t S_\theta E$, we can consider only $S_\theta\widetilde{Z}_m^k E_t$, since the other term involving $(1+t)^{-1}E$ is similar and simpler. The C terms in $\widetilde{Z}_m^k E_t$ are

$$f\sigma_1^{-1}\widetilde{Z}_m^{k+1}u, \quad f\sigma_1^{-1}(\widetilde{Z}_m^k a - \widetilde{Z}_m^k\widetilde{a})L_1 u,$$
$$\widetilde{Z}_m^k\partial_t\widetilde{Z}_m u, \quad (\widetilde{Z}_m^k a_t - \widetilde{Z}_m^k\partial_t\widetilde{a})L_1 u, \quad (\widetilde{Z}_m^k a - \widetilde{Z}_m^k\widetilde{a})\partial_t L_1 u, (a-\widetilde{a})\widetilde{Z}_m^k\partial_t L_1 u.$$

Except for the last two terms, they are respectively bounded in weighted L^2 norm by

$$C\varepsilon(1+t)^{-1}\phi'_{k+1},\quad C\varepsilon A_{k+1}+C\varepsilon(1+t)^{-1}\phi'_{k+1},$$
$$C\varepsilon(1+t)^{C\varepsilon}+C\varepsilon(1+t)^{-1}\phi'_{k+1},\quad C\varepsilon A_{k+1}+C\varepsilon(1+t)^{-1}\phi'_{k+1}.$$

We now write

$$|\partial_t L_1 u|\leqslant C|LL_1u|+C|L_1^2u|\leqslant C\varepsilon\sigma_1^{\mu-1}(1+t)^{-2+C\varepsilon}+C|L_1^2u|.$$

Thus, using the Poincaré Lemma, and Lemma IV.4.2, we obtain

$$|(\widetilde{Z}_m^k a-\widetilde{Z}_m^k\widetilde{a})\partial_t L_1 u|_0\leqslant C\varepsilon(1+t)^{-1}\phi'_{k+1}+C\varepsilon A_{k+1}+C\varepsilon(1+t)^{C\varepsilon}.$$

For the last term, we write

$$|\widetilde{Z}_m^k\partial_t L_1u|_0\leqslant C|\widetilde{Z}_m^{k+1}\partial u|_0+C\varepsilon(1+t)^{C\varepsilon},\quad |a-\widetilde{a}|\leqslant C\theta_1^{-1}|M_2|.$$

Choosing β_1 big enough in the sense we have already explained will give $|a-\widetilde{a}|\leqslant C$, which finishes the estimate of $|S_\theta\widetilde{Z}_m^k E_t|_0$. Now, as explained before, if β_1 has been chosen big enough, the bracket terms

$$[\widetilde{Z}_m^k,\quad S_\theta]E_t,[\widetilde{Z}_m^k,\quad s_\theta](1+t)^{-1}E$$

generate terms having the same bound, except the terms involving M_{k+1}. This terms will be bounded by

$$C\theta_1^{-1}\varepsilon(1+t)^{-1+C\varepsilon}\phi_{k+1}\leqslant C\theta_1^{-1}\varepsilon(1+t)^{-1+C\varepsilon}\phi'_{k+1},$$

which have the desired bound if β_1 is big enough.

b. The term $\varepsilon/(1+t)^2 s_\theta a$ is much better than $\varepsilon/(1+t)s_\theta a_t$, and similarly the terms $u/(1+t)s_\theta a_r,(1+t)^{-1}s_\theta u a_r$ are much better than $u_tS_\theta a_r, S_\theta u_t a_r$. Considering only

$$\widetilde{Z}_m^l(\varepsilon/(1+t)s_\theta a_t+u_tS_\theta a_r-S_\theta u_t a_r+S_\theta u_r a_t),$$

we see that all terms are SC, except when $l=k$. In this latter case, the C terms may come only from

$$\varepsilon/(1+t)\widetilde{Z}_m^k s_\theta a_t+u_t\widetilde{Z}_m^k S_\theta a_r-\widetilde{Z}_m^k S_\theta u_t a_r+\widetilde{Z}_m^k S_\theta u_r a_t.$$

Ignoring first the bracket terms, we obtain the desired bound $C\varepsilon A_{k+1}$ for the C terms, and the bound $C\varepsilon(1+t)^{C\varepsilon}$ for the SC terms. For the bracket terms, we proceed as before, getting the same bound plus $C\varepsilon(1+t)^{-1}\phi'_{k+1}$.

c. The term $\theta_1[u,s_\theta]a_t$, being already amplified by θ_1, is the most delicate to handle. We write

$$\widetilde{Z}_m^l([u,s_\theta]a_t)=\sum_{l_1+l_2=l,l_1\geqslant 1}\widetilde{Z}_m^{l_1}u\widetilde{Z}_m^{l_2}s_\theta a_t+u\widetilde{Z}_m^l s_\theta a_t-\widetilde{Z}_m^l s_\theta u a_t.$$

If $l\leqslant k-1$, all terms are SC and bounded as desired. If $l=k$, we use Lemma IV.5.4 to express $\widetilde{Z}_m^k s_\theta$. This Lemma displays terms of three types:

i) A term $fs_\theta f\widetilde{Z}_m^k$, critical with no amplification,

ii) Subcritical terms,

iii) Possibly critical terms accompanied by a factor θ_2^{-1}.

We obtain

$$\theta_1 \widetilde{Z}_m^k[u, s_\theta]a_t = \theta_1 f[u, s_\theta] f \widetilde{Z}_m^k a_t + \theta_1 \times \text{SC terms} + \theta_1\theta_2^{-1}(\dots).$$

The first term is bounded in weighted L^2 norm by

$$C(|u_r|_{L^\infty} + \theta_1/\theta_2 |Ru|_{L^\infty})|\widetilde{Z}_m^k a_t|_0.$$

Having chosen β_1, we can choose β_2 big enough with respect to β_1 to obtain (on the support of a)

$$\theta_1/\theta_2 |Ru|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1}.$$

The SC terms are bilinear in u and a and bounded by $C\varepsilon(1+t)^{C\varepsilon}$. The terms containing $\theta_1\theta_2^{-1}$ are either SC, or involve $\widetilde{Z}_m^k a_t$ or M_{k+1}. We handle them as usual, choosing $\beta_2 - \beta_1$ big enough if necessary, and get the bound

$$C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon A_{k+1} + C\varepsilon(1+t)^{-1}\phi'_{k+1}.$$

C.3. Consider now the term $\widetilde{Z}_m^q(f(1+t)^{-1}L\widetilde{a}\partial\widetilde{Z}_m^p u)$. All terms are SC, except if $p=0$, $q=k$ the only term

$$f(1+t)^{-1}\widetilde{Z}_m^k L\widetilde{a}\partial u.$$

According to the estimates of **C.1**, the weighted L^2 norm of these terms is bounded by

$$C\varepsilon(1+t)^{-1-\gamma} + C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1} + C\varepsilon(1+t)^{-1-\gamma}A_{k+1}.$$

C.4. We use the estimates of **C.2** to handle the term $\widetilde{Z}_m^q(fAL\widetilde{a}\partial\widetilde{Z}_m^p u)$. We obtain right away the bound

$$C\varepsilon^2(1+t)^{-1+C\varepsilon} + C\varepsilon^2(1+t)^{-2}\phi'_{k+1} + C\varepsilon^2(1+t)^{-1}A_{k+1}.$$

C.5. We consider finally

$$\widetilde{Z}_m^q(\widetilde{A}L_1^2\widetilde{Z}_m^p u) = \sum_{q_1+q_2=q}(\widetilde{Z}_m^{q_1}\widetilde{A})\widetilde{Z}_m^{q_2}L_1^2\widetilde{Z}_m^p u.$$

a. For $1 \leqslant q_1 \leqslant k-1$, all terms are SC. Remembering that $L_1 = f\widetilde{Z}_m$, we write

$$(\widetilde{Z}_m^{q_1}L\widetilde{a})\widetilde{Z}_m^{q_2} f\widetilde{Z}_m f\partial\widetilde{Z}_m^p u.$$

Using the estimates of **C.1**, we see that these SC terms are bounded as desired. For the other part of $\widetilde{A}$, we write, since $L_1^2 = f\partial^2 + f/(1+t)\partial$,

$$\widetilde{Z}_m^{q_1+1}u\widetilde{Z}_m^{q_2}L_1^2\widetilde{Z}_m^p u = M_l(1+t)^{-1}(\widetilde{Z}_m^{q_1+1}u)\widetilde{Z}_m^s\partial u + M_l(\widetilde{Z}_m^{q_1+1}u)\partial\widetilde{Z}_m^s\partial u,$$

where in both sums $l-1+s \leqslant p+q_2$. Using $\partial = M_1\sigma_1^{-1}\widetilde{Z}_m$ in the last term, we see that all these SC terms are bounded by $C\varepsilon^2(1+t)^{-1+C\varepsilon}$.

b. If $q_1 = k$, we have the term $\widetilde{Z}_m^k\widetilde{A}L_1^2 u$, which gives (apart from trivial SC terms) $\widetilde{Z}_m^k L\widetilde{a}L_1^2 u, \widetilde{Z}_m^{k+1}uL_1^2 u$. Using Poincaré Lemma, we see that the last term is bounded by

$$C\varepsilon/(1+t)|\partial\widetilde{Z}_m^{k+1}u|_0,$$

which is the desired bound. Similarly, the first term is bounded by

$$C\varepsilon/(1+t)|\partial_r \widetilde{Z}_m^k L\widetilde{a}|_0.$$

Here arises a slight technical difficulty: the commutation of ∂_r with $\widetilde{Z}_m^k$ yields non radial derivatives, and our **C.2** estimates are only for $A = \partial_t$ or $A = \partial_r$. We have easily, in the spirit of Lemma 4.2,

$$[\widetilde{Z}_m^k, \partial_r] = \sum_{l \leqslant k-1} M_{k-l} \widetilde{Z}_m^l \partial.$$

Hence

$$\partial_r \widetilde{Z}_m^k L\widetilde{a} = \widetilde{Z}_m^k \partial_r L\widetilde{a} + \sum M_{k-l} \widetilde{Z}_m^l \partial L\widetilde{a}.$$

If, in the last term, $\partial = \partial_t$, we use **C.2**. If not, we write

$$\partial_i = \omega_i \partial_r + f/(1+t)R = f\partial_r + M_1/(1+t)\widetilde{Z}_m,$$

and $\widetilde{Z}_m^l \partial L\widetilde{a}$ yields either SC terms involving $\widetilde{Z}_m^{l'} \partial_r L\widetilde{a}$ that we have already handled (in **C.4**), or terms

$$\sum_{l' \leqslant l \leqslant k-1} (1+t)^{-1} M_{l-l'+1} \widetilde{Z}_m^{l'+1} L\widetilde{a}$$

that we have already handled in **C.3**.

c. Finally, if $q_1 = 0$, apart from already discussed terms, we are left with

$$\widetilde{A} L_1^2 \widetilde{Z}_m^k u.$$

We proceed now exactly as in the proof of Proposition IV.1, e), writing

$$\sigma_1 L_1 = f\widetilde{Z}_m + f(\widetilde{a}/\sigma_1)\widetilde{Z}_m + f\widetilde{a}^2/\sigma_1 L_1,$$

so our term is

$$(\widetilde{A}/\sigma_1)(f\widetilde{Z}_m L_1 \widetilde{Z}_m^k u + f(\widetilde{a}/\sigma_1)\widetilde{Z}_m L_1 \widetilde{Z}_m^k u + f(\widetilde{a}^2/\sigma_1)\widetilde{Z}_m L_1 \widetilde{Z}_m^k u).$$

Using the estimate of Lemma 1.4, we obtain the bound

$$C\varepsilon^2(1+t)^{-1+C\varepsilon} + C\varepsilon^2(1+t)^{-2-\varepsilon}\phi'_{k+1}.$$

D. Taking $\overline{\chi}$ into account now, using Lemma 2.1, we obtain

$$[\widetilde{Z}_m^{k+1}, P] = \sum \widetilde{Z}_m^q [\widetilde{Z}_m, P] \widetilde{Z}_m^p$$

as the sum of the main term

$$\overline{\chi} \sum \widetilde{Z}_m^q [\widetilde{Z}_m, P] \widetilde{Z}_m^p,$$

and terms supported on the support of $1 - \overline{\chi}$. The main term has been analyzed in **A, B, C** using the expression of $[\widetilde{Z}_m, P]$ given in Proposition 1.1. The other terms are easily analyzed using formula (1.2) of Lemma 1.1, and yield terms bounded by

$$C\varepsilon(1+t)^{-2-\gamma}\phi_{k+1}. \qquad \square$$

VII.2. L^2 estimates of a. — We estimate now the perturbation coefficients.

Proposition 2. — *We have the estimate*

$$(1+t)^{-1}|\widetilde{Z}_m^k \partial a|_0 \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-1+C\varepsilon}|\partial \widetilde{Z}_m^{k+1} u|_0$$
$$+ C\int_0^t \varepsilon A_{k+1} ds/(1+s) + C\int_0^t A_{k+1}(1+s)^{-1-\gamma} ds + C\int_0^t \varepsilon \phi'_{k+1}(s) ds/(1+s)^2.$$

Proof. — The new difficulty here is that the fields H_i do not commute very well with L: we have to use H_0 instead.

1. We construct a calculus just as in IV.3. We define f as before, and keep the fields

$$\widetilde{R}_i^m = R_i + \widetilde{a}(R_i)L_1, \quad \widetilde{S}^m = S + \widetilde{a}(S)L_1, \quad \widetilde{K} = L + L_1.$$

We replace $\widetilde{H}_i^m = H_i + \widetilde{a}(H_i)L_1$ by $\overline{H}_m = H_0 - \widetilde{a}(S)L_1$. We denote by $\overline{Z}_m$ any one of the fields

$$\widetilde{R}_i^m, \ \widetilde{S}^m, \ \widetilde{K}, \ H_0 - \widetilde{a}(S)L_1,$$

and by $\overline{N}_k$ any of the quantities

$$(1+t)\varepsilon^{-1}\sigma_1^{-1}\overline{Z}_m^k u, \ (1+t)\varepsilon^{-1}\overline{Z}_m^k \partial u, \ \sigma_1^{-1}\overline{Z}_m^{k-1}\widetilde{a}, \ \overline{Z}_m^{k-1}\widetilde{a}, \ \overline{Z}_m^{k-1}\partial\widetilde{a},$$

where $\widetilde{a} = \widetilde{a}(R_i)$ or $\widetilde{a} = \widetilde{a}(S)$. We have for these fields the usual calculus Lemmas: Lemma 4.1 is straightforward. Note also

$$r + ct/\sqrt{c}L = H_0 + S = \overline{H}_m + S^m = \overline{Z}_m, \quad L_1 = \widetilde{K} - L = f\overline{Z}_m.$$

The analogue to Lemma 3.1 is also true:

$$R = f\overline{Z}_m + f\overline{N}_1\overline{Z}_m, \quad \sigma_1 L_1 = f\overline{Z}_m + f\overline{N}_1\overline{Z}_m.$$

We define as before quantities

$$\overline{M}_k = \sum f\overline{N}_1^l \overline{N}_{k_1} \cdots \overline{N}_{k_i}, \quad k_j \geqslant 2, \ \sum(k_j - 1) \leqslant k-1.$$

The following Lemma gives the relations between the fields $\widetilde{Z}_m$ and $\overline{Z}_m$.

Lemma 1. — *We have the formula*

i) $$\overline{Z}_m^k = \sum f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}\widetilde{Z}_m^p, \quad p \geqslant 1, \ \sum k_i + p \leqslant k,$$

ii) $$\widetilde{Z}_m^k = \sum f\overline{N}_1^l\overline{N}_{k_1}\cdots\overline{N}_{k_i}\overline{Z}_m^{p+1}.$$

Here, $k_j \geqslant 2, \ \sum(k_j-1) + p \leqslant k-1.$

iii) $$\overline{N}_k = \sum_{(\sum k_j \leqslant k)} f\widetilde{N}_{k_1}\cdots\widetilde{N}_{k_i}, \quad \overline{M}_k = M_k,$$

iv) $$\widetilde{Z}_m^k = (f + f\sigma_1/(1+t)\widetilde{a})^k\overline{Z}_m^k + \sum_{0\leqslant p\leqslant k-2}\overline{M}_{k-p}\overline{Z}_m^{p+1}.$$

Note that $|\sigma_1/(1+t)\widetilde{a}| \leqslant C$.

Proof

a. We have

$$\overline{H}_m = H_0 - \widetilde{a}(S)L_1 = \sum \omega_i \widetilde{H}_i^m - (\sum \omega_i \widetilde{a}(H_i) + \widetilde{a}(S))L_1 = \sum \omega_i \widetilde{Z}_m,$$

which proves i) for $k=1$. Conversely,

$$\begin{aligned}\widetilde{H}_i^m &= H_i + \widetilde{a}(H_i)L_1 = \omega_i H_0 - ct/r(\omega \wedge R)_i + \widetilde{a}(H_i)L_1\\ &= \omega_i(H_0 - \widetilde{a}(S)L_1) - ct/r(\omega \wedge \widetilde{R})_i - (r-ct)/r(\omega \wedge \widetilde{a}(R))_i L_1\\ &= f\overline{Z}_m + f\sigma_1/(1+t)\widetilde{a}\overline{Z}_m,\end{aligned}$$

which proves ii), iii) and iv) for $k=1$.

b. Formula i) is immediate by induction. Formula ii) can be written

$$\widetilde{Z}_m^k = \sum_{p \leqslant k-1} \overline{M}_{k-p} \overline{Z}_m^{p+1}.$$

Hence the calculus on $\overline{M}_l$ proves ii) for all k, and the same reasoning applies to prove iv). Finally, iii) follows from i) and ii) by the very definitions of the quantities, since

$$M_1 = \overline{M}_1, \quad \overline{N}_k = M_k, \quad \widetilde{N}_k = \overline{M}_k. \qquad \square$$

2. We have the following commutation Lemma.

Lemma 2. — *We have*

i) $$[\overline{Z}_m, L_1] = (f + f\overline{N}_1)L + (f + f\overline{N}_1)L_1,$$

ii) $$[\overline{Z}_m, L] = f\overline{d}L_1 + (f + f\overline{N}_1)L.$$

Here, $\overline{d}$ means one of the three quantities

$$\overline{d} = L\widetilde{a}(R_i) + \widetilde{R}_i^m u/2c, \quad \overline{d} = L\widetilde{a}(S) + \widetilde{S}^m u/2c, \quad \overline{d} = L_1 u.$$

Thus the critical quantity $\overline{d}$ is just $\widetilde{A}$ (or $L_1 u$). We have, with $Lw = g$, the formula

iii) $$\begin{aligned}[L, \overline{Z}_m^k]w &= \sum f\overline{N}_{l_1}\cdots\overline{N}_{l_i}\overline{Z}_m^{l_{i+1}} g + \sum f\overline{Z}_m^{q_1}\overline{d}\ldots\overline{Z}_m^{q_i}\overline{d}\overline{N}_{k_1}\cdots\overline{N}_{k_j}L_1\overline{Z}_m^{k_{j+1}} w\\ &+ \sum(1+t)^{-1} f\overline{Z}_m^{q_1}\overline{d}\ldots\overline{Z}_m^{q_i}\overline{d}\overline{N}_{k_1}\cdots\overline{N}_{k_j}\overline{Z}_m^{k_{j+1}} w = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 .\end{aligned}$$

In $\sum_1$, $\sum l_j \leqslant k$, $l_{i+1} \leqslant k-1$. In $\sum_2$, $i \geqslant 1$, $i + \sum q_j + \sum k_i \leqslant k$, $k_{j+1} \leqslant k-1$. In $\sum_3$, $i \geqslant 1$, $i + \sum q_j + \sum k_i \leqslant k+1$, $1 \leqslant k_{j+1} \leqslant k-1$.

Proof. — Since i) is clear, we need only prove ii), the proof of iii) following then exactly as in Lemma III.3.3. We have

$$\begin{aligned}[\widetilde{R}_i^m, L] &= (f + f\overline{N}_1)L - (L\widetilde{a}(R_i) + \widetilde{R}_i^m u/2c)L_1,\\ [\widetilde{S}^m, L] &= (f + f\overline{N}_1)L - (L\widetilde{a}(S) + \widetilde{S}^m u/2c)L_1,\\ [\overline{H}_m, L] &= f + f\overline{N}_1)L + (L\widetilde{a}(S) + \widetilde{S}^m u/2c)L_1,\\ [\widetilde{K}, L] &= Lu/2cL - L_1 u/2cL_1.\end{aligned} \qquad \square$$

3. We write now

$$La = -\chi/(2c)(Zu + aL_1u) = -\chi/(2c)(\overline{Z}_m u + (a-\widetilde{a})L_1u),$$
$$LL_1a = g_1 + L_1u/(2c)L_1a = G_1, \quad LR/ra = g_2 - \sqrt{c}Ra/r^2 = G_2,$$

with

$$g_i = f/\sigma_1\overline{Z}_m u + f\partial\overline{Z}_m u + f\partial u(a/\sigma_1) + f\partial u\partial a + f\partial u(\widetilde{a}/\sigma_1) + f\partial u\partial\widetilde{a} + f(a-\widetilde{a})\partial L_1u.$$

Using the structure of the g_i, we see that all terms in $\overline{Z}_m^l g_i$ are SC (in the sense of Proposition 1) for $l \leqslant k-1$ and

$$|\overline{Z}_m^l g_i|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad |\overline{Z}_m^l g_i|_{L^\infty} \leqslant C\varepsilon(1+t)^{-1+C\varepsilon}.$$

For G_1, we have the same estimates as for g_1. If $l=k$, we can replace the fields $\overline{Z}_m$ by $\widetilde{Z}_m$ in the critical terms of $\overline{Z}_m^k g_i$, this substitution generating only SC terms with the already seen estimate. Hence

$$|\overline{Z}_m^k g_i|_0 + |\overline{Z}_m^k G_1|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{-1}\phi'_{k+1} + C\varepsilon A_{k+1}.$$

The delicate part is the estimation of $\overline{Z}_m^l(\sqrt{c}Ra/r^2)$ in $\overline{Z}_m^l G_2$. We write

$$\overline{Z}_m^l(\sqrt{c}Ra/r^2) = \sqrt{c}/r\overline{Z}_m^l(Ra/r) + \sum_{1\leqslant l_1\leqslant l}(1+t)^{-1}M_{l_1}\overline{Z}_m^{l-l_1}(Ra/r).$$

Now $Ra = \overline{M}_1\overline{Z}_m a$,

$$\overline{Z}_m^{l-l_1}(M_1(1+t)^{-1}\overline{Z}_m a) = \sum_{0\leqslant l_2\leqslant l-l_1}(1+t)^{-1}M_{1+l_2}\overline{Z}_m^{l-l_1-l_2+1}a,$$
$$\overline{Z}_m^l(\sqrt{c}Ra/r^2) = \sqrt{c}/r\overline{Z}_m^l(Ra/r) + \sum_{(l_1\geqslant 1, l_1+l_2\leqslant l)}(1+t)^{-2}M_{l_1+l_2}\overline{Z}_m^{l-l_1-l_2+1}a.$$

If $l=k$, we keep the first term as it is, the second sum being bounded by

$$C(1+t)^{-\gamma} + C(1+t)^{-\gamma}A_{k+1}.$$

If $l = k-1$, we compute the first term as before, and obtain for the whole of $\overline{Z}_m^{k-1}(\sqrt{c}Ra/r^2)$ the above bound. If $l \leqslant k-2$, the bound is the same as before, without the critical part containing A_{k+1}.

4. With $w = L_1a$ or $w = Ra/r$ and $Lw = G$, we write the result of Lemma 2 in the form

$$L\overline{Z}_m^k w = \overline{Z}_m^k G + \sum_{l\leqslant k-1}\overline{M}_{k-l}\overline{Z}_m^l G + f\overline{d}L_1\overline{Z}_m^{k-1}w$$
$$+ \sigma_1^{-1}\sum_{1\leqslant q\leqslant k-1}\overline{M}_1\overline{Z}_m^q\overline{d}\overline{Z}_m^{k-q}w + \sigma_1^{-1}\sum_{\substack{p\geqslant 1, q\geqslant 1\\ p+q\leqslant k-1}}\overline{M}_{k-p-q}\overline{Z}_m^q\overline{d}\overline{Z}_m^p w.$$

All terms of the second line are SC terms, and we see using the **C.1** estimates of Proposition 1 that they are bounded by $C\varepsilon(1+t)^{C\varepsilon}$. We also have, using the estimates of **3.**,

$$\sum_{l\leqslant k-1} |\overline{M}_{k-l}\overline{Z}_m^l G|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C(1+t)^{-\gamma} + C(1+t)^{-\gamma}A_{k+1}.$$

We handle the critical term $\overline{d}L_1\overline{Z}_m^{k-1}w$ exactly as we have done with the term $\widetilde{A}L_1^2\widetilde{Z}_m^k u$ in **C.5** of the proof of Proposition 1. Using the energy inequality for L, we finally get

$$(1+t)^{-1}(|\overline{Z}_m^k L_1 a|_0 + |\overline{Z}_m^k(Ra/r)|_0) \leqslant C(1+t)^{C\varepsilon}$$
$$+ C\int_0^t \varepsilon\phi_{k+1}ds/(1+s)^2 + C\varepsilon\int_0^t A_{k+1}ds/(1+s) + C\int_0^t A_{k+1}(1+s)^{-1-\gamma}ds.$$

From the very definition of a, we obtain

$$|\overline{Z}_m^k La|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon(1+t)^{C\varepsilon}A_{k+1} + C(1+t)^{C\varepsilon}|\partial\widetilde{Z}_m^{k+1}u|_0.$$

Adding this to the preceding estimate, we get

$$(1+t)^{-1}|\overline{Z}_m^k\partial a|_0 \leqslant C(1+t)^{C\varepsilon} + C(1+t)^{-1+C\varepsilon}|\partial\widetilde{Z}_m^{k+1}u|_0$$
$$+ C\int_0^t \varepsilon\phi'_{k+1}ds/(1+s)^2 + C\varepsilon\int_0^t A_{k+1}ds/(1+s) + C\int_0^t A_{k+1}(1+s)^{-1-\gamma}ds.$$

Now, using Lemma 1, we can replace the fields $\overline{Z}_m$ by $\widetilde{Z}_m$ in the above estimate, obtaining the desired result. □

VII.3. End of the proof of the main result

a. We first use the energy inequality and Proposition 1. In doing so, we have to take care of the special quantity

$$E = \int_0^t (1+t')^{-1+C\varepsilon}|\partial\widetilde{Z}_m^{k+1}u|_0|T_i\widetilde{Z}_m^{k+1}u|_0 dt'$$

arising from

$$\iint e^p|P\widetilde{Z}_m^{k+1}u||\partial_t\widetilde{Z}_m^{k+1}u|dxdt'.$$

It is understood here, in accordance with Proposition 1, that the integral of $T_i\widetilde{Z}_m^{k+1}u$ is taken only on

$$\sigma_1 \leqslant C(1+t')^{C_0\varepsilon}.$$

Using Cauchy-Schwarz inequality, we obtain, with $\alpha > 0, \beta > 0$ to be chosen,

$$E \leqslant \alpha\int_0^t (1+t')^{-\beta\varepsilon}|T_i\widetilde{Z}_m^{k+1}u|_0^2 dt' + 1/(4\alpha)\int_0^t (1+t')^{-2+2C\varepsilon+\beta\varepsilon}|\partial\widetilde{Z}_m^{k+1}u|_0^2 dt'.$$

Since the energy inequality gives us a control of

$$\int_0^t\int_{r\geqslant t'/2} e^p(\tau+1)b'(\psi)\sum(T_i\widetilde{Z}_m^{k+1}u)^2 dxdt',$$

and $b'(\psi) = B\nu|\psi|^{-\nu-1}$, we have, by Lemma II.3.4, a control of

$$\int_0^t \int_{\sigma_1 \leqslant C(1+t')^{C_0\varepsilon}} e^p(1+t')^{-C_2\varepsilon} \sum (T_i \widetilde{Z}_m^{k+1} u)^2 dx dt'.$$

Taking $\beta = C_2$ and α small enough, we see that the first term of E is absorbed in the left-hand side of the inequality, while the second is smaller than terms already there.

b. We have now

$$\begin{aligned}
|\partial \widetilde{Z}_m^{k+1} u|_0 &\leqslant C\varepsilon + C\varepsilon \int_0^t dt'/(1+t')|\partial \widetilde{Z}_m^{k+1} u|_0 + C\int_0^t |P\widetilde{Z}_m^{k+1} u|_0 dt' \\
&\leqslant C\varepsilon(1+t)^{C\varepsilon} + C\varepsilon \int_0^t \phi'_{k+1} dt'/(1+t')^{2+\gamma} + C\varepsilon^2 \int_0^t \phi'_{k+1} dt'/(1+t')^2 \\
&\quad + C\varepsilon \int_0^t A_{k+1} dt'/(1+t')^{1+\gamma} + C\varepsilon^2 \int_0^t A_{k+1} dt'/(1+t').
\end{aligned}$$

We set here for convenience

$$E_{k+1} = \varepsilon^{-1}|\partial \widetilde{Z}_m^{k+1} u|_0 + A_{k+1}.$$

We use now the formula

$$|\widetilde{Z}_m^k \partial \widetilde{a}|_0 \leqslant C(1+t)^{1+C\varepsilon} + C|\widetilde{Z}_m^k \partial a|_0 + C\varepsilon^{-1}(1+t)|\partial \widetilde{Z}_m^{k+1} u|_0.$$

To prove it, we go back to the formula

$$\widetilde{Z}_m^k \partial \widetilde{a} = \widetilde{Z}_m^k [\partial, S_\theta] a + [\widetilde{Z}_m^k, S_\theta]\partial a + S_\theta \widetilde{Z}_m^k \partial a.$$

As before, the first two terms in the right-hand side involve

i) Terms already bounded by the induction hypothesis,

ii) Terms bounded by $|\widetilde{Z}_m^k \partial a|_0$ with a coefficient of the form $\theta_1^{-1} C_2 (1+t)^{C_3\varepsilon}$, where C_2 and C_3 do not depend on θ_1.

iii) Terms involving $\widetilde{N}_{k+1}$, with a coefficient of the same form as in ii).

The part of $\widetilde{N}_{k+1}$ involving $\widetilde{a}$ will be absorbed in the left-hand side by choosing β_1 and θ_1^0 big enough. Keeping the part involving derivatives of u, we obtain the formula. Using it, we obtain

$$(1+t)^{-1}\phi'_{k+1} \leqslant C(1+t)^{C\varepsilon} + CE_{k+1}.$$

With these notations, the control of $\partial \widetilde{Z}_m^{k+1} u$ given by the energy inequality for P and the control of A_{k+1} given by the energy inequality for L, added together, give

$$E_{k+1} \leqslant C(1+t)^{C\varepsilon} + C\int_0^t E_{k+1} dt'/(1+t')^{1+\gamma} + C\varepsilon \int_0^t E_{k+1} dt'/(1+t'),$$

which yields by Gronwall Lemma $E_{k+1} \leqslant C(1+t)^{C\varepsilon}$. This proves the induction hypothesis for $l = k+1$

$$|\widetilde{N}_{k+1}|_0 \leqslant C(1+t)^{1+C\varepsilon}, \quad A_{k+1} \leqslant C(1+t)^{C\varepsilon}.$$

c. It remains now to obtain, for the standard fields $Z_0 = R_i, S, h_i, \partial$,

$$|Z_0^k \partial u|_{L^2} \leqslant C\varepsilon(1+t)^{C\varepsilon}, \quad k \leqslant 2(s_0 - 4).$$

First, we obtain

$$Z_0 = f \widetilde{N}_1^r \widetilde{Z}_m, \quad r \leqslant 1.$$

Next, exactly as in Lemma 2, we get

$$Z_0^k = \sum f \widetilde{N}_1^l \widetilde{N}_{k_1} \cdots \widetilde{N}_{k_i} \widetilde{Z}_m^{p+1},$$

with

$$k_j \geqslant 2, \quad \sum (k_j - 1) + p \leqslant k - 1.$$

Applying this identity to ∂u, we obtain finally

$$|Z_0^k \partial u|_0 \leqslant C\varepsilon(1+t)^{C\varepsilon}.$$

Since we have the inequality

$$|w|_0 \leqslant C(1+t)^{C\varepsilon} |w|_{L^2},$$

this gives the result.

d. From Klainerman's inequality, we obtain now

$$|Z_0^k \partial u| \leqslant C\varepsilon \sigma_1^{-1/2} (1+t)^{-1+C\varepsilon}, \quad k \leqslant 2(s_0 - 4) - 2.$$

Assuming that

$$2(s_0 - 4) - 2 \geqslant s_0,$$

for instance, $s_0 = 10$, we obtain the same control as the induction hypothesis, with η replaced by $C\varepsilon$.

Fix now $\overline{\tau} > 0$: we know from Theorem II.1 that, for ε small enough, there exists a smooth solution for $\tau \leqslant \overline{\tau} = \varepsilon \log(1 + \overline{t})$ with

$$|Z_0^k \partial u| \leqslant C^{(1)} \varepsilon \sigma_1^{-1/2} (1+t)^{-1}, \quad k \leqslant s_0.$$

In particular, u exists as a smooth function for $t < T'$ (with $T' > \overline{t}$), and satisfies for $t < T \leqslant T'$ (with $T > \overline{t}$) the inequality (say $\eta = 10^{-2}$)

$$|Z_0^k \partial u| \leqslant C^{(1)} \varepsilon \sigma_1^{-1/2} (1+t)^{-1+\eta}.$$

If $T < T'$, we obtain from this hypothesis, as we have seen, for $t \leqslant T$,

$$|Z_0^k \partial u| \leqslant C^{(2)} \varepsilon \sigma_1^{-1/2} (1+t)^{-1+C\varepsilon}.$$

If ε is small enough to verify $C\varepsilon \leqslant \eta/2$, we deduce from this

$$|Z_0^k \partial u| \leqslant C^{(2)} (1+\overline{t})^{-\eta/2} \varepsilon \sigma_1^{-1/2} (1+t)^{-1+\eta}, \quad \overline{t} \leqslant t \leqslant T.$$

If ε is such that

$$C^{(2)} (1+\overline{t})^{-\eta/2} \leqslant C^{(1)}/2,$$

we see that the supremum of such T cannot be strictly less than T', hence $T' = +\infty$ and our estimates are true for all t, which finishes the proof. □

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MICROLOCAL ANALYSIS, BILINEAR ESTIMATES AND CUBIC QUASILINEAR WAVE EQUATION

by

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Abstract. — In this paper, we study the local wellposedness of a cubic quasilinear wave equation. The Strichartz estimate used for the solutions of linear variable coefficients wave equations are not relevant here. We prove bilinear estimates for solutions of linear wave equations with variable coefficients. The main tools are Bony's paradifferential calculus and the microlocalization in the sense of Weyl-Hörmander calculus.

***Résumé* (Analyse microlocale et équation d'onde quasilinéaire cubique).** — Dans cet article, nous étudions l'existence et l'unicité locale de solutions pour une équation d'onde quasilinéaire cubique. Les classiques estimations de Strichartz ne sont pas adaptées dans ce cas. Nous démontrons des estimations bilinéaires pour des solutions d'équations d'ondes à coefficients variables. Les deux outils principaux sont le calcul paradifférentiel de Bony et la microlocalisation au sens du calcul pseudodifférentiel de Weyl-Hörmander.

Introduction

In this paper, our interest is to prove local solvability for equations of the type

$$(EC)\begin{cases} \partial_t^2 u - \Delta u - \displaystyle\sum_{1\leqslant j,k\leqslant d} g^{j,k}\partial_j\partial_k u = 0 \\ \qquad\qquad\qquad\qquad \Delta g^{j,k} = Q_{j,k}(\partial u, \partial u) \\ \qquad\qquad\qquad (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

where $Q_{j,k}$ are quadratic forms on $\mathbf{R}^{d+1}$. In all this work, we shall state, for a real valued function u on $[0,T]\times\mathbf{R}^d$,

$$\nabla u \stackrel{\text{def}}{=} (\partial_1 u, \cdots, \partial_d u)\,,\ \partial u \stackrel{\text{def}}{=} (\partial_t u, \partial_1 u, \cdots, \partial_d u) \quad \text{and} \quad g\cdot\nabla^2 u \stackrel{\text{def}}{=} \sum_{1\leqslant j,k\leqslant d} g^{j,k}\partial_j\partial_k u.$$

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When no confusion is possible, we shall also state

$$\gamma \stackrel{\text{def}}{=} (\nabla u_0, u_1).$$

This problem of course is a model one. The general problem consists in considering equations of the type

$$\begin{cases} \partial_t^2 u - \Delta u - \displaystyle\sum_{1\leqslant j,k\leqslant d} g^{j,k}\partial_j\partial_k u = \sum_{1\leqslant j,k\leqslant d} \widetilde{Q}_{j,k}(\partial g^{j,k}, \partial u) \\ \qquad\qquad\qquad\qquad \Delta g^{j,k} = Q_{j,k}(\partial u, \partial u) \\ \qquad\qquad\qquad (u, \partial_t u)_{|t=0} = (u_0, u_1). \end{cases}$$

where $\widetilde{Q}_{j,k}$ are quadratic form on $\mathbf{R}^{d+1}$ and where all the quadratic forms are supposed to be smooth functions of u. This simply complicates a little the estimates without any relevant new phenomenon. In the frame work of equation (EC), it makes sense to work with small data and this simplifies the proofs.

Energy methods allow to prove local wellposedness for initial data (u_0, u_1) in $H^{\frac{d}{2}+\frac{1}{2}} \times H^{\frac{d}{2}-\frac{1}{2}}$. More precisely, we have the following theorem.

Theorem 0.1. — *If $d \geqslant 3$, let (u_0, u_1) be in $H^{\frac{d}{2}+\frac{1}{2}} \times H^{\frac{d}{2}-\frac{1}{2}}$ such that $\|\gamma\|_{\dot{H}^{\frac{d}{2}-1}}$ is small enough. Then, a positive time T exists such that a unique solution u of (EC) exists in $C([0,T]; H^{\frac{d}{2}+\frac{1}{2}}) \cap C^1([0,T]; H^{\frac{d}{2}-\frac{1}{2}})$. Moreover, a constant C exists (which of course does not depend on the initial data) such that*

$$T \geqslant C\|\gamma\|^{-2}_{\dot{H}^{\frac{d}{2}-\frac{1}{2}}}.$$

Let us recall that H^s is the usual Sobolev space on $\mathbf{R}^d$ and that $\dot{H}^s$ is the homogeneous one and we shall state

$$\|f\|_s^2 \stackrel{\text{def}}{=} \int_{\mathbf{R}^d} |\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi.$$

This is an Hilbert space when $s < d/2$.

The goal of this paper is to go below the regularity $H^{d/2+1/2}$ for the initial data. Let us have a look to the scaling properties of equation (EC). If u is a solution of (EC), then $u_\lambda(t,x) \stackrel{\text{def}}{=} u(\lambda t, \lambda x)$ is also a solution of (EC). The space which is invariant under this scaling is $\dot{H}^{d/2}$. So the above theorem appears to require $1/2$ derivative more than the scaling. The goal of this work is to try to go as closed as possible to the scaling invariant regularity.

Some results in that direction have been proved by the authors (see [**4**] and [**5**]) and also by D. Tataru (see [**27**] and [**28**]) for quasilinear wave equations of the following type

$$(E)\begin{cases} \partial_t^2 u - \Delta u - G(u)\cdot\nabla^2 u = F(u)Q(\partial u, \partial u) \\ \qquad\qquad (u, \partial_t u)_{|t=0} = (u_0, u_1) \end{cases}$$

where G is a smooth function vanishing at 0 and with value in K such that $\mathrm{Id}+K$ is a convex compact subset of the set of positive symmetric matrices. Let us recall this results. Let us notice that the scaling of the two equations (E) and (EC) is the same.

Theorem 0.2. — *If $d \geqslant 3$, let (u_0, u_1) be in $H^s \times H^{s-1}$ for $s > s_d$ with $s_d = \frac{d}{2} + \frac{1}{2} + \frac{1}{6}\cdot$ Then, a positive time T exists such that a unique solution u exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2([0,T]; L^\infty).$$

Moreover, a constant C exists such that

$$T^{\frac{2}{3}+(s-s_d)} \geqslant C\|\gamma\|^{-1}_{\dot{H}^{s-1}}.$$

This theorem has been proved with 1/4 instead than 1/6 in [**4**] and then improved a little bit in [**5**] and proved with 1/6 by D. Tataru in [**28**]. Strichartz estimates for quasilinear equations are the key point of the proofs. Recently, S. Klainerman and S. Rodnianski have announced a better index. Their proof is based on very different methods. In this case, the energy methods give the classical index $s > d/2 + 1$ and

$$T \geqslant C\|\gamma\|^{-1}_{H^{s-1}}.$$

The goal of this work is to do the analogous in the case of Equation (EC). The result will be the following.

Theorem 0.3. — *If $d \geqslant 5$, let (u_0, u_1) be in $H^s \times H^{s-1}$ with $s > \frac{d}{2} + \frac{1}{6}$ such that $\|\gamma\|_{\dot{H}^{\frac{d}{2}-1}}$ is small enough. Then, a positive time T exists such that a unique solution u of (EC) exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2_T(\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2})$$

where $\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}$ denotes the Besov space defined in Definition 1.1. Moreover, for any positive α, a constant C_α exists such that

$$T^{\frac{1}{6}+\alpha} \geqslant C_\alpha\|\gamma\|^{-1}_{\dot{H}^{\frac{d}{2}-\frac{5}{6}+\alpha}}.$$

The case of dimension 4 is a little bit different. The theorem is the following.

Theorem 0.4. — *If $d = 4$, let (u_0, u_1) be in $H^s \times H^{s-1}$ with $s > 2 + \frac{1}{6}$ such that $\|\gamma\|_{\dot{H}^1}$ is small enough. Then, a positive time T exists such that a unique solution u of (EC) exists such that*

$$\partial u \in C([0,T]; H^{s-1}) \cap L^2_T(\dot{B}^{1/6}_{6,2}) \quad \text{and} \quad \partial g \in L^1_T(L^\infty)$$

where $\dot{B}^{\frac{d}{6}-\frac{1}{2}}_{6,2}$ denotes the Besov space defined in Definition 1.1. Moreover, for any positive α, a constant C_α exists such that

$$T^{\frac{1}{6}+\alpha} \geqslant C_\alpha\|\gamma\|^{-1}_{\dot{H}^{\frac{d}{2}-\frac{5}{6}+\alpha}}.$$

Remarks

– If we think in term of small data (i.e. of initial data of the type $\varepsilon(u_0, u_1)$), then energy methods give a life span in ε^{-2}. The above theorem gives a life span of order $\varepsilon^{-6+\alpha}$ for any positive α.

– As we shall see, the case when $d \geqslant 5$ can be treated only with Strichartz estimates simply because laws of product in Besov spaces imply that if ∂u belongs to $L^2_T(\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2})$ then ∂g is in $L^1_T(L^\infty)$.

– The case when $d = 4$ requires bilinear estimates. This fact appears in the statement of Theorem 0.4 through the following phenomenon: the fact that ∂u is in $L^2_T(\dot{B}^{1/6}_{6,2})$ does not imply that the time derivative of g belongs to $L^1_T(L^\infty)$. Of course this condition is crucial in particular to get the basic energy estimate. But we have been unable to exibit a Banach space $\mathcal{B}$ which contains the solution u and such that if a function a is contained in $\mathcal{B}$, then $\partial\Delta^{-1}(a^2)$ belongs to $L^1_T(L^\infty)$.

– In all that follows, the dimension d will supposed to be greater or equal than 4.

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1. Method of the proof and structure of the paper

As we shall use Littlewood-Paley theory all along this work, let us begin by recalling some basic facts and definitions related to it.

1.1. Some basic facts in Littlewood-Paley theory. — Let us denote by $\mathcal{C}$ the ring of center 0, of small radius 3/4 and of big radius 8/3. Let us choose two non negative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0, 4/3))$ and $\mathcal{D}(\mathcal{C})$ such that

$$\chi(\xi) + \sum_{q\in\mathbf{N}} \varphi(2^{-q}\xi) = \sum_{q\in\mathbf{Z}} \varphi(2^{-q}\xi) = 1, \tag{1}$$

$$|p-q| \geqslant 2 \Rightarrow \operatorname{Supp}\varphi(2^{-q}\cdot) \cap \operatorname{Supp}\varphi(2^{-p}\cdot) = \varnothing, \tag{2}$$

$$q \geqslant 1 \Rightarrow \operatorname{Supp}\chi \cap \operatorname{Supp}\varphi(2^{-q}\cdot) = \varnothing, \tag{3}$$

and if $\widetilde{\mathcal{C}} = B(0, 2/3) + \mathcal{C}$, then $\widetilde{\mathcal{C}}$ is a ring and we have

$$|p-q| \geqslant 5 \Rightarrow 2^p\widetilde{\mathcal{C}} \cap 2^q\mathcal{C} = \varnothing. \tag{4}$$

Notations

$$h = \mathcal{F}^{-1}\varphi \quad \text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi,$$

$$\Delta_q u = \varphi(2^{-q}D)u = 2^{qd}\int h(2^q y)u(x-y)dy,$$

$$S_q u = \sum_{p\leqslant q-1} \Delta_p u = \chi(2^{-q}D)u = 2^{qd}\int \widetilde{h}(2^q y)u(x-y)dy.$$

We shall often denote $\Delta_q u$ by u_q. Let us recall the definition of Besov spaces.

Definition 1.1. — Let s be a real number, and (p,r) in $[1,\infty]^2$. Let us state

$$\|u\|_{\dot B^s_{p,r}} \stackrel{\text{def}}{=} \big\|\big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{q\in\mathbf{Z}}\big\|_{\ell^r(\mathbf{Z})}.$$

If $s < d/p$ then the closure of the compactly smooth functions with respect to this norm is a Banach space and we have that $\dot H^s = \dot B^s_{2,2}$ and the norm $\|\cdot\|_{\dot B^s_{2,2}}$ is equivalent to $\|\cdot\|_s$.

Notation. — We shall also state

$$\|a\|_s \stackrel{\text{def}}{=} \|a\|_{\dot B^s_{2,2}}\,,\ \ \|b\|_{L^p_I(E)} \stackrel{\text{def}}{=} \|b\|_{L^p(I;E)}\,,\ \ \|b\|_{L^p_T(E)} \stackrel{\text{def}}{=} \|b\|_{L^p([0,T];E)}$$
$$\text{and}\quad \|b\|_{T,s} \stackrel{\text{def}}{=} \|b\|_{L^\infty_T(\dot B^s_{2,2})}.$$

Here we want to explain the problems we have to solve in order to prove Theorem 0.4. As in the case of Equation (E), the basic fact is energy estimates. This implies the control of

$$\int_0^T \|\partial g(t,\cdot)\|_{L^\infty}dt.$$

In the case of Equation (E), it is obtained by Strichartz estimates. This will be the case here when $d \geqslant 5$ but this will not be the case when $d = 4$. Let us have a look on a model problem to understand this difficulty. Here we essentially follow ideas of S. Klainerman and D. Tataru (see [**22**]).

Let us assume that u is the solution of the constant coefficient wave equation and let us estimate

$$\int_0^T \|\partial\Delta^{-1}\big(\partial_j u(t,\cdot)\partial_k u(t,\cdot)\big)\|_{L^\infty}dt.$$

As

$$\partial_t\Delta^{-1}\big(\partial_j u(t,\cdot)\partial_k u(t,\cdot)\big) = \Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big) + \Delta^{-1}\big(\partial_j u\partial_t\partial_k u(t,\cdot)\big),$$

we have to control expression of the type

$$\int_0^T \|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{L^\infty}dt.$$

When $d \geqslant 3$, we have (see Lemma 2.1) that

$$\|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{\dot B^{d/2}_{2,1}} \leqslant C\|\partial u(t,\cdot)\|^2_{\frac{d}{2}-\frac12}.$$

So we get that

$$\int_0^T \|\Delta^{-1}\big(\partial_t\partial_j u\partial_k u(t,\cdot)\big)\|_{L^\infty}dt \leqslant T\|\partial u\|^2_{T,\frac{d}{2}-\frac12}.$$

Then the proof of Theorem 0.1 is routine. If we want to go below this $H^{\frac{d}{2}+\frac{1}{2}}$ regularity of the initial data, we shall use Strichartz estimates. Let us introduce Bony's decomposition which consists in writing

$$ab = \sum_q S_{q-1} a \Delta_q b + \sum_q S_{q-1} b \Delta_q a + \sum_{\substack{-1\leqslant j\leqslant 1\\ q}} \Delta_q a \Delta_{q-j} b.$$

When $d \geqslant 4$, we have

$$\|\partial^k u_q\|_{L^2_T(L^\infty)} \leqslant C 2^{q\left(\frac{d}{2}-\frac{1}{2}+k-1\right)} \|\gamma_q\|_{L^2}.$$

Then it is not difficult to prove that

$$\Big\| \Delta^{-1}\Big(\sum_q S_{q-1}\partial^2 u \partial u_q\Big)\Big\|_{L^1_T(L^\infty)} \leqslant C\|\gamma\|^2_{\frac{d}{2}-1}.$$

The symmetric term can be treated exactly along the same lines. The so called remainder term

$$\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q}} \partial^2 u_q \partial u_{q-j}\Big)$$

is much more difficult to treat particulary in dimension 4. The reason why is the following. When d is greater or equal to 5, the Strichartz estimates tells us that

$$\|\partial^k u_q\|_{L^2_T(L^4)} \leqslant 2^{q\left(\frac{d}{4}-\frac{1}{2}+k-1\right)} \|\gamma_q\|_{L^2}.$$

So thanks to Bernstein inequality, we infer that

$$\begin{aligned}\Big\| \Delta_p \Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}} \Delta_q \partial^2 u \Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} &\leqslant C 2^{p\left(\frac{d}{2}-2\right)} \sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}} 2^{qd/2} \|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}\\ &\leqslant C \sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}} 2^{-(q-p)\left(\frac{d}{2}-2\right)} 2^{q(d-2)} \|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}.\end{aligned}$$

Convolution and Cauchy-Schwarz inequalities implies that

$$\big\|\Delta^{-1}\big(\partial^2 u \partial u\big)\big\|_{L^1_T(L^\infty)} \leqslant C\|\gamma\|^2_{\frac{d}{2}-1}.$$

The case of dimension 4 is much more delicate. In dimension 4, the Strichartz estimate is

$$\|\partial^k u_q\|_{L^2_T(L^6)} \leqslant 2^{q\left(\frac{4}{3}-\frac{1}{2}+k-1\right)} \|\gamma_q\|_{L^2}.$$

So the series $\partial^2 u_q \partial u_{q-j}$ does not converge in $L^1_T(L^3)$ because the only estimate we have is

$$\begin{aligned}\|\partial^2 u_q \partial u_{q-j}\|_{L^1_T(L^3)} &\leqslant C 2^{q8/3}\|\gamma_q\|^2_{L^2}\\ &\leqslant C 2^{q2/3} d_q \|\gamma\|^2_1 \quad \text{with} \quad \sum_q d_q = 1.\end{aligned}$$

To overcome this difficulty, we follow an idea of S. Klainerman and D. Tataru: the precised Strichartz estimate which will allow to prove bilinear estimates.

1.2. Bilinear estimates and precised Strichartz estimates. — To explain the basic ideas of bilinear estimates, let us consider the case of constant coefficient case. In this paragraph, we essentially follow the ideas of [**22**]. What a bilinear estimates lookes like is described by the following proposition.

Proposition 1.1. — *Let u_1 and u_2 two solutions of*

$$\begin{cases} \partial_t^2 u_j - \Delta u_j = 0 \\ \quad (\partial u_j)_{|t=0} = \gamma_j. \end{cases}$$

Then, if $d \geqslant 4$, we have

$$\|\partial \Delta^{-1} Q(\partial u_1 \partial u_2)\|_{L^1_T(L^\infty)} \leqslant C_{\varepsilon,T} \|\gamma_1\|_{\frac{d}{2}-1+\varepsilon} \|\gamma_2\|_{\frac{d}{2}-1+\varepsilon}.$$

Remark. — We find a gain of half a derivative about the regularity of the initial data compared with purely Strichartz methods.

The precised Strichartz estimates is described by the following proposition proved in [**22**].

Proposition 1.2. — *A constant C exists such that for any T and any $h \leqslant 1$, if* $\operatorname{Supp} \widehat{u}_j$ *and* $\operatorname{Supp} \mathcal{F}(\Box\, u(t,\cdot))$ *are included in a ball of radius h and in the ring $\mathcal{C}$, we have*

$$\|u\|_{L^2_T(L^\infty)} \leqslant C\big(h^{d-2} \log(e+T)\big)^{1/2} \big(\|u(0)\|_{L^2} + \|\partial_t u(0)\|_{L^2} + \|\Box\, u\|_{L^1_T(L^2)}\big).$$

To prove Proposition 1.1, let us recall that we want to estimate the

$$\Big\| \Delta_p \Delta^{-1} \Big(\sum_{\substack{-1 \leqslant j \leqslant 1 \\ q \geqslant p - N_0}} \Delta_q \partial^2 u \Delta_{q-j} \partial u \Big) \Big\|_{L^1_T(L^\infty)}.$$

With a rescaling of the equation, we can assume that $q = 1$ and let us state $h = 2^{p-q}$. Let us define $(\phi_\nu)_{1 \leqslant \nu \leqslant N_h}$ a partition of unity of the ring $\mathcal{C}$ such that

$$\operatorname{Supp} \phi_\nu \subset B(\xi_\nu, h).$$

Then, using the fact that the support of the Fourier transform of the product of two functions is included in the sum of the supports of their Fourier transform, a family of function $(\widetilde{\phi}_\nu)_{1 \leqslant \nu \leqslant N_h}$ exists such that $\operatorname{Supp} \widetilde{\phi}_\nu \subset B(-\xi_\nu, 2h)$ and

$$\chi(h^{-1}D)(\partial^2 v \partial v) = \sum_{\nu=1}^{N_h} \chi(h^{-1}D) \big(\partial^2 \widetilde{\phi}_\nu(D) v \partial \phi_\nu(D) v\big). \tag{5}$$

Applying Proposition 1.2 gives

$$\|\chi(h^{-1}D)(\partial^2 v \partial v)\|_{L^1_T(L^\infty)} \leqslant C h^{d-2} \log(e+T) \sum_{\nu=1}^{N_h} \|\widetilde{\phi}_\nu(D)\gamma\|_{L^2} \|\phi_\nu(D)\gamma\|_{L^2}.$$

The Cauchy Schwarz inequality implies that

$$\|\chi(h^{-1}D)(\partial^2 v\partial v)\|_{L^1_T(L^\infty)} \leqslant Ch^{d-2}\log(e+T)\Big(\sum_{\nu=1}^{N_h}\|\widetilde{\phi}_\nu(D)\gamma\|^2_{L^2}\Big)^{1/2}\Big(\sum_{\nu=1}^{N_h}\|\phi_\nu(D)\gamma\|^2_{L^2}\Big)^{1/2}.$$

The almost orthogonality of $(\widetilde{\phi}_\nu(D)\gamma_1)_{1\leqslant\nu\leqslant N_h}$ and $(\phi_\nu(D)\gamma_2)_{1\leqslant\nu\leqslant N_h}$ implies that

$$\|\chi(h^{-1}D)(\partial^2 v\partial v)\|_{L^1_T(L^\infty)} \leqslant Ch^{d-2}\log(e+T)\|\gamma\|_{L^2}\|\gamma\|_{L^2}. \tag{6}$$

So after rescaling, we get that

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} \leqslant 2^{p(d-4)}\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}\log(e+2^qT)2^{2q}\|\gamma_q\|_{L^2}\|\gamma_{q-j}\|_{L^2}.$$

If $\gamma\in\dot H^{\frac d2-1+\varepsilon}$ then we have

$$\Big\|\Delta_p\Delta^{-1}\Big(\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}\Delta_q\partial^2 u\Delta_{q-j}\partial u\Big)\Big\|_{L^1_T(L^\infty)} \leqslant (2^pT)^{-\varepsilon}\sum_{\substack{-1\leqslant j\leqslant 1\\ q\geqslant p-N_0}}2^{-(q-p)(d-4+\varepsilon)} \times 2^{q\left(\frac d2-1\right)}(2^qT)^\varepsilon\|\gamma_q\|_{L^2}2^{(q-j)\left(\frac d2-1\right)}(2^qT)^\varepsilon\|\gamma_{q-j}\|_{L^2}.$$

So the series convergences in $L^1_T(L^\infty)$ for large p. The case when p is small (low frequencies) is nothing but Sobolev embeddings.

The real problem we have to solve in this work is to prove this bilinear estimate in the context of quasilinear wave equation. To do this, we follow the lines of [**4**] and [**5**].

As we shall use geometrical optics technics, we need to deal with smooth functions in time also. This leads to the following iterative scheme introduced in [**5**]. Let us define the sequence $(u^{(n)})_{n\in\mathbf N}$ by the first term $u^{(0)}$ satisfying

$$\begin{cases}\partial_t^2u^{(0)}-\Delta u^{(0)}=0\\ (u^{(0)},\partial_tu^{(0)})_{|t=0}=(S_0u_0,S_0u_1),\end{cases}$$

and by the following induction

$$(\mathcal R_n)\begin{cases}\partial_t^2u^{(n+1)}-\Delta u^{(n+1)}-G_{n,T}\cdot\nabla^2u^{(n+1)}=0\\ \qquad (u^{(n+1)},\partial_tu^{(n+1)})_{|t=0}=(S_{n+1}u_0,S_{n+1}u_1)\end{cases}$$

with

$$G_{n,T}\stackrel{\rm def}{=}\theta(T^{-1})G_n\quad\text{with}\quad G_n^{j,k}\stackrel{\rm def}{=}\Delta^{-1}Q_{j,k}(\partial u^{(n)},\partial u^{(n)}).$$

where θ is a function of $\mathcal D(]-1,1[)$ whose value is 1 near 0. Let us point out that the sequence $(u^{(n)})_{n\in\mathbf N}$ does depend on T. We introduce some notations which will be used all along this work. If α is a (small) positive number, let us define

$$s_\alpha \stackrel{\text{def}}{=} \frac{d}{2} + \frac{1}{6} + \alpha \quad \text{and} \quad N_T^\alpha(\gamma) \stackrel{\text{def}}{=} T^{\frac{1}{6}+\alpha}\|\gamma\|_{s_\alpha - 1}.$$

Let us introduce the assertions we are going to prove by induction.

– If $d \geqslant 5$,

$$(\mathcal{P}_n) \begin{cases} \|\partial u^{(n)}\|_{L^2([0,T];\dot B_{4,2}^{\frac{d}{4}-\frac{1}{2}})} \leqslant C_0 N_T^\alpha(\gamma) \\ \|\partial u^{(n)}\|_{T,s-1} \leqslant e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s \in \Big[s_\alpha - 1, \frac{d}{2} + \frac{1}{2}\Big]; \end{cases}$$

– if $d = 4$,

$$(\mathcal{P}_n) \begin{cases} \|\partial u^{(n)}\|_{L^2([0,T];\dot B_{6,2}^{\frac{d}{6}-\frac{1}{2}})} \leqslant C_0 N_T^\alpha(\gamma) \\ \|\partial G_{n,T}\|_{L^1([0,T];L^\infty)} \leqslant 2 \\ \|\partial u^{(n)}\|_{T,s-1} \leqslant e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s \in \Big[\frac{3}{2} + \alpha, \frac{d}{2} + \frac{1}{2}\Big]. \end{cases}$$

All what follows in this paper consists in proving that if

$$\|\gamma\|_{\dot H^{\frac{d}{2}-1}} + N_T^\alpha(\gamma)$$

is small enough, $(\mathcal{P}_0)$ is true and $(\mathcal{P}_n)$ implies $(\mathcal{P}_{n+1})$. Then the proof of Theorems 0.3 and 0.4 is pure routine of non linear partial differential equations.

To do this, we shall localize in frequency and transform equation $\mathcal{R}_n$ into an equation where the space-time frequencies of the metric which defines the d'Alembertian are very small with respect to the level frequencies we work with. This is the purpose of the second section.

In the third section, we show how the proof can be reduced to "microlocal" Strichartz and bilinear estimates. By microlocal estimates, we mean estimates that are valid only a time interval whose length depends on the size of the frequencies we work with. To prove the complete estimates (with a loose of course), we use D. Tataru's version of the method we introduced in [**4**] which consists in a decomposition of the interval $[0,T]$ on intervals where microlocal estimates are true.

In the forth section, we recall the method of approximation of solutions of (variable coefficients) wave equation by the method of geometrical optics. This is the opportunity to study precisely the link between the solutions of the Hamilton-Jacobi equation

$$\begin{cases} \partial_\tau \Phi(\tau,y,\eta) = F(\tau,y,\partial_y\Phi(\tau,y,\eta)) \\ \Phi(0,y,\eta) \;= (y|\eta) \end{cases}$$

and the flow of H_F and also properties of this flow which will be useful in the seventh section.

The fifth section is devoted to the following problem: in the proof of the equivalent of Inequality (6), we use the fact that the support of the Fourier transform is preserved by the flow of the constant coefficient wave equation; this is no longer true in the variable coefficient case. So this information is not relevant because it is not preserved by the flow of the equation. The purpose of this fifth section is to define the concept of microlocalized function near a point $X = (x, \xi)$ of the cotangent space $T^\star \mathbf{R}^d$ (the cotangent space of $\mathbf{R}^d$). This notion is due to J.-M. Bony ([**7**]) and means that the function is concentrated in space near the point x and in frequency near the point ξ with of course the limit on the uncertainty principle. The good framework of this is a simplified version of Weyl-Hörmander calculus which is also presented in this section. Properties of the product of microlocalized functions is also studied.

In the sixth section, we prove that for solutions of a variable coefficients wave equation, microlocalization properties propagates nicely along the Hamiltonian flows related to the wave operator.

In the seventh section, we apply the three previous sections to prove the microlocal bilinear estimates. This proof consists in a second microlocalization, which means that we have to decompose again the interval on which we work. The reason why is that interaction in the product and propagation of microlocalization are badly related.

2. Littlewood-Paley theory and Paralinearization of the equation

All along this work, we shall need to study the quadratic operator $\Delta^{-1}((Du)^2)$. Let us summarize now some basic properties of this operator in the following lemma.

Lemma 2.1. — *A constant C exists such that*

$$\|\Delta^{-1}(\partial a \partial b)\|_{\dot{B}^{d/2}_{2,1}} \leqslant C\|\partial a\|_{\dot{H}^{\frac{d}{2}-1}}\|\partial b\|_{\dot{H}^{\frac{d}{2}-1}} \quad \text{and}$$
$$\|\nabla\Delta^{-1}(\partial a \partial b)\|_{\dot{B}^{d/4}_{4,1}} \leqslant C\|\partial a\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}\|\partial b\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}.$$

Moreover, for any σ greater than $3/2$, a constant C exists such that

$$\|\Delta^{-1}(\partial a \partial b)\|_{\dot{H}^{\sigma+\frac{1}{2}}} \leqslant C\big(\|\partial a\|_{\dot{C}^{-1/2}}\|\partial b\|_{\dot{H}^{\sigma-1}} + \|\partial a\|_{\dot{H}^{\sigma-1}}\|\partial b\|_{\dot{C}^{-1/2}}\big).$$

And, for any σ greater than $\dfrac{3}{2}-\dfrac{d}{4}$, a constant C exists such that

$$\|\Delta^{-1}(\partial a \partial b)\|_{\dot{H}^{\sigma+\frac{1}{2}}} \leqslant C\big(\|\partial a\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}\|\partial b\|_{\dot{H}^{\sigma-1}} + \|\partial a\|_{\dot{H}^{\sigma-1}}\|\partial b\|_{\dot{B}^{\frac{d}{4}-\frac{1}{2}}_{4,2}}\big).$$

From this lemma, we give the following corollary.

Corollary 2.1. — *A constant C exists such that, if $(\mathcal{P}_n)$ holds, then*

$$\|G_{n,T}\|_{L^\infty} \leqslant C\|\gamma\|^2_{\frac{d}{2}-1}.$$

Moreover, if $d \geqslant 5$, then

$$\|\partial G_{n,T}\|_{L^1_T(L^\infty)} \leqslant C N^\alpha_T(\gamma)^2.$$

The proof of this lemma and its corollary is an exercice on Littlewood-Paley theory and we omit it.

Theorem 2.1. — *For any $s > 3/2$, a constant C exists which satisfies the following properties. Let us consider two functions u and v whose partial derivatives belong to the space $L^\infty_T(\dot H^{s-1}) \cap L^2_T(\dot C^{-1/2})$ and a function F in $L^1_T(\dot H^{s-1})$. Let us assume that*

$$G^{j,k}_{v,T} \stackrel{def}{=} \theta(T^{-1}\cdot)\Delta^{-1}Q_{j,k}(\partial v, \partial v) \in L^1_T(L^\infty)$$

and that

$$\partial_t^2 u - \Delta u - G_{v,T} \cdot \nabla^2 u = F.$$

Then we have

$$\partial_t^2 u_q - \Delta u_q - S_{q-1} G_{v,T} \nabla^2 u_q = R_q(\nabla u, \partial v) + F_q$$

with

$$\begin{aligned}\|R_q(\nabla u(t), \partial v(t))\|_{L^2} \leqslant C c_q(t) 2^{-q(s-1)} \big(&\|\nabla G_{v,T}(t)\|_{L^\infty} \|\nabla u(t)\|_{s-1} \\ &+ \|\partial v(t)\|_{s-1} \|\partial v(t)\|_{\dot C^{-1/2}} \|\nabla u(t)\|_{\dot C^{-1/2}}\big).\end{aligned}$$

with as in all that follows $\displaystyle\sum_q c_q^2(t) = 1$.

To prove this theorem, we use paradifferential calculus. More precisely, we apply Bony's decomposition which consists in writing

$$\begin{aligned} G_{v,T}(t)\nabla^2 u(t) &= \mathcal{R}_1(t) + \mathcal{R}_2(t) \quad \text{with} \\ \mathcal{R}_1(t) &\stackrel{\text{def}}{=} \sum_{q'} S_{q'-1} G_{v,T} \nabla^2 u_{q'} \quad \text{and} \\ \mathcal{R}_2(t) &\stackrel{\text{def}}{=} \sum_{q'} S_{q'+2} \nabla^2 u \Delta_{q'} G_{v,T}. \end{aligned} \tag{7}$$

The first term $\mathcal{R}_1(t)$ is easy to estimate. As the support of the Fourier transform of the function $S_{q'-1}G_{v,T}\nabla^2 u_{q'}$ is included in a ring of type $2^{q'}\widetilde{\mathcal{C}}$, we have

$$\begin{aligned} \Delta_q \mathcal{R}_1(t) &= \sum_{|q-q'|\leqslant N_1} \Delta_q\big(S_{q'-1}G_{v,T}\nabla^2 u_{q'}\big) \\ &= S_{q-1}G_{v,T}\nabla^2 u_q + \sum_{|q-q'|\leqslant N_1} \big[\Delta_q, S_{q'-1}G_{v,T}\big]\nabla^2 u_{q'} \\ &\qquad + \sum_{|q-q'|\leqslant N_1} (S_{q'-1}G_{v,T} - S_{q-1}G_{v,T})\nabla^2 \Delta_q u_{q'}. \end{aligned} \tag{8}$$

As for instance in [**4**], we have

$$\big\|\big[\Delta_q, S_{q'-1}G_{v,T}\big]\nabla^2 u_{q'}\big\|_{L^2} \leqslant Cc_q 2^{-q(s-1)}\|\nabla G_{v,T}(t)\|_{L^\infty}\|\nabla u(t)\|_{s-1} \quad \text{and}$$
$$\big\|(S_{q'-1}G_{v,T} - S_{q-1}G_{v,T})\nabla^2 u_{q'}\big\|_{L^2} \leqslant Cc_q 2^{-q(s-1)}\|\nabla G_{v,T}(t)\|_{L^\infty}\|\nabla u(t)\|_{s-1}.$$

So it turns out that

$$\big\|\Delta_q \mathcal{R}_1(t) - S_{q-1}G_{v,T}\nabla^2 u_q\big\|_{L^2} \leqslant Cc_q 2^{-q(s-1)}\|\nabla G_{v,T}(t)\|_{L^\infty}\|\nabla u(t)\|_{s-1}. \tag{9}$$

The second term is a little bit more delicate to estimate. Because the support of the Fourier transform of $S_{q'+1}\nabla^2 u\Delta_{q'}G_{v,T}$ is included in a ball of center 0 and radius $C2^{q'}$, we have that

$$\Delta_q \mathcal{R}_2(t) = \sum_{q'\geqslant q-N_1} \Delta_q\big(S_{q'+2}\nabla^2 u\Delta_{q'}G_{v,T}\big)$$

By definition of $\dot{C}^{1/2}$ and using Bernstein inequalities, it is obvious that

$$\|S_{q'+1}\nabla^2 u\|_{L^\infty} \leqslant 2^{3q'/2}\|\nabla u(t)\|_{\dot{C}^{-1/2}}.$$

Using Lemma 2.1, we get that

$$\|\Delta_{q'}G_{v,T}\|_{L^2} \leqslant Cc_{q'}(t)2^{-q'(s+\frac12)}\|\partial v(t)\|_{\dot{C}^{-1/2}}\|\partial v(t)\|_{\dot{H}^{s-1}}$$

when s is greater than $3/2$. So the theorem is proved.

Now we are going to state two corollaries of this theorem.

Corollary 2.2. — *If $(\mathcal{P}_n)$ is satisfied, then for any $s\in\,]3/2, s_\alpha]$, a constant C exists such that*

$$\|\partial u^{(n+1)}\|_{T,s-1} \leqslant e^2\|\gamma\|_{s-1}\Big(1 + CC_0 N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot{C}^{-1/2})}\Big).$$

To prove it, let us first deduce by standard energy estimates from Theorem 2.1 above applied with $u = u^{(n+1)}$ and $v = u^{(n)}$ that

$$\begin{aligned}\frac{d}{dt}\|\partial u_q^{(n+1)}(t)\|_{L^2}^2 \leqslant Cc_q^2(t)2^{-2q(s-1)}\Big(&\|\partial G_{n,T}(t)\|_{L^\infty}\|\partial u^{(n+1)}(t)\|_{s-1}^2\\ &+ C\|\gamma\|_{s-1}\|\partial u^{(n)}(t)\|_{\dot{C}^{-1/2}}\|\nabla u^{(n+1)}(t)\|_{\dot{C}^{-1/2}}\|\partial u^{(n+1)}(t)\|_{s-1}\Big).\end{aligned}$$

By multiplication by $2^{2q(s-1)}$ and summation we have that

$$\begin{aligned}\frac{d}{dt}\|\partial u^{(n+1)}(t)\|_{s-1}^2 \leqslant C\Big(&\|\partial G_{n,T}(t)\|_{L^\infty}\|\partial u^{(n+1)}(t)\|_{s-1}^2\\ &+ C\|\gamma\|_{s-1}\|\partial u^{(n)}(t)\|_{\dot{C}^{-1/2}}\|\nabla u^{(n+1)}(t)\|_{\dot{C}^{-1/2}}\|\partial u^{(n+1)}(t)\|_{s-1}\Big).\end{aligned}$$

Using Gronwall lemma, it turns out that

$$\begin{aligned}&\|\partial u^{(n+1)}(t)\|_{s-1}\exp\Big(-C\int_0^t\|\partial G_{n,T}(t')\|_{L^\infty}dt'\Big)\\ &\qquad\leqslant \|\gamma\|_{s-1} + C\|\gamma\|_{s-1}\int_0^t\|\partial u^{(n)}(t')\|_{\dot{C}^{-1/2}}\|\nabla u^{(n+1)}(t')\|_{\dot{C}^{-1/2}}dt'.\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\|\partial u^{(n+1)}(t)\|_{s-1} \exp\Big(-C\int_0^t \|\partial G_{n,T}(t')\|_{L^\infty} dt'\Big) \leqslant \|\gamma\|_{s-1} + C\|\gamma\|_{s-1}\|\partial u^{(n)}\|_{L^2_T(\dot C^{-1/2})}\|\nabla u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}.$$

Using $(\mathcal{P}_n)$, we get that

$$\|\partial u^{(n+1)}(t)\|_{s-1} \exp\Big(-C\int_0^t \|\partial G_{n,T}(t')\|_{L^\infty} dt'\Big) \leqslant \|\gamma\|_{s-1} + CC_0 N_T^\alpha(\gamma)\|\gamma\|_{s-1}\|\nabla u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}.$$

The fact that

$$\partial_{t'} G_{n,T}(t') = \frac{1}{T}\theta'\left(\frac{t'}{T}\right) G_n + \theta\partial_{t'} G_n$$

together with induction hypothesis and corollary 2.1 implies the result.

The second corollary treats the case of low frequencies.

Corollary 2.3. — *A constant C exists such that under the hypothesis $(\mathcal{P}_n)$, we have, for any $r \geqslant 2$,*

$$\|\partial S_q u^{(n+1)}\|_{L^2_T(\dot B^{\frac{d}{r}-\frac12}_{r,2})} \leqslant C(2^q T)^{\frac13-\alpha} N_T^\alpha(\gamma)\big(1 + CC_0 N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big).$$

Using Bernstein inequalities and Corollary 2.2, we get that

$$\begin{aligned} 2^{2p\left(\frac{d}{r}-\frac12\right)}\|\partial u_p^{(n+1)}\|^2_{L^2_T(L^r)} &\leqslant CT2^{p(d-1)}\|\partial u_p^{(n+1)}\|^2_{L^\infty_T(L^2)} \\ &\leqslant C(2^pT)^{\frac23-2\alpha}T^{\frac13+2\alpha}\|\partial u^{(n+1)}\|^2_{T,s_\alpha-1} \\ &\leqslant C(2^pT)^{\frac23-2\alpha}N_T^\alpha(\gamma)^2\big(1+CC_0N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big)^2. \end{aligned}$$

Thus as

$$\|\partial S_q u^{(n+1)}\|^2_{L^2_T(\dot B^{\frac{d}{r}-\frac12}_{r,2})} \leqslant C\sum_{p\leqslant q-1} 2^{2p\left(\frac{d}{r}-\frac12\right)}\|\partial u_p^{(n+1)}\|^2_{L^2_T(L^r)},$$

we have proved the corollary.

Let us now do a precised paralinearization in the spirit of [**4**].

Theorem 2.2. — *A constant C exists which satisfies the following properties. Let us consider two functions u and v whose partial derivatives belong to $L^\infty_T(\dot H^{s_\alpha-1}) \cap L^2_T(\dot C^{-1/2})$ and a function F in $L^1_T(\dot H^{s-1})$. Let us assume that $\partial G_{v,T}$ belongs to $L^1_T(L^\infty)$ and that*

$$\partial_t^2 u - \Delta u - G_{v,T}\cdot\nabla^2 u = F.$$

Then for any $\delta \in [0,1]$, we have

$$\partial_t^2 u_q - \Delta u_q - S_q^\delta(G_{v,T})\cdot\nabla^2 u_q = R_q^\delta(\nabla u, \partial v) + F_q$$

with

$$S_q^\delta b \overset{def}{=} S^{(1+d)}_{q\delta-(1-\delta)\log_2 T - N_0} b \quad \textit{and}$$

$$\|R_q^\delta(\nabla u,\partial v)\|_{L^1_T(L^2)} \leqslant C2^{-q(s-1)}(1+(2^qT)^{1-\delta})\big(\|\partial G_{v,T}\|_{L^1_T(L^\infty)}\|\nabla u\|_{L^\infty_T(\dot H^{s-1})} + \|\partial v\|_{L^\infty_T(\dot H^{s-1})}\|\partial v\|_{L^2_T(\dot C^{-1/2})}\|\nabla u\|_{L^2_T(\dot C^{-1/2})}\big).$$

The proof of this theorem is based essentially on Theorem 2.1 and Corollary 2.2. Using Theorem 2.1, it is obvious that

$$R_q^\delta(\nabla u,\partial v) = R_q(\nabla u,\partial v) + (S_q^\delta - S_{q-1})(G_{v,T})\cdot\nabla^2 u_q.$$

As we have

$$\|(S_q^\delta - S_{q-1})G_{v,T}\|_{L^1_T(L^\infty)} \leqslant C2^{-q}(2^qT)^{1-\delta}\|\partial G_{v,T}\|_{L^1_T(L^\infty)},$$

we get the theorem applying Theorem 2.1.

As a corollary, we have

Corollary 2.4. — *A constant C exists such that under the hypothesis $(\mathcal{P}_n)$ we have for any δ in the interval $[0,1]$,*

$$(ECP_{T,q})\begin{cases}\partial_t^2 u_q^{(n+1)} - \Delta u_q^{(n+1)} - S_q^\delta(G_{n,T})\cdot\nabla^2 u_q^{(n+1)} = R_q^\delta(n)\\ \partial u_q^{(n+1)}{}_{|t=0} = \gamma_q^{(n+1)}\end{cases}$$

with

$$S_q^\delta b \overset{def}{=} S^{(1+d)}_{q\delta-(1-\delta)\log_2 T - N_0} b \quad and$$

$$\|R_q^\delta(n)\|_{L^1_T(L^2)} \leqslant C2^{-q(\frac d2-1)}(2^qT)^{-\frac16-\alpha}\big(1+(2^qT)^{1-\delta}\big)N_T^\alpha(\gamma) \times\Big(1+CC_0N_T^\alpha(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\Big).$$

3. Reduction to microlocalized estimates

By microlocalization of the estimates, we mean that we shall prove estimates that are valid on time intervals whose length depend on the frequency parameter. These techniques have been introduced in [**4**] and used in [**5**] and improved by D. Tataru in [**28**]. For technical reasons, we prefer to work at frequencies of size 1.

3.1. The statement of the microlocal estimates. — In all that follows, we shall consider a family of smooth functions $\mathcal{G} = (G_\Lambda)_{\Lambda\geqslant\Lambda_0}$ defined on $I_\Lambda\times\mathbf{R}^d$ such that G_Λ is small enough and such that, for any $k\geqslant 0$, the following quantities

$$(10)\qquad \|\mathcal{G}\|_0 \overset{\text{def}}{=} \sup_{\Lambda\geqslant\Lambda_0}\|\nabla G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} + |I_\Lambda|\,\|\nabla^2 G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad\text{and}$$

$$(11)\qquad \|\mathcal{G}\|_k \overset{\text{def}}{=} \sup_{\Lambda\geqslant\Lambda_0}|I_\Lambda|\Lambda^k\|\nabla^{k+2}G_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)} \quad\text{for}\quad k\geqslant 1.$$

are finite. Let us denote by P_Λ the operator

$$P_\Lambda v \overset{\text{def}}{=} \partial_\tau^2 v - \Delta v - \sum_{k,\ell} G_\Lambda^{k,\ell}\partial_k\partial_\ell v.$$

Theorem 3.1. — *Let $\mathcal{C}$ be a ring of $\mathbf{R}^d$ and ε_0 a positive real number. Let us consider two families of smooth metrics $\mathcal{G}^{(j)} \stackrel{def}{=} (G_\Lambda^{(j)})_{\Lambda\geqslant\Lambda_0}$ such that for any k, $\|\mathcal{G}^{(j)}\|_k$ is finite and such that $\|\mathcal{G}^{(j)}\|_0$ is small enough. For any positive real number $\varepsilon \leqslant \varepsilon_0$, a constant C_ε exists which satisfies the following properties. Let f_1 and f_2 be two functions in $L^1_{I_\Lambda}(L^2)$ and γ_1 and γ_2 two functions of L^2; let us assume that the Fourier transform of those functions have their support included in $\mathcal{C}$. Let us assume that*

$$|I_\Lambda| \leqslant \Lambda^{2-\varepsilon}.$$

Then if $v_{1,\Lambda}$ and $v_{2,\Lambda}$ are solutions of

$$(E_\Lambda)\begin{cases} P_\Lambda^{(j)} v_{j,\Lambda} &= f_j \\ \nabla v_{j,\Lambda|\tau=0} &= \gamma_j \end{cases}$$

we shall have the following properties:

– *if $d \geqslant 5$, we have*

$$\|v_{j,\Lambda}\|_{L^2_{I_\Lambda}(L^4)} \leqslant C(\|\gamma_j\|_{L^2} + \|f_j\|_{L^1_{I_\Lambda}(L^2)}).$$

– *if $d = 4$,*

$$\|v_{j,\Lambda}\|_{L^2_{I_\Lambda}(L^6)} \leqslant C(\|\gamma_j\|_{L^2} + \|f_j\|_{L^1_{I_\Lambda}(L^2)}).$$

– *if $d \geqslant 4$, then we have, for any $h \leqslant 1$ and any $\varepsilon > 0$,*

$$\|\chi(h^{-1}D)(v_{1,\Lambda}v_{2,\Lambda})\|_{L^1_{I_\Lambda}(L^\infty)} \leqslant C_\varepsilon h^{d-2-\varepsilon}\log(e+|I_\Lambda|) \times \big(\|\gamma_1\|_{L^2} + \|f_1\|_{L^1_{I_\Lambda}(L^2)}\big)\big(\|\gamma_2\|_{L^2} + \|f_2\|_{L^1_{I_\Lambda}(L^2)}\big).$$

Let us point out at this step that when h is small enough, this estimate is nothing but the Sobolev embedding. Using Bernstein inequality, we can write

$$\begin{aligned}\|\chi(h^{-1}D)(v_{1,\Lambda}v_{2,\Lambda})\|_{L^1_{I_\Lambda}(L^\infty)} &\leqslant h^d\|v_{1,\Lambda}v_{2,\Lambda}\|_{L^1_{I_\Lambda}(L^1)} \\ &\leqslant h^d|I_\Lambda|\,\|v_{1,\Lambda}\|_{L^\infty_{I_\Lambda}(L^2)}\|v_{2,\Lambda}\|_{L^\infty_{I_\Lambda}(L^2)} \\ &\leqslant h^d|I_\Lambda|\big(\|\gamma_1\|_{L^2} + \|f_1\|_{L^1_{I_\Lambda}(L^2)}\big)\big(\|\gamma_2\|_{L^2} + \|f_2\|_{L^1_{I_\Lambda}(L^2)}\big).\end{aligned}$$

So when $h^d|I_\Lambda| \leqslant h^{d-2-\varepsilon}$, the inequality of above Theorem 3.1 is proved. In all that follows, we shall assume that

$$|I_\Lambda| \geqslant h^{-2-\varepsilon}. \tag{12}$$

The proof of this theorem will be the purpose of sections 4 to 7 and this is in fact the core of this work.

3.2. The local estimates. — From this microlocal statement, let us deduce now the following local result.

Theorem 3.2. — *Let $(G^{(j)})_{1\leqslant j\leqslant 2}$ be two metrics such that $\|\partial G^{(j)}\|_{L^1_T(L^\infty)}\leqslant C_0$. For any ε, a constant C_ε exists (which of course depends on d) such that if $u_{j,q}$ are functions whose Fourier transform is supported in a ring $2^q\mathcal{C}$ and are solutions of*

$$(E_\Lambda)\begin{cases}\partial_t^2 u_q^{(j)}-\Delta u_q^{(j)}-\widetilde{G}^{(j)}\cdot\nabla^2 u_q^{(j)} &= f_{j,q}\\ \nabla u_q^{(j)}{}_{|t=0} &= \gamma_{j,q}\end{cases}$$

where $\widetilde{G}^{(j)}\stackrel{def}{=}S_q^{2/3}G^{(j)}$ and where $\gamma_{j,q}$ and $f_{j,q}$ have Fourier transform supported in a ring $2^q\mathcal{C}$, then we have

– *if $d\geqslant 5$,*

$$2^{q\left(\frac{d}{4}-\frac{1}{2}-k\right)}\|\partial^{1+k}u_q^{(j)}\|_{L^2_T(L^4)}\leqslant C_\varepsilon 2^{q\left(\frac{d}{2}-1\right)}(2^qT)^{\frac{1}{6}+\varepsilon}\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big);$$

– *if $d=4$,*

$$2^{q\left(\frac{1}{6}-k\right)}\|\partial^{1+k}u_q^{(j)}\|_{L^2_T(L^6)}\leqslant C_\varepsilon 2^q(2^qT)^{\frac{1}{6}+\varepsilon}\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big);$$

– *if $d\geqslant 4$, for any $p\leqslant q$,*

$$\begin{aligned}\big\|\chi(2^{-p}D)\big(\partial^{1+k}u_q^{(1)}\partial u_q^{(2)}\big)\big\|_{L^1_T(L^\infty)}&\\ \leqslant C_\varepsilon 2^{p(d-2)}2^{q(1+k)}(2^qT)^{\frac{1}{3}+\varepsilon}&\big(\|\partial u_q^{(1)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{1,q}\|_{L^1_T(L^2)}\big)\\ &\times\big(\|\partial u_q^{(2)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{2,q}\|_{L^1_T(L^2)}\big).\end{aligned}$$

To start with, let us observe that after a rescaling of the above Theorem 3.1, we get that, for any subinterval $I=(t^-,t^+)$ of $[0,T]$ such that

$$|I|\leqslant T(2^qT)^{1-2\delta-\varepsilon}\quad\text{and}\quad\|\nabla G_\delta^{(j)}\|_{L^1_I(L^\infty)}+|I|\,\|\nabla^2 G_\delta^{(j)}\|_{L^1_I(L^\infty)}\leqslant\varepsilon_0\,,\tag{13}$$

we have

– if $d\geqslant 5$,

$$\|\partial^{1+k}u_q^{(j)}\|_{L^2_I(L^4)}\leqslant C2^{q\left(\frac{d}{4}-\frac{1}{2}+k\right)}\big(\|\partial u_q^{(j)}(t^-)\|_{L^2}+\|f_{j,q}\|_{L^1_I(L^2)}\big).\tag{14}$$

– if $d=4$,

$$\|\partial^{1+k}u_q^{(j)}\|_{L^2_I(L^6)}\leqslant C2^{q\left(\frac{5}{6}+k\right)}\big(\|\partial u_q^{(j)}(t^-)\|_{L^2}+\|f_{j,q}\|_{L^1_I(L^2)}\big).\tag{15}$$

– if $d\geqslant 3$, for any $p\leqslant q$ and any $\varepsilon>0$,

$$\begin{aligned}\big\|\chi(2^{-p}D)\big(\partial^{1+k}u_q^{(1)}\cdot\partial u_q^{(2)}\big)\big\|_{L^1_I(L^\infty)}&\leqslant C_\varepsilon 2^{p(d-2)}(2^qT)^\varepsilon 2^{q(1+k)}\\ \times\big(\|\partial u_q^{(1)}(t^-)\|_{L^2}+\|f_{1,q}\|_{L^1_I(L^2)}\big)&\big(\|\partial u_q^{(2)}(t^-)\|_{L^2}+\|f_{2,q}\|_{L^1_I(L^2)}\big).\end{aligned}\tag{16}$$

Let us observe that in the case when $2^pT \leqslant 1$, the above inequality (16) is obtained by Bernstein inequality.

Then the method consists in a decomposition of the interval $[0,T]$ in subintervals I on which the above microlocalized estimates are true. The key point is a careful counting of the number of such intervals. This method has been introduced by the authors in [**4**] and improved by D. Tataru in [**28**].

Let us state $G_\delta^{(j)} \stackrel{\text{def}}{=} S_q^\delta G^{(j)}$. Using the fact that

$$\|\nabla^2 G_\delta^{(j)}\|_{L^1_I(L^\infty)} \leqslant \frac{1}{T}(2^qT)^\delta \|\nabla G_\delta^{(j)}\|_{L^1_I(L^\infty)},$$

Condition (13) becomes

$$|I| \leqslant T(2^qT)^{1-2\delta-\varepsilon} \quad \text{and} \quad \frac{|I|}{T}(2^qT)^\delta \, \|\nabla G_\delta^{(j)}\|_{L^1_I(L^\infty)} \leqslant \varepsilon_0. \tag{17}$$

But as seen in Corollary 2.4, there is a loose on the remainder. The decomposition is the opportunity to compensate this loose. To do so, let us consider a parameter λ in the interval $[0,1]$ which will be determined later on. We impose on the interval I that

$$\|f_{j,q}\|_{L^1_I(L^2)} \leqslant \lambda \|f_{j,q}\|_{L^1_T(L^2)}.$$

This constraint joint to the condition (17) can be sum up by

$$\begin{aligned}\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}} \int_I dt + \frac{1}{\lambda \|f_{j,q}\|_{L^1_T(L^2)}} \int_I \|f_{j,q}(t)\|_{L^2} dt \\ + \frac{|I|}{T}(2^qT)^\delta \int_I \|\nabla G_\delta^{(j)}(t)\|_{L^\infty} dt \leqslant \varepsilon_0.\end{aligned} \tag{18}$$

We shall prove that such a finite decomposition exists (and also control the number of intervals) by induction. Let us assume that an increasing sequence $(t_j)_{0\leqslant j\leqslant k}$ of points of $[0,T]$ such that $t_n < T$ and, for any $j \leqslant n-1$,

$$\begin{aligned}\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t_{j+1}-t_j) + \frac{1}{\lambda \|f_{j,q}\|_{L^1_T(L^2)}} \int_{t_j}^{t_{j+1}} \|f_{j,q}(t)\|_{L^2} dt \\ + \frac{t_{j+1}-t_j}{T}(2^qT)^\delta \int_{t_j}^{t_{j+1}} \|\nabla G_\delta^{(j)}(t)\|_{L^\infty} dt = \varepsilon_0\end{aligned}$$

As the function

$$\begin{aligned}F_k(t) \stackrel{\text{def}}{=} \frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t-t_k) + \frac{1}{\lambda \|f_{j,q}\|_{L^1_T(L^2)}} \int_{t_k}^{t} \|f_{j,q}(t')\|_{L^2} dt' \\ + \frac{t-t_k}{T}(2^qT)^\delta \int_{t_k}^{t} \|\nabla G_\delta^{(j)}(t')\|_{L^\infty} dt'\end{aligned}$$

is a increasing function on the interval $[t_k,T]$, either the interval $[t_k,T]$ satisfies Condition (18), or a unique t_{k+1} exists in the interval $]t_k,T[$ sucht that $F_k(t_{k+1}) = \varepsilon_0$.

Now let us estimate the number of intervals. At least one of the three terms of the left inside of the above inequality is greater or equal to $\varepsilon_0/3$. So either

$$\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t_{j+1}-t_j)\geqslant\frac{\varepsilon_0}{3},$$

or

$$\frac{1}{\lambda\|f_{j,q}\|_{L^1_T(L^2)}}\int_{t_j}^{t_{j+1}}\|f_{j,q}(t)\|_{L^2}dt\geqslant\frac{\varepsilon_0}{3},$$

or

$$\frac{t_{j+1}-t_j}{T}(2^qT)^{\delta}\int_{t_j}^{t_{j+1}}\|\nabla G^{(j)}_\delta(t)\|_{L^\infty}dt\geqslant\frac{\varepsilon_0}{3}\cdot$$

In the third case, we get that for any positive real number A,

$$\frac{3}{\varepsilon_0}(t_{j+1}-t_j)A+(2^qT)^{\delta}\frac{1}{AT}\int_{t_j}^{t_{j+1}}\|\nabla G^{(j)}_\delta(t')\|_{L^\infty}dt'\geqslant 1.$$

It turns out that in any case,

$$\frac{1}{T(2^qT)^{1-2\delta-\varepsilon}}(t_{j+1}-t_j)+\frac{1}{\lambda\|f_{j,q}\|_{L^1_T(L^2)}}\int_{t_j}^{t_{j+1}}\|f_{j,q}(t)\|_{L^2}dt$$
$$+(t_{j+1}-t_j)A+(2^qT)^{\delta}\frac{\varepsilon_0}{3AT}\int_{t_j}^{t_{j+1}}\|\nabla G^{(j)}_\delta(t)\|_{L^\infty}dt\geqslant\frac{\varepsilon_0}{3}\cdot$$

So by summation we infer that the number N of intervals is finite and that

$$N\leqslant\frac{C}{\varepsilon_0}(2^qT)^{2\delta-1+\varepsilon}+\frac{1}{\lambda\varepsilon_0}+\frac{3AT}{\varepsilon_0^2}+(2^qT)^{\delta}\frac{1}{AT}\|\nabla G^{(j)}_\delta\|_{L^1_T(L^\infty)}.$$

As usual, the best choice in the above inequality is the one that ensures that all the terms are (almost) equivalent. So here, we choose

$$AT=(2^qT)^{\delta/2},\ \lambda=(2^qT)^{-\delta/2}\quad\text{and}\quad\delta=\frac{2}{3}\cdot$$

So the number of intervals N is less than $C(2^qT)^{\frac13+\varepsilon}$. So let us denote by $(I_{q,\ell})_{1\leqslant\ell\leqslant N}$ the partition of the interval $[0,T]$ and state $I_{q,\ell}=(t_{q,\ell},t_{q,\ell+1})$. Using (18) and (14), we can write

$$2^{2q\left(\frac d4-\frac12-k\right)}\|\partial^{1+k}u_q^{(j)}\|^2_{L^2_T(L^4)}=C2^{2q\left(\frac d4-\frac12-k\right)}\sum_{\ell=1}^N\|\partial^{1+k}u_q^{(j)}\|^2_{L^2_{I_{q,\ell}}(L^4)}$$
$$\leqslant C2^{2q\left(\frac d2-1\right)}N\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big)^2$$

As N is less than $C(2^qT)^{\frac13+\varepsilon}$, we have, when $d\geqslant 5$,

$$2^{q\left(\frac d4-\frac12-k\right)}\|\partial^{1+k}u_q^{(j)}\|_{L^2_T(L^4)}$$
$$\leqslant C_\varepsilon 2^{q\left(\frac d2-1\right)}(2^qT)^{\frac16+\varepsilon}\big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\big).$$

The case when $d=4$ can be treated exactly along the same lines and is thus omitted. In order to prove the bilinear estimate, let us write, using (16) and (18),

$$\begin{aligned}\|\chi(2^{-p}D)(\partial^{1+k}u_q^{(1)}u_q^{(2)})\|_{L^1_T(L^\infty)} &\leqslant \sum_{\ell=1}^{N}\|\chi(2^{-p}D)(\partial^{1+k}u_q^{(1)}u_q^{(2)}\|_{L^1_{I_{q,\ell}}(L^\infty)}\\ &\leqslant C_\varepsilon 2^{p(d-2)}2^{q(1+k)}(2^qT)^{\varepsilon}N\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big)\\ &\qquad\times\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big)\\ &\leqslant C_\varepsilon 2^{p(d-2)}2^{q(1+k)}(2^qT)^{\frac13+2\varepsilon}\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big)\\ &\qquad\times\Big(\|\partial u_q^{(j)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|f_{j,q}\|_{L^1_T(L^2)}\Big).\end{aligned}$$

So Theorem 3.2 is proved. Let us state the following corollary.

Corollary 3.1. — *If $N^\alpha_T(\gamma)$ is small enough and C_0 large enough, then assertion $(\mathcal{P}_n)$ implies assertion $(\mathcal{P}_{n+1})$.*

Let us first investigate the case when $d\geqslant 5$. Assertion $(\mathcal{P}_n)$ and Theorem 3.2 imply that

$$\begin{aligned}2^{q\left(\frac d4-\frac12\right)}\|\partial u_q^{(n+1)}\|_{L^2_T(L^4)} &\leqslant C_\varepsilon 2^{q\left(\frac d2-1\right)}(2^qT)^{\frac16+\varepsilon}\\ &\qquad\times\big(\|\partial u_q^{(n+1)}\|_{L^\infty_T(L^2)}+(2^qT)^{-1/3}\|R_q^\delta(n)\|_{L^1_T(L^2)}\big).\end{aligned}$$

Corollaries 2.2 and 2.4 imply that, if $2^qT\geqslant C_1$,

$$2^{q\left(\frac d4-\frac12\right)}\|\partial u_q^{(n+1)}\|_{L^2_T(L^4)}\leqslant C_\varepsilon(2^qT)^{\varepsilon-\alpha}N^\alpha_T(\gamma)\big(1+CC_0N^\alpha_T(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big).$$

When $2^qT\leqslant C_1$, we use Corollary 2.3 to write that

$$\|\partial u^{(n+1)}\|_{L^2_T(\dot B^{\frac d4-\frac12}_{4,2})}\leqslant CN^\alpha_T(\gamma)\big(1+CC_0N^\alpha_T(\gamma)\|\partial u^{(n+1)}\|_{L^2_T(\dot C^{-1/2})}\big).$$

As the space $\dot B^{\frac d4-\frac12}_{4,2}$ is continuously embedded in $\dot C^{-1/2}$, we have, if $N^\alpha_T(\gamma)$ is small enough and C_0 large enough,

$$\|\partial u^{(n+1)}\|_{L^2_T(\dot B^{\frac d4-\frac12}_{4,2})}\leqslant C_0N^\alpha_T(\gamma)$$

and so using Corollary 2.3 we get $(\mathcal{P}_{n+1})$ for $d\geqslant 5$.

In the case $d=4$, following exactly the same lines we obtain that

$$\|\partial u^{(n+1)}\|_{L^2_T(\dot B^{1/6}_{6,2})}\leqslant C_0N^\alpha_T(\gamma). \tag{19}$$

We have to control $\|\partial G_{n+1,T}\|_{L^1_T(L^\infty)}$. Let us use Bony's decomposition as in the introduction. We get

$$\begin{aligned}
\Delta_p \partial G_{n+1,T} &= \Delta_p \Delta^{-1}(\partial \nabla u^{(n+1)} \nabla u^{(n+1)})\\
&= \sum_{j=1}^{3} \Delta_p^{(j)} \quad \text{with}\\
\Delta_p^{(1)} &\stackrel{\text{def}}{=} \Delta_p \Delta^{-1} \sum_q S_{q-1} \partial \nabla u^{(n+1)} \nabla u_q^{(n+1)},\\
\Delta_p^{(2)} &\stackrel{\text{def}}{=} \Delta_p \Delta^{-1} \sum_q S_{q-1} \nabla u^{(n+1)} \partial \nabla u_q^{(n+1)} \quad \text{and}\\
\Delta_p^{(3)} &\stackrel{\text{def}}{=} \Delta_p \Delta^{-1} \sum_{\substack{q\\ -1\leqslant \ell \leqslant 1}} \partial \nabla u_q^{(n+1)} \nabla u_{q-\ell}^{(n+1)}.
\end{aligned}$$

To estimate the norm $\|\cdot\|_{L^1_T(L^\infty)}$ of $\Delta_p^{(1)}$, let us observe that we have, for $k \in \{0,1\}$,

$$\begin{aligned}
\|S_{q-1}\partial^k \nabla u^{(n+1)}\|_{L^2_T(L^\infty)} &\leqslant \sum_{q'\leqslant q-2} 2^{q'k} \|\Delta_{q'} \nabla u^{(n+1)}\|_{L^2_T(L^\infty)}\\
&\leqslant \sum_{q'\leqslant q-2} 2^{q'\left(k+\frac{2}{3}\right)} \|\Delta_{q'} \nabla u^{(n+1)}\|_{L^2_T(L^6)}.
\end{aligned}$$

So by convolution inequality on the series, we get that

$$2^{-q\left(k+\frac{1}{2}\right)} \|S_{q-1}\partial^k \nabla u^{(n+1)}\|_{L^2_T(L^\infty)} \in \ell^2(\mathbf{Z}).$$

As the support of the Fourier transform of $S_{q-1}\partial \nabla u^{(n+1)} \nabla u_q^{(n+1)}$ is included in a ring of the type $2^q \widetilde{C}$, we get that

$$\sum_p \|\Delta_p^{(1)}\|_{L^1_T(L^\infty)} \leqslant C C_0 N_T^\alpha(\gamma)^2.$$

The term $\Delta_p^{(2)}$ can be estimated exactly in the same way. As seen in the introduction, the remainder term will required the use of bilinear estimates. Using the fact that the support of the Fourier transform of $\partial \nabla u_q^{(n+1)} \nabla u_{q-\ell}^{(n+1)}$ is included in ball of the type $2^q B$, we have that

$$\|\Delta_p^{(3)}\|_{L^1_T(L^\infty)} \leqslant C_\varepsilon (2^p T)^{-2(\alpha-\varepsilon)} N_T^\alpha(\gamma)^2 \sum_{q\geqslant p-N_0} (2^{q-p}T)^{-2(\alpha-\varepsilon)}.$$

So choosing for instance $\varepsilon = \alpha/2$, we have that

$$\sum_{p\,/\,2^p T\geqslant C} \|\Delta_p^{(3)}\|_{L^1_T(L^\infty)} \leqslant C_\alpha N_T^\alpha(\gamma)^2.$$

But, for low frequencies in p, we simply observe that, by Bernstein inequality and Corollary 2.2, we have

$$\begin{aligned}
\|\Delta_p^{(3)}\|_{L^1_T(L^\infty)} &\leqslant CT2^{2p} \sum_{q\geqslant p-N_0} \|\partial\nabla u_q^{(n+1)}\|_{L^\infty_T(L^2)}\|\nabla u_{q-\ell}^{(n+1)}\|_{L^\infty_T(L^2)}\\
&\leqslant CT2^{2p}\|\gamma\|^2_{s_\alpha-1}\sum_{q\geqslant p-N_0} 2^{-q\left(1+\frac{1}{3}+2\alpha\right)}\\
&\leqslant C(2^pT)^{\frac{2}{3}-2\alpha}N_T^\alpha(\gamma)^2.
\end{aligned}$$

So we have

$$\|\partial G_{n+1,T}\|_{L^1_T(L^\infty)} \leqslant CN_T^\alpha(\gamma)^2$$

and Corollary 3.1 is proved.

Now the proof of Theorem 0.3 (i.e. the case of dimension greater or equal to 5) is pure routine of non linear hyperbolic partial differential equations.

3.3. The existence and uniqueness when $d = 4$. — The case of dimension 4 requires some attention. Let us first assume that γ belongs to $H^{\frac{d}{2}-\frac{1}{2}}$. So it is clear that on an interval $[-T,T]$ the length of which depends only on $\|\gamma\|_{\dot H^{\frac{d}{2}-1}}$ and $\|\gamma\|_{\dot H^{\frac{d}{2}-1+\frac{1}{6}+\alpha}}$, the sequence $(\partial u^{(n)})_{n\in\mathbf{N}}$ is bounded in $L^\infty_T(H^{\frac{d}{2}-\frac{1}{2}})$. So energy methods (because the initial data is more regular) allow to claim that a solution u exists on $[-T,T]$ such that

$$\partial u \in L^\infty_T(H^{\frac{d}{2}-\frac{1}{2}}).$$

Moreover, we have on this interval the following estimates:

$$\begin{aligned}
&\|\partial u\|_{L^2([0,T];\dot B^{\frac{d}{6}-\frac{1}{2}}_{6,2})} \leqslant C_0N_T^\alpha(\gamma)\\
&\|\partial G_u\|_{L^1([0,T];L^\infty)} \leqslant 2\\
&\|\partial u\|_{T,s-1} \leqslant e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s\in\Big[\frac{3}{2}+\alpha, 2+\frac{1}{6}+\alpha\Big].
\end{aligned}$$

This solution is unique because of the result based on energy methods. Now let us consider intial data (u_0,u_1) which satisfy the hypothesis of Theorem 0.4. So if we consider initial data (S_nu_0,S_nu_1), a solution $\widetilde{u}^{(n)}$ associated to (S_nu_0,S_nu_1), exists on an interval $[-T,T]$ such that

$$\|\partial\widetilde{u}^{(n)}\|_{L^2([0,T];\dot B^{\frac{d}{6}-\frac{1}{2}}_{6,2})} \leqslant C_0N_T^\alpha(\gamma) \tag{20}$$

$$\|\partial G_{\widetilde{u}^{(n)}}\|_{L^1([0,T];L^\infty)} \leqslant 2 \tag{21}$$

$$\|\partial\widetilde{u}^{(n)}\|_{T,s-1} \leqslant e^3\|\gamma\|_{s-1} \quad \text{for any} \quad s\in\Big[\frac{3}{2}+\alpha, 2+\frac{1}{6}+\alpha\Big]. \tag{22}$$

In order to prove that $(\widetilde{u}^{(n)})_{n\in\mathbf{N}}$ is a Cauchy sequence and thus the uniqueness part of Theorem 0.4, we shall prove the following lemma which clearly concludes the proof.

Lemma 3.1. — *Let $u^{(j)}$ be two solutions of (EC) on the interval $[-T_0, T_0]$ such that*

$$\partial u^{(j)} \in C([-T_0,T_0]; H^{s_\alpha -1}) \cap L^2_{T_0}(\dot B^{1/6}_{6,2}) \quad \textit{and} \quad \partial g_{u^{(j)}} \in L^1_{T_0}(L^\infty).$$

Then if T is small enough, we have that

$$\|\partial u^{(1)} - \partial u^{(2)}\|_{L^\infty_T(\dot H^{s_\alpha-2})} \leqslant 2\|\gamma^{(1)} - \gamma^{(2)}\|_{\dot H^{s_\alpha-2}}.$$

As in the iterative scheme, let us introduce a time cut-off. Let θ be a smooth function such that $\operatorname{Supp}\theta \subset]-2,2[$ and θ has value 1 near $[-1,1]$. So on the interval $[-T,T]$, the function $u^{(j)}$ is the solution of

$$(EC)\begin{cases} \partial_t^2 u^{(j)} - \Delta u^{(j)} - \displaystyle\sum_{1\leqslant k,\ell\leqslant d} G^{k,\ell}_{u^{(j)},T}\partial_k\partial_\ell u^{(j)} = 0 \\ \qquad\qquad\qquad\qquad (u,\partial_t u)_{|t=0} = (u_0,u_1). \end{cases}$$

with

$$G^{k,\ell}_{u^{(j)},T} = \theta\Big(\frac{t}{T}\Big) g^{k,\ell}_{u^{(j)}} \quad \text{with} \quad \Delta g^{k,\ell}_{u^{(j)}} = Q_{k,\ell}(\partial u^{(j)}, \partial u^{(j)}).$$

From now on in this section, we shall always work in the interval $[-T,T]$ with $2T \leqslant T_0$. Let us define $w = u^{(1)} - u^{(2)}$. Then on the interval $[-T,T]$, w is the solution of

$$\begin{cases} \partial_t^2 w - \Delta w - \displaystyle\sum_{1\leqslant k,\ell\leqslant d} G^{k,\ell}_{u^{(1)},T}\partial_k\partial_\ell w = F_{1,2} \\ \qquad\qquad\qquad (w,\partial_t w)_{|t=0} = (u_0^{(1)} - u_0^{(2)}, u_1^{(1)} - u_1^{(2)}) \end{cases}$$

with

$$F_{1,2} \stackrel{\text{def}}{=} (G_{u^{(2)}} - G_{u^{(1)}})\cdot\nabla^2 u^{(2)}.$$

We shall use the fact all the time in this paragraph that the two solutions $u^{(j)}$ satisfies

$$\|\partial u^{(j)}\|_{L^2_T(\dot C^{-1/2})} \leqslant C\|\partial u^{(j)}\|_{L^2_T(\dot B^{1/6}_{6,2})} \leqslant N^\alpha_T(\gamma^{(j)}) \quad \text{and} \quad \|\partial G_{u^{(j)},T}\|_{L^1_T(L^\infty)} \leqslant C_0.$$

Moreover, we state

$$\Gamma \stackrel{\text{def}}{=} \|\gamma^{(1)}\|_{s_\alpha-1} + \|\gamma^{(2)}\|_{s_\alpha-1}\,,\ \Gamma_T \stackrel{\text{def}}{=} N^\alpha_T(\gamma^{(1)}) + N^\alpha_T(\gamma^{(2)}) \quad \text{and} \quad \underline{\gamma} \stackrel{\text{def}}{=} \gamma^{(1)} - \gamma^{(2)}.$$

Let us use computations done during the proof of the paralinearization theorem 2.1. Thanks to Formulas (7) and (8), we get that the function $w_q = \Delta_q w$ is solution of

$$\partial_t^2 w_q - \Delta w_q - S_{q-1}G_{u^{(1)},T}\nabla^2 w_q = R_q(t)$$

with

$$
\begin{aligned}
R_q &= \Delta_q F_{1,2}(t) + \sum_{j=1}^{4} R_q^{(j)} \quad \text{where}\\
R_q^{(1)} &\stackrel{\text{def}}{=} \sum_{|q-q'|\leqslant N_1} \big[\Delta_q, S_{q'-1}G_{u^{(1)},T}\big]\nabla^2 w_{q'}\\
R_q^{(2)} &\stackrel{\text{def}}{=} \sum_{|q-q'|\leqslant N_1} (S_{q'-1}G_{u^{(1)},T} - S_{q-1}G_{u^{(1)},T})\nabla^2\Delta_q u_{q'}\\
R_q^{(3)} &\stackrel{\text{def}}{=} \Delta_q \sum_{\substack{q'\geqslant q-N_1\\ \ell\in\{-1,0,1\}}} \nabla^2 w_{q'}\Delta_{q'+\ell}G_{u^{(1)},T}\\
R_q^{(4)} &\stackrel{\text{def}}{=} \Delta_q \sum_{|q'-q|\leqslant N_1} S_{q'-1}\nabla^2 w_{q'}\Delta_{q'}G_{u^{(1)},T}.
\end{aligned}
$$

It is obvious that, if $s_\alpha - 2 > 1$, we have for any $j \in \{1,2,3\}$,

$$
\|R_q^{(j)}(t)\|_{L^2} \leqslant c_q(t) C 2^{-q(s_\alpha-2)} \|\nabla G_{u^{(1)},T}(t)\|_{L^\infty} \|\partial w(t)\|_{s_\alpha-2}.
$$

Using Lemma 2.1, we have

$$
\|\Delta_{q'} G_{u^{(1)},T}(t)\|_{L^2} \leqslant C c_{q'}(t) 2^{-q'(s_\alpha-\frac{3}{2})} \|\partial u^{(1)}(t)\|_{\dot C^{-1/2}} \|\partial u^{(1)}(t)\|_{\dot H^{s_\alpha-1}}.
$$

Thus

$$
\|R_q^{(4)}(t)\|_{L^2} \leqslant C c_q(t) 2^{-q(s_\alpha-2)} \|\partial w(t)\|_{\dot C^{-3/2}} \|\partial u^{(1)}(t)\|_{\dot C^{-1/2}} \|\partial u^{(1)}(t)\|_{s_\alpha-1}.
$$

Using the properties of $u^{(j)}$ on the interval $[-T,T]$ imply that

$$
\begin{aligned}
(23)\quad \|R_q(t)\|_{L^2} \leqslant c_q(t) C 2^{-q(s_\alpha-2)} \big(&\|\nabla G_{u^{(1)},T}(t)\|_{L^\infty} \|\partial w(t)\|_{s_\alpha-2}\\
&+ \|\gamma^{(1)}\|_{s_\alpha-1} \|\partial w(t)\|_{\dot C^{-3/2}} \|\partial u^{(1)}(t)\|_{\dot C^{-1/2}} + \|F_{1,2}(t)\|_{s_\alpha-2}\big).
\end{aligned}
$$

So using Gronwall lemma, we infer that for any t in $[-T,T]$,

$$
\begin{aligned}
\|\partial w\|_{L^\infty_T(\dot H^{s_\alpha-2})} \leqslant \big(&\|\underline{\gamma}\|_{s_\alpha-2} + \|\gamma^{(1)}\|_{s_\alpha-1} \|\partial w\|_{L^2_T(\dot C^{-3/2})} \|\partial u^{(1)}\|_{L^2_T(\dot C^{-1/2})}\\
&+ \|F_{1,2}\|_{L^1_T(H^{s_\alpha-2})}\big) \exp\big(\|\partial G_{u^{(1)},T}\|_{L^1_T(L^\infty)}\big).
\end{aligned}
$$

So the properties of the solution $u^{(1)}$ imply that

$$
(24)\quad \|\partial w\|_{L^\infty_T(\dot H^{s_\alpha-2})} \leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma\Gamma_T \|\partial w\|_{L^2_T(\dot C^{-3/2})} + \|F_{1,2}\|_{L^1_T(H^{s_\alpha-2})}\big) \quad \text{and}
$$

$$
(25)\quad \|R_q\|_{L^1_T(L^2)} \leqslant C 2^{-q(s_\alpha-2)}\big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma\Gamma_T \|\partial w\|_{L^2_T(\dot C^{-3/2})} + \|F_{1,2}\|_{L^1_T(H^{s_\alpha-2})}\big).
$$

Because the L^2 norm in time with value in $\dot C^{-3/2}$ of w appears in the right side of the above inequality, we have to use the Strichartz estimates. Applying Theorem 2.2 with $\delta = 2/3$, and (25), it turns out that w_q is solution of

$$
\partial_t^2 w_q - \Delta w_q - \widetilde G_{u^{(1)},T}\nabla^2 w_q = \widetilde R_q(t)
$$

with $\widetilde{G}_{u^{(1)},T} \stackrel{\text{def}}{=} S_q^{2/3} G_{u^{(1)},T}$ and (dropping the case of low frequencies)

$$\begin{aligned}(26)\quad &\|\widetilde{R}_q\|_{L^1_T(L^2)} \\ &\leqslant C2^{-q(s_\alpha-2)}(2^qT)^{1/3}\big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma\Gamma_T\|\partial w\|_{L^2_T(\dot{C}^{-3/2})} + \|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).\end{aligned}$$

Now thanks to Theorem 3.2 applied with $\varepsilon = \alpha/2$, we infer that

$$\begin{aligned}&2^{-q5/6}\|\partial w_q\|_{L^2_T(L^6)} \\ &\quad\leqslant (2^qT)^{-\alpha/2}\big(T^{\frac16+\alpha}\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma_T^2\|\partial w\|_{L^2_T(\dot{C}^{-3/2})} + T^{\frac16+\alpha}\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).\end{aligned}$$

As $2^{-q3/2}\|\partial w_q\|_{L^2_T(L^\infty)} \leqslant 2^{-q5/6}\|\partial w_q\|_{L^2_T(L^6)}$ it turns out that, if Γ_T is small enough, (dropping the case of low frequencies),

$$(27)\qquad \|\partial w\|_{L^2_T(\dot{C}^{-3/2})} \leqslant CT^{\frac16+\alpha}\big(\|\underline{\gamma}\|_{s_\alpha-2} + C\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big)$$

and thus with (24),

$$(28)\qquad \|\partial w\|_{L^\infty_T(\dot{H}^{s_\alpha-2})} \leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2} + \|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).$$

The estimate of the term $F_{1,2}$ is more delicate than the others. Using Bony's decomposition, we get that

$$\begin{aligned}F_{1,2} &= \sum_{j=1}^4 F^{(j)} \quad \text{with}\\ F^{(1)} &\stackrel{\text{def}}{=} T_{\nabla^2 u^{(2)}}\Delta^{-1}Q(\partial w, \partial u^{(1)} + \partial u^{(2)})\,,\\ F^{(2)} &\stackrel{\text{def}}{=} R\big(\nabla^2 u^{(2)}, \Delta^{-1}Q(\partial w, \partial u^{(1)} + \partial u^{(2)}\big)\,,\\ F^{(3)} &\stackrel{\text{def}}{=} T_{\Delta^{-1}Q(T_{\partial w},(\partial u^{(1)}+\partial u^{(2)})+Q(T_{\partial u^{(1)}+\partial u^{(2)}},\partial w)}\nabla^2 u^{(2)} \quad \text{and}\\ F^{(4)} &\stackrel{\text{def}}{=} T_{\Delta^{-1}(QR(\partial w,\partial u^{(1)})+QR(\partial w,\partial u^{(2)}))}\nabla^2 u^{(2)}.\end{aligned}$$

The terms $F^{(j)}$ with $j \leqslant 3$ will require only Strichartz inequalities to be controled. So law of product in Besov spaces implies that

$$\begin{aligned}\|Q(\partial w, \partial u^{(1)} + \partial u^{(2)})(t)\|_{s_\alpha-\frac52} &\leqslant C\big(\|\partial w(t)\|_{\dot{B}^{-5/6}_{6,\infty}}(\|\partial u^{(1)}(t)\|_{s_\alpha-1}\\ &+ \|\partial u^{(2)}(t)\|_{s_\alpha-1}) + \|\partial w(t)\|_{s_\alpha-2}(\|\partial u^{(1)}(t)\|_{\dot{B}^{1/6}_{6,\infty}} + \|\partial u^{(2)}(t)\|_{\dot{B}^{1/6}_{6,\infty}})\big)\end{aligned}$$

Using the properties of $u^{(1)}$, we get that

$$\begin{aligned}\|F^{(1)}(t)\|_{s_\alpha-2} &\leqslant C\Gamma_T\|\partial w(t)\|_{\dot{B}^{-5/6}_{6,\infty}}\|\partial u^{(2)}(t)\|_{\dot{B}^{1/6}_{6,\infty}}\\ &\quad + C\|\partial w(t)\|_{s_\alpha-2}\big(\|\partial u^{(1)}(t)\|^2_{\dot{B}^{-5/6}_{6,\infty}} + \|\partial u^{(2)}(t)\|^2_{\dot{B}^{-5/6}_{6,\infty}}\big).\end{aligned}$$

By integration, using the properties of the two solutions and (24), (27) and (28), we get that, if T is small enough,

$$(29)\qquad \|F^{(1)}\|_{L^1_T(\dot{H}^{s_\alpha-2})} \leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2} + \Gamma_T\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})}\big).$$

The term $F^{(2)}$ is estimated exactly along the same lines. The term $F^{(3)}$ is the analog to the paraproduct term in the first section. Let us write that

$$\begin{aligned}\|T_{\partial w}(\partial u^{(1)}+\partial u^{(2)})\|_{\dot B^{-2}_{\infty,1}} &\leqslant C\|\partial w\|_{\dot B^{-3/2}_{\infty,2}}(\|\partial u^{(1)}\|_{\dot B^{-1/2}_{\infty,2}}+\|\partial u^{(2)}\|_{\dot B^{-1/2}_{\infty,2}})\\ &\leqslant C\|\partial w\|_{\dot B^{-5/6}_{6,2}}(\|\partial u^{(1)}\|_{\dot B^{1/6}_{6,2}}+\|\partial u^{(2)}\|_{\dot B^{1/6}_{6,2}}).\end{aligned}$$

As the same estimate is true for $T_{\partial u^{(1)}+\partial u^{(2)}}\partial w$, using the estimate (27), it turns out after times integration, and if T is small enough, that

$$\|F^{(3)}\|_{L^1_T(\dot H^{s_\alpha-2})}\leqslant C\big(\|\underline{\gamma}\|_{s_\alpha-2}+\Gamma_T\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big). \tag{30}$$

The estimate of the term $F^{(4)}$ requires the use of the bilinear estimate stated in Theorem 3.2. The key point is obviously to estimate

$$\Delta_{p,q}\overset{\text{def}}{=}\|\Delta_p\Delta^{-1}(\partial w_q\partial u_{q-j})\|_{L^1_T(L^\infty)}.$$

Theorem 3.2 applied with $\varepsilon=\alpha$ and $f=\widetilde R_q$ implies that

$$\begin{aligned}\Delta_{p,q}\leqslant C_\alpha 2^q\Big((2^qT)^{\frac16+\frac\alpha2}\|\partial w_q\|_{L^\infty_T(L^2)}+(2^qT)^{-\frac16+\frac\alpha2}\|\widetilde R_q\|_{L^1_T(L^2)}\Big)&\\ \times\Big((2^qT)^{\frac16+\frac\alpha2}\|\partial u^{(j)}_q\|_{L^\infty_T(L^2)}+(2^qT)^{-\frac16+\frac\alpha2}\|R_q^{\frac23-\frac\varepsilon2}(u^{(j)})\|_{L^1_T(L^2)}\Big)&.\end{aligned}$$

As $s_\alpha=2+\dfrac16+\alpha$, Theorem 2.2 and properties of the solution $u^{(1)}$ imply that

$$(2^qT)^{\frac16+\frac\alpha2}\|\partial u^{(j)}_q\|_{L^\infty_T(L^2)}+(2^qT)^{-\frac16+\frac\alpha2}\|R_q^{\frac23-\frac\alpha2}(u^{(j)})\|_{L^1_T(L^2)}\leqslant C_\alpha 2^{-q}(2^qT)^{-\alpha/2}\Gamma_T.$$

Theorem 2.2 and estimation (28) imply that

$$\begin{aligned}(2^qT)^{\frac16+\frac\alpha2}\|\partial w_q\|_{L^\infty_T(L^2)}+(2^qT)^{-\frac16+\frac\alpha2}\|\widetilde R_q\|_{L^1_T(L^2)}&\\ \leqslant C(2^qT)^{-\alpha/2}(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})})&.\end{aligned}$$

So it turns out that

$$\Delta_{p,q}\leqslant C_\varepsilon(2^qT)^{-\alpha}\Gamma_T T^{\frac16+\alpha}\big(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big).$$

So dropping the case of low frequencies (treated exactly along the same lines as in the proof of Corollary 2.3), we get that

$$\|\Delta^{-1}R(\partial w,\partial u)\|_{L^1_T(L^\infty)}\leqslant C\Gamma_T T^{\frac16+\alpha}\big(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big). \tag{31}$$

Using the properties of the solution $u^{(2)}$ and the properties of the action of the paraproduct, we deduce from the above inequality that

$$\|F^{(4)}\|_{L^1_T(\dot H^{s_\alpha-2})}\leqslant C\Gamma_T\big(\|\underline{\gamma}\|_{s_\alpha-2}+\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\big).$$

Together with the inequalities (29) and (30), we get that

$$\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}\leqslant C\|\underline{\gamma}\|_{s_\alpha-2}+C\Gamma_T\|F_{1,2}\|_{L^1_T(\dot H^{s_\alpha-2})}.$$

So if Γ_T is small enough, we have

$$\|F_{1,2}\|_{L^1_T(\dot{H}^{s_\alpha-2})} \leqslant C\|\underline{\gamma}\|_{s_\alpha-2}.$$

Plugging this estimate into (28) implies that

$$\|\partial w\|_{L^\infty_T(\dot{H}^{s_\alpha-2})} \leqslant C\|\underline{\gamma}\|_{s_\alpha-2}.$$

So uniqueness (and in fact stability) is proved.

4. Approximation of the solution and geometrical optics

4.1. The Hamilton-Jacobi equation. — The following proposition (and its proof) is a small modification of Proposition 6.1 of [**4**].

Proposition 4.1. — *Let F be a real valued smooth function on $\mathbf{R}^d \times \mathbf{R}^N$ bounded as all its derivatives such that*

$$F(\zeta, G) = \pm\big(|\zeta|^2 + G(\zeta,\zeta)\big)^{1/2} \quad \textit{for all} \quad \zeta \in \widetilde{\mathcal{C}}.$$

For any positive real number ε, a positive real number α exists such that, if $\|\mathcal{G}\|_0 \leqslant \alpha$ and $\Lambda \geqslant \alpha^{-1}$, for any η, a solution Φ_Λ of the equation

$$(\widetilde{HJ_\Lambda}) \begin{cases} \partial_\tau \Phi_\Lambda(\tau,y,\eta) = F_\Lambda(\tau,y,\partial_y\Phi_\Lambda(\tau,y,\eta)) \\ \Phi_\Lambda(0,y,\eta) \;= (y|\eta) \end{cases} \quad \textit{with } F_\Lambda(\tau,z,\zeta) \stackrel{def}{=} F(\zeta, G_\Lambda(\tau,z)).$$

exists and is smooth on $I_\Lambda \times \mathbf{R}^d \times \mathbf{R}^d$. Moreover, the family defined by $\Phi \stackrel{def}{=} (\Phi_\Lambda)_{\Lambda\geqslant\Lambda_0}$ satisfies the following properties: for any couple of integer (k,ℓ), a constant $C_{k,\ell}$ (independent of ε) exists such that

$$\sup_{\Lambda\geqslant\Lambda_0} \|(\partial_y\partial_\eta\Phi_\Lambda - \mathrm{Id})\|_{L^\infty(I_\Lambda\times\mathbf{R}^{2d})} \leqslant C\,\varepsilon\,, \tag{32}$$

$$\sup_{\Lambda\geqslant\Lambda_0} |I_\Lambda|\Lambda^k \|\partial_\eta^\ell \nabla^{2+k}\Phi_\Lambda\|_{L^\infty(I_\Lambda\times\mathbf{R}^{2d})} \leqslant C_{k,\ell}\,\varepsilon \quad \textit{and} \tag{33}$$

$$\sup_{\Lambda\geqslant\Lambda_0} \|\partial_\eta^{\ell+2}\Phi_\Lambda\|_{L^\infty(I_\Lambda\times\mathbf{R}^{2d})} \leqslant C|I_\Lambda|. \tag{34}$$

In section 6, we shall use the link between the solution of the above Hamilton-Jacobi equation and the Hamiltonian flow of the function $-F_\Lambda$ on $T^\star\mathbf{R}^d$. This link is classical but here we need precise estimates with respect to the metric G_Λ. It is described by the two following lemmas.

Lemma 4.1. — *Let Φ_Λ be the solution of the above Hamilton-Jacobi equation $(\widetilde{HJ_\Lambda})$ and Ψ_Λ the Hamiltonian flow of $-F_\Lambda(\tau,Y)$ i.e. the solution of*

$$\begin{cases} \dfrac{d\Psi_\Lambda}{d\tau}(\tau,y,\eta) = -H_{F_\Lambda}(\tau,\Psi_\Lambda(Y)) \\ \Psi_\Lambda(0,y,\eta) = (y,\eta). \end{cases}$$

Then we have

$$(\partial_\eta \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta) = y \quad \textit{and}$$
$$(\partial_y \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta) = \Psi^\eta_\Lambda(\tau,y,\eta).$$

To prove this, we have simply to remember that by construction of the solution of Hamilton-Jacobi equations (see for instance [**3**]), we have

$$\Big\{\big(\tau, \Psi^y_\Lambda(\tau,y,\eta), \Psi^\eta_\Lambda(\tau,y,\eta)\big)\,,\ \tau\in I_\Lambda\Big\} = \Big\{\big(\tau, \widetilde{y}, (\partial_{\widetilde{y}}\Phi_\Lambda)(\tau,\widetilde{y},\eta)\big)\,,\ \tau\in I_\Lambda\Big\}.$$

So we deduce immediately that

$$(\partial_y \Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta) = \Psi^\eta_\Lambda(\tau,y,\eta). \tag{35}$$

Now let us compute

$$A_j \stackrel{\text{def}}{=} \frac{d}{d\tau}\big((\partial_{\eta_j}\Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta)\big).$$

The chain rule implies that

$$A_j = (\partial_\tau \partial_{\eta_j}\Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta) + \sum_{k=1}^d (\partial_{\eta_j}\partial_{y_k}\Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta)\frac{d\Psi^{y_k}_\Lambda}{d\tau}(\tau,y,\eta).$$

By differentiation of $(\widetilde{HJ}_\Lambda)$ with respect to η, we get that, for any $\widetilde{y}\in\mathbf{R}^d$,

$$\partial_\tau\partial_{\eta_j}\Phi_\Lambda(\tau,\widetilde{y},\eta) = \sum_{k=1}^d (\partial_{\zeta_k}F_\Lambda)(\tau,\widetilde{y},\partial_{\widetilde{y}}\Phi_\Lambda(\tau,\widetilde{y},\eta))\partial_{\widetilde{y}_k}\partial_{\eta_j}\Phi_\Lambda(\tau,\widetilde{y},\eta).$$

Applying this identity with $\widetilde{y} = \Psi^y_\Lambda(\tau,y,\eta)$, we get

$$\begin{aligned}(\partial_\tau\partial_{\eta_j}\Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta) = \sum_{k=1}^d (\partial_{\zeta_k}F_\Lambda)\big(\tau, \Psi^y_\Lambda(\tau,y,\eta), (\partial_{\widetilde{y}}\Phi_\Lambda)(\tau,\Psi^y_\Lambda(\tau,y,\eta),\eta)\big)\\ \times (\partial_{\widetilde{y}_k}\partial_{\eta_j}\Phi_\Lambda)(\tau,\Psi^y_\Lambda(\tau,y,\eta),\eta).\end{aligned}$$

Using identity (35), we infer that

$$\begin{aligned}(\partial_\tau\partial_{\eta_j}\Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta) = \sum_{k=1}^d (\partial_{\zeta_k}F_\Lambda)\big(\tau, \Psi^y_\Lambda(\tau,y,\eta), \Psi^\eta_\Lambda(\tau,y,\eta)\big)\\ \times (\partial_{\widetilde{y}_k}\partial_{\eta_j}\Phi_\Lambda)(\tau,\Psi^y_\Lambda(\tau,y,\eta),\eta).\end{aligned}$$

Then we deduce that

$$A_j = \sum_{k=1}^d (\partial_{\eta_j}\partial_{y_k}\Phi_\Lambda)(\tau, \Psi^y_\Lambda(\tau,y,\eta),\eta)\bigg(\frac{d\Psi^{y_k}_\Lambda}{d\tau}(\tau,y,\eta) + (\partial_{\zeta_k}F_\Lambda)(\tau,\Psi_\Lambda(\tau,y,\eta)\bigg).$$

As for $\tau=0$, we have $\partial_\eta\Phi_\Lambda(0,\Psi^y_\Lambda(0,y,\eta),\eta) = \partial_\eta\Phi_\Lambda(0,y,\eta) = \partial_\eta(y|\eta) = y$, the first lemma is proved.

The second lemma is more technical and is related to properties of the hamiltonian flow with respect to a large class of metrics (i.e. of positive quadratic forms) on $T^\star \mathbf{R}^d$. It will be crucial in section 6.

Lemma 4.2. — *A constant C_0 exists such that for any couple of positive numbers (r,h) such that $|I_\Lambda| \geqslant h^{-2}$ we have the following properties. If*

$$g_a(dy^2, d\eta^2) \overset{def}{=} \frac{dy^2}{K^2} + \frac{d\eta^2}{h^2} \quad \text{with} \quad K = C|I_\Lambda| h$$

then, provided we choose C large enough, we have:

– for any couple (Y,Z) and for any $\tau \in I_\Lambda$, we have

$$\frac{1}{C_0}\, g_a(Y-Z) \leqslant g_a(\Psi_\Lambda(\tau,Y) - \Psi_\Lambda(\tau,Z)) \leqslant C_0\, g_a(Y-Z); \tag{36}$$

– for any couple of points (Y_0, Z_τ) of $T^\star \mathbf{R}^d$ such that

$$g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} \geqslant C_0 r$$

if $(z,\eta) \in B_{g_a}(Y_0, r)$ and if $(y,\zeta) \in B_{g_a}(Z_\tau, r)$ then

$$g_a\big(\nabla_\eta \Phi_\Lambda(\tau,y,\eta) - z, \nabla_y \Phi_\Lambda(\tau,y,\eta) - \zeta\big) \geqslant \frac{1}{C_0} g_a\big(Z_\tau - \Psi_\Lambda(\tau, Y_0)\big).$$

Remark. — The choice of the metric g_a will become clearer in section 6. But anyway, it is essentially the only choice of a metric such that the above inequalities are true.

Let us prove the first point of this lemma. By differentiation of the equation of the Hamiltonian flow, we have

$$\frac{d}{d\tau}\big(D\Psi_\Lambda(\tau,Y) - \mathrm{Id}\big) = -DH_{F_\Lambda} \cdot \big(D\Psi_\Lambda(\tau,Y) - \mathrm{Id}\big) - DH_{F_\Lambda}.$$

By Gronwall lemma, we get, for any $\tau \in I_\Lambda$,

$$\begin{aligned} \|D\Psi_\Lambda(\tau,Y) - \mathrm{Id}\,\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} \leqslant & \int_{I_\Lambda} \sup_{Y\in T^\star \mathbf{R}^d} \|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} d\tau \\ & \times \exp \int_{I_\Lambda} \sup_{Y\in T^\star \mathbf{R}^d} \|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} d\tau \end{aligned}$$

where

$$\|A\|_{\mathcal{L}_{g_a}(T^\star \mathbf{R}^d)} \overset{\mathrm{def}}{=} \sup_{\substack{Z\in T^\star \mathbf{R}^d \\ g_a(Z)\leqslant 1}} g_a(A\cdot Z)^{1/2}.$$

By definition of the Hamiltonian of F_Λ and of the metric g_a, we infer that, if $Z=(z,\zeta)$,

$$\begin{aligned}
g_a(DH_{F_\Lambda}(\tau,Y)\cdot Z) &\leqslant \frac{C}{K^2}\big(\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}|z|^2+|\zeta|^2\big)\\
&\qquad + \frac{C}{h^2}\big(\|\nabla^2 G_\Lambda(\tau,\cdot)\|^2_{L^\infty}|z|^2+\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}|\zeta|^2\big)\\
&\leqslant \frac{C|z|^2}{K^2}\bigg(\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}+\frac{K^2}{h^2}\|\nabla^2 G_\Lambda(\tau,\cdot)\|^2_{L^\infty}\bigg)\\
&\qquad + \frac{C|\zeta|^2}{h^2}\bigg(\frac{h^2}{K^2}+\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}\bigg)\\
&\leqslant \bigg(\frac{h^2}{K^2}+\|\nabla G_\Lambda(\tau,\cdot)\|^2_{L^\infty}+\frac{K^2}{h^2}\|\nabla^2 G_\Lambda(\tau,\cdot)\|^2_{L^\infty}\bigg)g_a(Z).
\end{aligned}$$

So it turns out that

$$\sup_{Y\in T^\star\mathbf{R}^d}\|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}\leqslant C\bigg(\frac{h}{K}+\|\nabla G_\Lambda(\tau,\cdot)\|_{L^\infty}+\frac{K}{h}\|\nabla^2 G_\Lambda(\tau,\cdot)\|_{L^\infty}\bigg).$$

By integration and by definition (10) of $\|\mathcal{G}\|_0$, we get that

$$\int_{I_\Lambda}\sup_{Y\in T^\star\mathbf{R}^d}\|DH_{F_\Lambda}(\tau,Y)\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}d\tau\leqslant C\bigg(\frac{|I_\Lambda|h}{K}+\|\mathcal{G}\|_0\Big(1+\frac{K}{h|I_\Lambda|}\Big)\bigg)\cdot$$

If ε is any positive real number, let us choose

$$K=\frac{4C}{\varepsilon}|I_\Lambda|h\quad\text{and } \|\mathcal{G}\|_0 \text{ such that}\quad \|\mathcal{G}\|_0\Big(1+\frac{4C}{\varepsilon}\Big)\leqslant\frac{\varepsilon}{4C}\cdot \tag{37}$$

Then, we have, for ε small enough,

$$\sup_{(\tau,Y)\in I_\Lambda\times T^\star\mathbf{R}^d}\|D\Psi_\Lambda(\tau,Y)-\mathrm{Id}\,\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}\leqslant\varepsilon.$$

Using Taylor formula, we write that

$$\begin{aligned}
g_a\big(\psi_\Lambda(\tau,Y)-Y-\psi_\Lambda(\tau,Z)+Z\big)^{1/2}&\\
&\leqslant \sup_{\substack{Y\in T^\star\mathbf{R}^d\\ \tau\in I_\Lambda}}\|D\Psi_\Lambda(\tau,Y)-\mathrm{Id}\,\|_{\mathcal{L}_{g_a}(T^\star\mathbf{R}^d)}g_a(Y-Z)^{1/2}\\
&\leqslant \varepsilon g_a(Y-Z)^{1/2}.
\end{aligned}$$

Using the inequality of the triangle and choosing $\varepsilon=1/2$, we get that, for any $\tau\in I_\Lambda$, any couple (Y,Z) of points of $T^\star\mathbf{R}^d$, we have

$$\frac12 g_a(Y-Z)^{1/2}\leqslant g_a\big(\Psi_\Lambda(\tau,Y)-\Psi_\Lambda(\tau,Z)\big)^{1/2}\leqslant\frac32 g_a(Y-Z)^{1/2}.$$

To prove the second point of this lemma let us write, with of course the obvious notation $Y_0=(y_0,\eta_0)$ and $Z_\tau=(z_\tau,\zeta_\tau)$, that

$$\begin{aligned}
\frac1K\big|\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\eta_0)\big|&\\
&\leqslant\frac1K\|\nabla_z\nabla_\eta\Phi_\Lambda\|_{L^\infty}|y-z_\tau|+\frac1K\|\nabla^2_\eta\Phi_\Lambda\|_{L^\infty}|\eta-\eta_0|.
\end{aligned}$$

Estimates (33) and (34) imply that

$$\begin{aligned}\frac{1}{K}\big|\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\eta_0)\big| &\leqslant \frac{C|y-z_\tau|}{K}+\frac{|I_\Lambda|}{K}|\eta-\eta_0|\\ &\leqslant Cr.\end{aligned}$$

Along the same lines, we have

$$\frac{1}{h}|\nabla_y\Phi_\Lambda(\tau,y,\eta)-\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)|\leqslant Cr.$$

So using the inequality of the triangle and the fact that (z,η) is in $B_{g_a}(Y_0,r)$ and (y,ζ) in $B_{g_a}(Z_\tau,r)$, we infer that

$$\begin{aligned}(38)\quad g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}&\\ \geqslant g_a\big(\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\eta_0)-y_0,\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)-\zeta_\tau\big)^{1/2}-4r.&\end{aligned}$$

Let us define $Z_0\stackrel{\text{def}}{=}\Psi_\Lambda^{-1}(\tau,Z_\tau)=(z_0,\zeta_0)$ and let us assume that

$$g_a(0,\zeta_0-\eta_0)\leqslant\beta^2 g_a(Z_0-Y_0)$$

for some β in the interval $]0,1[$ that will determine later on. Then, using estimates (32)–(34) as above, we obtain that

$$\begin{aligned}&g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\\ &\quad\geqslant g_a\big(\nabla_\eta\Phi_\Lambda(\tau,z_\tau,\zeta_0)-y_0,\nabla_y\Phi_\Lambda(\tau,z_\tau,\zeta_0)-\zeta_\tau\big)^{1/2}-Cr-C\beta g_a(Z_0-Y_0)^{1/2}.\end{aligned}$$

Using Lemma 4.1, we infer that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant g_a(z_0-y_0,0)^{1/2}-Cr-C\beta g_a(Z_0-Y_0)^{1/2}.$$

But, as $g_a(z_0-y_0,0)\geqslant(1-\beta^2)g_a(Z_0-Y_0)$, we get that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant\big((1-\beta^2)^{1/2}-C\beta\big)g_a(Z_0-Y_0)^{1/2}-Cr.$$

Let us choose for instance β so small that

$$(1-\beta^2)^{1/2}-C\beta\geqslant\frac{1}{2}\cdot$$

Then, if $g_a(Z_0-Y_0)^{1/2}\geqslant C_0r$ with C_0 large enough, we have that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant\frac{1}{4}g_a(Z_0-Y_0)^{1/2}.$$

Now let us assume that

$$g_a(0,\zeta_0-\eta_0)\geqslant\beta^2 g_a(Z_0-Y_0).$$

Going back to Inequality (38) and using Lemma 4.1, we claim that

$$\begin{aligned}&g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\\ &\qquad\geqslant g_a\big(0,\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)-\nabla_y\Phi_\Lambda(\tau,z_\tau,\zeta_0)\big)^{1/2}-Cr.\end{aligned}$$

Using estimate (32) and choosing ε small enough in it, we have that

$$\begin{aligned}\big|\nabla_y\Phi_\Lambda(\tau,z_\tau,\eta_0)-\nabla_y\Phi_\Lambda(\tau,z_\tau,\zeta_0)\big| &\geqslant \big(1-\|\nabla_y\nabla_\eta\Phi_\Lambda-\mathrm{Id}\,\|_{L^\infty(I_\Lambda\times T^\star\mathbf{R}^d)}\big)|\zeta_0-\eta_0|\\ &\geqslant \frac{1}{2}|\zeta_0-\eta_0|.\end{aligned}$$

So by definition of the metric g_a it turns out that

$$g_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)^{1/2}\geqslant\frac{\beta}{2}g_a(Z_0-Y_0)^{1/2}-Cr.$$

This concludes the proof of the lemma if C_0 is large enough. To be able to handle interactions between pair of points of type (x,ξ)–$(x,-\xi)$, we shall need to control the time variation of the Hamiltonian flow. This will be crucial in section 7. The following lemma determines the subintervals of I_Λ such that the flow does vary very few.

Lemma 4.3. — *Let J be any subinterval of I_Λ. Then, we have*

$$\sup_{\substack{(\tau,\tau')\in J^2\\ Y\in T^\star\mathbf{R}^d}} g_a(\Psi_\Lambda(\tau,Y)-\Psi_\Lambda(\tau',Y))^{1/2}\leqslant C\bigg(\frac{|J|}{h|I_\Lambda|}+\frac{1}{h}\|\nabla G_\Lambda\|_{L^1_J(L^\infty)}\bigg).$$

To prove this, let us observe that by definition of the Hamiltonian flow, we have

$$\Psi_\Lambda(\tau',Y)-\Psi_\Lambda(\tau,Y)=-\int_\tau^{\tau'}\mathcal{H}_{F_\Lambda}(\tau'',\Psi_\Lambda(\tau'',Y))d\tau''.$$

So we immediately get that, for any $(\tau,\tau')\in J$,

$$g_a(\Psi_\Lambda(\tau',Y)-\Psi_\Lambda(\tau,Y))^{1/2}\leqslant\int_J\sup_{Y\in T^\star\mathbf{R}^d}g_a(\mathcal{H}_{F_\Lambda}(\tau'',Y))^{1/2}d\tau''.$$

But by definition of the Hamiltonian vector field and the metric g_a, we have

$$g_a(\mathcal{H}_{F_\Lambda}(\tau,Y))=\frac{1}{K^2}|\partial\eta F_\Lambda(\tau,Y)|^2+\frac{1}{h^2}|\partial_y F_\Lambda(\tau,Y)|^2.$$

By definition of F_Λ and of K, we infer that

$$g_a(\mathcal{H}_{F_\Lambda}(\tau,Y))^{1/2}\leqslant C\bigg(\frac{1}{h|I_\Lambda|}+\frac{1}{h}\|\nabla G_\Lambda(\tau,\cdot)\|_{L^\infty}\bigg).$$

So an immediat integration concludes the proof of the lemma.

4.2. The approximation of the solution. — Before stating the theorem, let us recall the concept of symbols we introduced in [**4**].

Definition 4.1. — Let us denote by S^{-N} the set of families of functions $\sigma=(\sigma_\Lambda)_{\Lambda\geqslant\Lambda_0}$ such that

– the function σ_Λ is smooth on $I_\Lambda\times\mathbf{R}^d\times\mathcal{C}$ in $\mathbf{C}$;

– for any integer k, the quantity defined by

$$q_k^{(N)}(\sigma) \stackrel{\text{def}}{=} \sup_{\substack{j+j'\leqslant k\\ \Lambda\geqslant\Lambda_0}} \Lambda^{N+j}\|\partial_\eta^{j'}\nabla^j\sigma_\Lambda\|_{L^\infty(I_\Lambda\times\mathbf{R}^d\times\mathcal{C})}$$

is finite.

– An element of S^{-N} is a symbol of order $-N$.

Now we are able to state the approximation theorem.

Theorem 4.1. — *Let us assume that $\|\mathcal{G}\|_0$ is small enough. Then, for any integer N, two symbols $\sigma^\pm$ (with value in $\mathbf{R}^2$) belonging to S^0 and a constant C exists such that the following properties are satisfied.*

Let $(v_\Lambda)_{\Lambda\geqslant\Lambda_0}$ be the family of solutions of (E_Λ) with $f=0$ and with initial data $\gamma=(\gamma^0,\gamma^1)$; if we state

$$\mathcal{I}_\Lambda^\pm(\gamma) \stackrel{def}{=} \int_{\mathbf{R}^d} e^{i\Phi_\Lambda^\pm(\tau,y,\eta)}\sigma_\Lambda^\pm(\tau,y,\eta)\cdot\widehat{\gamma}_\pm(\eta)d\eta, \tag{39}$$

then, if

$$|I_\Lambda| \leqslant \Lambda^{2-\varepsilon} \tag{40}$$

we have

$$\|\nabla(v_\Lambda - \mathcal{I}_\Lambda^+(\gamma) - \mathcal{I}_\Lambda^-(\gamma))\|_{L^\infty_{I_\Lambda}(L^2)} \leqslant C\Lambda^{-N}\|\gamma\|_{L^2}. \tag{41}$$

The proof of this is done in [**4**] and [**5**].

4.3. The precised Strichartz estimate. — The theorem is the following.

Theorem 4.2. — *Let $\mathcal{C}$ be a ring of $\mathbf{R}^d$ and let us assume that $\|\mathcal{G}\|_0$ is small enough. For any positive real number ε, a constant C_ε exists which satisfies the following properties. Let f be a function in $L^1_{I_\Lambda}(L^2)$ and γ a function of L^2; let us assume that those two functions have their support included in $\mathcal{C}$ and of diameter less than h. Let us assume*

$$|I_\Lambda| \leqslant \Lambda^{2-\varepsilon}.$$

Then if v_Λ is the solution of

$$(E_\Lambda)\begin{cases} P_\Lambda v_\Lambda &= f\\ \partial v_{\Lambda|\tau=0} &= \gamma.\end{cases}$$

we have

$$\|\nabla v_\Lambda\|_{L^2_{I_\Lambda}(L^\infty)} \leqslant Ch^{(d-2)/2}\big(\log(e+|I_\Lambda|)\big)^{1/2}\big(\|\gamma\|_{L^2}+\|f\|_{L^1_{I_\Lambda}(L^2)}\big).$$

To prove this theorem, we shall use the classical $TT^\star$ method. Following [**5**] and using the fact that the support of the Fourier transform of $\widehat{\gamma}$ is included in the ball of center ξ_0 and radius h denoted by $B(\xi_0,h)$, let us write that, for any $f\in\mathcal{D}(I_\Lambda\times\mathbf{R}^d)$, we have

$$\begin{aligned}\langle\mathcal{I}_\Lambda(\gamma),f\rangle &= \langle\widehat{\gamma},A_\Lambda f\rangle \quad\text{with}\\ A_\Lambda f &\stackrel{\text{def}}{=} \int e^{i\Phi_\Lambda(\tau,x,\xi)}\sigma_\Lambda(\tau,x,\xi)\chi\Big(\frac{\xi-\xi_0}{h}\Big)f(\tau,x)d\tau dx.\end{aligned}$$

where χ is a function of $\mathcal{D}(\mathbf{R}^d)$.

$$\langle\mathcal{I}_\Lambda(\gamma),f\rangle\leqslant\|\gamma\|_{L^2}\|A_\Lambda f\|_{L^2(B(\xi_0,h))}.$$

By definition of A_Λ we have

$$|A_\Lambda f(\xi)|^2=\int e^{i(\Phi_\Lambda(\tau,x,\xi)-\Phi_\Lambda(\tau',y,\xi))}\widetilde{\sigma}_{\Lambda,h}(\tau,\tau',x,y,\xi)f(\tau,x)f(\tau',y)d\tau d\tau' dxdy$$

where

$$\widetilde{\sigma}_{\Lambda,h}(\tau,\tau',x,y,\xi)\stackrel{\text{def}}{=}\sigma_\Lambda(\tau,x,\xi)\overline{\sigma}_\Lambda(\tau',y,\xi)\chi\Big(\frac{\xi-\xi_0}{h}\Big)\overline{\chi}\Big(\frac{\xi-\xi_0}{h}\Big).$$

First, let us decompose A_Λ as follows

$$\begin{aligned}|A_\Lambda f(\xi)|^2 &= B_\Lambda f(\xi)+C_\Lambda f(\xi)\quad\text{with}\\ B_\Lambda f(\xi) &\stackrel{\text{def}}{=}\int_{|\tau-\tau'|h^2\geqslant 1} e^{i(\Phi_\Lambda(\tau,x,\xi)-\Phi_\Lambda(\tau',y,\xi))}\widetilde{\sigma}_\Lambda(\tau,\tau',x,y,\xi)f(\tau,x)f(\tau',y)d\tau d\tau' dxdy.\end{aligned}$$

The estimate about $C_\Lambda f$ is very easy. As the support of $C_\Lambda f$ is included in the ball $B(\xi_0,h)$ we have

$$\begin{aligned}\int|C_\Lambda f(\xi)|d\xi &\leqslant Ch^d\sup_\xi|C_\Lambda f(\xi)|\\ &\leqslant Ch^d\int_{|\tau-\tau'|h^2\leqslant 1}\|f(\tau,\cdot)\|_{L^1(\mathbf{R}^d)}\|f(\tau',\cdot)\|_{L^1(\mathbf{R}^d)}d\tau d\tau'\\ &\leqslant Ch^{d-2}\int\frac{h^2}{1+(\tau-\tau')^2h^4}\|f(\tau,\cdot)\|_{L^1(\mathbf{R}^d)}\|f(\tau',\cdot)\|_{L^1(\mathbf{R}^d)}d\tau d\tau'\\ &\leqslant Ch^{d-2}\|f\|^2_{L^2_{I_\Lambda}(L^1(\mathbf{R}^d))}.\end{aligned}$$

Now we shall assume that $|\tau-\tau'|h^2\geqslant 1$. Let us follow [**5**]. Using Taylor formula, we write that

$$\Phi_\Lambda(\tau,x,\xi)-\Phi_\Lambda(\tau',y,\xi)=(x-y)\Theta_\Lambda(\tau,\tau',x,y,\xi)+(\tau-\tau')\widetilde{\Psi}_\Lambda(\tau,\tau',x,y,\xi)\ \text{with}$$

$$\widetilde{\Psi}_\Lambda(\tau,\tau',x,y,\xi)=\int_0^1\frac{\partial\Phi_\Lambda}{\partial\tau}(\tau'+t(\tau-\tau'),y+t(x-y),\xi)dt \tag{42}$$

$$\Theta_\Lambda(\tau,\tau',x,y,\xi)=\int_0^1\frac{\partial\Phi_\Lambda}{\partial x}(\tau'+t(\tau-\tau'),y+t(x-y),\xi)dt.$$

Stating the change of variables

$$\eta=\Theta_\Lambda(\tau,\tau',x,y,\xi),$$

we get, denoting by F_Λ the inverse of the above diffeomorphism,

$$\widetilde{K}_\Lambda(\tau,\tau',x,y) = \int e^{i(x-y)\eta} e^{i(\tau-\tau')\Psi_\Lambda(\tau,\tau',x,y,\eta)} \sigma_{\Lambda,h}(\tau,\tau',x,y,\eta)d\eta \quad \text{with}$$

$$\sigma_{\Lambda,h}(\tau,\tau',x,y,\eta) \stackrel{\text{def}}{=} \widetilde{\sigma}_{\Lambda,h}(\tau,\tau',x,y,F_\Lambda(\tau,\tau',x,y,\eta))J_\Lambda(\tau,\tau',x,y,F_\Lambda(\tau,\tau',x,y,\eta)))$$

and

$$\Psi_\Lambda(\tau,\tau',x,y,\eta) = \widetilde{\Psi}_\Lambda(\tau,\tau',x,y,F_\Lambda(\tau,\tau',x,y,\eta)).$$

where J_Λ denotes the Jacobian of this change of variables.

Now let us change the variable

$$\eta = \eta_0 + h\zeta \quad \text{with} \quad \eta_0 \stackrel{\text{def}}{=} \Theta_\Lambda(\tau,\tau',x,y,\xi_0).$$

Then we have

$$\begin{aligned} \widetilde{K}_\Lambda(\tau,\tau',x,y) &= h^d e^{ih(x-y)\eta_0} K_\Lambda(\tau,\tau',x,y) \quad \text{with} \\ K_\Lambda(\tau,\tau',x,y) &\stackrel{\text{def}}{=} \int e^{ih(x-y)\zeta} e^{i(\tau-\tau')\Psi_\Lambda(\tau,\tau',x,y,\eta_0+h\zeta)} \sigma_{\Lambda,h}(\tau,\tau',x,y,\zeta)d\zeta \end{aligned}$$

where all derivatives of $\sigma_{\Lambda,h}$ are bounded with respect to ζ. Let us study of the form of the function Ψ_Λ. Using Taylor formula, we can write (dropping the fact that Ψ_Λ depends on τ, τ', x and y)

$$\psi_\Lambda(\eta_0+h\zeta) = \psi_\Lambda(\eta_0) + h(\nabla_\eta\Psi_\Lambda(\eta_0)|\zeta) + h^2\int_0^1 D^2\Psi_\Lambda(\eta_0+sh\zeta)ds(\zeta,\zeta).$$

Using the inequalities (32) and (33) it turns out that, for any s, h and ζ, we have

$$\forall\, \theta\in\mathbf{R}^d\,,\ \left|D^2\Psi_\Lambda(\eta_0+sh\zeta)(\theta,\theta) - |p_{(\eta_0+sh\zeta)^\perp}\theta|^2\right| \leqslant \varepsilon|\theta|^2.$$

As ζ belongs to the unit ball of $\mathbf{R}^d$, and h can be choosen small enough, we have that the quadratic form

$$Q(h\zeta) \stackrel{\text{def}}{=} \int_0^1 D^2\Psi_\Lambda(\eta_0+sh\zeta)ds$$

is a non negative quadratic form of rank greater or equal to $d-1$. Then stating $x-y = (\tau-\tau')z$, we can write the phase

$$i(\tau-\tau')h(z+\nabla\Psi_\Lambda(\eta_0)|\zeta) + i(\tau-\tau')h^2 Q(h\zeta)(\zeta,\zeta).$$

Then we can choose coordinates such that the phase function is

$$i(\tau-\tau')h(z+\nabla\Psi_\Lambda(\eta_0))\zeta_1 + i(\tau-\tau')h^2\sum_{j=1}^d a_j(h\zeta)\zeta_j^2$$

where for any $j\geqslant 2$, the functions a_j are smooth with bounded derivatives and $1/2\leqslant a_j\leqslant 2$ except possibily one of them. Then following the basic proof of the stationnary phase theorem we get that

$$\begin{aligned}|\widetilde{K}_\Lambda(\tau,\tau',x,y)| &\leqslant C\frac{h^d}{(|\tau-\tau'|h^2)^{(d-2)/2}}\\ &\leqslant C\frac{h^2}{|\tau-\tau'|^{(d-2)/2}}\,\cdot\end{aligned}$$

As $d \geqslant 4$, we have that

$$\begin{aligned}\int |B_\Lambda f(\xi)|d\xi \leqslant C\int_{|\tau-\tau'|h^2\geqslant 1}\frac{h^2}{|\tau-\tau'|^{(d-2)/2}}\|f(\tau,\cdot)\|_{L^1(\mathbf{R}^d)}\|f(\tau',\cdot)\|_{L^1(\mathbf{R}^d)}d\tau d\tau'\\ \leqslant Ch^{d-2}\log(e+|I_\Lambda|)\|f\|^2_{L^2_{I_\Lambda}(L^1(\mathbf{R}^d))}.\end{aligned}$$

Thus Theorem 4.2 holds.

5. The concept of microlocalized functions

In this section, we present the concept of microlocalized functions introduced by J.-M. Bony in [**7**]. This concept is related to the Weyl-Hörmander calculus (see [**11**], [**9**]). But the problem we investigate here allows us to use a simplified version of it.

5.1. A simplified version of pseudo-differential calculus. — In this paragraph, we shall consider a positive quadratic form g on $T^\star\mathbf{R}^d$ such that the symplectic conjugate quadratic form g^σ defined by

$$g^\sigma(T) \stackrel{\text{def}}{=} \sup_{W\neq 0}\frac{[T,W]^2}{g(W)}$$

satisfies the uncertainty principle

$$g^\sigma \geqslant g.$$

Here $[\cdot,\cdot]$ denotes the basic symplectic form on $T^\star\mathbf{R}^d$ defined by

$$[(x,\xi),(y,\eta)] = \sum_{j=1}^d \big(\xi^j y_j - \eta^j x_j\big).$$

In all this paper, we are going to be in the case when

$$g(dx,d\xi) = \frac{dx^2}{K^2} + \frac{d\xi^2}{h^2}\,\cdot$$

In this case, we have

$$g^\sigma = \lambda^2 g \quad \text{with} \quad \lambda = Kh.$$

The uncertainty principle means that $\lambda \geqslant 1$.

We shall mesure the length of derivatives of smooth functions on $T^\star \mathbf{R}^d$ with respect to this metric g. More precisely, let us define, for any smooth function φ on $T^\star \mathbf{R}^d$,

$$\|\varphi\|_{j,g} \stackrel{\text{def}}{=} \sup_{\substack{k\leqslant j\\ X\in T^\star \mathbf{R}^d}} \sup_{\substack{(T_\ell)_{1\leqslant \ell\leqslant k}\\ g(T_\ell)\leqslant 1}} |D^k\varphi(X)(T_1,\cdot,T_k)|.$$

Now, to a function φ in $\mathcal{D}(T^\star \mathbf{R}^d)$, we associate the operator φ^D defined by

$$(\varphi^D u)(x) = (2\pi)^{-d}\int_{T^\star \mathbf{R}^d} e^{i(x-y|\xi)}\varphi(y,\xi)u(y)dyd\xi.$$

The choice of this quantization process makes the computation of section 6 simpler. Let us remark that if the function $\varphi(x,\xi)$ is equal to $\varphi_1(x)\varphi_2(\xi)$, then

$$\varphi^D u = \mathcal{F}^{-1}\big(\varphi_2(\mathcal{F}(\varphi_1 u)\big).$$

Moreover we have

$$\mathcal{F}(\varphi^D u)(\xi) = \int_{\mathbf{R}^d} e^{-i(y|\xi)}\varphi(y,\xi)u(y)dy.$$

Later on in this paper we shall need to decompose L^2 functions whose Fourier transform is supported in the ring $\mathcal{C}$ using these operators φ^D. Let us state the following lemma which will be useful.

Lemma 5.1. — *A sequence $(X_\nu)_{\nu\in\mathcal{Z}}$ exists such that two sequencies $(\varphi_\nu)_{\nu\in\mathcal{Z}}$ and $(\psi_\nu)_{\nu\in\mathcal{Z}}$ exist which satisfy the following properties.*

– the support of φ_ν is included in a ball $B_\nu \stackrel{def}{=} B_g(X_\nu, r)$,

– A sequence $(C_j)_{j\in\mathbf{N}}$ exists (which depends only on r and not in the parameters K and h) such that

$$\forall\, \nu\in\mathcal{Z}\,,\ \|\varphi_\nu\|_{j,g}\leqslant C_j,$$

– the functions ψ_ν are not supported in B_ν but confined, which means that a sequence $(C_N)_{N\in\mathbf{N}}$ exists such that

$$\forall\,\nu\in\mathcal{Z},\ \|\psi_\nu\|_{N,g,X} \stackrel{def}{=} \sup_{\substack{k\leqslant N\\ X\in T^\star \mathbf{R}^d}} (1+\lambda^2 g(X-B_\nu))^N \sup_{\substack{(T_\ell)_{1\leqslant \ell\leqslant k}\\ g(T_\ell)\leqslant 1}} |D^k\psi_\nu(X)(T_1,\cdot,T_k)|\leqslant C_N,$$

– For any function u of L^2 whose Fourier transform has a support included in $\mathcal{C}$, we have

$$\sum_{\nu\in\mathcal{Z}}\varphi_\nu^D\psi_\nu^D u = \sum_{\nu\in\mathcal{Z}}\varphi_\nu^D u = u.$$

Such partitions of unity are "compatible" with L^2 in the following sense.

Lemma 5.2. — *A constant C exists such that*

$$C^{-1}\|u\|_{L^2}^2 \leqslant \sum_\nu \|\varphi_\nu^D u\|_{L^2}^2 \leqslant\ C\|u\|_{L^2}^2 \quad \textit{and} \quad \sum_\nu \|\psi_\nu^D u\|_{L^2}^2 \leqslant C\|u\|_{L^2}^2.$$

Those two lemmas are proved in [**8**].

Lemma 5.3. — *For any N, a constant C_N and an integer k_N exist which satisfy the following properties. Let ϕ and $\widetilde{\phi}$ be two functions on $\mathcal{S}(T^\star \mathbf{R}^d)$ and Y and $\widetilde{Y}$ two points of $T^\star \mathbf{R}^d$. Then a function θ exists in $\mathcal{S}(T^\star \mathbf{R}^d)$ such that*

$$\theta^D = \phi^D \widetilde{\phi}^D \quad \text{and} \quad \|\theta\|_{N,g,Y} + \|\theta\|_{N,g,\widetilde{Y}} \leqslant C_N \|\phi\|_{N,g,Y} \|\widetilde{\phi}\|_{N,g,\widetilde{Y}}$$

This lemma is proved in [**9**].

Of course, the operators φ^D does not completely fit with any L^p space when $p \neq 2$. Nevertheless we have the following lemma.

Lemma 5.4. — *Let φ be a function of $\mathcal{S}(T^\star \mathbf{R}^d)$. The operator φ^D maps L^p into L^p for any p in $[1,\infty]$. More precisely, a constant C and an integer N exists such that*

$$\forall\, X_0 \in T^\star \mathbf{R}^d\,,\ \|\varphi^D a\|_{L^p} \leqslant C \|\varphi\|_{N,g,X_0} \|a\|_{L^p}.$$

This lemma can be seen as a corollary of Lemma 4.3 of [**10**]. For the convenience of the reader, we give here a self contained proof based of course on integrations by part. We have

$$\begin{aligned}
\varphi^D a(x) &= \int_{T^\star \mathbf{R}^d} e^{i(x-y|\xi)} \varphi(y,\xi) a(y)\, dy d\xi \\
&= \int_{T^\star \mathbf{R}^d} e^{i(x-y|\xi)} \big(1 + h^2|x-y|^2\big)^{-d} (\mathrm{Id} - h^2\Delta_\xi)^d \varphi(y,\xi) a(y)\, dy d\xi.
\end{aligned}$$

So it turns out that

$$\begin{aligned}
|\varphi^D a(x)| \leqslant C \|\varphi\|_{4d,g,X_0} \bigg(& \int_{\{(y,\xi)\,/\,\frac{|\xi-\xi_0|}{h} \leqslant r\}} \big(1 + h^2|x-y|^2\big)^{-d} |a(y)|\, dy d\xi \\
& + \int_{T^\star \mathbf{R}^d} \big(1 + h^2|x-y|^2\big)^{-d} \big(1 + K^2|\xi-\xi_0|^2\big)^{-d} |a(y)|\, dy d\xi \bigg).
\end{aligned}$$

So the lemma is proved, as thanks to the uncertainty principle, Kh is greater or equal to 1.

Remark. — The points X_ν are exactly the points of the lattice

$$\mathcal{Z} \stackrel{\text{def}}{=} (c_d r K\, \mathbf{Z}^d) \times (c_d r h\, \mathbf{Z}^d \cap \mathcal{C}). \tag{43}$$

Now we can defined the concept of microlocalized function.

Definition 5.1. — Let X_0 be a point of $T^\star \mathbf{R}^d$ and (C_0, r) a couple of positive real numbers. A function u in $L^2(\mathbf{R}^d)$ is said to be (C_0,r)-microlocalized in X_0 if a sequence of integers $(k_N)_{N\in\mathbf{N}}$ exists such that, for any integer N, the quantities

$$\mathcal{M}^{C_0,r}_{X_0,N}(u) \stackrel{\text{def}}{=} \sup_{g(X-X_0)^{1/2} \geqslant C_0 r} \lambda^{2N} g(X-X_0)^N \sup_{\substack{\varphi \in \mathcal{D}(B_g(X,r)) \\ \|\varphi\|_{k_N,g} \leqslant 1}} \|\varphi^D u\|_{L^2}$$

are finite. Here, $B_g(X,r)$ denotes as in all that follows the set of points of $T^\star \mathbf{R}^d$ such that $g(Y-X)^{1/2} \leqslant r$.

A basic example of microlocalized functions is given by the following proposition.

Proposition 5.1. — *A sequence of integers $(k_N)_{N\in\mathbf{N}}$ and a sequence of positive real numbers $(C_N)_{N\in\mathbf{N}}$ exist such that the following properties are satisfied. Let X_0 be a point of $T^\star\mathbf{R}^d$, φ_0 a function in $\mathcal{D}(B_g(X_0,r))$ and u a function of $L^2(\mathbf{R}^d)$. Then the function $\varphi_0^D u$ is $(3,r)$-microlocalized in X_0 and, for any N, we have*

$$\mathcal{M}^{3,r}_{X_0,N,g}(\varphi_0^D u)\leqslant C_N\|\varphi_0\|_{k_N,g}\|u\|_{L^2}.$$

This proposition can be seen as an immediat corollary of the general theory of Weyl-Hörmander calculus, for instance as a corollary of Theorem 2.2.1. of [**9**]. But as a warm up for the next section, we are going to give a proof of it in our particular situation.

By definition of φ_0^D, we have, for any function φ belonging to $\mathcal{D}(B_g(X,r))$,

$$\mathcal{F}\big(\varphi^D\varphi_0^D u\big)(\xi)=(2\pi)^{-d}\int_{\mathbf{R}^{3d}}e^{-i(y|\xi-\eta)-i(z|\eta)}\varphi(y,\xi)\varphi_0(z,\eta)u(z)dzdyd\eta.$$

Let us do some integrations by part with respect to some derivatives of g-length less than 1. It is obvious that

$$(K^2\Delta_y+h^2\Delta_\eta)\big(e^{-i(y|\xi-\eta)-i(z|\eta)}\big)=-\lambda^2 g(y-z,\xi-\eta)e^{-i(y|\xi-\eta)-i(z|\eta)}.$$

Using the fact that derivatives in (y,η) of g-length less than 1 of $g(y-z,\xi-\eta)$ is less than $g(y-z,\xi-\eta)^{1/2}$ it turns out that

$$\mathcal{F}\big(\varphi^D\varphi_0^D u\big)(\xi)=(2\pi)^{-d}\int_{\mathbf{R}^{3d}}e^{-i(y|\xi-\eta)-i(z|\eta)}\mathcal{K}(y,z,\xi,\eta)u(z)dzdyd\eta\quad\text{with}$$

$$|\mathcal{K}(y,z,\xi,\eta)|\leqslant C_{r,N}\big(1+\lambda^2 g(X-X_0)\big)^{-N}\|\varphi\|_{2N+N_0}\|\varphi_0\|_{2N+N_0}\times\frac{1}{\big(r^2+h^2|y-z|^2+K^2|\xi-\eta|^2\big)^{N_0}}\cdot$$

Using the fact that $\lambda=Kh\geqslant 1$ and convolution inequalities, we get that

$$\big\|\mathcal{F}\big(\varphi^D\varphi_0^D u\big)\|_{L^2}\leqslant C_{r,N}\big(1+\lambda^2 g(X-X_0)\big)^{-N}\|\varphi\|_{2N+N_0}\|\varphi_0\|_{2N+N_0}\|u\|_{L^2}.$$

This concludes the proof of the proposition.

In all that follows, the concept of uniformly microlocalized families of functions will be a basic tool.

Definition 5.2. — Let $g\overset{\text{def}}{=}(g_a)_{a\in A}$ be a family of metrics, $\mathcal{X}\overset{\text{def}}{=}(X_a)_{a\in A}$ a family of points of $T^\star\mathbf{R}^d$ and (C_0,r) a pair of positive real numbers. A family of functions $U\overset{\text{def}}{=}(u_a)_{a\in A}$ in $L^2(\mathbf{R}^d)$ is said to be uniformly (C_0,r)-microlocalized in $\mathcal{X}$ with respect to g if, for any integer N,

$$\mathcal{M}^{C_0,r}_{N,X,g}(U)\overset{\text{def}}{=}\sup_{a\in A}\mathcal{M}^{C_0,r}_{X_a,N,g_a}(u_a)<\infty.$$

5.2. A lemma about the product. — We want here to study the interaction between two (typical examples of) microlocalized functions. More precisely we are going to prove the following lemma.

Lemma 5.5. — *A constant C_0 exists such that, for any integer N, a constant C_N and an integer k_N exist which satisfy the following properties.*

If u_1 and u_2 are two L^2 functions on $\mathbf{R}^d$, if χ is a function of $\mathcal{D}(\mathbf{R}^d)$ supported in an euclidian ball of radius r, if φ_1 and φ_2 are two functions of $\mathcal{D}(T^\star\mathbf{R}^d)$ respectively supported in $B_g(Y_1,r)$ and in $B_g(Y_2,r)$, then if

$$g(\check{Y}_1-Y_2)^{1/2}\geqslant C_0r,$$

for any N, we have

$$\begin{aligned}\big\|\chi(h^{-1}D)(\varphi_1^Du_1\varphi_2^Du_2)\big\|_{L^1}\\ \leqslant C_N\|\varphi_1\|_{k_N,g}\|\varphi_2\|_{k_N,g}\big(1+\lambda^2g(\check{Y}_1-Y_2)\big)^{-N}\|u_1\|_{L^2}\|u_2\|_{L^2}\end{aligned}$$

where $\check{Y}\overset{def}{=}(y,-\eta)$ if $Y=(y,\eta)$.

Let us suppose first that

$$\frac{|\eta_1+\eta_2|}{h}\geqslant\frac{1}{2}g(\check{Y}_1-Y_2)^{1/2}.$$

By definition of the operator φ_j^D, the support of the Fourier transform of $\varphi_j^Du_j$ is included in the (euclidian) ball of center η_j and radius rh. So, it is clear that, if C_0 is large enough,

$$\operatorname{Supp}\mathcal{F}\big(\varphi_1^Du_1\varphi_2^Du_2\big)\subset\{\eta\in\mathbf{R}^d\ /\ |\eta|\geqslant 2rh\}.$$

So it turns out that

$$\chi(h^{-1}D)(\varphi_1^Du_1\varphi_2^Du_2)=0.$$

Now we have to study the case when

$$\frac{|y_1-y_2|}{K}\geqslant\frac{1}{2}g(\check{Y}_1-Y_2)^{1/2}.$$

By definition of the operator φ_j^D, we have

$$\begin{aligned}(\varphi_1^Du_1\varphi_2^Du_2)(x)=(2\pi)^{-2d}\int_{B_g(Y_1,r)\times B_g(Y_2,r)}e^{i(x-y|\eta)+i(x-z|\zeta)}\\ \times\varphi_1(Y)\varphi_2(Z)u_1(y)u_2(z)dYdZ.\end{aligned}\tag{44}$$

The fact that

$$(\operatorname{Id}-h^2\Delta_\eta)e^{i(x-y|\eta)}=(1+h^2|x-y|^2)e^{i(x-y|\eta)}$$

So by repeated integration by parts, we get

$$\big|(\varphi_1^D u_1 \varphi_2^D u_2)(x)\big| \leqslant \int_{B_g(Y_1,r)\times B_g(Y_2,r)} (1+h^2|x-y|^2)^{-N}(1+h^2|x-z|^2)^{-N}$$
$$|(\mathrm{Id}-h^2\Delta_\eta)^N\varphi_1(Y)|\,|(\mathrm{Id}-h^2\Delta_\zeta)^N\varphi_2(Z)|\,|u_1(y)u_2(z)|dYdZ.$$

The inequality of the triangle implies that

$$|x-y_1|+|x-y_2|\geqslant |y_1-y_2| \quad\text{and}\quad |x-y|+|x-z|\geqslant |y_1-y_2|-2rK.$$

So, if C_0 is greater than 12, we have that

$$|x-y|+|x-z|\geqslant \frac{1}{2}|y_1-y_2|+rK.$$

So we infer that, for any N,

$$\big|(\varphi_1^D u_1 \varphi_2^D u_2)(x)\big| \leqslant C_N(1+h^2|y_1-y_2|^2)^{-N}\|\varphi_1\|_{2N+2d}\|\varphi_2\|_{2N+2d}\,\mathcal{I}(x) \quad \text{with}$$
$$\mathcal{I}(x) \stackrel{\text{def}}{=} C_d h^{2d}\int (1+h^2|x-y|^2)^{-d}(1+h^2|x-z|^2)^{-d}|u_1(y)u_2(z)|dydz.$$

By Cauchy-Schwarz inequality, we have that

$$|\mathcal{I}(x)|^2 \leqslant C_d h^{4d}\prod_{j=1}^{2}\int (1+h^2|x-y|^2)^{-d}(1+h^2|x-z|^2)^{-d}|u_j(z)|^2dydz$$

So using again Cauchy-Schwarz inequality, we get that

$$\|\mathcal{I}\|_{L^1(\mathbf{R}^d)} \leqslant C_d\|u_1\|_{L^2}\|u_2\|_{L^2}.$$

So the lemma is proved.

6. The propagation theorem

One of the important point of this study is that the (approximate) flow of the operator P_Λ preserves the microlocalization of functions. The aim of this section is to state and prove a theorem of propagation of microlocalization.

Theorem 6.1. — *A constant C_0 exists which satisfies the following property.*

Let us consider a point $Y_0=(y_0,\eta_0)$ of $T^\star\mathbf{R}^d$ such that η_0 belongs to $\mathcal{C}$, a smooth function ϕ supported in $B_{g_a}(Y_0,r)$ and a function γ of L^2. Then $\mathcal{I}_\Lambda^\pm(\phi^D\gamma)(\tau,\cdot)$ is (C_0,r)-microlocalized near $\Psi_\Lambda^\pm(\tau,Y_0)$. Moreover, for any integer N, a constant C and an integer k exist (which depend only on N) such that

$$\mathcal{M}^{C_0,r}_{\Psi_\Lambda^\pm(\tau,Y_0),N,g_a}\big(\mathcal{I}_\Lambda^\pm(\phi^D\gamma)(\tau,\cdot)\big)\leqslant C\,\|\phi\|_{k,g_a}\|\gamma\|_{L^2}.$$

In the following proof of this theorem, we shall drop the exponent $\pm$ for sake of simplicity of the notations. By definition of the microlocalized functions, we have to estimate the following quantity

$$\mathcal{J}\stackrel{\text{def}}{=}\mathcal{F}\big(\varphi_{Z_\tau}^D\mathcal{I}_\Lambda(\phi^D\gamma)(\tau,\cdot)\big)$$

where Z_τ is a point of $T^\star \mathbf{R}^d$ such that $g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} \geqslant C_0 r$. By definition, we have

$$\begin{aligned}\mathcal{J}(\zeta) &= \int_{\mathbf{R}^d} \mathcal{K}(\zeta, z)\gamma(z)dz \quad \text{with}\\ \mathcal{K}(\zeta, z) &\stackrel{\text{def}}{=} \int_{\mathbf{R}^{2d}} e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)dyd\eta.\end{aligned}$$

The proof consists in integrations by parts in the above integral. Let us define the vector Θ by

$$\Theta = (\Theta^y, \Theta^\eta) \stackrel{\text{def}}{=} \Big(|I_\Lambda|^{-1/2}\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta) - z\big), |I_\Lambda|^{1/2}\big(\nabla_y\Phi_\Lambda(\tau,y,\eta) - \zeta\big)\Big)$$

and the vector field $\mathcal{L}$ by

$$\mathcal{L}f \stackrel{\text{def}}{=} \frac{1}{1+|\Theta|^2}\bigg(f - i|I_\Lambda|^{-1/2}\Theta_y\partial_\eta f - i|I_\Lambda|^{1/2}\Theta_\eta\partial_y f\bigg).$$

It is obvious that

$$\mathcal{L}\big(e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}\big) = e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}.$$

So as usual, we have, for any integer N,

$$\mathcal{K}(\zeta, z) = \int_{\mathbf{R}^{2d}} e^{-i(y|\zeta)+i\Phi_\Lambda(\tau,y,\eta)-i(z|\eta)}({}^t\mathcal{L})^N\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)dyd\eta.$$

Let us state the following technical lemma which will allow us to estimate the repeated action of the differential operator ${}^t\mathcal{L}$.

Lemma 6.1. — *For any integer N, a family of functions $(L_{\alpha,N})_{|\alpha|\leqslant N}$ exists such that $L_{\alpha,N}(Y,\mathcal{Y})$ is a smooth function from $T^\star \mathbf{R}^d \times (T^\star \mathbf{R}^d)^{M_N}$ and such that*

$$\|\partial_Y^\beta L_{\alpha,N}(Y,\cdot)\|_{L^\infty((T^\star \mathbf{R}^d)^{M_N})} \leqslant C_{N,|\beta|}(1+|Y|^2)^{-(N+|\beta|)/2}. \tag{45}$$

Moreover, they satisty

$$({}^t\mathcal{L})^N f = \sum_{|\alpha|\leqslant N} L_{\alpha,N}(\Theta, (\partial^\beta\Theta)_{|\beta|\leqslant N})\widetilde{\partial}^\alpha f$$

where $\widetilde{\partial}$ denotes differentiation of length 1 for the metric $\widetilde{g}_a$ defined by

$$\widetilde{g}_a(dy^2, d\eta^2) \stackrel{\text{def}}{=} |I_\Lambda|^{-1}dy^2 + |I_\Lambda|d\eta^2 = \lambda g_a(dy^2, d\eta^2).$$

The metric $\widetilde{g}_a$ is the interpolation between g_a and $g_a^\sigma = \lambda^2 g_a$. To prove this lemma, let us notice that the two vectors

$$|I_\Lambda|^{-1/2}\partial_\eta \quad \text{and} \quad |I_\Lambda|^{1/2}\partial_y$$

are of $\widetilde{g}_a$-length 1. Proposition 4.1 and the fact that $|I_\Lambda| \leqslant \Lambda^{2-\varepsilon}$ implies that, for any positive integer k, a constant c_k (which depends only on constants of Proposition 4.1) such that

$$\|\widetilde{\partial}^k\Theta\|_{L^\infty(I_\Lambda\times T^\star \mathbf{R}^d)} \leqslant c_k. \tag{46}$$

Now, we write that ${}^t\mathcal{L}f = \mathcal{L}f + L_0 f$ with

$$L_0 \stackrel{\text{def}}{=} i\Big(|I_\Lambda|^{-1/2}\sum_{j=1}^d \partial_{\eta_j}\Big(\frac{\Theta_{y_j}}{1+|\Theta|^2}\Big) + |I_\Lambda|^{1/2}\sum_{j=1}^d \partial_{y_j}\Big(\frac{\Theta_{\eta_j}}{1+|\Theta|^2}\Big)\Big)\cdot$$

So thanks to (46), we have

$$\|L_0(\Theta,\cdot)\|_{L^\infty} \leqslant \frac{C}{1+|\Theta|^2}\,\cdot$$

But, by definition of $\mathcal{L}$, it is obvious that $\mathcal{L}$ is of the form

$$\sum_{|\alpha|\leqslant 1} L_{\alpha,1}(\Theta)\widetilde{\partial}^\alpha f$$

where $\mathcal{L}_{\alpha,1}$ satisfy (45) for $N=1$. So the lemma is proved for $N=1$. The lemma follows by an omitted (and straightforward) induction.

Now, let us go back to the proof of the propagation theorem. The point is to prove that derivatives of $\widetilde{g}_a$-length 1 of

$$\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)$$

are bounded uniformly to the involved parameters. Thanks to Leibnitz formula, we have

$$\widetilde{\partial}_y^\alpha\widetilde{\partial}_\eta^\beta\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big) = \sum_{\substack{\alpha_1\leqslant\alpha\\ \beta_1\leqslant\beta}} C_{\alpha,\beta}^{\alpha_1,\beta_1}\widetilde{\partial}_y^{\alpha-\alpha_1}\varphi_{Z_\tau}(y,\zeta)\widetilde{\partial}_y^{\alpha_1}\widetilde{\partial}_\eta^{\beta_1}\sigma_\Lambda(\tau,y,\eta)\widetilde{\partial}_\eta^{\beta-\beta_1}\phi(z,\eta).$$

The metric $\widetilde{g}_a$ is choosen such that it is greater than the metric g_a and the metric g_Λ defined by

$$g_\Lambda(dy^2,d\eta^2) \stackrel{\text{def}}{=} \frac{dy^2}{\Lambda^2} + d\eta^2.$$

Then, it is obvious that, for any integer k, we have

$$\sup_{\substack{|\alpha+\beta|\leqslant k\\ (\tau,Y,Z)\in I_\Lambda\times(T^\star\mathbf{R}^d)^2}} \big|\widetilde{\partial}_y^\alpha\widetilde{\partial}_\eta^\beta\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)\big| \leqslant C_k.$$

So thanks to Lemma 6.1, it turns out that, for any N, a constant C_N exists such that

$$\Big\|({}^t\mathcal{L})^N\big(\varphi_{Z_\tau}(y,\zeta)\sigma_\Lambda(\tau,y,\eta)\phi(z,\eta)\big)\Big\| \leqslant \frac{C_N}{(1+|\Theta|^2)^{N/2}}\,\cdot$$

So by definition of Θ, we infer that, for any integer N, a constant C_N exists such that

$$|\mathcal{K}(\zeta,z)| \leqslant C_N\int_{\substack{|y-z_\tau|\leqslant rK\\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1+\widetilde{g}_a\big(\nabla_\eta\Phi_\Lambda(\tau,y,\eta)-z,\nabla_y\Phi_\Lambda(\tau,y,\eta)-\zeta\big)\Big)^N}\,,$$

with of course

$$|z-y_0|\leqslant rK \quad\text{and}\quad |\zeta-\zeta_\tau|\leqslant rh.$$

But, as $K = C|I_\Lambda|h$ and $\lambda = Kh$, we have

$$\widetilde{g}_a(dy^2, d\eta^2) \geqslant c\lambda g_a(dy^2, d\eta^2).$$

So we have that

$$|\mathcal{K}(\zeta, z)| \leqslant C_N \int_{\substack{|y-z_\tau|\leqslant rK\\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1 + \lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\big)\Big)^N}\cdot$$

Now let us apply Lemma 4.2. As $g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0))^{1/2}$ is greater than $C_0 r$, as (z, η) belongs to $B_{g_a}(Y_0, r)$ and (y, ζ) to $B_{g_a}(Z_\tau, r)$ then

$$g_a(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta) \geqslant \frac{1}{C_0} g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0)).$$

So we have, if $g_a(Z_\tau, \Psi_\Lambda(\tau, Y_0))^{1/2}$ is greater than $C_0 r$,

$$\begin{aligned}|\mathcal{K}(\zeta, z)| \leqslant{}& \frac{C_N}{\big(1 + \lambda g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))\big)^N}\\ &\times \int_{\substack{|y-z_\tau|\leqslant rK\\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1 + \lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\big)\Big)^N}\cdot\end{aligned}$$

As (y, ζ) belongs to $B_{g_a}(Z_\tau, r)$, we have that

$$\begin{aligned}g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} &\geqslant g_a((y, \zeta) - \Psi_\Lambda(\tau, Y_0))^{1/2} - r\\ &\geqslant \frac{|\zeta - \Psi^\eta_\Lambda(\tau, Y_0)|}{h} - r.\end{aligned}$$

Stating $Z_0 \stackrel{\text{def}}{=} \Psi_\Lambda^{-1}(\tau, Z_\tau)$, we have, thanks to the assertion (36) of Lemma 4.2 and as (z, η) belongs to $B_{g_a}(Y_0, r)$,

$$\begin{aligned}g_a(Z_\tau - \Psi_\Lambda(\tau, Y_0))^{1/2} &\geqslant C g_a(Z_0 - Y_0)^{1/2}\\ &\geqslant C\frac{|z - z_0|}{K} - Cr.\end{aligned}$$

So it turns out that

$$\begin{aligned}|\mathcal{K}(\zeta, z)| \leqslant{}& C_N\Big(1 + \lambda g_a\big(Z_\tau - \Psi_\Lambda(\tau, Y_0)\big)\Big)^{-N}\Big(1 + \lambda g_a\big((z, \zeta) - (z_0, \Psi^\eta_\Lambda(\tau, Y_0))\big)\Big)^{-N}\\ &\times \int_{\substack{|y-z_\tau|\leqslant rK\\ |\eta-\eta_0|\leqslant rh}} \frac{dyd\eta}{\Big(1 + \lambda g_a\big(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z, \nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\big)\Big)^N}\cdot\end{aligned}$$

Let us state the change of variables

$$\begin{cases} y' = |I|^{-1/2}\big(\nabla_\eta \Phi_\Lambda(\tau, y, \eta) - z\big)\\ \eta' = |I|^{1/2}\big(\nabla_y \Phi_\Lambda(\tau, y, \eta) - \zeta\big).\end{cases}$$

Using estimates (32), we infer that the jacobian of this change of variables is closed to 1. Then it turns out that

$$|\mathcal{K}(\zeta, z)| \leqslant C_N\Big(1 + \lambda g_a\big(Z_\tau - \Psi_\Lambda(\tau, Y_0)\big)\Big)^{-N}\Big(1 + \lambda g_a\big((z, \zeta) - (z_0, \Psi^\eta_\Lambda(\tau, Y_0))\big)\Big)^{-N}.$$

But Schur's lemma implies that

$$\|\mathcal{J}\|_{L^2}^2 \leqslant \left(\sup_\zeta \int |\mathcal{K}(\zeta,z)|dz\right)\left(\sup_z \int |\mathcal{K}(\zeta,z)|d\zeta\right)\|\gamma\|_{L^2}^2.$$

Immediate integrations imply that

$$\int |\mathcal{K}(\zeta,z)|dz \leqslant C_N\big(1+\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))\big)^{-N}|I|^{d/2} \quad \text{and}$$
$$\int |\mathcal{K}(\zeta,z)|d\zeta \leqslant C_N\big(1+\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))\big)^{-N}|I|^{-d/2}.$$

So, for any N, we have

$$\|\mathcal{J}\|_{L^2} \leqslant C_N\big(1+\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))\big)^{-N}\|\gamma\|_{L^2}.$$

As $g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))^{1/2}$ is greater that $C_0 r$, then

$$\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0)) \geqslant C_0 r\lambda g_a(Z_\tau-\Psi_\Lambda(\tau,Y_0))^{1/2}.$$

So Theorem 6.1 is proved.

In the next section, the following corollary will be useful.

Corollary 6.1. — *A constant C_0 exists which satisfies the following property.*

Let us consider a point $Y_0=(y_0,\eta_0)$ of $T^\star\mathbf{R}^d$ such that η_0 belongs to $\mathcal{C}$, a smooth function ϕ supported in $B_{g_a}(Y_0,r)$ and a function γ of L^2.

For any integer N, a constant C and an integer k exist (which depend only on N) such that, for any a, if $g_a(\Psi_\Lambda(\tau,Y_0)-Y)\geqslant C_0 r$, for any function ψ in $\mathcal{S}(T^\star\mathbf{R}^d)$, we have

$$\|\psi^D\big(\mathcal{I}_\Lambda^\pm(\phi^D\gamma)(\tau,\cdot)\big)\|_{L^2} \leqslant C\lambda^{-N}\big(1+g_a(\Psi_\Lambda(\tau,Y_0)-Y)\big)^{-N}\,\|\psi\|_{k,g,Y}\,\|\phi\|_{k,g_a}\|\gamma\|_{L^2}.$$

The proof of this corollary is a simple combination of Theorem 6.1 and Lemmas 5.3 and 5.4.

7. The conclusion of the proof

This section is the conclusion of the proof of theorem 3.1. The strategy is the following. First, we apply Lemma 5.5 about the product and the propagation theorem 6.1 to concentrate on real interaction (see the proof in the constant coefficient case). Because of the fact that variable coefficents do not respect the localization in frequency space, we need at this step of the proof to decompose the interval I_Λ.

In this section, we shall state

$$\mathcal{J}(\tau,y) \stackrel{\text{def}}{=} \chi(h^{-1}D)\big(\mathcal{I}_\Lambda^{(1)}(\gamma_1)(\tau,y)\mathcal{I}_\Lambda^{(2)}(\gamma_2)(\tau,y)\big).$$

The equivalent of Identity (5) that appears in the constant coefficient case is the following lemma.

Lemma 7.1. — *Let $J=(\tau_J,\tau_J^+)$ be a subinterval of I_Λ such that*

$$|J|\leqslant h|I_\Lambda| \quad \textit{and} \quad \|\nabla G^{(j)}_\Lambda\|_{L^1_J(L^\infty)}\leqslant h\|\nabla G^{(j)}_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)}.$$

Then two families (ϕ_μ) and (θ_μ) of confined symbols exist such that, for any integer N,

$$\forall\,\mu\,,\ \|\phi_\mu\|_{N,g_a,\Psi^{(1)}_\Lambda(\tau_J,Y_\mu)}+\|\theta_\mu\|_{N,g_a,\check\Psi^{(2)}_\Lambda(\tau_J,Y_\mu)}\leqslant C_N$$

and, for any N, a constant C_N exists such that

$$\|\mathcal{J}-\underline{\mathcal{J}}\|_{L^1_J(L^\infty)}\leqslant C_N h\lambda^{-N}(|I_\Lambda|h^2)h^{d-2}\|\gamma_1\|_{L^2}\|\gamma_2\|_{L^2}.$$

with

$$\underline{\mathcal{J}}(\tau)\stackrel{def}{=}\sum_{\substack{\mu,\mu'\\ \mu'\in A_\mu}}\chi(h^{-1}D)\Big(\phi^D_\mu\mathcal{I}^{(1)}_\Lambda\big(\varphi^D_\mu\psi^D_\mu\gamma_1\big)(\tau,\cdot)\times\theta^D_\mu\mathcal{I}^{(2)}_\Lambda\big(\varphi^D_{\mu'}\psi^D_{\mu'}\gamma_2\big)(\tau,\cdot)\Big)\quad \textit{and}$$

$$A_\mu\subset\big\{\mu'\,/\,g_a\big(\Psi^{(2)}_\Lambda(\tau_J,Y_{\mu'})-\check\Psi^{(1)}_\Lambda(\tau_J,Y_\mu)\big)^{1/2}\leqslant Cr\big\}.$$

Let us admit this lemma for a while. Let us simply notice that the number of elements of A_μ is finite and bounded indepently on μ and J.

Now we shall decompose the interval I_Λ on subintervals J such that the above lemma can be applied. To do this let us introduce the following function on the interval I_Λ

$$H(\tau)\stackrel{\rm def}{=}\bigg(\sum_\mu\|\mathcal{I}^{(1)}_\Lambda(\varphi^D_\mu\psi^D_\mu\gamma_1)(\tau,\cdot)\|^2_{L^\infty}\bigg)^{1/2}\bigg(\sum_\mu\|\mathcal{I}^{(2)}_\Lambda(\varphi^D_\mu\psi^D_\mu\gamma_2)(\tau,\cdot)\|^2_{L^\infty}\bigg)^{1/2}.$$

Using precised Strichartz estimates, we get that

$$\|\mathcal{I}^{(j)}_\Lambda(\varphi^D_\mu\psi^D_\mu\gamma_j)\|_{L^2_{I_\Lambda}(L^\infty)}\leqslant C\big(\log(e+|I_\Lambda|)\big)^{1/2}h^{(d-2)/2}\|\psi^D_\mu\gamma_j\|_{L^2}.$$

So, using Cauchy-Schwartz inequality, we get that

$$\int_{I_\Lambda}H(\tau)d\tau\leqslant C\big(\log(e+|I_\Lambda|)\big)h^{d-2}\bigg(\sum_\mu\|\psi^D_\mu\gamma_1\|^2_{L^2}\bigg)^{1/2}\bigg(\sum_\mu\|\psi^D_\mu\gamma_2\|^2_{L^2}\bigg)^{1/2}.$$

Lemma 5.2 implies that

$$\int_{I_\Lambda}H(\tau)d\tau\leqslant C\big(\log(e+|I_\Lambda|)\big)h^{d-2}\|\gamma_1\|^2_{L^2}\|\gamma_2\|^2_{L^2}.$$

As in section 3, we decompose I_Λ in intervals J such that

$$|J|\leqslant h|I_\Lambda|\,,\ \|\nabla G^{(j)}_\Lambda\|_{L^1_J(L^\infty)}\leqslant h\|\nabla G^{(j)}_\Lambda\|_{L^1_{I_\Lambda}(L^\infty)}\quad\text{and}\quad\int_J H(\tau)d\tau\leqslant h\int_{I_\Lambda}H(\tau)d\tau.$$

Let us estimate $\|\underline{\mathcal{J}}\|_{L^1_J(L^\infty)}$. Lemma 5.4 implies that

$$\|\underline{\mathcal{J}}(\tau)\|_{L^\infty}\leqslant\sum_{\substack{\mu,\mu'\\ \mu'\in A_\mu}}\big\|\mathcal{I}^{(1)}_\Lambda\big(\varphi^D_\mu\psi^D_\mu\gamma_1\big)(\tau,\cdot)\big\|_{L^\infty}\big\|\mathcal{I}^{(2)}_\Lambda\big(\varphi^D_{\mu'}\psi^D_{\mu'}\gamma_2\big)(\tau,\cdot)\big\|_{L^\infty}.$$

By Cauchy-Schwarz inequality, we infer that

$$\|\underline{\mathcal{J}}(\tau)\|_{L^\infty} \leqslant H(\tau).$$

So by construction of J, we get that

$$\|\underline{\mathcal{J}}\|_{L^1_J(L^\infty)} \leqslant Ch\big(\log(e+|I_\Lambda|)\big)h^{d-2}\|\gamma_1\|_{L^2}\|\gamma_2\|_{L^2}.$$

Exactly along the same lines as in section 3, the number of intervals J is less than Ch^{-1}. As $|I_\Lambda| \geqslant h^{-2+\varepsilon}$ and $\lambda = |I_\Lambda|h^2$, the theorem is proved if we apply Lemma 7.1 with N large enough.

But we have to prove Lemma 7.1. First, let us write

$$\mathcal{J}(\tau) = \sum_{\nu,\nu',\mu,\mu'} \mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau) \quad \text{with}$$

$$\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau) \stackrel{\text{def}}{=} \chi(h^{-1}D)\Big(\varphi^D_\nu\psi^D_\nu\mathcal{I}^{(1)}_\Lambda(\varphi^D_\mu\psi^D_\mu\gamma_1)(\tau,\cdot)\varphi^D_{\nu'}\psi^D_{\nu'}\mathcal{I}^{(2)}_\Lambda(\varphi^D_{\mu'}\psi^D_{\mu'}\gamma_2)(\tau,\cdot)\Big).$$

Propagation theorem 6.1 and its corollary 6.1 imply that, if

$$g_a\big(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J, Y_\mu)\big)^{1/2} \geqslant C_0 r,$$

then

$$\|\varphi^D_\nu\psi^D_\nu\mathcal{I}^{(j)}_\Lambda(\varphi^D_\mu\psi^D_\mu\gamma_j)(\tau)\|_{L^2} \leqslant C_N\lambda^{-N}\big(g_a(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J,Y_\mu))\big)^{-d-1}\|\psi^D_\mu\gamma_j\|_{L^2}.$$

So using Bernstein inequality and integrating on the interval J, if

$$g_a\big(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J, Y_\mu)\big)^{1/2} \geqslant C_0 r,$$

we get that

$$\begin{aligned}(47)\quad \|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\|_{L^1_J(L^\infty)} \leqslant C_N h(|I_\Lambda|h^2)h^{d-2}\lambda^{-N}\big(1+g_a(Y_\nu - \Psi^{(j)}_\Lambda(\tau_J,Y_\mu))\big)^{-d-1}\\ \times\|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2}.\end{aligned}$$

Lemma 5.5 implies that if $g_a(\check{Y}_\nu - Y_{\nu'})^{1/2} \geqslant C_0 r$, then for any N we have

$$\|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau)\|_{L^1(\mathbf{R}^d)} \leqslant C_N\lambda^{-N}\big(1+g_a(\check{Y}_\nu - Y_{\nu'})\big)^{-d-1}\|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2}.$$

Using Bernstein inequality, we get by integration that

$$(48)\qquad \|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}(\tau)\|_{L^\infty} \leqslant C_N\lambda^{-N}h^d\big(1+g_a(\check{Y}_\nu - Y_{\nu'})\big)^{-d-1}\|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2}.$$

Let us define

$$\Delta_{g_a}(X) \stackrel{\text{def}}{=} \begin{cases} 1+g_a(X) & \text{if } g_a(X)^{1/2} \geqslant Cr\\ \quad 1 & \text{if } g_a(X)^{1/2} \leqslant Cr.\end{cases} \quad \text{and}$$

$$\begin{aligned}A \stackrel{\text{def}}{=} \big\{(\nu,\nu',\mu,\mu')\,/\, g_a(Y_\nu - \Psi^{(1)}_\Lambda(\tau_J,Y_\mu))^{1/2} \leqslant Cr \quad \text{and}\\ g_a(Y_{\nu'} - \Psi^{(2)}_\Lambda(\tau_J,Y_{\mu'}))^{1/2} \leqslant Cr \quad \text{and} \quad g_a(\check{Y}_\nu - Y_{\nu'})^{1/2} \leqslant Cr\big\}.\end{aligned}$$

Thanks to Inequality (36) of Lemma 4.2, and thanks to the fact that the point (X_ν) are the points of the lattice $\mathcal{Z}$ defined in (43), the number of indices ν such that

$$g_a(Y_\nu - \Psi^{(1)}_\Lambda(\tau_J,Y_\mu))^{1/2} \leqslant Cr$$

is finite and independent of the interval J. So plugging the estimates (47) and (48) together, we get, if $(\nu,\nu',\mu,\mu')\notin A$,

$$\|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\|_{L^1_J(L^\infty)} \leqslant K_N(\nu,\nu',\mu,\mu')\lambda^{-N}h(|I_\Lambda|h^2)h^{d-2}\|\psi^D_\mu\gamma_1\|_{L^2}\|\psi^D_{\mu'}\gamma_2\|_{L^2} \quad \text{with}$$
$$K_N(\nu,\nu',\mu,\mu') \stackrel{\text{def}}{=} C_N\Delta_{g_a}(Y_\nu-\Psi^{(1)}_\Lambda(\tau_J,Y_\mu))^{-d-1}\Delta_{g_a}(Y_{\nu'}-\Psi^{(2)}_\Lambda(\tau_J,Y_{\mu'}))^{-d-1} \times \Delta_{g_a}(\check{Y}_\nu-Y_{\nu'})^{-d-1}.$$

But we have that
$$\sup_X\sum_\nu \Delta_{g_a}(X-Y_\nu)^{-d-1}<\infty.$$
Applying Schur's lemma and then Lemma 5.2, it turns out that
$$\begin{aligned}\Big\|\sum_{(\nu,\nu',\mu,\mu')\notin A}\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\Big\|_{L^1_J(L^\infty)} &\leqslant \sum_{(\nu,\nu',\mu,\mu')\notin A}\|\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}\|_{L^1_J(L^\infty)}\\ &\leqslant C_N\lambda^{-N}h(|I_\Lambda|h^2)h^{d-2}\Big(\sum_\mu\|\psi^D_\mu\gamma_1\|^2_{L^2}\Big)^{1/2}\Big(\sum_\mu\|\psi^D_\mu\gamma_2\|^2_{L^2}\Big)^{1/2}\\ &\leqslant C_N\lambda^{-N}h(|I_\Lambda|h^2)h^{d-2}\|\gamma_1\|_{L^2}\|\gamma_2\|_{L^2}.\end{aligned}$$
Now let us state
$$\underline{\mathcal{J}}\stackrel{\text{def}}{=}\sum_{(\nu,\nu',\mu,\mu')\in A}\mathcal{J}^{\nu,\nu'}_{\mu,\mu'}$$
and check that it satisfies the conclusions of the lemma. Let us define
$$\begin{aligned}B^{(j)}_\mu &\stackrel{\text{def}}{=}\big\{\nu\,/\,g_a\big(Y_\nu-\Psi^{(j)}_\Lambda(\tau_J,Y_\mu)\big)^{1/2}\leqslant Cr\big\},\\ C_\mu &\stackrel{\text{def}}{=}\big\{\nu'\,/\exists\nu\in B^{(1)}_\mu\,/g_a(\check{Y}_\nu-Y_{\nu'})^{1/2}\leqslant Cr\big\},\\ A_\mu &\stackrel{\text{def}}{=}\big\{\mu'\,,/\exists\nu'\in C_\mu\cap B^{(2)}_{\mu'}\big\}.\end{aligned}$$
Let us notice that, thanks to Inequality (36) of Lemma 4.2, we have
$$A_\mu\subset\big\{\mu'\,/\,g_a\big(\Psi^{(1)}_\Lambda(\tau_J,Y_\mu)-\check{\Psi}^{(2)}_\Lambda(\tau_J,Y_{\mu'})\big)^{1/2}\leqslant Cr\big\}.$$
Now let us state
$$\phi^D_\mu\stackrel{\text{def}}{=}\sum_{\nu\in B^{(1)}_\mu}\varphi^D_\nu\psi^D_\nu\quad\text{and}\quad\theta^D_\mu\stackrel{\text{def}}{=}\sum_{\nu'\in C_\mu}\varphi^D_{\nu'}\psi^D_{\nu'}.$$
We apply Lemma 5.3 to conclude the proof.

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MICROLOCAL STUDY OF IND-SHEAVES I: MICRO-SUPPORT AND REGULARITY

by

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Abstract. — We introduce the notions of micro-support and regularity for ind-sheaves, and prove their invariance by quantized contact transformations. We apply these results to the ind-sheaves of temperate holomorphic solutions of $\mathcal{D}$-modules. We prove that the micro-support of such an ind-sheaf is the characteristic variety of the corresponding $\mathcal{D}$-module and that the ind-sheaf is regular if the $\mathcal{D}$-module is regular holonomic. We finally calculate an example of the ind-sheaf of temperate solutions of an irregular $\mathcal{D}$-module in dimension one.

Résumé **(Étude microlocale des Ind-faisceaux I: micro-support et régularité)**
Nous introduisons les notions de micro-support et régularité pour les ind-faisceaux et prouvons leur invariance par transformations de contact quantifiées. Nous appliquons ces résultats aux ind-faisceaux des solutions holomorphes tempérées des $\mathcal{D}$-modules. Nous prouvons que le micro-support d'un tel ind-faisceau est la variété caractéristique du $\mathcal{D}$-module correspondant et que le ind-faisceau est régulier si le $\mathcal{D}$-module est holonome régulier. Nous calculons enfin un exemple du ind-faisceau des solutions tempérées d'un $\mathcal{D}$-module irrégulier en dimension un.

1. Introduction

Recall that a system of linear partial differential equations on a complex manifold X is the data of a coherent module $\mathcal{M}$ over the sheaf of rings $\mathcal{D}_X$ of holomorphic differential operators. Let F be a complex of sheaves on X with $\mathbb{R}$-constructible cohomologies (one says an $\mathbb{R}$-constructible sheaf, for short). The complex of "generalized functions" associated with F is described by the complex $R\mathcal{H}om(F, \mathcal{O}_X)$, and the complex of solutions of $\mathcal{M}$ with values in this complex is described by the complex

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om(F, \mathcal{O}_X)).$$

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One may also microlocalize the problem by replacing $R\mathcal{H}om(F, \mathcal{O}_X)$ with $\mu hom(F, \mathcal{O}_X)$. In [**4**] one shows that most of the properties of this complex, especially those related to propagation or Cauchy problem, are encoded in two geometric objects, both living in the cotangent bundle T^*X, the characteristic variety of the system $\mathcal{M}$, denoted by $\mathrm{char}(\mathcal{M})$, and the micro-support of F, denoted by $SS(F)$.

The complex $R\mathcal{H}om(F, \mathcal{O}_X)$ allows us to treat various situations. For example if M is a real analytic manifold and X is a complexification of M, by taking as F the dual $D'(\mathbb{C}_M)$ of the constant sheaf on M, one obtains the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions. If Z is a complex analytic hypersurface of X and $F = \mathbb{C}_Z[-1]$ is the (shifted) constant sheaf on Z, one obtains the sheaf of holomorphic functions with singularities on Z. However, the complex $R\mathcal{H}om(F, \mathcal{O}_X)$ does not allow us to treat sheaves associated with holomorphic functions with growth conditions. So far this difficulty was overcome in two cases, the temperate case including Schwartz's distributions and meromorphic functions with poles on Z and the dual case including C^∞-functions and the formal completion of $\mathcal{O}_X$ along Z. The method was two construct specific functors, the functor $T\mathcal{H}om$ of [**2**] and the functor $\overset{\mathrm{w}}{\otimes}$ of [**5**].

There is a more radical method, which consists in replacing the too narrow framework of sheaves by that of ind-sheaves, as explained in [**6**]. For example, the presheaf of holomorphic temperate functions on a complex manifold X (which, to a subanalytic open subset of X, associates the space of holomorphic functions with temperate growth at the boundary) is clearly not a sheaf. However it makes sense as an object (denoted by $\mathcal{O}_X^t$) of the derived category of ind-sheaves on X. Then it is natural to ask if the microlocal theory of sheaves, in particular the theory of micro-support, applies in this general setting.

In this paper we give the definition and the elementary properties of the micro-support of ind-sheaves as well as the notion of regularity.

We prove in particular that the micro-support $SS(\cdot)$ and the regular micro-support $SS_{\mathrm{reg}}(\cdot)$ of ind-sheaves behave naturally with respect to distinguished triangles and that these micro-supports are invariant by "quantized contact transformations" (in the framework of sheaf theory, as explained in [**4**]).

When X is a complex manifold and $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, we study the ind-sheaf $Sol^t(\mathcal{M}) := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^t)$. We prove that

(i) $SS(Sol^t(\mathcal{M})) = \mathrm{char}(\mathcal{M})$,
(ii) if $\mathcal{M}$ is regular holonomic, then $Sol^t(\mathcal{M})$ is regular.

Finally, we treat an example: we calculate the ind-sheaf of the temperate holomorphic solutions of an irregular differential equation.

This paper is the first one of a series. In Part II, we shall introduce the microlocalization functor for ind-sheaves, and in Part III we shall study the functorial behavior of micro-supports.

2. Notations and review

We will mainly follow the notations in [**4**] and [**6**].

Geometry. — In this paper, all manifolds will be real analytic (sometimes, complex analytic). Let X be a manifold. One denotes by $\tau\colon TX \to X$ the tangent bundle to X and by $\pi\colon T^*X \to X$ the cotangent bundle.

For a smooth submanifold Y of X, $T_Y X$ denotes the normal bundle to Y and $T^*_Y X$ the conormal bundle. In particular, $T^*_X X$ is identified with X, the zero-section. For a submanifold Y of X and a subset S of X, we denote by $C_Y(S)$ the Whitney normal cone to S along Y, a conic subset of $T_Y X$.

One denotes by $a\colon T^*X \to T^*X$ the antipodal map. If $S \subset T^*X$, one denotes by $\dot{S}$ the set $S \setminus T^*_X X$, and one denotes by S^a the image of S by the antipodal map. In particular, $\dot{T}^*X = T^*X \setminus X$, the set T^*X with the zero-section removed. One denotes by $\dot{\pi}\colon \dot{T}^*X \to X$ the projection.

If S is a locally closed subset of T^*X, we say that S is $\mathbb{R}^+$-conic (or simply "conic", for short) if it is locally invariant under the action of $\mathbb{R}^+$. If S is smooth, this is equivalent to saying that the Euler vector field on T^*X is tangent to S.

Let $f\colon X \to Y$ be a morphism of real manifolds. One has two natural maps

$$T^*X \underset{f_d}{\longleftarrow} X \times_Y T^*Y \underset{f_\pi}{\longrightarrow} T^*Y \tag{2.1}$$

(In [**4**], f_d is denoted by ${}^t f'$.) We denote by q_1 and q_2 the first and second projections defined on $X \times Y$.

Sheaves. — Let k be a field. We denote by $\mathrm{Mod}(k_X)$ the abelian category of sheaves of k-vector spaces and by $D^b(k_X)$ its bounded derived category.

We denote by $\mathbb{R}\text{-C}(k_X)$ the abelian category of $\mathbb{R}$-constructible sheaves of k-vector spaces on X, and by $D^b_{\mathbb{R}-\mathrm{c}}(k_X)$ (resp. $D^b_{\mathrm{w}-\mathbb{R}-\mathrm{c}}(k_X)$) the full triangulated subcategory of $D^b(k_X)$ consisting of objects with $\mathbb{R}$-constructible (resp. weakly $\mathbb{R}$-constructible) cohomology. On a complex manifold, one defines similarly the categories $D^b_{\mathbb{C}-\mathrm{c}}(k_X)$ and $D^b_{\mathrm{w}-\mathbb{C}-\mathrm{c}}(k_X)$ of $\mathbb{C}$-constructible and weakly $\mathbb{C}$-constructible sheaves.

If Z is a locally closed subset of X and if F is a sheaf on X, recall that F_Z is a sheaf on X such that $F_Z|_Z \simeq F|_Z$ and $F_Z|_{X \setminus Z} \simeq 0$. One writes k_{XZ} instead of $(k_X)_Z$ and one sometimes writes k_Z instead of k_{XZ}.

If $f\colon X \to Y$ is a morphism of manifolds, one denotes by $\omega_{X/Y}$ the relative dualizing complex on X and if $Y = \{\mathrm{pt}\}$ one simply denotes it by ω_X. Recall that

$$\omega_X \simeq \mathrm{or}_X[\dim_{\mathbb{R}} X]$$

where or_X is the orientation sheaf and $\dim_R X$ is the dimension of X as a real manifold. We denote by D'_X and D_X the duality functors on $D^b(k_X)$, defined by

$$D'_X(F) = R\mathcal{H}om\,(F, k_X),\ \ D_X(F) = R\mathcal{H}om\,(F, \omega_X).$$

If F is an object of $D^b(k_X)$, $SS(F)$ denotes its micro-support, a closed conic involutive subset of T^*X. For an open subset U of T^*X, one denotes by $D^b(k_X;U)$ the localization of the category $D^b(k_X)$ with respect to the triangulated subcategory consisting of sheaves F such that $SS(F)\cap U=\varnothing$.

We shall also use the functor μhom as well as the operation $\widehat{+}$ and refer to loc. cit. for details.

$\mathcal{O}$-modules and $\mathcal{D}$-modules. — On a complex manifold X we consider the structural sheaf $\mathcal{O}_X$ of holomorphic functions and the sheaf $\mathcal{D}_X$ of linear holomorphic differential operators of finite order.

We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the abelian category of coherent $\mathcal{D}_X$-modules. We denote by $D^b(\mathcal{D}_X)$ the bounded derived category of left $\mathcal{D}_X$-modules and by $D^b_{\mathrm{coh}}(\mathcal{D}_X)$ (resp. $D^b_{\mathrm{hol}}(\mathcal{D}_X)$), $D^b_{\mathrm{rh}}(\mathcal{D}_X)$) its full triangulated category consisting of objects with coherent cohomologies (resp. holonomic cohomologies, regular holonomic cohomologies).

Categories. — In this paper, we shall work in a given universe $\mathcal{U}$, and a category means a $\mathcal{U}$-category. If $\mathcal{C}$ is a category, $\mathcal{C}^\wedge$ denotes the category of functors from $\mathcal{C}^{\mathrm{op}}$ to **Set**. The category $\mathcal{C}^\wedge$ admits inductive limits, however, in case $\mathcal{C}$ also admits inductive limits, the Yoneda functor $h^\wedge\colon \mathcal{C}\to\mathcal{C}^\wedge$ does not commute with such limits. Hence, one denotes by $\varinjlim$ the inductive limit in $\mathcal{C}$ and by "$\varinjlim$" the inductive limit in $\mathcal{C}^\wedge$.

One denotes by $\mathrm{Ind}(\mathcal{C})$ the category of ind-objects of $\mathcal{C}$, that is the full subcategory of $\mathcal{C}^\wedge$ consisting of objects F such that there exist a small filtrant category I and a functor $\alpha\colon I\to\mathcal{C}$, with

$$F\simeq \text{“}\varinjlim\text{”}\,\alpha,\ \text{i.e., } F\simeq \text{“}\varinjlim_{i\in I}\text{”}\,F_i,\ \text{with } F_i\in\mathcal{C}.$$

The category $\mathcal{C}$ is considered as a full subcategory of $\mathrm{Ind}(\mathcal{C})$.

If $\varphi\colon\mathcal{C}\to\mathcal{C}'$ is a functor, it defines a functor $I\varphi\colon \mathrm{Ind}(\mathcal{C})\to\mathrm{Ind}(\mathcal{C}')$ which commutes with "$\varinjlim$".

If $\mathcal{C}$ is an additive category, we denote by $C(\mathcal{C})$ the category of complexes in $\mathcal{C}$ and by $K(\mathcal{C})$ the associated homotopy category. If $\mathcal{C}$ is abelian, one denotes by $D(\mathcal{C})$ its derived category. One defines as usual the full subcategories $C^*(\mathcal{C}), K^*(\mathcal{C}), D^*(\mathcal{C})$, with $*=+,-,b$. One denotes by Q the localization functor:

$$Q\colon K^*(\mathcal{C})\longrightarrow D^*(\mathcal{C}).$$

We keep the same notation Q to denote the composition $C^*(\mathcal{C})\to K^*(\mathcal{C})\to D^*(\mathcal{C})$.

Let $a,b\in\mathbb{Z}$ with $a\leqslant b$. One denotes by $C^{[a,b]}(\mathcal{C})$ the full subcategory of $C(\mathcal{C})$ consisting of objects $F^\bullet$ satisfying $F^i=0$ for $i\notin[a,b]$. There is a natural equivalence

$$\mathrm{Ind}(C^{[a,b]}(\mathcal{C}))\xrightarrow{\sim} C^{[a,b]}(\mathrm{Ind}(\mathcal{C})).$$

Ind-sheaves. — Here, X is a Hausdorff locally compact space with a countable base of open sets and k is a field. One denotes by $\mathrm{I}(k_X)$ the abelian category of ind-sheaves of k-vector spaces on X, that is, $\mathrm{I}(k_X) = \mathrm{Ind}(\mathrm{Mod}^c(k_X))$, the category of ind-objects of the category $\mathrm{Mod}^c(k_X)$ of sheaves with compact support on X. We denote by $D^b(\mathrm{I}(k_X))$ the bounded derived category of $\mathrm{I}(k_X)$.

There is a natural fully faithful exact functor

$$\begin{aligned}\iota_X \colon \mathrm{Mod}(k_X) &\longrightarrow \mathrm{I}(k_X),\\ F &\longmapsto \text{“}\varinjlim_{U\subset\subset X}\text{”}\, F_U \ (U \text{ open}).\end{aligned}$$

Most of the time, we shall not write this functor and identify $\mathrm{Mod}(k_X)$ with a full abelian subcategory of $\mathrm{I}(k_X)$ and $D^b(k_X)$ with a full triangulated subcategory of $D^b(\mathrm{I}(k_X))$.

The category $\mathrm{I}(k_X)$ admits an internal hom denoted by $\mathcal{I}hom$ and this functor admits a left adjoint, denoted by $\otimes$. If $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$ and $G \simeq \text{“}\varinjlim_j\text{”}\, G_j$, then

$$\begin{aligned}\mathcal{I}hom\,(G,F) &\simeq \varprojlim_j \text{“}\varinjlim_i\text{”}\, \mathcal{H}om\,(G_j,F_i)\\ G\otimes F &\simeq \text{“}\varinjlim_i\text{”}\ \text{“}\varinjlim_j\text{”}(G_j\otimes F_i).\end{aligned}$$

The functor ι_X admits a left adjoint

$$\alpha_X \colon \mathrm{I}(k_X) \longrightarrow \mathrm{Mod}(k_X),$$

To $F = \text{“}\varinjlim_{i\in I}\text{”}\, F_i$, this functor associates $\alpha_X(F) = \varinjlim_{i\in I} F_i$. This functor also admits a left adjoint

$$\beta_X \colon \mathrm{Mod}(k_X) \longrightarrow \mathrm{I}(k_X),$$

and both functors α_X and β_X are exact. The functor β_X is not so easy to describe. For example, for an open subset U and a closed subset Z, one has;

$$\begin{aligned}\beta_X(k_{XU}) &\simeq \text{“}\varinjlim_{V\subset\subset U}\text{”}\, k_{XV} \ (V \text{ open}),\\ \beta_X(k_{XZ}) &\simeq \text{“}\varinjlim_{Z\subset V}\text{”}\, k_{X\overline{V}} \ (V \text{ open}).\end{aligned}$$

One sets

$$\mathcal{H}om\,(G,F) = \alpha_X \mathcal{I}hom\,(G,F) \in \mathrm{Mod}(k_X).$$

One has

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(G,F) = \Gamma(X;\mathcal{H}om\,(G,F)).$$

The functors $\mathcal{I}hom$ and $\mathcal{H}om$ are left exact and admit right derived functors $R\mathcal{I}hom$ and $R\mathcal{H}om$.

Let $f\colon X \to Y$ be a morphism of topological spaces (Y satisfies the same assumptions as X). There are natural functors

$$\begin{aligned} f^{-1}\colon \mathrm{I}(k_Y)) &\longrightarrow \mathrm{I}(k_X) \\ f_*\colon \mathrm{I}(k_X) &\longrightarrow \mathrm{I}(k_Y) \\ f_{!!}\colon \mathrm{I}(k_X) &\longrightarrow \mathrm{I}(k_Y). \end{aligned}$$

The proper direct image functor is denoted by $f_{!!}$ instead of $f_!$ because it does not commute with ι, that is $\iota_Y f_! \neq f_{!!}\iota_X$ in general..

These functors induce derived functors, and moreover the functor $Rf_{!!}$ admits a right adjoint denoted by $f^!$:

$$\begin{aligned} f^{-1}\colon D^b\mathrm{I}(k_Y)) &\longrightarrow D^b(\mathrm{I}(k_X)), \\ Rf_*\colon D^b(\mathrm{I}(k_X)) &\longrightarrow D^b(\mathrm{I}(k_Y)), \\ Rf_{!!}\colon D^b(\mathrm{I}(k_X)) &\longrightarrow D^b(\mathrm{I}(k_Y)), \\ f^!\colon D^b(\mathrm{I}(k_Y)) &\longrightarrow D^b(\mathrm{I}(k_X)). \end{aligned}$$

Let $a_X\colon X \to \{\mathrm{pt}\}$ denote the canonical map. We also introduce a notation. We set

$$\begin{aligned} \mathrm{I}\Gamma(X;\cdot) &= a_{X*}(\cdot), \\ R\mathrm{I}\Gamma(X;\cdot) &= Ra_{X*}(\cdot). \end{aligned}$$

Ind-sheaves on real manifolds. — Let X be a real analytic manifold. Among all ind-sheaves, there are those which are ind-objects of the category of $\mathbb{R}$-constructible sheaves, and we shall encounter them in our applications.

We denote by $\mathbb{R}\text{-}\mathrm{C}^c(k_X)$ the full abelian subcategory of $\mathbb{R}\text{-}\mathrm{C}(k_X)$ consisting of $\mathbb{R}$-constructible sheaves with compact support. We set

$$\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X) = \mathrm{Ind}(\mathbb{R}\text{-}\mathrm{C}^c(k_X))$$

and denote by $D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$ the full subcategory of $D^b(\mathrm{I}(k_X))$ consisting of objects with cohomology in $\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X)$. (Note that in [**6**], $\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X)$ was denoted by $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(k_X)$.)

Theorem 2.1. — *The natural functor* $D^b(\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X)) \to D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$ *is an equivalence.*

There is an alternative construction of $\mathrm{I}\mathbb{R}{-}\mathrm{c}(k_X)$, using Grothendieck topologies. Denote by Op_X the category of open subsets of X (the morphisms $U \to V$ are the inclusions), and by by $\mathrm{Op}_{X_{sa}}$ its full subcategory consisting of open subanalytic subsets of X. One endows this category with a Grothendieck topology by deciding that a family $\{U_i\}_i$ in $\mathrm{Op}_{X_{sa}}$ is a covering of $U \in \mathrm{Op}_{X_{sa}}$ if for any compact subset K of X, there exists a finite subfamily which covers $U \cap K$. In other words, we consider families which are locally finite in X. One denotes by X_{sa} the site defined by this topology.

Sheaves on X_{sa} are easy to construct. Indeed, consider a presheaf F of k-vector spaces defined on the subcategory $\mathrm{Op}^c_{X_{sa}}$ of relatively compact open subanalytic subsets of X and assume that the sequence

$$0 \longrightarrow F(U \cup V) \longrightarrow F(U) \oplus F(V) \longrightarrow F(U \cap V)$$

is exact for any U and V in $\mathrm{Op}^c_{X_{sa}}$. Then there exists a unique sheaf $\widetilde{F}$ on X_{sa} such that $\widetilde{F}(U) \simeq F(U)$ for all $U \in \mathrm{Op}^c_{X_{sa}}$. Sheaves on X_{sa} define naturally ind-sheaves on X. Indeed:

Theorem 2.2. — *There is a natural equivalence of abelian categories*

$$\mathrm{I\!R}\text{-c}(k_X) \xrightarrow{\sim} \mathrm{Mod}(k_{X_{sa}}),$$

given by

$$\mathrm{I\!R}\text{-c}(k_X) \ni F \longmapsto \big(\mathrm{Op}^c_{X_{sa}} \ni U \mapsto \mathrm{Hom}_{\mathrm{I\!R}\text{-c}(k_X)}(k_{XU}, F)\big).$$

As usual, we denote by $\mathcal{C}^\infty_X$ the sheaf of complex-valued functions of class $\mathcal{C}^\infty$, by $\mathcal{D}b_X$ (resp. $\mathcal{B}_X$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions), and by $\mathcal{D}_X$ the sheaf of analytic finite-order differential operators.

Let U be an open subset of X. One sets $\mathcal{C}^\infty_X(U) = \Gamma(U; \mathcal{C}^\infty_X)$.

Definition 2.3. — Let $f \in \mathcal{C}^\infty_X(U)$. One says that f has *polynomial growth* at $p \in X$ if it satisfies the following condition. For a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,. \qquad (2.2)$$

It is obvious that f has polynomial growth at any point of U. We say that f is *tempered* at p if all its derivatives have polynomial growth at p. We say that f is tempered if it is tempered at any point.

For an open subanalytic set U in X, denote by $\mathcal{C}^{\infty,t}_X(U)$ the subspace of $\mathcal{C}^\infty_X(U)$ consisting of tempered functions. Denote by $\mathcal{D}b^t_X(U)$ the space of tempered distributions on U, defined by the exact sequence

$$0 \longrightarrow \Gamma_{X \setminus U}(X; \mathcal{D}b_X) \longrightarrow \Gamma(X; \mathcal{D}b_X) \longrightarrow \mathcal{D}b^t_X(U) \longrightarrow 0.$$

It follows from the results of Lojasiewicz [**8**] that $U \mapsto \mathcal{C}^\infty_X(U)$ and $U \mapsto \mathcal{D}b^t_X(U)$ are sheaves on the subanalytic site X_{sa}, hence define ind-sheaves.

Definition 2.4. — We call $\mathcal{C}^{\infty,t}_X$ (resp. $\mathcal{D}b^t_X$) the ind-sheaf of tempered $\mathcal{C}^\infty$-functions (resp. tempered distributions).

One can also define the ind-sheaf of Whitney $\mathcal{C}^\infty$-functions, but we shall not recall here its construction. These ind-sheaves are well-defined in the category $\mathrm{Mod}(\beta_X \mathcal{D}_X)$. Roughly speaking, it means that if P is a differential operator defined on the closure $\overline{U}$ of an open subset U, then it acts on $\mathcal{C}^{\infty,t}_X(U)$ and $\mathcal{D}b^t_X(U)$.

Let now X be a complex manifold. We denote by $\overline{X}$ the complex conjugate manifold and by $X^{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X\times\overline{X}$. We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic differential operators, not to be confused with $\mathcal{D}_{X^{\mathbb{R}}}$. We set

$$\mathcal{O}_X^t := R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X^{\mathbb{R}}}).$$

One can prove that the natural morphism

$$R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_{X^{\mathbb{R}}}) \longrightarrow R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X^{\mathbb{R}}})$$

is an isomorphism. One calls $\mathcal{O}_X^t$ the ind-sheaf of tempered holomorphic functions. One shall be aware that in fact, $\mathcal{O}_X^t$ is not an ind-sheaf but an object of the derived category $D^b(\mathrm{I}(\mathbb{C}_X))$, or better, of $D^b(\beta_X\mathcal{D}_X)$. It is not concentrated in degree 0 as soon as $\dim X > 1$.

Let $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. It follows from the construction of $\mathcal{O}_X^t$ that:

$$R\mathcal{H}om\,(G, \mathcal{O}_X^t) \simeq T\mathcal{H}om\,(G, \mathcal{O}_X),$$

where $T\mathcal{H}om\,(\cdot, \mathcal{O}_X)$ denotes the functor of temperate cohomology of [**2**] (see also [**5**] for a detailed construction and [**1**] for its microlocalization).

3. Complements of homological algebra

The results of this section are extracted from [**7**]. Let $\mathcal{C}$ denote an abelian category. We shall study some links between the derived category $D^b(\mathrm{Ind}(\mathcal{C}))$ and the category $\mathrm{Ind}(D^b(\mathcal{C}))$.

Definition 3.1. — Let $\mathcal{C}$ be an abelian category. A system of strict $\mathcal{U}$-generators in $\mathcal{C}$ is a $\mathcal{U}$-small family $\{G_i; i \in I\}$ of objects of $\mathcal{C}$ such that for all $X \in \mathcal{C}$ and all $i \in I$, the object $G_i^{\bigoplus \mathrm{Hom}_{\mathcal{C}}(G_i,X)}$ exists and for all $X \in \mathcal{C}$, there exists $i \in I$ such that the morphism $G_i^{\bigoplus \mathrm{Hom}_{\mathcal{C}}(G_i,X)} \to X$ is an epimorphism.

In this section, we shall shall always make the hypothesis

(3.1) $\qquad \mathcal{C}$ has enough injectives and a system of strict generators.

This implies in particular that $D^b(\mathcal{C})$ is a $\mathcal{U}$-category.

We define the functor $J\colon D^b(\mathrm{Ind}(\mathcal{C})) \to (D^b(\mathcal{C}))^\wedge$ by setting for $F \in D^b(\mathrm{Ind}(\mathcal{C}))$ and $G \in D^b(\mathcal{C})$

(3.2) $$J(F)(G) = \mathrm{Hom}_{D^b(\mathrm{Ind}(\mathcal{C}))}(G, F).$$

Theorem 3.2

(i) *The functor J takes its values in* $\mathrm{Ind}(D^b(\mathcal{C}))$.

(ii) *Consider a small and filtrant category I, integers $a \leqslant b$ and a functor $I \to C^{[a,b]}(\mathcal{C})$, $i \mapsto F_i$. If $F \in D^b(\mathrm{Ind}(\mathcal{C}))$, $F \simeq Q(\text{“}\varinjlim_i\text{”}\, F_i)$ and $G \in D^b(\mathcal{C})$, then:*

(a) $J(F) \simeq \text{“}\varinjlim_i\text{”}\, Q(F_i)$,

(b) $\operatorname{Hom}_{D^b(\operatorname{Ind}(\mathcal{C}))}(G, F) \simeq \varinjlim_i \operatorname{Hom}_{D^b(\mathcal{C})}(G, F_i)$.

(iii) *For each $k \in \mathbb{Z}$, the diagram below commutes.*

$$\begin{array}{ccc} D^b(\operatorname{Ind}(\mathcal{C})) & \xrightarrow{\quad J \quad} & \operatorname{Ind}(D^b(\mathcal{C})) \\ & {\scriptstyle H^k} \searrow \quad \swarrow {\scriptstyle IH^k} & \\ & \operatorname{Ind}(\mathcal{C}) & \end{array}$$

Lemma 3.3. — *Assume that $\mathcal{C}$ has finite homological dimension. Let $\varphi\colon X \to Y$ be a morphism in $\operatorname{Ind}(D^b(\mathcal{C}))$ and assume that φ induces an isomorphism*

$$IH^k(\varphi)\colon IH^k(X) \xrightarrow{\sim} IH^k(Y)$$

for every $k \in \mathbb{Z}$. Then φ is an isomorphism.

Theorem 3.4. — *Let $\psi\colon D^b(\operatorname{Ind}(\mathcal{C})) \to D^b(\operatorname{Ind}(\mathcal{C}'))$ be a triangulated functor which satisfies: if $F \in D^b(\operatorname{Ind}(\mathcal{C}))$, $F \simeq Q(\text{“}\varinjlim_i\text{”}\, F_i)$ with $F_i \in C^{[a,b]}(\mathcal{C})$, then $H^k\psi(F) \simeq \text{“}\varinjlim_i\text{”}\, H^k\psi(Q(F_i))$. Assume moreover that the homological dimension of $\mathcal{C}'$ is finite. Then there exists a unique functor $J\psi\colon \operatorname{Ind}(D^b(\mathcal{C})) \to \operatorname{Ind}(D^b(\mathcal{C}'))$ which commutes with “$\varinjlim$” and such that the diagram below commutes:*

$$\begin{array}{ccc} D^b(\operatorname{Ind}(\mathcal{C})) & \xrightarrow{\ \psi\ } & D^b(\operatorname{Ind}(\mathcal{C}')) \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \operatorname{Ind}(D^b(\mathcal{C})) & \xrightarrow{\ J\psi\ } & \operatorname{Ind}(D^b(\mathcal{C}')) . \end{array}$$

Remark 3.5. — The functor $J\colon D^b(\operatorname{Ind}(\mathcal{C})) \to \operatorname{Ind}(D^b(\mathcal{C}))$ is neither full nor faithful. Indeed, let $\mathcal{C} = \operatorname{Mod}^c(k_X)$ and let $F \in \operatorname{Mod}(k_X)$ considered as a full subcategory of $\mathrm{I}(k_X)$. Then

$$\operatorname{Hom}_{D^b(\mathrm{I}(k_X))}(k_X, F[n]) \simeq H^n(X; F).$$

On the other hand,

$$\operatorname{Hom}_{\operatorname{Ind}(D^b(\operatorname{Mod}^c(k_X)))}(J(k_X), J(F[n])) \simeq \varprojlim_{U \subset\subset X} H^n(U; F).$$

Let $\mathcal{T}$ be a full triangulated subcategory $D^b(\mathcal{C})$. One identifies $\operatorname{Ind}(\mathcal{T})$ with a full subcategory of $\operatorname{Ind}(D^b(\mathcal{C}))$.

Let $F \in D^b(\operatorname{Ind}(\mathcal{C}))$. Let us denote by $\mathcal{T}_F$ the category of arrows $G \to F$ in $D^b(\operatorname{Ind}(\mathcal{C}))$ with $G \in \mathcal{T}$. The category $\mathcal{T}_F$ is filtrant.

Lemma 3.6. — *For $F \in D^b(\operatorname{Ind}(\mathcal{C}))$, the conditions below are equivalent.*

(i) *$J(F) \in \operatorname{Ind}(\mathcal{T})$,*

(ii) *for each $k \in \mathbb{Z}$, one has $H^k(F) \simeq \text{“}\varinjlim_{G \to F \in \mathcal{T}_F}\text{”}\ H^k(G)$.*

Definition 3.7. — Let $\mathcal{T}$ be a full triangulated subcategory of $D^b(\mathcal{C})$. One denotes by $J^{-1}\mathrm{Ind}(\mathcal{T})$ the full subcategory of $D^b(\mathrm{Ind}(\mathcal{C}))$ consisting of objects $F \in D^b(\mathrm{Ind}(\mathcal{C}))$ such that $J(F) \in \mathrm{Ind}(\mathcal{T})$.

Proposition 3.8. — *The category $J^{-1}\mathrm{Ind}(\mathcal{T})$ is a triangulated subcategory of $D^b(\mathrm{Ind}(\mathcal{C}))$.*

We will apply these results to the category $\mathrm{I}(k_X) = \mathrm{Ind}(\mathrm{Mod}^c(k_X))$. Hence J is the functor:

$$J\colon D^b(\mathrm{I}(k_X)) \longrightarrow \mathrm{Ind}(D^b(\mathrm{Mod}^c(k_X))).$$

By the definition one has

$$J(F) \simeq \text{“}\varinjlim_{U \subset\subset X}\text{”}\, J(F_U) \quad \text{for any } F \in D^b(\mathrm{I}(k_X)).$$

As a corollary of Theorem 3.4, one gets:

Proposition 3.9. — *For $G \in D^b(k_X)$ and $F \in D^b(\mathrm{I}(k_X))$, assume that $J(F) \simeq \text{“}\varinjlim_i\text{”}\, J(F_i)$ with $F_i \in D^b(k_X)$. Then there are natural isomorphisms:*

$$J(G \otimes F) \simeq \text{“}\varinjlim_i\text{”}\, J(G \otimes F_i), \tag{3.3}$$

$$J(R\mathcal{I}hom\,(G, F)) \simeq \text{“}\varinjlim_i\text{”}\, J(R\mathcal{I}hom\,(G, F_i)). \tag{3.4}$$

4. Micro-support and regularity

Let γ be a closed convex proper cone in an affine space X. One denotes by γ° its polar cone,

$$\gamma^\circ = \{\xi \in X^*; \langle x, \xi\rangle \geqslant 0 \text{ for all } x \in \gamma\}.$$

Let $W \subset X$ be an open subset. We introduce the functor $\Phi_{\gamma,W}\colon D^b(\mathrm{I}(k_X)) \to D^b(\mathrm{I}(k_X))$ as follows. Denote by $q_1, q_2\colon X \times X \to X$ the first and second projections and denote by $s\colon X \times X \to X$ the map $(x, y) \mapsto x - y$. One sets

$$\Phi_{\gamma,W}(F) = Rq_{1!!}(k_{s^{-1}\gamma \cap q_1^{-1}W \cap q_2^{-1}W} \otimes q_2^{-1}F).$$

One writes Φ_γ instead of $\Phi_{\gamma,X}$. Define the functor $\Phi^-_{\gamma,W}$ by replacing the kernel $k_{s^{-1}\gamma \cap q_1^{-1}W \cap q_2^{-1}W}$ with the complex $k_{s^{-1}\gamma \cap q_1^{-1}W \cap q_2^{-1}W} \to k_{s^{-1}(0)}$ in which $k_{s^{-1}(0)}$ is situated in degree 0. We have a distinguished triangle in $D^b(\mathrm{I}(k_X))$

$$\Phi_{\gamma,W}(F) \longrightarrow F \longrightarrow \Phi^-_{\gamma,W}(F) \xrightarrow{+1}.$$

Note that if $F \in D^b(k_X)$, then

$$\begin{cases} \mathrm{supp}(\Phi_{\gamma,W}(F)) \subset \overline{W}, \\ \Phi_\gamma(F) \to F \text{ is an isomorphism on } X \times \mathrm{Int}\gamma^\circ, \\ SS(\Phi_\gamma(F)) \subset X \times \gamma^\circ, \\ SS(\Phi^-_{\gamma,W}(F)) \bigcap W \times \mathrm{Int}\gamma^\circ = \varnothing. \end{cases}$$

Lemma 4.1. — *Let $F \in D^b(\mathrm{I}(k_X))$ and let $p \in T^*X$. The conditions* (1a)–(4b) *below are all equivalent. Moreover, if $F \in D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(k_X))$, these conditions are equivalent to* (5a).

(1a) *Assume that for a small and filtrant category I, integers $a \leqslant b$ and a functor $I \to C^{[a,b]}(\mathrm{Mod}(k_X))$, $i \mapsto F_i$ one has $F \simeq Q(\text{“}\varinjlim_{i\in I}\text{”}\, F_i)$. Then there exists a conic open neighborhood U of p in T^*X such that for any $i \in I$ there exists a morphism $i \to j$ in I which induces the zero-morphism $0 : F_i \to F_j$ in $D^b(k_X; U)$.*

(1b) *There exist a conic open neighborhood U of p in T^*X, a small and filtrant category I, integers $a \leqslant b$ and a functor $I \to C^{[a,b]}(\mathrm{Mod}(k_X))$, $i \mapsto F_i$, such that $SS(F_i) \cap U = \varnothing$ and $F \simeq Q(\text{“}\varinjlim\text{”}\, F_i)$ in a neighborhood of $\pi(p)$.*

(2a) *Assume that for a small and filtrant category I, integers $a \leqslant b$ and a functor $I \to D^{[a,b]}(k_X)$, $i \mapsto F_i$ one has $J(F) \simeq \text{“}\varinjlim_{i\in I}\text{”}\, J(F_i)$. Then there exists a conic open neighborhood U of p in T^*X such that for any $i \in I$ there exists a morphism $i \to j$ in I which induces the zero-morphism $0 : F_i \to F_j$ in $D^b(k_X; U)$.*

(2b) *There exist a conic open neighborhood U of p in T^*X, a small and filtrant category I, integers $a \leqslant b$, a functor $I \to D^b(k_X)$, $i \mapsto F_i$ and F' isomorphic to F in neighborhood of $\pi(p)$ such that $SS(F_i) \cap U = \varnothing$ and $J(F') \simeq \text{“}\varinjlim_{i}\text{”}\, J(F_i)$.*

(3a) *There exists a conic open neighborhood U of p in T^*X such that for any $G \in D^b(k_X)$ with $\mathrm{supp}(G) \subset\subset \pi(U)$, $SS(G) \subset U \cup T^*_X X$, one has $\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(G, F) = 0$.*

(3b) *There exists a conic open neighborhood U of p in T^*X such that for any $G \in D^b(k_X)$ with $\mathrm{supp}(G) \subset\subset \pi(U)$, $SS(G) \subset U^a \cup T^*_X X$, one has $R\mathrm{I}\Gamma(X; G \otimes F) = 0$.*

Assume now that X is an affine space and let $p = (x_0; \xi_0)$.

(4a) *There exist a relatively compact open neighborhood W of x_0 and a closed convex proper cone γ with $\xi_0 \in \mathrm{Int}\gamma^\circ$ such that $\Phi_{\gamma,W}(F) \simeq 0$.*

(4b) *There exist $F' \in D^b(\mathrm{I}(k_X))$ with $F' \simeq F$ in a neighborhood of x_0 and F' has compact support, and a closed convex proper cone γ as in* (4a) *such that $\Phi_\gamma(F') \simeq 0$ in a neighborhood of x_0.*

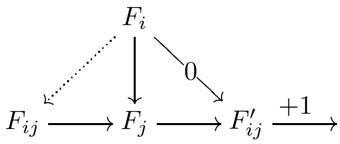
(5a) *Same condition as* (3a) *with $G \in D^b_{\mathbb{R}-c}(k_X)$.*

Proof. — The plan of the proof is as follows:

$$\begin{array}{ccccc}
(2a) & \Longleftarrow & (3a) & \Longleftarrow & (2b) \\
\Downarrow & & \Downarrow & & \Uparrow \\
(1a) & & (5a) & & (1b) \\
\Downarrow & \swarrow & & & \Uparrow \\
(3b) & \Longrightarrow & (4a) & \Longrightarrow & (4b)
\end{array}$$

(2a) $\Rightarrow$ (1a) follows from $F \simeq Q(\text{“}\varinjlim_i\text{”}\, F_i) \Rightarrow J(F) \simeq \text{“}\varinjlim_i\text{”}\, J(Q(F_i))$.

(1a) $\Rightarrow$ (3b). Let $F \simeq Q(\text{“}\varinjlim_i\text{”}\, F_i)$ and let $i \in I$. There exists $i \to j$ such that the morphism $F_i \to F_j$ in $D^b(k_X)$ is zero in $D^b(k_X; U)$. Hence, there exists a morphism $F_j \to F'_{ij}$ in $D^b(k_X)$ which is an isomorphism on U and such that the composition $F_i \to F_j \to F'_{ij}$ is the zero-morphism in $D^b(k_X)$. Consider the commutative diagram in which the row on the bottom is a distinguished triangle in $D^b(k_X)$ and $SS(F_{ij}) \cap U = \varnothing$:



Since the arrow $F_i \to F'_{ij}$ is zero, the dotted arrow may be completed, making the diagram commutative. Hence, we may assume from the beginning that for any $i \in I$ there exists $i \to j$ such that the morphism $F_i \to F_j$ factorizes as $F_i \to F_{ij} \to F_j$ with $SS(F_{ij}) \cap U = \varnothing$.

We may assume X is affine and $U = W \times \lambda$ where W is open and relatively compact and λ is an open convex cone. Then $SS(G \otimes F_{ij}) \cap U = \varnothing$, and the sheaf $G \otimes F_{ij}$ has compact support. Hence, $R\Gamma(X; G \otimes F_{ij}) \simeq 0$ which implies $H^j R\mathrm{I}\Gamma(X; G \otimes F) \simeq \text{“}\varinjlim_i\text{”}\, H^j R\Gamma(X; G \otimes F_i) \simeq 0$ for all j. We conclude therefore $R\mathrm{I}\Gamma(X; G \otimes F) \simeq 0$.

(3b) $\Rightarrow$ (4a). Let $F = Q(\text{“}\varinjlim_i\text{”}\, F_i)$, with $F_i \in C^{[a,b]}(\mathrm{Mod}(k_X))$. Set

$$H_\varepsilon = \{x; \langle x - x_0; \xi_0 \rangle > -\varepsilon\}$$

and let $K \subset\subset \pi(U)$ be a compact neighborhood of x_0. Then there exist an open convex cone γ and an open neighborhood W of x_0 satisfying the following conditions:

$$\begin{cases} W \subset H_\varepsilon \cap K, \\ (x + \gamma) \cap H_\varepsilon \subset W \text{ for all } x \in W, \\ \overline{W} \times \gamma^\circ \subset U \cup T^*_X X. \end{cases}$$

Set

$$G_x = k_{(x+\gamma^a)\cap H_\varepsilon}, \quad G = \bigoplus_{x \in W} G_x.$$

Since $\mathrm{supp}(G) \subset\subset \pi(U)$ and $SS(G) \subset \overline{W} \times \gamma^{\circ a}$, we get by the hypothesis:

$$\text{“}\varinjlim_i\text{”}\, H^k R\Gamma(X; G \otimes F_i) \simeq 0.$$

Hence,

$$\text{“}\varinjlim_i\text{”} \Big(\bigoplus_{x \in W} H^k R\Gamma(X; G_x \otimes F_i) \Big) \simeq 0.$$

Hence one obtains:

$$\begin{cases}\text{for any } i \in I, \text{ there exists } i \to j \text{ such that } H^k R\Gamma(X; G_x \otimes F_i) \to \\ H^k R\Gamma(X; G_x \otimes F_j) \text{ is zero for any } x \in W \text{ and any } k \in \mathbb{Z}.\end{cases}$$

On the other-hand,

$$H^k(\Phi_{\gamma,W}(F_i))_x \simeq H^k R\Gamma(X; G_x \otimes F_i).$$

Therefore, for any $i \in I$ there exists $i \to j$ such that for any $k \in \mathbb{Z}$, the morphism $H^k(\Phi_{\gamma,W}(F_i)) \to H^k(\Phi_{\gamma,W}(F_j))$ is the zero morphism, and this implies

$$H^k(\Phi_{\gamma,W}(F)) \simeq \text{“}\varinjlim_i\text{”}\, H^k\Phi_{\gamma,W}(F_i) \simeq 0.$$

This gives the desired result: $\Phi_{\gamma,W}(F) = 0$.

(4a) $\Rightarrow$ (4b) is obvious by taking F_W as F'.

(4b) $\Rightarrow$ (1b). Let W be an open relatively compact neighborhood of x_0 such that $F|_W \simeq F'|_W$ and $\Phi_\gamma(F')|_W \simeq 0$.

Then one has a distinguished triangle:

$$Rq_{1!!}(k_{s^{-1}(\gamma\smallsetminus\{0\})\cap q_1^{-1}W} \otimes q_2^{-1}F') \longrightarrow \Phi_\gamma(F')_W \longrightarrow F'_W \xrightarrow{+1},$$

and hence one obtains $Rq_{1!!}(k_{s^{-1}(\gamma\smallsetminus\{0\})\cap q_1^{-1}W}[1] \otimes q_2^{-1}F') \simeq F'_W$. Let $F' = Q(\text{“}\varinjlim_i\text{”}\, F_i)$ with $F_i \in C^{[a,b]}(\mathrm{Mod}(k_X))$, and take a finite injective resolution I of $k_{s^{-1}(\gamma\smallsetminus\{0\})\cap q_1^{-1}W}[1]$. Since $I \otimes F_i$ is a finite complex of soft sheaves, $Rq_{1!}(k_{s^{-1}(\gamma\smallsetminus\{0\})\cap q_1^{-1}W}[1] \otimes q_2^{-1}F_i)$ is represented by $F'_i := q_{1!}(I \otimes q_2^{-1}F_i)$. Hence one has

$$Rq_{1!!}(k_{s^{-1}(\gamma\smallsetminus\{0\})\cap q_1^{-1}W} \otimes q_2^{-1}F') \simeq Q(\text{“}\varinjlim_i\text{”}\, F'_i).$$

Since $SS(F'_i) \cap W \times \mathrm{Int}\gamma^\circ = \varnothing$, we obtain the desired result.

(1b) $\Rightarrow$ (2b) is obvious.

(2b) $\Rightarrow$ (3a). Let $J(F) \simeq \text{“}\varinjlim_i\text{”}\, J(F_i)$. If $G \in D^b(k_X)$, we get the isomorphism:

$$\mathrm{Hom}_{D^b(\mathrm{I}(k_X))}(G, F) \simeq \varinjlim_i \mathrm{Hom}_{D^b(k_X)}(G, F_i).$$

We may assume that X is affine and $U = W \times \lambda$ where W is open and λ is an open convex cone. Then the micro-support of $R\mathcal{H}om(G, F_i)$ is contained in $SS(F_i) + \overline{\lambda}^a$ and this set does not intersect $X \times \lambda$. Since $R\mathcal{H}om(G, F_i)$ has compact support, $\mathrm{Hom}(G, F_i)$ is zero.

(3a) $\Rightarrow$ (2a). We may assume that X is affine, $p = (x_0; \xi_0)$ and $U = X' \times \mathrm{Int}\gamma^\circ$, with $\xi_0 \in \mathrm{Int}\gamma^\circ$ for a neighborhood X' of x_0. Let V be an open neighborhood of x_0 and let $W = \{x; \langle x - x_0; \xi_0\rangle > -\varepsilon\}$. Then by taking V and ε small enough, the sheaf $\Phi_\gamma(H_W)_V$ satisfies the condition in (3a) for any $H \in D^b(k_X)$. Let $J(F) = \text{“}\varinjlim_i\text{”}\, J(F_i)$.

Then $\varinjlim_i \operatorname{Hom}_{D^b(k_X)}(G, F_i) \simeq 0$ for any $G = \Phi_\gamma(H_W)_V$. Let $i \in I$ and choose $H = F_i$. There exists $i \to j$ such that the composition $(\Phi_\gamma(F_{iW}))_V \to F_i \to F_j$ is zero. The morphism $(\Phi_\gamma((F_{iW}))_V \to F_i$ is an isomorphism on $U' := (V \cap W) \times \operatorname{Int}\gamma^\circ$. Therefore, $F_i \to F_j$ is zero in $D^b(k_X; U')$.

(3a) $\Rightarrow$ (5a) is obvious.

(5a) $\Rightarrow$ (3b). (Assuming $F \in D^b_{\mathrm{I}\,\mathbb{R}-c}(\mathrm{I}(k_X))$.) Let (2a-rc) denote the condition (2a) in which one asks moreover that $F_i \in D^{[a,b]}_{\mathbb{R}-c}(k_X)$. Define similarly (1a-rc). Then the same proof of (3a) $\Rightarrow$ (2a) $\Rightarrow$ (1a) $\Rightarrow$ (3b) can be applied to show (5a) $\Rightarrow$ (2a-rc) $\Rightarrow$ (1a-rc) $\Rightarrow$ (3b). q.e.d.

Definition 4.2. — Let $F \in D^b(\mathrm{I}(k_X))$. The micro-support of F, denoted by $SS(F)$, is the closed conic subset of T^*X whose complementary is the set of points $p \in T^*X$ such that one of the equivalent conditions in Lemma 4.1 is satisfied.

Proposition 4.3

(i) *For* $F \in D^b(\mathrm{I}(k_X))$, *one has* $SS(F) \cap T^*_X X = \operatorname{supp}(F)$.

(ii) *Let* $F \in D^b(k_X)$. *Then* $SS(\iota_X F) = SS(F)$.

(iii) *Let* $F \in D^b(\mathrm{I}(k_X))$. *Then* $SS(\alpha_X F) \subset SS(F)$.

(iv) *Let* $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ *be a distinguished triangle in* $D^b(\mathrm{I}(k_X))$. *Then* $SS(F_i) \subset SS(F_j) \cup SS(F_k)$ *if* $\{i, j, k\} = \{1, 2, 3\}$.

Proof

(i) $\operatorname{supp}(F) \subset SS(F)$ follows for example from (1b) of Lemma 4.1. The other inclusion is obvious.

(ii) The inclusion $SS(F) \subset SS(\iota_X F)$ follows from (2a) since $J(F)$ is "$\varinjlim$" F. The converse inclusion follows from (1b).

(iii) is obvious, using condition (3b).

(iv) is obvious by (3b). q.e.d.

Remark 4.4. — Let $F \in D^b(\mathrm{I}(k_X))$. It is possible to define another micro-support of F, denoted by $SS_0(F)$, as follows. Let $p \in T^*X$. Then $p \notin SS_0(F)$ if there exist a small and filtrant category I, integers $a \leqslant b$ and a functor $I \to C^{[a,b]}(\mathrm{Mod}(k_X))$, $i \mapsto F_i$, such that $p \notin SS(F_i)$ and $F \simeq Q(\text{“}\varinjlim_i\text{”}\, F_i)$ in a neighborhood of $\pi(p)$. Clearly, $SS_0(F) \subset SS(F)$. This inclusion is strict in general. (As an example, consider the ind-sheaf given in Proposition 7.3 below.) One checks easily that Proposition 4.3 (iv) still holds when $SS(F)$ is replaced with $SS_0(F)$.

Definition 4.5. — Let $\Lambda_i, i \in I$ be a family of closed conic subsets of T^*X, indexed by the objects of a small and filtrant category I. One sets

$$\mathop{lim}_{i} \Lambda_i = \bigcap_{J \subset I} \overline{\bigcup_{j \in J} \Lambda_j}, \quad \text{where } J \text{ ranges over the family of cofinal subcategories of } I.$$

In other words, $p \in T^*X$ does not belong to $\underset{i}{lim}\, \Lambda_i$ if there exists an open neighborhood U of p and a cofinal subset J of I such that $\Lambda_j \cap U = \varnothing$ for every $j \in J$.

It follows immediately from the definition that if $J(F) \simeq$ "$\varinjlim_i$" $J(F_i)$, then

$$SS(F) \subset \underset{i}{lim}\, SS(F_i). \tag{4.1}$$

It follows from Proposition 3.9 that if $G \in D^b(k_X)$, one has the inclusions

$$\begin{cases} SS(G \otimes F) \subset \underset{i}{lim}\,(SS(G) \widehat{+} SS(F_i)), \\ SS(R\mathcal{I}hom\,(G,F)) \subset \underset{i}{lim}\,(SS(G)^a \widehat{+} SS(F_i)). \end{cases} \tag{4.2}$$

Example 4.6. — Let $X = \mathbb{R}^2$ endowed with coordinates (x,y) and denote by $(x,y;\xi,\eta)$ the associated coordinates on T^*X. Let

$$\begin{aligned} Y &= \{(x,y); y = 0\}, \\ U &= \{(x,y); x^2 < y\}, \\ Z_\varepsilon &= \{(x,y); x^2 < y \leqslant \varepsilon^2\}. \end{aligned}$$

Set $F_\varepsilon = k_{Z_\varepsilon}$ and $F = k_U \otimes \beta_X(k_{\{0\}}) \simeq$ "$\varinjlim_\varepsilon$" F_ε. Then

$$\begin{aligned} SS(k_Y) &= T^*_Y X = \{(x,y;\xi,\eta)\,;\, y = \xi = 0\}, \\ SS(F_\varepsilon) &= \{(x,y;0,0)\,;\, x^2 \leqslant y \leqslant \varepsilon^2\} \\ &\quad \bigcup \{(x,y;\xi,\eta)\,;\, y = x^2, |x| \leqslant \varepsilon,\ \xi = -2x\eta,\ \eta \leqslant 0\} \\ &\quad \bigcup \{(x,y;\xi,\eta)\,;\, y = \varepsilon^2, |x| \leqslant \varepsilon,\ \xi = 0,\ \eta \leqslant 0\} \\ &\quad \bigcup \{(\pm\varepsilon,\varepsilon^2;\xi,\eta)\,;\, 0 \leqslant \pm\xi \leqslant -2\varepsilon\eta,\ \eta \leqslant 0\}, \\ SS(F) &= \{(x,y;\xi,\eta); x = y = \xi = 0, \eta \leqslant 0\}. \end{aligned}$$

On the other-hand, one has

$$\begin{aligned} &SS(F) = \underset{\varepsilon}{lim}\, SS(F_\varepsilon), \\ &R\mathcal{H}om\,(k_Y, F) \simeq k_{\{0\}}\,[-2]\,, \\ &\underset{\varepsilon}{lim}\,(T^*_Y X \widehat{+} SS(F_\varepsilon)) = T^*_{\{0\}}X, \\ &T^*_Y X \widehat{+} SS(F) = \{(x,y;\xi,\eta); x = y = \xi = 0\} \\ &\qquad\qquad\qquad\qquad \subsetneqq SS(R\mathcal{H}om\,(k_Y, F)). \end{aligned}$$

Note that $SS(F)$ is not involutive.

Recall that subanalytic isotropic subsets of T^*X are defined in [**4**]. Let us say for short that a conic locally closed subset Λ of T^*X is isotropic if Λ is contained in a conic locally closed subanalytic isotropic subset.

Definition 4.7

(i) We denote by $D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$ the full triangulated subcategory of $D^b_{\mathrm{I}\,\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$ consisting of objects F such that $SS(F)$ is isotropic. We call an object of this category a weakly $\mathbb{R}$-constructible ind-sheaf.

(ii) We denote by $D^b_{\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$ the full triangulated subcategory of $D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$ consisting of objects F such that $R\mathcal{H}om\,(G, F) \in D^b_{\mathbb{R}\mathrm{-c}}(k_X)$ for any $G \in D^b_{\mathbb{R}\mathrm{-c}}(k_X)$. We call an object of this category an $\mathbb{R}$-constructible ind-sheaf.

Note that the functor α_X induces functors

$$\alpha_X\colon D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X)) \longrightarrow D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(k_X),$$
$$\alpha_X\colon D^b_{\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X)) \longrightarrow D^b_{\mathbb{R}\mathrm{-c}}(k_X).$$

The last property follows from $\alpha_X(F) = R\mathcal{H}om\,(\mathbb{C}_X, F)$.

Conjecture 4.8. — *Let $F \in D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$ and let $G \in D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(k_X)$. Then $R\mathcal{I}hom\,(G, F)$ and $G \otimes F$ belong to $D^b_{\mathrm{w-}\mathbb{R}\mathrm{-c}}(\mathrm{I}(k_X))$.*

Example 4.6 shows that the knowledge of $SS(F)$ and $SS(G)$ does not allow us to estimate the micro-support of $R\mathcal{H}om\,(G, F)$ by the one for sheaves, and that is one reason for the definition below.

Definition 4.9. — Let $F \in D^b(\mathrm{I}(k_X))$.

(i) Let $S \subset T^*X$ be a locally closed conic subset and let $p \in T^*X$. We say that F is regular along S at p if there exist F' isomorphic to F in a neighborhood of $\pi(p)$, an open neighborhood U of p with $S \cap U$ closed in U, a small and filtrant category I and a functor $I \to D^{[a,b]}(k_X), i \mapsto F_i$ such that $J(F') \simeq$ "$\varinjlim\limits_i$" $J(F_i)$ and $SS(F_i) \cap U \subset S$.

(ii) If U is an open subset of T^*X and F is regular along S at each $p \in U$, we say that F is regular along S on U.

(iii) Let $p \in T^*X$. We say that F is regular at p if F is regular along $SS(F)$ at p. If F is regular at each $p \in SS(F)$, we say that F is regular.

(iv) We denote by $SS_{\mathrm{reg}}(F)$ the conic open subset of $SS(F)$ consisting of points p such that F is regular at p, and we set

$$SS_{\mathrm{irr}}(F) = SS(F) \setminus SS_{\mathrm{reg}}(F).$$

Note that $SS_{\mathrm{irr}}(F) = SS(F)$ for F in Example 4.6.

Proposition 4.10

(i) *Let $F \in D^b(\mathrm{I}(k_X))$. Then F is regular along any locally closed set S at each $p \notin SS(F)$.*

(ii) *Let $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ be a distinguished triangle in $D^b(\mathrm{I}(k_X))$. If F_j and F_k are regular along S, so is F_i for $\{i, j, k\} = \{1, 2, 3\}$.*

(iii) *Let $F \in D^b(k_X)$. Then $\iota_X F$ is regular.*

Proof. — (i) and (iii) are obvious and the proof of (ii) is similar to that of Proposition 4.3 (iv). q.e.d.

It is possible to localize the category $D^b(\mathrm{I}(k_X))$ with respect to the micro-support, exactly as for usual sheaves.

Let V be a subset of T^*X and let $\Omega = T^*X \setminus V$. We shall denote by $D^b_V(k_X)$ the full triangulated subcategory of $D^b(k_X)$ consisting of objects F such that $SS(F) \subset V$, and by $D^b(k_X;\Omega)$ the localization of $D^b(k_X)$ by $D^b_V(k_X)$.

Similarly, we denote by $D^b_V(\mathrm{I}(k_X))$ the full triangulated subcategory of $D^b(\mathrm{I}(k_X))$ consisting of objects F such that $SS(F) \subset V$.

Definition 4.11. — One sets

$$D^b(\mathrm{I}(k_X;\Omega)) = D^b(\mathrm{I}(k_X))/D^b_V(\mathrm{I}(k_X)),$$

the localization of $D^b(\mathrm{I}(k_X))$ by $D^b_V(\mathrm{I}(k_X))$.

Let F_1 and F_2 are two objects of $D^b(\mathrm{I}(k_X))$ whose images in $D^b(\mathrm{I}(k_X;\Omega))$ are isomorphic. There exist a third object $F_3 \in D^b(\mathrm{I}(k_X;\Omega))$ and distinguished triangles in $D^b(\mathrm{I}(k_X))$: $F_i \to F_3 \to G_i \xrightarrow{+1}$ $(i = 1,2)$ such that $SS(G_i) \cap \Omega = \varnothing$. It follows that $SS(F_1) \cap \Omega = SS(F_3) \cap \Omega = SS(F_2) \cap \Omega$.

Therefore if $F \in D^b(\mathrm{I}(k_X;\Omega))$, the subsets $SS(F)$ and $SS_{\mathrm{irr}}(F)$ of Ω are well-defined.

5. Invariance by contact transformations

It is possible to define contact transformations on ind-sheaves. We shall follow the notations in [**4**] Chapter VII.

We denote by p_1 and p_2 the first and second projections defined on $T^*(X \times Y) \simeq T^*X \times T^*Y$, and we denote by p_2^a the composition of p_2 with the antipodal map on T^*Y.

We denote by $r\colon X \times Y \to Y \times X$ the canonical map and we keep the same notation to denote its inverse.

By a kernel K on $X \times Y$ we mean an object of $D^b(k_{X\times Y})$. To a kernel K one associates the kernel on $Y \times X$

$$K^* := r_* R\mathcal{H}om\,(K, \omega_{X\times Y/Y}).$$

One defines the functor

$$\begin{aligned}(5.1)\qquad \Phi_K\colon D^b(k_Y) &\to D^b(k_X)\\ G &\mapsto Rq_{1!}(K \otimes q_2^{-1}G).\end{aligned}$$

Consider another manifold Z and a kernel L on $Y \times Z$. One defines the projection q_{12} from $X \times Y \times Z$ to $X \times Y$, and similarly with q_{23}, q_{13}.

One sets

$$K \circ L = Rq_{13!}(q_{12}^{-1}K \otimes q_{23}^{-1}L). \tag{5.2}$$

Choosing $Z = \{\mathrm{pt}\}$, one has $\Phi_K(G) = K \circ G$ for $G \in D^b(k_Y)$.

Let Ω_X and Ω_Y be two conic open subsets of T^*X and T^*Y, respectively. One denotes by $N(\Omega_X, \Omega_Y)$ the full subcategory of $D^b(k_{X\times Y}; \Omega_X \times T^*Y)$ of objects K satisfying;

$$\begin{cases} SS(K) \cap (\Omega_X \times T^*Y) \subset \Omega_X \times \Omega_Y^a, \\ p_1 \colon SS(K) \cap (\Omega_X \times T^*Y) \to \Omega_X \text{ is proper.} \end{cases} \tag{5.3}$$

Let us recall some results of loc. cit.

(i) Let $K \in N(\Omega_X, \Omega_Y)$. Then the functor Φ_K induces a well-defined functor: $\Phi_K^\mu \colon D^b(k_Y; \Omega_Y) \to D^b(k_X; \Omega_X)$.

(ii) Let $L \in N(\Omega_Y, \Omega_Z)$. Then $K \circ L \in N(\Omega_X, \Omega_Z)$. Moreover, the two functors $\Phi_{K\circ L}^\mu$ and $\Phi_K^\mu \circ \Phi_L^\mu$ from $D^b(k_Z; \Omega_Z)$ to $D^b(k_X; \Omega_X)$ are isomorphic.

We construct the functor analogous to the functor Φ_K for ind-sheaves by defining

$$\begin{aligned} \widetilde{\Phi}_K \colon D^b(\mathrm{I}(k_Y)) &\to D^b(\mathrm{I}(k_X)) \\ G &\mapsto Rq_{1!!}(K \otimes q_2^{-1}G). \end{aligned} \tag{5.4}$$

Applying Theorem 3.4, we get:

Lemma 5.1. — *Let $G \in D^b(\mathrm{I}(k_Y))$ and assume that $J(G) \simeq$ "$\varinjlim_i$" $J(G_i)$, with I small and filtrant and $G_i \in D^b(k_Y)$. Then $J(\widetilde{\Phi}_K(G)) \simeq$ "$\varinjlim_i$" $J(\Phi_K(G_i))$.*

Now assume that $\dim X = \dim Y$ and that there exists a smooth conic Lagrangian submanifold $\Lambda \subset \Omega_X \times \Omega_Y^a$ such that $p_1 \colon \Lambda \to \Omega_X$ and $p_2^a \colon \Lambda \to \Omega_Y$ are isomorphisms. In other words, Λ is the graph of a homogeneous symplectic isomorphism $\chi \colon \Omega_Y \xrightarrow{\sim} \Omega_X$.

Let K be a kernel satisfying the assumptions of Theorem 7.2.1 of loc. cit., that is:

$$\begin{cases} K \text{ is cohomologically constructible,} \\ (p_1^{-1}(\Omega_X) \cup p_2^{a\,-1}(\Omega_Y)) \cap SS(K) \subset \Lambda, \\ k_\Lambda \xrightarrow{\sim} \mu hom(K, K) \quad \text{on } \Omega_X \times \Omega_Y^a. \end{cases} \tag{5.5}$$

Theorem 5.2. — *Assume* (5.5).

(i) *The functor $\widetilde{\Phi}_K$ induces a well-defined functor:*

$$\widetilde{\Phi}_K^\mu \colon D^b(\mathrm{I}(k_Y; \Omega_Y)) \longrightarrow D^b(\mathrm{I}(k_X; \Omega_X)).$$

Similarly, the functor $\widetilde{\Phi}_{K^*}$ induces a well-defined functor:

$$\widetilde{\Phi}_{K^*}^\mu \colon D^b(\mathrm{I}(k_X; \Omega_X)) \longrightarrow D^b(\mathrm{I}(k_Y; \Omega_Y)).$$

(ii) *The functor*

$$\widetilde{\Phi}^{\mu}_{K}\colon D^b(\mathrm{I}(k_Y;\Omega_Y)) \longrightarrow D^b(\mathrm{I}(k_X;\Omega_X))$$

and the functor

$$\widetilde{\Phi}^{\mu}_{K^*}\colon D^b(\mathrm{I}(k_X;\Omega_X)) \longrightarrow D^b(\mathrm{I}(k_Y;\Omega_Y))$$

are equivalences of categories inverse one to each other.

(iii) *If $G \in D^b(\mathrm{I}(k_Y))$, then $SS(\widetilde{\Phi}_K(G)) \cap \Omega_X = \chi(SS(G) \cap \Omega_Y)$.*

(iv) *If G is regular at $p \in \Omega_Y$, then $\widetilde{\Phi}_K(G)$ is regular at $\chi(p) \in \Omega_X$. In other words, $SS_{\mathrm{irr}}(\Phi_K(G)) \cap \Omega_X = \chi(SS_{\mathrm{irr}}(G) \cap \Omega_Y)$.*

Proof

(i) Let $G \in D^b(\mathrm{I}(k_Y))$ and assume that $SS(G) \cap \Omega_Y = \varnothing$. Let us prove that $SS(\widetilde{\Phi}_K(G)) \cap \Omega_X = \varnothing$. Let $p_X \in \Omega_X$ and let $p_Y = \chi^{-1}(p_X)$. There exist an open neighborhood U_Y of p_Y in Ω_Y and an inductive system such that $J(G) \simeq$ "$\varinjlim_{i\in I}$" $J(G_i)$, and for any $i \in I$ there exists $i \to j$ such that the morphism $G_i \to G_j$ is zero in $D^b(k_Y;U_Y)$. Applying Lemma 5.1 we find that $J(\widetilde{\Phi}_K(G)) \simeq$ "$\varinjlim_{i}$" $J(\Phi_K(G_i))$. Since the morphism $\Phi_K(G_i) \to \Phi_K(G_j)$ is zero in $D^b(k_X;U_X)$, the result follows.

(ii) One has the isomorphism $K \circ K^* \simeq k_{\Delta_X}$ in $N(\Omega_X,\Omega_X)$ and the isomorphism $K^* \circ K \simeq k_{\Delta_Y}$ in $N(\Omega_Y,\Omega_Y)$. Hence, it is enough to remark that

$$\widetilde{\Phi}^{\mu}_{K} \circ \widetilde{\Phi}^{\mu}_{K^*} \simeq \widetilde{\Phi}^{\mu}_{K\circ K^*}, \tag{5.6}$$

which follows from the fact that the two functors $\widetilde{\Phi}_K \circ \widetilde{\Phi}_{K^*}$ and $\widetilde{\Phi}_{K\circ K^*}$, from $D^b(\mathrm{I}(k_X))$ to $D^b(\mathrm{I}(k_X))$ are isomorphic.

(iii) For an open subset $U_Y \subset \Omega_Y$, set $U_X = \chi(U_Y)$. Then $K \in N(U_X,U_Y)$ and K satisfies (5.5) with Ω replaced with U. Let $G \in D^b(\mathrm{I}(k_Y))$ with $SS(G) = \varnothing$ in a neighborhood of $p_Y \in \Omega_Y$. By the proof of (i), $SS(\widetilde{\Phi}_K(G)) = \varnothing$ in a neighborhood of $\chi(p_Y)$.

(iv) The proof is similar to that of (iii). q.e.d.

6. Ind-sheaves and $\mathcal{D}$-modules

Let now X be a complex manifold and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We set for short

$$Sol(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X),$$
$$Sol^t(\mathcal{M}) = R\mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M},\mathcal{O}^t_X).$$

Theorem 6.1. — *One has*

$$SS(Sol^t(\mathcal{M})) = \mathrm{char}(\mathcal{M}).$$

Proof
(i) The inclusion $\mathrm{char}(\mathcal{M}) \subset SS(Sol^t(\mathcal{M}))$ follows from

$$SS(Sol(\mathcal{M})) = \mathrm{char}(\mathcal{M}), \quad \alpha_X(Sol^t(\mathcal{M})) \simeq Sol(\mathcal{M}).$$

and Proposition 4.3 (ii).

(ii) Let us prove the converse inclusion using condition (5a) of Lemma 4.1. Assume that $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ satisfy $SS(G) \cap \mathrm{char}(\mathcal{M}) \subset T^*_X X$. One has the morphisms

$$\begin{aligned} R\mathcal{H}om\,(G, R\mathcal{I}hom_{\beta_X \mathcal{D}_X}(\beta_X \mathcal{M}, \mathcal{O}^t_X)) &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T\mathcal{H}om\,(G, \mathcal{O}_X)) \\ &\to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om\,(G, \mathcal{O}_X)). \end{aligned}$$

It follows from [**1**, Corollary 4.2.5] that the second morphism is an isomorphism. Hence the result follows from $SS(Sol(\mathcal{M})) = \mathrm{char}(\mathcal{M})$ and Lemma 4.1 (5a). q.e.d.

The following conjecture is a consequence of Conjecture 4.8.

Conjecture 6.2. — *If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, then $Sol^t(\mathcal{M})$ belongs to $D^b_{\mathbb{R}-\mathrm{c}}(\mathrm{I}(\mathbb{C}_X))$.*

Theorem 6.3. — *If $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, then $Sol^t(\mathcal{M}) \to Sol(\mathcal{M})$ is an isomorphism.*

Proof. — This is a reformulation of a result of [**2**] which asserts that for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, the natural morphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T\mathcal{H}om\,(G, \mathcal{O}_X)) \longrightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\mathcal{H}om\,(G, \mathcal{O}_X))$$

is an isomorphism. q.e.d.

We conjecture the following statement in which "only if " part is a consequence of the theorem above.

Conjecture 6.4. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathcal{M}$ is regular holonomic if and only if $Sol^t(\mathcal{M})$ is regular.*

7. An example

In this section $X = \mathbb{C}$ endowed with the holomorphic coordinate z, and we shall study the ind-sheaf of temperate holomorphic solutions of the $\mathcal{D}_X$-module $\mathcal{M} := \mathcal{D}_X \exp(1/z) = \mathcal{D}_X/\mathcal{D}_X(z^2\partial_z + 1)$. We set for short

$$\begin{aligned} \mathcal{S}^t &:= H^0(Sol^t(\mathcal{M})) \simeq \mathcal{I}hom_{\beta_X \mathcal{D}_X}(\beta_X \mathcal{M}, \mathcal{O}^t_X), \\ \mathcal{S} &:= H^0(Sol(\mathcal{M})) \;\simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X). \end{aligned}$$

Notice first that $\mathcal{O}^t_X$ is concentrated in degree 0 (since $\dim X = 1$), and it is a sub-ind-sheaf of $\mathcal{O}_X$. It follows that the morphism $\mathcal{S}^t \to \mathcal{S}$ is a monomorphism.

Moreover,

$$\mathcal{S} \simeq \mathbb{C}_{X, X\smallsetminus\{0\}} \cdot \exp(1/z).$$

Lemma 7.1. — *Let $V \subset X$ be a connected open subset. Then $\Gamma(V;\mathcal{S}^t) \neq 0$ if and only if $V \subset X \smallsetminus \{0\}$ and $\exp(1/z)|_V$ is tempered.*

Proof. — The space $\Gamma(V;\mathcal{S})$ has dimension one and is generated by the function $\exp(1/z)$. Hence, the subspace $\Gamma(V;\mathcal{S}^t) \simeq \Gamma(V;\mathcal{S}) \cap \Gamma(V;\mathcal{O}^t)$ is not zero if and only if $\exp(1/z) \in \Gamma(V;\mathcal{O}^t_X)$, that is, if and only if $\exp(1/z)|_V$ is tempered. q.e.d.

Let us set $z = x + iy$.

Lemma 7.2. — *Let W be an open subanalytic subset of $\mathbb{P}^1(\mathbb{C})$ with $\infty \notin W$. Assume that there exist positive constants C and A such that*

$$\exp(x) \leqslant C(1 + x^2 + y^2)^N \text{ on } W. \tag{7.1}$$

Then there exists a constant B such that $x \leqslant B$ on W.

Proof. — We shall compactify $\mathbb{C} \subset \mathbb{P}^1(\mathbb{C})$ by $(\mathbb{R} \sqcup \{\infty\})^2$. If x is not bounded on W, then there exists a real analytic curve $\gamma\colon [0,\varepsilon[\to (\mathbb{R} \sqcup \{\infty\})^2$ such that $\operatorname{Re}\gamma(0) = \infty$ and $\gamma(t) \in W$ for $t > 0$. Writing $\gamma(t) = (x(t), y(t))$, one has

$$y(t) = cx(t)^q + \mathrm{O}(x(t)^{q-\varepsilon}).$$

for some $q \in \mathbb{Q}$, $c \in \mathbb{R}$ and $\varepsilon > 0$. Then (7.1) implies that $\exp(x)$ has a polynomial growth when $x \to \infty$, which is a contradiction. q.e.d.

Let $\overline{B}_\varepsilon$ denote the closed ball with center $(\varepsilon, 0)$ and radius ε and set $U_\varepsilon = X \smallsetminus \overline{B}_\varepsilon$.

Proposition 7.3. — *One has the isomorphism*

$$\text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{XU_\varepsilon} \xrightarrow{\sim} \mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X). \tag{7.2}$$

Proof. — It follows from Lemma 7.2 that $\exp(1/z)$ is temperate (in a neighborhood of 0) on an open subanalytic subset $V \subset X \smallsetminus \{0\}$ if and only if $\operatorname{Re}(1/z)$ is bounded on V, that is, if and only if $V \subset U_\varepsilon$ for some $\varepsilon > 0$.

Let V be a connected relatively compact subanalytic open subset of $X \smallsetminus \{0\}$. Then a morphism $\mathbb{C}_V \to \mathbb{C}_{X\smallsetminus\{0\}} \cdot \exp(1/z)$ factorizes through a morphism $\mathbb{C}_V \to \mathcal{S}^t$ if and only if it factorizes through $\mathbb{C}_{U_\varepsilon}$. Hence we get the isomorphism (7.2) by Theorem 2.2. q.e.d.

Remark 7.4. — In fact one can show

$$H^1(Sol^t(\mathcal{M})) \xrightarrow{\sim} H^1(Sol(\mathcal{M})) \simeq \mathbb{C}_0.$$

The isomorphism $H^1(Sol(\mathcal{M})) = \mathcal{O}_X/(z^2\partial_z + 1)\mathcal{O}_X \xrightarrow{\sim} \mathbb{C}_0$ is given by

$$(\mathcal{O}_X)_0 \ni v(z) \longmapsto \oint v(z) z^{-2} \exp(-1/z)\, dz.$$

Note that $\varphi(z){:=}z^{-2}\exp(-1/z)$ is a solution to the adjoint equation

$$(-\partial_z z^2 + 1)\varphi(z) = 0.$$

The distinguished triangle

$$\mathcal{S}^t \longrightarrow Sol^t(\mathcal{M}) \longrightarrow H^1(Sol^t(\mathcal{M}))[-1] \xrightarrow{+1}$$

gives a non-zero element of $Ext^2(\mathbb{C}_0, \mathcal{S}^t) \xrightarrow{\sim} Ext^2(\mathbb{C}_0, \mathbb{C}_X) \simeq \mathbb{C}$.

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REGULARITY OF $\mathcal{D}$-MODULES ASSOCIATED TO A SYMMETRIC PAIR

by

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À Jean-Michel Bony, à l'occasion de son 60^e anniversaire.

***Abstract.* —** The invariant eigendistributions on a reductive Lie algebra are solutions of a holonomic $\mathcal{D}$-module which has been proved to be regular by Kashiwara-Hotta. We solve here a conjecture of Sekiguchi saying that in the more general case of symmetric pairs, the corresponding module is still regular.

***Résumé* (Régularité des $\mathcal{D}$-modules associés à une paire symétrique). —** Sur une algèbre de Lie réductive, les distributions invariantes qui sont vecteurs propres des opérateurs différentiels bi-invariants sont les solutions d'un système holonome. Il a été démontré par Kashiwara-Hotta que ce module est régulier. Nous résolvons ici une conjecture de Sekiguchi en montrant que ce résultat est encore vrai dans le cas plus général des paires symétriques.

Introduction

Let G be a semi-simple Lie group. An irreducible representation of G has a character which is an *invariant eigendistribution*, that is a distribution on G which is invariant under the adjoint action of G and which is an eigenvalue of every biinvariant differential operator on G. A celebrated theorem of Harish-Chandra [**2**] says that all invariant eigendistributions are locally integrable functions on G.

After transfer to the Lie algebra $\mathfrak{g}$ of G by the inverse of the exponential map, an invariant eigendistribution is a solution of a $\mathcal{D}_\mathfrak{g}$-module $\mathcal{M}_\lambda^F$ for some $\lambda \in \mathfrak{g}^*$. Kashiwara and Hotta studied in [**4**] these $\mathcal{D}_\mathfrak{g}$-modules $\mathcal{M}_\lambda^F$, in particular they proved that they are holonomic and, using a modified version of the result of Harish-Chandra, proved that they are regular holonomic. This shows in particular that any hyperfunction solution of a module $\mathcal{M}_\lambda^F$ is a distribution, hence that any invariant eigenhyperfunction is a distribution.

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In [**15**], Sekiguchi extended the definition of the modules $\mathcal{M}_\lambda^F$ to a symmetric pair. A symmetric pair is a decomposition of a reductive Lie algebra into a direct sum of an even and an odd part, and the group associated to the even part has an action on the odd part (see section 2.1 for the details). In the diagonal case where even and odd part are identical, it is the action of a group on its Lie algebra. Sekiguchi defined a subclass of symmetric pairs ("nice pairs"), for which he proved a kind of Harish-Chandra theorem, that is that there is no hyperfunction solution of a module $\mathcal{M}_\lambda^F$ supported by a hypersurface. He also conjectured that these modules are regular holonomic.

In [**11**] and [**12**], Levasseur and Stafford give new proofs of the Harish-Chandra theorem in the original case (the "diagonal" case) and in the Sekiguchi case ("nice pairs"). In [**1**], we show that both theorems may be deduced from results on the roots of the b-functions associated to $\mathcal{M}_\lambda^F$.

The aim of this paper is to prove Sekiguchi's conjecture, that is the regularity of $\mathcal{M}_\lambda^F$, in the general case of symmetric pairs. Our proof do not use Harish-Chandra's theorem or its generalization, so we do not need to ask here the pairs to be "nice".

In the first section of the paper we study the regularity of holonomic $\mathcal{D}$-modules. In the definition of Kashiwara-Kawaï [**6**], a holonomic $\mathcal{D}$-module is regular if it is microlocally regular along each irreducible component of its characteristic variety. We had proven in [**9**], that the microlocal regularity may be connected to some microcharacteristic variety. We show here that an analogous result is still true if homogeneity is replaced by some quasi-homogeneity.

In the second section, we prove Sekiguchi's conjecture in theorem 2.2.1. First by standard arguments, we show that outside of the nilpotent cone, the result may be proved by reduction to a Lie algebra of lower dimension. Then on the nilpotent cone we use the results of the first section to show that the module is microlocally regular along the conormals to the nilpotent orbits.

1. Bifiltrations of $\mathcal{D}$-modules

1.1. V-filtration and microcharacteristic varieties. — In this section, we recall briefly the definitions of the V-filtration and microcharacteristic varieties. Details may be found in [**10**] (see also [**5**], [**8**], [**13**]).

Let X be a complex manifold, $\mathcal{O}_X$ be the sheaf of holomorphic functions on X and $\mathcal{D}_X$ be the sheaf of differential operators with coefficients in $\mathcal{O}_X$. Let Y be a submanifold of X. The ideal $\mathcal{I}_Y$ of holomorphic functions vanishing on Y defines a filtration of the sheaf $\mathcal{O}_X|_Y$ of functions on X defined on a neighborhood of Y by $F_Y^k\mathcal{O}_X = \mathcal{I}_Y^k$. The associate graduate, $\mathrm{gr}_Y\mathcal{O}_X = \bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k+1}$ is isomorphic to the sheaf $\lambda_*\mathcal{O}_{[T_YX]}$ where $\lambda : T_YX \to Y$ is the normal bundle to Y in X and $\mathcal{O}_{[T_YX]}$ the sheaf of holomorphic functions on T_YX which are polynomial in the fibers of λ. For f a function of $\mathcal{O}_X|_Y$ we will denote by $\sigma_Y(f)$ its image in $\mathrm{gr}_Y\mathcal{O}_X$.

If $\mathcal{I}$ is the ideal of definition of an analytic subvariety Z of X, then $\sigma_Y(\mathcal{I}) = \{\sigma_Y(f) \mid f \in \mathcal{I}\}$ is an ideal of $\mathcal{O}_{[T_Y X]}$ which defines the tangent cone to Z along Y [**17**].

In local coordinates (x,t) such that $Y = \{t = 0\}$, $\mathcal{I}_Y^k$ is, for $k \geqslant 0$, the sheaf of functions

$$f(x,t) = \sum_{|\alpha|=k} f_\alpha(x,t) t^\alpha$$

and if k is maximal with $f \in \mathcal{I}_Y^k$, we have $\sigma_Y(f)(x,\widetilde{t}) = \sum_{|\alpha|=k} f_\alpha(x,0)\widetilde{t}^\alpha$.

Consider now the conormal bundle to Y denoted by $\Lambda = T_Y^*X$ as a submanifold of T^*X. If f is a function on T^*X, $\sigma_\Lambda(f)$ is a function on the normal bundle $T_\Lambda(T^*X)$. The hamiltonian isomorphism $TT^*X \simeq T^*T^*X$ associated to the symplectic structure of T^*X identifies $T_\Lambda(T^*X)$ with the the cotangent bundle $T^*\Lambda$ and thus $\sigma_\Lambda(f)$ may be considered as a function on $T^*\Lambda$.

The sheaf $\mathcal{D}_X$ is provided with the filtration by the usual order of operators denoted by $(\mathcal{D}_{X,m})_{m\geqslant 0}$ and that we will call the "usual filtration". The graduate associated to this filtration is $\mathrm{gr}\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}$ where $\pi : T^*X \to X$ is the cotangent bundle and $\mathcal{O}_{[T^*X]}$ is the sheaf of holomorphic functions polynomial in the fibers of π. We have also $\mathrm{gr}^m\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}[m]$ where $\mathcal{O}_{[T^*X]}[m]$ is the sheaf of holomorphic functions polynomial homogeneous of degree m in the fibers of π. If P is a differential operator of $\mathcal{D}_X|_Y$, its principal symbol is a function $\sigma(P)$ on T^*X defined in a neighborhood of $\Lambda = T_Y^*X$ and $\sigma_\Lambda(\sigma(P))$ is a function on $T^*\Lambda$ (denoted by $\sigma_\Lambda\{1\}(P)$ in the notations of [**10**]).

The sheaf $\mathcal{D}_X|Y$ of differential operators on a neighborhood of Y is also provided with the the V-filtration of Kashiwara [**5**]:

$$V_k\mathcal{D}_X = \{P \in \mathcal{D}_X \mid \forall\, j \in \mathbb{Z},\ P\mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\},$$

where $\mathcal{I}_Y^j = \mathcal{O}_X$ if $j \leqslant 0$.

In local coordinates (x,t), the operators x_i and $D_{x_i} := \partial/\partial x_i$ have order 0 for the V-filtration while the operators t_i have order -1 and $D_{t_i} := \partial/\partial t_i$ order $+1$.

Remark that the V-filtration induces a filtration on $\mathrm{gr}\mathcal{D}_X \simeq \pi_*\mathcal{O}_{[T^*X]}$ which is nothing but the filtration F_Λ associated the conormal bundle $\Lambda = T_Y^*X$. In coordinates, $\Lambda = \{(x,t,\xi,\tau) \in T^*X \mid t = 0,\, \xi = 0\}$, a function of $\mathcal{O}_{[T^*X]}[m] \cap \mathcal{I}_\Lambda^{m-k}$ is a function $f(x,t,\xi,\tau)$ which is polynomial homogeneous of degree m in (ξ,τ) and vanishes at order at least $m-k$ on $\{t = 0,\, \xi = 0\}$.

The two filtrations of $\mathcal{D}_X$ define a bifiltration $F_{kj}\mathcal{D}_X = \mathcal{D}_{X,j} \cap V_k\mathcal{D}_X$. The associated bigraduate is defined by $\mathrm{gr}_F\mathcal{D}_X = \oplus \mathrm{gr}_F^{kj}\mathcal{D}_X$ with

$$\mathrm{gr}_F^{kj}\mathcal{D}_X = F_{kj}\mathcal{D}_X \,/(F_{k-1,j}\mathcal{D}_X + F_{k,j-1}\mathcal{D}_X)$$

and is isomorphic to $\mathrm{gr}_\Lambda \mathrm{gr}\mathcal{D}_X$ that is to the sheaf $\pi_*\mathcal{O}_{[T^*\Lambda]}$ of holomorphic functions on $T^*\Lambda$ polynomial in the fibers of $\pi : T^*\Lambda \to Y$. The image of a differential operator P in this bigraduate will be denoted by $\sigma_{\Lambda(\infty,1)}(P)$ and may be defined as follows:

If the order of P for the V-filtration is equal to the order of its principal symbol $\sigma(P)$ for the induced V-filtration then $\sigma_{\Lambda(\infty,1)}(P) = \sigma_\Lambda(\sigma(P))$ and if the order of $\sigma(P)$ is strictly lower then $\sigma_{\Lambda(\infty,1)}(P) = 0$.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. A good filtration of $\mathcal{M}$ is a filtration which is locally finitely generated that is locally of the form:

$$\mathcal{M}_m = \sum_{j=1,\dots,N} \mathcal{D}_{X,m+m_j} u_j,$$

where $u_1, \dots, u_N$ are (local) sections of $\mathcal{M}$ and $m_1, \dots, m_N$ integers.

It is well known that if $(\mathcal{M}_m)$ is a good filtration of $\mathcal{M}$, the associated graduate $\mathrm{gr}\mathcal{M}$ is a coherent $\mathrm{gr}\mathcal{D}_X$-module and defines the characteristic variety of $\mathcal{M}$ which is a subvariety of T^*X. This subvariety is involutive for the canonical symplectic structure of T^*X and a $\mathcal{D}_X$-module is said to be holonomic if its characteristic variety is lagrangian that is of minimal dimension.

In the same way, a good bifiltration of $\mathcal{M}$ is a bifiltration which is locally finitely generated. Then the associated bigraduate is a coherent $\mathrm{gr}_F\mathcal{D}_X$-module which defines a subvariety $\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ of $T^*\Lambda$. It is a homogeneous involutive subvariety of $T^*\Lambda$ but it is not necessarily lagrangian even if $\mathcal{M}$ is holonomic.

If $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_X$ then:

$$\mathrm{Ch}(\mathcal{M}) = \{\xi \in T^*X \mid \forall\, P \in \mathcal{I},\ \sigma(P)(\xi) = 0\}$$
$$\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M}) = \{\zeta \in T^*\Lambda \mid \forall\, P \in \mathcal{I},\ \sigma_{\Lambda(\infty,1)}(P)(\zeta) = 0\}$$

Regular holonomic $\mathcal{D}_X$-modules have been defined by Kashiwara and Kawaï in [**6**, Definition 1.1.16.]. A holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is regular if it has regular singularities along the smooth part of each irreducible component of its characteristic variety. It is proved in [**6**] that the property of regular singularities is generic, that is it suffices to prove it on a dense open subset of Λ, in particular we may assume that Λ is the conormal bundle to a smooth subvariety of X. The definition of regular singularities along a smooth lagrangian variety is given in [**6**, Definition 1.1.11.] but in this paper, we will use the following characterization which we proved in [**9**, Theorem 3.1.7.]:

Proposition 1.1.1. — *A coherent $\mathcal{D}_X$-module has regular singularities along a lagrangian manifold Λ if and only if $\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ is contained in the zero section of $T^*\Lambda$.*

1.2. Weighted V-filtration. — The V-filtration is associated to the Euler vector field of the normal bundle T_YX which in coordinates is equal to $\sum \widetilde{t}_i D_{\widetilde{t}_j}$. We want to define a new filtration associated to a vector field $\sum m_i \widetilde{t}_i D_{\widetilde{t}_j}$. As this is not invariant under coordinate transform, we have first to give an invariant definition.

Let us consider the fiber bundle $p : T_Y X \to Y$. The sheaf $\mathcal{D}_{[T_Y X/Y]}$ of relative differential operators is the subsheaf of the sheaf $\mathcal{D}_{T_Y X}$ of differential operators on $T_Y X$ commuting with all functions of $p^{-1}\mathcal{O}_Y$. A differential operator P on $T_Y X$ is homogeneous of degree 0 if for any function f homogeneous of degree k in the fibers of p, Pf is homogeneous of degree k.

In particular, a vector field $\widetilde{\eta}$ on $T_Y X$ which is a relative differential operator homogeneous of degree 0 defines a morphism from the set of homogeneous functions of degree 1 into itself which commutes with the action of $p^{-1}\mathcal{O}_Y$, that is a section of

$$\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{O}_Y}(\mathcal{O}_{T_Y X}[1], \mathcal{O}_{T_Y X}[1]).$$

Let (x,t) be coordinates of X such that $Y = \{(x,t) \in X \mid t = 0\}$. Let $(x,\widetilde{t})$ be the corresponding coordinates of $T_Y X$. Then $\widetilde{\eta}$ is written as:

$$\widetilde{\eta} = \sum a_{ij}(x)\widetilde{t}_i D_{\widetilde{t}_j}$$

and the matrix $A = (a_{ij}(x))$ is the matrix of the associated endomorphism of $\mathcal{O}_{T_Y X}[1]$ which is a locally free $p^{-1}\mathcal{O}_Y$-module of rank $d = \mathrm{codim}_X Y$. Its conjugation class is thus independent of the choice of coordinates (x,t). When the morphism is the identity, $\widetilde{\eta}$ is by definition the Euler vector field of $T_Y X$.

Definition 1.2.1. — A vector field $\widetilde{\eta}$ on $T_Y X$ is *definite positive* if it is a relative differential operator homogeneous of degree 0 whose eigenvalues are strictly positive rational numbers and which is locally diagonalizable as an endomorphism of $\mathcal{O}_{T_Y X}[1]$.

A structure of *local fiber bundle of* X *over* Y is an analytic isomorphism between a neighborhood of Y in X and a neighborhood of Y in $T_Y X$. For example a local system of coordinates defines such an isomorphism.

Definition 1.2.2. — A vector field η on X is *definite positive with respect to* Y if:

(i) η is of degree 0 for the V-filtration associated to Y and the image $\sigma_Y(\eta)$ of η in $\mathrm{gr}^0_V \mathcal{D}_X$ is *definite positive* as a vector field on $T_Y X$.

(ii) There is a structure of local fiber bundle of X over Y which identifies η and $\sigma_Y(\eta)$.

It is proved in [**10**, proposition 5.2.2] that if $\sigma_Y(\eta)$ is the Euler vector field of $T_Y X$ the condition (ii) is always satisfied and the local fiber bundle structure of X over Y is unique for a given η, but this is not true in general.

We will now assume that X is provided with such a vector field η. Let $\beta = a/b$ the rational number with minimum positive integers a and b such that the eigenvalues of $\beta^{-1}\eta$ are positive relatively prime integers. Let $\mathcal{D}_X[k]$ be the sheaf of differential operators Q satisfying the equation $[Q,\eta] = \beta k Q$ and let $V^\eta_k \mathcal{D}_X$ be the sheaf of differential operators Q which are equal to a series $Q = \sum_{l \leqslant k} Q_l$ with Q_l in $\mathcal{D}_X[l]$ for each $l \in \mathbb{Z}$.

By definition of a definite positive vector field, we may find local coordinates (x, t) such that $\eta = \sum m_i t_i D_{t_i}$ and we may assume that the m_i are relatively prime integers after multiplication of η by β^{-1}. In this situation, the operators x_j and D_{x_j} have order 0 while the operators t_i have order $-m_i$ and D_{t_i} order $+m_i$. This shows in particular that any monomial $x^\alpha t^\beta D_x^\gamma D_t^\delta$ is in some $\mathcal{D}_X[k]$ and thus that $\mathcal{D}_X$ is the union of all $V_k^\eta \mathcal{D}_X$. This defines a filtration V^η of the sheaf of rings $\mathcal{D}_X$.

The principal symbol of $[Q, \eta]$ is the Poisson bracket $\{\sigma(P), \sigma(\eta)\}$ which is equal to $H_\eta(\sigma(P))$ where H_η is a vector field on T^*X, the Hamiltonian of η. The V^η-filtration on $\mathcal{D}_X$ induces a filtration on the graduate of $\mathcal{D}_X$ that is on $\mathcal{O}_{[T^*X]}$. A function f of $\mathcal{O}_{[T^*X]}$ will be in $V_k^\eta \mathcal{O}_{[T^*X]}$ if it is a series of functions f_l for $l \geqslant k$ with $H_\eta f = -lf$. In this case we set $\sigma_k^\eta(f) = f_k$.

We are now in a situation analog to that of section 1.1 with two filtrations on $\mathcal{D}_X$, the usual filtration and the V^η-filtration. The sheaf $\mathcal{D}_X$ is thus provided with a bifiltration by $F_{kj}^\eta \mathcal{D}_X = \mathcal{D}_{X,j} \cap V_k^\eta \mathcal{D}_X$ and this defines a symbol $\sigma^{\eta}{}_{(\infty,1)}(P)$ which is a function on T^*X. By definition, $\sigma^{\eta}{}_{(\infty,1)}(P)$ is equal to $\sigma_k^\eta(\sigma(P))$ where k is the order of P for the V^η-filtration. This symbol is thus equal to 0 if the order of $\sigma(P)$ is strictly less than k.

If $\mathcal{M}$ is a coherent $\mathcal{D}_X$-module, we define a good bifiltration and a microcharacteristic variety $\mathrm{Ch}^{\eta}{}_{(\infty,1)}(\mathcal{M})$. If $\mathcal{M} = \mathcal{D}_X / \mathcal{I}$ we will have:

$$\mathrm{Ch}^{\eta}{}_{(\infty,1)}(\mathcal{M}) = \{\zeta \in T^*X \mid \forall\, P \in \mathcal{I},\ \sigma^{\eta}{}_{(\infty,1)}(P)(\zeta) = 0\}.$$

The difference with the previous situation is the local identification of $T_Y X$ with X which defines isomorphisms $T^*T_Y^*X \simeq T^*T_Y X \simeq T^*X$ and make $\sigma^\eta(\sigma(P))$ a function on T^*X. Especially, if $\widetilde{\eta}$ is the Euler vector field of $T_Y X$ and η a vector field on X with $\sigma_V(\eta) = \widetilde{\eta}$, the definitions of this section coincide with the definitions of the previous one except for this identification.

1.3. Direct image of V-filtration. — Let $\varphi : Y \to X$ be a morphism of complex analytic manifolds. A vector field u on Y is said to be *tangent to the fibers of* φ if $u(f \circ \varphi) = 0$ for all f in $\mathcal{O}_X$. A differential operator P is said to be *invariant under* φ if there exists a $\mathbb{C}$-endomorphism A of $\mathcal{O}_X$ such that $P(f \circ \varphi) = A(f) \circ \varphi$ for all f in $\mathcal{O}_X$. If we assume from now that φ has a dense range in X, A is uniquely determined by P and is a differential operator on X. We will denote by $A = \varphi_*(P)$ the image of P in $\mathcal{D}_X$ under this ring homomorphism.

Let Z be a submanifold of Y and T a submanifold of X. Let η be a vector field on Y invariant under φ. We assume that η is definite positive with respect to Z and that $\eta' = \varphi_*(\eta)$ is definite positive with respect to T. We also multiply η by an integer so that its eigenvalues and those of η' are integers.

Example 1.3.1. — Let Y be a complex vector space and $\varphi : Y \to X = \mathbb{C}^d$ given by $\varphi = (\varphi_1, \dots, \varphi_d)$ where $\varphi_1, \dots, \varphi_d$ are holomorphic functions on Y homogeneous of degree $m_1, \dots, m_d$. Let $Z = \{0\}$ and η be the Euler vector field of Y, so that the V^η-filtration is the V-filtration along $\{0\}$. Then $\eta' = \varphi_*(\eta)$ is equal to $\sum m_i t_i D_{t_i}$ on X and is definite positive with respect to $\{0\}$. Remark that we do not assume that φ is defined in a neighborhood of Z.

In the general case, we can choose local coordinates (y, t) on X so that $\eta' = \sum m_j t_j D_{t_j}$, then the map φ is given by $y_i = \varphi_i(x)$ and $t_j = \psi_j(x)$ where the functions $\varphi_i(x)$ is homogeneous of degee 0 for η while the function $\psi_j(x)$ is homogeneous of degee m_j for η.

The sheaf $\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{\varphi^{-1}\mathcal{O}_X} \varphi^{-1}\mathcal{D}_X$ is a $(\mathcal{D}_Y,\varphi^{-1}\mathcal{D}_X)$-bimodule with a canonical section $1 \otimes 1$ denoted by $1_{Y\to X}$. If we choose coordinates $(x_1, \dots, x_n)$ of X and coordinates $(y_1, \dots, y_p)$ of Y and if $\varphi = (\varphi_1, \dots, \varphi_p)$, then the sections of $\mathcal{D}_{Y\to X}$ are represented by finite sums $\sum f_\alpha(y) \otimes D_x^\alpha$ and the left action of $\mathcal{D}_Y$ is given by

$$D_{y_i}\Big(\sum_\alpha f_\alpha(y) \otimes D_x^\alpha\Big) = \sum_\alpha \frac{\partial f_\alpha}{\partial y_i}(y) \otimes D_x^\alpha + \sum_{\alpha,j} f_\alpha(y)\frac{\partial \varphi_j}{\partial y_i}(y) \otimes D_{x_j} D_x^\alpha$$

If $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module, its inverse image under φ is the $\mathcal{D}_Y$-module $\varphi^*\mathcal{N} = \mathcal{D}_{Y\to X} \otimes_{\varphi^{-1}\mathcal{D}_X} \varphi^{-1}\mathcal{N}$. In general, $\varphi^*\mathcal{N}$ is not coherent but if $\mathcal{N}$ is holonomic, $\varphi^*\mathcal{N}$ is holonomic (hence coherent).

Let $\mathcal{D}_{Y\to X}[k]$ be the set of sections satisfying $\eta \cdot u - u \cdot \varphi_*\eta = -\beta\beta' k u$ where β (resp. β') is the g.c.d. of the eigenvalues of η (resp. $\varphi_*\eta$). (We may assume that $\beta = 1$ or $\beta' = 1$ but not both in general). We define $V_k\mathcal{D}_{Y\to X}$ as the subsheaf of $\mathcal{D}_{Y\to X}$ of the sections which may be written as series $\sum_{l\geqslant k} u_l$ with u_l in $\mathcal{D}_{Y\to X}[l]$. Remark that $1_{Y\to X}$ satisfies $\eta \cdot 1_{Y\to X} = 1_{Y\to X} \cdot \varphi_*\eta$ hence is of order 0.

If $\mathcal{N}$ is a coherent $\mathcal{D}_X$-module provided with a $V^{\eta'}$-filtration we define a filtration on its inverse image by:

$$V_k^\eta \varphi^*\mathcal{N} = \sum_{k=\beta' i+\beta j} V_i\mathcal{D}_{Y\to X} \otimes_{\varphi^{-1}V_0^{\eta'}\mathcal{D}_X} \varphi^{-1}V_j^{\eta'}\mathcal{N}.$$

The sheaf $\mathcal{D}_{Y\to X}$ is also provided with a filtration $(\mathcal{D}_{Y\to X})_j$ induced by the usual filtration of $\mathcal{D}_X$ hence of a bifiltration $F^\eta\mathcal{D}_{Y\to X}$. If $\mathcal{N}$ is bi-filtrated, we define in the same way a bifiltration on $\varphi_*\mathcal{N}$.

Proposition 1.3.2. — *Let $\mathcal{I}$ be an ideal of $\mathcal{D}_Y$ which is generated by all the vector fields tangent to the fibers of φ and by a finite set $(P_1, \dots, P_l)$ of differential operators invariant under φ. Let $\mathcal{J}$ be the ideal of $\mathcal{D}_X$ generated by $(\varphi_*(P_1), \dots, \varphi_*(P_l))$. Let $\mathcal{M} = \mathcal{D}_Y/\mathcal{I}$ and $\mathcal{N} = \mathcal{D}_X/\mathcal{J}$ and put on $\mathcal{M}$ and $\mathcal{N}$ the bifiltrations induced by $F^\eta\mathcal{D}_Y$ and $F^{\eta'}\mathcal{D}_X$.*

Then, there exists a canonical morphism of $\mathcal{D}_Y$-modules $\mathcal{M} \to \varphi^*\mathcal{N}$ which is a morphism of bi-filtrated $F^\eta\mathcal{D}_Y$-modules and an isomorphism at the points where φ is a submersion.

Proof. — There is a canonical morphism $\mathcal{D}_Y \to \mathcal{D}_{Y\to X}$ given by $P \mapsto P\cdot 1_{Y\to X}$. The vector fields tangent to the fibers cancel $\mathcal{D}_{Y\to X}$ and a differential operator invariant under φ satisfy $P\cdot 1_{Y\to X} = 1_{Y\to X}\cdot\varphi_*(P)$ hence this morphism defines a morphism $\mathcal{M}\to\varphi_*\mathcal{N}$ which is a morphism of left $V^\eta\mathcal{D}_Y$-modules by the definitions.

In a neighborhood of a point where φ is a submersion, we may choose local coordinates $(x_1,\dots,x_p,y_1,\dots,y_{n-p})$ such that $\varphi(x,y) = x$. Then $\mathcal{D}_{Y\to X}$ is the sheaf of operators $P(x,y,D_x)$, the vector fields tangent to the fibers are generated by $D_{y_1},\dots,D_{y_{n-p}}$ and the differential operators invariant under φ are of the form $P(x,D_x)$ modulo (D_{y_i}), so $\mathcal{M}\to\varphi_*\mathcal{N}$ is an isomorphism. □

Let $S = \varphi^{-1}(T)$ and x be a point of S where φ is a submersion. In a neighborhood of x, Y is isomorphic to $X\times S$ and if we fix such an isomorphism, η' which is a vector field on X may be considered as a vector field on Y, definite positive relatively to S. Remark that η' differ from η by a vector field tangent to φ. Then proposition 1.3.2 gives:

Corollary 1.3.3

The microcharacteristic variety $\mathrm{Ch}^{\eta}{}_{(\infty,1)}(\mathcal{M})$ *is equal to* $\mathrm{Ch}^{\eta'}{}_{(\infty,1)}(\mathcal{M})$ *in a neighborhood of x.*

1.4. Weighted V-filtration and regularity

Definition 1.4.1. — Let Z be a submanifold of X and η be a vector field which is definite positive with respect to Z. A coherent $\mathcal{D}_X$-module has *η-weighted regular singularities* along the lagrangian manifold $\Lambda = T^*_Z X$ if there is a dense open subset Ω of Λ such that $\mathrm{Ch}^{\eta}{}_{(\infty,1)}(\mathcal{M})\subset\Lambda$ in a neighborhood of Ω.

If $\sigma_Z(\eta)$ is the Euler vector field of $T_Z X$, proposition 1.1.1 shows that this definition coincide with the definition of Kashiwara-Kawaï.

Let $X = \mathbb{C}^n$ with coordinates $(x_1,\dots,x_{n-p},t_1,\dots,t_p)$ and $Z = \{t=0\}$, let $Y = \mathbb{C}^n$ with coordinates $(x_1,\dots,x_{n-p},y_1,\dots,y_p)$ and $Z' = \{y=0\}$. Let $m_1,\dots,m_p$ be strictly positive integers , we define the map $\varphi: Y\to X$ by $\varphi(x,y) = (x,y_1^{m_1},\dots,y_p^{m_p})$ and the vector field $\eta = \sum_{i=1\dots p} m_i t_i D_{t_i}$.

Lemma 1.4.2. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module with η-weighted regular singularities along $T^*_Z X$, then $\varphi^*\mathcal{M}$ is a holonomic $\mathcal{D}_Y$-module with regular singularities along $T^*_{Z'}Y$.*

Proof. — We may assume that $\mathcal{M}$ is equal to $\mathcal{D}_X/\mathcal{I}$ for some coherent ideal $\mathcal{I}$ of $\mathcal{M}$. The inverse image of $\mathcal{M}$ by φ is, by definition:

$$\varphi^*\mathcal{M} = \mathcal{D}_{Y\to X} \otimes_{\varphi^{-1}\mathcal{D}_X} \varphi^{-1}\mathcal{M} = \mathcal{D}_{Y\to X}/\mathcal{D}_{Y\to X}\mathcal{I}.$$

The sections of $\mathcal{D}_{Y\to X}$ are represented by $P(x,y,D_x,D_t) = \sum a_{\alpha\beta}(x,y)D_x^\alpha D_t^\beta$ and we define the filtration $V^\eta\mathcal{D}_{Y\to X}$ in the same way than in the previous section. For this filtration $x^\gamma y^\delta D_x^\alpha D_t^\beta$ is of order $<m,\beta>-|\delta|$. We also define the usual filtration on $\mathcal{D}_{Y\to X}$, that is the filtration by the order in (D_x, D_t). In this way, $\mathcal{D}_{Y\to X}$ is provided with a bifiltration $F^\eta\mathcal{D}_{Y\to X}$ which is compatible with the bifiltration $F^\eta\mathcal{D}_X$, that is an operator P of $F^\eta_{kl}\mathcal{D}_X$ sends $F^\eta_{ij}\mathcal{D}_{Y\to X}$ into $F^\eta_{i+k,j+l}\mathcal{D}_{Y\to X}$.

Let $\mathcal{D}_{Y\to X}[N]$ be the sub-$\mathcal{D}_Y$-module of $\mathcal{D}_{Y\to X}$ generated by D_t^β for $|\beta| \leqslant N$. If $\mathcal{M}$ is holonomic, $\varphi^*\mathcal{M}$ is holonomic hence coherent. The images of the morphisms $\mathcal{D}_{Y\to X}[N] \to \varphi^*\mathcal{M}$ make an increasing sequence of coherent submodules of $\varphi^*\mathcal{M}$ which is therefore stationary, so there exists some N_0 such that $\mathcal{D}_{Y\to X}[N] \to \varphi^*\mathcal{M}$ is surjective for all $N \geqslant N_0$. The bifiltration induced by $F^\eta\mathcal{D}_{Y\to X}$ on $\mathcal{D}_{Y\to X}[N]$ is a good $F\mathcal{D}_Y$-filtration which induces a good filtration on $\varphi^*\mathcal{M}$ if $N \geqslant N_0$, we will denote it by $F[N]\varphi^*\mathcal{M}$.

The associate graduate is denoted by $\mathrm{gr}[N]\varphi^*\mathcal{M}$ and, as $F[N]$ is a good bifiltration, the analytic cycle of T^*Y associated to $\mathrm{gr}[N]\varphi^*\mathcal{M}$ is independent of N [**10**, Prop 3.2.3.]. For $N \geqslant N_0$, the canonical morphism $\mathrm{gr}[N_0]\varphi^*\mathcal{M} \to \mathrm{gr}[N]\varphi^*\mathcal{M}$ induces an isomorphism on the associated cycles hence $\mathrm{gr}[N_0]\varphi^*\mathcal{M}$ and $\mathrm{gr}[N]\varphi^*\mathcal{M}$ have the same support and the kernel and cokernel of the morphism have a support of dimension strictly lower.

An operator P of $F^\eta_{kl}\mathcal{D}_X$ sends $F^\eta_{ij}\mathcal{D}_{Y\to X}[N_0]$ into $F^\eta_{i+k,j+l}\mathcal{D}_{Y\to X}[N_0+l]$. If P annihilates a section u of $F_{ij}[N_0]\varphi^*\mathcal{M}$, its class in $\mathrm{gr}_{kl}\mathcal{D}_X$ that is the function $\sigma^\eta{}_{(\infty,1)}(P)$ annihilates the image of u in $\mathrm{gr}[N+l]\varphi^*\mathcal{M}$. Let ζ be a point of $\Lambda = T^*_Z X$ such that $\mathrm{Ch}^\eta{}_{(\infty,1)}(\mathcal{M}) \subset T^*_Z X$ in a neighborhood of ζ. By the hypothesis, there is a dense open subset Ω of such points in Λ. There is a differential operator P which annihilates u and such that $\sigma^\eta{}_{(\infty,1)}(P) = t_1^M\mu$ where μ is a function invertible at ζ. Hence there exists some l such that the image of u in $\mathrm{gr}[N+l]\varphi^*\mathcal{M}$ is annihilated by $t_1^M = y_1^{Mm_1}$ hence is supported by $y_1 = 0$. As $\mathrm{gr}[N_0]\varphi^*\mathcal{M}$ is finitely generated, there exists some $N_1 \geqslant N_0$ such that the image of $\mathrm{gr}[N_0]\varphi^*\mathcal{M}$ in $\mathrm{gr}[N_1]\varphi^*\mathcal{M}$ is contained in $y_1 = 0$.

We can do the same for the other equations of $T^*_{Z'}Y$ and show that there exists some $N_2 \geqslant N_0$ such that the image of $\mathrm{gr}[N_0]\varphi^*\mathcal{M}$ in $\mathrm{gr}[N_2]\varphi^*\mathcal{M}$ is contained in $T^*_{Z'}Y$. This shows that $\mathrm{gr}[N_0]\varphi^*\mathcal{M}$ is supported by the union of $T^*_{Z'}Y$ and of a set W of dimension strictly lower than the dimension of $T^*_{Z'}Y$. But we know that this support is involutive hence all its component have a dimension at least that dimension, so $\mathrm{gr}[N_0]\varphi^*\mathcal{M}$ is supported in $T^*_{Z'}Y$ in a neighborhood of $\varphi^{-1}(\zeta)$. By definition $\mathrm{gr}[N_0]\varphi^*\mathcal{M}$ is equal to $\mathrm{Ch}_{T^*_{Z'}Y}{}_{(\infty,1)}(\varphi^*\mathcal{M})$, hence $\varphi^*\mathcal{M}$ has regular singularities along $T^*_{Z'}Y$. □

Theorem 1.4.3. — *Let X be a complex manifold, $\pi : T^*X \to X$ the projection, Z a submanifold of X and η a vector field on X which is definite positive with respect to Z. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. We assume that:*

(1) *$\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module on $X - Z$,*

(2) *$\mathcal{M}$ has η-weighted regular singularities along $T^*_Z X$,*

(3) *The dimension of $\mathrm{Ch}(\mathcal{M}) \cap T^*_Z X$ is equal to the dimension of X.*

Then $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module.

Proof. — We fix local coordinates $(x_1, \dots, x_{n-p}, t_1, \dots, t_p)$ of X so that $Z = \{t = 0\}$ and $\eta = \sum_{i=1\dots p} m_i t_i D_{t_i}$. We define a map $\varphi : Y \to X$ by $\varphi(x, y) = (x, y_1^{m_1}, \dots, y_p^{m_p})$ where Y is a neighborhood of 0 in $\mathbb{C}^n$. If Z' is the set $\{y = 0\}$, lemma 1.4.2 shows that $\varphi^*\mathcal{M}$ has regular singularities along $T^*_{Z'}Y$.

The third condition means that the characteristic variety of $\mathcal{M}$ has no irreducible component contained in $\pi^{-1}(Z)$ except $T^*_Z X$. The same is true for $\varphi^*\mathcal{M}$ on Z'. This may be proved as in lemma 1.4.2 but with the usual filtration replacing the bifiltration. This may also be proved easily with the definition of the characteristic variety in terms of microdifferential operators.

By hypothesis, $\mathcal{M}$ is regular on $X \smallsetminus Z$ hence by [**6**, Cor 5.4.8.] $\varphi^*\mathcal{M}$ is regular holonomic on $Y \smallsetminus Z'$. So, $\varphi^*\mathcal{M}$ has regular singularities along each irreducible component of its characteristic variety, hence by definition, it is a regular holonomic $\mathcal{D}_Y$-module.

Then by [**6**, theorem 6.2.1.], the direct image $\varphi_*\varphi^*\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module. By definition

$$\varphi_*\varphi^*\mathcal{M} = \mathbb{R}\varphi_*(\mathcal{D}_{X\leftarrow Y} \otimes^{\mathbb{L}}_{\mathcal{D}_Y} \mathcal{D}_{Y\to X} \otimes^{\mathbb{L}}_{\varphi^{-1}\mathcal{D}_Y} \varphi^{-1}\mathcal{M})$$

and the morphism $\mathcal{D}_X \to \mathcal{D}_{X\leftarrow Y} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ is injective hence $\mathcal{M}$ is a submodule of $\varphi_*\varphi^*\mathcal{M}$ hence a regular holonomic $\mathcal{D}_X$-module. □

The following corollary is the generalization of the definition of regular holonomic $\mathcal{D}$-modules and of proposition 1.1.1. It is proved from the previous theorem by descending induction on the dimension of the strata.

Corollary 1.4.4. — *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Assume that there is a stratification $X = \bigcup X_\alpha$ such that $\mathrm{Ch}(\mathcal{M}) \subset \bigcup T^*_{X_\alpha}X$ and for each α there is a vector field η_α positive definite along X_α such that $\mathcal{M}$ has η_α-weighted regular singularities along $T^*_{X_\alpha}X$.*

Then $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module.

2. Symmetric pairs

2.1. Definitions. — Let us briefly recall what is a symmetric pair. For the details we refer to [**15**] and [**12**]. Let G be a connected complex reductive algebraic group with

Lie algebra $\mathfrak{g}$. Fix a non-degenerate, G-invariant symmetric bilinear form κ on the reductive Lie algebra $\mathfrak{g}$ such that κ is the Killing form on the semi-simple Lie algebra $[\mathfrak{g},\mathfrak{g}]$. Fix an involutive automorphism ϑ of $\mathfrak{g}$ preserving κ and set $\mathfrak{k} = \mathrm{Ker}(\vartheta - I)$, $\mathfrak{p} = \mathrm{Ker}(\vartheta + I)$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the pair $(\mathfrak{g},\mathfrak{k})$ or $(\mathfrak{g},\vartheta)$ is called a symmetric pair. Recall that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to κ and that $\mathfrak{k}$ is a reductive Lie subalgebra of $\mathfrak{g}$. Denote by K the connected reductive subgroup of G with Lie algebra $\mathfrak{k}$. The group K acts on $\mathfrak{p}$ via the adjoint action.

Let $\mathfrak{p}^*$ be the dual of $\mathfrak{p}$, $\mathcal{O}(\mathfrak{p}) = S(\mathfrak{p}^*)$ the ring of regular functions on $\mathfrak{p}$ ($S(\mathfrak{p}^*)$ is the symmetric algebra), $\mathcal{O}(\mathfrak{p}^*) = S(\mathfrak{p})$ the ring of regular functions on $\mathfrak{p}^*$ and $\mathcal{D}(\mathfrak{p})$ the ring of differential operators on $\mathfrak{p}$ with coefficients in $\mathcal{O}(\mathfrak{p})$. The ring of functions $\mathcal{O}(\mathfrak{p})$ is naturally embedded in $\mathcal{D}(\mathfrak{p})$ and we embed $\mathcal{O}(\mathfrak{p}^*) = S(\mathfrak{p})$ in $\mathcal{D}(\mathfrak{p})$ as differential operators with constant coefficients. That is we associate to an element u of the vector space $\mathfrak{g}$ the derivation in the direction of u

$$D_u(f)(x) = \frac{d}{dt} f(x+tu)|_{t=0}$$

and we extend to the symmetric algebra $S(\mathfrak{p})$. Remark that this embedding is compatible with the filtration by the degree in $S(\mathfrak{p})$ and the filtration by the order in $\mathcal{D}(\mathfrak{p})$.

Notice that K has an induced action on $S(\mathfrak{p})$, $S(\mathfrak{p}^*)$ and $\mathcal{D}(\mathfrak{p})$ and we have natural embeddings of the invariant subrings $S(\mathfrak{p})^K \subset \mathcal{D}(\mathfrak{p})^K$ and $S(\mathfrak{p}^*)^K \subset \mathcal{D}(\mathfrak{p})^K$. The ring $S(\mathfrak{p})^K$ is equal to the ring of polynomials $\mathbb{C}[p_1,\dots,p_r]$ for some $p_1,\dots,p_r$ in $S(\mathfrak{p})^K$ and in the same way $S(\mathfrak{p}^*)^K$ is equal to a ring of polynomials $\mathbb{C}[q_1,\dots,q_r]$ [**7**].

The differential of the action of K on $\mathfrak{p}$ induces a Lie algebra homomorphism $\tau : \mathfrak{k} \to \mathrm{Der}\, S(\mathfrak{p}^*)$ hence an embedding $\tau : \mathfrak{k} \to \mathcal{D}(\mathfrak{p})$ defined by

$$(\tau(a)\cdot f)(v) = \frac{d}{dt} f(e^{-ta}\cdot v)|_{t=0}, \quad \text{for } a \in \mathfrak{k},\ f \in \mathcal{O}(\mathfrak{p}),\ v \in \mathfrak{p}.$$

As a section of the tangent bundle, $\tau(A)$ is the map $\mathfrak{p} \to T\mathfrak{p} = \mathfrak{p}\times\mathfrak{p}$ given by $\tau(A)(X) = (X,[X,A])$.

We denote by $\mathbf{N}(\mathfrak{p})$ the nilpotent cone of $\mathfrak{p}$, that is the set of nilpotent elements of $\mathfrak{g}$ which lie in $\mathfrak{p}$, it is also the subvariety of $\mathfrak{p}$ defined by the set of K-invariant functions $S(\mathfrak{p}^*)^K$. In the same way we consider the nilpotent cone $\mathbf{N}(\mathfrak{p}^*)$ which is the subvariety of $\mathfrak{p}^*$ defined by $S(\mathfrak{p})^K$. An important result is that the nilpotent cone $\mathbf{N}(\mathfrak{p})$ is a finite union of K-orbits [**7**, theorem 2].

The cotangent bundle $T^*\mathfrak{p}$ is equal to $\mathfrak{p}\times\mathfrak{p}^*$. The non-degenerate form κ on $\mathfrak{g}$ defines a non-degenerate symmetric bilinear form on $\mathfrak{p}$ and an isomorphism $\mathfrak{p} \simeq \mathfrak{p}^*$. We identify $T^*\mathfrak{p} = \mathfrak{p}\times\mathfrak{p}^* \simeq \mathfrak{p}\times\mathfrak{p}$. Let $\mathcal{C}(\mathfrak{p}) = \{(x,y)\in\mathfrak{p}\times\mathfrak{p} \mid [x,y]=0\}$, then the dimension of $(\mathfrak{p}\times\mathbf{N}(\mathfrak{p}))\cap\mathcal{C}(\mathfrak{p})$ is equal to the dimension of $\mathfrak{p}$ [**12**, lemma 2.2.].

The characteristic variety of $\mathcal{D}_\mathfrak{p}/\mathcal{D}_\mathfrak{p}\tau(\mathfrak{k})$ is equal to $\mathcal{C}(\mathfrak{p})$. Let F be an ideal of finite codimension of $S(\mathfrak{p})^K$, its graduate is a power of $S(\mathfrak{p})^K$ hence the characteristic variety of the $\mathcal{D}_\mathfrak{p}$-module $\mathcal{D}_\mathfrak{p}/\mathcal{D}_\mathfrak{p}F$ is $\mathfrak{p}\times\mathbf{N}(\mathfrak{p})$. Finally, if $\mathcal{I}$ be the left ideal of $\mathcal{D}_\mathfrak{p}$

generated by F and $\tau(\mathfrak{k})$, the characteristic variety of $\mathcal{M}_F = \mathcal{D}_\mathfrak{p}/\mathcal{I}$ is contained in $(\mathfrak{p}\times\mathbf{N}(\mathfrak{p}))\cap\mathcal{C}(\mathfrak{p})$ hence $\mathcal{M}_F$ is a holonomic $\mathcal{D}_\mathfrak{p}$-module.

As a special case, we have the diagonal case where $G = G_1\times G_1$ with $\vartheta(x,y) = (y,x)$ for some reductive group G_1. Thus $(\mathfrak{g},\mathfrak{k}) = (\mathfrak{g}_1\oplus\mathfrak{g}_1,\mathfrak{g}_1)$ and $K = G_1$ with its adjoint action on $\mathfrak{p} = \mathfrak{g}_1$. Let $\lambda\in\mathfrak{p}^*$ and $F_\lambda = \{P - P(\lambda) \mid P\in S(\mathfrak{p})^K\}$, then the corresponding module $\mathcal{M}_\lambda^F = \mathcal{D}_\mathfrak{p}/\mathcal{D}_\mathfrak{p}\tau(\mathfrak{k}) + \mathcal{D}_\mathfrak{p}F_\lambda$ is the module of Kashiwara-Hotta [**4**].

2.2. The conjecture of Sekiguchi

Theorem 2.2.1. — *Let F be an ideal of finite codimension of $S(\mathfrak{p})^K$ and $\mathcal{M}_F = \mathcal{D}_\mathfrak{p}/\mathcal{I}$ where $\mathcal{I}$ is the left ideal of $\mathcal{D}_\mathfrak{p}$ generated by F and $\tau(\mathfrak{k})$.*

Then $\mathcal{M}_F$ is a regular *holonomic $\mathcal{D}_\mathfrak{p}$-module.*

The proof of this theorem will be made in several steps. First we will reduce to the semi-simple case (lemma 2.2.3), then prove by induction on the dimension of the Lie algebra, that the result is true outside of the nilpotent cone (lemma 2.2.4) and the key point of the proof is the case of a nilpotent orbit (lemma 2.2.6).

Lemma 2.2.2. — *Let Y be a complex manifold and $X = Y\times\mathbb{C}$. Let $P(t,D_t)$ be a differential operator on $\mathbb{C}$ with principal symbol independent of t and $\mathcal{I}$ be a coherent ideal of $\mathcal{D}_X$ which contains P.*

Let $\mathcal{M}_Y$ be the inverse image of $\mathcal{M} = \mathcal{D}_X/\mathcal{I}$ on Y by the immersion $Y\to X$, then $\mathcal{M}$ is isomorphic to the inverse image of $\mathcal{M}_Y$ by the projection $q: X\to Y$, that is

$$\mathcal{M} = \mathcal{D}_{X\to Y}\otimes_{q^{-1}\mathcal{D}_Y} q^{-1}\mathcal{M}_Y = \mathcal{M}_Y\widehat{\otimes}\mathcal{O}_\mathbb{C}.$$

In particular, $\mathcal{M}$ is regular holonomic if and only if $\mathcal{M}_Y$ is regular holonomic.

Proof. — This lemma is a (very) special case of [**14**, theorem 5.3.1. ch II]. The first step is to prove that $\mathcal{D}_\mathbb{C}/\mathcal{D}_\mathbb{C}P$ is isomorphic to $(\mathcal{D}_\mathbb{C}/\mathcal{D}_\mathbb{C}D_t)^N$. The proof is the same than that of [**14**, theorem 5.2.1. ch II], but as there is only one variable, the proof is very simple and use only functions instead of differential operators of infinite order. Then we can follow the proof of [**14**] but with finite order operators instead of infinite order operators.

Remark that if P were a differential operator in several variables, for example, $P = D_t^2 + D_x$, this result would be true only with the sheaf $\mathcal{D}_X^\infty$ of differential operators with infinite order.

As $X = Y\times\mathbb{C}$, the inverse image of $\mathcal{M}_Y$ by q is isomorphic to the external product of $\mathcal{D}$-modules $\mathcal{M}_Y\widehat{\otimes}\mathcal{O}_\mathbb{C}$. □

Assume that $\mathfrak{p} = \mathfrak{p}_0\oplus\mathfrak{p}_1$, the action of K on $\mathfrak{p}_0$ being trivial. Then $S(\mathfrak{p})^K = S(\mathfrak{p}_0)\otimes S(\mathfrak{p}_1)^K$, this defines a morphism $\delta: S(\mathfrak{p})^K\to S(\mathfrak{p}_1)^K$ by restriction and $F_1 = \delta(F)$ is an ideal of finite codimension of $S(\mathfrak{p}_1)^K$. Let $\mathcal{M}_{F_1} = \mathcal{D}_{\mathfrak{p}_1}/\mathcal{I}_1$ where $\mathcal{I}_1$ is the ideal of $\mathcal{D}_{\mathfrak{p}_1}$ generated by $\tau_{\mathfrak{p}_1}(\mathfrak{k})$ and F_1.

Lemma 2.2.3

(1) *The module* $\mathcal{M}_F$ *is isomorphic to* $\mathcal{O}_{\mathfrak{p}_0}\widehat{\otimes}(\mathcal{M}_F)_{\mathfrak{p}_1}$ *where* $(\mathcal{M}_F)_{\mathfrak{p}_1}$ *is the restriction of* $\mathcal{M}_F$ *to* $\mathfrak{p}_1$.

(2) $(\mathcal{M}_F)_{\mathfrak{p}_1}$ *(hence* $\mathcal{M}_F$*) is regular if* $\mathcal{M}_{F_1}$ *is regular.*

Proof. — By induction on the dimension of $\mathfrak{p}_0$, we may assume that $\mathfrak{p}_0 = \mathbb{C}$ and choose linear coordinates (x,t) of $\mathfrak{p}$ such that $\mathfrak{p}_0 = \{(x,t) \in \mathfrak{p} \mid x = 0\}$. The action of K is trivial on $\mathfrak{p}_0$ hence $S(\mathfrak{p})^K$ contains $S(\mathfrak{p}_0)$ and as F is finite codimensional in $S(\mathfrak{p})^K$ it contains a polynomial in D_t. Lemma 2.2.2 shows the first part of the lemma.

We assume now that $\mathcal{M}_{F_1}$ is regular. Recall that $(\mathcal{M}_F)_{\mathfrak{p}_1} = \mathcal{M}_F/t\mathcal{M}_F$ is a holonomic $\mathcal{D}_{\mathfrak{p}_1}$-module generated by the classes of $1,\dots,D_t^{m-1}$. Let $\mathcal{M}'$ be the submodule of $(\mathcal{M}_F)_{\mathfrak{p}_1}$ generated by the class u of D_t^{m-1}. The vector fields of $\tau(\mathfrak{k})$ are independent of (t, D_t) hence u is annihilated by $\tau(\mathfrak{k})$. If P is an element of F, as an operator of $\mathcal{D}_{\mathfrak{p}}$ it is equal to $\delta(P) + AD_t$ hence $\delta(P)$ annihilates u. So u is annihilated by $\tau(\mathfrak{k})$ and by F_1 and $\mathcal{M}'$ is a quotient of $\mathcal{M}_{F_1}$. So $\mathcal{M}'$ is regular.

Consider now $\mathcal{M}''$ which is the submodule of $\mathcal{M}$ generated by the classes D_t^{m-1} and D_t^{m-2}. The quotient $\mathcal{M}''/\mathcal{M}'$ is generated by the class v of D_t^{m-2} which is annihilated by $\tau(\mathfrak{k})$ and by F_1, so it is regular. We have an exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M}'' \longrightarrow \mathcal{M}''/\mathcal{M}' \longrightarrow 0,$$

where two terms are regular hence $\mathcal{M}''$ is regular. Continuing the same argument, we get that $(\mathcal{M}_F)_{\mathfrak{p}_1}$ is regular. □

Let b be a semisimple element of $\mathfrak{p}$. Then $\mathfrak{p} = \mathfrak{p}^b \oplus [\mathfrak{k}, b]$ and $\mathfrak{g}^b = \mathfrak{k}^b \oplus \mathfrak{p}^b$ defines a symmetric pair. Let δ be the restriction map $\delta : S(\mathfrak{p})^K \to S(\mathfrak{p}^b)^{K^b}$, this map is injective and if F is an ideal of finite codimension of $S(\mathfrak{p})^K$ then $\delta(F)$ is an ideal of finite codimension of $S(\mathfrak{p}^b)^{K^b}$ [**3**, lemma 19]. Let $\mathcal{I}_b$ be the left ideal of $\mathcal{D}_{\mathfrak{p}^b}$ generated by $\delta(F)$ and $\tau(\mathfrak{k}^b)$ and $\mathcal{M}_b = \mathcal{D}_{\mathfrak{p}^b}/\mathcal{I}_b$.

Lemma 2.2.4. — *In a neighborhood of* b, $\mathcal{M}_F$ *is isomorphic to the external product of the holomorphic functions on the orbit of* b *by a quotient of* $\mathcal{M}_b$. *In particular,* $\mathcal{M}_F$ *is regular if* $\mathcal{M}_b$ *is regular.*

Proof. — Let V be a linear subspace of $\mathfrak{k}$ such that $\mathfrak{k} = V \oplus \mathfrak{k}^b$. The map $f : V \times \mathfrak{p}^b \to \mathfrak{p}$ given by $f(y,Z) = \exp(y)\cdot Z$ is a local isomorphism. If $(x_1,\dots,x_{n-r})$ are linear coordinates of V and $(t_1,\dots,t_r)$ are linear coordinates of $\mathfrak{p}^b$, the map f defines local coordinates $(x_1,\dots,x_{n-r},t_1,\dots,t_r)$ of $\mathfrak{p}$ in a neighborhood of b. Lemma 3.7 of [**15**] shows that in these coordinates, the orbit Kb is $\{(x,t) \mid t = 0\}$, $\mathfrak{p}^b = \{(x,t) \mid x = 0\}$ and the differential operators $D_{x_1},\dots,D_{x_{n-r}}$ belong to $\tau(\mathfrak{k})$. Hence $\mathcal{M}$ is the product of $\mathcal{O}_{Kb}$ by a module $\mathcal{N}$.

If Z is an element of $\mathfrak{k}^b$, $\tau_{\mathfrak{p}}(Z)$ is by definition the vector field on $\mathfrak{p}$ with value $[Z,A]$ at a point A of $\mathfrak{p}$. The value of $\tau_{\mathfrak{p}^b}(Z)$ at a point A of $\mathfrak{p}^b$ is the projection of $[Z,A]$ on $\mathfrak{p}^b$, hence $\tau_{\mathfrak{p}^b}(\mathfrak{k}^b)$ is equal to $\tau_{\mathfrak{p}}(\mathfrak{k})$ modulo $D_{x_1},\dots,D_{x_{n-r}}$. On the other hand, let

$P \in F$, as the coordinates $(t_1, \dots, t_r)$ are linear coordinates of $\mathfrak{p}^b$, the value of P on a function of t is the restriction of P to $S(\mathfrak{p}^b)^{K^b}$. Hence $\mathcal{N}$ is a quotient of $\mathcal{M}_b$. □

Lemma 2.2.5. — *Let Λ be the conormal to 0 in $\mathfrak{p}$. The microcharacteristic variety $\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M}_F)$ is contained in $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$.*

Proof. — Let E be the Euler vector field of the vector space $\mathfrak{p}$. It is clear on the definition, that the vector fields of $\tau(\mathfrak{k})$ preserve the homogeneity of functions hence that they commute with E. So they are homogeneous of degree 0 for the V-filtration at 0. On the other hand, they are homogeneous of degree 1 for the usual filtration as any vector field. So if $u \in \tau(\mathfrak{k})$, $\sigma_{\Lambda(\infty,1)}(u) = \sigma(u)$.

On differential operators with constant coefficients, the V-filtration at $\{0\}$ and the usual filtration coincide, hence we have also $\sigma^E{}_{(\infty,1)}(P) = \sigma(P)$ for these operators.

So, $\mathrm{Ch}_{\Lambda(\infty,1)}(\mathcal{M}_F)$ is contained in the set of points where the symbols of the operators of $\tau(\mathfrak{k})$ and of F vanish that is in $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$. □

Lemma 2.2.6. — *For each nilpotent orbit S of $\mathbf{N}(\mathfrak{p})$, there is a vector field η which is positive definite with respect to S and such that $\mathcal{M}_F$ has η-weighted regular singularities along $T^*_S\mathfrak{p}$.*

Proof. — Let S be one of these orbits, r the codimension of S and $X \in S$. As in [**12**, § 3] (see also [**16**, Part I, § 5.6]) we can choose a normal $\mathfrak{sl}_2$-triple (H, X, Y) in $\mathfrak{p}$ which generates a Lie algebra isomorphic to $\mathfrak{sl}_2$ and acting on $\mathfrak{p}$ by the adjoint representation. Then $\mathfrak{p}$ splits into a direct sum of irreducible submodules of dimensions $\lambda_i + 1$ for $i = 1 \dots r$. Moreover $\mathfrak{p} = \mathfrak{p}^Y \oplus [X, \mathfrak{k}]$, $\dim \mathfrak{p}^Y = r$ and we can select a basis $(Y_1, \dots, Y_r)$ of $\mathfrak{p}^Y$ such that $[H, Y_i] = -\lambda_i Y_i$. Let V be a linear subspace of $\mathfrak{k}$ such that $\mathfrak{k} = V \oplus \mathfrak{k}^X$. If $(b_1, \dots, b_{n-r})$ is a basis of V, the map $F : \mathbb{C}^n \to \mathfrak{p}$ given by

$$F(x_1, \dots, x_{n-r}, t_1, \dots, t_r) = \exp(x_1 b_1) \dots \exp(x_{n-r} b_{n-r}) \cdot (X + \textstyle\sum t_i Y_i)$$

is a local isomorphism hence defines local coordinates (x, t) of $\mathfrak{p}$ in a neighborhood of X. In these coordinates, $S = \{(x, t) \mid t = 0\}$, $\mathfrak{p}^Y = \{(x, t) \mid x = 0\}$, and the differential operators $D_{x_1}, \dots, D_{x_{n-r}}$ are in the ideal generated by $\tau(\mathfrak{k})$ [**15**, lemma 3.7].

Let E be the Euler vector field of the vector space $\mathfrak{p}$. A standard calculation [**16**, Part I, § 5.6] shows that $E(t_i)|_{x=0} = m_i t_i$ with $m_i = \frac{1}{2}\lambda_i + 1$. Moreover, if $b_{n-r} = H$, we proved in [**1**, lemma 3.4.1] that E is equal to $\eta + w$ where $\eta = \sum_{j=1}^r m_j t_j D_{t_j}$ and $w = 1/2 D_{x_{n-r}}$. By definition, η is positive definite with respect to S.

Define a map $\varphi : \mathfrak{p} \to V = \mathbb{C}^r$ by $\varphi(x, t) = t$. Let $\eta' = \sum m_j t_j D_{t_j}$ on V. The functions $t_1, \dots, t_r$ satisfy $E(t_i) = \eta'(t_i) = m_i t_i$ hence they are homogeneous and the map φ is defined in a conic neighborhood of X. This also shows that E is invariant under φ and that $\eta' = \varphi_*(E)$.

The module $\mathcal{M}_F$ is equal to $\mathcal{D}_\mathfrak{p}/\mathcal{I}$ where $\mathcal{I}$ is a coherent ideal of $\mathcal{D}_\mathfrak{p}$ which contains the derivations $D_{x_1}, \dots, D_{x_{n-r}}$ hence $\mathcal{I}$ is generated by $D_{x_1}, \dots, D_{x_{n-r}}$ and a finite set of differential operators $Q_1(t, D_t), \dots, Q_N(t, D_t)$ depending only of (t, D_t). (This result is standard and also a special case of lemma 2.2.2).

The module $\mathcal{M}_F$ satisfies the hypothesis of corollary 1.3.3 hence $\mathrm{Ch}^E{}_{(\infty,1)}(\mathcal{M}_F)$ is equal to $\mathrm{Ch}^\eta{}_{(\infty,1)}(\mathcal{M}_F)$ and by lemma 2.2.5 it is contained in $(\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$.

Assume now that $T^*_S\mathfrak{p}$ is an irreducible component of the characteristic variety $\mathrm{Ch}(\mathcal{M}_F)$ and let x^* be a generic point of $T^*_S\mathfrak{p}$, that is a point which does not belong to other irreducible components of $\mathrm{Ch}(\mathcal{M}_F)$. We have

$$T^*_S\mathfrak{p} \subset \mathrm{Ch}(\mathcal{M}_F) \subset (\mathfrak{p} \times \mathbf{N}(\mathfrak{p})) \cap \mathcal{C}(\mathfrak{p})$$

and as they have the same dimension, they are equal generically. So $\mathrm{Ch}^\eta{}_{(\infty,1)}(\mathcal{M}_F) = T^*_S\mathfrak{p}$ generically on $T^*_S\mathfrak{p}$ and we are done. □

Proof of theorem 2.2.1. — We will argue by induction on the dimension of $\mathfrak{g}$ and first, we reduce to the semi-simple case. Set $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{g}_1$, $\mathfrak{p}_1 = \mathfrak{p} \cap \mathfrak{g}_1$, $\mathfrak{z}$ the center of $\mathfrak{g}$ and $\mathfrak{p}_0 = \mathfrak{z} \cap \mathfrak{p}$. We have $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$ and by lemma 2.2.3, it suffices to prove the theorem for $\mathfrak{p}_1$. As $\mathfrak{z} \cap \mathfrak{k}$ acts trivially we may assume that $\mathfrak{g}$ is semisimple.

Let x be a non-nilpotent element of $\mathfrak{p}$. It decomposes as $x = b + n$ where b is non zero and semisimple, n is nilpotent and $[b, n] = 0$. As $\mathfrak{g}$ is semisimple, $\mathfrak{p}^b$ is of dimension strictly less than $\mathfrak{p}$, hence we may assume by the induction hypothesis that the theorem is true for $\mathfrak{p}^b$. Lemma 2.2.4 shows that $\mathcal{M}_F$ is regular in a neighborhood of b. As $\mathcal{M}_F$ is constant on the orbits, it is regular on the orbits whose closure contains b, in particular at x.

We proved that $\mathcal{M}_F$ is regular outside of the nilpotent cone. As the nilpotent cone is a finite union of orbits, we will now argue by descending induction on the dimension of these orbits. So let x be a nilpotent point of $\mathfrak{p}$, Kx its orbit and assume that $\mathcal{M}_F$ is regular on $\mathfrak{p} - Kx$ in a neighborhood of x. Lemma 2.2.6 shows that $\mathcal{M}_F$ has η-weighted regular singularities along $T^*_{Kx}\mathfrak{p}$ hence theorem 1.4.3 shows that $\mathcal{M}_F$ is regular at x. □

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BOHR-SOMMERFELD QUANTIZATION CONDITION FOR NON-SELFADJOINT OPERATORS IN DIMENSION 2

by

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Abstract. — For a class of non-selfadjoint h-pseudodifferential operators in dimension 2, we determine all eigenvalues in an h-independent domain in the complex plane and show that they are given by a Bohr–Sommerfeld quantization condition. No complete integrability is assumed, and as a geometrical step in our proof, we get a KAM–type theorem (without small divisors) in the complex domain.

Résumé **(Condition de quantification de Bohr-Sommerfeld pour des opérateurs non-autoadjoints en dimension 2)**
Pour une classe d'opérateurs h-pseudodifférentiels non-autoadjoints, nous déterminons toutes les valeurs propres dans un domaine complexe indépendant de h et nous montrons que ces valeurs propres sont données par une condition de quantification de Bohr-Sommerfeld. Aucune condition d'integrabilité complète est supposée, et une étape géométrique de la démonstration est donnée par un théoreme du type KAM dans le complexe (sans petits dénominateurs).

0. Introduction

In [**MeSj**] we developed a variational approach for estimating determinants of pseudodifferential operators in the semiclassical setting, and we obtained many results and estimates of some aesthetical and philosophical value. The original purpose of the present work was to continue the study in a somewhat more special situation (see [**MeSj**], section 8) and show in that case, that our methods can lead to optimal results. This attempt turned out to be successful, but at the same time the results below are of independent interest, so the relation to the preceding work, will only be hinted upon here and there.

Let $p(x,\xi)$ be bounded and holomorphic in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4 = \mathbf{C}^2_x \times \mathbf{C}^2_\xi$. (The assumptions near ∞ will be of importance only in the quantized case,

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and can then be be varied in many ways.) Assume that

$$\mathbf{R}^4 \cap p^{-1}(0) \neq \varnothing \text{ is connected,} \tag{0.1}$$

$$\text{on } \mathbf{R}^4 \text{ we have } |p(x,\xi)| \geqslant \frac{1}{C}, \text{ for } |(x,\xi)| \geqslant C, \tag{0.2}$$

for some $C > 0$,

$$d\,\mathrm{Re}\,p(x,\xi),\ d\,\mathrm{Im}\,p(x,\xi) \text{ are linearly independent for all } (x,\xi) \in p^{-1}(0) \cap \mathbf{R}^4. \tag{0.3}$$

It follows that $p^{-1}(0) \cap \mathbf{R}^4$ is a compact (2-dimensional) surface. Also assume that

$$|\{\mathrm{Re}\,p, \mathrm{Im}\,p\}| \text{ is sufficiently small on } p^{-1}(0) \cap \mathbf{R}^4. \tag{0.4}$$

Here

$$\{a, b\} = \sum_1^2 \Big(\frac{\partial a}{\partial \xi_j} \frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j} \frac{\partial b}{\partial \xi_j} \Big) = H_a(b)$$

is the Poisson bracket, and we adopt the following convention: We assume that p varies in some set of functions that are uniformly bounded in some fixed tube as above and satisfy (0.2), (0.3) uniformly. Then we require $|\{\mathrm{Re}\,p, \mathrm{Im}\,p\}|$ to be bounded on $p^{-1}(0) \cap \mathbf{R}^4$ by some constant > 0 which only depends on the class.

If we strengthen (0.4) to requiring that $\{\mathrm{Re}\,p, \mathrm{Im}\,p\} = 0$ on $p^{-1}(0) \cap \mathbf{R}^4$, then the latter manifold becomes Lagrangian and will carry a complex elliptic vector field $H_p = H_{\mathrm{Re}\,p} + iH_{\mathrm{Im}\,p}$. It is then a well-known topological fact (and reviewed from the point of view of analysis in appendix B of section 1) that $p^{-1}(0) \cap \mathbf{R}^4$ is (diffeomorphic to) a torus. If we only assume (0.1)–(0.4), then H_p is close to being tangent to $p^{-1}(0) \cap \mathbf{R}^4$ and the orthogonal projection of this vector field to $p^{-1}(0) \cap \mathbf{R}^4$ is still elliptic. So in this case, we have still a torus, which in general is no more Lagrangian.

In section 1 we will establish the following result:

Theorem 0.1. — *There exists a smooth 2-dimensional torus $\Gamma \subset \mathbf{C}^4$, close to $p^{-1}(0) \cap \mathbf{R}^4$ such that $\sigma_{|_\Gamma} = 0$ and $I_j(\Gamma) \in \mathbf{R}$, $j = 1, 2$. Here $I_j(\Gamma) = \int_{\gamma_j} \xi \cdot dx$ are the actions along the two fundamental cycles $\gamma_1, \gamma_2 \subset \Gamma$, and $\sigma = \sum_1^2 d\xi_j \wedge dx_j$ is the complex symplectic $(2,0)$-form.*

If we form

$$L = \{\exp \widehat{tH_p}(\rho);\ \rho \in \Gamma,\ t \in \mathbf{C},\ |t| < 1/C\},$$

where $\widehat{tH_p} = tH_p + \overline{tH_p}$ is the real vector field associated to tH_p, then, as we shall see, L is a complex Lagrangian manifold $\subset p^{-1}(0)$ and L will be uniquely determined near $p^{-1}(0) \cap \mathbf{R}^4$ contrary to Γ. As a matter of fact, we will show that there is a smooth family of 2-dimensional torii $\Gamma_a \subset p^{-1}(0)$ with $\sigma_{|_{\Gamma_a}} = 0$, depending on a complex parameter a, such that the corresponding L_a form a holomorphic foliation of $p^{-1}(0)$ near $p^{-1}(0) \cap \mathbf{R}^4$. The L_a depend holomorphically on a and so do the corresponding actions $I_j(\Gamma_a)$. We can even take one of the actions to be our complex parameter a.

It then turns out that $\mathrm{Im}\,\frac{dI_2}{dI_1} \neq 0$, and this implies the existence of a unique value of a for which $I_j(\Gamma_a) \in \mathbf{R}$ for $j = 1, 2$.

Theorem 0.1 can be viewed as a complex version of the KAM theorem, in a case where no small denominators are present. As pointed out to us by D. Bambusi and S. Graffi, the absence of small divisors for certain dynamical systems in the complex has been exploited by Moser [**Mo**], Bazzani–Turchetti [**BaTu**] and by Marmi–Yoccoz.

The proof we give in section 1 finally became rather simple. Using special real symplectic coordinates, we reduce the construction of the Γ_a to that of multivalued functions with single-valued gradient (from now on grad-periodic functions) on a torus, that satisfy a certain Hamilton-Jacobi equation. In suitable coordinates, this becomes a Cauchy-Riemann equation with small non-linearity and can be solved in non-integer C^m-spaces by means of a straight-forward iteration.

The fact that $I_j(\Gamma) \in \mathbf{R}$ implies that there exists an IR-manifold $\Lambda \subset \mathbf{C}^4$ (i.e. a smooth manifold for which $\sigma_{|_\Lambda}$ is real and non-degenerate) which is close to $\mathbf{R}^4$ and contains Γ. The reality of the actions $I_j(\Gamma)$ is an obvious necessary condition and the sufficiency will be established in section 1. When $p(x,\xi) \to 1$ sufficiently fast at ∞, Λ will be a critical point of the functional

$$\Lambda \longmapsto I(\Lambda) := \int_\Lambda \log|p(x,\xi)|\mu(d(x,\xi)), \tag{0.5}$$

where μ is the symplectic volume element on Λ. This was discussed in [**MeSj**] and in section 8 of that paper we also discussed the linearized problem corresponding to finding such a critical point. The reason for studying the functional (0.5) is that $I(\Lambda)$ enters in a general asymptotic upper bound on the determinant of an h-pseudodifferential operator with symbol p. Our quantum result below implies that this bound is essentially optimal.

Now let $p(x,\xi,z)$ be a uniformly bounded family of functions as above, depending holomorphically on a parameter $z \in \mathrm{neigh}\,(0,\mathbf{C})$ (some neighborhood of 0 in $\mathbf{C}$). Let $P(z) = p^w(x,hD,z)$ be the corresponding h-Weyl quantization of p, given by

$$p^w(x,hD,z)u(x) = \frac{1}{(2\pi h)^2}\iint e^{\frac{i}{h}(x-y)\cdot\theta} p\Big(\frac{x+y}{2},\theta,z\Big)u(y)dyd\theta. \tag{0.6}$$

It is well known (see for instance [**DiSj**]) that $P(z)$ is bounded: $L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$, uniformly with respect to (z,h). Moreover, the ellipticity near infinity, imposed by (0.2), implies that it is a Fredholm operator (of index 0 as will follow from the contructions below). Let us say that z is an eigen-value if $p^w(x,hD,z)$ is not bijective. The main result of our work is that the eigen-values are given by a Bohr-Sommerfeld quantization condition. We here state a shortened version (of Theorem 6.3). Let $I(z) = (I_1(z), I_2(z))$, where $I_j(z) = I_j(\Gamma(z)) \in \mathbf{R}$ and $\Gamma(z) \subset p^{-1}(0,z)$ is given by Theorem 0.1. $I(z)$ depends smoothly on z, since $\Gamma(z)$ can be chosen with smooth z-dependence.

Theorem 0.2. — *Under the above assumptions, there exists* $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ *and* $\theta(z;h) \sim \theta_0+\theta_1(z)h+\theta_2(z)h^2+\cdots$ *in* $C^\infty(\mathrm{neigh}\,(0,\mathbf{C});\mathbf{R}^2)$, *such that for* z *in an* h*-independent neighborhood of* 0 *and for* $h>0$ *sufficiently small, we have*
1) z *is and eigen-value iff we have*

$$\frac{I(z)}{2\pi h} = k - \theta(z;h), \text{ for some } k \in \mathbf{Z}^2. \tag{BS}$$

2) When I *is a local diffeomorphism, then the eigen-values are simple (in a natural sense) and form a distorted lattice.*

Classically, the Bohr-Sommerfeld quantization condition describes the eigen-values of self-adjoint operators in dimension 1. See for instance [**HeRo**], [**GrSj**] exercise 12.3. In higher dimension Bohr-Sommerfeld conditions can still be used in the (quantum) completely integrable case for self-adjoint operators and can give all eigen-values in some interval independent of h. See for instance [**Vu**] and further references given there. This case is also intimately related to the development of Fourier integral operator theory in the version of Maslov's canonical operator theory, [**Mas**].

When dropping the integrability condition, one can still justify the BS condition and get families of eigen-values for self-adjoint operators by using quantum and classical Birkhoff normal forms, sometimes in combination with the KAM theorem, but to the authors' knowledge, no result so far describes all the eigen-values in some h-independent non-trivial interval in the self-adjoint case. See Lazutkin [**La**], Colin de Verdière [**Co**], [**Sj4**], Bambusi–Graffi–Paul [**BaGrPa**] Kaidi-Kerdelhué [**KaKe**], Popov [**Po1, Po2**]. It therefore first seems that Theorem 0.2 (6.3) is remarkable in that it describes all eigen-values in an h-independent domain and that the non-self-adjoint case (for once!) is easier to handle than the self-adjoint one. The following philosophical remark will perhaps make our result seem more natural: In dimension 1, the BS-condition gives a sequence of eigen-values that are separated by a distance $\sim h$. In higher dimension $n \geqslant 2$, this cannot hold in the self-adjoint case, since an h-independent interval will typically contain $\sim h^{-n}$ eigen-values by Weyl asymptotics, so the average separation between eigen-values is $\sim h^n$. In dimension 2 however, we can get a separation of $\sim h$ between neighboring eigen-values for non-self-adjoint operators, since the number of eigen-values in some bounded open h-independent complex domain can be bounded from above by $\mathcal{O}(h^{-2})$ by general methods.

In section 7, we study resonances of a Schrödinger operator, generated by a saddle point of the potential and apply Theorem 6.3 and its proof. In this case, the resonances in a disc of radius Ch around the corresponding critical value of the potential were determined in [**Sj2**] for every fixed $C>0$, and this result was extended by Kaidi–Kerdelhué [**KaKe**] to a description of all resonances in a disc of radius h^δ, with $\delta>0$ arbitrary but independent of h. We show that the description of [**KaKe**] extends to give all resonances in some h-independent domain.

To prove Theorem 6.3, we use the machinery of FBI (here Bargman-) transformations and the corresponding calculus of pseudodifferential operators and Fourier integral operators on weighted L^2-spaces of holomorphic functions (see [**Sj1, Sj3**], [**HeSj**], [**MeSj**]). This allows us to define spaces $H(\Lambda)$ when Λ is an IR-manifold close to $\mathbf{R}^4$ in such a way that $H(\mathbf{R}^4)$ becomes the usual $L^2(\mathbf{R}^2)$ with the usual norm. Viewing p^w as an operator: $H(\Lambda) \to H(\Lambda)$, the corresponding leading symbol becomes $p_{|_\Lambda}$. We apply this to the IR-manifold $\Lambda(z)$ which contains $\Gamma(z)$ and get a reduction to the case when the characteristics of p (in $\Lambda(z)$) is a Lagrangian torus.

Contents of the paper

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1. Construction of complex Lagrangian torii in $p^{-1}(0)$ in dimension 2

We shall work in $\mathbf{R}^4 = T^*\mathbf{R}^2$ and its complexification $\mathbf{C}^4$, equipped with the standard symplectic form $\sigma = \sum_{j=1}^2 d\xi_j \wedge dx_j$. Let $\Gamma \subset \mathbf{R}^4$ be a smooth two-dimensional manifold, and assume that there exist real-valued real-analytic functions p_1 and p_2 defined in some tubular real neighborhood of Γ, which vanish on Γ and have linearly independent differentials at every point of Γ. We shall assume that

$$\sigma_{|_\Gamma} \text{ is small,} \tag{1.1}$$

in the sense that $|\langle \sigma, t \wedge s\rangle| \leqslant \varepsilon$ for all $\rho \in \Gamma$ and all $t, s \in T_\rho(\Gamma)$ with $|t|, |s| \leqslant 1$, where $\varepsilon > 0$ is sufficiently small. Here we use the standard norm on $\mathbf{R}^4$. It is tacitly assumed that nothing else degenerates when ε tends to 0; the tubular neighborhood is independent of ε, and p_j and all their derivatives satisfy uniform bounds there. Moreover

$|p_1| + |p_2|$ is bounded from below by a strictly positive constant near the boundary of the tubular neighborhood and we have a fixed positive lower bound on $|\lambda_1 dp_1 + \lambda_2 dp_2|$ uniformly in λ_1, λ_2 with $|\lambda_1|^2 + |\lambda_2|^2 = 1$. Under these additional uniformity assumptions, (1.1) is equivalent to saying that the Poisson bracket $\{p_1, p_2\} = \langle \sigma, H_{p_1} \wedge H_{p_2} \rangle$ is small ($\mathcal{O}(\varepsilon)$) on Γ. Indeed, if $\rho \in \Gamma$, then the symplectic orthogonal space to $T_\rho\Gamma$ is the space spanned by H_{p_1}, H_{p_2} and to say that the Poisson bracket is very small is equivalent to saying that the tangent space and its symplectic orthogonal are close to each other. (Alternatively, we may notice that there is a new symplectic form σ_ε in a tubular neighborhood of Γ with $\sigma_\varepsilon - \sigma = \mathcal{O}(\varepsilon)$, $\sigma_{\varepsilon}{}_{|_\Gamma} = 0$.) In what follows we extend p_1 and p_2 to holomorphic functions in a complex neighborhood of Γ and complexify Γ (the complexification is sometimes denoted $\Gamma_{\mathbf{C}}$). Then $\sigma_{|_{\Gamma_{\mathbf{C}}}} = \mathcal{O}(\varepsilon)$ in a full complex neighborhood of the original real manifold and with a new ε that we can take equal to the square root of the previous one. Since the complex vector field $H_p = H_{p_1} + iH_{p_2}$ is close to be tangent to Γ and H_{p_1}, H_{p_2} are linearly independent, it can be projected to an elliptic vector field on Γ. It is then a well-known fact (that we recall in Appendix B) that Γ is (diffeomorphic to) a torus.

We shall say that a multi-valued smooth function is grad-periodic if its differential is single-valued. Let x_1, x_2 be grad-periodic, real and real-analytic on Γ such that (x_1, x_2) induces an identification between the original torus and $\mathbf{R}^2/L$ for some lattice $L = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$. Extend x_1 to a real-analytic, grad-periodic (and real-valued) function in a tubular neighborhood of Γ in $\mathbf{R}^4$ in such a way that dx_1 vanishes on the orthogonal plane of $T_\rho\Gamma$ (w.r.t. the standard scalar product on $\mathbf{R}^4$) at every point $\rho \in \Gamma$. (We could even get a unique extension by requiring that x_1 be constant on the sets L_ρ of points in the (small) tubular neighborhood, which are closer to $\rho \in \Gamma$ than to any other point in Γ.) If $\sigma_{|_\Gamma}$ is sufficiently small, then $|H_{p_1}x_1| + |H_{p_2}x_1| \neq 0$, so H_{x_1} is transversal to Γ. Let $H \subset \mathbf{R}^4$ be a real-analytic closed hypersurface in a tubular neighborhood of Γ which contains Γ and is everywhere transversal to H_{x_1}. Extend x_2 real-analytically first to a grad periodic function on H, and then to a full tubular neighborhood in $\mathbf{R}^4$, by requiring that

$$\{x_1, x_2\} = 0. \tag{1.2}$$

We further extend x_1 and x_2 to grad-periodic holomorphic functions in a complex neighborhood of Γ. This will allow us to identify $\Gamma_{\mathbf{C}}$ with a complex neighborhood in $\mathbf{C}^2/L$ of $\mathbf{R}^2/L$. We notice that $\sigma_{|_{\Gamma_{\mathbf{C}}}} = f(x)dx_1 \wedge dx_2$, where $f(x) = \mathcal{O}(\varepsilon)$ and f is holomorphic in a full complex neighborhood of $\mathbf{R}^2/L$ in $\mathbf{C}^2/L$. Since σ is exact and $\mathcal{O}(\varepsilon)$ when restricted to Γ, there are real-analytic functions γ_1 and γ_2 on Γ, with values in $\mathbf{R}$ (hence single-valued) such that

$$\sigma_{|_\Gamma} = d(\gamma_1 dx_1 + \gamma_2 dx_2), \quad \gamma_1, \gamma_2 = \mathcal{O}(\varepsilon), \tag{1.3}$$

in the C^∞-sense. Since the Hamilton fields H_{x_1} and H_{x_2} commute in view of (1.2) and Jacobi's identity and span a space transversal to Γ at every point of Γ, we may find real-valued and real-analytic functions ξ_1 and ξ_2 in a neighborhood of Γ in $\mathbf{R}^4$ such that

$$\xi_{j}{}_{|_\Gamma} = \gamma_j,\ H_{x_j}\xi_k = -\delta_{jk}. \tag{1.4}$$

Proposition 1.1. — *(x,ξ) are symplectic coordinates for $\mathbf{R}^4$ in a neighborhood of Γ.*

Proof. — Locally we may find $(\widetilde{\xi_1},\widetilde{\xi_2})$ such that $(x,\widetilde{\xi})$ are symplectic coordinates. Since $H_{x_j}\widetilde{\xi_k} = -\delta_{jk} = H_{x_j}\xi_k$, it follows that $\xi_j - \widetilde{\xi_j} = g_j(x)$ is a function of x only. Then

$$\sum_1^2 d\xi_j \wedge dx_j - \sum_1^2 d\widetilde{\xi_j} \wedge dx_j = \sum_1^2 d(g_j(x)) \wedge dx_j. \tag{1.5}$$

Since the restriction to Γ of the left-hand side vanishes in view of (1.3) and (1.4) it follows that $\sum_1^2 d(g_j(x)) \wedge dx_j = 0$. Hence $\sum_1^2 d\xi_j \wedge dx_j = \sigma$. Since we know already that (x_1, x_2) is a coordinate system for Γ it follows that (x,ξ) is a coordinate system in a tubular neighborhood. □

In the coordinates (x,ξ), Γ takes the form

$$\xi = \gamma(x),\ \gamma = \mathcal{O}(\varepsilon),\ x \in \mathbf{R}^2/L, \tag{1.6}$$

where we view γ also as an L-periodic function in $\mathbf{R}^2$. Considering $p = p_1 + ip_2$ as a function in the new coordinates we get

$$\begin{aligned} p(x,\xi) &= p_1(x,\xi) + ip_2(x,\xi) \\ &= \sum_1^2 a_j(x)(\xi_j - \gamma_j(x)) + \sum_{j,k} b_{j,k}(x,\xi)(\xi_j - \gamma_j(x))(\xi_k - \gamma_k(x)) \\ &= \sum_1^2 a_j(x)\xi_j + \mathcal{O}(|\xi - \gamma(x)|^2) - r(x), \quad r(x) = \sum a_j(x)\gamma_j(x) = \mathcal{O}(\varepsilon) \end{aligned} \tag{1.7}$$

in the sense of holomorphic functions in a fixed tubular complex neighborhood of $\mathbf{R}^2_x \times \{\xi = 0\}$. With this point of view p is L-periodic in x.

We look for torii Γ_ϕ in a complex neighborhood of Γ of the form

$$\Gamma_\phi : \xi = \phi'(x), \quad x \in \mathbf{R}^2/L, \tag{1.8}$$

where ϕ is complex-valued and grad-periodic with $\nabla\phi \in \mathbf{C}^m$ for some $0 < m \in \mathbf{R} \smallsetminus \mathbf{N}$. We want $\Gamma_\phi \subset p^{-1}(0)$, so ϕ has to satisfy the Hamilton-Jacobi equation

$$p(x, \phi'(x)) = 0. \tag{1.9}$$

Using (1.7) we can write this as

$$Z\phi + F(x, \phi'(x) - \gamma(x)) - r(x) = 0, \tag{1.10}$$

where $Z = \sum_1^2 a_j(x)\frac{\partial}{\partial x_j}$ and $F(x,\xi) = \mathcal{O}(\xi^2)$, $r = \mathcal{O}(\varepsilon)$. Look for ϕ in the form

$$\phi = \widetilde{\varepsilon}\psi, \quad \varepsilon \ll \widetilde{\varepsilon} \ll 1. \tag{1.11}$$

Then ψ has to solve

$$Z\psi + \frac{1}{\widetilde{\varepsilon}}F(x, \widetilde{\varepsilon}(\psi' - \frac{\gamma}{\widetilde{\widetilde{\varepsilon}}})) - \frac{r(x)}{\widetilde{\widetilde{\varepsilon}}} = 0. \tag{1.12}$$

We look for solutions ψ with $\psi' = \mathcal{O}(1)$, and we rewrite (1.12) as

$$Z\psi + \widetilde{\varepsilon}G(x, \psi' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\widetilde{\varepsilon}}} = 0, \tag{1.13}$$

where

$$G(x, \xi; \widetilde{\varepsilon}) = \frac{1}{\widetilde{\varepsilon}^2}F(x, \widetilde{\varepsilon}\xi) \tag{1.14}$$

is holomorphic and uniformly bounded with respect to $\widetilde{\varepsilon}$ when $|\mathrm{Im}\, x|, |\xi| = \mathcal{O}(1)$.

Changing the x-coordinates and L conveniently, we may (by applying Theorem B.6), assume that

$$Z = A(x)\frac{\partial}{\partial \overline{x}}, \quad x = x_1 + ix_2, \tag{1.15}$$

where A is real-analytic and non-vanishing. (We now view L as a lattice in $\mathbf{C}$.) After division by $A(x)$, (1.13) becomes

$$\frac{\partial \psi}{\partial \overline{x}} + \widetilde{\varepsilon}G(x, \psi' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\widetilde{\varepsilon}}} = 0 \tag{1.16}$$

with new functions $G = G_{\mathrm{new}}$, $r = r_{\mathrm{new}}$, obtained from the earlier ones by division by $A(x)$ (and therefore satisfying the same estimates as before).

We look for solutions ψ of the form

$$\psi = \psi_{\mathrm{per}} + ax + b\overline{x}, \tag{1.17}$$

where ψ_{per} is periodic with respect to L and $a, b \in \mathbf{C}$. We shall apply an iteration procedure and get a corresponding solution for each a in the unit disc $D(0,1)$. So, let $u(x) = \psi_{\mathrm{per}} + b\overline{x}$ belong to the space of grad-periodic functions on $\mathbf{C}/L$ with antiholomorphic linear part. Then (1.16) becomes

$$\frac{\partial u}{\partial \overline{x}} + \widetilde{\varepsilon}G_a(x, u' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\widetilde{\varepsilon}}} = 0, \tag{1.18}$$

where

$$G_a(x, \xi; \widetilde{\varepsilon}) = G(x, \xi + a dx; \widetilde{\varepsilon}),$$

and dx denotes the complex cotangent vector given by the differential of x. Notice that G_a depends holomorphically on a.

Fix $m \in \mathbf{R}_+ \setminus \mathbf{N}$, and solve (1.18) for $u' \in C^m$ by the natural iteration procedure $u_0 = 0$,

$$\frac{\partial u_{j+1}}{\partial \overline{x}} + \widetilde{\varepsilon}G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}} = 0, \quad j \geqslant 0. \tag{1.19}$$

Write $u_j(x) = u_{j,\mathrm{per}}(x) + b_j\overline{x}$. If u_j has already been determined, then considering the Fourier series expansion of $u_{j+1,\mathrm{per}}$, we see that

$$b_{j+1} = -\mathcal{F}(\widetilde{\varepsilon}G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}})(0), \tag{1.20}$$

where $\mathcal{F}v(0)$ denotes the 0:th Fourier coefficient of the function v with respect to L. We see that $u_{j+1,\mathrm{per}}$ is uniquely determined modulo a constant through the equation

$$\frac{\partial u_{j+1,\mathrm{per}}}{\partial \overline{x}} + \widetilde{\varepsilon}G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - \frac{r(x)}{\widetilde{\varepsilon}} + b_{j+1} = 0. \tag{1.21}$$

For $j = 0$, we get $b_1 = \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}})$. Applying a basic result about the boundedness in $C^m(\mathbf{R}^2/L)$ of Calderon–Zygmund operators (see [**BeJoSc**]) and considering also Fourier expansions, we get the bound

$$\|u_{1,\mathrm{per}}'\|_{C^m} \leqslant \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}}).$$

For $j \geqslant 1$, we write

$$b_{j+1} - b_j + \widetilde{\varepsilon}\mathcal{F}(G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - G_a(x, u_{j-1}' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}))(0) = 0 \tag{1.22}$$

and

$$\frac{\partial}{\partial \overline{x}}(u_{j+1,\mathrm{per}} - u_{j,\mathrm{per}}) + \widetilde{\varepsilon}(G_a(x, u_j' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon}) - G_a(x, u_{j-1}' - \frac{\gamma}{\widetilde{\varepsilon}}; \widetilde{\varepsilon})) + (b_{j+1} - b_j) = 0. \tag{1.23}$$

From (1.22) we get

$$|b_{j+1} - b_j| \leqslant \mathcal{O}(\widetilde{\varepsilon})(\|u_{j,\mathrm{per}}' - u_{j-1,\mathrm{per}}'\|_{C^m} + |b_j - b_{j-1}|),$$

and using this in (1.23) together with (1.22), we get

$$\|u_{j+1,\mathrm{per}}' - u_{j,\mathrm{per}}'\|_{C^m} + |b_{j+1} - b_j| \leqslant \mathcal{O}(\widetilde{\varepsilon})(\|u_{j,\mathrm{per}}' - u_{j-1,\mathrm{per}}'\|_{C^m} + |b_j - b_{j-1}|). \tag{1.24}$$

So, if $\widetilde{\varepsilon}$ (and ε) is small enough, our procedure converges to a solution

$$u = u_{\mathrm{per}} + b\overline{x} \tag{1.25}$$

of (1.18) with

$$\|u_{\mathrm{per}}'\|_{C^m} + |b| = \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}}). \tag{1.26}$$

Summing up we have for a given m:

Proposition 1.2. — *Let $C \geqslant 1$ be large enough. For $0 < \varepsilon \ll \widetilde{\varepsilon}$ small enough and for $|a| < 1$, the equation (1.18) has a solution u of the form (1.25) with $|b| + \|u_{\mathrm{per}}'\|_{C^m} \leqslant 1/C$. This solution is unique modulo constants and satisfies (1.26).*

Proof of the uniqueness. — Let $u_{\mathrm{per}} + b\overline{x}$ and $\widetilde{u} = \widetilde{u}_{\mathrm{per}} + \widetilde{b}\overline{x}$ be two solutions of (1.18). Then as above, we have

$$\|u_{\mathrm{per}}' - \widetilde{u}_{\mathrm{per}}'\|_{C^m} + |b - \widetilde{b}| \leqslant \mathcal{O}(\widetilde{\varepsilon})(\|u_{\mathrm{per}}' - \widetilde{u}_{\mathrm{per}}'\|_{C^m} + |b - \widetilde{b}|),$$

and the uniqueness follows. ☐

This means that we have solved (1.9) with

$$\phi = \widetilde{\varepsilon}(u_{\mathrm{per}} + ax + b\overline{x}), \quad 0 < \varepsilon \ll \widetilde{\varepsilon} \ll 1, \tag{1.27}$$

where $|a| < 1$, and b, u_{per} depend on the choice of a (and of $\widetilde{\varepsilon}$). In (1.27) it is further assumed that x_1, x_2 are chosen so that (1.15) holds.

We next show that ϕ' depends holomorphically on a, and for that we again consider (1.18), where we recall that G_a depends holomorphically on a. This is actually immediate because the preceding iteration argument trivially extends to the case of functions of a: $u = u_{\mathrm{per}}(x,a) + b(a)\overline{x}$, with

$$D(0,1) \ni a \longmapsto (u'_{\mathrm{per}}(\cdot\,,a), b(a)) \in C^m \times \mathbf{C} \tag{1.28}$$

holomorphic. Hence (after imposing the extra condition that $\mathcal{F}u_{\mathrm{per}}(0) = 0$) we have

Proposition 1.3. — *u_{per}, b and hence ϕ depend holomorphically on a.*

Now let $p = p_z$ depend holomorphically on a spectral parameter $z \in D(0,1)$ and assume that $p_z = \mathcal{O}(1)$ uniformly in some fixed tubular neighborhoood of $\mathbf{R}^4$. Assume that p_0 fulfills the assumptions of p above. Choose coordinates (x,ξ) as above for $p = p_0$. We now look for $\Gamma_\phi \subset p_z^{-1}(0)$ of the form (1.8), and (1.10) becomes:

$$Z\phi + F(x, \phi'(x) - \gamma(x); z) - r(x,z) = 0, \tag{1.29}$$

where $F(x,\xi;z)$, $r(x,z)$ depend holomorphically on z. If we restrict the attention to $|z| < \mathcal{O}(\widetilde{\varepsilon})$, then the previous considerations go through and we get a solution

$$\phi = \phi_a = \phi_{a,z}(x) = \widetilde{\varepsilon}(u_{\mathrm{per}}(x,z,a) + ax + b(z,a)\overline{x}) \tag{1.30}$$

depending holomorphically on z, a with $|z| < \frac{\widetilde{\varepsilon}}{C}$, $|a| < \frac{1}{C}$, and

$$\|u'_{\mathrm{per}}(\cdot\,, z, a)\|_{C^m} + |b| = \mathcal{O}(1). \tag{1.31}$$

We shall now extend ϕ to the complex domain in x. Let $\widetilde{\phi}(x) \in C^{k+1}(\mathbf{C}^2)$ denote an almost holomorphic extension of ϕ, where k is a positive integer and m has been chosen larger than k. (Here we consider $\widetilde{\phi}$ as a grad-periodic function in $\mathbf{R}^4$.) Then $p(x, \partial_x\widetilde{\phi}(x))$ vanishes to the order k on $\mathbf{R}^2$, and the corresponding manifold $\Lambda_{\widetilde{\phi}} = \{(x, \partial_x\widetilde{\phi}(x));\ x \in \mathbf{C}^2\}$ is to that order a complex Lagrangian manifold at the points of intersection with $\mathbf{R}^2_x \times \mathbf{C}^2_\xi$. This intersection is nothing else but Γ_ϕ in (1.8).

The complex Hamilton field H_p is transversal to $\mathbf{R}^2_x \times \mathbf{C}^2_\xi$ at the points of Γ_ϕ and we form the flow out

$$\Lambda_\phi = \{\exp \widehat{tH_p}(\rho);\ \rho \in \Gamma_\phi,\ t \in \mathbf{C},\ |t| < 1/\mathcal{O}(1)\}. \tag{1.32}$$

Here $\widehat{tH_p} = tH_p + \overline{tH_p}$ is the real vectorfield (in the complex domain) which has the same action as tH_p as differential operators acting on holomorphic functions. Since $\widehat{tH_p}$ is tangential to $\Lambda_{\widetilde{\phi}}$ to the order k at Γ_ϕ, we see that Λ_ϕ is tangential to $\Lambda_{\widetilde{\phi}}$ there. In particular $T_\rho\Lambda_\phi$ is a complex Lagrangian space for every $\rho \in \Gamma_\phi$. Since $\exp \widehat{tH_p}$ are complex canonical transformations, the same fact is true for the tangent spaces

$T_{\exp \widehat{tH_p}(\rho)}\Lambda_\phi = (\exp \widehat{tH_p})_* T_\rho \Lambda_\phi$ at an arbitrary point $\exp \widehat{tH_p}(\rho) \in \Lambda_\phi$. Hence Λ_ϕ is a complex Lagrangian manifold. Restricting the size of $|t|$ in (1.32) we see also that the projection $\Lambda_\phi \ni (x,\xi) \mapsto x$ is a holomorphic diffeomorphism, so Λ_ϕ is of the form $\{\xi = \phi'(x);\ |\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)}\}$ for a function ϕ which is a holomorphic extension of the previously constructed one.

Let $\Lambda \subset p^{-1}(0)$ be a relatively closed complex Lagrangian manifold in a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$ and assume that Λ contains a torus Γ which is $\widehat{\varepsilon}$-close to $p^{-1}(0) \cap \mathbf{R}^4$ in C^1, for $\varepsilon \ll \widehat{\varepsilon} \ll 1$. Let (x,ξ) be the coordinates constructed above. If $\rho \in \Gamma$, we know that $T_\rho\Gamma$ is $\widehat{\varepsilon}$-close to $\mathbf{R}^2_x \times \{\xi = 0\}$, so $T_\rho\Lambda$ is $\widehat{\varepsilon}$-close to $\mathbf{C}^2_x \times \{\xi = 0\}$. Using that Λ is locally invariant under the $\widehat{\mathbf{C}H_p}$-flow, we see that Λ is of the form $\{(x,\phi'(x));\ |\mathrm{Im}\, x| < \frac{1}{C}\}$ in a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$, where ϕ is holomorphic and grad-periodic with $\phi' = \mathcal{O}(\widetilde{\varepsilon})$, $\widetilde{\varepsilon} = \mathcal{O}(\widehat{\varepsilon}^{1/2})$. Moreover, we have the eiconal equation $p(x,\phi'(x)) = 0$ and restricting it to $\mathbf{R}^2$, we get (1.9), and Proposition 1.2 shows that $\phi = \phi_a$ for some a. Hence in a neighborhood of $p^{-1}(0) \cap \mathbf{R}^4$, Λ coincides with Λ_ϕ in (1.32).

The parameter dependence of ϕ in (1.27) behaves as expected: Clearly the holomorphic extension $\phi(x,a,z)$ depends in a C^1-fashion of a (and possibly z), and we know that it is holomorphic in a and z when x is real. Then $\frac{\partial\phi}{\partial\overline{a}}$, $\frac{\partial\phi}{\partial\overline{z}}$ are holomorphic in x and vanish for real x. Consequently they vanish for all x. Summing up we have shown:

Proposition 1.4. — *The function ϕ in (1.27) depends holomorphically on (x,a,z) in a domain*

$$|\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)},\ |\alpha| < \frac{1}{C},\ |z| < \frac{\varepsilon}{\mathcal{O}(1)}.$$

We shall next show (in the z-independent case) that the Λ_{ϕ_a} form a complex fibration of $p^{-1}(0)$ in a region where $|\xi| < \frac{\widetilde{\varepsilon}}{\mathcal{O}(1)}$, $|\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)}$. Let first x be real. From Propositions 1.2 and 1.3, we see that

$$\frac{\partial}{\partial a}u'_{\mathrm{per}},\ \frac{\partial}{\partial a}b = \mathcal{O}(\widetilde{\varepsilon} + \frac{\varepsilon}{\widetilde{\varepsilon}}), \tag{1.33}$$

and consequently for ϕ in (1.27), we get for the x-differential $\phi'_x = d_x\phi$:

$$\frac{\partial}{\partial a}\phi'_x = \widetilde{\varepsilon}dx + \mathcal{O}(\varepsilon + \widetilde{\varepsilon}^2). \tag{1.34}$$

In order to treat the case of complex x, we notice that the geometric arguments leading to Proposition 1.4 together with the form $\sum a_j(x)\xi_j + \widetilde{\varepsilon}^{-1}F(x,\xi - \frac{\gamma}{\widetilde{\varepsilon}};\widetilde{\varepsilon}) - r(x)/\widetilde{\varepsilon}$ for the Hamiltonian for ψ, show that $u'_{\mathrm{per}} = \mathcal{O}(\widetilde{\varepsilon} + \varepsilon/\widetilde{\varepsilon})$ also in the complex domain, and hence by the Cauchy inequality (in a) that (1.34) holds also for $|\mathrm{Im}\, x| < \frac{1}{\mathcal{O}(1)}$. This shows that

$$a \longmapsto \phi'_x \in (p(x,\cdot))^{-1}(0) \tag{1.35}$$

is a local diffeomorphism and hence that the Λ_{ϕ_a} form a foliation of $p^{-1}(0)\cap\{(x,\xi):|\xi|<\frac{\widetilde{\varepsilon}}{\mathcal{O}(1)},\ |\mathrm{Im}\,x|<\frac{1}{\mathcal{O}(1)}\}$ in the natural sense. (Recall that $\widetilde{\varepsilon}$ can be close to a fixed constant so we get a foliation in $\{(x,\xi):|\xi|<\frac{1}{\mathcal{O}(1)},\ |\mathrm{Im}\,x|<\frac{1}{\mathcal{O}(1)}\}$.)

We next consider the actions associated to a torus. Let $\gamma_j(a)$ be a closed curve in Γ_{ϕ_a} (assuming $\widetilde{\varepsilon}>0$ fixed) which corresponds to e_j in the natural way, where we recall that $Z=A(x)\frac{\partial}{\partial\overline{x}}$, $x\in\mathbf{C}/L$, and take $L=\mathbf{Z}e_1\oplus\mathbf{Z}e_2$. If ω is a $(1,0)$-form with holomorphic coefficients, such that $d\omega=\sigma$ near Γ_{ϕ_a}, then we define the actions

$$I_j(\Gamma_{\phi_a},\omega)=\int_{\gamma_j(a)}\omega. \tag{1.36}$$

These only depend on the homotopy class of $\gamma_j(a)$ in Γ_{ϕ_a}, and we can even deform $\gamma_j(a)$ from this set into the complex, provided that we stay inside the complex Lagrangian manifold Λ_{ϕ_a}. Also notice that if $\widetilde{\omega}$ is another $(1,0)$-form with the same properties, then

$$\int_\gamma\omega-\int_\gamma\widetilde{\omega}$$

only depends on the homotopy class of γ as a closed curve in the intersection of the domains of definition of ω and $\widetilde{\omega}$. In particular,

$$I_j(\Gamma_{\phi_a},\omega)-I_j(\Gamma_{\phi_a},\widetilde{\omega})=C_j \tag{1.37}$$

is a constant which is independent of a (and of z if we let p depend holomorphically on z). If ω and $\widetilde{\omega}$ are both real in the real domain then C_j in (1.37) is real.

For the special x-coordinates above, we let ξ be the corresponding coordinates constructed in the beginning of this section and we choose

$$\widetilde{\omega}=\sum_1^2\xi_j dx_j. \tag{1.38}$$

Then

$$I_j(\Gamma_{\phi_a},\widetilde{\omega})=\phi_a(e_j)-\phi_a(0)$$

depends holomorphically on a, and from (1.11), (1.17) and Proposition 1.2 we get

$$I_j(\Gamma_{\phi_a},\widetilde{\omega})=\widetilde{\varepsilon}(ae_j+b\overline{e_j})=\widetilde{\varepsilon}ae_j+\mathcal{O}(\varepsilon+\widetilde{\varepsilon}^2). \tag{1.39}$$

For ω we can choose the fundamental 1-form in the original coordinates on $\mathbf{R}^4$ (formally given by the right-hand side of (1.38) for these original coordinates (x,ξ)). Thus

$$I_j(\Gamma_{\phi_a},\omega)=C_j+\widetilde{\varepsilon}ae_j+\mathcal{O}(\varepsilon+\widetilde{\varepsilon}^2). \tag{1.40}$$

From this we see that we can use, say, $I_1(\Gamma_{\phi_a},\omega)\in C_j+D(0,\widetilde{\varepsilon}/\mathcal{O}(1))$ as a new holomorphic parameter instead of a. In the z-dependent case, we can replace the parameters (a,z) by $(I_1,z)=(I_1(\Gamma_{\phi_a},\omega),z)$ and the correspondence $(a,z)\mapsto(I_1,z)$ is biholomorphic.

The advantage of using I_1 instead of a as a parameter, is that the family Λ_{ϕ_a} is now independent of the way we choose the coordinates (x, ξ) in the beginning of this section, so we get an intrinsic parametrisation. From (1.40) it follows that

$$\frac{dI_2}{dI_1} = \frac{e_2}{e_1} + \mathcal{O}(\widetilde{\varepsilon} + \varepsilon/\widetilde{\varepsilon}), \tag{1.41}$$

so $\operatorname{Im} \frac{dI_2}{dI_1} \neq 0$, and we have a unique value $a = \mathcal{O}(\widetilde{\varepsilon} + \varepsilon/\widetilde{\varepsilon})$ for which I_1 and I_2 are both real.

There are two related reasons why we want to select Γ_{ϕ_a}, with both I_1 and I_2 real. The first reason is geometric: Γ_{ϕ_a} is a small deformation of a real torus $\Gamma \subset \mathbf{R}^4$ and we want to find an I-Lagrangian manifold $\Lambda \subset \mathbf{C}^4$ which is a small deformation of $\mathbf{R}^4$ and which contains Γ_{ϕ_a}. If we have such a Λ, the cycles $\gamma_j(\Gamma_{\phi_a})$, $j = 1, 2$ become boundaries of some 2-dimensional discs $D_j \subset \Lambda$ and we get

$$I_j(\Gamma_{\phi_a}, \omega) = \int_{\gamma_j} \omega = \int_{D_j} \sigma \in \mathbf{R},$$

since $\sigma_{|\Lambda}$ is real.

Conversely, let $\widetilde{\Gamma}$ be a two-dimensional torus which is a small perturbation of Γ with

$$\sigma_{|\widetilde{\Gamma}} = 0, \ \operatorname{Im} I_j(\widetilde{\Gamma}, \omega) = 0, \ j = 1, 2. \tag{1.42}$$

We can construct an I-Lagrangian manifold $\Lambda \supset \widetilde{\Gamma}$ as a small perturbation of $\mathbf{R}^4$ in the following way: After applying a complex linear canonical transformation, we may replace $\mathbf{R}^4$ by Λ_{Φ_0}: $\xi = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)$, $x \in \mathbf{C}^2$, where Φ_0 is a strictly plurisubharmonic quadratic form (see [**Sj1, Sj3**]), so that $\widetilde{\Gamma}$ becomes a small perturbation of a torus $\Gamma \subset \Lambda_{\Phi_0}$. The canonical 1-form ω is now transformed into some other globally defined 1-form $\widetilde{\omega}$ with holomorphic coefficients satisfying $d\widetilde{\omega} = \sigma$, but the actions $I_j(\widetilde{\Gamma}, \widetilde{\omega})$ do not change if we replace $\widetilde{\omega}$ by $\xi \cdot dx$, so

$$\int_{\gamma_j(\widetilde{\Gamma})} \xi \cdot dx \in \mathbf{R}, \ j = 1, 2. \tag{1.43}$$

We can write this as

$$\int_{\gamma_j(\widetilde{\Gamma})} (-\operatorname{Im}(\xi \cdot dx)) = 0, \tag{1.44}$$

where $-\operatorname{Im} \xi \cdot dx$ is a primitive of $-\operatorname{Im} \sigma$, so $-\operatorname{Im} \xi \cdot dx_{|\widetilde{\Gamma}}$ is closed. (1.44) then implies that it is exact:

$$-\operatorname{Im} \xi \cdot dx_{|\widetilde{\Gamma}} = d\phi, \tag{1.45}$$

where ϕ is a smooth real-valued function on $\widetilde{\Gamma}$. We now view ϕ as a function on the x-space projection $\pi_x(\widetilde{\Gamma})$ of $\widetilde{\Gamma}$, which is also a smooth torus and represent $\widetilde{\Gamma}$ by $\xi = \widetilde{\xi}(x)$, $x \in \pi_x(\widetilde{\Gamma})$. Then with the obvious identifications, (1.45) becomes

$$-\operatorname{Im}(\widetilde{\xi}(x) \cdot dx)_{|\pi_x(\widetilde{\Gamma})} = d\phi, \ \text{on } \pi_x(\widetilde{\Gamma}). \tag{1.46}$$

We can find real smooth extensions Φ of ϕ to $\mathbf{C}_x^2$ with an arbitrary prescription of the conormal part of the derivative, so we can choose Φ satisfying

$$-\mathrm{Im}\,(\widetilde{\xi}(x)\cdot dx) = d\Phi(x),\ \forall x\in\pi_x(\widetilde{\Gamma}). \tag{1.47}$$

This means that

$$-\frac{1}{2i}\widetilde{\xi}(x)dx + \frac{1}{2i}\overline{\widetilde{\xi}}(x)\overline{dx} = d\Phi,\ x\in\pi_x(\widetilde{\Gamma}),$$

or that

$$\widetilde{\xi}(x) = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x),\ x\in\pi_x(\widetilde{\Gamma}). \tag{1.48}$$

Since $\widetilde{\Gamma}$ is close to Γ, $\frac{\partial\Phi}{\partial x}(x)$ is close to $\frac{\partial\Phi_0}{\partial x}$ on $\pi_x(\widetilde{\Gamma})$, and we may choose the extension Φ so that $\frac{\partial\Phi}{\partial x}-\frac{\partial\Phi_0}{\partial x}$ is small everywhere. The I-Lagrangian manifold $\Lambda=\Lambda_\Phi$ given by $\xi=\frac{2}{i}\frac{\partial\Phi}{\partial x}$ then has the desired properties when $\mathbf{R}^4$ is replaced by Λ_{Φ_0}, and applying the inverse of the above mentioned complex linear canonical transformation, we get the desired Λ in terms of the original problem.

The second reason, why we want $I_1(\Gamma_{\phi_a},\omega)$ and $I_2(\Gamma_{\phi_a},\omega)$ to be real comes from the Bohr-Sommerfeld, Einstein, Keller, Maslov quantization condition. The actions $I_j(\Gamma_{\phi_a},\omega)$ coincide with the corresponding actions $I_j(\Lambda_{\phi_a},\omega)$, and if we want Λ_{ϕ_a} to correspond to an eigenstate of some pseudodifferential operator with leading symbol p and eigenvalue $o(h)$, it is natural to impose a quantization condition of the type

$$I_j(\Lambda_{\phi_a},\omega) = 2\pi k_j h,\ k_j\in\mathbf{Z}, \tag{1.49}$$

where we choose to ignore the Maslov indices, and where $h>0$ is the small semi-classical parameter. Since Λ_{ϕ_a} are not real Lagrangian manifolds (even after introducing Λ as a new real phase space), the quantization condition (1.49) will need an entirely new justification.

Consider the case when p depends on z and choose $w=I_1(\Lambda_{\phi_{a,z}},\omega)$ so that we can use the simplified notation $\Lambda_{(z,w)}$ for $\Lambda_{\phi_{a,z}}$. Also write $\nu=(z,w)$. Recall that

$$\mathrm{Im}\,\frac{dI_2}{dI_1}\neq 0 \tag{1.50}$$

when z is kept constant. It follows that there is a unique smooth function $z\mapsto w(z)\in\mathbf{C}$ such that $I_j(z,w(z))$ are real for $j=1,2$, where we write $I_j(z,w)=I_j(\Lambda_{(z,w)},\omega)$. We will be interested in the property

$$z\longmapsto I(z,w(z)) = (I_1(z,w(z)),I_2(z,w(z)))\in\mathbf{R}^2 \text{ is a local diffeomorphism.} \tag{1.51}$$

This is equivalent to the property

$$\nu\longmapsto (I_1(\Lambda_\nu),I_2(\Lambda_\nu))\in\mathbf{C}^2 \text{ is locally biholomorphic.} \tag{1.52}$$

In fact, if $\delta_z\in\mathbf{C}$ belongs to the kernel of the differential of the map (1.51) at some point, then (δ_z,δ_w) with $\delta_w=\frac{\partial w}{\partial z}\delta_z+\frac{\partial w}{\partial\overline{z}}\overline{\delta_z}$ will belong to the kernel of the differential of (1.52) at the corresponding point. Conversely if (δ_z,δ_w) is in the kernel of the differential of (1.52) at some point (z,w) with w real (so that $w = w(z)$), then necessarily $\delta_w = \frac{\partial w}{\partial z}\delta_z + \frac{\partial w}{\partial \overline{z}}\overline{\delta_z}$ for some δ_z in the kernel of the differential of (1.51).

Example. — Let $p = p_1(x_1,\xi_1) + ip_2(x_2,\xi_2)$, where p_j is real with $p_j^{-1}(0)$ being a closed curve in $\mathbf{R}^2$ on which $dp_j \neq 0$. For E in a small complex neighborhood of 0, we put

$$A_j(E) = \int_{p_j^{-1}(E)} \xi_j dx_j$$

and notice that these one dimensional actions are real for real E and that $A_j'(E) \neq 0$. With $z, w \in \mathbf{C}$ close to 0, we get the complex fibration

$$\Lambda_{z,w} = \{(x,\xi) \in \mathbf{C}^4; p_1(x_1,\xi_1) = w,\ ip_2(x_2,\xi_2) = z - w\}.$$

Then

$$I_1(\Lambda_{z,w}) = A_1(w),\ I_2(\Lambda_{z,w}) = A_2\Big(\frac{z-w}{i}\Big),$$

and we see that (1.51) and (1.52) hold.

Appendix A: Reduction of elliptic vector fields on a torus

Let Z be a smooth complex elliptic vector field on $\mathbf{T}^2 = (\mathbf{R}/\mathbf{Z})^2$. After left multiplication by a non-vanishing function and possibly reversal of one of the coordinates, we may assume that with $z = x_1 + ix_2$:

$$Z = \frac{\partial}{\partial \overline{z}} + g\frac{\partial}{\partial z},\ \|g\|_\infty < 1,\ g \in C^\infty. \tag{A.1}$$

Let

$$\mathcal{H}^1 = \{u = a\overline{z} + v;\ a \in \mathbf{C},\ v \in H^1_{\mathrm{per}},\ \widehat{v}(0) = 0\}, \tag{A.2}$$

where

$$H^k_{\mathrm{per}} = \{v \in H^k_{\mathrm{loc}}(\mathbf{R}^2); v(x+\gamma) = v(x), \forall\gamma \in \mathbf{Z}^2\}$$

and $\widehat{v}(k)$ is the kth Fourier coefficient and H^s is the standard Sobolev space. Let $\mathcal{H}^0 = H^0_{\mathrm{per}}$, and let $\|\cdot\|$ denote the L^2 norm on the torus (i.e. the H^0_{per} norm) if nothing else is specified. We choose the norm in $\mathcal{H}^1$ with

$$\|u\|^2_{\mathcal{H}^1} = |a|^2 + \Big\|\frac{\partial v}{\partial z}\Big\|^2 = |a|^2 + \Big\|\frac{\partial v}{\partial \overline{z}}\Big\|^2, \tag{A.3}$$

for $u = a\overline{z} + v \in \mathcal{H}^1$. Since $\frac{\partial}{\partial \overline{z}}(a\overline{z} + v) = a + \frac{\partial v}{\partial \overline{z}}$ (orthogonal sum), we see that

$$\Big\|\frac{\partial u}{\partial \overline{z}}\Big\|_{\mathcal{H}^0} = \|u\|_{\mathcal{H}^1}. \tag{A.4}$$

Moreover $\frac{\partial}{\partial \overline{z}} : \mathcal{H}^1 \to \mathcal{H}^0$ is surjective, so in view of (A.4) it is unitary. It is also clear that $\frac{\partial}{\partial z} : \mathcal{H}^1 \to \mathcal{H}^0$ is of norm 1:

$$\Big\|\frac{\partial u}{\partial z}\Big\|_{\mathcal{H}^0} \leqslant \|u\|_{\mathcal{H}^1}. \tag{A.5}$$

Since $\|g\frac{\partial}{\partial z}\|_{\mathcal{H}^1\to\mathcal{H}^0} < 1$, we see that $Z : \mathcal{H}^1 \to \mathcal{H}^0$ is bijective with inverse Z^{-1} satisfying

$$\|Z^{-1}\|_{\mathcal{H}^0\to\mathcal{H}^1} \leqslant \frac{1}{1-\|g\|_\infty}.$$

Consider the function

$$u = z - Z^{-1}(g) \in z + \mathcal{H}^1. \tag{A.6}$$

It is clear that

$$Zu = 0, \tag{A.7}$$

and u is the unique function in $z+\mathcal{H}^1$ which is annihilated by Z. It follows that the kernel of Z, acting on $\{u = \text{linear function} + v;\ v \in H^1_{\text{per}},\ \widehat{v}(0) = 0\}$, is of dimension 1.

Lemma A.1. — $\overline{Z}u \neq 0$ *everywhere.*

Proof. — $\overline{Z}u$ cannot be identically zero since otherwise we would have both $Zu = 0$ and $\overline{Z}u = 0$, implying that u is constant; which is impossible.

We have

$$[Z,\overline{Z}] = \overline{a}Z - a\overline{Z} \tag{A.8}$$

for some $a \in C^\infty_{\text{per}}$. Then $Z\overline{Z}u = -a\overline{Z}u$, so

$$(Z+a)(\overline{Z}u) = 0. \tag{A.9}$$

It is well known that if v is a null solution of a 1st order elliptic equation on a connected domain and v is not identically zero, then v cannot vanish to infinite order at any point, and (by looking at Taylor expansions) the zeros are all isolated. We can apply this to $v = \overline{Z}u$. We also see that the argument variation of $\overline{Z}u$, along a small positively oriented circle around a zero is equal to $2\pi k$ for some finite integer $k > 0$. Let $\Gamma = \partial\Omega$, where $\Omega = z_0 + (]0,1[+i]0,1[)$ and z_0 is chosen so that $\overline{Z}u$ has no zeros on Γ. If $\overline{Z}u$ has at least one zero in $\mathbf{T}^2$, then it has a zero in Ω and $\operatorname{var}\arg_\Gamma(\overline{Z}u) > 0$. This is in contradiction with the fact that $\overline{Z}u$ is periodic and hence that $\operatorname{var}\arg_\Gamma(\overline{Z}u) = 0$. It follows that

$$\overline{Z}u \neq 0. \tag{A.10}$$

□

If we view u as a map $\mathbf{R}^2 \to \mathbf{R}^2$, it follows from (A.7,10), that the corresponding Jacobian is everywhere $\neq 0$. It follows that $u = u_1 + iu_2$ is a diffeomorphism from $\mathbf{C}$ to $\mathbf{C}$. Let

$$u(z+1) - u(z) =: e_1,\ u(z+i) - u(z) =: e_2. \tag{A.11}$$

Then e_1, e_2 are $\mathbf{R}$-linearly independent, and we let $L = \mathbf{Z}e_1 + \mathbf{Z}e_2$ be the corresponding lattice. Using that $u : \mathbf{C} \to \mathbf{C}$ is a diffeomorphism, we see that the induced map $[u] : \mathbf{T}^2 \to \mathbf{C}/L$ is bijective. (Only the injectivity needs to be checked: Let $x, y \in \mathbf{T}^2$ with $[u](x) = [u](y) := u_0$. We can find corresponding points $\widetilde{x}, \widetilde{y}, \widetilde{u}_0 \in \mathbf{C}$, such that

$u(\widetilde{x})=\widetilde{u}_0$, $u(\widetilde{y})=\widetilde{u}_0+k_1e_1+k_2e_2$, $k_j\in\mathbf{Z}$. Then $u(\widetilde{y}-k_1-k_2i)=\widetilde{u}_0$, so by the injectivity of u, we have $\widetilde{x}=\widetilde{y}-k_1-k_2i$ and hence $x=y$.)

If $f(w)$ is a C^1 function on $\mathbf{C}$, then

$$Z(f(u(z)))=\frac{\partial f}{\partial w}Zu+\frac{\partial f}{\partial\overline{w}}Z(\overline{u})=Z(\overline{u})\frac{\partial f}{\partial\overline{w}}.$$

In other words, if we let lower $*$ indicate push forward of vector fields, then

$$u_*(Z)=Z(\overline{u})\frac{\partial}{\partial\overline{w}},\ [u]_*(Z)=Z(\overline{u})\frac{\partial}{\partial\overline{w}}. \tag{A.12}$$

Conversely, if $\widetilde{L}$ is some lattice and $[t]:\mathbf{T}^2\to\mathbf{C}/\widetilde{L}$ a diffeomorphism corresponding to a grad periodic function t with

$$t_*(Z)=F\frac{\partial}{\partial\overline{w}},\ F\neq 0 \text{ everywhere}, \tag{A.13}$$

then

$$Z(t)=0. \tag{A.14}$$

Since $t\in\mathbf{C}z+\mathcal{H}^1$, we know that $\exists\, 0\neq\alpha\in\mathbf{C}$ such that $t=\alpha u$. Consequently,

$$\widetilde{L}=\alpha L. \tag{A.15}$$

Actually, we can see this more directly, by considering the biholomorphic map $[u][t]^{-1}$.

It follows from our constructions that if Z depends smoothly (real-analytically) on an additional parameter w, then so does u.

Appendix B: 2-dimensional manifolds with elliptic vector fields

Let M be a smooth compact connected 2-dimensional manifold with an elliptic (complex) vector field Z. We shall see that M is diffeomorphic to a torus $\mathbf{C}/L$ in such a way that Z maps to a multiple of $\frac{\partial}{\partial\overline{z}}$. Clearly $Z:H^1(M)\to H^0(M)$ is a Fredholm operator. Let $\operatorname{ind}Z=\dim\mathcal{N}(Z)-\operatorname{codim}\mathcal{R}(Z)=\dim\mathcal{N}(Z)-\dim\mathcal{N}(Z^*)$ be the index, where Z^* denotes the adjoint of Z with respect to some positive density on M. Recall that the kernels $\mathcal{N}(Z)$, $\mathcal{N}(Z^*)$ are contained in $C^\infty(M)$, since Z and Z^* are elliptic.

Lemma B.1. — $\operatorname{ind}Z=0$.

Proof. — Clearly $\operatorname{ind}Z^*=-\operatorname{ind}Z$. On the other hand $Z^*=-\overline{Z}+f$ for some $f\in C^\infty(M)$ and since the index is stable under changes of the lower order part:

$$\operatorname{ind}Z^*=\operatorname{ind}(-\overline{Z})=\operatorname{ind}\overline{Z}=\operatorname{ind}Z.$$

Here the last equality follows from the fact that $\mathcal{N}(\overline{Z})=\overline{\mathcal{N}(Z)}$, $\mathcal{R}(\overline{Z})=\overline{\mathcal{R}(Z)}$. Then $\operatorname{ind}Z=-\operatorname{ind}Z^*=-\operatorname{ind}Z$, and hence $\operatorname{ind}Z=0$. □

Because of the ellipticity, there is a unique $a\in C^\infty(M)$, such that

$$[Z,\overline{Z}]=\overline{a}Z-a\overline{Z}. \tag{B.1}$$

Lemma B.2. — *$P := -(Z+a)\overline{Z}$ is a real differential operator.*

Proof. — $\overline{P} - P = (Z+a)\overline{Z} - (\overline{Z}+\overline{a})Z = [Z,\overline{Z}] - (\overline{a}Z - a\overline{Z}) = 0$. □

Let us identify M with the zero section in T^*M and let $p = p_1 + ip_2$ be the principal symbol of Z. Then p_j are linear in ξ and dp_1, dp_2 are independent at the points of $M \subset T^*M$. Let $\lambda(dx)$ be the Liouville measure on M induced by p_1, p_2, so that

$$\lambda(dx) \wedge dp_1 \wedge dp_2 = dxd\xi \text{ at the points of } M, \tag{B.2}$$

where $dxd\xi$ denotes the symplectic volume. The principal symbol of $\overline{Z}$ is $\overline{p(x,-\xi)} = -\overline{p(x,\xi)}$, so if we take the principal symbols of (B.1), we get

$$\{p,\overline{p}\} = \overline{ia}p - ia\overline{p}. \tag{B.3}$$

We use this to compute the Lie derivative $\mathcal{L}_{H_p}(\lambda(dx))$: Since $\mathcal{L}_{H_p}(dxd\xi) = 0$, we get from (B.2), (B.3) at $\xi = 0$:

$$\mathcal{L}_{H_p}(\lambda) \wedge dp \wedge d\overline{p} + \lambda \wedge dp \wedge \mathcal{L}_{H_p} d\overline{p} = 0,$$

$$\mathcal{L}_{H_p}(\lambda) \wedge dp \wedge d\overline{p} + \lambda \wedge dp \wedge d\{p,\overline{p}\} = 0,$$

$$\mathcal{L}_{H_p}(\lambda) \wedge dp \wedge d\overline{p} - ia\lambda \wedge dp \wedge d\overline{p} = 0.$$

Hence

$$\mathcal{L}_{H_p}(\lambda) = ia\lambda \text{ on } \xi = 0. \tag{B.4}$$

But the restriction of H_p to $\xi = 0$, can be identified with iZ, so (B.4) gives

$$\mathcal{L}_Z(\lambda) = a\lambda \text{ on } M. \tag{B.5}$$

Let A^* and ${}^tA$ denote the adjoint and the transpose of A in $L^2(M, \lambda(dx))$. From (B.5), we get

Lemma B.3. — *$Z^* = -(\overline{Z} + \overline{a})$, ${}^tZ = -(Z+a)$.*

Proof. — We start with the general fact that

$$\int_M \mathcal{L}_Z(u\lambda(dx)) = 0,$$

for all $u \in C^\infty(M)$. Using (B.5), we get

$$\int_M (Z+a)u\lambda(dx) = 0. \tag{B.6}$$

Replace u by uv:

$$\int_M ((Zu)v + u(Z+a)v)\lambda(dx) = 0. \tag{B.7}$$

It follows that ${}^tZ = -(Z+a)$, $Z^* = \overline{{}^tZ} = -(\overline{Z}+\overline{a})$. □

We also have $\overline{Z}^* = -(Z+a)$. Lemma B.2 gave us the real operator

$$P = -(Z+a)\overline{Z} = -(\overline{Z}+\overline{a})Z. \tag{B.8}$$

Lemma B.3 shows that the operator is self-adjoint and $\geqslant 0$:

$$P = \overline{Z}^*\overline{Z} = Z^*Z. \tag{B.9}$$

Moreover it is an elliptic 2nd order operator. From (B.9) it is easy to see that

$$\mathcal{N}(P) = \mathcal{N}(Z) = \mathcal{N}(\overline{Z}) = \mathbf{C}1. \tag{B.10}$$

The last equality follows from the other equalities since $Zu = 0$, $\overline{Z}u = 0$ implies that u is constant.

By a more direct argument, we have

Proposition B.4. — *Let $f \in C^\infty(M)$. If $u \not\equiv 0$, $(Z+f)u = 0$, then $u(x) \neq 0$ for every $x \in M$. We have* $\dim \mathcal{N}(Z+f) \leqslant 1$.

Proof. — Applying a classical result of Aronsjajn about the strong uniqueness of nullsolutions of second order elliptic equations, we know that u cannot vanish to ∞ order at any point. Let x_0 be a zero and choose local coordinates x_1, x_2 centered at x_0, such that

$$Z = \frac{1}{2}\Big(\frac{\partial}{\partial x_1} - \frac{1}{i}\frac{\partial}{\partial x_2}\Big) + \mathcal{O}(|x|)\Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big).$$

Let m be the order of vanishing of u at x_0, so that $u(x) = u_m(x) + \mathcal{O}(|x|^{m+1})$, where $u_m(x)$ is a homogeneous polynomial of degree m. Then we get

$$\frac{\partial u_m}{\partial \overline{z}} = 0, \text{with } z = x_1 + ix_2,$$

so $u_m(x) = Cz^m$ for some $C \neq 0$. Hence x_0 is an isolated zero. Moreover, $\operatorname{var}\arg_\gamma u = 2\pi m$, if γ is a simple closed loop around x_0 (contained in the coordinate neighborhood) which is positively oriented with respect to the directions $(\operatorname{Re} Z, \operatorname{Im} Z)$. We can now triangulate M in such a way that every zero of u is in the interior of one of the triangles. If Δ is one of the triangles, then $\operatorname{var}\arg_{\partial\Delta} u \geqslant 0$ with strict inequality precisely when D contains a zero of u. Since every boundary segment is common to two different triangles, but with opposite orientations, we see that

$$\sum_\Delta \operatorname{var}\arg_{\partial\Delta} u = 0,$$

when we sum over all the triangles in the triangulation. It follows that u cannot have any zeros.

The second statement is now clear: Let $0 \neq u_0 \in \mathcal{N}(Z+f)$, so that u_0 is everywhere different from 0. Let $u \in \mathcal{N}(Z+f)$ and let $x_0 \in M$. Then $v(x) := u(x) - \frac{u(x_0)}{u_0(x_0)}u_0(x)$ belongs to $\mathcal{N}(Z+f)$ and vanishes at one point (x_0). The first part of the proposition implies that v vanishes identically, and hence that u is a constant multiple of u_0. This shows that the dimension of $\mathcal{N}(Z+f)$ is at most equal to 1. ∎

Proposition B.5. — *There exists a non-vanishing function* $b \in C^\infty(M)$ *such that* $[\overline{b}Z, b\overline{Z}] = 0$.

Proof. — We develop the commutation relation to solve and get:

$$\begin{aligned} 0 &= \overline{b}b[Z,\overline{Z}] + \overline{b}[Z,b]\overline{Z} + b[\overline{b},\overline{Z}]Z \\ &= \overline{b}b(\overline{a}Z - a\overline{Z}) + \overline{b}Z(b)\overline{Z} - b\overline{Z(b)}Z \\ &= (\overline{b}b\overline{a} - b\overline{Z(b)})Z - (\overline{b}ba - \overline{b}Z(b))\overline{Z} \\ &= b\overline{(ab - Z(b))}Z - \overline{b}(ab - Z(b))\overline{Z}, \end{aligned}$$

so b should solve

$$(Z - a)b = 0. \tag{B.11}$$

Notice that if (B.11) holds for some non-vanishing b, then

$$Z\frac{1}{b} = -\frac{1}{b^2}Z(b) = -a\frac{1}{b},$$

so

$$(Z + a)\frac{1}{b} = 0, \text{ i.e. } Z^*c = 0, \ c = \frac{1}{\overline{b}}. \tag{B.12}$$

Conversely, (B.12) implies (B.11).

We have seen that Z has index 0 and has a 1-dimensional kernel. Then the same holds for Z^* and Proposition B.4 shows that $\mathcal{N}(Z^*)$ is generated by a non-vanishing function c. It suffices to take $b = 1/\overline{c}$. ◻

Theorem B.6. — *There exists a diffeomorphism* $\kappa : \mathbf{C}/L \to M$ *such that* $\overline{b}Z$ *corresponds to* $\frac{\partial}{\partial \overline{z}}$. *Here* $L = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ *is a lattice (so that* $e_1, e_2 \in \mathbf{C}$ *are* $\mathbf{R}$*-linearly independent).*

Proof. — Write $\overline{b}Z = \frac{1}{2}(\nu_1 + i\nu_2)$, where ν_1, ν_2 are real *commuting* vector fields which are pointwise linearly independent. Fix a point $x_0 \in M$ and consider the map

$$K : \mathbf{C} \simeq \mathbf{R}^2 \ni x \longmapsto \exp(x_1\nu_1 + x_2\nu_2)(x_0) \in M.$$

Notice that $\exp(x_1\nu_1 + x_2\nu_2) = \exp(x_1\nu_1) \circ \exp(x_2\nu_2) = \exp(x_2\nu_2) \circ \exp(x_1\nu_1)$ by commutativity. Let

$$L = \{x \in \mathbf{R}^2; K(x) = x_0\}.$$

L is a discrete Abelian subgroup of $\mathbf{R}^2$ and hence of the form 0, $\mathbf{Z}e$ with $e \neq 0$, or a lattice $\mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ with e_1, e_2 $\mathbf{R}$-linearly independent. K induces a diffeomorphism $\kappa : \mathbf{R}^2/L \to M$, so $\mathbf{R}^2/L$ must be compact and hence L is a lattice. Clearly the inverse image of $\overline{b}Z$ is $\frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial \overline{z}}$ with $z = x_1 + ix_2$. ◻

2. Review of Fourier integral operators between H_Φ spaces

We shall not review all the aspects of Fourier integral operator calculus (see [**MeSj**] for a similar discussion), and for simplicity, we restrict the attention to the Toeplitz (or Bergman projection) point of view. Let Φ be a smooth real-valued function defined near some point $x_0 \in \mathbf{C}^n$. Assume that Φ is strictly plurisubharmonic (s.pl.s.h.). Then

$$\Lambda_\Phi := \{(x, \tfrac{2}{i}\tfrac{\partial \Phi}{\partial x}(x));\ x \in \operatorname{neigh}(x_0, \mathbf{C}^n)\} \tag{2.1}$$

is I-Lagrangian and R-symplectic. Assume that $\Gamma \subset \Lambda_\Phi$ is a smooth Lagrangian submanifold (i.e. Lagrangian for the real symplectic form $\sigma_{|\Lambda_\Phi}$). If we identify Γ with its projection $\pi_x\Gamma$ in $\mathbf{C}^n$ then on Γ the fundamental 1-form $\xi \cdot dx$ can be identified with $\omega = \frac{2}{i}\partial\Phi_{|\Gamma}$ and hence this is a closed one-form in Γ. Here

$$\operatorname{Im}\omega = -d\Phi, \tag{2.2}$$

so $\operatorname{Im}\omega$ is exact. We notice that $\pi_x\Gamma$ is totally real. In fact, if $u \in \mathbf{C}^n$ and u, iu are both tangential to $\pi_x\Gamma$ at a point x, then

$$U_1 = (u, \tfrac{2}{i}(\Phi''_{xx}(x)u + \Phi''_{x\overline{x}}(x)\overline{u})) \text{ and } U_2 = (iu, \tfrac{2}{i}(\Phi''_{xx}(x)iu + \Phi''_{x\overline{x}}(x)\overline{iu}))$$

are both tangential to Γ at $(x, \frac{2}{i}\frac{\partial\Phi(x)}{\partial x})$. It follows that

$$0 = \sigma(U_1, U_2) = \sigma(U_1, U_2 - iU_1) = 4\langle \Phi''_{x\overline{x}}(x)\overline{u}, u\rangle, \tag{2.3}$$

which implies that $u = 0$. Locally in $\pi_x\Gamma$ we may then find a primitive ϕ of ω and extend $\phi(x)$ to an almost analytic function in $\mathbf{C}^n$ so that $\overline{\partial}\phi(x) = \mathcal{O}(\operatorname{dist}(x, \pi_x\Gamma)^\infty)$. Then at the points of $\pi_x\Gamma$, we have $d\phi = \frac{2}{i}\partial\Phi$, so at those points, we get

$$d\operatorname{Im}\phi = \tfrac{1}{2i}(\tfrac{2}{i}\partial\Phi + \tfrac{2}{i}\overline{\partial}\Phi) = -d\Phi.$$

After modifying ϕ by an imaginary constant (assuming Γ connected) we have that $\operatorname{Im}\phi + \Phi$ vanishes to the second order on Γ. Since this function is s.pl.s.h. it follows that

$$\Phi(x) + \operatorname{Im}\phi(x) \sim \operatorname{dist}(x, \pi_x(\Gamma))^2 \text{ near } \pi_x(\Gamma). \tag{2.4}$$

Let $\widetilde{\Phi}(y)$ be a second smooth s.pl.s.h function defined near $y_0 \in \mathbf{C}^n$. Let $\xi_0 = \frac{2}{i}\frac{\partial\Phi}{\partial x}(x_0)$, $\eta_0 = \frac{2}{i}\frac{\partial\widetilde{\Phi}(y)}{\partial y}(y_0)$, and let $\kappa : \operatorname{neigh}((y_0,\eta_0), \Lambda_{\widetilde{\Phi}}) \to \operatorname{neigh}((x_0,\xi_0), \Lambda_\Phi)$ be a smooth canonical transformation (with Λ_Φ, $\Lambda_{\widetilde{\Phi}}$ considered as real symplectic manifolds).

On $\mathbf{C}^{2n}_{x,\xi} \times \mathbf{C}^{2n}_{y,\eta}$, we choose the complex structure for which holomorphic functions are holomorphic in $(x, \xi; \overline{y}, \overline{\eta})$ in the usual sense. A corresponding "holomorphic" symplectic form is then given by

$$d\xi \wedge dx - d\overline{\eta} \wedge d\overline{y}. \tag{2.5}$$

We notice that the form (2.5) and the more standard form $d\xi \wedge dx - d\eta \wedge dy$ have the same restriction to $\Lambda_\Phi \times \Lambda_{\widetilde{\Phi}}$, since $d\eta \wedge dy_{|\Lambda_{\widetilde{\Phi}}}$ is real. The manifold $\Lambda_\Phi \times \Lambda_{\widetilde{\Phi}}$ is

I-Lagrangian and R-symplectic for the form (2.5), and we can view it as a "Λ_F" for our non-standard structure, with $F = \Phi(x) + \widetilde{\Phi}(y)$, since it can be represented as

$$\xi = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x), \ -\overline{\eta} = \frac{2}{i}\frac{\partial \widetilde{\Phi}}{\partial \overline{y}}(y).$$

The earlier discussion for Lagrangian manifolds can then be applied with Γ equal to graph$\,(\kappa)$, and we conclude that there is a function $\psi(x, y)$ such that

$$\partial_{\overline{x},y}\psi \text{ vanishes to infinite order on } \pi_{x,y}(\Gamma), \tag{2.6}$$

$$\partial_x\psi(x,y) = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x), \ \partial_{\overline{y}}\psi(x,y) = \frac{2}{i}\frac{\partial \widetilde{\Phi}}{\partial \overline{y}}, \text{ for } (x,y) \in \pi_{x,y}(\Gamma), \tag{2.7}$$

$$\Phi(x) + \widetilde{\Phi}(y) + \operatorname{Im}\psi(x,y) \sim \operatorname{dist}\left((x,y), \pi_{x,y}(\Gamma)\right)^2. \tag{2.8}$$

When $\widetilde{\Phi} = \Phi$ and $\kappa = \mathrm{id}$ is the identity, we can choose $\psi(x, y)$ to be the unique function (up to $\mathcal{O}(|x-y|^\infty)$), which satisfies (2.6) and $\psi(x, x) = \frac{2}{i}\Phi(x)$. In the general case, we deduce from (2.6), (2.7) that on $\pi_{x,y}(\Gamma)$:

$$d\psi = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x)dx + \frac{2}{i}\frac{\partial \widetilde{\Phi}}{\partial \overline{y}}d\overline{y}. \tag{2.9}$$

If we restrict ψ to $\pi_{x,y}(\Gamma)$ and identify it with a function on Γ, we get

$$d(\psi_{|_\Gamma}) = \xi dx - \overline{\eta} d\overline{y}, \ (x, \xi; y, \eta) \in \Gamma. \tag{2.10}$$

Since ξdx and $\overline{\eta}d\overline{y}$ are primitives of $\sigma_{|_{\Lambda_\Phi}}$ and $\sigma_{|_{\Lambda_{\widetilde{\Phi}}}}$ respectively, we can interpret (2.10) as stating that $\psi_{|_\Gamma}$ is a generating function for κ. For the moment, we make a local discussion and all our domains can be assumed to be simply connected. Later this will no more be the case and we have to consider what happens when we follow the locally defined function ψ around a closed loop in Γ, of the form $\widehat{\gamma} = \{(\kappa(\rho), \rho);\ \rho \in \gamma\}$, where γ is a closed loop in the domain of κ in $\Lambda_{\widetilde{\Phi}}$. We have

$$\operatorname{Im}(\xi dx)_{|_{\Lambda_\Phi}} = \operatorname{Im}(\tfrac{2}{i}\partial\Phi) = -d\Phi,$$

which is exact, since we will always require Φ and $\widetilde{\Phi}$ to be single valued. Similarly $\operatorname{Im}(\overline{\eta}d\overline{y})_{|_{\Lambda_{\widetilde{\Phi}}}}$ is exact. Hence

$$\int_{\widehat{\gamma}} d\psi = \int_{\kappa\circ\gamma} \operatorname{Re}(\xi dx) - \int_\gamma \operatorname{Re}(\eta dy). \tag{2.11}$$

So the undeterminacy in ψ is real (as can also be seen from (2.8)) and following ψ around a closed loop as above, ψ changes by a real constant, which is the difference of two real actions.

The implementation of Fourier integral operators is now fairly routine, and we will not go into all the details. (See [**Sj1**].) Formally such an operator is of the form

$$Au(x) = h^{-n}\int e^{\frac{i}{h}\psi(x,y)}a(x,y;h)u(y)e^{-\frac{2}{h}\widetilde{\Phi}(y)}L(dy), \tag{2.12}$$

where $L(dy)$ is the Lebesgue measure and a is a symbol of order m in $1/h$:

$$\nabla^k_{x,y} a = \mathcal{O}_k(1)h^{-m}, \tag{2.13}$$

$$\partial_{\overline{x}} a,\ \partial_y a = \mathcal{O}(h^{-m}\mathrm{dist}\,((x,y),\pi_{x,y}(\Gamma))^\infty + h^\infty). \tag{2.14}$$

See also section 3 of [**MeSj**].

3. Formulation of the problem in H_Φ and reduction to a neighborhood of $\xi = 0$ in $T^*\Gamma_0$

Let Φ_0 be a s.pl.s.h. quadratic form on $\mathbf{C}^n$. Let $P(x,\xi;h)$ be holomorphic and bounded in a tubular neighborhood V of Λ_{Φ_0} and assume that

$$|P(x,\xi;h)| \geqslant \frac{1}{C},\ (x,\xi)\in V,\ |(x,\xi)| > C. \tag{3.1}$$

Also assume (for simplicity) that

$$P \sim \sum_0^\infty h^k p_k(x,\xi), \tag{3.2}$$

in the space of bounded holomorphic functions on V. Then $|p_0(x,\xi)| \geqslant 1/C$, $(x,\xi)\in V$, $|(x,\xi)| > C$.

If we take the Weyl quantization, we know (see [**Sj3**], [**MeSj**]), that

$$P^w(x,hD_x;h) = \mathcal{O}(1) : H_{\Phi_0} \longrightarrow H_{\Phi_0}, \tag{3.3}$$

where

$$H_{\Phi_0} := \mathrm{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n; e^{-2\Phi_0/h}L(dx)), \tag{3.4}$$

and $\mathrm{Hol}(\mathbf{C}^n)$ denotes the space of holomorphic (entire) functions on $\mathbf{C}^n$.

Since Φ_0 is a quadratic form, we can infer (3.3) solely from the fact that P is a symbol of class S^0 on Λ_{Φ_0}, i.e. from the fact that $\nabla^k(P_{|\Lambda_{\Phi_0}}) = \mathcal{O}_k(1)$ for every $k\in\mathbf{N}$. However the fact that P is bounded and holomorphic in a tubular neighborhood of Λ_{Φ_0} allows us to consider other weights as well. Let $\Phi \in C^{1,1}(\mathbf{C}^n;\mathbf{R})$ (the space of C^1 functions with Lipschitz gradient) with $\Phi-\Phi_0$ bounded and $\sup|\frac{\partial\Phi}{\partial x} - \frac{\partial\Phi_0}{\partial x}|$ small enough. Then,

$$P^w(x,hD_x;h) = \mathcal{O}(1) :\ H_\Phi \longrightarrow H_\Phi, \tag{3.5}$$

where H_Φ is defined as in (3.4). In fact, in the standard formula,

$$P^w(x,hD_x;h)u = \frac{1}{(2\pi h)^n}\iint_{(\frac{x+y}{2},\xi)\in\Lambda_{\Phi_0}} e^{\frac{i}{h}(x-y)\xi} P\Big(\frac{x+y}{2},\xi;h\Big)u(y)dyd\xi, \tag{3.6}$$

we deform to the contour

$$\xi = \frac{2}{i}\int_0^1 \frac{\partial\Phi}{\partial x}(tx+(1-t)y)dt + \frac{i}{C}\frac{\overline{x-y}}{\langle x-y\rangle},\ \langle x\rangle = (1+|x|^2)^{1/2}. \tag{3.7}$$

In the following, we also assume for simplicity that $\Phi \in C^\infty$, that $\nabla^k\Phi$ is bounded for every $k \geqslant 2$, and that Φ is uniformly s.pl.s.h. We also assume that $n = 2$ and that there is a smooth Lagrangian torus $\Gamma \subset \Lambda_\Phi$, such that $p_\Phi = p_{0|_{\Lambda_\Phi}}$ satisfies

$$p_\Phi^{-1}(0) = \Gamma, \tag{3.8}$$

$$dp_\Phi, d\overline{p_\Phi} \text{ are independent at every point of } \Gamma. \tag{3.9}$$

Let $\Gamma_0 = (\mathbf{R}/2\pi\mathbf{Z})^2$ be the standard 2 torus and view Γ_0 as a maximally totally real submanifold of $X := \Gamma_0 + i\mathbf{R}^2$. In $X \times \mathbf{C}^2$ (equipped with the standard symplectic form) we consider

$$\Lambda_{\Phi_1} : \xi = \frac{2}{i}\frac{\partial \Phi_1}{\partial x}, \ \Phi_1(x) = \frac{1}{2}(\operatorname{Im} x)^2. \tag{3.10}$$

Φ_1 is s.pl.s.h. so Λ_{Φ_1} is I-Lagrangian and R-symplectic. According to section 1 and the beginning of section 2, there is a smooth "real" canonical transformation:

$$\kappa : \operatorname{neigh}(\Gamma, \Lambda_\Phi) \longrightarrow \operatorname{neigh}(\Gamma_0 \times \{0\}, \Lambda_{\Phi_1}), \tag{3.11}$$

mapping Γ onto $\Gamma_0 \times \{0\}$. Let $\psi(x, y)$ be a corresponding function defined as in section 2 for (x, y) in a neighborhood of $\pi_{x,y}(\operatorname{graph}(\kappa))$. Strictly speaking, it is clear how to define ψ locally up to a constant and up to a function which vanishes to infinite order on $\pi_{x,y}(\operatorname{graph}(\kappa))$. To see that we can get a corresponding grad-periodic function, we first define ψ on the projection of the graph of κ with $d\psi$ single-valued, then let α denote a single-valued almost holomorphic extension of this differential. For $(x, y) \in \operatorname{neigh}(\pi_{x,y}(\operatorname{graph}(\kappa)))$, let $\gamma_{x,y} : [0, 1] \to \mathbf{C}^4$ be the shortest segment from a point $\gamma_{x,y}(0)$ in the projection of the graph to $\gamma_{x,y}(1) = (x, y)$, and put $\psi(x, y) = \psi(\gamma_{x,y}(0)) + \int_{\gamma_{x,y}} \alpha$. Then ψ is grad-periodic and $\operatorname{Im}\psi$ is single-valued.

Let γ_j, $j = 1, 2$ be fundamental cycles in Γ, so that $\kappa \circ \gamma_j$ are fundamental cycles in $\Gamma_0 \times \{0\}$. Define $\widehat{\gamma}_j = \{(\kappa(\rho), \rho);\ \rho \in \gamma_j\}$. Then (2.11) is applicable and gives:

$$\int_{\widehat{\gamma}_j} d\psi = -\int_{\gamma_j} \operatorname{Re}(\eta dy) = -I_j(\Gamma), \tag{3.12}$$

where the last equality defines the action $I_j(\Gamma)$, which does not depend on the choice of global primitive of $\sigma_{|_{\Lambda_\Phi}}$, since Λ_Φ is diffeomorphic to $\mathbf{R}^4$. Here as in (2.11) we view ψ as a function on graph(κ). Since $d\psi$ is single valued, this means that if we start from a point (x, y) close to some point $(x_0, y_0) \in (\Gamma_0 \times \pi_y(\Gamma)) \cap \pi_{x,y}(\operatorname{graph}(\kappa))$, and follow a closed curve $[0, 1] \ni t \mapsto (x(t), y(t))$ which remains close to $\pi_{x,y}(\operatorname{graph}(\kappa))$ and with $x(t)$ close to a fundamental cycle $\gamma_{0,j}$ in Γ_0, then we get a new value of $\psi(x, y)$: "$\psi(x(1), y(1))$" satisfying

$$\psi(x(1), y(1)) = \psi(x(0), y(0)) - I_j(\Gamma). \tag{3.13}$$

We now implement κ by a Fourier integral operator of the form (2.12) with a of class $S^0_{\rm cl}(\mathrm{neigh}(\pi_{x,y}(\mathrm{graph}(\kappa))))$:

$$a(x,y;h)\sim\sum_0^\infty a_j(x,y)h^j \text{ in } C^\infty(\mathrm{neigh}(\pi_{x,y}(\mathrm{graph}(\kappa)))),\tag{3.14}$$

with a_j of class C^∞, and

$$\partial_{\overline{x}}a_j,\ \partial_y a_j=\mathcal{O}(\mathrm{dist}((x,y),\pi_{x,y}(\mathrm{graph}(\kappa)))^\infty).\tag{3.15}$$

We also choose a elliptic, i.e. with a_0 non-vanishing. (Notice that unlike ψ, a is single valued.)

Let $U\subset\Lambda_\Phi$, $V\subset\Lambda_{\Phi_1}$ be small neighborhoods of Γ and $\Gamma_0\times\{0\}$ respectively, with $\kappa(U)=V$. Then putting a suitable cutoff in (2.12) (equal to 1 near the projection of the graph of κ and replacing $\widetilde{\Phi}$ by Φ), we get an operator

$$A=\mathcal{O}(1):\ L^2(\pi(U);e^{-2\Phi/h}L(dy))\longrightarrow L^2_h(\pi(V);e^{-2\Phi_1/h}L(dx)),$$

where the subscript h indicates that we have a space of multi-valued Floquet periodic functions v:

$$v(x(1))=e^{-iI_j(\Gamma)/h}v(x(0)),\tag{3.16}$$

if $[0,1]\ni t\mapsto x(t)$ is a closed curve which is close to the jth fundamental cycle in Γ_0. We also see that $\|\overline{\partial}Au\|_{L^2_h}\leqslant\mathcal{O}(h^\infty)\|u\|_{L^2}$.

The complex adjoint A^* will be a Fourier integral operator associated to κ^{-1} by the same general procedure, and it is a routine matter to see that a can be chosen so that A^*A, AA^* are formally the orthogonal projections

$$L^2(\pi(U);e^{-2\Phi/h}L(dy))\to H(\pi(U),\Phi),\quad L^2_h(\pi(V);e^{-2\Phi_1/h}L(dx))\to H_h(\pi(V)),\Phi_1),$$

where $H(\pi(U),\Phi):=\mathrm{Hol}(\pi(U))\cap L^2(\pi(U);e^{-2\Phi/h}L)$ and H_h is defined similarly. (See [**MeSj**].) This implies that if $u\in H(\pi(U),\Phi)$ and $\widetilde{U}\subset\subset\pi(U)$, then

$$\|A^*Au-u\|_{L^2(\widetilde{U},e^{-2\Phi/h}L(dy))}=\mathcal{O}(h^\infty)\|u\|_{L^2(\pi(U),e^{-2\Phi/h}L(dy))},$$

and similarly for AA^*.

We also have Egorov's theorem which permits us to find $Q\in S^0_{\rm cl}(V)$ such that if q_0 is the leading symbol, then

$$q_0\circ\kappa=p_0,\tag{3.17}$$

$$Q^w(x,hD_x)A\equiv AP^w,\ A^*Q^w\equiv P^wA^*,\tag{3.18}$$

in the sense that

$$\|(Q^wA-AP^w)u\|_{L^2_h(\widetilde{V};e^{-2\Phi/h}L(dx))}\leqslant\mathcal{O}(h^\infty)\|u\|_{H(U;\Phi)},$$

when $\widetilde{V}\subset\subset\pi(V)$, and similarly for the second relation. Here Q^w is defined as in (3.6) after replacing Q by $(\chi Q)(\frac{x+y}{2},\xi;h)$, where χ is suitable cut-off, and where we identify $\Gamma_0+i\mathbf{R}^2$ with $\mathbf{C}^2/(2\pi\mathbf{Z}^2)$.

Finally we shall take a unitary transform

$$B:\, H(\Gamma_0+i\mathbf{R}^2,\Phi_1)\longrightarrow L^2(\Gamma_0), \tag{3.19}$$

and similarly on the corresponding spaces of Floquet-periodic functions, that will be the inverse of a Bargman transform. Since Φ_1 only depends on $\operatorname{Im} z$, we may view this function also as a function on $\mathbf{C}^2$. We recall that the Bargman transform

$$Tu(z;h)=C_2h^{-3/2}\int e^{-\frac{1}{2h}(z-y)^2}u(y)dy=\int k(z-y;h)u(y)dy, \tag{3.20}$$

(with the last equality defining the kernel k in the obvious way) is unitary: $L^2(\mathbf{R}^2)\to H(\mathbf{C}^2,\Phi_1)$ for a suitable $C_2>0$. The inverse is given by $T^{-1}=T^*$, with

$$\begin{aligned}T^*v(x;h)&=C_2h^{-3/2}\int e^{-\frac{1}{2h}(\overline{z}-\overline{x})^2-\frac{2}{h}\Phi_1(z)}v(z)L(dz)\\ &=\int \overline{k(z-x;h)}e^{-\frac{2}{h}\Phi_1(z)}v(z)L(dz).\end{aligned} \tag{3.21}$$

If we identify $L^2_h(\Gamma_0)$ with the θ-Floquet periodic locally square integrable functions, for $\theta=(I_1(\Gamma)/2\pi h, I_2(\Gamma)/2\pi h)$ on $\mathbf{R}^2$, and view $H_h(\Gamma_0+i\mathbf{R}^2,\Phi_1)$ similarly, we see that T generates an operator B^* from $L^2_h(\Gamma_0)$ to θ-Floquet periodic holomorphic functions on $\mathbf{C}^2$, given by

$$B^*u(z)=\int_{\mathbf{R}^2}k(z-y;h)u(y)dy=\int_{y\in E}\sum_{\nu\in(2\pi\mathbf{Z})^2}k(z-y+\nu)e^{i\langle\theta,\nu\rangle}u(y)dy, \tag{3.22}$$

where $E\subset\mathbf{R}^2$ is a fundamental domain for $(2\pi\mathbf{Z})^2$ (so $u(z+\nu)=e^{-i\langle\theta,\nu\rangle}u(z)$, $\nu\in(2\pi\mathbf{Z})^2$). Put

$$\ell(z,y)=\sum_{\nu\in(2\pi\mathbf{Z})^2}k(z-y+\nu)e^{i\langle\theta,\nu\rangle} \tag{3.23}$$

so that

$$\ell(z+\nu,y)=e^{-i\langle\theta,\nu\rangle}\ell(z,\nu),\ \ell(z,y+\nu)=e^{i\langle\theta,\nu\rangle}\ell(z,y).$$

The adjoint B is given by

$$Bv(x)=\int_{z\in E+i\mathbf{R}^2}\overline{\ell(z,x)}e^{-2\Phi_1(z)/h}v(z)L(dz)=\int\overline{k(z-x)}v(z)e^{-2\Phi_1(z)/h}L(dz), \tag{3.24}$$

so B coincides with T^*. Recall that $T^*T=1$ on $L^2(\mathbf{R}^2)$. It is easy to see that this relation extends to $L^2_h(\Gamma_0)$ and we get

$$BB^*=1. \tag{3.25}$$

To check that B^*B is also the identity on $H_h(\Gamma_0+i\mathbf{R}^2,\Phi_1)$, we first recall that TT^* is the identity on $H(\mathbf{C}^2,\Phi_1)$ and when we compute TT^* in a straight forward manner, we get the orthogonal projection: $L^2(\mathbf{C}^2, e^{-2\Phi_1/h}L(dz)) \to H(\mathbf{C}^2, \Phi_1)$:

$$\begin{aligned} TT^*v(z) &= \iint k(z-y;h)\overline{k(w-y;h)}v(w)e^{-2\Phi_1(w)/h}L(dw)dy \\ &= \widetilde{C}h^{-2}\int e^{\frac{2}{h}\psi_1(z,w)}v(w)e^{-\frac{2}{h}\Phi_1(w)}L(dw), \end{aligned}$$

where

$$\psi_1(z,w) = -\frac{1}{8}(z-\overline{w})^2 \tag{3.26}$$

is the unique function which is holomorphic in z, antiholomorphic in w and satisfies $\psi_1(z,z) = \Phi_1(z)$. Recall that $-\Phi_1(z)+2\mathrm{Re}\,\psi_1(z,w)-\Phi_1(w) \sim -|z-w|^2$, so TT^* is a bounded operator on $H_h(\Gamma_0+i\mathbf{R}^2, \Phi_1)$. If u is a normalized element of this space, then by solving a correcting d-bar problem for $\chi(x/R)u(x)$, we see that there is a sequence of functions $u_R \in H(\mathbf{C}^2, \Phi_1)$, $R = 1, 2, \ldots$, with $\|u_R\|_{H(\mathbf{C}^2,\Phi_1)} = \mathcal{O}_h(1)R^{1/2}$, such that

$$\|u-u_R\|_{L^2(B(0,R/2),e^{-2\Phi_1/h}L(dx))} \leqslant \mathcal{O}_h(1)e^{-R/C_0h},$$

for some $C_0 > 0$. Using that $TT^*u_R = u_R$, we see that $TT^*u = u$. Hence $B^*B = 1$. We have then checked that $B^*B = 1$, $BB^* = 1$, so B is unitary.

We recall that B is associated to a canonical transformation from Λ_{Φ_1} to $T^*(\Gamma_0)$. This allows us to view the previously defined κ also from a neighborhood of Γ in Λ_Φ to a neighborhood of $\Gamma_0 \times \{0\}$ in $T^*\Gamma_0$. We therefore have a Egorov's theorem and using $U := BA$, we get an equivalence between classical h-pseudodifferential operators acting in $H(\pi(U), \Phi)$ and h-pseudodifferential operators microlocally defined near $\xi = 0$ in $T^*\Gamma_0$, acting on Floquet periodic functions $u(x)$, satisfying:

$$u(x+e_j) = e^{-iI_j(\Gamma)/h}u(x), \tag{3.27}$$

where $e_1 = (2\pi, 0)$, $e_2 = (0, 2\pi)$.

Let $L^2_\theta(\Gamma_0)$ be the subspace of $L^2_{\mathrm{loc}}(\mathbf{R}^2)$ of θ-Floquet periodic functions: $u(x-k) = e^{i\theta\cdot k}u(x)$, $k \in (2\pi\mathbf{Z})^2$, $\theta \in (\mathbf{R}/\mathbf{Z})^2$.

Proposition 3.1. — *Let $P^w = P^w(x, hD_x; h) : H_\Phi \to H_\Phi$ be defined as in the beginning of this section and assume that $\Gamma \subset \Lambda_\Phi$ is a Lagrangian torus satisfying (3.8), (3.9). Then there exists a smooth canonical diffeomorphism*

$$\kappa : \mathrm{neigh}\,(\Gamma, \Lambda_\Phi) \longrightarrow \mathrm{neigh}\,(\Gamma_0 \times \{0\}, T^*\Gamma_0)$$

with $\kappa(\Gamma) = \Gamma_0$, where $\Gamma_0 = (\mathbf{R}/2\pi\mathbf{Z})^2$ is the standard torus.

Moreover, there exists an operator $U : H_\Phi \to L^2_{I/2\pi h}(\Gamma_0), I = (I_1(\Gamma), I_2(\Gamma))$, with the following properties:

1) $\|U\|_{\mathcal{L}(H_\Phi, L^2_{I/2\pi h}(\Gamma_0))} = \mathcal{O}(1)$, uniformly, when $h \to 0$.

2) U is concentrated to $\overline{\mathrm{graph}\,(\kappa)}$ in the sense that if $N \in \mathbf{N}$ and $\chi_1 \in S(T^*\Gamma_0, 1)$, $\chi_2 \in C_b^\infty(\mathbf{C}^2)$ are independent of h and

$$\mathrm{supp}\,\chi_1 \times \mathrm{supp}\,\chi_2 \cap \overline{\{\kappa(y,\eta), y);\ (y,\eta) \in \mathrm{neigh}\,(\Gamma, \Lambda_\Phi)\}} = \varnothing,$$

then

$$\langle hD\rangle^N \chi_1^w(x,hD)\circ U\circ \Pi_\Phi\circ\chi_2 = \mathcal{O}(h^\infty): L^2(e^{-2\Phi/h}L(dx)) \longrightarrow L^2_{I/2\pi h}(\Gamma_0).$$

Here Π_Φ *is the orthogonal projection* $L^2(e^{-2\Phi/h}L(dx))\to H_\Phi$ *(see* [**MeSj**]*).*
3) U *is microlocally unitary: For every* $\chi_2\in C_0^\infty(\mathrm{neigh}\,(\pi_x(\Gamma),\mathbf{C}^2))$*, independent of* h*, we have* $(U^*U-1)\Pi_\Phi\chi_2=\mathcal{O}(h^\infty): L^2(e^{-2\Phi/h}L(dy))\to L^2(e^{-2\Phi/h}L(dy))$*. For every* $\chi_1\in C_0^\infty(\mathrm{neigh}\,(\Gamma_0\times\{0\},T^*\Gamma_0))$*, independent of* h*, we have* $(UU^*-1)\chi_1^w(x,hD)=\mathcal{O}(h^\infty): L^2_{I/2\pi h}(\Gamma_0)\to L^2_{I/2\pi h}(\Gamma_0)$*.*
4) We have a Egorov's theorem: $\exists Q(x,\xi;h)\sim q_0(x,\xi)+hq_1(x,\xi)+\cdots\in S(T^*\Gamma_0,1)$*, with* $q_0\circ\kappa=p_0$ *in* $\mathrm{neigh}\,(\Gamma,\Lambda_\Phi)$*, such that* $Q^wU=UP^w$ *microlocally, i.e.* $(Q^wU-UP^w)\Pi_\Phi\chi_2=\mathcal{O}(h^\infty)$*,* $\chi_1^w(Q^wU-UP^w)=\mathcal{O}(h^\infty)$*, for* χ_1,χ_2 *as in 3).*
5) If P,Φ *depend smoothly on* $z\in\mathrm{neigh}\,(0,\mathbf{C})$*, then we can find* U,κ *depending smoothly on* z *in a possibly smaller neighborhood of* 0*.*

4. Spectrum of elliptic first order differential operators on Γ_0

Let $P=Z+q$ be a first order elliptic differential operator on Γ_0 with smooth coefficients, Z denoting the vector field part. After applying a diffeomorphism, we may assume that

$$P=A(x)\frac{\partial}{\partial\overline{x}}+q(x), \tag{4.1}$$

on $\mathbf{C}/L$, $L=\mathbf{Z}e_1\oplus\mathbf{Z}e_2$, where e_1,e_2 are $\mathbf{R}$ linearly independent and $A\in C^\infty(\mathbf{C}/L)$ is non-vanishing. Further, $q\in C^\infty(\mathbf{C}/L)$, and this function will later depend on a spectral parameter. It will be convenient to introduce $B(x)=1/A(x)$. The equation $Pu=v$ becomes

$$\Big(\frac{\partial}{\partial\overline{x}}+Bq\Big)u=Bv. \tag{4.2}$$

Let $\phi\in C^\infty(\mathbf{C}/L)$ and conjugate by e^ϕ:

$$e^{-\phi}\Big(\frac{\partial}{\partial\overline{x}}+Bq\Big)e^{\phi}e^{-\phi}u=Be^{-\phi}v,$$

i.e.

$$\Big(\frac{\partial}{\partial\overline{x}}+\Big(\frac{\partial\phi}{\partial\overline{x}}+Bq\Big)\Big)(e^{-\phi}u)=Be^{-\phi}v. \tag{4.3}$$

Let ϕ be the periodic solution (unique up to a constant) of

$$\frac{\partial\phi}{\partial\overline{x}}+Bq=\widehat{Bq}(0), \tag{4.4}$$

where $\widehat{Bq}$ is the Fourier transform, defined on the dual lattice

$$L^*=\mathbf{Z}e_1^*\oplus\mathbf{Z}e_2^*,\ \langle e_j^*,e_k\rangle_{\mathbf{R}^2}=2\pi\delta_{j,k}. \tag{4.5}$$

Then (4.3) becomes

$$\left(\frac{\partial}{\partial\overline{x}} + \widehat{Bq}(0)\right)(e^{-\phi}u) = Be^{-\phi}v. \tag{4.6}$$

We want to solve (4.2) ($\Leftrightarrow$ (4.6)) in the space of θ-Floquet periodic functions, where $\theta \in \mathbf{C}/L^*$, that is in the space of functions satisfying the condition

$$u(x-\ell) = e^{i\langle \ell,\theta\rangle_{\mathbf{R}^2}}u(x),\ \forall \ell \in L. \tag{4.7}$$

Writing $\theta \equiv \theta_1 e_1^* + \theta_2 e_2^* \bmod L^*$, we get

$$u(x-e_j) = e^{2\pi i\theta_j}u(x), \tag{4.8}$$

so the relation between θ in (4.7) and the $I_j(\Gamma)$ in (3.27) is given by

$$\theta_j \equiv \frac{I_j(\Gamma)}{2\pi h} \bmod \mathbf{Z}. \tag{4.9}$$

Let $H_\theta^k(\mathbf{C}/L)$ denote the space of θ-Floquet periodic functions on $\mathbf{C}$, which are of class H_{loc}^k (standard Sobolev spaces). The Fourier series representation of such a function (with convergence at least in the sense of distributions) becomes

$$f(x) = \sum_{\nu\in L^*-\theta} \widehat{f}(\nu)e^{i\langle\nu,x\rangle_{\mathbf{R}^2}} = \sum_{\nu\in L^*-\theta}\widehat{f}(\nu)e^{\frac{i}{2}(\overline{\nu}x+\nu\overline{x})}, \tag{4.10}$$

where we used that $\langle\nu,x\rangle_{\mathbf{R}^2} = \mathrm{Re}\,\overline{\nu}x$ in the last step. The corresponding expression for $\partial f/\partial\overline{x}$ becomes:

$$\frac{\partial f}{\partial\overline{x}} = \sum_{\nu\in L^*-\theta}\frac{i}{2}\nu\widehat{f}(\nu)e^{\frac{i}{2}(\overline{\nu}x+\nu\overline{x})}. \tag{4.11}$$

We now consider (4.2), (4.6) for $u\in H_\theta^1$, $v\in H_\theta^0$, and identify Fourier coefficients,

$$(\tfrac{i}{2}\nu + \widehat{Bq}(0))(\widehat{e^{-\phi}u})(\nu) = \mathcal{F}(Be^{-\phi}v)(\nu),\ \nu\in L^*-\theta, \tag{4.12}$$

where we write $\mathcal{F}u = \widehat{u}$. We get,

Proposition 4.1
(a) If $\frac{2}{i}\widehat{Bq}(0)-\theta\notin L^*$, then P in (4.1) is bijective $H_\theta^1\to H_\theta^0$.
(b) If $\frac{2}{i}\widehat{Bq}(0)-\theta\in L^*$, then P in (4.1) is a Fredholm operator of index 0 with one-dimensional kernel given by

$$\mathrm{Ker}\,(P) = \mathbf{C}\exp[(\overline{\widehat{Bq}}(0)x - \widehat{Bq}(0)\overline{x}) + \phi(x)],$$

where ϕ solves (4.4).

Before continuing, let us compute e_1^*, e_2^*. We have

$$\begin{pmatrix}\overline{e}_1 & e_1\\ \overline{e}_2 & e_2\end{pmatrix}\begin{pmatrix}e_1^* & e_2^*\\ \overline{e}_1^* & \overline{e}_2^*\end{pmatrix} = 4\pi I,$$

so

$$\begin{pmatrix}e_1^* & e_2^*\\ \overline{e}_1^* & \overline{e}_2^*\end{pmatrix} = 2\pi\frac{2i}{\overline{e}_1e_2 - e_1\overline{e}_2}\frac{1}{i}\begin{pmatrix}e_2 & -e_1\\ -\overline{e}_2 & \overline{e}_1\end{pmatrix} = \frac{2\pi}{\mathrm{Im}\,(\overline{e}_1e_2)}\frac{1}{i}\begin{pmatrix}e_2 & -e_1\\ -\overline{e}_2 & \overline{e}_1\end{pmatrix}.$$

Hence

$$e_1^* = \frac{2\pi}{i\mathrm{Im}\,(\overline{e}_1 e_2)} e_2,\ e_2^* = -\frac{2\pi}{i\mathrm{Im}\,(\overline{e}_1 e_2)} e_1. \tag{4.13}$$

Next we introduce a complex spectral parameter z and let q be of the form

$$q(x,z) = q_0(x) + zr(x). \tag{4.14}$$

The z dependence is chosen to be linear, since the situation we examine in this section is the linearized case. Let us call the spectrum of P, the set of values z for which P is not invertible (case (b) in the proposition). Then the spectrum of P is the set of values z that satisfy

$$\frac{2}{i}\widehat{Bq_0}(0) + z\frac{2}{i}\widehat{Br}(0) - \theta \in L^*, \tag{4.15}$$

or equivalently

$$\frac{2}{i}\widehat{Bq}(0,z) \in \theta + L^*, \tag{4.15$'$}$$

and we get a non-degenerate (affine) lattice precisely when

$$\widehat{Br}(0) \neq 0. \tag{4.16}$$

5. Grushin problem near $\xi = 0$ in $T^*\Gamma_0$

In the original problem, we shall restrict the spectral parameter z to some small disc. Performing the reduction of section 3, we are led to the operator

$$Q = Q_z = Q_z^w(x, hD) = Q^w(x, hD, z)$$

on $\Gamma_0 = \mathbf{T}^2$ with semiclassical Weyl symbol:

$$Q(x,\xi,z;h) \sim q_0(x,\xi,z) + hq_1(x,\xi,z) + h^2 q_2(x,\xi,z) + \cdots,\ |\xi| \leqslant \mathcal{O}(1), \tag{5.1}$$

with

$$q_0(x,0,z) = 0, \tag{5.2}$$

and $q_0, q_1, q_2, \ldots$ depend smoothly on z. Further, we have the ellipticity property:

$$|q_0(x,\xi,z)| \sim |\xi|. \tag{5.3}$$

In the region $|\xi| \in]h^\delta, \mathcal{O}(1)]$, for $\delta > 0$ close to 0, we shall invert Q_z^w by ellipticity. In the region $|\xi| \leqslant h^\delta$, we shall use 2nd microlocalization, which here only amounts to considering our operators in the "$h = 1$" quantization, after a cosmetic multiplication by h^{-1}. The corresponding symbol (for the $h = 1$ quantization) is then

$$\frac{1}{h}Q(x,h\xi,z;h) \sim Q_0(x,\xi,z) + hQ_1(x,\xi,z) + h^2 Q_2(x,\xi,z) + \cdots, \tag{5.4}$$

where the RHS is obtained by Taylor expanding at $\xi = 0$ and regrouping terms according to powers of h. We get

$$Q_0(x,\xi,z) = \sum_{j=1}^{2} \frac{\partial q_0}{\partial \xi_j}(x,0,z)\xi_j + q_1(x,0,z), \tag{5.5}$$

while the higher Q_j will involve higher order Taylor expansions. Q_j is a polynomial of degree at most $j+1$ in ξ, and in particular,

$$Q_j \in S_{1,0}^{j+1}(T^*\Gamma_0). \tag{5.6}$$

The expression (5.4) shall be considered only in the region $|h\xi| \leqslant h^\delta$, i.e. for $|\xi| \leqslant h^{\delta-1}$, so (5.4) is a well-defined asymptotic sum for $h \to 0$ of symbols in $S^1_{1,0}$. The operator $Q_0(x, D_x, z)$ is precisely of the type studied in the preceding section, the ellipticity follows from (5.3).

From section 4 and Appendix A of section 1 we recall that $Q_0(x, D_x, z)$ can be reduced by a change of variable to $A(x,z)\frac{\partial}{\partial \overline{x}} + q(x,z)$ on $\mathbf{C}/L(z)$, where A, q, L depend smoothly on z, and that this operator: $H^1_\theta \to H^0_\theta$ is invertible when $\theta \notin \frac{2}{i}\mathcal{F}(B(\cdot,z)q(\cdot,z))(0) + L^*(z)$ (with $B = 1/A$) and otherwise it has one dimensional kernel and cokernel. It will also be useful to recall that Q_0 can be further simplified by conjugation to

$$Q_0(x,D_x,z) = Q_0 = \frac{\partial}{\partial \overline{x}} + \theta_0(z), \tag{5.7}$$

where

$$\theta_0(z) = \frac{2}{i}\widehat{Bq}(0,z). \tag{5.8}$$

A simplified version of the discussion below shows that $\frac{1}{h}Q_z : H^1_\theta \to H^0_\theta$ is invertible (microlocally in $|\xi| \leqslant \mathcal{O}(1)$), when $\operatorname{dist}(\theta, \theta_0(z) + L^*(z)) \geqslant 1/\mathcal{O}(1)$. We concentrate on the more interesting case when this distance is small. Since θ is really defined only modulo $L^*(z)$, we decide to think of θ as a complex number close to $\theta_0(z)$.

Let $e_\theta(x) = ce^{-i\theta\cdot x}$ with $\cdot$ indicating that we take the $\mathbf{R}^2$ scalar product, and $c = c(z)$ is chosen to normalize $e_\theta(x)$ in $H^0_\theta(\mathbf{C}/L(z))$. Then

$$\mathcal{Q}_0(\theta,z) = \begin{pmatrix} Q_0(z) & R_{-,\theta} \\ R_{+,\theta} & 0 \end{pmatrix} : H^1_\theta \times \mathbf{C} \longrightarrow H^0_\theta \times \mathbf{C} \tag{5.9}$$

is bijective, where

$$R_{+,\theta}u = (u|e_\theta),\ R_{-,\theta}u_- = u_-e_\theta. \tag{5.10}$$

We denote the inverse by

$$\mathcal{E}_0(\theta,z) = \begin{pmatrix} E^0(\theta,z) & E^0_+(\theta,z) \\ E^0_-(\theta,z) & E^0_{-+}(\theta,z) \end{pmatrix}. \tag{5.11}$$

This depends smoothly on z and analytically on θ. By Beals' lemma, we know that

$$E^0 \in \operatorname{Op}_1(S^{-1}_{1,0}). \tag{5.12}$$

Moreover,

$$E^0_+(\theta,z)v_+ = v_+e^0_+(\theta,z),\ E^0_-(\theta,z)v = (v|e^0_-(\theta,z)), \tag{5.13}$$

where $e^0_\pm \in C^\infty_\theta$, and $E^0_{-+} \in \mathbf{C}$ with

$$|E^0_{-+}(\theta,z)| \sim |\theta-\theta_0(z)|, \tag{5.14}$$

and with $\theta_0(z)$ defined in (5.8). More explictily, using (5.7), we have $e^0_+ = e^0_- = e_\theta$, $E^0_{-+}(\theta,z) = \frac{i\theta}{2} - \widehat{Bq}(0)$. Recall or notice that $Q_0(z) : H^1_\theta \to H^0_\theta$ is invertible precisely for $\theta \neq \theta_0$ and that the inverse is given by $E^0(\theta,z) - E^0_+(\theta,z)E^0_{-+}(\theta,z)^{-1}E_-(\theta,z)$.

Now put

$$\mathcal{Q}(\theta,z) = \begin{pmatrix} \frac{1}{h}Q_z(x,hD_x;h) & R_{-,\theta} \\ R_{+,\theta} & 0 \end{pmatrix} \tag{5.15}$$

formally as an operator $H^1_\theta \times \mathbf{C} \to H^0_\theta \times \mathbf{C}$, so that (in view of (5.4))

$$\mathcal{Q}(\theta,z) \sim \sum_0^\infty h^j\mathcal{Q}_j(\theta,z), \tag{5.16}$$

with

$$\mathcal{Q}_j(\theta,z) = \begin{pmatrix} Q_j(x,D_x,z) & 0 \\ 0 & 0 \end{pmatrix},\ j \geqslant 1. \tag{5.17}$$

For simplicity, we assume that the same conjugation that simplified Q_0 to the form (5.7) has been applied to $h^{-1}Q_z$. We invert $\mathcal{Q}$ formally by the asymptotic Neumann series

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_0 - \mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0 + \mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0 - \cdots \\ &= \sum_0^\infty (-1)^k\mathcal{E}_0((\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0)^k = \sum_0^\infty (-1)^k(\mathcal{E}_0(\mathcal{Q}-\mathcal{Q}_0))^k\mathcal{E}_0. \end{aligned} \tag{5.18}$$

Write $Q_h = \frac{1}{h}Q_z(x,hD_x;h)$. Then

$$(\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0 = \begin{pmatrix} (Q_h-Q_0)E^0 & (Q_h-Q_0)E^0_+ \\ 0 & 0 \end{pmatrix}, \tag{5.19}$$

and for $k \geqslant 1$:

$$((\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0)^k = \begin{pmatrix} ((Q_h-Q_0)E^0)^k & ((Q_h-Q_0)E^0)^{k-1}(Q_h-Q_0)E^0_+ \\ 0 & 0 \end{pmatrix}. \tag{5.20}$$

The general term in the series (5.18) becomes

$$(-1)^k\mathcal{E}_0((\mathcal{Q}-\mathcal{Q}_0)\mathcal{E}_0)^k = \begin{pmatrix} (-1)^kE^0((Q_h-Q_0)E^0)^k & (-1)^k(E^0(Q_h-Q_0))^kE^0_+ \\ (-1)^kE^0_-((Q_h-Q_0)E^0)^k & (-1)^kE^0_-((Q_h-Q_0)E^0)^{k-1}(Q_h-Q_0)E^0_+ \end{pmatrix}. \tag{5.21}$$

Here $(Q_h-Q_0)E^0$, $E^0(Q_h-Q_0)$ are $(h=1)$ pseudodifferential operators with symbols in $hS^1_{1,0} + h^2S^2_{1,0} + \cdots$. $((Q_h-Q_0)E^0)^k$, $(E^0(Q_h-Q_0))^k$ then have their symbols in $h^kS^k_{1,0} + h^{k+1}S^{k+1}_{1,0} + \cdots$. It follows that $E^0((Q_h-Q_0)E^0)^k$ has its symbol in $h^kS^{k-1}_{1,0} + h^{k+1}S^k_{1,0} + \cdots$. Moreover, $(E^0(Q_h-Q_0))^kE^0_+v_+ = v_+e^k_+$, with e_+^k in $h^k C_\theta^\infty + h^{k+1} C_\theta^\infty + \cdots$ and similarly for $E_-^0((Q_h - Q_0)E^0)^k$. Finally $E_-^0((Q_h - Q_0)E^0)^{k-1}(Q_h - Q_0)E_+^0$ belongs to $h^k\mathbf{C} + h^{k+1}\mathbf{C} + \cdots$. Using all this in the asymptotic series (5.18), we get

$$\mathcal{E} = \begin{pmatrix} E(\theta,z) & E_+(\theta,z) \\ E_-(\theta,z) & E_{-+}(\theta,z) \end{pmatrix}, \tag{5.22}$$

where
– $E(\theta,z)$ is a 1-pseudodifferential operator with symbol in $S_{1,0}^{-1} + hS_{1,0}^0 + \cdots$.
– $E_+v_+ = v_+e_+$, $E_-u = (u|e_-)$, with $e_\pm \in C^\infty + hC^\infty + h^2C^\infty + \cdots$.
– $E_{-+}(\theta,z) \in \mathbf{C} + h\mathbf{C} + h^2\mathbf{C} + \cdots$, more explicitly,

$$E_{-+}(\theta,z) \sim E_{-+}^0(\theta,z) + hE_{-+}^1(\theta,z) + \cdots \ . \tag{5.23}$$

Formally, the spectrum of P_z^w (acting on θ-Floquet functions) will be the set of values z for which $E_{-+}(\theta,z) = 0$.

We will now sum up the discussion of this section, and for that it will be convenient to return to the case of the standard torus Γ_0. Then the dual lattice "$L^*(z)$" is simply $\mathbf{Z}^2$ and $Q_0(z)$ in (5.5) will be invertible $H_\theta^1(\Gamma_0) \to H_\theta^0(\Gamma_0)$ precisely when

$$\theta \notin \theta_0(z) + \mathbf{Z}^2, \tag{5.24}$$

where $\theta_0(z) \in \mathbf{R}^2$ depends smoothly on z.

Proposition 5.1. — *Let $C > 0$ be a sufficiently large constant.*
1. For $\operatorname{dist}(\theta, \theta_0(z) + \mathbf{Z}^2) \geqslant 1/C$, $z \in \operatorname{neigh}(0, \mathbf{C})$, *there exists an operator $F(\theta, z; h) = \mathcal{O}(1) : H_\theta^0 \to H_\theta^0$ such that:*
1a) F is pseudolocal in the sense that $\langle hD\rangle^N \chi_1(x,hD) F \chi_2(x,hD) \langle hD\rangle^N = \mathcal{O}(h^N) : H_\theta^0 \to H_\theta^0$ for every $N \in \mathbf{N}$ and all $\chi_j \in C_b^\infty(T^*\Gamma_0)$, $j = 1,2$, independent of h with $(\operatorname{supp}\chi_1 \times \operatorname{supp}\chi_2) \cap (\operatorname{diag}(T^*\Gamma_0)^2 \cup (\Gamma_0 \times \{0\})^2) = \varnothing$.
1b) There is a neighborhood $V \subset T^*\Gamma_0$ of $\Gamma_0 \times \{0\}$ such that

$$\Big(\frac{1}{h}QF - 1\Big)\chi^w,\ \chi^w\Big(\frac{1}{h}QF - 1\Big) = \mathcal{O}(h^\infty) : H_\theta^0 \longrightarrow H_\theta^0,$$

for every $\chi \in C_0^\infty(V)$, independent of h. The same holds with $F\frac{1}{h}Q$ instead of $\frac{1}{h}QF$. (Notice that these compositions are welldefined mod $\mathcal{O}(h^\infty) : H_\theta^0 \to H_\theta^0$.)
2. For $\operatorname{dist}(\theta, \theta_0(z) + \mathbf{Z}^2) \leqslant 1/C$, *we may assume (by $\mathbf{Z}^2$-periodicity in θ) that $\theta \in \mathbf{R}^2$, $|\theta - \theta_0(z)| \leqslant 1/C$. Then we have rank one operators $R_{+,\theta}u = (u|e_{\theta,z})$, $R_{-,\theta}u_- = u_-f_{\theta,z}$, $R_{+,\theta} : H_\theta^0 \to \mathbf{C}$, $R_{-,\theta} : \mathbf{C} \to H_\theta^0$, with $e_{\theta,z}, f_{\theta,z} \in H_\theta^0 \cap C_b^\infty$ depending smoothly on θ, z, independent of h, and a bounded operator*

$$\mathcal{E} = \begin{pmatrix} E(\theta,z;h) & E_+(\theta,z;h) \\ E_-(\theta,z;h) & E_{-+}(\theta,z;h) \end{pmatrix} = \mathcal{O}(1) : H_\theta^0 \times \mathbf{C} \longrightarrow H_\theta^0 \times \mathbf{C},$$

with the following properties:
2a) E is pseudolocal as in 1a.

2b) If $\chi \in C_b^\infty(T^*\Gamma_0)$ is independent of h and $\Gamma_0 \times \{0\} \cap \operatorname{supp}\chi = \varnothing$, then for every $N \in \mathbf{N}$:

$$\langle hD\rangle^N \chi^w E_+ = \mathcal{O}(h^\infty) : \mathbf{C} \longrightarrow H_\theta^0, \quad E_-\chi^w \langle hD\rangle^N = \mathcal{O}(h^\infty) : H_\theta^0 \longrightarrow \mathbf{C}.$$

2c) E_{-+} has the asymptotic expansion (5.23) with $|E_{-+}^0(\theta,z)| \sim |\theta - \theta_0(z)|$.
2d) $\mathcal{E}$ is an inverse of $\mathcal{Q}$ in (5.15) in the sense that

$$(\mathcal{Q}\mathcal{E}-1)\begin{pmatrix}\chi^w & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}\chi^w & 0\\ 0 & 1\end{pmatrix}(\mathcal{Q}\mathcal{E}-1) = \mathcal{O}(h^\infty) : H_\theta^0 \times \mathbf{C} \longrightarrow H_\theta^0 \times \mathbf{C},$$

for all $\chi \in C_0^\infty(V)$, independent of h. Here V is as in 1b and we can replace $\mathcal{Q}\mathcal{E}$ by $\mathcal{E}\mathcal{Q}$ in the preceding estimates.

6. The main result

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbf{C}^2$ and let $P(x,\xi) = P(x,\xi,z;h)$ be a bounded holomorphic function in a tubular neighborhood of Λ_{Φ_0}, which depends holomorphically on $z \in \operatorname{neigh}(0,\mathbf{C})$, with the asymptotic expansion

$$P(x,\xi,z;h) \sim \sum_{k=0}^{\infty} p_k(x,\xi,z)h^k, \tag{6.1}$$

in the space of such functions. Later, it will be convenient to assume that the subprincipal symbol vanishes:

$$p_1(x,\xi,z) = 0. \tag{6.2}$$

Also assume ellipticity near infinity:

$$|p(x,\xi,z)| \geqslant 1/C,\ (x,\xi) \in \Lambda_{\Phi_0},\ |(x,\xi)| \geqslant C, \tag{6.3}$$

where $p = p_0$. (The boundedness assumption above could easily be replaced by some other symbol type condition, provided of course that we modify the ellipticity assumption accordingly.)

Assume for $z = 0$, that $\Sigma = p^{-1}(0) \cap \Lambda_{\Phi_0}$ is smooth, connected and that

$$dp_{\Lambda_{\Phi_0}}, d\overline{p}_{\Lambda_{\Phi_0}} \text{ are linearly independent on } \Sigma. \tag{6.4}$$

Further assume that

$$\{p_{\Lambda_{\Phi_0}}, \overline{p}_{\Lambda_{\Phi_0}}\} \text{ is small on } \Sigma, \tag{6.5}$$

where we adopt the convention of section 1, that we have uniformity in the other assumptions. Recall also from section 1, that this implies that Σ is a smooth torus. Notice that the assumptions above will also be fulfilled for $p = p(\cdot, z)$ when z is close enough to 0.

In section 1, we showed that $p(\cdot,z)^{-1}(0)$ contains a smooth torus $\Gamma(z)$, which is close to Σ and such that

$$\sigma_{|\Gamma(z)} = 0, \tag{6.6}$$

$$I_j(\Gamma(z),\omega)\in\mathbf{R},\ \ j=1,2, \tag{6.7}$$

where $\omega=\xi_1dx_1+\xi_2dx_2$ and $I_j(\Gamma(z),\omega)$ is the corresponding action along the jth fundamental cycle in $\Gamma(z)$. (Any other global primitive of σ gives the same actions.) $\Gamma(z)$ is not unique, but thanks to (6.7) its image in the quotient space $\mathcal{M}(z)$ of $p(\cdot,z)^{-1}(0)$ by the action of $H_{p(\cdot,z)}$, is unique. The full preimage of this image is a complex Lagrangian manifold $\widetilde{\Lambda}(z)$ which is also uniquely determined and which can be viewed as a complexification of the totally real manifold $\Gamma(z)$. This is Λ_ϕ in (1.32).

It is easy to see that $\Gamma(z)$ can be chosen to depend smoothly on z. Also thanks to (6.7), we have $\Gamma(z)\subset\Lambda_z$, where $\Lambda_z=\Lambda_{\Phi_z}$ is an IR-manifold close to Λ_{Φ_0} and we can view $\Gamma(z)$ as a Lagrangian submanifold of this real symplectic manifold. Λ_z can also be chosen to depend smoothly on z, and we may assume that $\Phi_z-\Phi_0=\mathcal{O}(1)$

Let $I_j(z)=I_j(\Gamma(z),\omega)$, $I(z)=(I_1(z),I_2(z))\in\mathbf{R}^2$. Let $P(z)=P^w(z)=P^w(x,hD_x,z;h)$ be the corresponding Weyl quantization which acts on H_{Φ_z}. Let $U=U(z)$, $Q=Q^w(z)$ be as in Proposition 3.1, and depend smoothly on $z\in\operatorname{neigh}(0,\mathbf{C})$. Then $Q_0=Q_0(x,D_x,z):H^1_\theta(\Gamma_0)\to H^0_\theta(\Gamma_0)$ (c.f. (5.4)) is invertible precisely when $\theta\notin\theta_0(z)+\mathbf{Z}^2$, where $\theta_0\in\mathbf{R}^2$ depends smoothly on z (cf. (5.14)). We also recall from section 3, that we will naturally have $\theta=I(z)/(2\pi h)$. We first consider the case when

$$\operatorname{dist}\Big(\frac{I(z)}{2\pi h},\theta_0(z)+\mathbf{Z}^2\Big)\geqslant\frac{1}{C}. \tag{6.8}$$

Let $\chi_2\in C_0^\infty(\operatorname{neigh}(\pi_x(\Gamma(0)),\mathbf{C}^2))$ be equal to 1 in a neighborhood of $\pi_x\Gamma(0)$. As an approximate right inverse to $h^{-1}P^w(z)$, we take

$$J:=h\Pi_\Phi G\Pi_\Phi(1-\chi_2)+\Pi_\Phi U^*FU\Pi_\Phi\chi_2, \tag{6.9}$$

where $F=F(z)$ is given by Proposition 5.1, with $\theta=I(z)/(2\pi h)$, and $G=G(z)$ is an asymptotic inverse to P^w away from $\pi_x(\Gamma)$ in the sense of Töplitz operators in section 3 of [**MeSj**], and Π_Φ is the orthogonal projection $L^2(e^{-2\Phi/h}L(dx))\to H_\Phi$. Then

$$P^w\Pi_\Phi G\Pi_\Phi(1-\chi_2)=\Pi_\Phi(1-\chi_2)+\mathcal{O}(h^\infty):H_\Phi\longrightarrow H_\Phi.$$

On the other hand, if we use local unitarity of U, the pseudolocality of F and 4) of Proposition 3.1, we get

$$\begin{aligned}\frac{1}{h}P^w\Pi_\Phi U^*FU\Pi_\Phi\chi_2&\equiv\Pi_\Phi U^*\frac{1}{h}Q^wFU\Pi_\Phi\chi_2\\&\equiv\Pi_\Phi U^*U\Pi_\Phi\chi_2\equiv\Pi_\Phi\chi_2\ \mathrm{mod}\,\mathcal{O}(h^\infty):H_\Phi\longrightarrow H_\Phi.\end{aligned}$$

It follows that

$$\frac{1}{h}P^w(z)J\equiv\Pi_\Phi=1\ \mathrm{mod}\,\mathcal{O}(h^\infty):H_\Phi\longrightarrow H_\Phi. \tag{6.10}$$

(Most of our operators as well as Φ depend on z, and this dependence is always smooth.)

In the same way we can show that

$$K = \Pi_\Phi(1-\chi_2)\Pi_\Phi hG + \Pi_\Phi\chi_2\Pi_\Phi U^* FU \tag{6.11}$$

satisfies

$$K\frac{1}{h}P^w \equiv 1 \bmod \mathcal{O}(h^\infty) : H_\Phi \longrightarrow H_\Phi. \tag{6.12}$$

We conclude that under the assumption (6.8), the operator $\frac{1}{h}P^w : H_\Phi \to H_\Phi$ has an inverse which is uniformly bounded, when $h \to 0$.

We now consider the case when

$$\operatorname{dist}\Big(\frac{I}{2\pi h}, \theta_0(z) + \mathbf{Z}^2\Big) \leqslant \frac{1}{C}, \tag{6.13}$$

for some large fixed $C > 0$. Let k be the point in $\mathbf{Z}^2$ such that

$$\Big|k + \frac{I}{2\pi h} - \theta_0(z)\Big| \leqslant \frac{1}{C}.$$

We apply the second part of Proposition 5.1 with $\theta = k + I/(2\pi h)$. Let $\mathcal{E}, R_+, R_-$ be as there. Consider

$$\mathcal{P}(z) = \begin{pmatrix} \frac{1}{h}P(z) & \Pi_\Phi \widetilde{R}_-(z) \\ \widetilde{R}_+(z) & 0 \end{pmatrix} : H_{\Phi_z} \times \mathbf{C} \longrightarrow H_{\Phi_z} \times \mathbf{C}, \tag{6.14}$$

with

$$\widetilde{R}_+ = R_+ U,\ \widetilde{R}_- = U^* R_-. \tag{6.15}$$

As an approximate right inverse to $\mathcal{P}$, we take (with $E, E_\pm, E_{-+}$ as in (5.22))

$$\widetilde{\mathcal{E}}_r = \begin{pmatrix} h\Pi_\Phi G\Pi_\Phi(1-\chi_2) + \Pi_\Phi U^* EU\Pi_\Phi\chi_2 & \Pi_\Phi U^* E_+ \\ E_- U & E_{-+} \end{pmatrix} =: \begin{pmatrix} \widetilde{E} & \widetilde{E}_+ \\ \widetilde{E}_- & \widetilde{E}_{-+} \end{pmatrix}. \tag{6.16}$$

We need to check that

$$\begin{cases} \frac{1}{h}P\widetilde{E} + \Pi_\Phi\widetilde{R}_-\widetilde{E}_- \equiv 1,\ \frac{1}{h}P\widetilde{E}_+ + \Pi_\Phi\widetilde{R}_-\widetilde{E}_{-+} \equiv 0, \\ \widetilde{R}_+\widetilde{E} \equiv 0,\ \widetilde{R}_+\widetilde{E_+} \equiv 1, \end{cases} \tag{6.17}$$

modulo terms that are $\mathcal{O}(h^\infty)$ in operator norm:

$$\begin{aligned} \frac{1}{h}P\widetilde{E} + \Pi_\Phi\widetilde{R}_-\widetilde{E}_- &\equiv \Pi_\Phi(1-\chi_2) + \frac{1}{h}P\Pi_\Phi U^* EU\Pi_\Phi\chi_2 + \Pi_\Phi U^* R_- E_- U \\ &\equiv \Pi_\Phi(1-\chi_2) + \Pi_\Phi U^*\frac{1}{h}QEU\Pi_\Phi\chi_2 + \Pi_\Phi U^* R_- E_- U \\ &\equiv \Pi_\Phi(1-\chi_2) + \Pi_\Phi U^*\frac{1}{h}QEU\Pi_\Phi\chi_2 + \Pi_\Phi U^* R_- E_- U\Pi_\Phi\chi_2 \\ &\equiv \Pi_\Phi(1-\chi_2) + \Pi_\Phi U^* U\Pi_\Phi\chi_2 \equiv \Pi_\Phi \\ &= 1, \end{aligned}$$

$$\frac{1}{h} P \widetilde{E}_+ + \Pi_\Phi \widetilde{R}_- \widetilde{E}_{-+} \equiv \frac{1}{h} P \Pi_\Phi U^* E_+ + \Pi_\Phi U^* R_- E_{-+}$$
$$\equiv \Pi_\Phi U^* (\frac{1}{h} Q E_+ + R_- E_{-+}) \equiv \Pi_\Phi U^* 0 = 0,$$

$$\widetilde{R}_+ \widetilde{E} \equiv R_+ U \Pi_\Phi \chi_2 (h \Pi_\Phi G \Pi_\Phi (1 - \chi_2) + \Pi_\Phi U^* E U \Pi_\Phi \chi_2)$$
$$\equiv 0 + R_+ U U^* E U \Pi_\Phi \chi_2 \equiv R_+ E U \Pi_\Phi \chi_2 \equiv 0,$$

$$\widetilde{R}_+ \widetilde{E}_+ \equiv R_+ U \Pi_\Phi U^* E_+ \equiv R_+ E_+ \equiv 1.$$

So,

$$\mathcal{P} \widetilde{\mathcal{E}}_r = 1 + \mathcal{O}(h^\infty). \tag{6.18}$$

Similarly, we check that

$$\widetilde{\mathcal{E}}_\ell \mathcal{P} = 1 + \mathcal{O}(h^\infty), \tag{6.19}$$

where

$$\widetilde{\mathcal{E}}_\ell = \begin{pmatrix} \Pi_\Phi (1 - \chi_2) \Pi_\Phi h G + \Pi_\Phi \chi_2 \Pi_\Phi U^* E U & \Pi_\Phi U^* E_+ \\ E_- U & E_{-+} \end{pmatrix}. \tag{6.20}$$

We sum up the discussion so far:

Proposition 6.1. — *Under the preceding assumption, there exists a smooth map* neigh $(0, \mathbf{C}) \mapsto \theta_0(z) \in \mathbf{R}^2$*, such that if we fix* $C > 0$ *large enough:*
1) For $\operatorname{dist}(I(z)/(2\pi h), \theta_0(z) + \mathbf{Z}^2) \geqslant (2C)^{-1}$, $h^{-1} P^w(z) : H_{\Phi_z} \to H_{\Phi_z}$ *has a uniformly bounded inverse.*
2) For

$$\operatorname{dist}(I(z)/(2\pi h), \theta_0(z) + \mathbf{Z}^2) < 1/C, \tag{6.21}$$

the operator $\mathcal{P}(z)$ *in (6.14) has a uniformly bounded inverse*

$$\mathcal{F}(z) = \begin{pmatrix} F(z) & F_+(z) \\ F_-(z) & F_{-+}(z) \end{pmatrix} : H_{\Phi_z} \times \mathbf{C} \longrightarrow H_{\Phi_z} \times \mathbf{C}. \tag{6.22}$$

Modulo terms that are $\mathcal{O}(h^\infty)$ *in operator norm, we have*

$$\begin{aligned} F_+(z) &\equiv U^*(z) E_+\Big(k + \frac{I(z)}{2\pi h}, z; h\Big), \\ F_-(z) &\equiv E_-\Big(k + \frac{I(z)}{2\pi h}, z; h\Big) U(z), \\ F_{-+}(z) &\equiv E_{-+}\Big(k + \frac{I(z)}{2\pi h}, z; h\Big), \end{aligned} \tag{6.23}$$

where $k \in \mathbf{Z}^2$ *is the point with* $|k - \theta_0(z) + I(z)/(2\pi h)| < 1/C$*, and* E_+, E_-, E_{-+} *are given in Proposition 5.1.*

From (6.23) and (5.23) we get the following asymptotic expansion in case 2) of the proposition:

$$F_{-+}(z;h) \sim E^0_{-+}\Big(k+\frac{I(z)}{2\pi h},z\Big)+hE^1_{-+}\Big(k+\frac{I(z)}{2\pi h},z\Big)+\cdots, \tag{6.24}$$

valid in the sense that

$$|R_N(z;h)| \leqslant C_N h^{N+1}, \tag{6.25}$$

where

$$R_N(z;h)=F_{-+}(z;h)-\sum_0^N h^j E^j_{-+}\Big(k+\frac{I(z)}{2\pi h},z\Big). \tag{6.26}$$

We shall next see that (6.24) can be differentiated with respect to z in the natural sense. Indeed, it is clear that $\nabla_z^j\mathcal{P}(z)=\mathcal{O}(h^{-j})$ in operator norm for $j=0,1,2,\dots$, so if we use that

$$\nabla_z\mathcal{F}(z)=-\mathcal{F}(z)\nabla_z\mathcal{P}(z)\mathcal{F}(z) \tag{6.27}$$

and similar more elaborate expressions for $\nabla_z^j\mathcal{F}(z)$, we see that

$$\nabla^j\mathcal{F}(z)=\mathcal{O}(h^{-j}) \tag{6.28}$$

in operator norm for $j=0,1,2,\dots$. In particular,

$$\nabla_z^j F_{-+}(z;h)=\mathcal{O}(h^{-j}), \tag{6.29}$$

and the same estimate holds for each of the terms in (6.24). It follows that

$$(h\nabla_z)^j R_N(z;h)=\mathcal{O}(1). \tag{6.30}$$

Now combine (6.25,30) with elementary convexity estimates for the derivatives to conclude that

$$(h\nabla_z)^j R_N(z;h)=\mathcal{O}(h^{N+1-\varepsilon}),$$

for every $\varepsilon>0$ (after an arbitrarily small increase of the constant C in (6.21)). Since

$$R_N(z;h)=h^{N+1}E^{N+1}_{-+}\Big(k-\frac{I(z)}{2\pi h},z\Big)+R_{N+1}(z;h),$$

we get

$$(h\nabla_z)^j R_N(z;h)=\mathcal{O}(h^{N+1}), \tag{6.31}$$

for every $j=0,1,2,\dots$. So we have proved that (6.24) can be differentiated with respect to z as many times as we want, in the natural way.

In this context, it may be of some interest to notice that F_{-+} is holomorphic in z after multiplication by a non-vanishing factor. Indeed, from (6.27) and the fact that $\partial_{\overline{z}}P^w(z)=0$, we get

$$\partial_{\overline{z}}F_{-+}+F_-(\partial_{\overline{z}}\Pi_\Phi\widetilde{R}_-(z))F_{-+}+F_{-+}(\partial_{\overline{z}}\widetilde{R}_+)F_+=0.$$

Since F_{-+} is scalar, this simplifies to

$$(\partial_{\overline{z}}+v(z))F_{-+}(z)=0,\ v(z)=F_-(\partial_{\overline{z}}\Pi_\Phi\widetilde{R}_-(z))+(\partial_{\overline{z}}\widetilde{R}_+(z))F_+. \tag{6.32}$$

If $\partial_{\overline{z}} V(z) = v(z)$ (and this equation can always be solved after increasing C in (6.21)), we get

$$\partial_{\overline{z}}(e^{V(z)} F_{-+}) = 0. \tag{6.33}$$

Since $\nabla_z^j v = \mathcal{O}(h^{-1-j})$, $j \geqslant 0$, we can restrict the attention to some disc of radius ch (after fixing k after (6.23)) and get

$$\nabla_z^j V = \mathcal{O}(h^{-j}),\ j \geqslant 0. \tag{6.34}$$

(Make the change of variable: $z = z_0 + hw$.)

We recall a general fact about Grushin problems, namely that $P^w(z)$ is invertible precisely when $F_{-+}(z)$ is. We will say that $z = z_0$ is an eigen-value of $z \mapsto P^w(z)$ if $P^w(z_0)$ is non-invertible. For such an eigen-value, we define the corresponding multiplicity $m(z_0)$ to be the order of z_0 as a zero of the holomorphic function $e^V F_{-+}$. In the appendix A to this section we show that this multiplicity does not depend on the way we construct the Grushin problem and also that it is the order of z_0 as a zero of $\det P^w(z)$ in case $P(z) - 1$ is of trace class.

We shall next use the assumption (6.2) and show that we have $\theta_0(z) = \text{Const.} \in (\frac{1}{2}\mathbf{Z})^2$. We shall do this by studying Floquet periodic WKB solutions in a neighborhood of $\pi_x(\Sigma)$, and we start by reviewing some facts for such solutions when working with the Weyl quantization for the corresponding pseudodifferential operators. (Cf. Appendix a in [**HeSj2**].)

Recall that the Weyl quantization of a symbol p on $\mathbf{R}^{2n}$ is given by:

$$p^w(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\theta} p\Big(\frac{x+y}{2}, \theta\Big) u(y) dy d\theta. \tag{6.35}$$

Let $\phi(x)$ be a smooth and real function. (The adaptation to the complex environment will be quite immediate.) Then

$$\begin{aligned} &e^{-\frac{i}{h}\phi(x)} p^w(x, hD_x) e^{\frac{i}{h}\phi(x)} u(x) \\ &\qquad = \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}((x-y)\cdot\theta - (\phi(x)-\phi(y)))} p\Big(\frac{x+y}{2}, \theta\Big) u(y) dy d\theta. \end{aligned} \tag{6.36}$$

Employ the Kuranishi trick: $\phi(x) - \phi(y) = (x-y)\cdot\Phi(x,y)$, with

$$\Phi(x,y) = \int_0^1 \frac{\partial\phi}{\partial x}(tx + (1-t)y) dt,$$

and notice that

$$\Phi(x,y) = \frac{\partial\phi}{\partial x}\Big(\frac{x+y}{2}\Big) + \mathcal{O}((x-y)^2).$$

Then,

$$e^{-\frac{i}{h}\phi(x)}p^w(x,hD_x)e^{\frac{i}{h}\phi(x)}u = \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot(\theta-\Phi(x,y))}p\Big(\frac{x+y}{2},\theta\Big)u(y)dyd\theta = \frac{1}{(2\pi h)^n}\iint e^{\frac{i}{h}(x-y)\cdot\theta}p\Big(\frac{x+y}{2},\theta+\Phi(x,y)\Big)u(y)dyd\theta. \tag{6.37}$$

Here

$$p\Big(\frac{x+y}{2},\theta+\Phi(x,y)\Big) = p\Big(\frac{x+y}{2},\theta+\frac{\partial\phi}{\partial x}\Big(\frac{x+y}{2}\Big)\Big)+\mathcal{O}((x-y)^2), \tag{6.38}$$

and it follows easily (by double integration by parts with respect to θ for the contribution from the remainder) that the h-Weyl symbol of $e^{-\frac{i}{h}\phi(x)}p^w(x,hD_x)e^{\frac{i}{h}\phi(x)}$ is equal to $p(x,\theta+\frac{\partial\phi}{\partial x}(x))+\mathcal{O}(h^2)$.

Suppose that ϕ solves the eikonal equation

$$p\Big(x,\frac{\partial\phi}{\partial x}(x)\Big)=0. \tag{6.39}$$

We look for a smooth function $a(x)$, independent of h, such that

$$e^{-\frac{i}{h}\phi(x)}p^w(x,hD_x)e^{\frac{i}{h}\phi(x)}a(x)=\mathcal{O}(h^2), \tag{6.40}$$

and get

$$p_\phi^w(x,hD_x)a(x)=\mathcal{O}(h^2), \tag{6.41}$$

where

$$p_\phi(x,\xi)=p\Big(x,\xi+\frac{\partial\phi}{\partial x}(x)\Big). \tag{6.42}$$

Write

$$p_\phi(x,\xi)=\sum_1^n\frac{\partial p_\phi}{\partial\xi_j}(x,0)\xi_j+\mathcal{O}(\xi^2). \tag{6.43}$$

The remainder will give an $\mathcal{O}(h^2)$ contribution to (6.41) and the Weyl quantization of the sum is

$$\frac{1}{2}\sum_1^n\Big(\frac{\partial p_\phi}{\partial\xi_j}(x,0)\circ hD_{x_j}+hD_{x_j}\circ\frac{\partial p_\phi}{\partial\xi_j}(x,0)\Big)=\frac{h}{i}\Big(\nu\Big(x,\frac{\partial}{\partial x}\Big)+\frac{1}{2}\mathrm{div}\,(\nu)\Big), \tag{6.44}$$

where $\nu(x,\frac{\partial}{\partial x})=\sum\frac{\partial p_\phi}{\partial\xi_j}(x,0)\frac{\partial}{\partial x_j}$ can be identified with the restriction of H_p to Λ_ϕ: $\xi=\phi'(x)$. The equation (6.41) therefore boils down to the transport equation

$$\Big(\nu\Big(x,\frac{\partial}{\partial x}\Big)+\frac{1}{2}\mathrm{div}\,(\nu)\Big)a=0. \tag{6.45}$$

As in [**DuHo**] the last equation can also be written in terms of the Lie derivative of ν acting on a half density:

$$\mathcal{L}_\nu(a(x)(dx_1\cdots dx_n)^{1/2})=0. \tag{6.46}$$

Recall that $\widetilde{\Lambda}(z) \subset p(\cdot,z)^{-1}(0)$ is a complex Lagrangian manifold which can be viewed as a complexification of $\Gamma(z)$. We can represent $\widetilde{\Lambda}(z)$ by

$$\xi = \frac{\partial \phi}{\partial x}(x,z),\ x \in \operatorname{neigh}(\pi_x(\Sigma)), \tag{6.47}$$

where ϕ is grad periodic, smooth in both variables, holomorphic in z and (cf. (2.4)) satisfies

$$\Phi(x,z) + \operatorname{Im}\phi(x,z) \sim \operatorname{dist}(x, \pi_x\Gamma(z))^2, \tag{6.48}$$

where $\Phi(\cdot,z) = \Phi_z$, $\Lambda_z = \Lambda_{\Phi_z}$. If $\gamma_1, \gamma_2 \subset \pi_x(\Gamma(z))$ are two fundamental cycles, we also have

$$\operatorname{var}_{\gamma_j}\phi(\cdot,z) = I_j(z),\ p(x, \frac{\partial\phi}{\partial x}(x,z), z) = 0, \tag{6.49}$$

with $I_j(z) = I_j(\Gamma(z),\omega)$. We look for a multivalued holomorphic symbol $a(x) = a(x,z)$ (being the leading term in an asymptotic expansion) such that

$$P^w(x, hD_x, z; h)(\frac{1}{h}a(x,z)e^{i\phi(x,z)/h}) = \mathcal{O}(h)e^{i\phi(x,z)/h}. \tag{6.50}$$

As reviewed above, (6.50) is equivalent to the transport equation

$$\mathcal{L}_\nu(a(x)(dx_1 \wedge dx_2)^{1/2}) = 0, \tag{6.51}$$

where $\nu \simeq H_{p|_{\widetilde{\Lambda}(z)}}$. We only want to solve (6.51) to infinite order on $\pi_x(\Gamma(z))$ which is maximally totally real, so we can restrict (6.51) to this torus by interpreting $\nu \simeq H_p$ as a complex vector field here. Once (6.51) is solved on the submanifold, we get it to infinite order there, by taking almost holomorphic extensions.

Recall from section 1 that there is a diffeomorphism

$$Q : \Gamma(z) \longrightarrow \mathbf{C}/L(z), \tag{6.52}$$

depending smoothly on z such that

$$\nu \simeq H_p = A\frac{\partial}{\partial\overline{Q}}, \tag{6.53}$$

where $A = A(Q,z)$ is smooth and non-vanishing.

Write $a(x)(dx_1 \wedge dx_2)^{1/2} = b(Q)(dQ_1 \wedge dQ_2)^{1/2}$, $Q = Q_1 + iQ_2$. We notice that

$$\frac{(dx_1 \wedge dx_2)^{1/2}}{(dQ_1 \wedge dQ_2)^{1/2}}$$

is not necessarily single valued, but θ_1-Floquet periodic for some $\theta_1 \in \frac{1}{2}L^*$. Then (6.51) becomes

$$\mathcal{L}_{A\frac{\partial}{\partial\overline{Q}}}(b(dQ_1 \wedge dQ_2)^{1/2}) = 0,$$

and more explicitly

$$A\frac{\partial}{\partial\overline{Q}}b + \frac{1}{2}\frac{\partial}{\partial\overline{Q}}(A)b = 0, \tag{6.54}$$

since div $A\frac{\partial}{\partial \overline{Q}} = \frac{\partial}{\partial \overline{Q}}A$. (6.54) can also be written

$$\frac{\partial}{\partial \overline{Q}}(A^{1/2}b) = 0, \tag{6.55}$$

where we notice that $A^{1/2}$ is α-Floquet periodic for some $\alpha \in \frac{1}{2}L^*$.

We restrict the attention to solutions $u = h^{-1/2}ae^{i\phi/h}$ of (6.50) which are multivalued but ω-Floquet periodic in the sense that

$$u(\Gamma_j^{-1}(x), z) = e^{2\pi i\omega_j}u(x,z),\ j = 1,2,\ \omega = (\omega_1, \omega_2) \in \mathbf{R}^2/\mathbf{Z}^2,$$

where Γ_j is the natural action of the fundamental cycle γ_j on the covering space of neigh $(\pi_x(\Gamma(z)), \mathbf{C}^2)$. Then,

$$a(\Gamma_j^{-1}(x), z) = e^{i(2\pi h\omega_j + I_j(z))/h}a(x,z),$$

so the restriction of $a(\cdot, z)$ to $\pi_x(\Gamma(z))$ is $\omega + I(z)/(2\pi h)$ Floquet periodic if we identify $\pi_x(\Gamma(z))$ with the standard torus Γ_0. Then $b(Q,z)$ is $\omega + \frac{I(z)}{2\pi h} + \theta_1$ Floquet periodic (as a function on Γ_0) and hence $A^{1/2}b$ is $\omega + \frac{I(z)}{2\pi h} - \theta_2$ Floquet periodic for some $\theta_2 \in (\frac{1}{2}\mathbf{Z})^2$. We now require that a be non-vanishing. Then from (6.55), we see that $A^{1/2}b$ is periodic and hence $\omega + \frac{I(z)}{2\pi h} - \theta_2 \equiv 0$, mod $\mathbf{Z}^2$:

$$\omega = -\frac{I(z)}{2\pi h} + \theta_2 \text{ in } \mathbf{R}^2/\mathbf{Z}^2. \tag{6.56}$$

Since $U(z)$ is pseudolocal, we can define $U(z)u$ mod $\mathcal{O}(h^\infty)$ as a θ_2-Floquet periodic function on Γ_0 which is microlocally concentrated to a small neighborhood of the zero-section of $T^*\Gamma_0$ with the property that $\|U(z)u\|_{H_{\theta_2}} \sim 1$. From (6.50), we get

$$Q^w(x, hD_x, z; h)(U(z)u) = \mathcal{O}(h^2) \text{ in } H_{\theta_2}.$$

This implies that we are not in the case 1) of Proposition 5.1 for any $C > 0$ and consequently (since θ_2, $\theta(z)$ are independent of h), that $\theta_2 \equiv \theta(z)$ mod $\mathbf{Z}^2$. We have proved under the assumptions above, in particular (6.2):

Proposition 6.2. — *θ_0 in Proposition 6.1 is independent of z and belongs to $(\frac{1}{2}\mathbf{Z})^2$.*

We have proved most of our main theorem below. The result will be most complete, under the additional assumption (1.51):

$$z \longmapsto (I_1(z), I_2(z)) \in \mathbf{R}^2 \text{ is a local diffeomorphism.} \tag{6.57}$$

Theorem 6.3. — *Let $P^w(z) : H_{\Phi_0} \to H_{\Phi_0}$ satisfy (6.1–5), where Φ_0 is a strictly plurisubharmonic quadratic form on $\mathbf{C}^2$, and define $I(z) = (I_1(z), I_2(z))$ as after (6.7). Let $\theta_0 \in \frac{1}{2}\mathbf{Z}^2$ be defined as above. There exists $\theta(z;h) \sim \theta_0 + \theta_1(z)h + \theta_2(z)h^2 + \cdots$ in $C^\infty(\text{neigh}\,(0,\mathbf{C}); \mathbf{R}^2)$, such that for z in an h-independent neighborhood of 0 and for $h > 0$ sufficiently small, we have:*

1) z is an eigen-value (i.e. P^w is non-bijective) iff we have

$$\frac{I(z)}{2\pi h} = \theta(z;h) - k, \text{ for some } k \in \mathbf{Z}^2. \tag{6.58}$$

2) If I is a local diffeomorphism then the eigenvalues form a distorted lattice and they are of the form $z(k;h) = z_0(k;h) + \mathcal{O}(h^2)$, $k \in \mathbf{Z}^2$, where $z_0(k;h)$ is the solution of the approximate BS-condition:

$$\frac{I(z_0(k;h))}{2\pi h} = \theta_0 - k. \tag{6.59}$$

These eigen-values have multiplicity 1 as defined after (6.34).

Let

$$Z_k = \{z \in \operatorname{neigh}(0,\mathbf{C});\ |\tfrac{I(z)}{2\pi h} + k - \theta_0| < 1/3\},\ k \in \mathbf{Z}^2,$$

so that the Z_k are mutually disjoint and all eigen-values have to belong to the union of the Z_k and so that every eigen-value in Z_k has to be a solution of (6.58) with the same value of k. Let $\widetilde{Z}_k$ be a connected component of Z_k.
3) Assume (for a given sufficiently small h) that not every point of $\widetilde{Z}_k$ is an eigen-value. Then the set of eigen-values in $\widetilde{Z}_k$ is discrete and the multiplicity of such an eigen-value z (solving (6.58)) is equal to $\operatorname{var}\arg_\gamma(\frac{I(w)}{2\pi h} + k - \theta(w;h)) \in \{1,2,\dots\}$, where γ is the oriented boundary of a sufficiently small disc centered at z. Here the orientation in the I is obtained from identifying the θ-plane with $\mathbf{C}$ so that we have the expression for $E^0_{-+}(\theta,z)$ after (5.14).

Proof. — For $k \in \mathbf{Z}^2$, let

$$\Omega_k(h) = \{z \in \operatorname{neigh}(0,\mathbf{C});\ |\tfrac{I(z)}{2\pi h} + k - \theta_0| < 1/C\},$$

for some fixed and sufficiently large $C > 0$. Then according to Proposition 6.1, all eigen-values of $P^w(z)$ are contained in the union of the $\Omega_k(h)$. Moreover the $\Omega_k(h)$ are mutually disjoint, and for $k \neq \ell$, we have that $\operatorname{dist}(\Omega_k(h), \Omega_\ell(h)) \geqslant c|k-\ell|h$, for some constant $c > 0$.

From (6.24) and the fact that this also holds in the C^∞-sense, we see that there exists a smooth function

$$E_{-+}(\theta,z;h) \sim E^0_{-+}(\theta,z) + hE^1_{-+}(\theta,z) + \cdots,\ h \longrightarrow 0,$$

defined for $\theta \in \operatorname{neigh}(\theta_0,\mathbf{C})$, such that

$$F_{-+}(z;h) = E_{-+}(k + \tfrac{I(z)}{2\pi h}, z; h),\ z \in \Omega_k(h). \tag{6.60}$$

As remarked after (5.14), we may assume, with a suitable identification of the I-plane and $\mathbf{C}$, that

$$E^0_{-+}(\theta,z) = \frac{i}{2}(\theta - \theta_0), \tag{6.61}$$

where $\theta_0 = \theta_0(z)$ now denotes the complex number which is identified with the previous θ_0. We equip the I-plane with the corresponding orientation.

Let $\theta(z;h)$ be the unique zero close to θ_0, of the function $\theta \mapsto E_{-+}(\theta,z;h)$. Then θ is smooth in z and has an asymptotic expansion as in the theorem. Clearly $z \in \Omega_k(h)$ is an eigen-value iff $k + \frac{I(z)}{2\pi h} = \theta(z;h)$, i.e. iff (6.58) holds. This proves 1).

The implicit function theorem gives everything in the statement 2) except perhaps that the eigen-values are simple. From (6.60) it is clear however that the eigen-values $z(k;h)$ must be simple zeros of the holomorphic function $e^{V(z;h)}F_{-+}(z;h)$ in (6.33), so 2) holds.

We now make the assumptions of 3) and identify the I-plane with $\mathbf{C}$ as in (6.61). In view of (6.60), and Taylor's formula for $\theta \mapsto E_{-+}(\theta, z; h)$, we get for $w \in \widetilde{Z}_k \cap \Omega_k$:

$$\begin{aligned} F_{-+}(w;h) &= E_{-+}(k + \tfrac{I(w)}{2\pi h}, w; h) - E_{-+}(\theta(w;h), w; h) \\ &= A(w;h)(k + \tfrac{I(w)}{2\pi h} - \theta(w;h)) + B(w;h)\overline{(k + \tfrac{I(w)}{2\pi h} - \theta(w;h))}, \end{aligned} \tag{6.62}$$

where A, B are smooth in w with bounded derivatives to all orders. Moreover $|A| \sim 1$, $|B| \ll |A|$. Let $z \in \widetilde{Z}_k$ be an eigen-value (necessarily in Ω_k, and let γ be as in 3). From (6.62) and the fact that A dominates over B, it follows that F_{-+} and $k + \frac{I(w)}{2\pi h} - \theta(w;h)$ have the same argument variation along γ, and 3) follows. □

We next compute the differential and the Jacobian of the map $z \mapsto (I_1(z), I_2(z))$ and show that (6.57) ((1.51)) is equivalent to the property (4.16). We fix some value of z, say $z = 0$. Choose grad-periodic coordinates Q_1, Q_2 on $\Gamma(0)$, so that

$$H_p = A(Q)\frac{\partial}{\partial \overline{Q}} \text{ on } \Gamma(0) \simeq \mathbf{C}/L,\ L = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2, \tag{6.63}$$

where $A(Q) \neq 0\ \forall Q$. Extend Q_1, Q_2 to grad-periodic functions in a neighborhood of $\Gamma(0)$ in $\Lambda_{\Phi_{z=0}}$, and let P_1, P_2 be corresponding "dual" coordinates, vanishing on $\Gamma(0)$, so that $(Q_1, Q_2; P_1, P_2)$ are symplectic coordinates near $\Gamma(0)$.

Then

$$p = \frac{1}{2}A(Q)(P_1 + iP_2) + zr(Q) + \mathcal{O}(P^2) + \mathcal{O}(z^2),$$

where $r = \frac{\partial p}{\partial z}(\cdot, 0)$. $\Gamma(z)$ can be represented by

$$P = \nabla_Q g(Q, z),$$

where $g = \mathcal{O}(z)$ is grad-periodic and

$$p(Q, \nabla_Q g, z) = 0,$$

so that

$$A(Q)\frac{\partial g}{\partial \overline{Q}} + zr(Q) = \mathcal{O}(z^2). \tag{6.64}$$

Let $J_j(z)$ be the actions in $\Gamma(z)$ with respect to $P_1 dQ_1 + P_2 dQ_2$. By Stokes' formula, $I_j - J_j$ is independent of z and since the difference is real for $z = 0$, we know that $J_j(z)$ are real. From this and (6.64) we see that

$$g = \overline{b(z)}Q + b(z)\overline{Q} + g_{\text{per}} + \mathcal{O}(z^2), \tag{6.65}$$

where g_{per} is periodic and

$$b(z) = -z\widehat{r/A}(0), \tag{6.66}$$

where the hat denotes Fourier transform on $\mathbf{C}/L(0)$: $\widehat{r/A} = \mathcal{F}(r/A)$. It follows that

$$J_j(z) = \overline{b(z)}e_j + b(z)\overline{e_j} + \mathcal{O}(z^2). \tag{6.67}$$

The map in (6.57) has the same differential as that of the map $z \mapsto (J_1(z), J_2(z))$, and we get for $z = 0$:

$$dI_1 \wedge dI_2 = (e_1 d\overline{b} + \overline{e}_1 db) \wedge (e_2 d\overline{b} + \overline{e}_2 db) = (e_1\overline{e}_2 - \overline{e}_1 e_2) d\overline{b} \wedge db,$$

so for $z = 0$:

$$\det \frac{\partial(I_1, I_2)}{\partial(z_1, z_2)} = 2i(e_1\overline{e}_2 - \overline{e}_1 e_2)|\mathcal{F}(\partial_z p/A)(0)|^2. \tag{6.68}$$

The equivalence of (6.57) and (4.16) follows.

For $z = 0$, let $\lambda_{p,0}$ be the Liouville measure on $\Gamma(0)$ defined by

$$\lambda_{p,0} \wedge d\mathrm{Re}\, p \wedge d\mathrm{Im}\, p = \mu,$$

where $\mu = \frac{1}{2}\sigma^2$ is the symplectic volume element on $\Lambda_{\Phi_{z=0}}$. In our special coordinates, we have

$$p = \frac{1}{2}A(Q)(P_1 + iP_2) + \mathcal{O}(P^2),$$

for $z = 0$, and the Liouville measure becomes $\lambda_{p,0} = 4|A|^{-2}L(dQ)$. The Hamilton field of H_p on $\Gamma(0)$ is $H_p = A(Q)\frac{\partial}{\partial \overline{Q}}$, which has the adjoints

$$H_p^* = -\frac{\partial}{\partial Q} \circ \overline{A}(Q), \; H_p^\dagger = -|A|^2 \frac{\partial}{\partial Q} \circ \frac{1}{A},$$

with respect to the measures $L(dQ)$ and $\lambda_{p,0}(dQ)$ respectively. Using that the volume of $\mathbf{C}/L = \Gamma(0)$ with respect to $L(dQ)$ is equal to $|\frac{i}{2}(e_1\overline{e}_2 - \overline{e}_1 e_2)|$, we see that the 1-dimensional kernel of $H_p^\dagger$ in $L^2(\Gamma(0), \lambda_{p,0}(d\rho))$ is spanned by the normalized element

$$f := |2i(e_1\overline{e}_2 - \overline{e}_1 e_2)|^{-1/2} A,$$

and a straight forward calculation from (6.68) gives for $z = 0$

$$\Big|\det \frac{\partial(I_1, I_2)}{\partial(z_1, z_2)}\Big| = \Big|\int \partial_z p \overline{f} \lambda_{p,0}(d\rho)\Big|^2 = \int |(1 - \Pi)\partial_z p|^2 \lambda_{p,0}(d\rho), \tag{6.69}$$

where in the last expression we used the notation of (8.38) in [**MeSj**], so that $1 - \Pi$ is the orthogonal projection onto the kernel of $H_p^\dagger$ in $L^2(\lambda_{p,0})$.

Assuming (6.57), the density of eigenvalues, given in 2) of the theorem, is

$$\frac{1}{(2\pi h)^2}\Big(\Big|\det \frac{\partial(I_1, I_2)}{\partial(z_1, z_2)}\Big| + o(1)\Big), \; h \longrightarrow 0.$$

Assume that $P(\cdot, z) \to 1$ sufficiently fast at ∞, so that $\det P^w$ is well defined. Since the eigenvalue $z(k; h)$ is a simple zero of this determinant and $\partial_z \partial_{\overline{z}} \log|z| = \frac{\pi}{2}\delta$, $z(k; h)$ will give the contribution $\frac{\pi}{2}\delta(z - z(k; h))$ to $\partial_z \partial_{\overline{z}} \log|\det P^w(z)|$ and hence in the sense of distributions (or even the weak measure sense), we have

$$\partial_z \partial_{\overline{z}} \log|\det P^w(z)| = \frac{1}{(2\pi h)^2}\Big(\frac{\pi}{2}\Big|\int_{p^{-1}(\cdot,z)(0)} \partial_z p \overline{f(z)} \lambda_{p,0}(d\rho)\Big|^2 + o(1)\Big), \tag{6.70}$$

where we now let f vary with z in the obvious sense. This is in perfect agreement with (8.38) of [**MeSj**], where we computed $\partial_z\partial_{\overline{z}}I(z)$ for an (infinitesimal) majorant $(2\pi h)^{-2}(I(z)+o(1))$ of $\log|\det P^w(z)|$.

Appendix A: Remark on multiplicities

Let $\Omega\subset\mathbf{C}$ be open and simply connected. Let $\mathcal{H}$ be a complex Hilbert space and let

$$\mathcal{P}(z)=\begin{pmatrix}P(z) & R_-(z)\\ R_+(z) & 0\end{pmatrix}:\mathcal{H}\times\mathbf{C}^N\longrightarrow\mathcal{H}\times\mathbf{C}^N$$

depend smoothly on $z\in\Omega$ and be bijective for all z. Assume that

$$dP(z)=\frac{\partial P}{\partial \mathrm{Re}\, z}d\mathrm{Re}\, z+\frac{\partial P}{\partial \mathrm{Im}\, z}d\mathrm{Im}\, z$$

is of trace class locally uniformly in z. Write

$$\mathcal{P}(z)^{-1}=\mathcal{E}(z)=\begin{pmatrix}E(z) & E_+(z)\\ E_-(z) & E_{-+}(z)\end{pmatrix}.$$

Recall that $P(z)$ is invertible precisely when $E_{-+}(z)$ is and that we have

$$P(z)^{-1}=E(z)-E_+(z)E_{-+}(z)^{-1}E_-(z). \tag{A.1}$$

Proposition. — *Let $\gamma\subset\Omega$ be a closed C^1-curve along which $P(z)$ (or equivalently $E_{-+}(z)$) is invertible. Then*

$$\mathrm{tr}\Big(\frac{1}{2\pi i}\int_\gamma P(z)^{-1}dP(z)\Big)=\mathrm{tr}\Big(\frac{1}{2\pi i}\int_\gamma E_{-+}(z)^{-1}dE_{-+}(z)\Big). \tag{A.2}$$

Proof. — From $d\mathcal{E}=-\mathcal{E}d\mathcal{P}\mathcal{E}$, we get

$$\begin{aligned}-dE&=EdPE+E_+dR_+E+EdR_-E_-,\\ -dE_+&=EdPE_++E_+dR_+E_++EdR_-E_{-+},\\ -dE_-&=E_-dPE+E_{-+}dR_+E+E_-dR_-E_-,\\ -dE_{-+}&=E_-dPE_++E_{-+}dR_+E_++E_-dR_-E_{-+}.\end{aligned} \tag{A.3}$$

We get,

$$\mathrm{tr}\, P^{-1}dP=\mathrm{tr}\,(EdP)-\mathrm{tr}\,(E_+E_{-+}^{-1}E_-dP).$$

Here by the cyclicity of the trace and the last equation in (A.3):

$$\begin{aligned}&-\mathrm{tr}\,(E_+E_{-+}^{-1}E_-dP)=-\mathrm{tr}\,(E_{-+}^{-1}E_-dPE_+)\\ &=\mathrm{tr}\,(E_{-+}^{-1}dE_{-+})+\mathrm{tr}\,(E_{-+}^{-1}E_{-+}dR_+E_+)+\mathrm{tr}\,(E_{-+}^{-1}E_-dR_-E_{-+})\\ &=\mathrm{tr}\,(E_{-+}^{-1}dE_{-+})+\mathrm{tr}\,(dR_+E_+)+\mathrm{tr}\,(E_-dR_-).\end{aligned}$$

It follows that

$$\mathrm{tr}\,(P^{-1}dP)=\mathrm{tr}\,(E_{-+}^{-1}dE_{-+})+\omega, \tag{A.4}$$

with

$$\omega = \mathrm{tr}\,(EdP) + \mathrm{tr}\,(dR_+E_+) + \mathrm{tr}\,(E_-dR_-). \tag{A.5}$$

If we assume that $\mathcal{P}$ is holomorphic, then $\mathcal{E}$ will be holomorphic and ω will be a (1,0)-form with holomorphic coefficients, hence closed, and the Proposition follows, since Ω is simply connected.

In the general case it suffices to verify that ω is still closed, as we shall now do. Using the obvious calculus of differential forms with operator coefficients, we get:

$$d\omega = \mathrm{tr}\,(dE \wedge dP) - \mathrm{tr}\,(dR_+ \wedge dE_+) + \mathrm{tr}\,(dE_- \wedge dR_-).$$

Use (A.3):

$$\begin{aligned} -d\omega =& \mathrm{tr}\,(EdPE \wedge dP) + \mathrm{tr}\,(E_+dR_+E \wedge dP) + \mathrm{tr}\,(EdR_-E_- \wedge dP) \\ &- \mathrm{tr}\,(dR_+ \wedge EdPE_+) - \mathrm{tr}\,(dR_+ \wedge E_+dR_+E_+) - \mathrm{tr}\,(dR_+ \wedge EdR_-E_{-+}) \\ &+ \mathrm{tr}\,(E_-dPE \wedge dR_-) + \mathrm{tr}\,(E_{-+}dR_+E \wedge dR_-) + \mathrm{tr}\,(E_-dR_-E_- \wedge dR_-). \end{aligned} \tag{A.6}$$

The cyclicity of the trace implies that if μ is an operator 1-form, with trace class coefficients, then $\mathrm{tr}\,\mu \wedge \mu = 0$. It follows that the 1st, 5th and 9th terms of the right hand side of (A.6) vanish:

$$\begin{aligned} \mathrm{tr}\,(EdPE \wedge dP) &= \mathrm{tr}\,(EdP \wedge EdP) = 0, \\ \mathrm{tr}\,(dR_+ \wedge E_+dR_+E_+) &= \mathrm{tr}\,(dR_+E_+ \wedge dR_+E_+) = 0, \\ \mathrm{tr}\,(E_-dR_-E_- \wedge dR_-) &= \mathrm{tr}\,(E_-dR_- \wedge E_-dR_-) = 0. \end{aligned}$$

The terms no 2 and 4, no 3 and 7 as well as no 6 and 8 cancel each other mutually, becauce the cyclicity of the trace implies that $\mathrm{tr}\,(\mu_1 \wedge \mu_2) = -\mathrm{tr}\,(\mu_2 \wedge \mu_1)$ for operator 1-forms with one factor of trace class, and hence

$$\begin{aligned} \mathrm{tr}\,(E_+dR_+E \wedge dP) &= \mathrm{tr}\,(E_+dR_+ \wedge EdP) = \mathrm{tr}\,(dR_+ \wedge EdPE_+) \\ \mathrm{tr}\,(EdR_- \wedge E_-dP) &= -\mathrm{tr}\,(E_-dP \wedge EdR_-), \\ \mathrm{tr}\,(dR_+ \wedge EdR_-E_{-+}) &= \mathrm{tr}\,(E_{-+}dR_+ \wedge EdR_-). \end{aligned}$$

Thus $d\omega = 0$ and we get the proposition in the general case. □

Now drop the assumption that $dP(z)$ be of trace class, but assume that there exists an invertible operator $Q(z)$ which depends smoothly on z such that $d(Q(z)P(z))$ is locally uniformly of trace class. Then we have the invertible Grushin operator:

$$\begin{pmatrix} Q(z)P(z) & Q(z)R_-(z) \\ R_+(z) & 0 \end{pmatrix}, \text{ with inverse } \begin{pmatrix} EQ^{-1} & E_+ \\ E_-Q^{-1} & E_{-+} \end{pmatrix}.$$

The equation (A.2) then holds, if we replace P by QP in the left hand side. Notice that if we add the assumption that $Q(z)P(z) - 1$ be of trace class, then (A.2) (with QP replacing P) gives

$$\mathrm{var}\,\arg_\gamma \det(Q(z)P(z)) = \mathrm{var}\,\arg_\gamma(\det(E_{-+}(z)). \tag{A.7}$$

Assume that $P(z)$ is invertible for $z_0 \neq z \in \operatorname{neigh}(z_0, \mathbf{C})$, but that $P(z_0)$ is not invertible. Then it is easy to see that there exists an operator K of finite rank such that $P(z_0) + K$ is invertible, and hence also that $P(z) + K$ is invertible for z in a small neighborhood of z_0. Put $Q(z) = (P(z) + K)^{-1}$. Then $Q(z)P(z) - 1 = -Q(z)K$ is of finte rank and hence of trace class, so (A.7) applies. Let

$$m(z_0) = \frac{1}{2\pi i} \operatorname{var} \arg_\gamma (\det Q(z)P(z)), \tag{A.8}$$

where γ is the oriented boundary of a small disc centered at z_0. (A.7) shows that this integer is independent both of the choice of Q and of the Grushin problem, and by the definition this will be the multiplicity of z_0 as an "eigen-value" of $z \mapsto P(z)$. In the main text, $P(z)$ depends holomorphically on z and then have $m(z_0) \geqslant 1$.

Appendix B: Modified $\overline{\partial}$-equation for $(I_1(z), I_2(z))$

We recall from section 1, that we have a holomorphic map

$$\operatorname{neigh}((0,0), \mathbf{C}^2) \ni (z, w) \longmapsto I(z, w) = (I_1(z, w), I_2(z, w)) \in \mathbf{C}^2, \tag{B.1}$$

with $I(0,0) \in \mathbf{R}^2$ and with

$$\operatorname{Im}(\partial_w I_1 \overline{\partial_w I_2}) \neq 0. \tag{B.2}$$

Let $(f_1(z, w), f_2(z, w))$ be holomorphic, non-vanishing such that

$$f_1(z, w)\partial_w I_1(z, w) + f_2(z, w)\partial_w I_2(z, w) = 0. \tag{B.3}$$

This implies that

$$f_1(z, w) dI_1 + f_2(z, w) dI_2 = g(z, w) dz, \tag{B.4}$$

where $g(z, w)$ is holomorphic.

From (B.2) it follows (as we saw in section 1) that there is a unique smooth function: $\operatorname{neigh}(0, \mathbf{C}) \ni z \mapsto z(w) \in \operatorname{neigh}(0, \mathbf{C})$, such that

$$I(z, w(z)) \in \mathbf{R}^2. \tag{B.5}$$

Indeed, this follows from the implicit function theorem, for if we formally make infinitesimal increments to z, w, we get

$$\partial_z I_j\, \delta_z + \partial_w I_j\, \delta_w = \overline{\partial_z I_j}\, \overline{\delta_z} + \overline{\partial_w I_j}\, \overline{\delta_w},$$

$$\begin{cases} \partial_w I_1\, \delta_w - \overline{\partial_w I_1}\, \overline{\delta_w} = -\partial_z I_1\, \delta_z + \overline{\partial_z I_1}\, \overline{\delta_z}, \\ \partial_w I_2\, \delta_w - \overline{\partial_w I_2}\, \overline{\delta_w} = -\partial_z I_2\, \delta_z + \overline{\partial_z I_2}\, \overline{\delta_z}, \end{cases} \tag{B.6}$$

and notice that

$$\det \begin{pmatrix} \partial_w I_1 & -\overline{\partial_w I_1} \\ \partial_w I_2 & -\overline{\partial_w I_2} \end{pmatrix} = -2i \operatorname{Im}(\partial_w I_1 \overline{\partial_w I_2}) \neq 0.$$

Treating $\delta_w, \overline{\delta_w}$ as independent variables, we see that (B.6) has a unique solution $(\delta_w, \overline{\delta_w}) \in \mathbf{C}^2$ for a given $\delta_z \in \mathbf{C}$, and it is easy to see that $\overline{\delta_w}$ has to be the complex conjugate of δ_w. The existence of the smooth function $w(z)$ in (B.5) therefore follows from the implicit function theorem.

Let $J_j(z) = I_j(z, w(z))$. (In the main text, we simply write $I_j(z) = I_j(z, w(z))$.) Restricting (B.4) to the submanifold, given by $w = w(z)$, we get

$$f_1 dJ_1 + f_2 dJ_2 = g dz, \tag{B.7}$$

with $f_j = f_j(z, w(z))$, $g = g(z, w(z))$. Taking the antilinear part of this relation, we get

$$f_1 \overline{\partial} J_1 + f_2 \overline{\partial} J_2 = 0,\ \overline{\partial} = \partial_{\overline{z}}. \tag{B.8}$$

This can also we written

$$\overline{\partial}(f_1(J_1 - J_1^0) + f_2(J_2 - J_2^0)) - ((\overline{\partial} f_1)(J_1 - J_1^0) + (\overline{\partial} f_2)(J_2 - J_2^0)) = 0, \tag{B.9}$$

where J_j^0 are arbitrary real constants. Put

$$u = f_1(J_1 - J_1^0) + f_2(J_2 - J_2^0). \tag{B.10}$$

Using (B.2) and (B.3), we see that the two real functions J_1, J_2 can be recovered from u by means of the formula,

$$\begin{cases} J_1 - J_1^0 = \frac{1}{2i\mathrm{Im}\,(f_1 \overline{f_2})}(\overline{f_2} u - f_2 \overline{u}), \\ J_2 - J_2^0 = \frac{1}{2i\mathrm{Im}\,(f_1 \overline{f_2})}(-\overline{f_1} u + f_1 \overline{u}). \end{cases} \tag{B.12}$$

(Notice that $(f_1, f_2) = a(\partial_w I_2, -\partial_w I_1)$ for some non-vanishing a, so that $\mathrm{Im}\,(f_1 \overline{f_2}) \neq 0$.) Then (B.9) gives

$$\overline{\partial} u + a u + b \overline{u} = 0, \tag{B.13}$$

for some smooth (and even real-analytic) functions a, b.

It follows from a classical result by Carleman [**Ca**] that if u solves (B.13) in a complex domain and vanishes to infinite order at some point then u vanishes identically. Since a and b are real-analytic we can show this differently: Treating u and $u^* = \overline{u}$ as independent functions, we get

$$\begin{cases} \overline{\partial} u + a(z) u + b(z) u^* = 0, \\ \partial u^* + \overline{b(z)} u + \overline{a(z)} u^* = 0, \end{cases} \tag{B.14}$$

which is an elliptic system with real-analytic coefficients. Hence u is real-analytic and cannot vanish to infinite order at any point without vanishing identically.

If u is not identically 0, let z_0 be a zero of u and write the Taylor expansion as

$$u(z) = p_m(z - z_0) + \mathcal{O}(|z - z_0|^{m+1}),$$

where $p_m \neq 0$ is a homogeneous polynomial of degree m. Substitution into (B.13) shows that p_m is holomorphic, so

$$u(z) = C(z - z_0)^m + \mathcal{O}(|z - z_0|^{m+1}),\ C \neq 0. \tag{B.15}$$

This means that the map

$$\operatorname{neigh}(0, \mathbf{C}) \longmapsto J(z) = (J_1(z), J_2(z)) \in \mathbf{R}^2 \tag{B.16}$$

is either constant or takes any given value J^0 only at isolated points, and if z_0 is such a point, then

$$|J(z) - J^0| \sim |z - z_0|^m. \tag{B.17}$$

Write (B.12) as

$$J(z) - J^0 = F(z) \begin{pmatrix} \operatorname{Re} u \\ \operatorname{Im} u \end{pmatrix}, \tag{B.18}$$

where F is a smooth invertible 2×2-matrix. Then from (B.15), we get

$$\frac{\partial J(z)}{\partial(\operatorname{Re} z, \operatorname{Im} z)} = F(z_0) \frac{\partial(\operatorname{Re} u, \operatorname{Im} u)}{\partial(\operatorname{Re} z, \operatorname{Im} z)} + \mathcal{O}(|z - z_0|^m), \tag{B.19}$$

where the first term to the right is $\mathcal{O}(|z - z_0|^{m-1})$ and has an inverse which is $\mathcal{O}(|z - z_0|^{1-m})$. From this, we see that the critical points of J are isolated if J is not identically constant, and that

$$\det \frac{\partial J(z)}{\partial(\operatorname{Re} z, \operatorname{Im} z)} \text{ is either } \geqslant 0, \forall z, \text{ or } \leqslant 0, \forall z. \tag{B.20}$$

This means that we can introduce a natural orientation on the (J_1, J_2)-plane such that the differential of J becomes orientation preserving. We can then define the multiplicity of a solution z_0 of $J(z) = J^0$ by

$$m(z_0) = \frac{1}{2\pi} \operatorname{var} \arg_\gamma (J(z) - J^0), \tag{B.21}$$

where γ is the positively oriented boundary of a small disc centered at z_0.

In the main text of section 6, we write $I_j(z)$ instead of $J_j(z)$. It is also clear from our discussion, that the orientation of the J-plane is the same as the one we got in the proof of Theorem 6.3 from (6.61).

7. Saddle point resonances

Consider the operator

$$P = -\frac{h^2}{2} \Delta + V(x),\ x \in \mathbf{R}^2, \tag{7.1}$$

where V is a real-valued analytic potential, which extends holomorphically to a set $\{x \in \mathbf{C}^2;\ |\operatorname{Im} x| < \frac{1}{C} \langle \operatorname{Re} x \rangle\}$, with $V(x) \to 0$, when $x \to \infty$ in that set. The resonances of P can be defined in an angle $\{z \in \mathbf{C};\ -2\theta_0 < \arg z \leqslant 0\}$ for some fixed $\theta_0 > 0$

as the eigen-values of $P_{|_{e^{i\theta_0}\mathbf{R}^n}}$. In [**HeSj**], they were also defined as the eigen-values of $P : H(\Lambda_G, 1) \to H(\Lambda_G, 1)$ with domain $H(\Lambda_G, \langle\xi\rangle^2)$, and below we shall have the occasion to recall some more about that approach. (Such a space consists of the functions u such that a suitable FBI-transformation Tu belongs to a certain exponentially weighted L^2 space.)

Let $E_0 > 0$. Let $p(x,\xi) = \xi^2 + V(x)$. We assume that the union of trapped H_p-trajectories in $p^{-1}(E_0) \cap \mathbf{R}^4$ (see [**GeSj**]) is reduced to a single point (x_0, ξ_0). Necessarily, $\xi_0 = 0$ and after a translation, we may also assume that $x_0 = 0$. (Recall for instance from [**GeSj**] that a trapped trajectory is a maximally extended trajectory which is contained in a bounded set.) It follows that 0 is a critical point for V and that $V(0) = E_0$. Assume,

(7.2) 0 is a non-degenerate critical point of V, of signature $(1, -1)$.

After a linear change of coordinates in x and a corresponding dual one in ξ, we may assume that

$$p(x,\xi) - E_0 = \frac{\lambda_1}{2}(\xi_1^2 + x_1^2) + \frac{\lambda_2}{2}(\xi_2^2 - x_2^2) + \mathcal{O}((x,\xi)^3),\ (x,\xi) \longrightarrow 0. \tag{7.3}$$

Under the assumptions above, but without any restriction on the dimension and without the assumption on the signature in (7.3), the second author ([**Sj2**]) determined all resonances in a disc $D(E_0, Ch)$ for any fixed $C > 0$, when $h > 0$ is small enough. (See also [**BrCoDu**] for the barrier top case.) Under the same assumptions plus a diophantine one on the eigen-values of $V''(0)$, Kaidi and Kerdelhué [**KaKe**] determined all resonances in a disc $D(E_0, h^\delta)$ for any fixed $\delta > 0$ and for $h > 0$ small enough. In the two dimensional case, their diophantine condition follows from (7.2), and we recall their result in that case.

***Theorem 7.1* ([KaKe]).** — *Under the assumptions from (7.1) to (7.2), let $\lambda_j > 0$ be defined in (7.3). Fix $\delta > 0$. Then for $h > 0$ small enough, the resonances in $D(E_0, h^\delta)$ are all simple and coincide with the values in that disc, given by:*

$$z = E_0 + f(2\pi h(k - \theta_0); h),\ k \in \mathbf{N}^2, \tag{7.4}$$

where $\theta_0 = (-\frac{1}{2}, \frac{1}{2}) \in (\frac{1}{2}\mathbf{Z})^2$ is fixed, and $f(\theta; h)$ is a smooth function of $\theta \in$ neigh$(0, \mathbf{R}^2)$, with

$$f(\theta; h) \sim f_0(\theta) + hf_1(\theta) + h^2 f_2(\theta) + \cdots,\ h \longrightarrow 0, \tag{7.5}$$

in the space of such functions. Further,

$$f_0(\theta) = \frac{1}{2\pi}(\lambda_1\theta_1 - i\lambda_2\theta_2) + \mathcal{O}(\theta^2).$$

The purpose of this section is to show that the description (7.4) extends to all resonances in a fixed disc $D(E_0, r_0)$ with $r_0 > 0$ small but independent of h, provided that we avoid arbitrarily small angular neighborhoods of $]0; +\infty[$ and $-i]0, +\infty[$. The main ingredient of the proof will be Theorem 6.3, that we will be able to apply after some reductions, using [**HeSj**], [**KaKe**].

As in [**KaKe**], we choose an escape function (in the sense of [**HeSj**]) G which is equal to $x_2\xi_2$ in a neighborhood of $(x,\xi) = (0,0)$ and such that $H_pG > 0$ on $p^{-1}(E_0) \setminus \{(0,0)\}$. Then for a small fixed $t > 0$, we take an FBI-transformation as in [**HeSj**] which is isometric:

$$T : \begin{cases} H(\Lambda_{tG}, 1) \to L^2(\mathbf{C}^2; e^{-2\phi/h}L(dx)), \\ H(\Lambda_{tG}, \langle\xi\rangle^2) \to L^2(\mathbf{C}^2, m^2 e^{-2\phi/h}L(dx)). \end{cases} \tag{7.6}$$

Here $m \geqslant 1 - \mathcal{O}(h)$ is a weight which is independent of h to leading order and $m \sim 1$ in any fixed compact set. Moreover, ϕ is a smooth real-valued function. For possibly only technical reasons, T has to take its values in $L^2(\cdots)\otimes\mathbf{C}^3$ rather than in $L^2(\cdots)$, but as noticed in [**KaKe**], we may modify the definition of T in such a way that the last two components of Tu vanish identically in a neighborhood Ω of $0 \in \mathbf{C}^2$, the point corresponding to $(x,\xi) = (0,0)$, and so that the first component of $Tu(x)$ is given by a standard Bargman transformation in that neighborhood and is consequently a holomorphic function of x. We can also arrange so that ϕ is a strictly plurisubharmonic quadratic form in Ω. Hence $Tu \in H_\phi(\Omega) := L^2_\phi(\Omega) \cap \mathrm{Hol}\,(\Omega)$, where $\mathrm{Hol}\,(\Omega)$ is the space of holomorphic functions on Ω and $L^2_\phi(\Omega) = L^2(\Omega; e^{-2\phi(x)/h}L(dx))$.

Kaidi and Kerdelhué showed that there exists a uniformly bounded operator

$$V : H_\phi(\Omega) \longrightarrow H_\psi(\widetilde{\Omega}),$$

which is a metaplectic operator, i.e. a Fourier integral operator as in [**Sj1**] with quadratic phase and constant amplitude, with an almost inverse (the lack of exactness being due to the fact that we do not work on all of $\mathbf{C}^2$ and consequently get cutoff errors) $U = \mathcal{O}(1) : H_\psi(\widetilde{\Omega}) \to H_\phi(\Omega)$ with the following properties:
(1) $\widetilde{\Omega}$ is a neighborhood of 0 and ψ is a strictly plurisubharmonic quadratic form.
(2) If $\phi_- \leqslant \phi \leqslant \phi_+$ are smooth, and $\phi_\pm$ are sufficiently close to ϕ in C^2 and equal to ϕ outside some neighborhood of 0, then there exist $\psi_- \leqslant \psi \leqslant \psi_+$ with analogous properties, such that

$$\begin{cases} 1 - UV = \mathcal{O}(1) : H_{\phi_+}(\Omega) \to H_{\phi_-}(\Omega), \\ 1 - VU = \mathcal{O}(1) : H_{\psi_+}(\widetilde{\Omega}) \to H_{\psi_-}(\widetilde{\Omega}). \end{cases} \tag{7.7}$$

(3) If we choose $\phi_\pm$ with $\phi_-(0) < \phi(0) < \phi_+(0)$, then $\psi_-(0) < \psi(0) < \psi_+(0)$.
(4) There exists an analytic h-pseudodifferential operator

$$Q^w(x, hD_x; h) = H_{\psi/\psi_+/\psi_-} \longrightarrow H_{\psi/\psi_+/\psi_-}$$

with symbol $Q(x,\xi;h) \sim q_0(x,\xi) + hq_1(x,\xi) + \cdots$, holomorphic in a neighborhood of the closure of $\{(x, \frac{2}{i}\frac{\partial\psi}{\partial x}(x));\ x \in \widetilde{\Omega}\}$ such that

$$Q^w VT - VTP = \mathcal{O}(1) : H(\Lambda_{G_+}, \langle\xi\rangle^2) \longrightarrow H_{\psi_-}(\widetilde{\Omega}). \tag{7.8}$$

Here we extend $\phi_\pm$ to be equal to ϕ outside Ω and define

$$H(\Lambda_{G_\pm},\langle\xi\rangle^2)=\{u\in H(\Lambda_G,\langle\xi\rangle^2);\ Tu\in L^2(\mathbf{C}^2;m^2e^{-2\phi_\pm/h}L(dx)).$$

The spaces $H(\Lambda_{G_\pm},1)$ are defined similarly. For simplicity, we have also introduced a new G; $G_{\rm new}=tG_{\rm old}$, so that $t=1$ from now on. Q^w is realized by means of choices of "good" integration contours as in [**Sj1**].

(5) We have

$$q_0(x,\xi)=i\lambda_1x_1\xi_1+\lambda_2x_2\xi_2+\mathcal{O}((x,\xi)^3). \tag{7.9}$$

Later on we shall also use that we have a local quasi-inverse S to T with

$$S=\mathcal{O}(1):H_{\phi/\phi_+/\phi_-}(\Omega)\longrightarrow H(\Lambda_{G/G_+/G_-},\langle\xi\rangle^2), \tag{7.10}$$

$$1-TS=\mathcal{O}(1):H_{\phi_+}(\Omega)\longrightarrow H_{\phi_-}(\Omega). \tag{7.11}$$

(6) A last feature of the reduction in [**KaKe**] is that there exists a strictly plurisubharmonic smooth function $\widetilde{\phi}$ on $\widetilde{\Omega}$, equal to ψ outside any previously given fixed neighborhood of 0, with

$$\widetilde{\phi}(x)=\frac{1}{2}|x|^2 \text{ in some neighborhood of } 0, \tag{7.12}$$

$$q_0\Big(x,\frac{2}{i}\frac{\partial\widetilde{\phi}}{\partial x}\Big)\neq 0,\ x\in\widetilde{\Omega}\smallsetminus\{0\}. \tag{7.13}$$

Moreover, $\widetilde{\phi}$ can be chosen with $\psi-\widetilde{\phi}$ arbitrarily small in C^1-norm.

Notice that for x in a region, where (7.12) holds, we have

$$q_0\Big(x,\frac{2}{i}\frac{\partial\widetilde{\phi}}{\partial x}\Big)=\lambda_1|x_1|^2-i\lambda_2|x_2|^2+\mathcal{O}(|x|^3). \tag{7.14}$$

We shall next discuss the invertibility of Q^w-z for $|z|$ small, by applying Theorem 6.3. For that, it will be convenient to globalize the problem. We recall that $\widetilde{\phi}=\psi=\psi_+=\psi_-=$ a quadratic form in $\widetilde{\Omega}\smallsetminus\mathrm{neigh}\,(0)$, and we extend these functions to all of $\mathbf{C}^2$, so that they keep the same properties. Extend Q to a symbol in $S^0(\Lambda_{\widetilde{\phi}})=C_b^\infty(\Lambda_{\widetilde{\phi}})$ with the asymptotic expansion

$$Q(x,\xi;h)\sim q_0(x,\xi)+hq_1(x,\xi)+\cdots \tag{7.15}$$

in that space, and so that

$$|q_0(x,\xi)|\geqslant\frac{1}{C}, \tag{7.16}$$

outside a small neighborhood of $(0,0)$.

Let $\chi\in C_b^\infty(\mathbf{C}^2\times\mathbf{C}^2)$ with $1_{|x-y|\leqslant\frac{1}{2C}}\prec\chi\prec 1_{|x-y|\leqslant\frac{1}{C}}$, where we write $f\prec g$ for two functions f,g, if $\mathrm{supp}\,f\cap\mathrm{supp}\,(1-g)=\varnothing$. Put

$$Q_\chi^w(x,hD_x;h)u=\frac{1}{(2\pi h)^2}\iint e^{\frac{i}{h}(x-y)\cdot\theta}Q\Big(\frac{x+y}{2},\theta;h\Big)\chi(x,y)u(y)dyd\theta, \tag{7.17}$$

where we integrate over a contour of the form

$$\theta=\frac{2}{i}\frac{\partial\widetilde{\phi}}{\partial x}\Big(\frac{x+y}{2}\Big)+iC(x)\overline{(x-y)},$$

where $C(x)\geqslant 0$ is a smooth function which is >0 near $x=0$ and with compact support in $\widetilde{\Omega}$. Then,

$$\begin{aligned}Q^w_\chi&=\mathcal{O}(1):H_{\widetilde{\phi}/\psi/\psi_+/\psi_-}(\mathbf{C}^2)\longrightarrow L^2_{\widetilde{\phi}/\psi/\psi_+/\psi_-}(\mathbf{C}^2),\\ \overline{\partial}Q^w_\chi&=\mathcal{O}(h^\infty):H_{\psi_+}\longrightarrow L^2_{\psi_-}.\end{aligned}\tag{7.18}$$

Let $\Pi_{\psi_-}=(1-\overline{\partial}^*(\Delta^{(1)}_{\psi_-})^{-1}\overline{\partial}):L^2_{\psi_-}(\mathbf{C}^2)\to H_{\psi_-}(\mathbf{C}^2)$ be the orthogonal projection (see [**MeSj**], [**Sj3**]), and put

$$Q^w=\Pi_{\psi_-}Q^w_\chi.\tag{7.19}$$

Then $Q^w=\mathcal{O}(1):H_{\widetilde{\phi}/\psi/\psi_+/\psi_-}\to H_{\widetilde{\phi}/\psi/\psi_+/\psi_-}$, $Q^w-Q^w_\chi=\mathcal{O}(h^\infty):H_{\psi_+}\to L^2_{\psi_-}$.

Consider the change of variables $x=\mu\widetilde{x}$, $h^\delta\leqslant\mu\leqslant 1$, for $0<\delta<\frac{1}{2}$. Formally, we get

$$\frac{1}{\mu^2}Q^w(x,hD_x;h)=\frac{1}{\mu^2}Q^w(\mu(\widetilde{x},\widetilde{h}D_{\widetilde{x}});h),\ \widetilde{h}=\frac{h}{\mu^2}.\tag{7.20}$$

The corresponding new symbol is

$$\frac{1}{\mu^2}Q(\mu(\widetilde{x},\widetilde{\xi});h)\sim\frac{1}{\mu^2}q_0(\mu(\widetilde{x},\widetilde{\xi}))+\widetilde{h}q_1(\mu(\widetilde{x},\widetilde{\xi}))+\mu^2\widetilde{h}^2q_2(\mu(\widetilde{x},\widetilde{\xi}))+\cdots.\tag{7.21}$$

Write $\widetilde{\phi}(x)/h=\widetilde{\phi}_\mu(\widetilde{x})/\widetilde{h}$, with

$$\widetilde{\phi}_\mu(\widetilde{x})=\frac{1}{\mu^2}\widetilde{\phi}(x)=\frac{1}{\mu^2}\widetilde{\phi}(\mu\widetilde{x}).\tag{7.22}$$

It follows that,

$$\Lambda_{\widetilde{\phi}_\mu}=\{\mu^{-1}(x,\xi);\ (x,\xi)\in\Lambda_{\widetilde{\phi}}\}.\tag{7.23}$$

The same change of variables in (7.17) gives (with $x=\mu\widetilde{x}$)

$$\begin{aligned}&Q^w_\chi(x,hD_x;h)u\\ &\quad=\frac{1}{(2\pi\widetilde{h})^2}\iint e^{\frac{i}{\widetilde{h}}(\widetilde{x}-\widetilde{y})\cdot\widetilde{\theta}}Q\Big(\mu(\frac{\widetilde{x}+\widetilde{y}}{2},\widetilde{\theta});h\Big)\chi(\mu(\widetilde{x},\widetilde{y}))u(\mu\widetilde{y})d\widetilde{y}d\widetilde{\theta},\end{aligned}\tag{7.24}$$

where the integration is now along the contour

$$\widetilde{\theta}=\frac{2}{i}\frac{\partial\widetilde{\phi}_\mu}{\partial\widetilde{x}}\Big(\frac{\widetilde{x}+\widetilde{y}}{2}\Big)+iC(\mu\widetilde{x})\overline{(\widetilde{x}-\widetilde{y})}.$$

Recall from (7.9) that q_0 vanishes to the 2nd order at $(0,0)$ and let $q_0(x,\xi)=q_{0,2}(x,\xi)+q_{0,3}(x,\xi)+\cdots$ be the Taylor expansion at $(0,0)$, so that $q_{0,j}$ is a homogenous polynomial of degree j. Then for $(\widetilde{x},\widetilde{\xi})$ in a μ-independent neighborhood of $(0,0)$, we get

$$\frac{1}{\mu^2}q_0(\mu(\widetilde{x},\widetilde{\xi}))=q_{0,2}(\widetilde{x},\widetilde{\xi})+\mu q_{0,3}(\widetilde{x},\widetilde{\xi})+\mu^2q_{0,4}(\widetilde{x},\widetilde{\xi})+\cdots.\tag{7.25}$$

This expansion actually holds in a μ^{-1}-neighborhood of (0,0), and outside such a neighborhood, we know that $\mu^{-2}|q_0(\mu\cdot)|$ is of the order of μ^{-2}, while $\nabla^k(\mu^{-2}q_0(\mu\cdot)) = \mathcal{O}(\mu^{-2+k})$. The sum of the other terms in the right hand side of (7.21) is $\mathcal{O}(\widetilde{h})$ together with all its derivatives.

From (7.22), we see that $\nabla^2\widetilde{\phi}_\mu$ varies in a bounded set in C_b^∞, when $\mu \to 0$, and in view of (7.12), we know that

$$\widetilde{\phi}_\mu(\widetilde{x}) = \frac{1}{2}|\widetilde{x}|^2, \tag{7.26}$$

for $\mu\widetilde{x}$ in a neighborhood of $(0,0)$. Consider the restriction of $q_{0,2}$ to $\Lambda_{\widetilde{\phi}_\mu} \cap \operatorname{neigh}((0,0))$. Let $w \in \mathbf{C}$ with

$$\frac{1}{2} < |w| < 2,\ -\frac{\pi}{2} + \varepsilon_0 < \arg w < -\varepsilon_0, \tag{7.27}$$

for some small but fixed $\varepsilon_0 > 0$. Then if $p_0 = q_{0,2|_{\Lambda_{\widetilde{\phi}_\mu}}}$, we see from (7.14) that

$$p_0(\widetilde{x},\widetilde{\xi}) - w = 0 \Longrightarrow \begin{cases} d\mathrm{Re}\, p_0, d\mathrm{Im}\, p_0 \text{ are independent,} \\ \text{and } \{\mathrm{Re}\, p_0, \mathrm{Im}\, p_0\} = 0, \text{ at } (\widetilde{x},\widetilde{\xi}). \end{cases} \tag{7.28}$$

Here the bracket is the Poisson bracket on the IR-manifold $\Lambda_{\widetilde{\phi}_\mu}$ and the linear independence is uniform with respect to μ.

Let $p = \mu^{-2}q_0(\mu(\cdot))_{|\Lambda_{\widetilde{\phi}_\mu}}$. Then from (7.25), (7.28), we get

$$p(\widetilde{x},\widetilde{\xi}) - w = 0 \Longrightarrow \begin{cases} d\mathrm{Re}\, p, d\mathrm{Im}\, p \text{ are independent,} \\ \text{and } \{\mathrm{Re}\, p, \mathrm{Im}\, p\} = \mathcal{O}(\mu), \text{ at } (\widetilde{x},\widetilde{\xi}). \end{cases} \tag{7.29}$$

Again the independence is uniform with respect to μ.

This means that we can apply Theorem 6.3 to $\mu^{-2}Q^w(x,hD_x;h) - w$, when μ is small and we use $\widetilde{h}$ as the new semi-classical parameter. Indeed, all the assumptions are then fulfilled in a fixed neighborhood of $(0,0)$. Outside such a neighborhood, the symbol is only defined on $\Lambda_{\widetilde{\phi}_\mu}$, but elliptic and of a sufficiently good class to guarantee invertibility there. We also need to recall how Theorem 6.3 is connected to a Grushin problem. (To have a better notational agreement with Theorem 7.1, we replaced θ, k by $-\theta, -k$ in (6.58).)

Proposition 7.2. — *For w in the domain (7.27), $\mu^{-2}Q^w(x,hD_x;h) - w : H_{\widetilde{\phi}_\mu} \to H_{\widetilde{\phi}_\mu}$ is non-invertible precisely when*

$$w = K(2\pi\widetilde{h}(k - \theta_0), \mu; \widetilde{h}), \tag{7.30}$$

for some $k \in \mathbf{Z}^2$. Here $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$ is fixed.

$$K(\theta,\mu;\widetilde{h}) \sim K_0(\theta,\mu) + \widetilde{h}^2K_2(\theta,\mu) + \widetilde{h}^3K_3(\theta,\mu) + \cdots, \tag{7.31}$$

where $K_0(\cdot,\mu)$ is the inverse of the action map

$$w \longmapsto I_0(w,\mu) \tag{7.32}$$

which is a diffeomorphism from a neighborhood of the closure of the domain (7.27) onto a neighborhood of its image. K_j *depend smoothly on* (θ,μ).

If

$$\operatorname{dist}(w, K(2\pi\widetilde{h}(\mathbf{Z}^2-\theta_0),\mu;\widetilde{h})) \geqslant \frac{\widetilde{h}}{2C}, \tag{7.33}$$

then the inverse of $\mu^{-2}Q^w - w$ *is of norm* $\leqslant \widetilde{h}^{-1}e^{\mathcal{O}(\mu)/\widetilde{h}}$. *More precisely, we can define weights* $\widetilde{\psi}_\mu$, *with* $\widetilde{\psi}_\mu - \widetilde{\phi}_\mu$ *of uniformly compact support in* $\mathbf{C}^2\setminus\{0\}$, *and* $=\mathcal{O}(\mu)$ *in* C^∞, *depending smoothly on* μ *(and also on* w*), such that*

$$(\tfrac{1}{\mu^2}Q^w - w)^{-1} = \mathcal{O}(1/\widetilde{h}) : H_{\widetilde{\psi}_\mu} \to H_{\widetilde{\psi}_\mu},$$

when (7.33) holds.

If

$$|w - K(2\pi\widetilde{h}(k-\theta_0),\mu;\widetilde{h})| < \frac{\widetilde{h}}{C},\ \textit{for some } k\in\mathbf{Z}^2, \tag{7.34}$$

then there exist operators

$$R_+(w,\mu:\widetilde{h}) : H_{\widetilde{\psi}_\mu} \longrightarrow \mathbf{C},\ R_-(w,\mu;\widetilde{h}) : \mathbf{C} \longrightarrow H_{\widetilde{\psi}_\mu}, \tag{7.35}$$

depending smoothly on w,μ, *such that the corresponding norms of* $\nabla_w^j R_\pm$ *are* $\mathcal{O}(\widetilde{h}^{-j})$ *and such that*

$$\begin{pmatrix} \frac{1}{\widetilde{h}}(\frac{1}{\mu^2}Q^w - w) & R_- \\ R_+ & 0 \end{pmatrix} : H_{\widetilde{\psi}_\mu}\times\mathbf{C} \longrightarrow H_{\widetilde{\psi}_\mu\times\mathbf{C}} \tag{7.36}$$

has a uniformly bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}. \tag{7.37}$$

Here $E_{-+}(w,\mu;\widetilde{h})$ *has an asymptotic expansion as in (6.24) where* $\theta\mapsto E^0_{-+}(\theta,w,\mu)$ *has a simple zero at 0.*

We notice that the eigen-values w are even functions of μ (if we make the change of variables also for negative μ) and to infinite order in $\widetilde{h}$, they are smooth in μ. Hence

$$K(2\pi h(k-\theta_0),\mu;h) = K(2\pi h(k-\theta_0),-\mu;h) + \mathcal{O}(h^\infty),$$

from which we deduce that $K_j(\theta,\mu) = K_j(\theta,-\mu)$, $j=0,1,2,\dots$.

Introduce the Taylor expansion in μ:

$$K_j(\theta,\mu) \sim \sum_{\ell=0}^{\infty} K_{j,\ell}(\theta)\mu^{2\ell}. \tag{7.38}$$

In (7.20) we put $\widetilde{x} = \lambda\widetilde{y}$, and obtain the isospectral operator

$$\lambda^2\frac{1}{(\mu\lambda)^2}Q^w(\mu\lambda(\widetilde{y},\tfrac{\widetilde{h}}{\lambda^2}D_{\widetilde{y}});h). \tag{7.39}$$

The eigen-values are given by

$$\lambda^2 K(2\pi \tfrac{\widetilde{h}}{\lambda^2}(k-\theta_0), \lambda\mu; \tfrac{\widetilde{h}}{\lambda^2}) + \mathcal{O}(h^\infty),$$

so we get

$$K(2\pi\widetilde{h}(k-\theta_0), \mu; \widetilde{h}) = \lambda^2 K(2\pi \tfrac{\widetilde{h}}{\lambda^2}(k-\theta_0), \lambda\mu; \tfrac{\widetilde{h}}{\lambda^2}) + \mathcal{O}(\widetilde{h}^\infty), \tag{7.40}$$

for $\lambda \sim 1$, $|\mu| \leqslant 1$, $k \in \mathbf{Z}^2$, and $w = K(2\pi\widetilde{h}(k-\theta_0), \mu; \widetilde{h})$ in the region (7.27). Combining this with (7.31), we get successively for $j = 0, 1, 2, \dots$:

$$\widetilde{h}^j K_j(\theta, \mu) = \lambda^2 K_j(\theta/\lambda^2, \lambda\mu)\big(\widetilde{h}/\lambda^2\big)^j,$$

for θ in a domain with $K_0(\theta, \mu)$ in the domain (7.27). Dividing by $\widetilde{h}^j$, we get

$$K_j(\theta, \mu) = (\lambda^2)^{1-j} K_j(\theta/\lambda^2, \lambda\mu). \tag{7.41}$$

This relation can be used to extend the definition to a domain

$$0 \leqslant |\theta|\mu^2 \leqslant \frac{1}{C}, \ |\theta| \neq 0, \tag{7.42}$$

with $K_0(\theta, 0)$ in the domain (7.27). Indeed, if (θ, μ) satisfies (7.42), then we can take $\lambda \sim |\theta|^{1/2}$ and notice that $|\lambda\mu| \leqslant 1$. Also notice that

$$K_j(\theta, 1) = \mu^{2(1-j)} K_j(\theta/\mu^2, \mu). \tag{7.43}$$

Combining (7.38), (7.41), we get

$$K_{j,\ell}(\theta) = \lambda^{2(1-j+\ell)} K_{j,\ell}(\theta/\lambda^2), \tag{7.44}$$

so $K_{j,\ell}$ is positively homogeneous of degree $1-j+\ell$.

The scaling argument above allows us to describe all eigen-values z of $Q^w(x, hD_x; h)$ in a domain

$$h^\delta < |z| < \frac{1}{C_1}, \ -\frac{\pi}{2} + \varepsilon_0 < \arg z < -\varepsilon_0, \tag{7.45}$$

for $0 < \delta < 1/2$, by

$$z = \mu^2 K(2\pi \tfrac{h}{\mu^2}(k-\theta_0), \mu; \tfrac{h}{\mu^2}) + \mathcal{O}(h^\infty), \tag{7.46}$$

where we choose $\mu > 0$ with $|z|/\mu^2 \sim 1$.

We now return to the operator P in (7.1). Let z be as in (7.45) and consider the most interesting case when

$$\Big|\frac{z}{\mu^2} - K(2\pi\widetilde{h}(k-\theta_0), \mu; \widetilde{h})\Big| \leqslant \frac{\widetilde{h}}{C}, \text{ for some } k \in \mathbf{Z}^2, \tag{7.47}$$

where μ is given as after (7.46) and $\widetilde{h} = h/\mu^2$. We shall need the Grushin problem evocated in Proposition 7.2, but now for simplicity for the unscaled operator

$$\frac{1}{h}(Q^w - z) = \frac{1}{\widetilde{h}}\Big(\frac{1}{\mu^2}Q^w - w\Big), \ w = \frac{z}{\mu^2} :$$

$$\begin{pmatrix} \frac{1}{h}(Q^w - z) & R_- \\ R_+ & 0 \end{pmatrix} : H_{\widetilde{\psi}} \times \mathbf{C} \longrightarrow H_{\widetilde{\psi}} \times \mathbf{C}. \tag{7.48}$$

This is the same as in (7.36) except that we work in the original unscaled variables $x = \mu\widetilde{x}$, so $\widetilde{\psi}(x) = \widetilde{\psi}_\mu(\widetilde{x})$. Then $\widetilde{\psi} = \widetilde{\phi} + \mathcal{O}(\mu^3)$ with $\widetilde{\psi} = \widetilde{\phi}$ outside a μ-neighborhood of 0. Recall that $\widetilde{\phi} = \psi$ outside a fixed neighborhood of 0. Also recall that $\widetilde{\psi}$ is a small perturbation of ψ and that $\widetilde{\psi} = \psi$ outside a small neighborhood of 0.

From the fact that (7.48) is globally bijective with a bounded inverse, we deduce that if $u \in H_{\widetilde{\psi}}(\widetilde{\Omega})$, $u_-, v_+ \in \mathbf{C}$ and

$$\frac{1}{h}(Q^w - z)u + R_- u_- = v, \text{ in } \widetilde{\Omega},\ R_+ u = v_+, \tag{7.49}$$

then

$$\|u\|_{H_{\widetilde{\psi}}(\widetilde{\Omega}_2)} + |u_-| \leqslant \mathcal{O}(1)(\|v\|_{H_{\widetilde{\psi}}(\widetilde{\Omega}_3)} + |v_+|) + \mathcal{O}(e^{-1/Ch})(\|u\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + |u_-|). \tag{7.50}$$

Here we let

$$\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$$

be neighborhoods of 0 and $\widetilde{\Omega}_j$ be the corresponding neighborhoods of 0 in $\mathbf{C}^2$ such that $\widetilde{\Omega}_j = \pi_x \kappa_V(\pi_x^{-1}\Omega_j \cap \Lambda_\phi)$, where κ_V is the canonical transformation associated to V. We may assume that $\widetilde{\psi}, \widetilde{\phi}, \psi$ coincide outside $\widetilde{\Omega}_1$. In (7.50) it is understood that we realize Q^w on $H_{\widetilde{\psi}(\widetilde{\Omega})}$ (see [**Sj1**]) and the last term in (7.50) takes into account the corresponding boundary effects.

We let $\widetilde{H}(1)$ be the space $H(\Lambda_G, 1)$ equipped with the norm

$$\|u\|_{\widetilde{H}(1)} = \|VTu\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + \|Tu\|_{L^2_\phi(\mathbf{C}^2 \smallsetminus \Omega_1)}. \tag{7.51}$$

We will see that this is a norm and that we get a uniformly equivalent norm if we replace Ω_1 by Ω_0 or Ω_2. We define $\widetilde{H}(\langle\xi\rangle^2)$ analogously.

We shall study the global Grushin problem

$$\begin{cases} \frac{1}{h}(P - z)u + SUR_- u_- = v \\ R_+ VTu = v_+, \end{cases} \tag{7.52}$$

for $u_-, v_+ \in \mathbf{C}$, $u \in \widetilde{H}(\langle\xi\rangle^2)$, $v \in \widetilde{H}(1)$.

Apply VT to the first equation,

$$\begin{cases} \frac{1}{h}(Q^w - z)VTu + R_- u_- = VTv + w, \\ R_+ VTu = v_+, \end{cases} \tag{7.53}$$

where

$$w = \frac{1}{h}(Q^w VT - VTP)u + (1 - VU)R_- u_- + V(1 - TS)UR_- u_-. \tag{7.54}$$

Here we notice that we may assume that

$$|\widetilde{\psi} - \psi| \leqslant \varepsilon_0, \tag{7.55}$$

for $\varepsilon_0 > 0$ fixed and arbitrarily small, provided that we restrict the spectral parameter to a sufficiently small h-independent disc. Combining (7.54) and the earlier estimates on $Q^w VT - VTP$, $1 - VU$, $1 - TS$, U, V, we see that

$$\|w\|_{H_\psi(\widetilde{\Omega}_3)} \leqslant \mathcal{O}(1)e^{-1/Ch}(\|Tu\|_{H_\phi(\Omega)} + |u_-|), \tag{7.56}$$

where C does not depend on ε_0 in (7.55). Applying this and (7.50) (with u, v replaced by VTu, VTv) to (7.53), we get the "interior" estimate

$$\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega}_2)} + |u_-| \leqslant \mathcal{O}(1)(\|VTv\|_{H_{\tilde{\psi}}(\widetilde{\Omega}_3)} + |v_+| + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega)}). \tag{7.57}$$

On the other hand, if we restrict the the spectral parameter to a sufficiently small (h-independent) disc, we get from [**HeSj**]:

$$\begin{aligned} &\|m^2Tu\|_{L^2_\phi(\mathbf{C}^2\smallsetminus\Omega_1)} \\ &\qquad \leqslant \mathcal{O}(1)(\|Tv\|_{H_\phi(\mathbf{C}^2\smallsetminus\Omega_0)} + e^{-1/Ch}|u_-| + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega_0)}). \end{aligned} \tag{7.58}$$

Indeed, we can apply the [**HeSj**] theory to the space $H(\langle\xi\rangle^2, \Lambda_{G_+})$, defined to be $H(\Lambda_G, \langle\xi\rangle^2)$ as a space, and with the norm $\|m^2Tu\|_{L^2_{\phi_+}}$, where $\phi_+ - \phi \geqslant 0$ is small in C^∞, strictly positive on $\overline{\Omega_0}$ and equal to 0 in a neighborhood of $\mathbf{C}^2 \smallsetminus \Omega_1$. We then see that $P - z$ is elliptic in this space away from a small neighborhood of $(0, 0)$, and (7.58) follows.

If we use

$$Tu = UVTu + (1 - UV)Tu,$$

we get

$$\|Tu\|_{H_\phi(\Omega_1)} \leqslant \mathcal{O}(1)(e^{\varepsilon_0/h}\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega)}). \tag{7.59}$$

Here the last term can be replaced by $e^{-1/Ch}\|Tu\|_{H_\phi(\Omega\smallsetminus\Omega_1)}$ when h is small. Moreover, it is clear that

$$\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega}\smallsetminus\widetilde{\Omega}_2)} = \|VTu\|_{H_\psi(\widetilde{\Omega}\smallsetminus\widetilde{\Omega}_2)} \leqslant \mathcal{O}(1)\|Tu\|_{H_\phi(\Omega\smallsetminus\Omega_1)}. \tag{7.60}$$

It follows that

$$\|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega}_2)} + \|Tu\|_{H_\phi(\Omega\smallsetminus\Omega_1)} \sim \|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + \|Tu\|_{H_\phi(\Omega\smallsetminus\Omega_1)}, \tag{7.61}$$

and similarly with $\|Tu\|_{H_\phi(\Omega\smallsetminus\Omega_1)}$ replaced by $\|m^2Tu\|_{H_\phi(\Omega\smallsetminus\Omega_1)}$. Now add (7.57), (7.58) and use (7.61):

$$\begin{aligned} &\|m^2Tu\|_{L^2_\phi(\mathbf{C}^2\smallsetminus\Omega_1)} + \|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + |u_-| \\ &\qquad \leqslant \mathcal{O}(1)(\|VTv\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + \|Tv\|_{L^2_\phi(\mathbf{C}^2\smallsetminus\Omega_2)} + |v_+| + e^{-1/Ch}\|Tu\|_{L^2_\phi(\mathbf{C}^2)}). \end{aligned} \tag{7.62}$$

Here we can absorb the contribution from $\|Tu\|_{L^2_\phi(\mathbf{C}^2\smallsetminus\Omega_1)}$ to the last term, and get

$$\begin{aligned} &\|m^2Tu\|_{L^2_\phi(\mathbf{C}^2\smallsetminus\Omega_1)} + \|VTu\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + |u_-| \\ &\qquad \leqslant \mathcal{O}(1)(\|VTv\|_{H_{\tilde{\psi}}(\widetilde{\Omega})} + \|Tv\|_{L^2_\phi(\mathbf{C}^2\smallsetminus\Omega_2)} + |v_+| + e^{-1/Ch}\|Tu\|_{H_\phi(\Omega_1)}). \end{aligned} \tag{7.63}$$

Now use (7.59) to estimate the last term. We can assume that $\varepsilon_0 < 1/2C$ and get with a new constant C:

$$\begin{aligned}(7.64)\qquad &\|m^2Tu\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_1)} + \|VTu\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + |u_-| \\ &\quad\leqslant \mathcal{O}(1)(\|VTv\|_{H_{\widetilde{\psi}}(\widetilde{\Omega})} + \|Tv\|_{L^2_\phi(\mathbf{C}^2\setminus\Omega_2)} + |v_+|).\end{aligned}$$

We have then proved:

Proposition 7.3. — *Let z be in the region (7.47) and (7.45) with $|z| < r$ and $r > 0$ small enough. Then the problem (7.52) has a unique solution $(u, u_-) \in \widetilde{H}(\langle\xi\rangle^2) \times \mathbf{C}$ for every $(v, v_+) \in \widetilde{H} \times \mathbf{C}$, satisfying*

$$(7.65)\qquad \|u\|_{\widetilde{H}(\langle\xi\rangle^2)} + |u_-| \leqslant \mathcal{O}(1)(\|v\|_{\widetilde{H}(1)} + |v_+|).$$

Indeed, it is clear that (7.52) is Fredholm of index 0 and (7.64) implies injectivity. (7.65) is just an equivalent form of (7.64).

Proposition 7.4. — *Under the assumptions of Proposition 7.3, let*

$$\mathcal{F} = \begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix}$$

be the inverse of

$$\begin{pmatrix} \frac{1}{h}(P-z) & SUR_- \\ R_+VT & 0 \end{pmatrix}.$$

Then

$$(7.66)\qquad \nabla_z^k(F_{-+} - E_{-+}) = \mathcal{O}(h^\infty) \text{ for every } k \in \mathbf{N}.$$

Proof. — It is easy to see that $\nabla_z^k F_{-+}$, $\nabla_z^k E_{-+}$ are $\mathcal{O}(h^{-N(k)})$, for every $k \in \mathbf{N}$ with some $N(k) \geqslant 0$, so it suffices to verify (7.66) for $k = 0$. Let $\widetilde{u} = E_+v_+$, $u_- = E_{-+}v_+$, $|v_+| = 1$, so that

$$(7.67)\qquad \frac{1}{h}(Q-z)\widetilde{u} + R_-u_- = 0,\ R_+\widetilde{u} = v_+.$$

Put $u = SU\widetilde{u}$. Then

$$\frac{1}{h}(P-z)u + SUR_-u_- = \frac{1}{h}(PSU - SUQ)\widetilde{u}.$$

Here in analogy with (7.8), we have

$$(7.68)\qquad PSU - SUQ = \mathcal{O}(1) : H_{\psi_+}(\widetilde{\Omega}) \longrightarrow H(\Lambda_{G_-}) \longrightarrow \widetilde{H}(1),$$

and $\widetilde{u}$ is exponentially small in $H_{\psi_+}(\widetilde{\Omega})$, so

$$(7.69)\qquad \frac{1}{h}(P-z)u + SUR_-u_- = v,\ \|v\|_{\widetilde{H}(1)} = \mathcal{O}(e^{-1/Ch}).$$

Similarly,

$$(7.70)\qquad R_+VTu = v_+ + R_+(VTSU - 1)\widetilde{u} =: v_+ + w_+$$

and $(VTSU-1)\widetilde{u}$ is exponentially small in $H_{\widetilde{\psi}}(\widetilde{\Omega})$, so $|w_+| = \mathcal{O}(e^{-1/Ch})$. It follows form this and Proposition 7.3 that

$$u_- = F_{-+}v_+ + F_{-+}w_+ + F_-v = \mathcal{O}(e^{-1/Ch}),$$

and the proposition follows since $u_- = E_{-+}v_+$. □

It is now clear that (7.46) describes all eigen-values of P in the domain (7.45).

If we further restrict the attention to

$$h^{\delta_2} < |z| < h^{\delta_1},\ -\frac{\pi}{2} + \varepsilon_0 < \arg z < -\varepsilon_0, \tag{7.71}$$

with $0 < \delta_1 < \delta_2 < 1/2$, then μ in (7.46) is $\mathcal{O}(h^{\delta_1/2})$ and we can apply the Taylor expansion (7.38). Then (7.46) becomes

$$z \sim \mu^2 \sum_1^\infty \sum_{\ell=0}^\infty \Big(\frac{h}{\mu^2}\Big)^j K_{j,\ell}(2\pi \tfrac{h}{\mu^2}(k-\theta_0))\mu^{2\ell}. \tag{7.72}$$

Now use that $K_{j,\ell}$ is homogeneous of degree $1-j+\ell$ to get the eigen-values in (7.71) on the form

$$z \sim \sum_{j=0}^\infty \sum_{\ell=0}^\infty K_{j,\ell}(2\pi h(k-\theta_0))h^j,\ k \in \mathbf{Z}^2. \tag{7.73}$$

From Theorem 7.1 (of [**KaKe**]) we know on the other hand that the eigen-values in (7.71) are given by

$$z \sim \sum_{j=0}^\infty h^j f_j(2\pi h(\widetilde{k}-\theta_0)),\ \widetilde{k} \in \mathbf{Z}^2, \tag{7.74}$$

where $f_j \in C^\infty(\mathrm{neigh}\,(0,\mathbf{R}^2))$ (with the same neighborhood for every j. Here $\widetilde{k}$ is not necessarily equal to k for the same eigen-value but if we start with some fixed small h and then let $h \to 0$, we see tht $\widetilde{k} = k + k_0$, where k_0 is constant. Approximating $f_j(\theta)$ for $\theta = 2\pi h(\widetilde{k}-\theta_0)$ by the Taylor expansion at $\theta = 2\pi h(k-\theta_0)$, we get a representation (7.74) with new f_js for $j \geqslant 1$, where we may assume that $\widetilde{k} = k$.

If we introduce the Taylor expansion of each f_j at 0, we see that (7.74) takes the form

$$z \sim \sum_{j=0}^\infty \sum_{\ell=0}^\infty h^j \widetilde{K}_{j,\ell}(2\pi h(k-\theta_0)), \tag{7.75}$$

where $\widetilde{K}_{j,\ell}$ is a homogeneous polynomial of degree $1-j+\ell$ (which vanishes for $1-j+\ell<0$).

Let $F_{j,\ell} = K_{j,\ell} - \widetilde{K}_{j,\ell}$, so that $F_{j,\ell}(\theta)$ is smooth and positively homogeneous of degree $1-j+\ell$ in the angle V, defined by $-\frac{\pi}{2} + \varepsilon_0 < \arg F_{0,0}(\theta) < -\varepsilon_0$. We then know that

$$\sum_{j=0}^\infty \sum_{\ell=0}^\infty h^j F_{j,\ell}(2\pi h(k-\theta_0)) = \mathcal{O}(h^\infty), \tag{7.76}$$

for $2\pi h(k-\theta_0) \in V$ with $h^{\delta_2} < |2\pi h(k-\theta_0)| < h^{\delta_1}$. We restrict the attention to the domain $|2\pi h(k-\theta_0)| \sim h^{\delta}$, where we are free to choose δ in $]0, \frac{1}{2}[$, and let $h \to 0$ for each fixed δ. We shall show that $F_{j,\ell} = 0$ by induction in alphabetical order in (j,ℓ). Assume that we already know that $F_{j,\ell} = 0$ for $j < j_0$ and for $j = j_0$, $\ell < \ell_0$. Here $(j_0, \ell_0) \in \mathbf{N}^2$. Then (7.76) gives

$$F_{j_0,\ell_0}(2\pi h(k-\theta_0)) = \mathcal{O}(1)\max(h^{\delta(2-j_0+\ell_0)}, h^{1-\delta j_0}), \tag{7.77}$$

for $k \in \mathbf{Z}^2$ with $\theta := 2\pi h(k-\theta_0)$ in V and $|\theta| \sim h^{\delta}$. In this region $\nabla F_{j_0,\ell_0} = \mathcal{O}(1)h^{\delta(-j_0+\ell_0)}$ and (7.77) implies that

$$F_{j_0,\ell_0}(\theta) = \mathcal{O}(1)\max(h^{1+\delta(\ell_0-j_0)}, h^{2\delta+\delta(\ell_0-j_0)}, h^{1-\delta j_0}) = \mathcal{O}(1)h^{\delta(2+\ell_0-j_0)}, \tag{7.78}$$

if $\delta > 0$ is small enough depending on (ℓ_0, j_0), and for $|\theta| \sim h^{\delta}$, $\theta \in V$. Since F_{j_0,ℓ_0} is homogeneous of degree $1+\ell_0-j_0$, we see that $F_{j_0,\ell_0} = 0$ in V. Consequently, we have

Proposition 7.5. — *$K_{j,\ell}(\theta)$ is a homogeneous polynomial of degree $1+\ell-j$ (equal to 0 for $1+\ell-j<0$).*

Using this, we get

Proposition 7.6. — *$K_j(\theta,1)$ extends to a smooth function in a j-independent neighborhood of 0.*

Proof. — We study the asymptotics when $V \ni \theta \to 0$, using (7.38), (7.41) and get with $\mu^2 \sim |\theta|$:

$$\begin{aligned} K_j(\theta,1) = \mu^{2(1-j)}K_j(\theta/\mu^2,\mu) &\sim \sum_{\ell \geqslant \max(0,j-1)} \mu^{2(1-j)}K_{j,\ell}(\theta/\mu^2)\mu^{2\ell} \\ &\sim \sum_{\ell \geqslant \max(0,j-1)} K_{j,\ell}(\theta), \ \theta \longrightarrow 0. \end{aligned}$$

This expansion is also valid after differentiation and since $K_{j,\ell}$ are polynomials, we see that (7.38) is the Taylor expansion of a smooth function in a neighborhood of 0. ☐

We now return to the description (7.46) of the resonances of P in (7.45) and use (7.41):

$$z \sim \sum_{j=0}^{\infty} \mu^2 K_j\Big(\frac{2\pi h(k-\theta_0)}{\mu^2}, \mu\Big)\frac{h^j}{\mu^{2j}} = \sum_{j=0}^{\infty} K_j(2\pi h(k-\theta_0),1)h^j. \tag{7.79}$$

With $f_j(\theta) = K_j(\theta,1)$, we get from this, Theorem 7.1 and the identification of the different ks in (7.73), (7.74):

Theorem 7.7. — *The description of the resonances in Theorem 7.1 extends to the set of z in (7.45), provided that C_1 there is sufficiently large as a function of $\varepsilon_0 > 0$ and that $h > 0$ is small enough.*

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LOGARITHMIC SOBOLEV INEQUALITY AND SEMI-LINEAR DIRICHLET PROBLEMS FOR INFINITELY DEGENERATE ELLIPTIC OPERATORS

by

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Abstract. — Let $X = (X_1, \ldots, X_m)$ be an infinitely degenerate system of vector fields, we prove firstly the logarithmic Sobolev inequality for this system on the associated Sobolev function spaces. Then we study the Dirichlet problem for the semilinear problem of the sum of square of vector fields X.

***Résumé* (Inégalité de Sobolev logarithmique et problèmes de Dirichlet semi-linéaires pour des opérateurs elliptiques infiniment dégénérés)**
Soit $X = (X_1, \ldots, X_m)$ un système de champs de vecteurs infiniment dégénérés. On montre d'abord l'inégalité de Sobolev logarithmique pour ce système de champs de vecteurs sur les espaces de fonctions associés, puis on étudie le problème de Dirichlet semi-linéaire pour des opérateurs somme de carrés de champs de vecteurs X.

1. Introduction

In this work, we consider a system of vector fields $X = (X_1, \ldots, X_m)$ defined on an open domain $\widetilde{\Omega} \subset \mathbb{R}^d$. We suppose that this system satisfies the following logarithmic regularity estimate,

$$\|(\log \Lambda)^s u\|^2_{L^2} \leqslant C \left\{ \sum_{j=1}^{m} \|X_j u\|^2_{L^2} + \|u\|^2_{L^2} \right\}, \quad \forall\, u \in C_0^\infty(\widetilde{\Omega}), \tag{1.1}$$

where $\Lambda = (e + |D|^2)^{1/2} = \langle D \rangle$. We shall give some sufficient conditions for this estimates in the Appendix, see also [**5, 10, 12, 14, 15, 21**]. The typical example is the system in $\mathbb{R}^2$ such as $X_1 = \partial_{x_1}, X_2 = e^{-|x_1|^{-1/s}} \partial_{x_2}$ with $s > 0$. Remark that if $s > 1$, the estimate (1.1) implies the hypoellipticity of the infinitely degenerate elliptic operators of second order $\triangle_X = \sum_{j=1}^m X_j^* X_j$, where X_j^* is the formal adjoint of X_j.

If Γ is a smooth surface of $\widetilde{\Omega}$, we say that Γ is non characteristic for the system of vector fields X, if for any point $x_0 \in \Gamma$, there exists at least one vector field of

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$X_1, \ldots, X_m$ which is transversal to Γ at x_0. Let now $\Gamma = \cup_{j\in J}\Gamma_j$ be the union of a family of smooth surface in $\widetilde{\Omega}$. We say that Γ is non characteristic for X, if for any point $x_0 \in \Gamma$, there exists at least one vector field of $X_1, \ldots, X_m$ which traverses Γ_j at x_0 for all $j \in J_0 = \{k \in J; x_0 \in \Gamma_k\}$. For this second case, the typical example is $X_1 = \partial_{x_1}, X_2 = \exp(-(x_1^2 \sin^2(\pi/x_1))^{-1/2s})\partial_{x_2}$, we have $\Gamma_j = \{x_1 = 1/j\}$, $j \in \mathbb{Z} \setminus \{0\}, \Gamma_0 = \{x_1 = 0\}$, and X_1 is transverse to all $\Gamma_j, j \in \mathbb{Z}$.

Associated with the system of vector fields $X = (X_1, \ldots, X_m)$, we define the following function spaces:

$$H^1_X(\widetilde{\Omega}) = \Big\{u \in L^2(\widetilde{\Omega}); X_j u \in L^2(\widetilde{\Omega}), j = 1, \ldots, m\Big\}.$$

Take now $\Omega \subset\subset \widetilde{\Omega}$, we suppose that $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X. We define $H^1_{X,0}(\Omega) = \big\{u \in H^1_X(\Omega); u|_{\partial\Omega} = 0\big\}$, we shall prove in the second section (see Lemma 2.1) that this is a Hilbert space.

Our first result is the following logarithmic Sobolev inequality.

Theorem 1.1. — *Suppose that the system of vector fields $X = (X_1, \ldots, X_m)$ verifies the estimate (1.1) for some $s > 1/2$. Then there exists $C_0 > 0$ such that*

$$\int_\Omega |v|^2 \log^{2s-1}\Big(\frac{|v|}{\|v\|_{L^2}}\Big) \leqslant C_0\Big\{\sum_{j=1}^m \|X_j v\|^2_{L^2} + \|v\|^2_{L^2}\Big\}, \tag{1.2}$$

for all $v \in H^1_{X,0}(\Omega)$.

Comparing this inequality with that of finite degenerate case of Hörmander's system, for example, for the system $X_1 = \partial_{x_1}, X_2 = x_1^k\partial_{x_2}$ on $\mathbb{R}^2$, we have (see [**4, 7, 24**])

$$\|v\|_{L^p} \leqslant C\left(\|\partial_1 v\|^2_{L^2} + \|x_1^k\partial_2 v\|^2_{L^2} + \|v\|^2_{L^2}\right)^{1/2}$$

for all $v \in C_0^\infty(\Omega)$, with $p = 2 + 4/k$. Consequently, if k go to infinity, we can only expect to gain the logarithmic estimates as (1.2). That means that we are not in the elliptic case of [**17**].

Similarly to the elliptic and subelliptic case (see [**3, 24**]), by using the Sobolev's inequality, we study the following semi-linear Dirichlet problems

$$\begin{aligned} &\triangle_X u = au\log|u| + bu, \\ &u|_{\partial\Omega} = 0, \end{aligned} \tag{1.3}$$

where $a, b \in \mathbb{R}$. We have the following theorem.

Theorem 1.2. — *We suppose that the system of vector fields $X = (X_1, \ldots, X_m)$ satisfies the following hypotheses:*

H-1) $\partial\Omega$ is C^∞ and non characteristic for the system of vector fields X ;

H-2) the system of vector fields X satisfies the finite type of Hörmander's condition on $\widetilde{\Omega}$ except an union of smooth surfaces Γ which are non characteristic for X.

H-3) the system of vector fields X verifies the estimate (1.1) for $s > 3/2$.

Suppose $a \neq 0$ in (1.3). Then the semi-linear Dirichlet problem (1.3) posses at least one non trivial weak solution $u \in H^1_{X,0}(\Omega) \cap L^\infty(\Omega)$. Moreover, if $a > 0$, we have $u \in C^\infty(\Omega \smallsetminus \Gamma) \cap C^0(\overline{\Omega} \smallsetminus \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \smallsetminus \Gamma$.

As in the elliptic case, we do not know the uniqueness of solutions (see [**3**]). The regularity of this weak solution near to the infinitely degenerate point of Γ is a more complicated problem, which will be studied in our future works.

The structure of the paper is as follows: The second section consists of the proof of Theorem 1.1. The third section is devoted to the proof for the existence of weak solution of Theorem 1.2, we introduce a variational problem and prove that the associated Euler-Lagrange equation is (1.3). In the fourth section we study the boundedness of weak solution of variational problems, which is a difficult step as in the classical case for the critical semilinear elliptic equations (see [**20**]). In the appendix we give some sufficient conditions for the logarithmic regularity estimates.

2. Logarithmic Sobolev inequality

We study now the function spaces $H^1_{X,0}(\Omega)$, see the similar results in [**22**].

Lemma 2.1. — *Suppose that $\partial\Omega$ is C^∞ and non characteristic for the system X, then $H^1_{X,0}(\Omega)$ is well-defined, and a Hilbert space. Moreover the extension of an element of $H^1_{X,0}(\Omega)$ by 0 belongs to $H^1_X(\widetilde{\Omega})$.*

Proof. — For the well-definedness, we need to prove the existence of trace for $v \in H^1_X(\Omega)$. We know that the trace problem is a local problem, so after the localization and straightened, we transfer the problem to the case: $v \in L^2(\mathbb{R}^d_+), \partial_{x_d} v \in L^2(\mathbb{R}^d_+)$ with support of v is a subset of $\{|(x', x_d)| < c, x_d \geqslant 0\}$, of course we can take the smooth function approximate to v, then we have

$$v(x', x_d) - v(x', c) = \int_c^{x_d} \partial_{x_d} v(x', t)dt,$$

which prove that

$$\|v(\cdot, x_d)\|^2_{L^2} \leqslant c\|\partial_{x_d} v\|^2_{L^2}, \tag{2.1}$$

for all $0 \leqslant x_d \leqslant c$. This shows that the trace $v(x', 0) \in L^2(\mathbb{R}^{d-1})$.

We shall prove now $H^1_{X,0}(\Omega)$ is a closed subspace of $H^1_X(\Omega)$. Let $\{v_j\}$ be a Cauchy sequence of $H^1_{X,0}(\Omega)$. Since it is also a Cauchy sequence of $H^1_X(\Omega)$, there exists a limit $v_0 \in H^1_X(\Omega)$, and so it suffices to show that $v|_{\partial\Omega} = 0$. Applying (2.1) to $v_j - v_0$, we have

$$\|v_j(\cdot, 0) - v_0(\cdot, 0)\|^2_{L^2} \leqslant c\|\partial_{x_d}(v_j - v_0)\|^2_{L^2},$$

which implies $\|v_0(\cdot, 0)\|_{L^2} = 0$. We have proved that $H^1_{X,0}(\Omega)$ is a Hilbert space. The extension problem is the same as classic case. This is also a local problem, if we extend v by 0 to $x_d < 0$ and denote that function by $\overline{v}$, then $v, \partial_{x_d} v \in L^2(\mathbb{R}^d_+), v|_{x_d=0} = 0$

implies that $\overline{v}, \partial_{x_d}\overline{v} \in L^2(\mathbb{R}^d)$, and the tangential derivation has nothing to change. So we have proved the Lemma.

Since $L\log L$ is not a normed space, we need the following Lemma, see also [**19**] for some detail of function space $L\log L$.

Lemma 2.2. — *Let $\sigma_2 > 0$, $B > 0$ and let $\{v_j, j \in \mathbb{N}\}$ be a sequence in L^2 satisfying*

$$\int |v_j|^2 |\log|v_j||^{\sigma_2} \leqslant B.$$

Then $\{|v_j|^2|\log|v_j||^{\sigma_1}\}$ is uniformly integrable for any $0 \leqslant \sigma_1 < \sigma_1$. Therefore there exists a convergent sub-sequence $\{v_{j_k}\}$ such that

$$\lim_{k\to\infty} \int |v_{j_k}|^2 |\log|v_{j_k}||^{\sigma_1} = \int |v_0|^2 |\log|v_0||^{\sigma_1},$$

and

$$\int |v_0|^2 |\log|v_0||^{\sigma_2} \leqslant B.$$

Proof. — We prove that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $E \subset \Omega$, $\mu(E) < \delta$, then

$$\int_E |v_j|^2 |\log|v_j||^{\sigma_1} < \varepsilon, \quad \forall\, j.$$

But for any $\varepsilon > 0$, there exists $t_0 > e^2$ such that

$$\frac{1}{\log^{\sigma_2-\sigma_1} t} < \varepsilon, \qquad \forall\, t \geqslant t_0.$$

Take now $\delta = \varepsilon(t_0^2 \log^{\sigma_1} t_0)^{-1}$, $\mu(E) < \delta$, and

$$A_j = E \cap \{|v_j| \leqslant t_0\},\ B_j = E \cap \{|v_j| > t_0\}.$$

then

$$\int_{A_j} |v_j|^2 |\log|v_j||^{\sigma_1} \leqslant t_0^2 \log^{\sigma_1} t_0 \mu(A_j) < \varepsilon,$$

and

$$\int_{B_j} |v_j|^2 |\log|v_j||^{\sigma_1} \leqslant \varepsilon \int_{B_j} |v_j|^2 |\log|v_j||^{\sigma_2} \leqslant \varepsilon M$$

where $M = \sup_j \int_\Omega |v_j|^2 |\log|v_j||^{\sigma_2}$. The proof of the Lemma is complete.

Proof of Theorem 1.1. — We are following the idea of [**4**]. Take $v \in H^1_{X,0}(\Omega)$, we use the same notation for the extension by 0, As in the classical case, there exists a mollifier family $\{\rho_\varepsilon, \varepsilon > 0\}$ such that $\rho_\varepsilon * v \in C_0^\infty$, $\lim_{\varepsilon\to 0} \rho_\varepsilon * v = v$ in L^2 and $\|X(\rho_\varepsilon * v)\|_{L^2} \leqslant C\{\|Xv\|_{L^2} + \|v\|_{L^2}\}$, $\|(\log\Lambda)^s(\rho_\varepsilon * v)\|_{L^2} \leqslant C\{\|(\log\Lambda)^s v\|_{L^2} + \|v\|_{L^2}\}$ with C independent on ε. By using (1.1) and Lemma 2.2, we need only to prove the following estimate:

$$\int_\Omega |v|^2 \log^{2s-1}\left(\frac{|v|}{\|v\|_{L^2}}\right) \leqslant C_0 \|(\log\Lambda)^s v\|^2_{L^2}, \tag{2.2}$$

for all for $v \in C_0^\infty(\Omega)$.

By the homogenization, we prove (2.2) for $v \in C_0^\infty(\Omega)$ and $\|v\|_{L^2} = 1$. Since $2s-1>0$, we have

$$\begin{aligned}\int_\Omega |v|^2|\log|v||^{2s-1} &\leqslant C|\Omega| + \int_{|v|\geqslant e} |v|^2\log^{2s-1}\langle|v|\rangle\\ &\leqslant C_0 + \int_\Omega |v|^2\log^{2s-1}\langle|v|\rangle.\end{aligned}$$

Since Ω is bounded, $v \in L^\infty(\Omega)$ and $2s-1>0$, we have by the definition of Lebesgue integration

$$\begin{aligned}\int_\Omega |v|^2\log^{2s-1}\langle|v|\rangle &= -\int_0^\infty \lambda^2\log^{2s-1}\langle\lambda\rangle d\mu\{|v|>\lambda\}\\ &= \int_0^\infty \Big(2\lambda\log^{2s-1}\langle\lambda\rangle + (2s-1)\frac{\lambda^3}{\langle\lambda\rangle^2}\log^{2s-2}\langle\lambda\rangle\Big)\mu(|v|>\lambda)d\lambda,\end{aligned}$$

where $\mu(\cdot)$ is the Lebesgue measure. Since $\lambda^3/\langle\lambda\rangle^2 \leqslant \lambda$, $\log\langle\lambda\rangle \geqslant 1$, we have that

$$\int_\Omega |v|^2|\log|v||^{2s-1} \leqslant C_0 + C_s\int_0^\infty \lambda\log^{2s-1}\langle\lambda\rangle\mu(|v|>\lambda)d\lambda. \tag{2.3}$$

So we need to estimate the second term of right hand side of (2.3). For $A>0$ we set $v = v_{1,A} + v_{2,A}$ with $\widehat{v}_{1,A} = \widehat{v}(\xi)\mathbf{1}_{\{|\xi|\leqslant e^A\}}$. Then

$$\mu\{|v|>\lambda\} \leqslant \mu\{|v_{1,A}|>\lambda/2\} + \mu\{|v_{2,A}|>\lambda/2\}.$$

For the first term we have

$$\|v_{1,A}\|_{L^\infty} \leqslant \|\widehat{v}_{1,A}\|_{L^1} \leqslant \|v\|_{L^2}\|\mathbf{1}_{\{|\xi|\leqslant e^A\}}\|_{L^2} \leqslant C_d e^{\frac{d}{2}A}.$$

Choose now $A_\lambda = \frac{2}{d}\log(\lambda/4C_d)$, we have $\mu\{|v_{1,A_\lambda}|>\lambda/2\} = 0$, hence

$$\begin{aligned}\int_0^\infty \lambda\log^{2s-1}\langle\lambda\rangle\mu(|v|>\lambda)d\lambda &\leqslant C_0 + C_s\int_e^\infty \lambda\log^{2s-1}\lambda\mu(|v|>\lambda)d\lambda\\ &\leqslant C_0 + C_s\int_e^\infty \lambda\log^{2s-1}\lambda\mu(|v_{2,A_\lambda}|>\lambda/2)d\lambda\\ &\leqslant C_0 + 2C_s\int_e^\infty \frac{\log^{2s-1}\lambda}{\lambda}\|v_{2,A_\lambda}\|_{L^2}^2 d\lambda\\ &\leqslant C_0 + 2C_s\int_e^\infty \frac{\log^{2s-1}\lambda}{\lambda}\int_{\{\xi\in\mathbb{R}^d;|\xi|\geqslant e^{A_\lambda}\}}|\widehat{v}(\xi)|^2 d\xi d\lambda.\end{aligned}$$

Now $|\xi| \geqslant e^{A_\lambda}$ implies that $\lambda \leqslant 4C_d\langle|\xi|\rangle^{d/2}$. By using Fubini theorem we have

$$\begin{aligned}\int_0^\infty \lambda\log^{2s-1}\langle\lambda\rangle\mu(|v|>\lambda)d\lambda &\leqslant C_0 + 2C_s\int_{\mathbb{R}^d}|\widehat{v}(\xi)|^2\int_e^{4C_d\langle|\xi|\rangle^{d/2}}\frac{\log^{2s-1}\lambda}{\lambda}d\lambda d\xi\\ &\leqslant C_0+2C_s\int_{\mathbb{R}^d}\log^{2s}(4C_d\langle|\xi|\rangle^{d/2})|\widehat{v}(\xi)|^2d\xi\\ &\leqslant C_s\int_{\mathbb{R}^d}\log^{2s}\langle|\xi|\rangle|\widehat{v}(\xi)|^2d\xi = C_s\|(\log\Lambda)^s v\|^2_{L^2(\Omega)}.\end{aligned}$$

Here we have used the fact

$$\int_{\mathbb{R}^d}\log^{2s}\langle|\xi|\rangle|\widehat{v}(\xi)|^2d\xi \geqslant \int_{\mathbb{R}^d}|\widehat{v}(\xi)|^2d\xi = 1.$$

Thus we have proved (2.2) by using (2.3).

In the proof of existence of weak solution for the variational problem of section 3, we need also the first Poincaré's inequality. We study the following Dirichlet eigenvalue problems:

$$\begin{aligned}&\triangle_X u = \lambda u,\\ &u|_{\partial\Omega} = 0.\end{aligned}\tag{2.4}$$

We have

Lemma 2.3. — *Under the hypotheses H-1), H-2) and H-3), the first eigenvalue λ_1 of problems (2.4) is strictly positive. This is equivalent to*

$$\|\varphi\|^2_{L^2} \leqslant \frac{1}{\lambda_1}\sum_{j=1}^m\|X_j\varphi\|^2_{L^2}, \quad \forall\,\varphi\in H^1_{X,0}(\Omega).\tag{2.5}$$

By using this lemma, in $H^1_{X,0}(\Omega)$, we can use $\|X\varphi\|_{L^2} = \Big(\sum_{j=1}^m\|X_j\varphi\|^2_{L^2}\Big)^{1/2}$ as norm.

Proof. — We set

$$\lambda_1 = \inf_{\|\varphi\|_{L^2}=1,\,\varphi\in H^1_{X,0}(\Omega)}\left\{\|X\varphi\|^2_{L^2}\right\}.$$

Suppose that $\lambda_1 = 0$, then there exists $\{\varphi_j\}\subset H^1_{X,0}(\Omega)$ such that $\|X\varphi_j\|_{L^2}\to 0$ and $\|\varphi_j\|_{L^2} = 1$. By using (1.1), $H^1_{X,0}(\Omega)$ is compactly embedding into $L^2(\Omega)$. The variational calculus deduce that there exists $\varphi_0\in H^1_{X,0}(\Omega), \|\varphi_0\|_{L^2}=1, \varphi_0\geqslant 0$ verifies

$$\triangle_X\varphi_0 = 0.$$

Since $\triangle_X$ is hypoelliptic on $\widetilde{\Omega}$ and $\partial\Omega$ is non characteristic for X, we have $\varphi_0\in C^\infty(\overline{\Omega}), \varphi_0|_{\partial\Omega} = 0$ (see [**6, 9, 11, 16**]). Under the hypothesis H-2), Bony's maximum principle (see [**2**]) implies that φ_0 has not the maximum point in $\Omega\smallsetminus\Gamma$, and the maximum of φ_0 propagates along the integral curves of $X_1,\dots,X_m$ in the interior of Ω. Since Γ is non characteristic for the system $X_1,\dots,X_m$, for any point of Γ, there exists at least one vector field of $X_1,\cdots,X_m$ which is transversal to Γ. Hence if the maximum of φ_0 attains at a point of Γ in the interior of Ω, then the maximum of φ_0 propagates along the integral curve of that vector field which traverses Γ, that means the maximum of φ_0 attains at a point of $\Omega \setminus \Gamma$, so it is impossible. Now it is only possible that the maximum of φ_0 attains at $\partial\Omega$, but $\varphi_0|_{\partial\Omega} = 0$, which implies that $\varphi_0 \equiv 0$ on Ω. This is impossible because $\|\varphi_0\|_{L^2} = 1$, so that we prove finally $\lambda_1 > 0$.

3. Variational problems

For $a \in \mathbb{R}$, we study now the following variational problems

$$I_a = \inf_{\|v\|_{L^2}=1,\, v\in H^1_{X,0}(\Omega)} I_a(v), \tag{3.1}$$

with

$$I_a(v) = \|Xv\|^2_{L^2(\Omega)} - a\int_\Omega |v|^2 \log|v|.$$

We have firstly the existence of minimizer of $I_a(v)$.

Proposition 3.1. — *Under the hypotheses H-1), H-2) and H-3), I_a is an attained minimum in $H^1_{X,0}(\Omega)$.*

Proof. — We prove firstly $I_a(v)$ is bounded below on $\{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$. Hypothesis H-3) and Theorem 1.1 give that

$$\int_\Omega |v|^2 \log^2\left(\frac{|v|}{\|v\|_{L^2}}\right) \leqslant C_0\left(\|Xv\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right), \tag{3.2}$$

for all $v \in H^1_{X,0}(\Omega)$. Now if $a = 0$, we have $I_0(v) \geqslant \lambda_1$ for all $v \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$. If $a \neq 0$, we have

$$a\int_\Omega |v|^2\,|\log|v|| \leqslant \frac{1}{2C_0}\int_\Omega |v|^2\,|\log|v||^2 + \frac{C_0|a|^2}{2} \leqslant \frac{1}{2}\|Xv\|^2_{L^2(\Omega)} + \left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right),$$

for all $v \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$. We have that

$$\begin{aligned} I_a(v) &= \|Xv\|^2_{L^2} - |a|\int_\Omega |v|^2|\log|v|| \\ &\geqslant \|Xv\|^2_{L^2} - \frac{1}{2}\|Xv\|^2_{L^2} - \left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right) \\ &\geqslant \frac{1}{2}\lambda_1 - \left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right), \end{aligned}$$

for all $v \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$.

Let now $\{v_j\} \subset \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$ be a minimizer sequence of I_a, then

$$\left(\frac{C_0}{2} + \frac{C_0|a|^2}{2}\right) + I_a(v_j) \geqslant \frac{1}{2}\|Xv_j\|^2_{L^2}.$$

It follows that $\{v_j\}$ is a bounded sequence in $H^1_{X,0}(\Omega)$. Then there exists a subsequence (denote still by $\{v_j\}$) such that $v_j \rightharpoonup v_0$ in $H^1_{X,0}(\Omega)$ and $v_j \to v_0$ in $L^2(\Omega)$ which give that

$$\liminf_{j\to\infty} \|Xv_j\|^2_{L^2(\Omega)} \geqslant \|Xv_0\|^2_{L^2(\Omega)}, \quad \lim_{j\to\infty} \|v_j\|_{L^2(\Omega)} = \|v_0\|_{L^2(\Omega)} = 1.$$

By using (3.2), $\{\int_\Omega |v_j|^2 \log^2 v_j\}$ is bounded, the Lemma 2.2 implies that there exists a subsequence of $\{v_j\}$ such that

$$\lim_{j\to\infty} \int_\Omega |v_j|^2 \log v_j = \int_\Omega |v_0|^2 \log v_0.$$

But we have also a direct proof of this convergence

$$\begin{aligned}
&\Big|\int_\Omega |v_j|^2 \log v_j - \int_\Omega |v_0|^2 \log v_0\Big|\\
&\quad = \Big|\int_\Omega (v_j - v_0) \int_0^1 v_t(2\log v_t + 1)dtdx\Big|\\
&\quad \leqslant C\|v_j - v_0\|_{L^2} \int_0^1 \Big(\int_\Omega |v_t|^2(\log^2|v_t| + 1)dx\Big)^{1/2} dt\\
&\quad \leqslant C\|v_j - v_0\|_{L^2} \int_0^1 \Big(\|v_t\|_{L^2} + \int_0^1 \Big(\int_\Omega |v_t|^2 \left|\log^2|v_t|\right| dx\Big)^{1/2}\Big)dt\\
&\quad \leqslant C\|v_j - v_0\|_{L^2} \int_0^1 \big(\|v_t\|_{L^2} + \big(\|Xv_t\|^2_{L^2} + \|v_t\|^2_{L^2} + \|v_t\|^2_{L^2}\log^2\|v_t\|^2_{L^2}\big)^{1/2}\big)dt,
\end{aligned}$$

where $v_t = v_j + t(v_j - v_0)$, and we have used (3.2) for the function $v_t \in H^1_{X,0}(\Omega)$. Since $\{v_j\}$ is a bounded sequence in $H^1_{X,0}(\Omega)$, and $\|v_j - v_0\|_{L^2} \to 0$, the right hand side of above estimate go to 0 if $j \to \infty$. We have proved finally Proposition 3.1.

We study now the Euler-Lagrange equation of variational problems (3.1).

Proposition 3.2. — *The minimizer u of variational problem (3.1) is a non trivial weak solution of the following semilinear Dirichlet problem*

$$\begin{aligned} \triangle_X u &= au\log|u| + I_a u,\\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{3.3}$$

Proof. — The minimizer u obtained in Proposition 3.1 is in $\{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$ and $u \geqslant 0$. u is a weak solution of (3.3) is equivalent to

$$\int_\Omega \sum_{j=1}^m X_j u X_j\varphi - a\int_\Omega u\varphi\log|u| - I_a\int_\Omega u\varphi = 0, \tag{3.4}$$

for all $\varphi \in H^1_{X,0}(\Omega)$. For fixed $\varphi \in H^1_{X,0}(\Omega)$ and $\varepsilon \in \mathbb{R}$ with $|\varepsilon|$ small enough, we put

$$u_\varepsilon = u + \varepsilon\varphi, \quad \widetilde{u}_\varepsilon = u_\varepsilon/\|u_\varepsilon\|_{L^2},$$

then $\widetilde{u}_\varepsilon \in \{v \in H^1_{X,0}(\Omega),\ \|v\|_{L^2} = 1\}$, so that

$$H(\varepsilon) = I_a(\widetilde{u}_\varepsilon) \geqslant I_a(u) = I_a,$$

and

$$H(\varepsilon) = \frac{1}{\|u_\varepsilon\|_{L^2}^2} I_a(u_\varepsilon) + a \log \|u_\varepsilon\|_{L^2}.$$

By direct calculus,

$$H'(\varepsilon) = -\frac{2}{\|u_\varepsilon\|_{L^2}^4} I_a(u_\varepsilon) \int_\Omega u_\varepsilon \varphi + \frac{a}{\|u_\varepsilon\|_{L^2}^2} \int_\Omega u_\varepsilon \varphi \\ + \frac{1}{\|u_\varepsilon\|_{L^2}^2} \left(2 \int_\Omega X u_\varepsilon X \varphi - 2a \int_\Omega u_\varepsilon \varphi \log |u_\varepsilon| - a \int_\Omega u_\varepsilon \varphi \right).$$

We have to prove the continuity of $H'(\varepsilon)$ at $\varepsilon = 0$, since $u_\varepsilon, X u_\varepsilon \in L^2(\Omega)$, we need only to prove

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon \varphi \log |u_\varepsilon| = \int_\Omega u \varphi \log |u|.$$

this can be deduced by Lebesgue dominant theorem if we use the fact $|t \log t| \leqslant t^2 + e^{-1}, \forall\, t \geqslant 0$ and φ can be approximated by bounded functions. So that we have, for any $\varepsilon \in \mathbb{R}$, with $|\varepsilon|$ small enough

$$I_a(\widetilde{u}_\varepsilon) = H(\varepsilon) = H(0) + H'(0)\varepsilon + \delta(\varepsilon)\varepsilon \geqslant I_a(u) = H(0),$$

where $\delta(\varepsilon) \to 0$ if $\varepsilon \to 0$. We get finally $H'(0) = 0$, this is true for all $\varphi \in H^1_{X,0}(\Omega)$, we have proved Proposition 3.2.

Theorem 3.1. — *Let $a, b \in \mathbb{R}, a \neq 0$, under the hypotheses H-1), H-2) and H-3), the Dirichlet problems (1.3) has at least one non trivial weak solution $u \in H^1_{X,0}(\Omega)$, $u \geqslant 0$, $\|u\|_{L^2} > 0$.*

In fact, if $\widetilde{u}$ is a weak solution of problem (3.3), for $c > 0$ we set $u = c\widetilde{u}$, then $\|u\|_{L^2} = c > 0$, $u \geqslant 0$, $u \in H^1_{X,0}(\Omega)$ and in the weak sense

$$\triangle_X u = au \log |u| + (I_a - \log c) u.$$

Choose $c = e^{I_a - b} > 0$, we get (1.3).

Following this direction, we can study the high order nonlinear eigenvalue problems. Suppose that we have the logarithmic Sobolev inequality

$$\int_\Omega |v|^2 \log^{k+1} \left(\frac{|v|}{\|v\|_{L^2}} \right) \leqslant C_0 \left(\|Xv\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \right).$$

For $a_1, \ldots, a_k \in \mathbb{R}$, we study the variational problems

$$I^k_{a_1,\ldots,a_k} = \inf_{\|v\|_{L^2}=1,\, v \in H^1_{X,0}(\Omega)} I^k_{a_1,\ldots,a_k}(v),$$

with

$$I^k_{a_1,\ldots,a_k}(v) = \|Xv\|^2_{L^2(\Omega)} - \sum_{j=1}^k a_j \int_\Omega |v|^2 \log^j |v|.$$

As in the proof of Proposition 3.1, we need to prove that there exists a subsequence of $\{v_j\}$ of minimizer sequence such that

$$\lim_{j\to\infty}\int_\Omega |v_j|^2\log^k v_j = \int_\Omega |v_0|^2\log^k v_0,$$

which was already shown in the Lemma 2.2.

By similar calculus as in Proposition 3.2, we can prove that for any $a_1,\dots,a_k\in\mathbb{R}$, there exists $I^k_{a_1\cdots,a_k}$ such that the following semilinear Dirichlet problems

$$\begin{aligned}&\triangle_X u = \textstyle\sum_{j=1}^k a_j u\log^j|u| + I^k_{a_1\cdots,a_k}u,\\ &u|_{\partial\Omega}=0,\end{aligned}$$

has at least one non trivial solution in $H^1_{X,0}(\Omega)$, with $u\geqslant 0$ and $\|u\|_{L^2}=1$. Moreover, we have similar regularity results as Theorem 1.2.

4. Boundedness and regularity of weak solutions

By using the interpolation inequality, the condition H-3) and the Logarithmic Sobolev inequality (1.2) give that, for any $N\geqslant 1$, there exists C_N such that

$$\int_\Omega v^2\log^2\left(\frac{|v|}{\|v\|_{L^2}}\right)\leqslant \frac{1}{N}\|Xv\|^2_{L^2}+C_N\|v\|^2_{L^2}, \tag{4.1}$$

for all $v\in H^1_{X,0}(\Omega)$.

Theorem 4.1. — *Let $u\in H^1_{X,0}(\Omega), u\geqslant 0, \|u\|_{L^2}\neq 0$ be a weak solutions of equation*

$$\triangle_X u = au\log u + bu. \tag{4.2}$$

Then $u\in L^\infty(\Omega)$.

It suffices to show that there exists $\overline{A}>0$ such that the estimate

$$\|u\|_{L^p}\leqslant \overline{A} \tag{4.3}$$

holds for any $p\geqslant 2$. In fact, if $\Omega_\varepsilon=\{x\in\Omega; |u(x)|\geqslant \overline{A}+\varepsilon\}$ for $\varepsilon>0$ then it follows from (4.3) that $|\Omega_\varepsilon|\leqslant\left(\frac{\overline{A}}{\overline{A}+\varepsilon}\right)^p\to 0\quad(p\to\infty)$ and hence we have $||u||_{L^\infty}\leqslant\overline{A}$.

We prove this by the following three propositions. To get the estimate as (4.3), we shall use u^{2p-1} or $u^{2p-1}\log^{2m}(u^p)$ as test function for the equation (4.2) for $p\geqslant 1$, $m\in\mathbb{N}$, but we don't know if $u^{2p-1}\log^{2m}(u^p)\in H^1_{X,0}(\Omega)$, so we replace the function u by $u_{(k)}$ with $u_{(k)}(x)=u(x)$ if $x\in\{x\in\Omega; |u(x)|<k\}$ and $u_{(k)}(x)=k$ if $x\in\{x\in\Omega; |u(x)|\geqslant k\}$ for $k>1$, $p>1$. Then it is easy to check (see [**22**] and Theorem 7.8 of [**8**]) that $u^{2p-1}_{(k)}\log^{2m}(u^p_{(k)})\in H^1_{X,0}(\Omega)$ for all $p>1$, $m\in\mathbb{N}$. If $p=1$, we use $u\,(\log^m u)^2_{(k)}\in H^1_{X,0}(\Omega)$ as test function. To simplify the notation, we shall drop the subscript and use $u^{2p-1}\log^{2m}(u^p)$ as test function.

Proposition 4.1. — *Let $u \in H^1_{X,0}(\Omega)$, $u \geqslant 0$, $\|u\|_{L^2} \neq 0$ be a weak solution of equation (4.2). Suppose that for some $p_0 \geqslant 1$, there exists A_0 such that*

$$\|u\|_{L^{2p_0}} \leqslant A_0.$$

Then

$$\int_\Omega |X(\widetilde{u})^{p_0}|^2 + \int_\Omega (\widetilde{u})^{2p_0} \log^2((\widetilde{u})^{p_0}) \leqslant 2C_2 + |a|^2 + 2p_0(|b| + \log A_0), \tag{4.4}$$

where the constant C_2 is given in (4.1) and $\widetilde{u} = u/\|u\|_{L^{2p_0}}$.

Proof. — We have $\widetilde{u} \in H^1_{X,0}(\Omega), \|\widetilde{u}\|_{L^{2p_0}} = 1$, and $\widetilde{u}$ is a weak solution of equation

$$\triangle_X \widetilde{u} = a\widetilde{u}\log\widetilde{u} + (b - \log\|u\|_{L^{2p_0}})\widetilde{u}. \tag{4.5}$$

Take u^{2p_0-1} as test function, we have

$$\frac{2p_0-1}{p_0^2}\int_\Omega |X\widetilde{u}^{p_0}|^2 = \frac{a}{p_0}\int_\Omega \widetilde{u}^{2p_0}\log\widetilde{u}^{p_0} + (b - \log\|u\|_{L^{2p_0}})\int_\Omega \widetilde{u}^{2p_0}$$

which shows that

$$\int_\Omega |X\widetilde{u}^{p_0}|^2 \leqslant \frac{1}{2}\int \widetilde{u}^{2p_0}\log^2\widetilde{u}^{p_0} + (\frac{1}{2}|a|^2 + p_0|b| + p_0\log A_0). \tag{4.6}$$

On the other hand, the logarithmic Sobolev inequality (4.1) gives

$$\int_\Omega (u^{p_0})^2 \log^2\left(\frac{|u^{p_0}|}{\|u^{p_0}\|_{L^2}}\right) \leqslant \frac{1}{2}\|X(u^{p_0})\|^2_{L^2} + C_2\|u^{p_0}\|^2_{L^2}.$$

Note that $\|u^{p_0}\|_{L^2} = \|u\|^{p_0}_{L^{2p_0}}$ and $\widetilde{u} = u/\|u\|_{L^{2p_0}}$, we have

$$\int_\Omega \widetilde{u}^{2p_0}\log^2(\widetilde{u}^{p_0}) \leqslant \frac{1}{2}\|X(\widetilde{u}^{p_0})\|^2_{L^2} + C_2. \tag{4.7}$$

Adding (4.6) and (4.7), we have the desired estimate (4.4).

Proposition 4.2. — *We have for any $m \in \mathbb{N}$*

$$\int_\Omega |X(\widetilde{u}^{p_0})|^2 \log^{2m-2}(\widetilde{u}^{p_0}) + \int_\Omega \widetilde{u}^{2p_0}\log^{2m}(\widetilde{u}^{p_0}) \leqslant M_1^{2m} P(m, p_0)(m!)^2, \tag{4.8}$$

where $P(m,p_0) = p_0^m$ if $m \leqslant \sqrt{p_0}$, $P(m,p_0) = p_0^{\sqrt{p_0}}$ if $m > \sqrt{p_0}$, and

$$M_1 \geqslant (2|\Omega| + 4C_2 + 2C_4 + 10 + 6|a|^2 + 8|b| + 8\log A_0)^{1/2}.$$

Proof. — For $m = 1$, this is (4.4). We prove now (4.8) by induction, suppose that (4.8) is true for some $m \in \mathbb{N}$, then we prove it for $m+1$. From now on we drop the tilde of u and subscript of p to simplify the notation. Take $u^{2p-1}\log^{2m}(u^p)$ as test function in (4.5), we have

$$\begin{aligned}\frac{2p-1}{p^2}\int_\Omega |Xu^p|^2\log^{2m}(u^p) + \frac{2m}{p}\int_\Omega |Xu^p|^2\log^{2m-1}(u^p)\\ = \frac{a}{p}\int_\Omega u^{2p}\log^{2m+1}(u^p) + (b - \log\|u\|_{L^{2p}})\int_\Omega u^{2p}\log^{2m}(u^p),\end{aligned}$$

which gives

$$\int_\Omega |Xu^p|^2 \log^{2m}(u^p) \leqslant \frac{1}{2}\int_\Omega |Xu^p|^2 \log^{2m}(u^p) + 2m^2 \int_\Omega |Xu^p|^2 \log^{2m-2}(u^p)$$
$$+ \frac{1}{4}\int_\Omega u^{2p} \log^{2m+2}(u^p) + (|a|^2 + p|b| + p\log A_0)\int_\Omega u^{2p}\log^{2m}(u^p)$$

so that

$$(4.9)\qquad \int_\Omega |Xu^p|^2 \log^{2m}(u^p) \leqslant \frac{1}{2}\int_\Omega u^{2p}\log^{2m+2}(u^p)$$
$$+ (4m^2 + 2(|a|^2 + p|b| + p\log A_0))M_1^{2m}P(m,p)(m!)^2.$$

We study now the term $\int_\Omega u^{2p}\log^{2m+2}(u^p)$, we cut $\Omega = \Omega_1 \cup \Omega_2^+ \cup \Omega_2^-$ with $\Omega_1 = \{x \in \Omega; u(x) \leqslant 1\}$ and

$$\Omega_2^+ = \{x \in \Omega; u(x) > 1, |\log^m(u(x)^p)| \leqslant \|u^p \log^m(u^p)\|_{L^2}\},$$
$$\Omega_2^- = \{x \in \Omega; u(x) > 1, |\log^m(u(x)^p)| > \|u^p \log^m(u^p)\|_{L^2}\}.$$

Then

$$\int_{\Omega_1} u^{2p}\log^{2m+2}(u^p) \leqslant |\Omega|((m+1)!)^2.$$

For the second term, (4.4) give

$$\int_{\Omega_2^+} u^{2p}\log^{2m+2}(u^p) \leqslant \|u^p\log^m(u^p)\|_{L^2}^2 \int_\Omega u^{2p}\log^2(u^p)$$
$$\leqslant (2C_2 + |a|^2 + 2p|b| + 2p\log A_0)M_1^{2m}P(m,p)(m!)^2,$$

and for the third term, we use the logarithmic Sobolev inequality (4.1) for $N=4$,

$$\int_{\Omega_2^-} u^{2p}\log^{2m+2}(u^p) \leqslant \int_{\Omega_2^-} (u^p\log^m u^p)^2 \log^2\left(\frac{u^p\log^m(u^p)}{\|u^p\log^m(u^p)\|_{L^2}}\right)$$
$$\leqslant \frac{1}{4}\|X(u^p\log^m u^p)\|_{L^2}^2 + C_4\|u^p\log^m u^p\|_{L^2}^2$$
$$\leqslant \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p) + m^2\int_\Omega |X(u^p)|^2\log^{2m-2}(u^p) + C_4\int_\Omega u^{2p}\log^{2m}(u^p)$$
$$\leqslant \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p) + (C_4 + m^2)M_1^{2m}P(m,p)(m!)^2.$$

Adding those three terms, we get

$$(4.10)\qquad \int_\Omega u^{2p}\log^{2m+2}(u^p) \leqslant \frac{1}{2}\int_\Omega |X(u^p)|^2\log^{2m}(u^p) + |\Omega|((m+1)!)^2$$
$$+ (2C_2 + C_4 + m^2 + |a|^2 + 2p|b| + 2p\log A_0)M_1^{2m}P(m,p)(m!)^2.$$

Adding (4.9) and (4.10), we get

$$\int_\Omega u^{2p}\log^{2m+2}(u^p)+\int_\Omega |X(u^p)|^2\log^{2m}(u^p)\leqslant (2|\Omega|+4C_2+2C_4 \\ +10+6|a|^2+8|b|+8\log A_0)M_1^{2m}P(m+1,p)((m+1)!)^2. \tag{4.11}$$

We have proved Proposition 4.2.

Proposition 4.3. — *Let $u\in H^1_{X,0}(\Omega), u\geqslant 0, \|u\|_{L^2}\neq 0$ be a weak solution of equation (4.2). Suppose that for some $p_0\geqslant 1$ and $A_0\geqslant e^{12}$ we have*

$$\|u\|_{L^{2p_0}}\leqslant A_0.$$

Then for

$$M_1\geqslant (2|\Omega|+4C_2+2C_4+10+6|a|^2+8|b|+8\log A_0)^{1/2},$$

and $\delta=1/2M_1$, we have

$$\int_\Omega u^{2p_0(1+\delta)}\leqslant A_0^{2p_0(1+\delta)\left(1+\left(\frac{1}{p_0(1+\delta)}\right)^{1/3}\right)}. \tag{4.12}$$

Proof. — For any $\delta>0$, the estimate (4.8) gives that

$$\begin{aligned}\Big(\int_\Omega |\widetilde u^{p_0(1+\delta)}|^2dx\Big)^{1/2}&=\Big(\int_\Omega |\widetilde u^{p_0}\widetilde u^{\delta p_0}|^2dx\Big)^{1/2}=\Big(\int_\Omega \Big|\widetilde u^{p_0}e^{\delta\log(\widetilde u^{p_0})}\Big|^2dx\Big)^{1/2}\\
&=\Big(\int_\Omega \Big|\widetilde u^{p_0}\sum_{m=0}^\infty\frac{(\delta\log(\widetilde u^{p_0}))^m}{m!}\Big|^2dx\Big)^{1/2}\leqslant\sum_{m=0}^\infty\Big(\int_\Omega \widetilde u^{2p_0}\frac{(\delta\log(\widetilde u^{p_0}))^{2m}}{(m!)^2}dx\Big)^{1/2}\\
&\leqslant\sum_{m=0}^\infty\frac{\delta^m}{m!}\Big(\int_\Omega \widetilde u^{2p_0}\log^{2m}(\widetilde u^{p_0})dx\Big)^{1/2}\leqslant\sum_{m=0}^\infty\delta^mM_1^mP(m,p_0)\leqslant p_0{}^{\sqrt{p_0}}\sum_{m=0}^\infty(\delta M_1)^m.\end{aligned}$$

For $\delta=1/2M_1$, we have finally

$$\int_\Omega u^{2p_0(1+\delta)}dx\leqslant 4p_0^{2\sqrt{p_0}}A_0^{2p_0(1+\delta)}.$$

Since for any $p_0>1$,

$$4p_0^{2\sqrt{p_0}}=4e^{2\sqrt{p_0}\log p_0}\leqslant \big(e^{12}\big)^{2p_0^{2/3}}.$$

We have proved (4.12) if $A_0\geqslant e^{12}$, and Proposition 4.3.

The same calculus give also

$$\int_\Omega |X(u^{p_0(1+\delta)})|^2dx\leqslant (1+\delta)^2(4M_1)^2A_0^{2p_0(1+\delta)\left(1+\left(\frac{1}{p_0(1+\delta)}\right)^{1/3}\right)}. \tag{4.13}$$

We put now for $k\in\mathbb N$,

$$p_k=p_0(1+\delta)^k, A_k=A_0^{1+p_0^{-1/3}\sum_{j=1}^k\left(\frac{1}{(1+\delta)}\right)^{j/3}},$$

then Proposition 4.3 implies that

$$\int_\Omega u^{2p_0(1+\delta)^{k+1}} a = \int_\Omega u^{2p_k(1+\delta)} \leqslant A_k^{2p_k(1+\delta)\left(1+\left(\frac{1}{p_k(1+\delta)}\right)^{1/3}\right)}$$
$$\leqslant A_0^{2p_0(1+\delta)^{k+1}\left(1+p_0^{-1/3}\sum_{j=1}^{k+1}\left(\frac{1}{(1+\delta)}\right)^{j/3}\right)},$$

with $\delta = 1/2M_1$ and

$$M_1 \geqslant \big(2|\Omega| + 4C_2 + 2C_4 + 10 + 6|a|^2 + 8|b| + 8\log A_k\big)^{1/2}. \tag{4.14}$$

We have now for $\delta = 1/2M_1 \leqslant 1/4$,

$$\frac{\log A_k}{\log A_0} = 1 + p_0^{-1/3}\sum_{j=1}^{k}\left(\frac{1}{(1+\delta)}\right)^{j/3} \leqslant 1 + p_0^{-1/3}\sum_{j=1}^{\infty}\left(\frac{1}{(1+\delta)}\right)^{j/3}$$
$$= 1 + p_0^{-1/3}\frac{\left(\frac{1}{1+\delta}\right)^{1/3}}{1-\left(\frac{1}{1+\delta}\right)^{1/3}} \leqslant 1 + 4p_0^{-1/3}M_1 \leqslant 5M_1.$$

So we can choose M_1 independent on k

$$M_1 = \big(2|\Omega| + 4C_2 + 2C_4 + 10 + 6|a|^2 + 8|b| + 40\log A_0\big). \tag{4.15}$$

We have proved for any $k \in \mathbb{N}$,

$$\int_\Omega u^{2p_0(1+\delta)^k} \leqslant \left(A_0^{5M_1}\right)^{2p_0(1+\delta)^k}.$$

For $p_0 = 1$, we have $A_0 = e^{12}$. So we have proved (4.3) with $\overline{A} = e^{60M_1}$ if $\|u\|_{L^2} = 1$.

Now the proof of the Theorem 4.1 is complete.

Theorem 4.2. — *Let $u \in H^1_{X,0}(\Omega), u \geqslant 0, \|u\|_{L^2} \neq 0$ be a weak solution of equation (4.2), suppose that $a > 0$, Γ and $\partial\Omega$ is non characteristic. Then $u \in C^\infty(\Omega \setminus \Gamma) \cap C^0(\overline{\Omega} \setminus \Gamma)$ and $u(x) > 0$ for all $x \in \Omega \setminus \Gamma$.*

Proof. — Suppose $x_0 \in \Omega \setminus \Gamma$, then there exists a neighborhood $V_0 \subset \Omega \setminus \Gamma$ of x_0, for $\varphi \in C_0^\infty(V_0)$ we shall prove that $v = \varphi u \in C^\infty(V_0)$. It follows from equation (4.2) that,

$$\triangle_X v = a\varphi u \log u + b\varphi u + \sum_{j=1}^m \varphi_j X_j u + \varphi_0 u = f_0 + \sum_{j=1}^m X_j f_j,$$

with $\varphi_j \in C^\infty(V_0), f_j \in L^\infty(V_0), j = 0, \ldots, m$. Since the system of vector fields X satisfies the finitely type Hörmander's condition on V_0, the regularity result of [**23**] (see also [**22, 24**]) implies that $u \in C^\varepsilon(V_0)$ for some $\varepsilon > 0$. If $u(x) \geqslant \alpha > 0$ for $x \in V_0$, we have $u \log u \in C^\varepsilon(V_0)$ since $t \log t \in C^\infty(t \geqslant \alpha)$. Then we prove by recurrence that $u \in C^\infty(V_0)$. For $x_0 \in \partial\Omega \setminus \Gamma$, we have also $u \in C^\varepsilon(V_0 \cap \overline{\Omega})$, but we know only $u \log u \in C^0(V_0 \cap \overline{\Omega})$, so we can't get the C^∞ regularity of u near to the boundary $\partial\Omega$. Now we finish the interior regularity of Theorem 4.2 by the following lemma.

Lemma 4.1. — *Suppose that* $u \in C^0(\Omega_1), u \geqslant 0$ *is a non trivial weak solution of equation (4.2) on an open set* $\Omega_1 \subset \Omega$*, let* $a > 0$*, then* $u(x) > 0$ *for all* $x \in \Omega_1$.

Proof. — Suppose that $u(x_0) = 0$ for some $x_0 \in \Omega_1$, then we have $f = au \log u + bu$ continuous on Ω_1, and $f(x_0) = 0$, then for any $\varepsilon > 0$, there exists a small neighborhood $U_0 \subset \Omega_1$ of x_0 such that $0 \leqslant u(x) \leqslant \varepsilon$ on U_0. Since $a > 0$, we have for ε small enough, $f(x) \leqslant 0$ on U_0, so that $\triangle_X u \leqslant 0$ on U_0, but x_0 is a minimum point of u, as in the proof of Lemma 2.3, the maximum principle of Bony ([**2**]) implies that $u \equiv 0$ on U_0, so that u is a trivial solution by continuous of u in Ω_1.

5. Appendix: Logarithmic regularity estimate

In this section we shall give sufficient conditions in order that the sum of squares of real vector fields

$$\triangle_X = \sum_{j=1}^{m} X_j^* X_j,$$

satisfies the logarithmic regularity estimate (1.1). We start by the following simple model operator in $\mathbb{R}^2$

$$L_0 = D_{x_1}^2 + D_{x_2}(g(x_1)D_{x_2}),$$

where $C^\infty \ni g(t) > 0$ if $t \neq 0$ and $g(0) = 0$. In what follows we do not require that $g(x)$ is written as $g = \varphi^2$ for some $\varphi \in C^\infty$, and we consider a little more general logarithmic regularity estimate than (1.1). The following proposition is essentially due to the device of Wakabayashi (see Example 5.1 of [**21**]).

Proposition 5.1. — *Let* $f(t)$ *and* $g(t)$ *be non-negative continuous functions and satisfy* $f(t), g(t) > 0$ *if* $t \neq 0$*. Assume that there exists an* $\varepsilon \geqslant 0$ *such that*

$$\limsup_{t\to 0} \left| \frac{\int_0^t f(\tau)d\tau}{\sqrt{f(t)}} \right|^{1/s} |\log g(t)| \leqslant \varepsilon. \tag{5.1}$$

Then for any compact set K *in* $\mathbb{R}^2$ *there exist constants* $C_0 > 0$ *independent of* ε *and* $C_\varepsilon > 0$ *such that*

$$||\sqrt{f(x_1)}(\log \Lambda)^s u||^2 \leqslant C_0 \varepsilon^{2s}(L_0 u, u) + C_\varepsilon ||u||^2 \tag{5.2}$$

for all $u \in C_0^\infty(K)$.

Remark. — The typical example satisfying (5.1) is $g(t) = \exp(-2|t|^{-1/s})$, stated in Introduction with $f \equiv 1$. It is known that (5.1) is also necessary for (5.2) with neglecting constant factor of ε if $f(t)$ and $g(t)$ are monotone in each half axis $\mathbb{R}_\pm$.

The necessity is shown by way of another sufficient condition for (5.1), given by Koike [**10**], as follows:

$$\limsup_{t\to 0} \mu(f;t)^{1/s}|\log g(t)| \leqslant \varepsilon,$$

where $\mu(f;t) = \sup_{0\leqslant \pm\tau \leqslant \pm t} \sqrt{f(\tau)}|t-\tau|$ if $\pm t > 0$. This condition is equivalent with (5.1) except for constant factor of ε under the monotonous condition. We refer [**14**] and references therein concerning details for the estimate (5.2).

Proof. — If $F(t) = \int_0^t f(\tau)d\tau$ then it follows from (5.1) that there exists a $t_0 > 0$ such that

$$g(t) < 1 \text{ and } |F(t)|(-\log g(t))^s \leqslant 2\varepsilon^s\sqrt{f(t)} \qquad \text{if } |t| < t_0. \tag{5.3}$$

Since $g(t) > 0$ for $t \neq 0$, one can find a $\lambda_0 > 0$ such that

$$\text{if } \lambda \geqslant \lambda_0 \text{ then } \Omega_\lambda := \{t; g(t)\lambda \leqslant 1\} \subset \{t; |t| < t_0\}. \tag{5.4}$$

Note that for $v(t) \in C_0^\infty(\mathbb{R}^1)$ we have

$$\begin{aligned}\|\sqrt{f(t)}(\log\lambda)^s v\|^2 &= ([D_t, F(t)](\log\lambda)^{2s}v, v)\\ &\leqslant 2|(D_t v, F(t)(\log\lambda)^{2s}v)|\\ &\leqslant 8\varepsilon^{2s}\|D_t v\|^2 + \frac{1}{8\varepsilon^{2s}}\|F(t)(\log\lambda)^{2s}v\|^2\end{aligned}$$

by the Schwartz inequality. Choosing another sufficiently large $\lambda_0 > 0$ if necessary, we may assume

$$\frac{1}{8\varepsilon^{4s}}F(t)^2(\log\lambda)^{4s} \leqslant \lambda \leqslant g(t)\lambda^2 \text{ in } \Omega_\lambda^c \cap \operatorname{supp} v \text{ if } \lambda \geqslant \lambda_0.$$

If $\lambda \geqslant \lambda_0$ then it follows from (5.3) and (5.4) that

$$F(t)^2(\log\lambda)^{4s} \leqslant F(t)^2(-\log g(t))^{2s}(\log\lambda)^{2s} \leqslant 4\varepsilon^{2s}f(t)(\log\lambda)^{2s} \quad \text{in } \Omega_\lambda.$$

Above two estimates give

$$\frac{1}{8\varepsilon^{2s}}||F(t)(\log\lambda)^{2s}v||^2 \leqslant \frac{1}{2}\int_{\Omega_\lambda} f(t)(\log\lambda)^{2s}|v|^2dt + \varepsilon^{2s}\int_{\Omega_\lambda^c} g(t)\lambda^2|v|^2dt.$$

Therefore we have

$$\|\sqrt{f(t)}(\log\lambda)^s v\|^2 \leqslant 16\varepsilon^{2s}(||D_t v||^2 + (g(t)\lambda^2 v, v))$$

if $\lambda \geqslant \lambda_0$. The estimate (5.2) is obvious if we consider the partial Fourier transform $v(x_1, \lambda)$ of $u(x_1, x_2)$ with respect to x_2 variable.

In the rest of this section we shall give a sufficient condition for general operator $\triangle_X$, by using Sawyer's lemma (see below), as in [**15**]. For the sake of simplicity, we confirm ourself to the logarithmic regularity estimate (1.1). Let X_J denote the repeated commutator

$$[X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}]\cdots]]]$$

for $J = (j_1, \ldots, j_k), j_i \in \{1, \ldots, m\}$, (and set $|J| = k$). For $k \geqslant 1$ put

$$G(x;k) = \min_{\xi \in \mathbb{S}^{d-1}} \sum_{|J| \leqslant k} |X_J(x,\xi)|^2, \qquad g(t;j,k,x_0) = G((\exp tX_j)(x_0);k),$$

where $(\exp tX_j)(x_0)$ is the integral curve of X_j starting from $x_0 \in \Gamma$. Here we recall that $\Gamma = \{x \in \widetilde{\Omega}\,;\, \exists \xi \in \mathbb{S}^{d-1}$ satisfying $X_J(x,\xi) = 0, \forall J\}$. Let $g_I^{j,k}(x_0)$ denote the mean value $\frac{1}{|I|}\int_I g(t;j,k,x_0)dt$ on the interval I. Then we have the following:

Proposition 5.2. — *If $s > 0$ and if there exists an $\varepsilon > 0$ such that*

$$\inf_{\substack{\delta>0, k\in N\\ \mu>0, 1\leqslant j\leqslant m}} \Big(\sup\Big\{|I|^{1/s}|\log g_I^{j,k}(x_0)|\,;\, I \subset (-\mu,\mu) \text{ and } g_I^{j,k}(x_0) < \delta\Big\}\Big) < \varepsilon \tag{5.5}$$

for any $x_0 \in \Gamma$, then there exist constants $C_0 > 0$ independent of ε and $C_\varepsilon > 0$ such that

$$\|(\log \Lambda)^s u\|_{L^2}^2 \leqslant C_0 \varepsilon^{2s}\,(\Delta_X u, u) + C_\varepsilon ||u||_{L^2}^2, \tag{5.6}$$

for any $u \in C_0^\infty(\widetilde{\Omega})$.

Remark. — The condition (5.5) admits the case where all integral curves of X_j intersect Γ in any small neighborhood of x_0, such as the following:

$$X_1 = \partial_{x_1}\ X_2 = \exp\big(-\big(x_1^2\sin^2(\pi/x_1)\big)^{-1/2s}\big)\partial_{x_2}.$$

In this example, Γ is composed of hypersurfaces $\Gamma_j = \{x_1 = 1/j\}$ $(j \in \mathbb{Z} \setminus \{0\})$ and $\Gamma_0 = \{x_1 = 0\}$. Since $|x_1 \sin \pi/x_1|$ is approximated to $\pi j|x_1 - 1/j|$ near Γ_j by Taylor's formula, (5.5) is satisfied for $x_0 \in \Gamma_j$. Let $x_0 \in \Gamma_0$. If the interval I contains the point $1/j$ and its length is larger than a half of $1/j$, then $g_I^{1,k}(x_0)$ is comparable to that with X_2 replaced by $\exp(-|x_1|^{-1/s})\partial_{x_2}$. If the length of I is not larger than a half of $1/j$, we can use the preceding result in the case of $x_0 \in \Gamma_j$.

Proof of Proposition 5.2. — It follows from (5.5) that there exist some $j \in \{1, \ldots, m\}$, $\delta > 0$, $k \in \mathbb{N}$ and $\mu > 0$ such that

$$\Big|\log g_I^{j,k}(x_0)\Big|^{2s} \leqslant (2\varepsilon)^{2s}|I|^{-2} \quad \text{ if } I \subset (-\mu,\mu) \text{ and } g_I^{j,k}(x_0) < \delta.$$

Take the new local coordinates $x = (x_1, x')$ in a neighborhood of x_0 such that $x_0 = (0,0)$ and the line $x' =$ constant vector in $\mathbb{R}^{d-1}$ is the integral curve of X_j starting from $(0,x')$. Since $G(x;k)$ is continuous, we have

$$\Big|\log g_I^{j,k}(0,x')\Big|^{2s} \leqslant (4\varepsilon)^{2s}|I|^{-2} \quad \text{ if } I \subset (-\mu,\mu)\, |x'| < \mu \text{ and } g_I^{j,k}(0,x') < \delta$$

by taking other small $\mu, \delta > 0$ if necessary. For a moment we consider x' as parameters. Let λ be a large parameter satisfying $\lambda \geqslant 1/\delta$. If $g_I^{j,k}(0,x')\lambda < 1$ then we have $-\log g_I^{j,k}(0,x') \geqslant \log \lambda$ and hence

$$(\log \lambda)^{2s} \leqslant (4\varepsilon)^{2s}(|I|^{-2} + g_I^{j,k}(0,x')\lambda^2) \text{ for } \forall\, I \subset (-\mu,\mu). \tag{5.7}$$

When $g_I^{j,k}(0,x')\lambda \geqslant 1$, this is also true for $\lambda \geqslant \lambda_0$ if λ_0 is chosen sufficiently large, depending on ε. By means of the following lemma of Sawyer, we see that (5.7) implies

$$\int (\log \lambda)^{2s}|v(t)|^2 dt \leqslant C_0 \varepsilon^{2s} \int (|D_t v(t)|^2 + G(t,x';k)\lambda^2 |v(t)|^2) dt, \tag{5.8}$$

for all $v(t) \in C_0^1((-\mu,\mu))$, where $C_0 > 0$ is a constant independent of ε.

***Sawyer's lemma* (see Remark 5 in [18])**. — *Let I_0 be an open interval in $\mathbb{R}_x^1$ and let $V(t), W(t) \geqslant 0$ belong to $L^1_{\mathrm{loc}}(I_0)$. Then we have the estimate*

$$\int_{I_0} V(t)|v(t)|^2 dt \leqslant C \int_{I_0} \big(W(t)|v(t)|^2 + |v'(t)|^2\big)\, dt \tag{5.9}$$

for all $v \in C_0^1(I_0)$ with a constant $C > 0$ if and only if

$$V_I \leqslant A\{3W_{3I} + 2|I|^{-2}\} \text{ for any interval } I \text{ with } 3I \subset I_0. \tag{5.10}$$

holds with a constant $A > 0$. Moreover, if C and A are the best constants (5.9) and (5.10) then $A < C < 100A$. Here $3I$ denotes the interval with the same center as I but with length three times.

In fact, if we set $V(t) = (\log\lambda)^{2s}$ and $W(t) = g(t;j,k,(0,x'))\lambda^2 = G(t,x';k)\lambda^2$, it is obvious that (5.8) follows from (5.7) if we replace $3I$ by I. It is well-known that

$$\sum_{|J|\leqslant k} \|\Lambda^{\sigma-1} X_J u\|^2 \leqslant C\{(\Delta_X u, u) + \|u\|^2\} \tag{5.11}$$

for some $0 < \sigma = \sigma(k) \leqslant 1/2$. If we set

$$q(x_1, x', \xi') = \Big(\sum_{|J|\leqslant k} |X_J(x,\xi)|^2 |\xi|^{-2+2\sigma}\Big)\Big|_{\xi_1=0},$$

in our local coordinates near x_0, then we have $q(x_1,x',\xi') - G(x;k) \geqslant 0$ on $\xi' \in \mathbb{S}^{d-2}$ and

$$\|D_t u\|^2 + (q^w(t,x',D')u,u) \leqslant C\{(\Delta_X u,u) + \|u\|^2\},$$

where q^w denotes the pseudo-differential operator of Weyl symbol in $\mathbb{R}^{d-1}_{x'}$. If $\widetilde{q}(t,x',\xi') = q(t,x',\xi')|\xi'|^{-2\sigma}$, then in view of the Littlewood-Paley decomposition in $\mathbb{R}^{d-1}_{\xi'}$ we may replace the second term by $(\widetilde{q}^w(t,x',D')\lambda^2 u,u)$, provided that the support of the partial Fourier transform of $u(t,x')$ with respect to x' is contained in $\{\lambda^{1/\sigma} \leqslant |\xi'| \leqslant 2\lambda^{1/\sigma}\}$. Though G is not smooth enough in general, the Wick approximation of $\widetilde{q}^w$ gives

$$(\widetilde{q}^w(t,x,D')\lambda^2 u,u) \geqslant (G(t,x';k)\lambda^2 u,u) - C\|u\|^2,$$

(see Proposition 2.1 of [**13**] and Proposition 1.1 of [**1**]). Hence (5.8) leads us to (5.6) for u with $\operatorname{supp} u$ contained in a small neighborhood of x_0. Finally, the usual covering argument shows (5.6) for the general u.

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GROUP VELOCITY AT SMOOTH POINTS OF HYPERBOLIC CHARACTERISTIC VARIETIES

by

Jeffrey Rauch

To my friend Jean-Michel Bony with best wishes and appreciation for what he has taught me of mathematics other things.

Abstract. — At a smooth point of the characteristic variety defined by a homogeneous hyperbolic polynomial, the tangent plane determines the group velocity. In this note an algebraic algorithm is derived for computing this tangent plane at a given point. This is interesting only where the differential of the polynomial vanishes.

***Résumé* (Vitesse de groupe aux points lisses de variétés caractéristiques hyperboliques)**
En un point lisse d'une variété caractéristique définie par un polynôme homogène hyperbolique, le plan tangent détermine la vitesse de groupe. Dans cet article, on en déduit un algorithme algébrique de calcul de ce plan tangent en un point donné. Il n'est intéressant que là où la différentielle du polynôme s'annule.

Suppose that $P(D)$ is a homogeneous hyperbolic polynomial of degree $m \geqslant 1$ with time-like covector θ. Here $D = \partial/i\partial y$ with $y \in \mathbb{R}^n$. The symbol $P(\eta)$ is a homogeneous polynomial on $(\mathbb{R}^n)^*$. Hyperbolicity with respect to $\theta \in (\mathbb{R}^n)^*$ means that for any $\eta \in (\mathbb{R}^n)^*$ the equation

$$P(\eta + s\theta) = 0 \tag{1}$$

has only real roots s. In particular, $P(\theta) \neq 0$. Dividing P by $P(\theta)$ we may suppose that P has real coefficients ([**H**, Thm 8.7.3]).

The characteristic variety

$$\operatorname{Char} P := \{\eta \in \mathbb{R}^n \setminus 0 \,:\, P(\eta) = 0\}$$

is a conic real algebraic variety in $(\mathbb{R}^n)^*$. Since the equation (1) has m complex roots (counting multiplicity), and they all are real, it follows that every real line $\eta + s\theta$ intersects the variety in at least one point and no more than m points which shows

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that the variety has codimension 1 in $(\mathbb{R}^n)^*$. The fundamental stratification of real algebraic geometry (see [**BR**]) asserts that except for a set of codimension at least 2, the variety consists of smooth points, that is points where locally the variety is equal to the zero set of a real analytic function with nonvanishing gradient.

Definitions. — If $\underline{\eta} \neq 0$ is a point of the characteristic variety then $Q_{\underline{\eta}}(\eta)$ is the homogeneous polynomial of degree $k \geqslant 1$ which is the leading term in the expansion of $P(\underline{\eta} + \eta)$ about $\underline{\eta}$,

$$P(\underline{\eta} + \eta) = Q_{\underline{\eta}}(\eta) + \text{higher order terms in } \eta, \qquad Q_{\underline{\eta}} \neq 0.$$

At a smooth point $\underline{\eta}$, the annihilator of the tangent space $T_{\underline{\eta}}(\operatorname{Char} P)$ is a one dimensional linear subspace $L_{\underline{\eta}} \in \big(T_{\underline{\eta}}(\operatorname{Char} P)\big)^* = \mathbb{R}^n$. The lines in $\mathbb{R}^n$ parallel to $L_{\underline{\eta}}$ are those moving with the *group velocity* (see [**AR**]).

This velocity describes the propagation of wave packets, pulses, and singularities associated with the frequencies $(\mathbb{R} \smallsetminus 0)\,\underline{\eta}$.

For variable coefficient operators, the above computations are performed in the tangent and cotangent spaces at a fixed point and P is the principal symbol at that point. They are pertinent for example for symmetric hyperbolic systems and points of the characteristic variety which are microlocally of constant multiplicity.

If $\underline{\eta} \in \operatorname{Char} P$ is a smooth point of multiplicity one, that is $P(\underline{\eta}) = 0$ and $dP(\underline{\eta}) \neq 0$, then $dP(\underline{\eta})$ is a basis for $L_{\underline{\eta}}$ and one has a simple way of recovering the velocity from the symbol.

In an analogous way, at a smooth point one can write the variety locally as $q = 0$ with $dq \neq 0$, then $dq(\underline{\eta})$ is a basis for $L_{\underline{\eta}}$. However, there is no algebraic algorithm to find a function q starting from the defining function P when the roots have multiplicity greater than one. The following two results provide a straightforward algorithm to compute the group velocity for hyperbolic operators. In a subsequent article, [**MR**], it is shown, by an independent calculation, that there are algebraic formulas yielding the entire germ of $\operatorname{Char} P$ at $\underline{\eta}$.

Theorem. — *If $\underline{\eta}$ is a smooth point of the characteristic variety and $Q_{\underline{\eta}}$ is as above, then there is a real linear form $\ell(\eta)$ so that the tangent plane at $\underline{\eta}$ to the characteristic variety of P is equal to $\{\ell(\eta - \underline{\eta}) = 0\}$, and, $Q_{\underline{\eta}}(\eta) = \operatorname{sign}(Q(\theta))\, \ell(\eta)^{\deg Q}$.*

Corollary. — *If $\underline{\eta}$ is a smooth point of the characteristic variety and $Q_{\underline{\eta}}$ and $L_{\underline{\eta}}$ are as above, then for all η which are not in the characteristic variety of $Q_{\underline{\eta}}$ (e.g. $\eta = \theta$), $dQ_{\underline{\eta}}(\eta)$ is a basis for $L_{\underline{\eta}}$.*

These proofs rely on the fundamental theorems concerning Local Hyperbolicity (see [**G**]). That theory is closely related to the ideas of microhyperbolicity introduced by Bony and Shapira in [**BS**] (see [**H**, § 8.7]).

Examples. — Consider $P(\eta_1,\eta_2)=(\eta_1^2-\eta_2^2)^2$, the square of the wave operator with time-like $\theta=(1,0)$. At the smooth point $\underline{\eta}=(1,1)$, the conormal is easy to compute directly by considering the reduced equation $\eta_1^2-\eta_2^2=0$. But, illustrating the above results, compute

$$P(\underline{\eta}+\eta)=\Big((1+\eta_1)^2-(1+\eta_2)^2\Big)^2=\Big(\eta_1^2+2\eta_1-\eta_2^2-2\eta_2\Big)^2=(2\eta_1-2\eta_2)^2+\text{ h.o.t.}$$

Thus $\ell=2\eta_1-2\eta_2$, $Q_{\underline{\eta}}=\ell^2$, $L_{\underline{\eta}}$ is generated by $(2,-2)$, and the group velocity is equal to -1.

The above example, the examples from [**H**] and [**C**], and the examples from mathematical physics that I know, all have the property that for any smooth point $\underline{\eta}$ there is an explicit hyperbolic factor of the symbol vanishing at $\underline{\eta}$ and with nonvanishing gradient. For those examples an appeal to the above algorithm can be avoided.

The proof of the Theorem begins with the fact from [**G**] that $Q_{\underline{\eta}}(\eta)$ is hyperbolic with time-like covector θ. Then for every real η the equation

$$Q_{\underline{\eta}}(\eta+s\theta)=0 \tag{2}$$

has only real roots s.

Lemma 1. — *For every real η the equation (2) has exactly one root s.*

Proof. — With $k:=$ the degree of $Q_{\underline{\eta}}$, one has as $\varepsilon\to 0$,

$$\varepsilon^{-k}P(\underline{\eta}+\varepsilon(\eta+s\theta))=Q_{\underline{\eta}}(\eta+s\theta)+O(\varepsilon). \tag{3}$$

If (2) had two roots s_1 and s_2, then Rouché's theorem would imply that the characteristic variety of P would have points near $\underline{\eta}+\varepsilon(\eta+s_j\theta)$ as $\varepsilon\to 0$ violating the smooth variety hypothesis. ☐

The next Lemma is then applied to $R=Q_{\underline{\eta}}$.

Lemma 2. — *If $R(\eta)$ is a homogeneous real polynomial hyperbolic with respect to the time-like covector θ and for all real η the equation $R(\eta+s\theta)=0$ has exactly one real root s, then there is a real linear form $\ell(\eta)$ such that*

$$R(\eta)=\operatorname{sign}(R(\theta))\,\ell(\eta)^{\deg R}.$$

Proof. — Introduce coordinates $(\tau,\xi_1,\dots,\xi_{n-1})$ in $(\mathbb{R}^n)^*$ so that $\theta=(1,0,\dots,0)$. Then

$$R(\tau,\xi)=R(1,0,\dots,0)\big(\tau^k+a_1(\xi)\tau^{k-1}+\cdots+a_{k-1}(\xi)\tau+a_k(\xi)\big)$$

with $a_j(\xi)$ homogeneous of degree j and $k=\deg R\geqslant 1$.

By hypothesis, for each real ξ the equation $R(\tau,\xi)=0$ has a unique root $\tau=r(\xi)$. Therefore

$$R(\tau,\xi)=R(1,0,\dots,0)\big(\tau-r(\xi)\big)^k.$$

Equating coefficients of τ^{k-1} shows that

$$-k\, r(\xi) = a_1(\xi),$$

so $r(\xi)$ is a homogeneous polynomial of degree 1. The Lemma follows with $\ell(\tau,\xi) = c(\tau - r(\xi))$ provided that c is chosen as

$$c := |R(1,0,\dots,0)|^{1/k} \quad \text{so } c^k = \operatorname{sign}(R(1,0,\dots,0))\, R(1,0,\dots,0). \qquad \square$$

Proof of Theorem. — Combining the above lemmas implies that

$$Q_{\underline{\eta}}(\eta) = \operatorname{sign}(Q(\theta))\, \ell(\eta)^k.$$

It remains to show that the tangent plane to the characteristic variety of P is given by the equation $\ell(\eta - \underline{\eta}) = 0$.

Use the local coordinates (τ,ξ) from the proof of Lemma 2. Since $\theta = (1,0,\dots,0)$ is noncharacteristic for P, the variety of P is given by the roots τ of $P(\tau,\xi) = 0$ with ξ ranging over $\mathbb{R}^n \setminus 0$.

The points near $\underline{\eta} = (\underline{\tau},\underline{\xi})$ are then given by the roots τ of

$$P(\underline{\tau} + \varepsilon\tau, \underline{\xi} + \varepsilon\xi) = 0, \tag{4}$$

with $|\xi| \leqslant 1$. Equation (2) takes the form

$$\varepsilon^{-k}\, P(\underline{\tau} + \varepsilon\tau, \underline{\xi} + \varepsilon\xi) = Q_{\underline{\eta}}(\tau,\xi) + O(\varepsilon). \tag{5}$$

Since $Q_{\underline{\eta}} = \ell^k$, the equation $Q_{\underline{\eta}}(\tau,\xi) = 0$ is equivalent to the equation $\ell(\tau,\xi) = 0$. Since $\ell(\theta)^k = Q_{\underline{\eta}}(\theta) \neq 0$, it follows that the solutions of $\ell(\tau,\xi) = 0$ are given by $\tau = \underline{x}\cdot\xi$ for an appropriate $\underline{x}$.

Rouché's Theorem applied to (5) shows that for $|\xi| < 1$ the roots of (4) are given by

$$\tau = \underline{x}\cdot\xi + O(\varepsilon).$$

The corresponding points $\eta = (\underline{\tau} + \varepsilon\tau, \underline{\xi} + \varepsilon\xi)$ of the characteristic variety of P differ from $\underline{\eta}$ by $O(\varepsilon)$ and satisfy

$$\ell(\eta - \underline{\eta}) = O(\varepsilon^2).$$

This shows that the equation of the tangent plane is $\ell(\eta - \underline{\eta}) = 0$. $\square$

Proof of Corollary. — Since $Q_{\underline{\eta}} = \pm \ell^k$ one has

$$dQ_{\underline{\eta}}(\eta) = \pm\, k\, \ell(\eta)^{k-1}\, d\ell(\eta).$$

Since ℓ is a linear form on $(\mathbb{R}^n)^*$, $d\ell(\eta)$ is a vector which does not depend on η. The Theorem implies that $d\ell$ is a basis for $L_{\underline{\eta}}$. Therefore, $dQ_{\underline{\eta}}(\eta)$ is a basis whenever it is nonvanishing. This holds exactly for η which satisfy $\ell(\eta) \neq 0$ which are exactly those η which are not in the characteristic variety of $Q_{\underline{\eta}}$. $\square$

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DISCRIMINATION ANALYTIQUE DES DIFFÉOMORPHISMES RÉSONNANTS DE $(\mathbb{C}, 0)$ ET RÉFLEXION DE SCHWARZ

par

Jean-Marie Trépreau

à Jean-Michel Bony,
pour son 60^e anniversaire

Résumé. — Nous montrons que des arguments géométriques très simples, basés sur la réflexion de Schwarz, permettent souvent de décider si deux paires d'arcs analytiques tangents en $0 \in \mathbb{C}$ sont analytiquement équivalentes au voisinage de 0. Nous en déduisons la construction de familles nombreuses de germes, formellement mais non analytiquement conjugués, de difféomorphismes analytiques résonnants de $(\mathbb{C}, 0)$.

***Abstract* (Resonant diffeomorphisms of $(\mathbb{C}, 0)$ and the Schwarz reflection).** — We show that simple geometric arguments, based on the Schwarz reflection, allow in many cases to decide whether two pairs of tangent analytic arcs at $0 \in \mathbb{C}$ are conformally equivalent in a small neighborhood of 0. As an application, we exhibit big families of germs of analytic resonant diffeomorphisms of $(\mathbb{C}, 0)$, which are formally, but not analytically conjugate.

Introduction

Paires d'arcs analytiques tangents...— On s'intéresse à la classification analytique des paires (A, B), où A et B sont deux germes, *tangents* en $0 \in \mathbb{C}$, d'arcs analytiques réguliers.

Deux telles paires, soit (A, B) et (C, D), sont *équivalentes* s'il existe un germe en 0 de difféomorphisme holomorphe ϕ (on dira plutôt *un difféomorphisme analytique de* $(\mathbb{C}, 0)$), tel que :

$$\phi(A) = C, \quad \phi(B) = D.$$

Elles sont *formellement équivalentes* si, pour tout $n \in \mathbb{N}$, il existe un difféomorphisme analytique ϕ_n de $(\mathbb{C}, 0)$, tel que $\phi_n(A)$ et $\phi_n(B)$ soient respectivement tangents à C et à D, à des ordres $\geqslant n$.

***Classification mathématique par sujets* (2000).** — 30 C 35, 30 D 05, 37 F 99.
Mots clefs. — Réflexion de Schwarz, transformations conformes, paires d'arcs analytiques.

Notons S_A la réflexion de Schwarz par rapport à un arc analytique régulier A. À la paire (A, B), on associe son *indicateur* :

$$f = S_A \circ S_B.$$

C'est un difféomorphisme analytique de $(\mathbb{C}, 0)$, tangent à l'identité. Il est évident que les indicateurs de deux paires équivalentes sont (analytiquement) conjugués.

La classification formelle des paires a été faite par Kasner [**15**]. Il apparaît que les paires (A, B) dont l'indicateur appartient à une classe de conjugaison formelle donnée, s'il en existe, se répartissent en une ou deux classes d'équivalence formelle, selon que l'ordre de contact entre les deux arcs A et B est impair ou pair.

Nakai [**19**] a montré que les paires analytiques, dont l'indicateur appartient à une classe de conjugaison analytique donnée, se répartissent en une, deux, ou quatre classes d'équivalence analytique, selon les cas.

...et difféomorphismes tangents à l'identité. — Un difféomorphisme analytique de $(\mathbb{C}, 0)$, tangent à l'identité, possède deux invariants formels, l'ordre de tangence et un nombre complexe, son *résidu*. Dans [**6**], publié en 1939, Birkhoff montre que deux germes, formellement conjugués, ne sont pas (analytiquement) conjugués en général. Mieux, il construit un système complet d'invariants analytiques, en associant à tout difféomorphisme analytique f de $(\mathbb{C}, 0)$, tangent à l'identité, un couple $(g, h) \in \mathcal{G} \times \mathcal{H}$ de fonctions, qu'il appelle les « fonctions de connexion » du germe f ; $\mathcal{G}$ et $\mathcal{H}$ sont des espaces parfaitement définis de fonctions holomorphes. Birkhoff montre que deux germes sont conjugués si et seulement s'ils ont les mêmes fonctions de connexion. Il pose aussi le problème de la synthèse : est-ce que tout $(g, h) \in \mathcal{G} \times \mathcal{H}$ est le couple des fonctions de connexion d'un difféomorphisme analytique de $(\mathbb{C}, 0)$, tangent à l'identité ? Cet article a été oublié [(1)].

Écalle [**12**], Malgrange [**18**] et Voronin [**27**] ont retrouvé les résultats de Birkhoff et résolu le problème de la synthèse. L'usage aujourd'hui est de parler des *invariants d'Écalle-Voronin.*

La classification analytique des germes de difféomorphismes tangents à l'identité est donc un problème résolu. Compte tenu de la relation entre paires d'arcs analytiques tangents et difféomorphismes, la classification analytique de ces paires trouve aussi sa solution.

Une construction géométrique d'invariants analytiques. — Notre point de vue, dans cet article, est différent. Rappelons d'abord que les invariants d'Écalle-Voronin ne se laissent pas facilement calculer !

Nous pensons démontrer qu'on gagne, pour étudier une paire d'arcs tangents, à ne pas se ramener à l'étude de son indicateur ; et même, que pour trouver des invariants

(1) Jean-Pierre Ramis a retrouvé « par hasard » cet article fondamental en 1995. Je le remercie de m'avoir donné des détails sur cette découverte.

analytiques d'un difféomorphisme, *qui soit un indicateur de paire*, on a avantage à étudier la paire. Mentionnons que les indicateurs de paires sont rares parmi les difféomorphismes tangents à l'identité : il y a des obstructions formelles (un indicateur de paire a un résidu imaginaire pur), mais surtout des obstructions analytiques [**19**] qui s'écrivent en terme des invariants d'Écalle-Voronin.

Venons-en à la description de la méthode (la méthode de la réflexion) que nous proposons. Soit (A,B) une paire d'arcs tangents en $0 \in \mathbb{C}$, soit f son indicateur. Il s'agit de trouver des invariants analytiques de la paire (A,B) qui, éventuellement, permettent de montrer qu'elle n'est pas équivalente à une autre paire donnée.

Concernant le difféomorphisme f de $(\mathbb{C},0)$, tangent à l'identité, nous utilisons le fait bien connu qu'il existe un voisinage épointé de 0 qui est recouvert par la réunion des bassins d'attraction de 0, respectivement pour f et f^{-1}. Soit Δ un domaine tel que $0 \in \partial\Delta$, que f soit définie et injective sur Δ et que $f(\Delta) \subset \Delta$.

Pour tout $n \geqslant 0$, on définit les arcs analytiques A_n et B_n en prolongeant analytiquement, autant qu'il est possible dans $\Delta \cup \{0\}$, les germes d'arcs $f^{(n)}(A)$ [(2)] et $f^{(n)}(B)$. Alors, tout événement géométrique (une intersection, un point singulier...), dont la définition fait intervenir la famille des courbes A_n, B_n, $n \geqslant 0$, donne lieu à un événement du même type dans $f(\Delta)$, à la translation près des indices. Tout événement de ce type est donc *asymptotique* ; il concerne en fait les germes de A et B en 0. Il peut permettre la discrimination analytique des paires.

Un exemple. — La Figure 1 devrait illustrer le caractère « évident » de la méthode de la réflexion. Chaque sous-figure représente une ellipse A, une tangente B à cette ellipse et le reflet B_1 de B par A. Dans (a), A est un cercle ; la figure obtenue est bien classique. Dans (b)–(f), A n'est pas un cercle. Dans (b) et (f), le point de contact est un des sommets de A. Les trois paires (a), (b) et (f) sont formellement équivalentes. Les deux paires (c) et (e) le sont aussi. On montre facilement que, *si l'excentricité de l'ellipse A est assez petite (par exemple, voir Lemme 5.8, si elle est inférieure à* $0,55$), la figure de l'ellipse et de l'intersection de l'ellipse pleine avec B_1 est asymptotique.

La différence entre (a), (b) et (f) est spectaculaire ; celle entre (c) et (e) aussi. *À l'œil*, on obtient le fait que les trois paires (a), (b) et (f) appartiennent à des classes analytiques distinctes.

Il y a mieux : l'angle au point double de B_1 dans (d)–(f), les angles aux intersections de B_1 avec A dans (b)–(f), sont aussi des invariants des germes (A,B). On peut ainsi obtenir, *à l'œil et au compas,* la discrimination entre des paires de ce type, obtenues en faisant varier l'excentricité, choisie assez petite, et le point de contact de la tangente. La Figure 2 donne un exemple de cela.

(2) Dans cet article, $f^{(n)}$ désignera toujours l'itéré de f à l'ordre n ; par exemple $f^{(2)} = f \circ f$... On notera souvent f^{-1} au lieu de $f^{(-1)}$

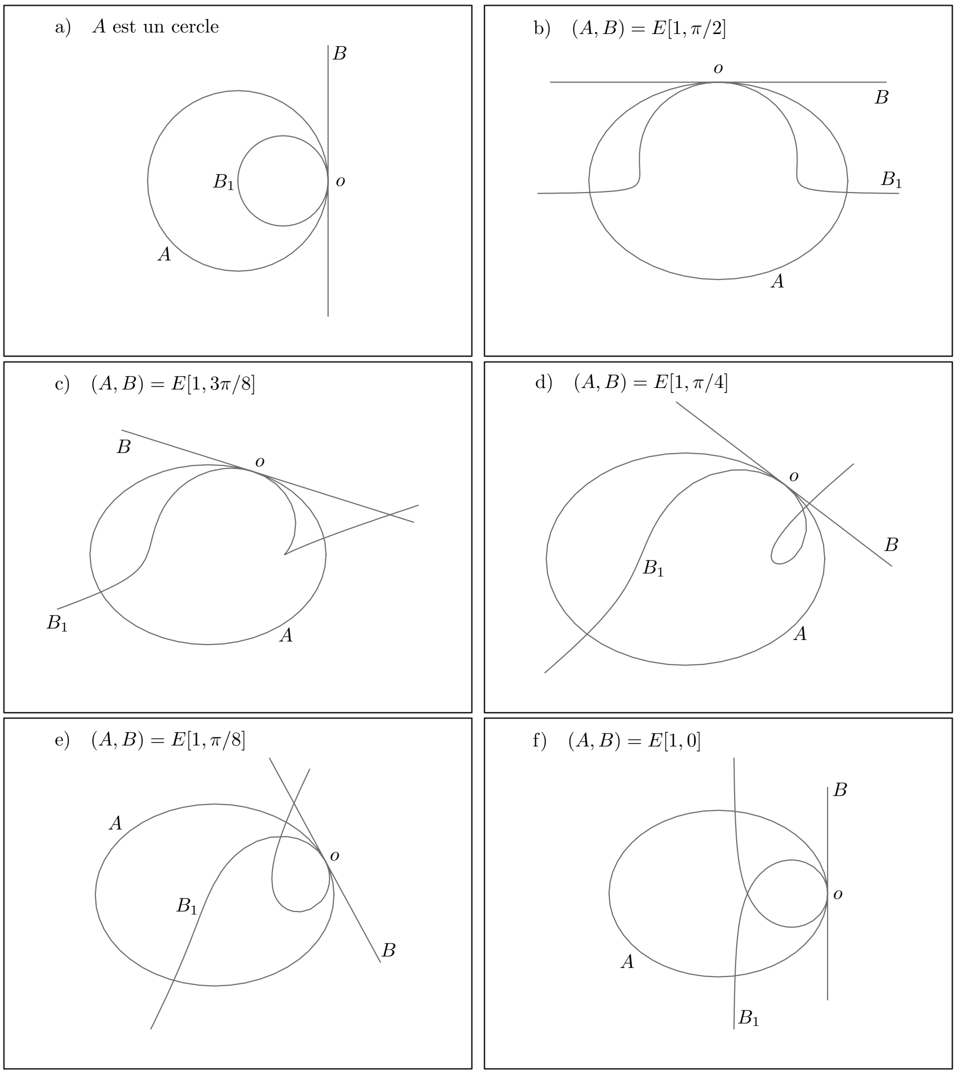

FIGURE 1. Le reflet d'une tangente par une ellipse

Contenu et plan de l'article. — Le cœur de l'article est constitué des Chapitres 3, 4 et 5, dans lesquels la méthode de la réflexion est appliquée. Ils peuvent être parcourus, pour la plus grande part, indépendamment du Chapitre 2. Pour simplifier, nous ne considérons dans ces chapitres que des paires d'arcs tangents à l'ordre 1, appelées *paires de type* 2, mais la méthode s'applique à des ordres de contact plus grands, avec quelques complications dues à la forme du bassin d'attraction du point

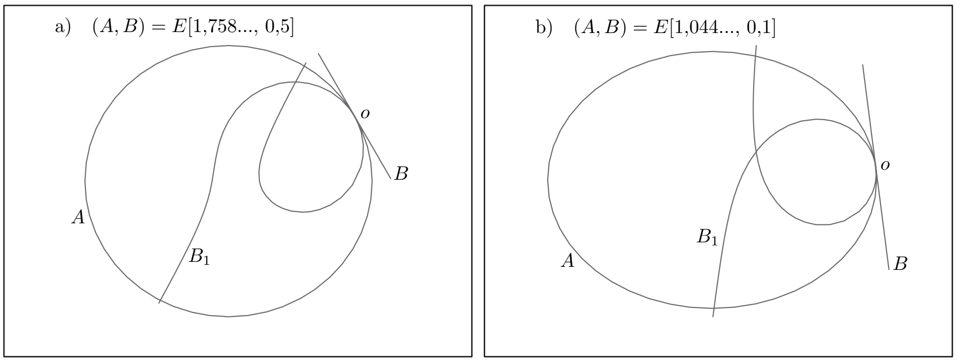

FIGURE 2. Deux paires formellement équivalentes qu'on discrimine au compas

fixe de l'indicateur quand l'ordre de contact est plus grand que 2 (la fleur de Leau a alors plus de pétales).

Dans le Chapitre 1, nous rappelons la définition de la réflexion de Schwarz et nous posons le problème des paires d'arcs. Nous introduisons l'indicateur d'une paire et la suite des paires engendrées.

Dans le Chapitre 2, nous rappelons la relation entre la classification des paires d'arcs et celle des difféomorphismes, dans le cas plus simple des paires de type 2.

Dans le Chapitre 3, nous appliquons la méthode de la réflexion à des paires d'arcs tangents qui sont des perturbations, en un sens qu'on précisera, d'une paire de cercles tangents. Dans le § 3.3, nous montrerons (avec les notations introduites plus haut) que :

Si une paire (A, B) est équivalente à une paire de cercles tangents, il existe un $n \in \mathbb{N}$ tel que

– A_n et B_n sont des courbes analytiques compactes lisses ;
– un isomorphisme (de Riemann) du domaine de bord A_n sur le disque unité envoie localement B_n sur un arc de cercle ou de droite.

Nous construirons aussi (voir le Théorème 3.2 et le Corollaire 3.4) une famille nombreuse de paires d'arcs, formellement équivalentes, deux à deux non équivalentes. Cette famille est paramétrée par l'ensemble des fonctions holomorphes et injectives sur le disque unité qui s'annulent à l'ordre 3 en 0. Birkhoff [**6**] et Elizarov [**13**] exhibent des familles à un paramètre de difféomorphismes non conjugués. Notre construction est tout à fait différente et très élémentaire.

Dans le Chapitre 4, nous esquissons une description des invariants géométriques que la méthode de la réflexion permet, dans les cas favorables, d'obtenir. Je ne sais pas dans quelle mesure les cas favorables sont « le cas général ». Par exemple, *peut-on toujours associer, par réflexions, une « figure asymptotique » à une paire d'arcs algébriques ?* Je n'en sais rien.

Dans le Chapitre 5, nous énonçons et nous commentons le théorème suivant :

Soit B et B' des droites tangentes aux ellipses A et A' respectivement. Si les paires (A, B) et (A', B'), considérées au voisinage des points de contact, sont équivalentes, elles sont semblables, i.e. on peut passer de l'une à l'autre par une similitude. En particulier, les deux ellipses ont la même excentricité.

Nous ne démontrons pas ce théorème, mais nous en présentons des cas particuliers.

Avec le souci d'être complet, nous rappelons dans l'Appendice la classification formelle des paires d'arcs analytiques dans le cas général, due à Kasner [**15**] et Pfeiffer [**22**], et la relation entre leur classification analytique et celle des difféomorphismes, étudiée par Nakai [**19**].

Les figures : légendes et notations. — Les figures de ce texte ont été obtenues en utilisant le logiciel Maple, ainsi que le logiciel CorelDraw pour certaines insertions de texte.

La plupart de ces figures concernent des paires (A, B), où B est une tangente à une conique A. On utilise les notations suivantes :

– $E[s, t]$ désigne (à une similitude près) la paire (A, B), où A est l'ellipse de foyers les points ± 1 qui passe par le point $\cosh(s + it)$, et où B est sa tangente en ce point. L'excentricité de l'ellipse A (rappelons que c'est le rapport entre la distance entre les foyers et la longueur du grand axe) vaut $1/\cosh s$. On peut montrer que le nombre $\sin 2t/\sinh 2s$ caractérise la classe formelle de la paire $E[s, t]$.

– $P[t]$ désigne la paire (A, B), où A est la parabole d'équation paramétrique $z(t) = (1 - t^2)/2 + it$ et B sa tangente au point $z(t)$;

– on note A_n et B_n les germes $(S_A \circ S_B)^{(n)}(A)$ et $(S_A \circ S_B)^{(n)}(B)$, ainsi que leurs prolongements analytiques ;

– les notations $A_n^{\pm}$ et $B_n^{\pm}$ sont introduites dans le § 4.1.

Remerciements. — Quand j'ai voulu connaître le reflet d'une droite par une ellipse, j'ai été rebuté par le calcul à faire, pourtant facile ! Je veux remercier Pierre-Vincent Koseleff qui m'a aidé à en obtenir sur Maple la première représentation graphique. Je tiens aussi à remercier Frank Loray, qui m'a appris l'existence de l'article de Nakai [**19**] ; sans cette information, cet article serait réduit à très peu de chose. J'ai aussi tiré profit de la lecture de ses notes de cours [**17**].

1. Paires d'arcs analytiques et réflexions

1.1. Notations. — Pour la discussion générale, il est commode de fixer un point de base ; on choisit $0 \in \mathbb{C}$ comme point de base.

On note **G** le groupe des difféomorphismes analytiques de $(\mathbb{C}, 0)$, c'est-à-dire des séries convergentes de la forme :

$$f(z) = a_1 z + \sum_{n=2}^{+\infty} a_n z^n \qquad (a_1 \neq 0). \tag{1}$$

Deux difféomorphismes f, $g \in \mathbf{G}$ sont conjugués s'il existe $h \in \mathbf{G}$ tel que :

$$f \circ h = h \circ g.$$

Le centralisateur $C(f)$ de $f \in \mathbf{G}$ est défini par :

$$C(f) = \{h \in \mathbf{G}, \quad f \circ h = h \circ f\}.$$

On appellera *réflexion* toute involution antiholomorphe, considérée au voisinage d'un point fixe. On note **S** l'ensemble des réflexions de $(\mathbb{C}, 0)$, c'est-à-dire des séries convergentes de la forme :

$$S(z) = s_1 \overline{z} + \sum_{n=2}^{+\infty} s_n \overline{z}^n, \tag{2}$$

telles que $S \circ S = I$, l'identité.

1.2. La réflexion de Schwarz. — La conjugaison complexe :

$$S_{\mathbb{R}}(z) = \overline{z}$$

est une réflexion dont $\mathbb{R}$ est l'ensemble des points fixes. C'est la seule transformation antiholomorphe qui ait cette propriété, même localement. En effet, si S est antiholomorphe au voisinage d'un point de $\mathbb{R}$ et laisse fixe tout point de $\mathbb{R}$, $S_{\mathbb{R}} \circ S$ est holomorphe et laisse fixe tout point de $\mathbb{R}$, donc $S_{\mathbb{R}} \circ S$ est l'identité, donc $S = S_{\mathbb{R}}$.

Soit maintenant C une courbe analytique régulière. Si $0 \in C$, en complexifiant une paramétrisation analytique locale :

$$\phi(t) = a_1 t + \sum_{n=2}^{+\infty} a_n t^n, \qquad (a_1 \neq 0),$$

de C, on obtient un élément de **G**,

$$\phi(z) = a_1 z + \sum_{n=2}^{+\infty} a_n z^n,$$

qui envoie localement $\mathbb{R}$ sur C. Alors :

$$S_C = \phi \circ S_{\mathbb{R}} \circ \phi^{-1}$$

est la seule réflexion qui induit l'identité sur C au voisinage de 0. En faisant varier le point de base, on obtient l'existence de la réflexion de Schwarz :

Définition 1.1. — Soit $C \subset \mathbb{C}$ une courbe analytique régulière. La réflexion de Schwarz par rapport à la courbe C est la seule involution antiholomorphe, ou *réflexion*, définie au voisinage de C, qui induit l'identité sur C. On la note S_C.

1.3. Le cas d'une courbe définie implicitement. — Si la courbe C est définie implicitement près de $0 \in C$ par

$$\sum_{m,n=0}^{+\infty} a_{m,n} z^m \overline{z}^n = 0, \qquad (a_{m,n} = \overline{a_{n,m}}, \quad a_{0,0} = 0, \quad a_{1,0} \neq 0), \tag{3}$$

l'équation :

$$\sum_{m,n=0}^{+\infty} a_{m,n} w^m \overline{z}^n = 0 \tag{4}$$

peut être résolue par rapport à w, et définit une fonction antiholomorphe

$$S(z) = s_1 \overline{z} + \sum_{n=2}^{+\infty} s_n \overline{z}^n, \qquad (s_1 \neq 0),$$

au voisinage de 0. Comme (3) est une équation de la courbe C, on a $S(z) = z$ pour tout $z \in C$, donc S est la réflexion de Schwarz par rapport à C.

Si C est une courbe algébrique, (4) montre que la réflexion S_C est une fonction algébrique de $\overline{z}$.

1.4. Covariance de la réflexion de Schwarz. — À tout germe C d'arc analytique régulier en 0, on a associé une réflexion $S_C \in \mathbf{S}$. La correspondance est biunivoque, par exemple parce que C est l'ensemble des points fixes de S_C. Il résulte aussi immédiatement de la définition qu'on a, pour tout $\phi \in \mathbf{G}$ et pour tout $\phi \in \mathbf{S}$:

$$S_{\phi(C)} = \phi \circ S_C \circ \phi^{-1}. \tag{5}$$

D'autre part, si la réflexion $S \in \mathbf{S}$ est donnée par (2), de $S(S(z)) \equiv z$, on tire $|s_1| = 1$. Soit u une racine carrée de s_1. La rotation $z \mapsto uz$ conjugue S avec :

$$S_1(z) := \frac{S(zu)}{u} = \overline{z} + \sum_{n=2}^{+\infty} b_n \overline{z}^n.$$

Si l'on pose $\phi(z) = z + S_{\mathbb{R}} \circ S_1(z)$, on obtient :

$$S_{\mathbb{R}} \circ \phi = \phi \circ S_1.$$

On en déduit que S est holomorphiquement conjuguée à $S_{\mathbb{R}}$. Si $\psi \in \mathbf{G}$ réalise cette conjugaison, $S = \psi \circ S_{\mathbb{R}} \circ \psi^{-1}$, on voit que S est la réflexion de Schwarz par rapport à l'arc $\psi(C)$. On a obtenu :

Lemme 1.2. — *La correspondance $C \mapsto S_C$ induit une bijection entre l'ensemble des germes d'arcs analytiques réguliers de $(\mathbb{C}, 0)$ et l'ensemble $\mathbf{S}$ des réflexions de $(\mathbb{C}, 0)$.*

1.5. Difféomorphisme, arc, conjugaison... formels. — On note $\widehat{\mathbf{G}}$ le groupe des *difféomorphismes formels* de $(\mathbb{C}, 0)$, c'est-à-dire des séries formelles de la forme (1). De même, une involution de la forme (2), sans condition de convergence, est appelée réflexion formelle et l'ensemble des réfexions formelles de $(\mathbb{C}, 0)$ est noté $\widehat{\mathbf{S}}$.

Les notions de conjugaison, de centralisateur, de germe d'arc régulier... ont leurs pendants dans le cadre formel. Les définitions sont évidentes. On note $\widehat{C}(f)$ le *centralisateur formel* de $f \in \widehat{\mathbf{G}}$. Le Lemme 1.2 est vrai dans le cas formel, avec la même démonstration.

1.6. Le problème des paires. — On appelle *paire*, respectivement *paire formelle*, tout couple (A, B) de germes en $0 \in \mathbb{C}$ d'arcs analytiques, respectivement formels, réguliers. Un difféomorphisme h agit sur les paires par :

$$(A, B) \longmapsto h(A, B) = (h(A), h(B)),$$

et sur les réflexions par conjugaison :

$$S \longmapsto h \circ S \circ h^{-1}.$$

Définition 1.3. — Deux paires (A, B) et (C, D) sont *équivalentes*, respectivement *formellement équivalentes*, s'il existe $h \in \mathbf{G}$, respectivement $h \in \widehat{\mathbf{G}}$, tel que $h(A, B) = (C, D)$.

Compte tenu des propriétés de la réflexion de Schwarz, le problème de la classification (analytique, formelle) des paires est équivalent au problème de la classification des paires de réflexions pour la conjugaison (analytique, formelle).

Dans la suite, on confondra la notion de paire d'arcs réguliers de $(\mathbb{C}, 0)$ *avec celle de paire de réflexions de* $(\mathbb{C}, 0)$.

1.7. L'indicateur d'une paire. — Ce qui est dit dans ce paragraphe vaut aussi dans le cas formel.

Définition 1.4. — *L'indicateur* d'une paire de réflexions $(S, S') \in \mathbf{S} \times \mathbf{S}$ est le difféomorphisme $S \circ S' \in \mathbf{G}$.

Si $f = S \circ S'$ est l'indicateur de la paire (S, S'), on a :

$$S' = S \circ f,$$

et :

$$S \circ f = f^{-1} \circ S. \tag{6}$$

Autrement dit, la réflexion S conjugue f à f^{-1}.

Lemme 1.5. — *Un difféomorphisme* $f \in \mathbf{G}$ *est l'indicateur d'une paire si et seulement s'il existe une réflexion* $S \in \mathbf{S}$ *vérifiant (6). Dans ce cas, l'application :*

$$S \longmapsto (S, S \circ f)$$

induit une bijection entre l'ensemble des réflexions $S \in \mathbf{S}$ qui vérifient (6) et l'ensemble des paires d'indicateur f.

Démonstration. — On a déjà vu que si f est l'indicateur de la paire (S, S'), S vérifie (6). Réciproquement, si $S \in \mathbf{S}$ vérifie (6), on a :

$$(S \circ f) \circ (S \circ f) = I,$$

donc $S \circ f$ est une réflexion et f est l'indicateur de la paire $(S, S \circ f)$. □

1.8. La suite des paires engendrées. — Ce qui est dit dans ce paragraphe vaut aussi dans le cas formel.

Définition 1.6. — Soit (A, B) une paire d'indicateur $f = S_A \circ S_B$. On appelle *suite des paires engendrées* par la paire (A, B) la famille de paires :

$$(A_n, B_n) := (f^{(n)}(A), f^{(n)}(B)), \qquad (n \in \mathbb{Z}). \tag{7}$$

Compte tenu de la covariance de la réflexion de Schwarz, on a :

Lemme 1.7. — *Soit $(u(A), u(B))$ l'image d'une paire (A, B) par un élément u de $\mathbf{G}$. Pour tout $n \in \mathbb{Z}$, on a :*

$$(u(A)_n, u(B)_n) = (u(A_n), u(B_n)).$$

Rappelons encore la formule (5). Comme l'indicateur f de la paire (A, B) « anticommute » avec S_A et avec S_B, on a $S_{A_n} \circ S_{B_n} = f$ pour tout $n \in \mathbb{Z}$. On a aussi $S_{S_A(B)} = S_A \circ S_B \circ S_A$. On en déduit :

Lemme 1.8. — *L'indicateur de la paire (A, B) est aussi l'indicateur de la paire $(S_A(B), A)$ et des paires (A_n, B_n), $n \in \mathbb{Z}$.*

1.9. Paires d'indicateur donné. — Ce qui est dit dans ce paragraphe vaut aussi dans le cas formel.

Soit $(S, S \circ f)$ une paire d'indicateur $f \in \mathbf{G}$. Si $h \in \mathbf{G}$, la paire équivalente $(h \circ S \circ h^{-1}, h \circ S \circ f \circ h^{-1})$ a pour indicateur $h \circ f \circ h^{-1}$. Elle a donc le même indicateur si et seulement si $h \in C(f)$.

D'autre part, si $(S', S' \circ f)$ est une autre paire d'indicateur f, $S \circ S'$ commute avec f, donc il existe $g \in C(f)$ tel que $S' = S \circ g$. De plus g doit vérifier $S \circ g = g^{-1} \circ S$. La réciproque est claire.

Ces remarques algébriques seront utilisées dans le Chapitre 2 et dans l'Appendice, quand il s'agira de classer les paires d'indicateur donné.

2. Classification des paires de type 2

Le but de ce chapitre est de mettre en perspective les résultats présentés dans les chapitres suivants, mais sa lecture n'est pas nécessaire à leur compréhension.

Une paire (A, B), éventuellement formelle, est dite *de type* $q+1$ si les arcs A et B sont tangents à l'ordre $q \geqslant 1$. Son indicateur, qui est tangent à l'ordre q à l'identité, est aussi dit *de type* $q+1$.

Nous allons rappeler quelques résultats classiques sur les difféomorphismes de type 2, et comment ils s'appliquent aux paires de type 2. Des rappels, dans un cadre plus général, seront faits dans l'Appendice.

2.1. La classification formelle. — En 1912, Kasner [**15**] a démontré le théorème suivant :

***Théorème 2.1* (Kasner).** — *Toute paire formelle de type* 2 *est formellement équivalente à une et une seule des paires* (A_l, B), $l \in \mathbb{R}$, *où :*

$$A_l = \{y = x^2 + lx^3\}; \qquad B = \{y = 0\}. \tag{8}$$

Un calcul formel donne facilement ce résultat. Plus bas, on le déduira, comme Nakai [**19**], de la classification formelle des difféomorphismes de type 2, qui est mieux connue, et d'intérêt plus général.

Calculons le développement à l'ordre 3 de l'indicateur $f_l = S_{A_l} \circ S_B$ de la paire (8). D'après le § 1.3, il est donné par $f_l(z) = w$, avec :

$$\frac{w-z}{2i} = \left(\frac{w+z}{2}\right)^2 + l\left(\frac{w+z}{2}\right)^3.$$

On en déduit $w = z + 2iz^2 + O(z^3)$, puis :

$$f_l(z) = z + 2iz^2 + (2i)^2(1 - i\frac{l}{2})z^3 + \cdots \tag{9}$$

2.2. Difféomorphismes formels de type 2. — Un élément $f \in \widehat{\mathbf{G}}$, de type 2, s'écrit :

$$f(z) = z + az^2 + a^2(1-r)z^3 + \cdots \qquad (a \neq 0). \tag{10}$$

Le nombre $r \in \mathbb{C}$ est invariant par conjugaison formelle et appelé le *résidu*(3) de f ; on note :

$$r = \text{rés}(f). \tag{11}$$

C'est le seul invariant formel :

Lemme 2.2. — *Deux éléments de type* 2 *de* $\widehat{\mathbf{G}}$ *sont formellement conjugués si et seulement s'ils ont le même résidu.*

(3) Le résidu d'Écalle est $r-1$. J'appelle résidu le *résidu normalisé* de [**19**]. L'avantage est qu'on a une formule simple pour $\text{rés}(f^{(t)})$.

On a :

$$\text{rés}(f^{(t)}) = \frac{\text{rés}(f)}{t}; \qquad \text{rés}(S \circ f \circ S) = \overline{\text{rés}(f)}, \tag{12}$$

pour tout $t \in \mathbb{Z}^*$ (et même $t \in \mathbb{C}^*$, voir ci-dessous) et pour tout $S \in \widehat{\mathbf{S}}$.

En particulier, si $S, S' \in \widehat{\mathbf{S}}$ et $f = S \circ S'$, on a $S \circ f \circ S = f^{(-1)}$ et les formules précédentes montrent que le résidu de f est imaginaire pur. Compte tenu du Lemme précédent et de (9), on a :

Lemme 2.3. — *Un élément de type* 2 *de* $\widehat{\mathbf{G}}$ *est un indicateur de paire formelle si et seulement si son résidu est imaginaire pur.*

Si $f(z) = z + az^2 + \cdots$, $a \neq 0$, un calcul formel simple montre que, pour tout $t \in \mathbb{C}$, il existe un et un seul difféomorphisme formel de la forme

$$z + taz^2 + \cdots$$

qui commute avec f. C'est *l'itéré d'ordre t de f* ; on le note $f^{(t)}$. Il coïncide avec l'itéré au sens usuel quand t est un entier.

En particulier, en choisissant $t = 0$, on obtient que l'identité est le seul élément de $\widehat{\mathbf{G}}$, tangent à un ordre $\geqslant 2$ à l'identité, qui commute avec f. Comme le commutateur de deux itérés de f a cette propriété, on obtient que l'application $t \mapsto f^{(t)}$ est un isomorphisme de $\mathbb{C}$ sur le groupe des itérés de f. Enfin, un calcul à l'ordre 3 montre que tout $g \in \widehat{C}(f)$ est tangent à l'identité, donc est un itéré de f :

Lemme 2.4. — *Si* $f \in \widehat{\mathbf{G}}$ *est de type* 2, *son centralisateur formel* $\widehat{C}(f)$ *est le groupe commutatif de ses itérés complexes* $f^{(t)}$, $t \in \mathbb{C}$.

On a aussi les formules suivantes :

$$h \circ f^{(t)} \circ h^{-1} = (h \circ f \circ h^{-1})^{(t)}; \qquad S \circ f^{(t)} \circ S = (S \circ f \circ S)^{(\overline{t})}, \tag{13}$$

pour tout $h \in \widehat{\mathbf{G}}$, pour tout $t \in \mathbb{C}$ et pour tout $S \in \widehat{\mathbf{S}}$.

Démonstration du Théorème de Kasner. — Il suffit de montrer que deux paires formelles de même indicateur f, soit $(S, S \circ f)$ et $(S', S' \circ f)$, sont formellement équivalentes. Compte tenu des remarques du § 1.9, on peut écrire $S' = S \circ f^{(t)}$, et (13) donne $S \circ f^{(t)} \circ S \circ f^{(t)} = f^{(t-\overline{t})}$, donc t est réel et $S' = f^{(-t/2)} \circ S \circ f^{(t/2)}$. □

La démonstration montre aussi qu'on a :

Lemme 2.5. — *Soit* $f \in \widehat{\mathbf{G}}$ *l'indicateur d'une paire formelle de type* 2. *Si t est imaginaire pur,* $f^{(t)}$ *conserve cette paire. Le groupe des itérés d'ordre réel de f opère transitivement et simplement sur l'ensemble des paires formelles d'indicateur f.*

Remarque 2.6. — En particulier, si f est l'indicateur de la paire formelle (A, B), on a :

$$A = f^{(1/2)}(B).$$

Si la paire est analytique, cette égalité est bien sûr correcte, bien que $f^{(1/2)}$ soit en général divergent.

2.3. Trois théorèmes classiques. — On les énonce dans le cas général des difféomorphismes tangents à l'identité, pour éviter des redites dans l'Appendice. On trouvera dans cet appendice la définition du résidu et de l'itération d'ordre complexe, dans ce cas général.

***Théorème 2.7* (Écalle, Liverpool)**. — *Si $f \in \mathbf{G}$ est tangent à un ordre fini à l'identité, le sous-groupe des $t \in \mathbb{C}$ tels que $f^{(t)} \in \widehat{\mathbf{G}}$ converge est soit égal à $\mathbb{C}$, soit de la forme $\mathbb{Z}\frac{1}{m}$ pour un $m \in \mathbb{N}^*$.*

Dans le premier cas, on dit que f est *pleinement itérable*, dans le second que m est *l'ordre maximal d'une racine itérative de f*. Le résultat suivant (je ne sais pas à qui l'attribuer) est aussi classique :

Théorème 2.8. — *Si deux éléments de $\mathbf{G}$, tangents à un ordre fini à l'identité, sont pleinement itérables et formellement conjugués, ils sont conjugués.*

Pour tout $q \geqslant 1$ et tout $r \in \mathbb{C}$, il existe un élément pleinement itérable $f_{q,r} \in \mathbf{G}$, de type $q + 1$ et de résidu r. Il est unique à conjugaison près d'après le théorème précédent. Des modèles sont déjà dans Birkhoff [**6**]. On peut prendre $f_{q,r} = \exp X_{q,r}$ où $X_{q,r}$ est le champ de vecteurs défini par :

$$X_{q,r} = iz^{q+1}(1 + irz^q)^{-1}\frac{d}{dz}. \tag{14}$$

On calcule facilement $f_{q,0}(z) = z(1 - iqz^q)^{-1/q}$. Si $r \neq 0$, (14) permet de trouver une équation vérifiée par $f_{q,r}$, mais il n'y a pas de modèle algébrique : le résultat suivant est dû à Baker [**4**] si $q = 1$, à Écalle [**10**] dans le cas général :

***Théorème 2.9* (Baker, Écalle)**. — *Soit $f \in \mathbf{G}$, tangent à un ordre fini à l'identité et de résidu non nul. Si f est le germe d'une fonction algébrique, f n'est pas pleinement itérable.*

2.4. Applications aux paires de type 2. — On a :

***Théorème 2.10* (Nakai)**. — *Soit (A, B) et (C, D) deux paires de type 2 et de même indicateur f. Si f est pleinement itérable, elles sont équivalentes. Sinon (C, D) est équivalente à une et une seule des deux paires (A, B) et $f^{(1/2m)}(A, B)$, où m est l'ordre maximal d'une racine itérative de f.*

Notez que, dans le second cas, $f^{(1/2m)}$ n'est pas analytique. Il n'empêche que $f^{(1/2m)}(A, B)$ l'est.

Démonstration. — On reprend la démonstration du Théorème de Kasner. On a $S_C = S_A \circ f^{(t)}$, avec t réel et $f^{(t)}$ analytique. Si f est pleinement itérable, on conclut comme dans cette démonstration.

Sinon, on écrit $t = (2p + \varepsilon)/m$, avec $p \in \mathbb{Z}$ et $\varepsilon \in \{0, 1\}$. L'itéré analytique $f^{(p/m)}$ conjugue $(S_C, S_C \circ f)$ à $(S_A \circ f^{(\varepsilon/m)}, S_A \circ f^{(\varepsilon/m)} \circ f)$. Les paires obtenues pour $\varepsilon = 0$ et $\varepsilon = 1$ ne sont pas conjuguées ; elles sont formellement conjuguées par $f^{(1/2m)}$. $\square$

Compte tenu du Lemme 2.5, les Théorèmes de Baker-Écalle et d'Écalle-Liverpool ont aussi la conséquence géométrique suivante :

Théorème 2.11. — *Soit (A, B) une paire de type 2. Si son indicateur n'est pas pleinement itérable, le seul automorphisme de cette paire est l'identité. C'est en particulier le cas si la paire est algébrique de résidu non nul.*

Pour finir, le Théorème de Nakai, compte tenu du Lemme 1.8, a la conséquence suivante :

Théorème 2.12. — *Soit (A, B) une paire de type 2, d'indicateur f. Les paires (A, B) et $(S_A(B), A)$ sont équivalentes si et seulement si f est un carré itératif, i.e. $f^{(1/2)}$ converge. Si f n'est pas un carré itératif, toute paire dont l'indicateur est conjugué à f est équivalente à une et une seule de ces deux paires.*

Dans les chapitres suivants, on verra que la méthode de la réflexion permet parfois de discriminer les paires (A, B) et $(S_A(B), A)$, donc de construire des difféomorphismes qui ne sont pas des carrés itératifs. Baker et Bhattacharyya [**5**], et aussi Ahern et Rosay [**1**], ont donné des exemples de difféomorphismes polynômiaux qui ne sont pas des carrés itératifs, ou qui plus généralement ne sont pas des puissances itératives. Notre construction est très différente.

2.5. Résidu « tangentiel » et aberration. — Dans le cas particulier où B est la tangente à l'arc A en $P \in A$ (on suppose A strictement convexe, ainsi la paire est de type 2), le résidu de la paire (A, B) est lié à *l'aberration* de la courbe A au point P, notion dont j'ai appris l'existence dans un article de Schot [**24**].

Soit A un arc régulier et strictement convexe de classe C^3. En un point P de A, on définit :

(1) *l'axe d'aberration de A* : c'est la position limite de la droite qui passe par P et le milieu d'une corde $P'P''$ parallèle à la tangente en P quand P' tend vers P ;

(2) *l'angle d'aberration de A* : c'est l'angle orienté entre la normale et l'axe d'aberration ;

(3) *l'aberration de A* : c'est la tangente de cet angle.

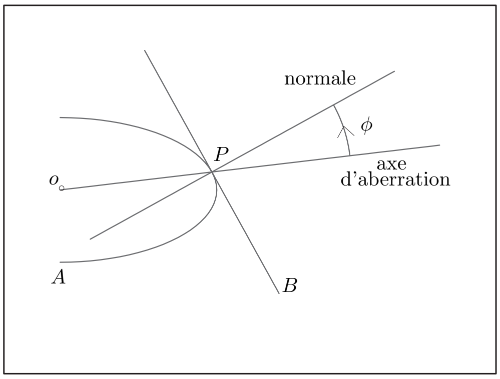

FIGURE 3. Le résidu de la paire (ellipse A, tangente B) est $i\tan\phi$

Schot donne la formule suivante pour l'aberration $\tan\delta$ d'une courbe définie par $y = y(x)$:

$$\tan\delta = y' - \frac{1+(y')^2}{3(y'')^2}\,y'''. \tag{15}$$

L'aberration de la courbe $y = x^2 + lx^3$ à l'origine vaut ainsi $\tan\delta = -l/2$.

Comme l'aberration est invariante par similitude, compte tenu de la formule (9) et de la définition (11) du résidu, on obtient :

$$\tan\delta = i\,\text{rés}((A,B)),$$

si B est la tangente en P à A et $\tan\delta$ l'aberration de A au point P.

Les courbes d'aberration constante sont les cercles et les spirales logarithmiques [**24**].

Si de plus A est une conique, un théorème classique dit que le milieu d'une corde de direction donnée décrit un diamètre de A (une demi-droite parallèle à son axe si A est une parabole). On en déduit que l'axe d'aberration d'une conique à centre A en $P \in A$ est la droite qui joint P au centre de A ; si A est une parabole, c'est la parallèle issue de P à l'axe de A.

On a donc le Lemme suivant (une démonstration directe serait plus rapide !) :

Lemme 2.13. — *Soit B la tangente à une conique A en $P \in A$. Le résidu de (A,B) est $i\tan\phi$, où ϕ est l'angle entre la droite qui passe par P et le centre de A (et est parallèle à l'axe de A si A est une parabole), et la normale en P à A.*

3. Perturbations d'une paire de cercles tangents

3.1. Figures d'une paire de cercles tangents. — Commençons par la paire la plus simple : deux cercles (distincts) tangents, ou un cercle et une droite tangents.

Toutes ces paires sont équivalentes par homographie. Le cercle A de centre a et qui passe par 0 est donné par $(z-a)(\overline{z}-\overline{a}) = a\overline{a}$, donc :

$$S_A(z) = \frac{a\,\overline{z}}{\overline{z}-\overline{a}}.$$

On prend l'exemple de la paire (A, B), où A est le cercle de diamètre $[0, 2i]$, et $B = \mathbb{R}$. Son indicateur est l'homographie :

$$f(z) = S_A \circ S_{\mathbb{R}}(z) = \frac{z}{1-iz},$$

un difféomorphisme pleinement itérable, de type 2 et de résidu nul, d'itérés complexes :

$$f^{(t)}(z) = \frac{z}{1-itz}\ , \qquad (t \in \mathbb{C}).$$

On sait (ou on vérifie) que $f^{(t)}$ conserve les cercles (et la droite) du faisceau défini par A et B si $t \in i\mathbb{R}$, tandis que $\{f^{(t)},\ t \in \mathbb{R}\}$ agit transitivement sur ce faisceau. En particulier, si $a > 0$, l'image par f du cercle de diamètre $[0, i/a]$ est le cercle de diamètre $[0, i/(a+1)]$.

On en déduit que, pout tout $n \geqslant 1$, A_n et B_n sont les (germes en $0 \in \mathbb{C}$ des) cercles de diamètres $[0, i/(n+1/2)]$ et $[0, i/n]$. Ces cercles « décroissent » vers 0, avec les relations d'emboîtements :

$$\overline{G_{n+1}} \subset H_{n+1} \cup \{0\}, \qquad \overline{H_{n+1}} \subset G_n \cup \{0\}, \tag{16}$$

entre les disques G_n et H_n de bords A_n et B_n.

Compte tenu de la convergence des itérés complexes de f, le lemme suivant est une conséquence des résultats du Chapitre 2. On peut aussi le démontrer facilement par un calcul formel.

Lemme 3.1. — *Les automorphismes d'une paire (de germes) de cercles tangents distincts sont les itérés de son indicateur, d'ordres imaginaires purs.*

En voici une autre démonstration, très différente. Il suffit de montrer qu'un automorphisme d'une telle paire est une homographie :

3.2. Une démonstration géométrique du Lemme 3.1. — Soit Δ un voisinage ouvert de 0, et g un isomorphisme $\Delta \to g(\Delta)$ qui conserve 0 et les germes des cercles A et B en 0.

D'après le Lemme 1.7, g conserve les germes de A_n et B_n pour tout $n \geqslant 0$. On choisit $n \in \mathbb{N}$ assez grand pour que le cercle A_n soit contenu dans Δ. Comme g envoie un arc de A_n dans A_n, par analyticité (réelle), g envoie le cercle A_n dans, donc sur lui-même. On en déduit que g induit un biholomorphisme du disque de bord A_n. C'est donc une homographie.

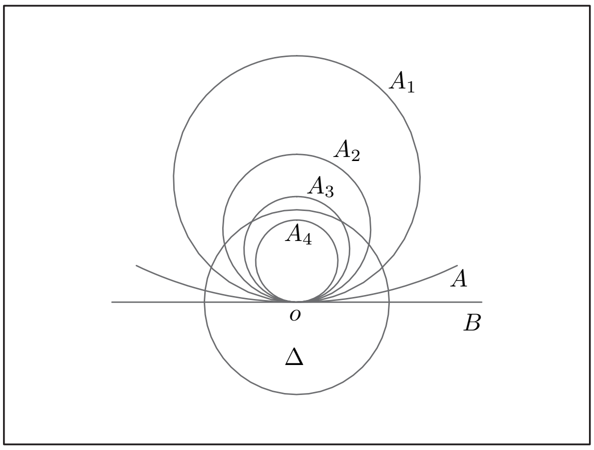

FIGURE 4. $A_n \subset \Delta$ si n est assez grand

3.3. Comment savoir si une paire est équivalente à une paire de cercles ?

L'argument précédent enfonçait une porte largement ouverte. Il apporte pourtant *une* réponse, que je crois nouvelle, à la question du titre.

Soit (C, D) une paire équivalente à une paire (A, B) de cercles tangents en $0 \in \mathbb{C}$. Soit Δ un voisinage ouvert de 0 et :

$$g : \Delta \longrightarrow g(\Delta), \qquad g(0) = 0,$$

un isomorphisme tel que $g(A, B) = (C, D)$. On note $(C_n, D_n)_{n \geqslant 0}$ la suite engendrée par la paire (C, D). Pour tout $n \in \mathbb{N}$, on a :

$$g(A_n) = C_n, \qquad g(B_n) = D_n$$

au voisinage de 0. Si n est choisi assez grand, les cercles A_n et B_n sont contenus dans Δ, et donc, en confondant les courbes avec leurs germes, $C_n = g(A_n)$ et $D_n = g(B_n)$ sont des courbes analytiques compactes. Elles bordent respectivement des domaines K_n et L_n qui vérifient des relations analogues à (16), et g induit des isomorphismes de G_n sur K_n et de H_n sur L_n. Compte tenu du théorème de Riemann, on a donc :

Si une paire (C, D) est équivalente à une paire de cercles tangents, il existe un $n \in \mathbb{N}$ tel que C_n et D_n sont (les germes de) deux courbes analytiques compactes, et qu'un isomorphisme du domaine de bord C_n sur le disque unité envoie D_n sur un arc de cercle (ou de droite).

On sent bien que cet énoncé contient le fait que l'équivalence de deux paires formellement équivalentes est exceptionnelle. Bien sûr, c'est une conséquence de la théorie des difféomorphismes tangents à l'identité. Je crois toutefois que la construction suivante, qui précise l'énoncé précédent, en est la démonstration la plus simple.

3.4. Une famille nombreuse de paires non équivalentes. — On note $\Delta = D(0,1)^{(4)}$ le disque unité et $I =]-1,+1[$. Rappelons la définition de la classe $\mathcal{S}$:

$$\mathcal{S} = \{u \in \mathcal{O}(\Delta),\ \text{injective},\ u(0)=0,\ u'(0)=1\}.$$

On a :

Théorème 3.2. — *Soit $0 < r < 1/40$ et A le cercle de diamètre $[0, 2ir]$. Quels que soient $u,\ v \in \mathcal{S}$, les paires $(A, u(I))$ et $(A, v(I))$ sont équivalentes en 0 si et seulement si elles sont équivalentes par homographie.*

La démonstration n'utilise que la propriété suivante du cercle A :

(17) $A \smallsetminus \{0\}$ *est contenu dans une composante connexe de $u(\Delta) \smallsetminus u(I)$.*

On montrera plus bas que cette propriété est vérifiée si $r < 1/40$.

Démonstration. — Soit $u \in \mathcal{S}$, et notons :

$$(A, B) = (A, u(I))$$

la paire associée. On note aussi G_0 le disque de bord A. On a :

$$S_B = u \circ S_{\mathbb{R}} \circ u^{-1}.$$

Comme $S_{\mathbb{R}}$ est une involution de Δ, de lieu fixe I, S_B est une involution de $u(\Delta)$, qui laisse fixe B et échange les deux composantes connexes de $u(\Delta) \smallsetminus B$.

La réflexion S_B est définie sur Δ, et envoie $\overline{G_0} \smallsetminus \{0\}$ dans son complémentaire, compte tenu de (17). Il en résulte que l'indicateur $f = S_A \circ S_B$ est holomorphe et injectif au voisinage du disque fermé $\overline{G_0}$ et vérifie :

$$f(\overline{G_0}) \subset G_0 \cup \{0\}.$$

De plus, A_1 est une courbe analytique compacte, qui borde $G_1 = f(G_0)$.

En itérant, on obtient que, pour tout $n \geqslant 0$, A_n est une courbe analytique compacte, qui borde un domaine G_n, avec :

$$\overline{G_{n+1}} \subset G_n \cup \{0\}, \tag{18}$$

et que f induit un isomorphisme :

$$f : G_n \smallsetminus \overline{G_{n+1}} \longrightarrow G_{n+1} \smallsetminus \overline{G_{n+2}}.$$

Si $v \in \mathcal{S}$, la même construction peut être appliquée à la la paire $(C, D) = (A, v(I))$, et à son indicateur g. On trouve que C_n est une courbe analytique compacte, qui borde un domaine K_n, avec des relations d'emboîtement similaires à (18), et que g induit des isomorphismes :

$$g : K_n \smallsetminus \overline{K_{n+1}} \longrightarrow K_{n+1} \smallsetminus \overline{K_{n+2}}.$$

(4) Dans cet article, on note $D(a, r)$ le disque de centre a et de rayon $r > 0$.

Si les paires (A, B) et (C, D) sont équivalentes et si $h \in \mathbf{G}$ réalise cette équivalence, par le même argument que dans le paragraphe précédent, on obtient que si $n \in \mathbb{N}$ est choisi assez grand, h induit un isomorphisme :

$$h : G_n \longrightarrow K_n.$$

Au voisinage de 0, on a la relation de conjugaison :

$$g \circ h = h \circ f.$$

Elle est vérifiée sur G_n par analyticité. Si $n \geqslant 1$, on prolonge h en un isomorphisme :

$$h : G_{n-1} \longrightarrow K_{n-1},$$

en posant, pour tout $z \in G_{n-1} \smallsetminus G_n$, $h(z) = g^{-1} \circ h \circ f(z)$. De proche en proche, on prolonge ainsi h en un automorphisme du disque $G_0 = K_0$; h est une homographie. □

Lemme 3.3. — *Si $0 < r < 0,03$, la propriété (17) est vérifiée.*

Démonstration. — Soit $u \in \mathcal{S}$. Si $0 < R < 1$, le domaine $u(D(0, R))$ contient le disque $D(0, R/4)$ d'après le Théorème de Koebe (voir par exemple [**8**]). On suppose maintenant :

$$r < R/8.$$

Notons $D = D(ir, r)$ le disque de bord A. On a donc $\overline{D} \subset u(\Delta)$ et :

$$|z| > R \implies u(z) \notin \overline{D}.$$

Il reste à vérifier que $u(I)$ ne coupe pas $A \smallsetminus \{0\}$; D'après ce qui précède, il suffit que :

$$t \in [-R, +R] \smallsetminus \{0\} \implies |\operatorname{Im} u(t)| < \frac{1}{2r}|u(t)|^2.$$

On utilise les inégalités de distorsion suivantes (voir [**8**], page 3) :

$$(19) \qquad |u(z)| \geqslant \frac{|z|}{(1+|z|)^2}, \qquad |u''(z)| \leqslant \frac{2(2+|z|)}{(1-|z|)^4}, \qquad (z \in \Delta).$$

Comme $\operatorname{Im} u(0) = \operatorname{Im} u'(0) = 0$, la formule de Taylor donne :

$$|\operatorname{Im} u(t)| \leqslant \frac{t^2}{2} \max_{|s| \leqslant |t|} |u''(s)|.$$

Compte tenu de (19), la condition (17) est donc vérifiée si $r < R/8$ et :

$$r \leqslant \frac{(1-R)^4}{2(2+R)(1+R)^4},$$

pour un $R < 1$. Tout $r < R/8$ convient si :

$$R \leqslant \frac{4}{(2+R)} \frac{(1-R)^4}{(1+R)^4}.$$

On vérifie à la main que $R = 1/5$, à la machine que $R = 0,24$, vérifie cette inégalité. □

Notons :

$$\mathcal{S}^* = \{ u \in \mathcal{S}, \ u''(0) = 0 \}.$$

Corollaire 3.4. — *Soit* $0 < r < 1/40$, *et* $A = \partial D(ir, r)$. *Les paires* $(A, u(I))$, $u \in \mathcal{S}^*$, *sont deux à deux non équivalentes en* 0.

Démonstration. — D'après le théorème, une équivalence h entre deux paires de cette forme est un automorphisme de $D(ir, r)$. Comme il conserve A et 0, il est tangent à l'identité en 0. Comme il envoie $u(I)$ sur $v(I)$, avec $u, \ v \in \mathcal{S}^*$, il est tangent à l'ordre 2 à l'identité. C'est donc l'identité. □

Corollaire 3.5. — *Soit* A *le cercle de centre* i *et de rayon* 1, B *un arc analytique régulier tangent à l'ordre* 2 *à* $\mathbb{R}$ *en* 0. *Les paires* (rA, B) *sont deux à deux non équivalentes en* 0 *si* $r > 0$ *est assez petit.*

Démonstration. — Soit $u \in \mathbf{G}$, $u(0) = 0, \ u'(0) = 1, \ u''(0) = 0$, un difféomorphisme qui envoie $\mathbb{R}$ sur B au voisinage de 0. L'homothétique B/r est paramétrée par $t \mapsto u(t)/r$, mais aussi par $t \mapsto u_r(t) = u(rt)/r$, avec encore $u_r(0) = 0, \ u_r'(0) = 1, \ u_r''(0) = 0$. Si r est assez petit, u_r appartient à la classe $\mathcal{S}^*$. □

Compte tenu du Théorème 2.10, dans l'un des deux corollaires précédents, si l'on prend trois paires distinctes de la forme prescrite, deux au moins de leurs trois indicateurs ne sont pas conjugués.

3.5. Les familles de Shcherbakov, Elizarov, Birkhoff. — L'article de Birkhoff [**6**] n'était pas connu des spécialistes des difféomorphismes analytiques de $(\mathbb{C}, 0)$ dans les années 1960. À cette époque, les premiers exemples de difféomorphismes tangents à l'identité, formellement mais non analytiquement conjugués, sont obtenus en relation avec l'étude de la convergence de leurs itérés complexes. Nous avons cité des résultats importants de Baker [**2**] [**4**], Écalle [**11**], Liverpool [**16**], dans le Chapitre 2. Ces articles contiennent d'autres résultats ; citons aussi Szekeres [**26**], Baker [**3**].

La méthode de Birkhoff est réinventée dans les années 1970 (Kimura, Écalle), complétée vers 1980 (Écalle, Voronin, Malgrange). Les invariants qu'elle décrit sont réputés difficiles à calculer. Il y a une exception notable, celle des perturbations du modèle pleinement itérable $f(z) = z(1-z)^{-1}$. Shcherbakov [**25**] a démontré le théorème suivant.

Il est commode de se placer au voisinage de l'infini (on pose $z = 1/w$). On considère une fonction holomorphe au voisinage de l'infini, de la forme :

$$F_\varepsilon(w) = w + 1 + \varepsilon G(w); \qquad G(w) = O(w^{-2}). \tag{20}$$

On note :

$$\gamma(w) = \sum_{n \in \mathbb{Z}} G(w+n). \tag{21}$$

C'est une fonction holomorphe pour $|\operatorname{Im} w|$ assez grand. On a :

***Théorème 3.6* (Shcherbakov).** — *Si $\gamma \not\equiv 0$, et si $\varepsilon > 0$ est assez petit, le difféomorphisme F_ε n'est pas conjugué à la translation $T(w) = w + 1$ au voisinage de l'infini.*

Elizarov [**13**] (voir aussi Il'yashenko [**14**]), en reprenant la démonstration de Shcherbakov, a obtenu le résultat plus fort suivant :

***Théorème 3.7* (Elizarov).** — *Si $\gamma \not\equiv 0$, et si $\varepsilon > \varepsilon' \geqslant 0$ sont assez petits, les difféomorphismes F_ε et $F_{\varepsilon'}$ ne sont pas conjugués au voisinage de l'infini.*

Les deux auteurs donnent l'exemple de $G(w) = 1/w^2$, pour lequel :

$$\gamma(w) = \sum_{n\in\mathbb{Z}} \frac{1}{(w+n)^2} = \frac{\pi^2}{\sin^2 \pi w}.$$

Le même exemple se trouve en fait dans [**6**].

Il est probable que le Corollaire 3.5 soit aussi un cas particulier du théorème précédent. D'autre part, la méthode de la démonstration de ce théorème permet bien sûr (ce n'est pas forcément facile) de donner une estimation des ε permis, en fonction de la non petitesse de la fonction γ. Il n'est donc pas impossible que la famille de paires considérée dans le Théorème 3.2 rentre dans le cadre des théorèmes précédents, convenablement quantifiés. La question est de savoir si une paire est « voisine » du modèle qu'est une paire de cercles tangents.

Quoi qu'il en soit, je pense que certains des exemples qu'on va donner dans les chapitres suivants sont très loin du modèle.

4. Invariants géométriques d'une paire

4.1. Introduction. — Les paires (A, B) considérées au Chapitre 3 avaient en commun que, pour n assez grand, A_n était une courbe compacte régulière, « décroissant » vers le point de contact. Comme on a vu, cette propriété est invariante par équivalence. C'est *une propriété asymptotique*, ou une *figure asymptotique* de la paire.

On va tenter, en en donnant d'autres exemples, de préciser ces deux notions.

Dans tout ce chapitre, (A, B) désigne une paire d'arcs tangents à l'ordre 1 (ou de type 2) en $0 \in \mathbb{C}$ (bien sûr, dans les exemples, le point de contact pourra être un autre point de $\mathbb{C}$) d'indicateur $f = S_A \circ S_B \in \mathbf{G}$. On note (A_n, B_n), $n \geqslant 0$, les paires engendrées.

Le fait que l'ordre de contact entre A et B est impair permet d'orienter canoniquement A et B. L'arc A est situé d'un côté de B au voisinage de 0. On oriente A et B, et plus généralement A_n et B_n pour tout $n \in \mathbb{N}$, par un vecteur tangent u en 0, tel que le vecteur normal iu pointe du côté de $\mathbb{C} \setminus B$ qui contient $A \setminus \{0\}$. Cette orientation est invariante par équivalence.

Par la suite, il sera utile de distinguer les deux brins des arcs A_n et B_n en 0. On pose :

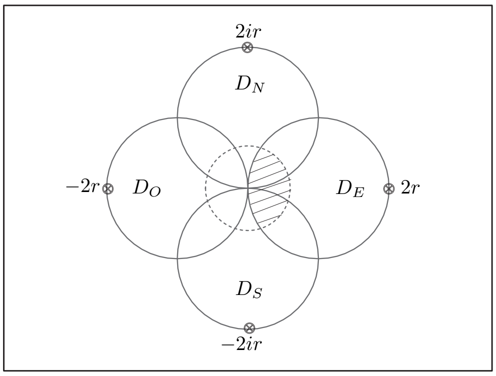

FIGURE 5. Lunule de Leau

Notation 4.1. — Le vecteur directeur u de A en 0 étant choisi comme on vient de dire, pour tout $n \geqslant 0$, on note A_n^+ et B_n^+ (respectivement A_n^- et B_n^-) les demi-arcs (ou brins) de A_n et B_n, de vecteur directeur u en 0 (respectivement de vecteur directeur $-u$ en 0).

4.2. Domaine de Leau d'une paire de type 2. — On choisit la définition suivante :

Définition 4.2. — Soit $f \in \mathbf{G}$, de type 2. On appelle *domaine de Leau de* f tout ouvert connexe $\Delta \subset \mathbb{C}$ tel que $0 \in \partial\Delta$ et :

(1) il existe un voisinage ouvert U de 0 tel que $\Delta \cap U$ soit un ouvert connexe de classe C^2 dans U ;

(2) f est holomorphe sur Δ, injective, et $f(\Delta) \subset \Delta$.

D'autres définitions sont possibles. En particulier, au voisinage de 0, un « pétale de Leau » est plus grand que ce nous appelons un domaine de Leau, voir par exemple [**8**]. L'important pour nous (c'est assuré par le première condition) est que Δ contienne un disque ouvert dont le bord passe par 0. Il en résulte en effet que les germes de A_n et B_n sont contenus dans $\Delta \cup \{0\}$ pour n assez grand, si f est l'indicateur de (A, B).

L'existence d'un domaine de Leau, aussi petit qu'on veut, est bien connue. On va démontrer :

Proposition 4.3. — *Soit* $f(z) = z - az^2 + \cdots$ *un élément de type* 2 *de* $\mathbf{G}$. *Pour tout* $r > 0$ *assez petit et pour tout* $0 < r' < r$, *la lunule* $D(0, r') \cap D(r\overline{a}, r|a|)$ *est un domaine de Leau de* f. *Si* Δ *et* Δ' *sont deux domaines de Leau de* f *et si* $K \subset \Delta \cup \{0\}$ *est compact,* $f^{(n)}(K)$ *est contenu dans* $\Delta' \cup \{0\}$ *pour tout* n *assez grand.*

C'est la conséquence des trois lemmes suivants. Une similitude permet de supposer :

$$f(z) = z - z^2 + \cdots,$$

donc :

$$\log \frac{f(z)}{z} = -z + O(z^2).$$

On fixe $r > 0$, tel que $f(z)/z$ soit holomorphe sur $D(0, 2r)$ et que :

$$\left|\frac{f(z)}{z} - 1\right| < \frac{1}{2}, \qquad \left|\log \frac{f(z)}{z} + z\right| \leqslant \frac{1}{2r}\,|z|^2.$$

On a donc, si $z = x + iy \in D(0, 2r)$:

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{6}, \tag{22}$$

et

$$\left|\log \frac{|f(z)|}{|z|} + x\right| \leqslant \frac{1}{2r}\,|z|^2, \qquad \left|\arg \frac{f(z)}{z} + y\right| \leqslant \frac{1}{2r}\,|z|^2. \tag{23}$$

On introduit les disques suivants, voir Figure 5 :

$$D_E = D(r, r), \quad D_N = D(ir, r), \quad D_O = D(-r, r), \quad D_S = D(-ir, r).$$

La première inégalité de (23) montre que $|f(z)| < |z|$ si $x > |z|^2/2r$, autrement dit si $z \in D_E$. De même, on obtient $|f(z)| > |z|$ si $z \in D_O$, et $|\arg(f(z))| < |\arg(z)|$ si $z \in D_N \cup D_S$. On en déduit d'abord l'existence des domaines de Leau :

Lemme 4.4. — *Avec les notations précédentes, pour tout $0 < r' < r$, la lunule $D(0, r') \cap D(r, r)$ est un domaine de Leau de f.*

Démonstration. — On doit montrer que, si z appartient à $D(0, r') \cap D_E$, $f(z)$ aussi. Comme $z \in D_E$, on a $|f(z)| < |z|$ donc $f(z) \in D(0, r')$. Si de plus $z \in D_N \cup D_S$, on a $|\arg(f(z)| < |\arg(z)|$, donc $f(z) \in D_E$. Si au contraire $z \notin D_N \cup D_S$, alors $|\arg(z)| < \pi/6$, donc $|\arg(f(z))| < \pi/3$, et encore $f(z) \in D_E$. □

Lemme 4.5. — *Avec les notations précédentes, pour tout $0 < r' < r$, il existe $n \in \mathbb{N}$ tel que $f^{(n)}(D(0, \rho) \setminus \overline{D}_O) \subset D(0, r') \cap D_E$ pour tout $\rho > 0$ assez petit.*

Démonstration. — La première inégalité de (23) donne $|\log(|f(z)|/|z|)| < 2|z|$, si $|z|$ est assez petit, donc :

$$f(D(0, \rho)) \subset D(0, \rho(1 - 3\rho)^{-1}) \tag{24}$$

pour tout $\rho > 0$ assez petit.

On fixe $A_0 > 0$ tel que, si $|z| < r$:

$$z \notin \overline{D}_O \implies x > -A_0 y^2; \qquad x > A_0 y^2 \implies z \in D_E.$$

Soit $-A_0 < A < +A_0$, et $z = x + iy$ tel que :

$$|z| < \rho, \qquad A y^2 < x < A_0 y^2.$$

Si $f(z) = Z = X + iY$, on a :

$$X - (A+1/2)Y^2 = (x - x^2 + y^2) - (A+1/2)(y-2xy)^2 + O(|z|^3),$$
$$= x - (A-1/2)y^2 + O(y^3)$$

où $|O(y^3)| \leqslant C|y|^3 \leqslant C\rho y^2$ et C ne dépend que de f et A_0. On donc :

$$X > (A+1/2)Y^2,$$

si ρ est assez petit. Si $f(z) \in D_E$, comme $D(0,r') \cap D_E$ est stable par le lemme précédent, $f^{(n)}(z) \in D(0,r') \cap D_E$ pour tout $n \geqslant 1$. Sinon, on a :

$$|Z| < \rho(1-3\rho)^{-1}, \qquad (A+1/2)Y^2 < X < A_0 Y^2.$$

On itère $n > 4A_0$ fois, avec $\rho < r'/(12n)$. □

Lemme 4.6. — *Avec les notations précédentes, si Δ et Δ' sont deux domaines de Leau de f, et si $K \subset \Delta \cup \{0\}$ est compact, $f^{(n)}(K)$ est contenu dans $\Delta' \cup \{0\}$ pour tout n assez grand.*

Démonstration. — Compte tenu de la définition des domaines de Leau, on peut supposer $\Delta' = D(0,r) \cap D(r,r)$. La suite $(f^{(n)})_{n\geqslant 0}$ est une famille normale sur Δ, donc $f^{(n)}$ converge vers 0, uniformément sur tout compact de Δ. Si $\varepsilon > 0$ et compte tenu des lemmes précédents, on en déduit que si $K \subset \Delta \cup \{0\}$ est compact, $f^{(n)}(K)$ est contenu dans $D(0,\varepsilon) \cap (\Delta \cup \{0\})$ pour n assez grand. Si on choisit $\varepsilon > 0$ assez petit, le Lemme 4.5 montre que $f^{(m+n)}(K) \subset \Delta' \cup \{0\}$ pour m assez grand. □

4.3. Préliminaire : prolongement analytique d'arcs. — Un arc paramétré (respectivement un arc paramétré analytique) est une fonction continue (respectivement analytique) *non constante* $c : I \to \mathbb{C}$, où $I \subset \mathbb{R}$ est un intervalle non réduit à un point. Deux arcs paramétrés $c_k : I_k \to \mathbb{C}$, $k = 1,2$, sont équivalents s'il existe un homéomorphisme *croissant* $\phi : I_2 \to I_1$ tel que $c_2 = c_1 \circ \phi$. Un arc est une classe d'équivalence d'arcs paramétrés. *Il s'agit donc d'arcs orientés.*

Ceci dit, la notation $c : I \to \mathbb{C}$ pourra désigner l'arc dont c est une paramétrisation et aussi une paramétrisation d'un arc c !

Définition 4.7. — Un arc $c : I \to \mathbb{C}$ est localement injectif (respectivement localement analytique) si pour tout $t \in I$, il existe un intervalle $J \subset I$, voisinage de t dans I, tel que $c_{|J}$ soit injectif (respectivement soit équivalent à un arc paramétré analytique).

On veut éviter les arcs qui, comme $c(t) = (\cos t)^2$, font des va-et-vient ; on veut aussi éviter la question de la paramétrisation analytique globale des arcs localement analytiques, dont la solution n'est pas élémentaire.

Proposition 4.8. — *Soit $c : [0,\varepsilon[\to \mathbb{C}$ un arc paramétré analytique. Il existe un arc localement injectif et analytique $\widehat{c} : [0,T[\to \mathbb{C}$, équivalent à c dans les germes en*

$0 \in \mathbb{R}^+$, *tel que de plus : si* $\widehat{d} : [0, S[\to \mathbb{C}$ *a aussi ces propriétés, il existe* $T' \in]0, T]$ *tel que* $\widehat{d}$ *soit équivalent à* $\widehat{c}_{|[0,T'[}$. *Comme arc,* $\widehat{c}$ *est unique.*

On dira que *l'arc* $\widehat{c}$ *est l'extension maximale de l'arc* c. Si $F \subset \mathbb{C}$ contient le germe de c en $0 \in \mathbb{R}^+$ et si I est le plus grand intervalle d'origine 0 tel que $\widehat{c}(I) \subset F$, on dit que *l'arc* $\widehat{c}_{|I}$ *est l'extension maximale de l'arc* c *dans* F.

Démonstration. — Un arc localement analytique est (localement) rectifiable. On en déduit facilement qu'il suffit de démontrer que si deux arcs $\widehat{c_k} : [0, T_k[\to \mathbb{C}$, $k = 1, 2$, paramétrés par la longueur d'arc, sont localement injectifs et analytiques et sont équivalents dans les germes en $0 \in \mathbb{R}^+$, et si $0 < T_0 < T_1 < T_2$, alors $\widehat{c_1} \equiv \widehat{c_2}$ sur $[0, T_0[$ implique $\widehat{c_1} \equiv \widehat{c_2}$ au voisinage de T_0. Cela résulte du lemme suivant. □

Lemme 4.9. — *Soit* u_1, $u_2 : (\mathbb{R}, 0) \to (\mathbb{C}, 0)$ *deux germes analytiques injectifs. Si* $u_{1\,|\mathbb{R}^+}$ *et* $u_{2\,|\mathbb{R}^+}$ *ont le même germe d'image en* 0, *les germes* u_1 *et* u_2 *sont équivalents en* 0.

Démonstration. — Nous noterons $u_1(\mathbb{R}^+)$ et $u_1(\mathbb{R})$ pour les germes des images de $u_{1\,|\mathbb{R}^+}$ et u_1 en 0... Si $u_k(t) = a_k t^{p_k} + \cdots$ $(a_k \neq 0)$, $k = 1, 2$, a_k donne la direction de la demi-tangente à $u_k(\mathbb{R}^+)$ en 0, donc $a_1/a_2 > 0$ et on se ramène par rotation et reparamétrisation au cas où :

$$u_k(t) = t^{p_k} + \cdots, \quad (p_k \geqslant 1,\ k = 1,\ 2).$$

Soit $p_k = dq_k$, $k = 1, 2$, avec q_1, $q_2 \in \mathbb{N}^*$ premiers entre eux ; $u_1(\mathbb{R}^+) = u_2(\mathbb{R}^+)$ est aussi paramétré par

$$u_1(t^{q_2}) = t^m + \cdots = h_1(t)^m, \text{ ou } \ u_2(t^{q_1}) = t^m + \cdots = h_2(t)^m,$$

où $m = dq_1q_2$; $h_1(t) = t + \cdots$ et $h_2(t) = t + \cdots$ sont deux arcs réguliers. Comme $h_1(\mathbb{R})$ et $h_2(\mathbb{R})$ sont tangents en 0 et ont même image par $z \mapsto z^m$, $h_1(\mathbb{R}) = h_2(\mathbb{R})$, donc h_1 et h_2 sont *analytiquement* équivalents : $h_2 = h_1 \circ \phi$, où ϕ est un difféomorphisme analytique de $(\mathbb{R}, 0)$. En remontant à u_1 et u_2, on obtient :

$$u_2(t^{q_1}) \equiv u_1((\phi(t)^{q_2})$$

au voisinage de 0. On peut supposer q_1 impair. Si q_2 était pair, on aurait $u_1(\mathbb{R}) \subset u_2(\mathbb{R}^+)$. Comme u_1 est injective, q_2 est impair et $u_1(\mathbb{R}) = u_2(\mathbb{R})$. □

4.4. Propriétés asymptotiques ; exemples. — On revient au problème des paires. Soit Δ un domaine de Leau de la paire (A, B) en $0 \in \mathbb{C}$, f son indicateur et $n \in \mathbb{N}$ un entier assez grand pour que les germes de $A_n^{\pm}$ et $B_n^{\pm}$ soient contenus dans $\Delta \cup \{0\}$.

Notation 4.10. — On note encore $A_n^{\pm}$ et $B_n^{\pm}$ les extensions maximales de ces germes dans $\Delta \cup \{0\}$.

Notation 4.11. — On adopte quelques conventions, à propos des arcs issus de $0 \in \mathbb{C}$.

– Un arc (orienté) A est un ensemble totalement ordonné, « directement » homéomorphe à un intervalle non trivial et muni d'une projection continue $\pi_A : A \to \mathbb{C}$. Dans la suite, quand on parlera d'un point $a \in A$, il faudra l'entendre en ce sens. En particulier, un point $a \in A$ est équipé du *germe de* A *en* a.

– Si a et b sont deux points de A, on appellera *arc* $[a, b]$ *de* A l'arc des points de A qui sont compris entre a et b au sens large. L'expression *arc* $]a, b[$ *de* A est définie de façon analogue. On notera o le point initial de A, au lieu de o_A.

– Une intersection de deux arcs A et B (pris dans cet ordre) est un couple $(a, b) \in A \times B$ tel que $\pi_A(a) = \pi_B(b)$.

Les propriétés suivantes sont des exemples de *propriétés asymptotiques* :

(1) A_n^+ est une variété analytique compacte ;

(2) A_n^+ a un point double ;

(3) A_n^+ a un point stationnaire (singularité analytique locale) ;

(4) A_n^+ a un point d'arrêt[5] dans Δ, *i.e.* A_n^+ a une paramétrisation $u : [0, 1[\to \Delta \cup \{0\}$ et $u(t) \to \alpha \in \Delta$ quand $t \to 1^-$;

(5) A_{n+k}^+, $k \geqslant 1$ donné, coupe A_n^+ en dehors de (o, o) ;

(6) A_{n+k}^+, $k \geqslant 1$ donné, coupe B_n^+ en dehors de (o, o).

La notion de *propriété asymptotique* est précisée par l'énoncé suivant, dans lequel, pour fixer les idées, on traite la Propriété 5 avec $k = 1$.

Théorème 4.12. — *Soit* (A, B) *et* (C, D) *deux paires équivalentes de type* 2 *en* $0 \in \mathbb{C}$. *Soit* Δ *et* Λ *des domaines de Leau associés et* $n \in \mathbb{N}$ *assez grand. On suppose que l'extension maximale de* A_{n+1}^+ *dans* $\Delta \cup \{0\}$ *coupe celle de* A_n^+ *en dehors de* (o, o). *Alors, pour tout entier* k *assez grand, l'extension maximale de* C_{k+1}^+ *dans* $\Lambda \cup \{0\}$ *coupe celle de* C_k^+ *en dehors de* (o, o).

Démonstration. — On utilise la Notation 4.10. Soit $h \in \mathbf{G}$ un difféomorphisme tel que $h(A, B) = (C, D)$. On peut choisir des lunules de Leau L et M (voir Proposition 4.3) telles que h induise un isomorphisme de L sur $h(L) \subset M$.

Soit α et β deux sous-arcs compacts de A_{n+1}^+ et A_n^+, qui vont de l'origine à une intersection autre que (o, o) (l'un des deux arcs peut être constant). On remarque que A_{n+k+1}^+ et A_{n+k}^+ sont des prolongements de $f^{(k)}(\alpha)$ et $f^{(k)}(\beta)$.

Compte tenu de la Proposition 4.3, quitte à remplacer n par $n + k$, k assez grand, on se ramène au cas où α et β sont contenus dans $L \cup \{0\}$. Alors $h(\alpha)$ et $h(\beta)$ se coupent dans $M \cup \{0\}$ en dehors de (o, o), et ce sont des sous-arcs de C_{n+1}^+ et C_n^+. □

Au fond, si on se donne un domaine de Leau Δ de la paire (A, B), $N \in \mathbb{N}$ assez grand, et qu'on ne considère que la famille $\mathcal{F}_N$ des arcs $A_n^\pm$ et $B_n^\pm$ pour $n \leqslant N$, la seule propriété de A_N^+ qui ne soit pas *a priori* asymptotique est la suivante :

[5] Compte tenu de la nature de la singularité d'un germe d'ensemble analytique dans $\mathbb{R}^2$ (développements de Puiseux), il s'agit d'une « singularité essentielle ».

A_N^+ quitte Δ par un point $\neq 0$, sans singularité et sans rencontrer un autre membre de la famille $\mathcal{F}_N$ en dehors de l'origine.

Cette propriété « inutile » pourrait-elle avoir lieu quel que soit N ?

4.5. Figures asymptotiques ; exemples. — On se place encore dans la situation décrite au début du § 4.4, dont on conserve les notations. Reprenons la liste des propriétés asymptotiques qu'on y a donnée.

La Propriété 1 et la Propriété 4 méritent le nom de *figures asymptotiques* de la paire (A, B). Si (C, D) est une paire équivalente à la paire (A, B) et si (A, B) a la Propriété 1 (respectivement la Propriété 4), pour tout entier k assez grand, A_k^+ et C_k^+ seront des courbes fermées lisses (respectivement auront *un* point d'arrêt). L'équivalence locale entre les paires (A_k, B_k) et (C_k, D_k) (éventuellement pour k plus grand encore) est alors nécessairement la restriction d'un isomorphisme global du domaine de bord A_k^+ sur le domaine de bord C_k^+ (respectivement transformera *la* singularité essentielle terminale de A_k^+ en celle de C_k^+).

Les autres propriétés ne définissent pas *a priori* des « figures » : les points doubles de A_n^+, ses singularités analytiques, ses intersections avec A_{n+k}^+ ou avec B_n^+, peuvent croître en nombre avec n. Il s'agit alors de dire quel point double... de C_n^+ correspond à tel point double... de A_n^+ pour tout n assez grand, si les paires (A, B) et (C, D) sont équivalentes, autrement dit de numéroter les points doubles ...

La Figure 6 est censée démontrer qu'il est impossible de numéroter les points doubles de A_n de façon invariante [(6)]. C'est pour cette raison qu'on a introduit les brins $A_n^\pm$. Pour la même raison, on ne peut pas construire une figure asymptotique à partir de la Propriété 6.

Dans les trois définitions qui suivent, on se place dans la situation décrite au début du § 4.4. Le numérotage des points doubles de A_n^+ est évident :

Définition 4.13. — Si A_n^+ a un point double, le *premier point double de* A_n^+ est le couple (a, b) de points distincts de A_n^+, où b est le premier point de A_n^+ de même image dans $\mathbb{C}$ qu'un point, soit a, de l'arc $[o, b[$ de A_n^+.

Le numérotage des points stationnaires est encore plus évident.

En ce qui concerne les intersections autres que (o, o) de A_n^+ et A_{n+k}^+, $k \geqslant 0$, on utilise le fait que $f^{(k)}$ induit un isomorphisme de A_n^+ sur un sous-arc de A_{n+k}^+ pour les ordonner. On se contente de le faire pour $k = 1$. Il y a lieu de distinguer deux types d'intersection :

[(6)] sauf bien sûr en utilisant une figure asymptotique déjà construite comme repère, comme dans la Figure 2 de l'Introduction : le fait que A_n est une courbe fermée lisse, bordant un domaine Ω_n, permet de définir *l'unique point double* du sous-arc maximal de B_{n+1} contenu dans Ω_n ; voir aussi le § 4.7.

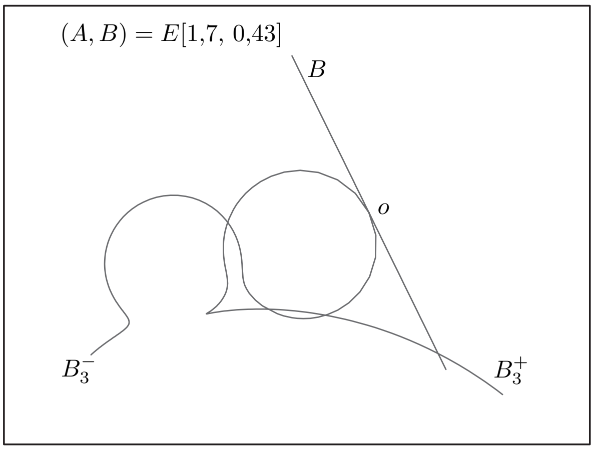

FIGURE 6. Quel est le premier point double de B_3 ?

Définition 4.14. — Soit $(a,b) \in A_n^+ \times A_{n+1}^+$ une intersection autre que (o,o). On dit que (a,b) est une *intersection de type* I si $f(a)$ appartient à l'arc $[o,b]$ de A_{n+1}^+, que c'est une *intersection de type* II dans le cas contraire.

Si $(a,b) \in A_n^+ \times A_{n+1}^+$ est une intersection de type I, autre que (o,o), $(a',b') := (f(a), f(b)) \in A_{n+1}^+ \times A_{n+2}^+$ aussi. Elle vérifie de plus qu'il existe un point c de A_{n+1}^+ (à savoir b), tel que $f(c) = b' \in A_{n+2}^+$ coïncide dans $\mathbb{C}$ avec un point (à savoir a') de l'arc $[o,c]$ de A_{n+1}^+.

On peut numéroter les intersections de type I. Contentons nous de définir la première. Rappelons qu'on utilise la Notation 4.10.

Définition 4.15. — On suppose que A_n^+ et A_{n+1}^+ ont une intersection de type I. Quitte à remplacer n par $n+1$, il existe un premier point $c \in A_n^+ \smallsetminus \{o\}$ tel que $b := f(c) \in A_{n+1}^+$ ait même image dans $\mathbb{C}$ qu'un point, soit a, de l'arc $[o,c]$ de A_n^+. La première intersection de type I de A_n^+ et A_{n+1}^+ est le couple (a,b). C'est aussi la paire au point image formée par les germes de A_n^+ en a et de A_{n+1}^+ en b.

Il est assez clair que la première intersection de type I se comporte bien vis-à-vis de l'action de l'indicateur, et qu'une démonstration analogue à celle du Théorème 4.12 donne le résultat suivant :

Théorème 4.16. — *Soit (A,B) et (C,D) deux paires de type 2, équivalentes en $0 \in \mathbb{C}$. Soit Δ et Λ des domaines de Leau associés et $n \in \mathbb{N}$ assez grand. On suppose que les extensions maximales de A_n^+ et A_{n+1}^+ dans $\Delta \cup \{0\}$ ont une première intersection de type* I (a,a'). *Alors, quel que soit k assez grand, les extensions maximales de C_{n+k}^+ et C_{n+k+1}^+ dans $\Lambda \cup \{0\}$ ont une première intersection de type* I (c_k, c'_k). *La paire des* *germes de* C^+_{n+k} *en* c_k *et de* C^+_{n+k+1} *en* c'_k *(considérée au point image) est équivalente à la paire des germes de* A^+_n *en* a *et de* A^+_{n+1} *en* a' *(considérée au point image).*

Le théorème associe à une paire (A, B), sous l'hypothèse que A^+_n a une intersection de type I avec A^+_{n+1}, une autre paire (A', B'), définie à équivalence près. D'autre part, si l'on introduit (avec les notations de l'énoncé) la réunion Ω_n des composantes connexes bornées du complémentaire de $\pi_{A^+_n}([o, a]) \cup \pi_{A^+_{n+1}}([o, a'])$, on obtient un ouvert, défini à équivalence *globale* près. Toute équivalence de paire de (A, B) avec (C, D) induit, pour n assez grand, un isomorphisme de Ω_n sur Π_n, si Π_n est défini de façon analogue avec la paire (C, D).

C'est un autre exemple de ce qu'on entend par « figure asymptotique » de la paire (A, B).

4.6. Exemples. — Soit A la parabole d'équation paramétrique

$$z(t) = \frac{1 - t^2}{2} + it, \quad (t \in \mathbb{R}) \tag{25}$$

et B sa tangente au point $z(t)$. Le résidu de (A, B) est it.

Quel que soit $t \in \mathbb{R}$, le tracé sur ordinateur montre que B^-_n a une intersection de type I avec B^-_{n+1} pour n grand : $n = 0$ convient si $t > 1,732\ldots$ (en fait $t > \sqrt{3}$), $n = 1$ convient si $1,732 \cdots > t > 0,07$ environ, $n = 2$ convient si $0,07 > t \geqslant 0$. Si $t \leqslant 0$, on a des résultats analogues par symétrie (anti-holomorphe).

La Figure 7[(7)] montre quelques exemples de ce qu'on obtient. Dans chaque cas, avec $n \in \{0, 1, 2\}$ le plus petit possible, on a représenté 1) en pointillé, des arcs arbitraires de $B^-_0, \ldots B^-_{n-1}$, qui n'interviennent pas dans la définition de la figure et 2) en trait plein, l'arc $[o, a]$ de B^-_n, et son image par $S_A \circ S_B$, où a est le premier point autre que o tel que $f(a)$ soit sur l'arc $[o, a[$ de B^-_n.

On voit que la première intersection de type I peut être précédée par une ou plusieurs intersections de type II ou par une auto-intersection. Si l'on suit pas à pas la figure de la première intersection de type I de B^-_n avec B^-_{n+1} quand t décroît de $+\infty$ à 0 on trouve :

– pour $+\infty > t > \sqrt{3}$, une figure homéomorphe à *a)* ; si $t = \sqrt{3}$, B^-_1 a un cusp ;
– pour $\sqrt{3} > t > 1,2$ environ, une figure homéomorphe à b) ;

(7) Fin 1998, j'ai cherché à savoir si la paire (A, B) était équivalente à une paire de cercles quand $t = 0$, par le calcul. On se ramène à étudier la convergence d'un champ de vecteurs formel en $0 \in \mathbb{C}$, le générateur du groupe des itérés formels de f. Les coefficients sont donnés par une relation de récurrence assez simple. Le calcul sur Maple (je remercie une nouvelle fois P.V. Koseleff) des premiers coefficients montre une croissance Gevrey typique. Guidé par un calcul d'Oshima [**20**], j'ai pu démontrer la divergence. La Figure 7–f donne le résultat et en fait beaucoup plus (sauf la classe de Gevrey du générateur !) ; il est très facile d'en déduire une « démonstration rigoureuse ».

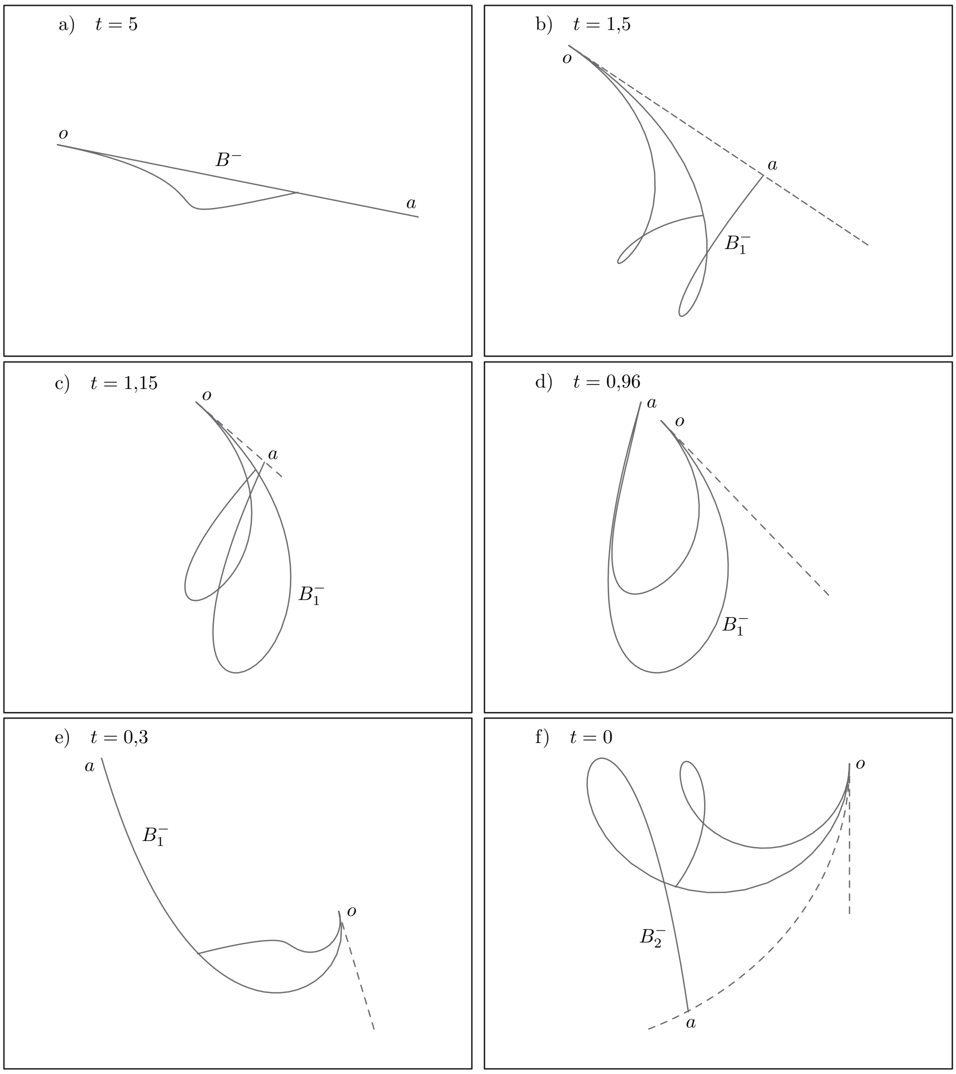

FIGURE 7. Première intersection de type I

– pour $1,2 > t > 0,96$ environ, l'évolution est très rapide ; des intersections de type II naissent puis disparaissent ; l'auto-intersection migre vers le point de contact (si $t = 1$ elle est au point de contact) puis disparaît ;

– pour $0,96 > t > 0,07$ environ, on retrouve une figure homéomorphe à a) ; pour $t = 0,07$ environ, un cusp apparaît ;

– pour $0,07 > t \geqslant 0$ on retrouve une figure homéomorphe à b).

Vérifier, dans chaque cas, que la figure est asymptotique, se fait aussi graphiquement, en choisissant un demi-disque Δ qui la contienne, centré au point de contact et limité d'un côté par un segment de la tangente, et en traçant l'image de son bord par $S_A \circ S_B$.

4.7. Application : une démonstration par ordinateur

« Théorème » 4.17. — *Soit B une tangente à une parabole A. L'indicateur $f = S_A \circ S_B$ (considéré au point de contact) n'est pas un carré itératif, i.e. l'unique difféomorphisme formel g tel que $g \circ g = f$ (voir § 2.3) diverge.*

Esquisse d'une démonstration graphique. — On choisit encore la parabole paramétrée par (25) et on paramètre les paires (A, B) par $t \in \mathbb{R}$, tel que le point de contact soit $z(t)$. Par symétrie, on peut encore se restreindre à $t \in [0, +\infty[$.

Compte tenu du Théorème 2.12, il suffit de montrer que les paires (A, B) et $(C, D) = (S_A(B), A)$ ne sont pas équivalentes. Rappelons qu'elles ont le même indicateur f. L'égalité $f = S_A \circ S_B$ donne les relations :

$$C_n^{\pm} = B_{n+1}^{\pm}; \qquad D_n^{\pm} = A_n^{\pm}.$$

On a « montré » dans le paragraphe précédent que B_n^- avait une intersection de type I avec B_{n+1}^- pour n assez grand, quelque soit $t \in \mathbb{R}$. Une étude graphique analogue « montre » qu'il en va de même pour A_n^- et A_{n+1}^-.

On compare graphiquement la figure de la première intersection de type I de B_n^- avec B_{n+1}^- à celle de $D_n^- = A_n^-$ avec $D_{n+1}^- = A_{n+1}^-$. On trouve que ces deux figures permettent la discrimination entre les deux paires sauf quand (il s'agit de valeurs approchées) :

$$t \in \left]0,07\ ,\ 0,247\right[\cup \left]0,55\ ,\ 0,96\right[\cup \left]1,732\ ,\ 6,5\right[. \tag{26}$$

Plus précisément, quand t n'appartient pas à cette réunion d'intervalles, on observe que l'intersection de type I est précédée par une auto-intersection ou une intersection de type II dans un cas et pas dans l'autre.

Quand t vérifie (26), on observe dans les deux cas que la première intersection de type I n'est précédée par aucune intersection de type II ni aucune auto-intersection. On enrichit alors la figure de la première intersection de type I de la façon suivante. Dans le premier cas, les arcs de B_n^- et de B_{n+1}^-, arrêtés au premier point d'intersection de type I, délimitent un domaine G_n. Le germe de A_n^- à l'origine rentre dans G_n. On regarde comment il en sort. On définit ainsi une figure asymptotique. Par le même procédé, on associe une figure asymptotique *de même construction* à la paire (C, D).

On observe que A_n^- sort de G_n en coupant B_{n+1}^- si $0,07 < t < 0,247$ ou si $1,732 < t < 6,5$, et en coupant B_n^- si $0,55 < t < 0,96$.

Pour chacun des intervalles de (26), on observe pour la paire (C, D) tout le contraire de ce qu'on a observé avec la paire (A, B). Un exemple est illustré par la Figure 8. On obtient donc le résultat. □

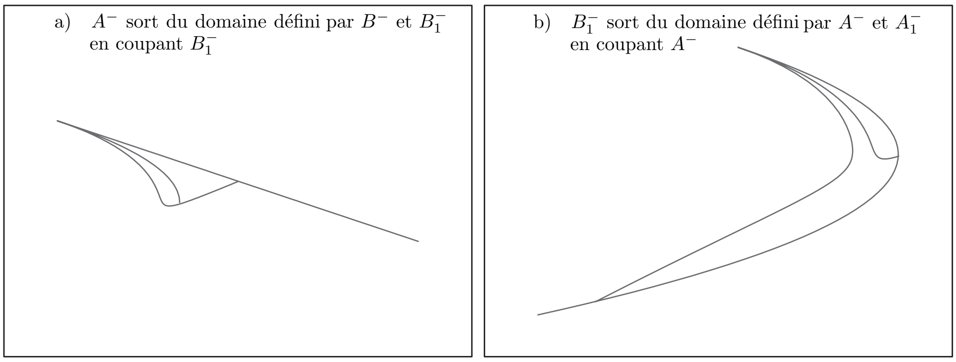

FIGURE 8. $t = 3$; les paires (A, B) et $(S_A(B), A)$ ne sont pas équivalentes

Exercice 4.18. — Soit A une parabole, $C = \{x^2 - y^2 = 1\}$ une hyperbole équilatère et D une tangente à C.

(1) Quelle est l'image de (C, D) par $z \mapsto z^2$?

(2) En déduire qu'il n'existe pas de tangente B à A telle que (A, B) soit équivalente à (C, D). (On a le droit d'utiliser le « Théorème 4.17 ».)

5. Des exemples, avec une ellipse et une droite

5.1. Introduction. — Dans ce chapitre, les paires (A, B) qu'on considère sont de la forme suivante :

(27) *A est une ellipse et B est une droite tangente à A.*

L'équivalence entre deux telles paires signifie toujours l'équivalence analytique dans les germes, au voisinage des points de contact. Je sais démontrer le résultat suivant :

Théorème 5.1. — *Deux paires (A, B) et (A', B') de la forme (27) sont équivalentes si et seulement si elles sont semblables. Si A n'est pas un cercle, le seul automorphisme de la paire (A, B) est l'identité.*

En particulier, si A est un cercle, A' aussi, et les difféomorphismes qui réalisent l'équivalence sont des homographies. Si A n'est pas un cercle, le théorème permet de supposer $A' = A$; alors B' est égal à B ou au symétrique de B par rapport au centre de A. La deuxième assertion de l'énoncé est plus facile à obtenir que la première par la méthode de la réflexion. Dans le cas où le résidu de la paire n'est pas nul, c'est aussi une conséquence d'un résultat général, le Théorème 2.11.

La démonstration du Théorème 5.1 est basée sur l'existence, pour chaque paire de la forme (27), d'une figure asymptotique qui la discrimine des paires qui ne lui sont pas semblables. Le calcul est pénible, mais possible parce qu'on arrive toujours à construire une telle figure avec (A, B), (A_1, B_1) et (A_2, B_2). Si l'on prend pour A une hyperbole, une difficulté apparaît quand l'angle entre ses asymptotes tend vers π : le nombre d'itérations nécessaires pour obtenir une figure asymptotique tend alors vers l'infini.

D'autre part, il serait plus intéressant de classer les paires (A, B) formées d'une conique A et d'un cercle (ou d'une droite) B, car il existe des paires de ce type qui sont équivalentes et qui ne sont pas semblables.

Une paire de paires non équivalentes est une chose aussi merveilleuse qu'un trèfle à trois feuilles, même si le comptage des feuilles est difficile. Je ne donnerai donc pas la démonstration du Théorème 5.1 ici. J'en présenterai un exemple intéressant, le cas où B est une tangente parallèle au grand axe de A ; un argument voisin de celui qu'on a utilisé dans le § 3.4 permet en effet la discrimination, sans calcul. À part cela, je me contenterai de justifier les illustrations de l'Introduction, et de faire quelques commentaires sur le cas plus facile des petites excentricités.

5.2. La réflexion par rapport à une ellipse. — La réflexion de Schwarz par rapport à une ellipse échange les ellipses homofocales et conserve les hyperboles homofocales. C'est une propriété remarquable dont j'ai trouvé l'énoncé dans un article de Webster [**29**].

On travaillera avec la famille $E(s)$ des ellipses de foyers -1 et $+1$:

$$E(s): \qquad \frac{x^2}{\cosh^2 s} + \frac{y^2}{\sinh^2 s} = 1.$$

On a $E(s) = E(-s)$ si $s \neq 0$, et on note aussi $E(0) = [-1, +1]$. En notation complexe, $E(s)$ a pour équation :

$$2(\cosh 2s) z\overline{z} - (z^2 + \overline{z}^2) - \sinh^2 2s = 0. \tag{28}$$

Bien sûr, toute ellipse du plan, autre qu'un cercle, est semblable à une et une seule de ces ellipses. L'excentricité de $E(s)$ vaut $1/\cosh s$.

À partir de l'équation (28), on obtient l'équation de la réflexion par rapport à l'ellipse $E(s)$ comme on a dit au § 1.3 :

$$\zeta^2 - 2(\cosh 2s)\, \overline{z}\zeta + (\overline{z}^2 + \sinh^2 2s) = 0. \tag{29}$$

Cette équation définit ζ comme fonction algébrique de $\overline{z}$, ramifiée autour des foyers ± 1. En notant $z \mapsto z^{1/2}$ la racine holomorphe sur $\mathbb{C} \setminus]-\infty, 0]$ qui vaut 1 en 1, on obtient deux branches uniformes sur $\mathbb{C} \setminus [-1, +1]$:

Notation 5.2. — Pour tout $s > 0$, on note :

$$S_{E(s)}(z) = \overline{z} \cosh 2s - \overline{z}\Big(1 - \frac{1}{\overline{z}^2}\Big)^{1/2} \sinh 2s. \tag{30}$$

C'est une fonction antiholomorphe sur $(\mathbb{C} \cup \{\infty\}) \setminus [-1, +1]$, qui coïncide au voisinage de $E(s)$ avec la réflexion de Schwarz par rapport à $E(s)$.

Si on remplace s par $-s$ dans le second membre de (30), on obtient l'autre solution de (29). On a $S_{E(s)}(\infty)=\infty$, et le comportement asymptotique :

$$S_{E(s)}(z)=\overline{z}\,\mathrm{e}^{-2s}+\frac{\sinh(2s)}{2\overline{z}}+O(\frac{1}{\overline{z}^2}),\qquad (z\longrightarrow\infty). \tag{31}$$

En particulier, *le reflet d'une tangente par une ellipse (autre qu'un cercle !) a une branche infinie.*

5.3. Le reflet d'une tangente par une ellipse. — Fixons $s>0$ et notons $A=E(s)$. Soit B une tangente à A et $B_1=S_A(B)$, où $S_A=S_{E(s)}$ est donné par (30). Soit $z(t)=a+bt$, $t\in\mathbb{R}$, une paramétrisation de B. B_1 est une courbe analytique paramétrée par $z_1(t)=S_A(a+bt)$. La paramétrisation est régulière, sauf si B passe par un des « cofoyers » $\pm\cosh(2s)$. Dans ce cas B_1 a une singularité de type cusp en un des foyers.

En remplaçant z par $z(t)$ dans l'équation de la réflexion (29) et en éliminant t, on obtient une équation du quatrième degré en ζ et $\overline{\zeta}$. Donc B_1 est contenu dans une quartique $\mathcal{B}_1$. Selon les cas (on laisse la vérification au lecteur) la quartique a un point double ou un point réel isolé.

La formule (31) montre que B_1 est asymptote aux deux bouts de la droite $\mathrm{e}^{-2s}S_{\mathbb{R}}(B)$. On suppose maintenant que B n'est parallèle ni à $\mathbb{R}$ ni à $i\mathbb{R}$. L'asymptote de B_1 est alors transverse à B. On en déduit que B_1, qui est tangente à B au point de base $A\cap B$, recoupe B au moins une fois, éventuellement encore au point de base. Mais B coupe aussi l'autre branche de la quartique $\mathcal{B}_1$ pour la même raison : elle est asymptote aux deux bouts de la droite $\mathrm{e}^{2s}S_{\mathbb{R}}(B)$ qui est parallèle à l'asymptote de B_1, mais distincte. On a obtenu quatre points intersections de la droite B avec $\mathcal{B}_1$. Il ne peut pas y en avoir d'autre. En résumé :

Lemme 5.3. — *Soit $s>0$ et $B_1=S_{E(s)}(B)$ le reflet d'une tangente B par rapport à $E(s)$. B_1 est une courbe analytique fermée dans $\mathbb{C}$, asymptote aux deux bouts de la droite $\mathrm{e}^{-2s}S_{\mathbb{R}}(B)$. Elle est régulière, sauf si B passe par un des points $\pm\cosh(2s)$. Elle a au plus un point double. Enfin, si B n'est pas parallèle à un des axes de $E(s)$, elle recoupe B une et une seule fois.*

Pour montrer que les illustrations de l'Introduction ont un caractère général, il reste à déterminer la nature de l'intersection de B_1 avec $E(s)$; c'est le contenu du Corollaire 5.7 ci-dessous.

5.4. La réflexion en coordonnées elliptiques. — Tout point $z\in\mathbb{C}$ peut s'écrire sous la forme :

$$z=\cosh(s+it),\qquad s\in\mathbb{R},\;\; t\in\mathbb{R}/2\pi.$$

On se réfère au couple $[s,t]$ comme aux coordonnées elliptiques de z. Un point z de coordonnées elliptiques $[s,t]$ a aussi les coordonnées elliptiques $[-s,-t]$. À ceci près, elles sont uniques.

Si on fixe $s \neq 0$, $t \mapsto \cosh(s+it)$ est une paramétrisation de $E(s)$ par $\mathbb{R}/2\pi$. Si on fixe $t \in]0, \frac{\pi}{2}[$, $\mathbb{R} \ni s \mapsto \cosh(s+it)$ est une paramétrisation de la branche de l'hyperbole :

$$H(t): \qquad \frac{x^2}{\cos^2 t} - \frac{y^2}{\sin^2 t} = 1,$$

située dans le demi-plan $x > 0$. On obtient l'autre branche en remplaçant t par $\pi - t$.

Soit $s > 0$. Le reflet d'un point par rapport à l'ellipse $E(s)$ est donné par une formule remarquablement simple en coordonnées elliptiques. Comme $h : w \mapsto \cosh w$ envoie la droite $L := \{\operatorname{Re} w = s\}$ sur $E(s)$, on obtient par conjugaison :

$$S_{E(s)}(\cosh w) = \cosh(2s - \overline{w}),$$

quand Rew est voisin de s. Si l'on prend en compte la définition (30) de $S_{E(s)}$ qu'on a choisie, on obtient :

Lemme 5.4. — *Soit $s > 0$. Pour tout $t \in [0, \pi]$, on a :*

$$\begin{aligned} S_{E(s)}(\cosh(s'+it)) &= \cosh(-s'+2s+it), \quad \text{si} \quad s' > 0, \\ &= \cosh(-s'-2s+it), \quad \text{si} \quad s' < 0. \end{aligned}$$

Cette formule a l'interprétation géométrique suivante : $S_{E(s)}$ envoie $E(s')$ sur $E(2s - s')$ pour tout $s' > 0$ et l'image d'un point non réel d'une branche d'une hyperbole homofocale $H(t)$ est situé sur cette même branche.

Notation 5.5. — Si $s \neq 0$, on note $\widehat{E}(s)$ le domaine de bord $E(s)$. On appelle parfois $\widehat{E}(s)$ une *ellipse pleine.*

Le Lemme 5.4 implique, voir la Figure 9 :

Lemme 5.6. — *Quel que soit $s > 0$, $S_{E(s)}$ induit une involution de $\widehat{E}(2s) \setminus [-1, +1]$. De plus :*

$$S_{E(s)}(\widehat{E}(3s) \setminus \overline{\widehat{E}(s)}) \subset \widehat{E}(s). \tag{32}$$

Une tangente B à $E(s)$ coupe l'ellipse $E(3s)$ en deux points distincts, transversalement. Le Lemme 5.6 a donc la conséquence suivante :

Corollaire 5.7. — *Quel que soit $s > 0$ et la tangente B à $E(s)$, chacun des deux brins $B_1^{\pm} = S_{E(s)}(B^{\pm})$ recoupe $E(s)$ en un et un seul point, transversalement.*

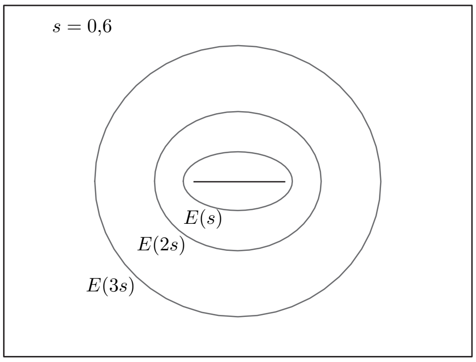

FIGURE 9

5.5. Petites excentricités. — Les invariants qu'on utilise pour démontrer le Théorème 5.1 proviennent des angles aux intersections de $B_1 = S_A(B)$ avec A, avec B et avec soi-même. Ceci ne signifie pas que ces intersections soient toutes et toujours contenues dans un domaine de Leau, mais que, dans tous les cas, on parvient à définir une figure asymptotique avec A, B, A_1, B_1, A_2, B_2, et qu'au moins un des angles décrits ci-dessus peut être réinterprété dans le cadre de cette figure, et fournir un invariant numérique de la paire.

Le cas des petites excentricités est plus facile. On a en effet :

Lemme 5.8. — *Si $s \geqslant 1,2$ — si l'excentricité est $< 0,55$ — l'ellipse pleine $\widehat{E}(s)$ est un domaine de Leau de la paire $(E(s), B)$, quelle que soit la tangente B.*

Ce n'est pas toujours le cas. La Figure 10–d montre un exemple où l'ellipse pleine, sans être un domaine de Leau, est contenue dans un domaine de Leau. La Figure 10–b montre un exemple où A_1 a une singularité asymptotique ; l'ellipse pleine n'est donc pas contenue dans un domaine de Leau. La Figure 10–c suggère (sans le démontrer) qu'il est possible qu'aucun A_n ne soit contenu dans un domaine de Leau.

Démonstration. — Remarquons d'abord que :

$$D(0, \sinh s) \subset \widehat{E}(s) \subset D(0, \cosh s)$$

quel que soit $s > 0$. Comme $S_{E(s)}$ envoie injectivement $\widehat{E}(2s) \smallsetminus \overline{\widehat{E}(s)}$ dans $\widehat{E}(s)$, il suffit que $S_B(\widehat{E}(s)) \subset \widehat{E}(2s)$ pour que $\widehat{E}(s)$ soit un domaine de Leau. Puisque

$$S_B(\widehat{E}(s)) \subset S_B(D(0, \cosh s) \subset D(0, 3\cosh s),$$

c'est vrai dès que $3\cosh s < \sinh 2s$; $s > 1,2$ convient. □

Si s, $s' > 1,2$, toute équivalence $(E(s), B) \to (E(s'), B')$ entre paires du type (27) se prolonge en un isomorphisme $\widehat{E}(s) \to \widehat{E}(s')$, par le même argument qu'on a employé dans le § 3.4. D'une certaine façon, la méthode de la réflexion a rempli son

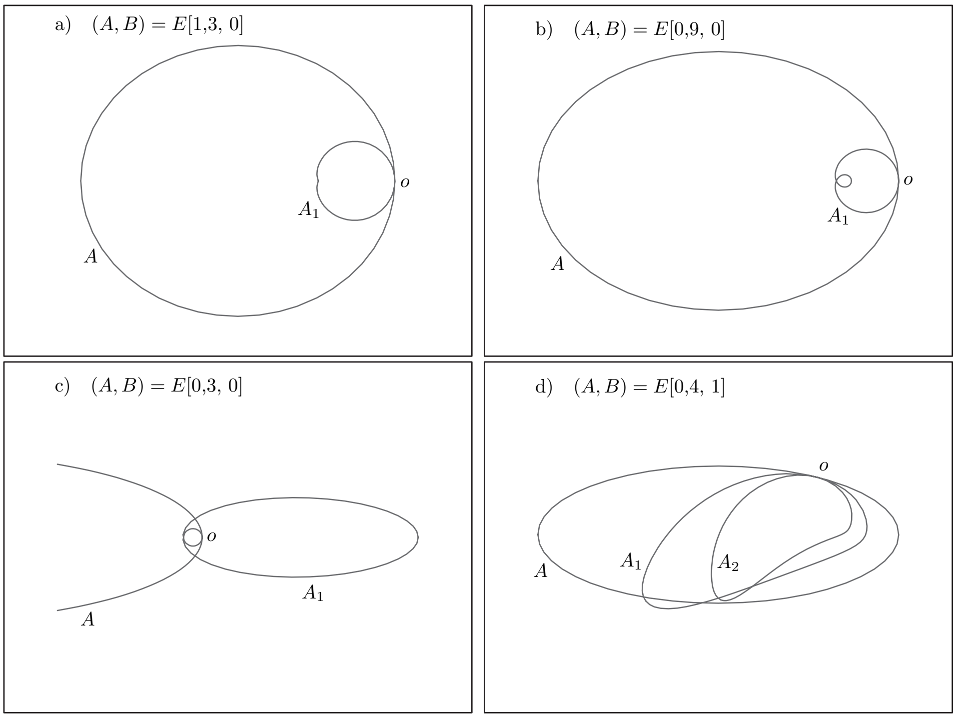

FIGURE 10

rôle. Il reste toutefois à montrer qu'un isomorphisme d'ellipses pleines, qui envoie *une* tangente de la première ellipse sur *une* tangente de la deuxième, est une similitude. Il peut exister des méthodes plus souples que celle de la réflexion pour traiter des problèmes de ce type !

Quoi qu'il en soit, sous l'hypothèse précédente, les intersections de $B_1^\pm$ avec A et le point double de B_1 définissent des figures asymptotiques, au sens du Chapitre 4. On obtient donc une démonstration du Théorème 5.1, *restreint aux petites excentricités*, en montrant que l'application

$$]1, 2, +\infty[\times[0, \pi/2] \ni (s, t) \longmapsto (\text{rés}(s, t), \beta^+(s, t), \beta^-(s, t), \gamma(s, t))$$

est injective, où $\cosh(s + it)$ est le point de contact de B avec $A = E(s)$. On a noté $\text{rés}(s, t)$ le résidu de la paire (le seul invariant formel, voir le Chapitre 2), $\beta^\pm(s, t)$ l'angle à l'intersection de $B_1^\pm$ avec A, et $\gamma(s, t)$ l'angle au point double de B_1 s'il existe, $\gamma = \varnothing$ s'il n'existe pas. C'est vrai, mais la vérification est pénible.

5.6. Quand l'excentricité tend vers 0. — Le fait que le reflet de Schwarz d'une droite par une ellipse soit si différent du reflet d'une droite par un cercle, aussi petite soit l'excentricité de l'ellipse, a quelque chose de fascinant.

Fixons $t \in [0, \pi/2]$. Pour tout $s > 0$, notons $(A(s), B(s))$ la paire de la forme (27) définie par :

$$A(s) = \mathrm{e}^{-s}\, E(s); \qquad B(s) \cap A(s) = \mathrm{e}^{-s}\, \cosh(s + it). \tag{33}$$

Quand $s \to +\infty$, l'ellipse $A(s)$ tend vers le cercle unité C et la droite $B(s)$ vers la tangente $\beta = \beta(t)$ au cercle en e^{it}. Notons g l'indicateur de la paire (C, β) et ρ la rotation de centre 0 et d'angle $-2t$.

En utilisant la formule (31), on calcule facilement la limite :

$$\beta_1 = \lim_{s \to +\infty} B_1(s).$$

C'est la réunion du cercle $g(\beta)$ et de la droite $\lambda_1 \ni 0$ d'angle polaire $-2t$.

En itérant, on voit que pour tout $n \in \mathbb{N}$:

$$\beta_n = \lim_{s \to +\infty} B_n(s)$$

est la réunion du bouquet (ou chapelet, ou grappe) de cercles $g(\beta_{n-1})$ et de la droite $\rho(\lambda_{n-1})$.

La configuration dépend de t ; par exemple, si t/π est rationnel, la droite λ_n est périodique (en n).

La Figure 11 montre comment B_n « approxime » une droite et un bouquet de n cercles pour $n = 1 \dots 4$ quand $s = 2$ (l'excentricité vaut $0,266...$) et $t = 0,6$. La Figure 12 montre mieux B_4. Il est peut-être intéressant de préciser que, si l'on prend le grand axe de l'ellipse comme unité, on a représenté l'image d'un segment de la tangente B, de longueur 80 unités pour B_1, 5 000 pour B_2, 300 000 pour B_3 et 12 millions pour B_4 !

5.7. Le cas d'une tangente parallèle au grand axe.

— C'est un cas exceptionnel. Le lemme suivant n'est vrai que quand B est une tangente parallèle au grand axe de A :

Lemme 5.9. — *Soit $s > 0$, $A = E(s)$ et B la tangente à A au point $m = i \sinh s$. Le demi-plan Δ de bord B qui contient $A \setminus \{m\}$ est un domaine de Leau de (A, B) ; $\overline{\Delta} \setminus \{m\}$ est contenu dans un domaine de Leau de (A, B).*

Démonstration. — La formule du Lemme 5.4 montre que S_A est injective sur $S_B(\overline{\Delta})$, et que l'image $S_A(z)$ d'un point $z \in S_B(\overline{\Delta})$ est située strictement au-dessous de B, sauf si $z = m$. Autrement dit $S_A \circ S_B(\overline{\Delta} \setminus \{m\}) \subset \Delta$. □

On considère Δ comme un disque de la sphère de Riemann $\mathbb{C} \cup \{\infty\}$. L'indicateur $f = S_A \circ S_B$ est holomorphe au voisinage du disque fermé $\overline{\Delta}$ et a deux points fixes $\in \partial\Delta$:

$$f(m) = m; \qquad f(\infty) = \infty.$$

La démonstration précédente montre que f est injective sur $\overline{\Delta}$ et que :

$$f(\overline{\Delta}) \subset \Delta \cup \{m, \infty\}.$$

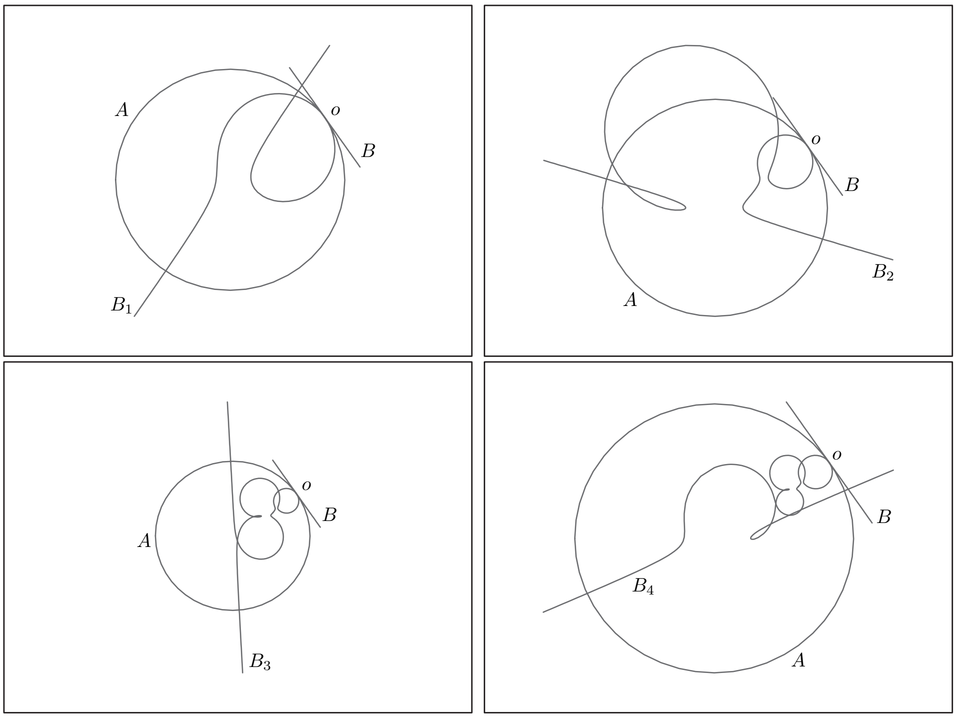

FIGURE 11. $(A, B) = E[2,\ 0,6]$

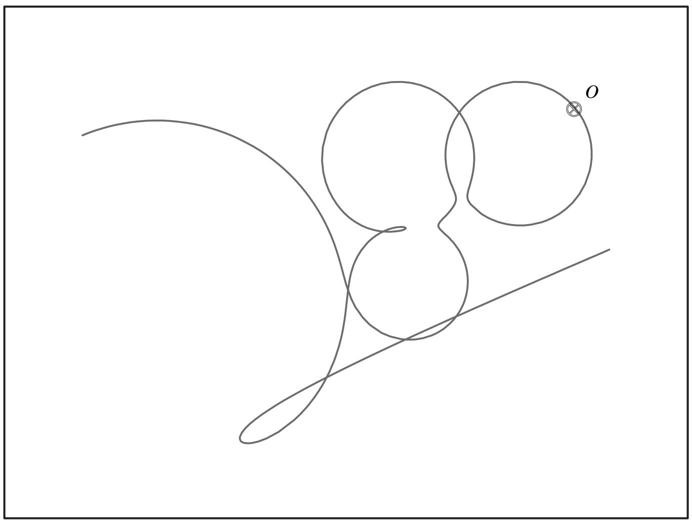

FIGURE 12. Un détail de B_4

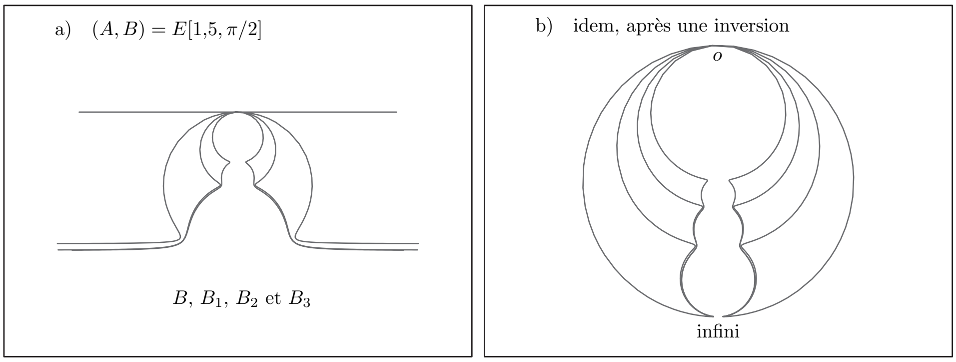

FIGURE 13

Considérons maintenant l'extension maximale de B_n, $n \geqslant 1$, dans $\overline{\Delta}$. C'est une courbe compacte lisse, contenue dans Δ, sauf pour les points m et ∞, où elle est tangente au cercle $\partial\Delta$. Notons $G_n \subset \Delta$ le domaine de bord B_n. On a les relations d'emboîtement :

$$\overline{G_{n+1}} \subset G_n \cup \{m, \infty\},$$

et f induit un isomorphisme :

$$f : G_n \setminus \overline{G_{n+1}} \longrightarrow G_{n+1} \setminus \overline{G_{n+2}}. \tag{34}$$

Si $A' = E(s')$ et si B' est la tangente en $i \sinh s'$ à A', la même construction s'applique : on obtient des domaines emboîtés $G'_n \subset \Delta'$, de bords B'_n, avec des propriétés analogues de l'indicateur f'.

On a alors facilement le cas très particulier suivant du Théorème 5.1 :

Lemme 5.10. — *Soit B et B' les tangentes à $A = E(s)$ et $A' = E(s')$, en $i \sinh s$ et $i \sinh s'$ respectivement. Si ϕ est une équivalence $(A,B) \to (A',B')$, ϕ est l'identité.*

Démonstration. — C'est une variante de la démonstration du § 3.4. On conserve les notations qui précèdent l'énoncé. En utilisant le fait que Δ est contenu dans le bassin d'attraction du point fixe $m = i \sinh s$ de f, la propriété analogue de Δ' et le fait qu'une équivalence ϕ de (A,B) sur (A',B') envoie, sur un voisinage fixé de m, les courbes B_n sur les courbes B'_n, on montre que ϕ se prolonge en un isomorphisme

$$\phi : \Delta \longrightarrow \Delta'.$$

Une dernière réflexion par rapport aux cercles $\partial\Delta$ et $\partial\Delta'$ permet de prolonger ϕ en un automorphisme de $\mathbb{C} \cup \{\infty\}$, qui conserve $\mathbb{C}$: ϕ est une similitude, donc l'identité. □

Exercice 5.11. — Avec les notations du Lemme 5.10, montrer que l'indicateur de la paire (A,B) n'est pas un carré itératif. En déduire que les indicateurs des paires (A,B) et (A',B') ne sont pas conjugués si $A \neq A'$.

Appendice : complément au Chapitre 2

A.1. Introduction. — Dans cet appendice, nous faisons un bilan des résultats connus sur la classification des paires d'arcs analytiques. C'est peut-être utile, car la classification formelle donnée par Kasner [**15**] et Pfeiffer [**22**] comporte (je crois) une (légère) erreur, et Nakai [**19**] ne traite qu'en quelques mots le cas des paires transverses, résonnantes ou non. La présentation, qui consiste à se ramener à la classification bien établie des difféomorphismes, via les indicateurs, est empruntée [**19**][(8)]. Antérieurement, Voronin [**28**] avait utilisé une méthode analogue pour classer les paires d'involutions *holomorphes* de $(\mathbb{C}, 0)$.

A.2. Classification formelle des difféomorphismes de $(\mathbb{C}, 0)$**.** — On la rappelle sans démonstration ; voir [**6**], [**14**], [**17**], [**19**].

Soit $f \in \widehat{\mathbf{G}}$. Le *multiplicateur* $\mu(f)$ de f est le coefficient $a_1 \in \mathbb{C}^*$ dans le développement :

$$f(z) = a_1 z + \sum_{n=2}^{+\infty} a_n z^n.$$

C'est un invariant formel. On a :

$$\mu(f \circ g) = \mu(f)\mu(g); \qquad \mu(S \circ f \circ S) = \overline{\mu(f)} \quad \text{si } S \in \widehat{\mathbf{S}}. \tag{35}$$

On dit que f est *résonnant* si $\mu(f)$ est une racine de l'unité, *non résonnant* sinon.

Si $f \in \widehat{\mathbf{G}}$ est résonnant et n'est pas d'ordre fini, son multiplicateur $\mu(f)$ est une racine primitive m-ième de l'unité, $m \in \mathbb{N}^*$, et $f^{(m)}$ est tangent à l'identité à un ordre fini $q \geqslant 1$. Le *type* $\tau(f)$ de f est défini par $\tau(f) = q + 1$; c'est un invariant formel. On remarque que q est un multiple de m, donc que $\mu(f)$ est une racine q-ième, pas forcément primitive, de l'unité.

On montre que tout $f \in \widehat{\mathbf{G}}$, de type $q + 1 \geqslant 2$, est formellement conjugué à un difféomorphisme de la forme :

$$f(z) = \mathrm{e}^{i2\pi k/q} z(1 + z^q - \kappa z^{2q} + \cdots) \tag{36}$$

Le nombre $\kappa \in \mathbb{C}$ est invariant par conjugaison formelle. J'appellerai *résidu* de f le nombre :

$$\text{rés}(f) = \frac{q+1}{2} + \kappa. \tag{37}$$

C'est le résidu normalisé de [**19**]. On a :

$$\text{rés}(S \circ f \circ S) = \overline{\text{rés}(f)} \quad \text{si } S \in \widehat{\mathbf{S}}. \tag{38}$$

Théorème A.1. — *Deux éléments de $\widehat{\mathbf{G}}$ sont formellement conjugués si et seulement si 1) ils ont le même multiplicateur et 2) dans le cas où ils sont résonnants et d'ordre infini, ils ont le même type et le même résidu.*

(8) L'exposé de [**19**] comporte plusieurs énoncés « évidemment faux ». D'autre part, je n'ai pas repris sa notion de « relation associative » qui ne me semble pas simplifier les choses.

A.3. Paires d'ordre fini. — C'est un cas qu'il vaut mieux écarter d'emblée. Ce qui est dit dans ce paragraphe vaut aussi dans le cas formel.

Proposition A.2. — *Toute paire d'ordre fini est équivalente à une paire de droites.*

Démonstration. — Soit (A, B) une paire d'ordre q, de multiplicateur $v = \mathrm{e}^{i2\pi k/q}$, $k \in \{0, \ldots, q-1\}$. Soit u son indicateur. On peut supposer $A = \mathbb{R}$. La méthode classique pour linéariser u consiste à introduire :

$$h = \frac{1}{q}\sum_{l=0}^{q-1} v^{-l} u^{(l)}.$$

En effet, on a $\mu(h) = 1$ et $h \circ u = vh$. Compte tenu de $S_{\mathbb{R}} \circ u = u^{-1} \circ S_{\mathbb{R}}$, on a :

$$S_{\mathbb{R}} \circ h \circ S_{\mathbb{R}} = \frac{1}{q}\sum_{l=0}^{q-1} v^{l} u^{(-l)} = h,$$

donc h est réelle et $(h(A), h(B)) = (\mathbb{R}, h(B))$ a pour indicateur vI : $h(B)$ est la droite qui fait l'angle $-\pi p/q$ modulo π avec $\mathbb{R}$. □

Une paire d'ordre fini a un « gros » groupe d'automorphismes. Toutes les transformations réelles de la forme $h(z) = z(1 + \sum_{k\geqslant 1} h_k z^{kq})$ conservent la paire de droites $(\mathbb{R}, \mathrm{e}^{-i\pi p/q}\mathbb{R})$. C'est un cas exceptionnel.

A.4. Difféomorphismes modèles. — Chaque classe formelle a des représentants analytiques, mais certains sont distingués par le fait qu'ils ont le maximum de « symétries » analytiques. On donne ci-dessous les modèles les plus couramment cités.

Un difféomorphisme formel non résonnant, ou résonnant et d'ordre fini, est formellement conjugué à la similitude :

$$f_u(z) := uz$$

de même multiplicateur. Pour $q \in \mathbb{N}^*$, on notera :

$$\omega_q = f_{\mathrm{e}^{i2\pi/q}}.$$

Pour tout $q \in \mathbb{N}^*$ et tout $r \in \mathbb{C}$, on introduit le champ de vecteurs :

$$X_{q,r} := iz^{q+1}(1 + irz^q)^{-1}\frac{d}{dz}.$$

On vérifie que :

$$(f_u)_* X_{q,r} = u^{-q} X_{q,ru^{-q}}. \tag{39}$$

Un difféomorphisme formel, tangent à l'ordre $q \geqslant 1$ à l'identité et de résidu r, est formellement conjugué à :

$$f_{q,r} := \exp X_{q,r}.$$

Compte tenu de (39), on a $\omega_q^{(k)} \circ f_{q,r} = f_{q,r} \circ \omega_q^{(k)}$ quel que soit $k \in \mathbb{Z}$. Un difféomorphisme formel de type $q+1$, de multiplicateur $e^{i2\pi k/q}$ et de résidu r, est formellement conjugué à :

$$f_{q,k,r} := \omega_q^{(k)} \circ f_{q,r} = f_{q,r} \circ \omega_q^{(k)}.$$

Pour tout $t \in \mathbb{C}$, on définit l'itéré d'ordre t de $f_{q,r}$ par :

$$f_{q,r}^{(t)} = \exp(tX_{q,r}).$$

En posant $t = u^{-q}$ dans (39), on obtient $f_{q,rt}^{(t)} = f_u \circ f_{q,r} \circ f_u^{-1}$, donc :

$$\text{rés}(f_{q,rt}^{(t)}) = \text{rés}(f_{q,r}), \quad (t \in \mathbb{C}^*).$$

Si $f \in \widehat{\mathbf{G}}$ est tangent à l'identité, de type $q+1$ et de résidu r, on peut écrire $f = \phi \circ f_{q,r} \circ \phi^{-1}$, $\phi \in \widehat{\mathbf{G}}$. Pour tout $t \in \mathbb{C}$, on définit *l'itéré d'ordre* t de f par

$$f^{(t)} = \phi \circ f_{q,r}^{(t)} \circ \phi^{-1};$$

c'est indépendant du choix de ϕ. L'avant-dernière formule donne :

$$\text{rés}(f^{(t)}) = \frac{\text{rés}(f)}{t}, \quad (t \in \mathbb{C}^*). \tag{40}$$

Si $f \in \mathbf{G}$ est tangent à l'identité, on dit que f est *pleinement itérable* si $f^{(t)} \in \widehat{\mathbf{G}}$ converge pour tout $t \in \mathbb{C}$. On a rappelé dans le § 2.3 trois théorèmes importants à ce sujet.

Le lemme suivant complète le Théorème A.1 :

Lemme A.3. — *Si u n'est pas une racine de l'unité, $\widehat{C}(f_u) = C(f_u)$; c'est le groupe des similitudes. Pour tout $q \geqslant 1$, tout $k \in \{0, \dots, q-1\}$ et tout $r \in \mathbb{C}$, $\widehat{C}(f_{q,k,r}) = C(f_{q,k,r})$; c'est le produit direct du groupe cyclique engendré par ω_q et du groupe des itérés d'ordre complexe de $f_{q,r}$.*

A.5. Classification formelle des paires. — On utilise les notations du § A.4. Remarquons que pour tout $q \in \mathbb{N}^*$, tout $r \in \mathbb{C}$, tout $t \in \mathbb{C}$:

$$(S_{\mathbb{R}})_* tX_{q,r} = -\overline{t} X_{q,-\overline{r}}.$$

On a donc les formules :

$$S_{\mathbb{R}} \circ f_u \circ S_{\mathbb{R}} = f_{\overline{u}}\,; \quad S_{\mathbb{R}} \circ \omega_q^{(k)} \circ f_{q,r}^{(t)} \circ S_{\mathbb{R}} = \omega_q^{(-k)} \circ f_{q,-\overline{r}}^{(-\overline{t})}\,. \tag{41}$$

Lemme A.4. — *Un élément f de $\widehat{\mathbf{G}}$ est un indicateur de paire formelle si et seulement 1) $|\mu(f)| = 1$ et 2) le cas échéant* $\text{rés}(f) \in i\mathbb{R}$.

Ce résultat est faux dans le cas analytique ; voir [**19**] pour les paires tangentes, et les remarques du § A.7.

Démonstration. — Les formules (35), (38) et (40) montrent que les conditions sont nécessaires.

Les formules (41) donnent la réciproque. En effet, si $|u|=1$, si r est imaginaire pur et pour $t=1$, on obtient :

$$S_{\mathbb{R}} \circ f_u \circ S_{\mathbb{R}} = f_u^{-1}; \quad S_{\mathbb{R}} \circ f_{q,k,r} \circ S_{\mathbb{R}} = f_{q,k,r}^{-1}.$$

Il en résulte que, pour $f = f_u$ ou $f = f_{q,k,r}$, f est l'indicateur de la paire $(S_{\mathbb{R}}, S_{\mathbb{R}} \circ f)$. □

Deux paires formelles formellement équivalentes ont des indicateurs formellement conjugués. La réciproque est seulement presque vraie :

Théorème A.5. — *Les paires formelles dont les indicateurs appartiennent à une classe de conjugaison formelle donnée forment une classe d'équivalence formelle, sauf si elles sont résonnantes de type impair. Dans ce cas-ci, elles se répartissent en deux classes d'équivalence formelle.*

Démonstration. — On utilise les remarques du § 1.9 et le Lemme A.3. On écarte le cas des paires d'ordre fini, déjà traité. Il suffit de classer formellement les paires d'indicateur donné f, quand f est un des difféomorphismes modèles du § A.4.

Supposons d'abord $f = f_{e^{i2\pi t}}$, $t \in \mathbb{R} \setminus \mathbb{Q}$. Compte-tenu de (41), les paires formelles d'indicateur f sont les paires $(S, S\circ f)$ avec $S = S_{\mathbb{R}} \circ f_{e^{it'}}$, $t' \in \mathbb{R}$. Les paires formellement équivalentes à $(S_{\mathbb{R}}, S_{\mathbb{R}} \circ f)$ sont les paires $(S, S\circ f)$ avec

$$S = f_u \circ S_{\mathbb{R}} \circ f_u^{-1} = f_{\overline{u}/u}.$$

Autrement dit, les rotations opèrent simplement et transitivement sur l'ensemble des paires d'indicateur f et les homothéties les conservent.

Soit maintenant $q \in \mathbb{N}^*$, $k \in \{0,\dots,q-1\}$, $r \in i\mathbb{R}$ et :

$$f = f_{q,k,r} = \omega_q^{(k)} \circ g; \quad g = f_{q,r}.$$

Comme r est imaginaire pur, (41) donne, si $l \in \{0,\dots,q-1\}$ et $t \in \mathbb{C}$:

$$S_{\mathbb{R}} \circ \omega_q^{(l)} \circ g^{(t)} \circ S_{\mathbb{R}} = \omega_q^{(-l)} \circ g^{(-\overline{t})}.$$

Les paires formelles d'indicateur f sont donc les paires $(S, S\circ f)$ avec

$$S = S_{\mathbb{R}} \circ \omega_q^{(l)} \circ g^{(t)}, \quad (l \in \mathbb{Z}, \quad t \in \mathbb{R}).$$

Les paires formellement équivalentes à cette paire sont les paires :

$$\omega_q^{(m)} \circ g^{(s)} \circ S_{\mathbb{R}} \circ \omega_q^{(l)} \circ g^{(t)} \circ \omega_q^{(-m)} \circ g^{(-s)} = S_{\mathbb{R}} \circ \omega_q^{(l-2m)} \circ g^{(t-(s+\overline{s}))},$$

où m décrit $\mathbb{Z}$ et s décrit $\mathbb{C}$.

Étant donnés $l' \in \mathbb{Z}$ et $t' \in \mathbb{R}$, l'équation $t-(s+\overline{s}) = t'$ est résoluble en $s \in \mathbb{C}$. L'équation $l - 2m = l'$ modulo q est résoluble en m si q est impair. Si q est pair, elle n'est résoluble que si $l - l'$ est pair. □

La démonstration montre de plus que le groupe des automorphismes formels d'une paire formelle d'indicateur f est indépendant de cette paire. Il est engendré par les itérés de g, d'ordres imaginaires purs, et *dans le cas où q est pair*, l'unique involution de $\widehat{C}(f)$.

Les modèles de paires qu'on vient de décrire ne sont pas tous explicites, mais ils ont le plus de symétries analytiques possibles. Les modèles plus simples suivants n'ont pas cette propriété :

***Théorème A.6* (Kasner, Pfeiffer).** — *Toute paire formelle est formellement équivalente à une paire de droites ou à une et une seule des paires suivantes :*

$$A := \{y = \varepsilon x^{q+1} + l x^{2q+1}\}, \quad B := \{\sin(k\pi/q)\, x - \cos(k\pi/q)\, y = 0\},$$

où $q \in \mathbb{N}^*$, $k \in \{0, \dots, q-1\}$, $l \in \mathbb{R}$ et $\varepsilon = 1$ si q est impair, $\varepsilon = \pm 1$ si q est pair.

Je n'ai pas repéré dans [**15**] ni [**22**] l'invariant $\varepsilon \in \{\pm 1\}$ quand l'ordre de contact est pair .

Démonstration. — Un calcul analogue à celui du § 2.1 montre que :

$$S_A(z) = \overline{z} \left(1 + 2i\varepsilon \overline{z}^q + (2i\varepsilon)^2 (\frac{q+1}{2} - i\frac{l}{2}) \overline{z}^{2q} + \cdots \right).$$

On a $S_B(z) = \mathrm{e}^{-2i\pi k/q} S_{\mathbb{R}}$, d'où l'indicateur

$$f(z) = \mathrm{e}^{-2i\pi k/q} z \left(1 + 2i\varepsilon z^q + (2i\varepsilon)^2 (\frac{q+1}{2} - i\frac{l}{2}) z^{2q} + \cdots \right).$$

Si $\varepsilon = \pm 1$ est fixé, on obtient bien tous les indicateurs de type fini possibles, à conjugaison formelle près, une et une seule fois.

Notons, q, k et l étant fixés, $(A_\pm, B)$ la paire correspondant à $\varepsilon = \pm 1$; elles ont des indicateurs différents mais formellement conjugués. Si q est impair, $z \mapsto -z$ induit une équivalence entre (A_+, B) et (A_-, B). Supposons maintenant q pair et soit

$$z = \sum_{n=1}^{+\infty} a_n Z^n$$

une substitution qui transforme (A_+, B) en (A_-, B). La conservation de B donne :

$$\mathrm{Im}(a_n \mathrm{e}^{i(n-1)\pi k/q}) = 0 \tag{42}$$

pour tout n. En particulier, $\mathrm{Im}\, a_1 = \mathrm{Im}\, a_{q+1} = 0$. On calcule maintenant l'équation de l'image de A_+. Modulo $o(Y)$ et $o(X^{q+1})$, on obtient :

$$a_1 Y + \sum_{n=2}^{q+1} \mathrm{Im}(a_n) X^n = a_1^{q+1} X^{q+1} + \cdots .$$

On a donc $\mathrm{Im}(a_n) = 0$ puis, compte tenu de (42), $a_n = 0$ pour $n = 2 \dots q$. Comme $\mathrm{Im}(a_{q+1}) = 0$, l'équation se réduit à $Y = a_1^q X^{q+1} + \dots$; comme q est pair, $a_1^q > 0$ et la substitution cherchée n'existe pas. □

A.6. Classification des paires de type fini. — Les Théorèmes 2.7 et 2.8 ont la conséquence suivante :

***Théorème A.7* (Nakai).** — *Soit $f \in \mathbf{G}$ un indicateur de paire de type $q+1$ et $N(f) \geqslant 1$ le nombre des classes d'équivalence des paires d'indicateurs conjugués à f. Si $f^{(q)}$ est pleinement itérable, $N(f) = 1$ si q est impair et $N(f) = 2$ si q est pair. Si $f^{(q)}$ n'est pas pleinement itérable, $N(f) = 2$ si q est impair ; si q est pair $N(f) = 2$ ou 4 selon les cas.*

Démonstration. — Si $f^{(q)}$ est pleinement itérable, $f^{(q)}$ est conjugué à l'un des modèles du § A.4 d'après le Théorème 2.8, donc aussi f. Alors $C(f) = \widehat{C}(f)$ et le Théorème A.5 donne le résultat.

On suppose maintenant que $f^{(q)}$ n'est pas pleinement itérable. D'après le Théorème d'Écalle-Liverpool, le sous-groupe des éléments de $C(f)$ tangents à l'identité est engendré par un élément h de type $q+1$. L'ensemble $M(f) = \{\mu(\phi), \quad \phi \in C(f)\}$ est un groupe cyclique d'ordre ρ, un diviseur de q. Soit $\psi \in C(f)$ de multiplicateur $\mathrm{e}^{i2\pi/\rho}$. Le groupe $C(f)$ est engendré par ψ et h.

Soit $(S, S \circ f)$ une paire d'indicateur f. Tout élément ϕ de type fini de $C(f)$ a un résidu $\in i\mathbb{R}$. On en déduit (la vérification est formelle, voir alors la démonstration du Théorème A.5) que $S \circ \phi \circ S = \phi^{-1}$ pour tout $\phi \in C(f)$, donc que

$$P(\phi) = (S \circ \phi, S \circ \phi \circ f)$$

est une paire, qu'on obtient ainsi toutes les paires d'indicateur f, et que ϕ_1 et ϕ_2 définissent des paires équivalentes si et seulement si $\phi_1 \circ \phi_2^{-1}$ est un carré dans $C(f)$.

Tout $\phi \in C(f)$ s'écrit $\phi = \psi^{(m)} \circ h^{(n)}$, m, $n \in \mathbb{Z}$. En réduisant modulo les carrés, selon la parité de m et n, on voit que toute paire d'indicateur f est équivalente à l'une des suivantes :

$$P(I), \quad P(h), \quad P(\psi), \quad P(\psi \circ h)\,.$$

Si ρ est impair (c'est le cas si q est impair), $\mu \mapsto \mu^2$ est un automorphisme de $M(f)$. En particulier $\mu(\psi)$ est un carré dans $M(f)$ et on peut écrire $\psi = \phi^{(2)} \circ h^{(n)}$; selon la parité de n, ψ ou $\psi \circ h$ est un carré. La liste se réduit à $P(I)$, $P(h)$. Mais h n'est un carré dans $C(f)$ puisqu'il n'a pas de racine itérative et que $-1 \notin M(f)$: $N(f) = 2$, les deux classes provenant de la même classe formelle.

Si q et ρ sont pairs, $-1 \in M(f)$. Si h est un carré de $C(f)$, $N(f) = 2$; sinon $N(f) = 4$. □

A.7. Remarques sur les paires non résonnantes. — Il s'agit des paires (A, B) de multiplicateur $\mathrm{e}^{i2\pi t}$ où $t \in \mathbb{R} \setminus \mathbb{Q}$; les deux arcs A et B font un angle $-\pi t$ modulo π, et c'est le seul invariant formel de ces paires.

Le problème de la classification analytique est bien plus délicat. Pfeiffer [**23**] a montré le premier l'existence de difféomorphismes non résonnants qui ne sont pas linéarisables, *i.e.* qui ne sont pas conjugués à une similitude. Il a montré dans le même article qu'il en existait qui étaient des indicateurs de paires. La théorie des difféomorphismes non résonnants a fait depuis de grands progrès, mais il ne semble pas qu'on l'ait appliquée aux paires. On se contentera de quelques commentaires.

Si $t \in \mathbb{R} \smallsetminus \mathbb{Q}$ est un nombre de Bruno, un difféomorphisme de multiplicateur $\mathrm{e}^{i2\pi t}$ est linéarisable (Théorème de Bruno, voir [**30**]), donc une paire de multiplicateur $\mathrm{e}^{i2\pi t}$ est équivalente à une paire de droites.

Si $t \in \mathbb{R} \smallsetminus \mathbb{Q}$ n'est pas un nombre de Bruno, l'ensemble des difféomorphismes de multiplicateur $\mathrm{e}^{i2\pi t}$ se scinde en une infinité non dénombrable de classes de conjugaison analytique (Théorème de Yoccoz, voir [**30**]).

Soit encore $t \in \mathbb{R} \smallsetminus \mathbb{Q}$ et $f \in \mathbf{G}$, de multiplicateur $\mathrm{e}^{i2\pi t}$. Supposons que f n'est pas linéarisable, et notons :

$$\mathrm{M}(f) = \{\mu(g), \quad g \in C(f)\}.$$

C'est une partie du cercle unité car, si $g \in C(f)$ et $|\mu(g)| \neq 1$, g est linéarisable (en vertu du Théorème de Koenigs, voir par exemple [**8**]), donc aussi f. C'est un sous-groupe propre du cercle unité car il ne contient aucun nombre de Bruno (irrationnel). Si de plus f est l'indicateur d'une paire $(S, S \circ f)$, on voit facilement, selon un raisonnement qu'on a fait plusieurs fois et compte-tenu du fait que $|\mu(g)| = 1$ si $g \in C(f)$, que $(S \circ g, S \circ g \circ f)$ est une paire pour tout $g \in C(f)$, qu'on obtient ainsi toutes les paires d'indicateur f, et que g_1, $g_2 \in C(f)$ définissent des paires équivalentes si et seulement si $g_1 \circ g_2^{-1}$ est un carré dans $C(f)$. L'ensemble des classes d'équivalence des paires dont l'indicateur est conjugué à f est donc en bijection avec :

$$\mathrm{M}(f) \,/\, \{u^2, \ u \in \mathrm{M}\,(f)\}.$$

Pérez-Marco [**21**] a montré qu'il existe $t \in \mathbb{R} \smallsetminus \mathbb{Q}$ et $f \in \mathbf{G}$ de multiplicateur $\mathrm{e}^{i2\pi t}$ tel que $\mathrm{M}(f)$ ne soit pas dénombrable.

Les résultats de Yoccoz et de Pérez-Marco qu'on a cités suggèrent que les questions suivantes ont des réponses positives ; les spécialistes des méthodes de [**30**] et [**21**] devraient pouvoir se prononcer.

(1) *Si $t \in \mathbb{R} \smallsetminus \mathbb{Q}$ n'est pas un nombre de Bruno, existe-t-il $f \in \mathbf{G}$, de multiplicateur $\mathrm{e}^{i2\pi t}$, qui ne soit pas un indicateur de paire ?*

(2) *Si $t \in \mathbb{R} \smallsetminus \mathbb{Q}$ n'est pas un nombre de Bruno, existe-t-il une infinité non dénombrable de paires de multiplicateur $\mathrm{e}^{i2\pi t}$, deux à deux non équivalentes ?*

(3) *Existe-t-il $f \in \mathbf{G}$ et une infinité non dénombrable de paires d'indicateur f, deux à deux non équivalentes ?*

A.8. Remarque historique. — Le problème de la classification formelle des paires a été étudié d'abord par Kasner, voir [**15**] dans les comptes rendus du 5ème Congrès International des Mathématiciens (1912). Il traite le cas non résonnant et le cas des paires tangentes, qu'il nomme « horn angles ». Il considère aussi le cas résonnant général, mais sans le résoudre. Dans tous les cas, il pose le problème de la convergence des transformations normalisantes. Le cas transverse résonnant est traité par Pfeiffer [**22**].

Birkhoff connaissait et appréciait les travaux de Kasner, voir [**7**], pages 309 et 310, voir aussi Davis [**9**], pages 212 et 213. On peut aussi remarquer que le premier exemple de difféomorphisme formellement mais non analytiquement linéarisable a été donné par Pfeiffer [**23**] en 1915, à propos d'un des problèmes posés par Kasner. Cet article est cité dans la plupart des livres de dynamique holomorphe. Dans cet article, Pfeiffer dit suivre une suggestion de Birkhoff !

Finalement, en 1939, Birkhoff résout le problème de la classification analytique des difféomorphismes résonnants, comme en passant, dans un article dont le titre « Sur les fonctions auto-équivalentes ... » n'évoque ni la géométrie conforme, ni la dynamique, ni la classification des difféomorphismes. Ni Birkhoff, ni Kasner, ni personne n'appliquera les résultats de cet article au problème des paires d'arcs tangents. En 1995, Nakai [**19**] applique la théorie d'Écalle-Voronin au problème.

Qui a lu l'article de Birkhoff avant 1995 ? En tout cas, il n'a pas trouvé son lecteur avant cette date.

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