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ARITHMETIC DIVISORS ON ORTHOGONAL AND UNITARY SHIMURA VARIETIES

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Abstract. — The three papers in this volume concern the modularity of generating series of divisors on integral models of orthogonal and unitary Shimura varieties.

***Résumé* (Diviseurs arithmétiques sur les variétés orthogonales et unitaires de Shimura).** — Les trois articles de ce volume traitent de modularité des séries génératrices des diviseurs sur les modèles entiers de variétés orthogonales et unitaires de Shimura.

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PREFACE

by

Jan H. Bruinier, Benjamin Howard, Stephen S. Kudla,
Keerthi Madapusi Pera, Michael Rapoport & Tonghai Yang

The formation of modular generating series whose coefficients are geometric cycles began with the work of Hirzebruch-Zagier [**14**], who constructed divisors on compactified Hilbert modular surfaces over $\mathbb{C}$, and showed that their cohomology classes formed the coefficients of a weight 2 modular form.

An extensive study of the modularity of generating series for cohomology classes of special cycles in Riemannian locally symmetric spaces $M = \Gamma\backslash X$ was undertaken in a series of papers [**21, 22, 23**] of Kudla and Millson. The main technical tool was a family of Siegel type theta series valued in the de Rham complex of M, from which modularity was inherited by the image in cohomology.

The special cycles used by Kudla-Millson are given by an explicit geometric construction, and so, in the cases where M is (the complex fiber of) a Shimura variety, it is natural to ask whether the analogous generating series for special cycle classes in the Chow group is likewise modular. In the case of Shimura varieties of orthogonal type, this question was raised in [**19**]. In some special cases modularity of the Chow group-valued generating series can be deduced from modularity of the cohomology-valued generating series; see [**27, 26**] for example.

The generating series for Heegner points in the Jacobian of a modular curve was proved to be modular by Gross-Kohnen-Zagier [**12**]. Motivated by their work, Borcherds [**2, 3**] proved the modularity of the generating series of Heegner (= special) divisors in the Chow groups of Shimura varieties of orthogonal type. His method depended on the miraculous construction of Borcherds products: meromorphic modular forms on orthogonal Shimura varieties, constructed via a regularized theta lift, whose explicitly known divisors provide enough relations among special divisors to prove modularity.

The three papers in this volume are concerned with similar modularity results, but now for generating series of divisors on integral models of orthogonal and unitary Shimura varieties; more precisely, of generating series with coefficients in the codimension one arithmetic Chow groups of Gillet-Soulé.

The first results in this arithmetic direction were obtained in [**20**], which dealt with arithmetic divisors on quaternionic Shimura curves (a special case of orthogonal Shimura varieties). Still in the Shimura curve setting, quite complete results on the modularity of generating series were obtained in the book [**24**]. There the case of arithmetic 0-cycles is also treated and the corresponding generating series is shown to coincide with the central derivative of a weight $3/2$ Siegel genus 2 incoherent Eisenstein series.

The Green functions used in [**20, 24**] are derived from the Kudla-Millson theta series, and a similar construction can be used to obtain Green functions for special divisors on all orthogonal Shimura varieties. On the other hand, Bruinier [**4**] generalized the regularized theta lift of Borcherds by allowing harmonic Maass forms as inputs. This provides a different construction of Green functions for special divisors, with the advantage that one can try to use the method of Borcherds to establish modularity of the corresponding generating series with coefficients in the arithmetic Chow group. In the case of Hilbert modular surfaces (once again, a special case of orthogonal Shimura varieties), this was done in [**5**].

The main obstruction to extending the method of Borcherds to integral models is that the divisor of a Borcherds product is, a priori, only known on the generic fiber of the Shimura variety. To obtain modularity of the generating series with coefficients in the codimension one arithmetic Chow group, one must compute the divisor of a Borcherds product on the integral model, where the divisor may contain vertical components.

The first paper [**6**] of this volume deals with arithmetic divisors on compactified unitary Shimura varieties of signature $(n-1,1)$, and the main result is the modularity of the corresponding generating series with coefficients in the arithmetic Chow group. The proof follows the method of Borcherds, with the essential new ingredient being the calculation of the vertical components and boundary components appearing in the divisor of a unitary Borcherds product.

The second paper [**7**] of this volume contains applications of the modularity result just stated. One can form the Petersson inner product of the generating series of arithmetic divisors against a cusp form g of the appropriate weight and level. This defines a class in the codimension one arithmetic Chow group of the unitary Shimura variety, called the arithmetic theta lift of g. On the other hand, taking Zariski closures of CM points yields cycles of dimension one on the integral model, which one can then intersect with the arithmetic theta lift. The main results show that such intersections are equal to central derivatives of (generalized) L-functions, somewhat in the spirit of the Gross-Zagier theorem [**13**] on heights of Heegner points. These results complete, in some sense, the series of papers [**16, 17, 8**], which contain the bulk of the intersection calculations.

The second paper also proves special cases of Colmez's conjecture [**10**] on the periods of CM abelian varieties. These special cases can actually be deduced from the averaged version of the conjecture [**1, 25**], but the proofs given here yield new information about the arithmetic of unitary Shimura varieties, which we hope is of independent interest.

Another application of the modularity result on unitary Shimura varieties has been found by W. Zhang [**29**], who has used it in his proof of the Arithmetic Fundamental Lemma.

The third paper [**18**] proves the modularity of generating series of arithmetic divisors on integral models of orthogonal type Shimura varieties. As in the unitary case, the new ingredient in the proof of modularity is the calculation of divisors of Borcherds products on integral models. This extends results of Hörmann [**15**], who does such calculations only after inverting all primes where the integral model has nonsmooth reduction. Hörmann must assume that the Shimura variety has cusps (so that one can study the Borcherds product using its q-expansion), an assumption that is removed here using an arithmetic version of the embedding trick of Borcherds.

With the results of this volume in hand, it is natural to ask about the modularity of generating series of arithmetic special cycles in higher codimension. Although the reader will find no such results in this volume, there is progress along these lines. The modularity of generating series of higher codimension cycles in the Chow group of the generic fiber of an orthogonal Shimura variety has been proved by Bruinier and Raum [**9**], building on the unpublished thesis of W. Zhang [**28**]. An extension of this result to cycles in the Chow groups of the integral model will appear in forthcoming work of Howard and Madapusi Pera, but extending the result further to arithmetic Chow groups remains an open problem. The recent construction of Green currents for higher codimension special cycles by Garcia-Sankaran [**11**] is a significant step in this direction.

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MODULARITY OF GENERATING SERIES OF DIVISORS ON UNITARY SHIMURA VARIETIES

by

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Abstract. — We form generating series, valued in the Chow group and the arithmetic Chow group, of special divisors on the compactified integral model of a Shimura variety associated to a unitary group of signature $(n-1,1)$, and prove their modularity. The main ingredient in the proof is the calculation of vertical components appearing in the divisor of a Borcherds product on the integral model.

Résumé **(Modularité des séries génératrices de diviseurs sur les variétés de Shimura unitaires)**
Nous formons des séries génératrices, à valeurs dans le groupe de Chow et dans le groupe de Chow arithmétique, formées des diviseurs spéciaux sur le modèle intégral compact d'une variété de Shimura associée à un groupe unitaire de signature $(n-1,1)$, et prouvons leur modularité. L'ingrédient principal de la preuve est le calcul des composantes verticales apparaissantes dans le diviseur d'un produit de Borcherds sur le modèle intégral.

1. Introduction

The goal of this paper is to prove the modularity of a generating series of special divisors on the compactified integral model of a Shimura variety associated to a unitary group of signature $(n-1,1)$. The special divisors in question were first studied on the open Shimura variety in **[33, 34]**, and then on the toroidal compactification in **[24]**.

This generating series is an arithmetic analogue of the classical theta kernel used to lift modular forms from $\mathrm{U}(2)$ and $\mathrm{U}(n)$. In a similar vein, our modular generating

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series can be used to define a lift from classical cuspidal modular forms of weight n to the codimension one Chow group of the unitary Shimura variety.

1.1. Statement of the main result. — Fix a quadratic imaginary field $\boldsymbol{k} \subset \mathbb{C}$ of odd discriminant $\mathrm{disc}(\boldsymbol{k}) = -D$. We are concerned with the arithmetic of a certain unitary Shimura variety, whose definition depends on the choices of $\boldsymbol{k}$-hermitian spaces W_0 and W of signature $(1,0)$ and $(n-1,1)$, respectively, where $n \geq 3$. We assume that W_0 and W each admit an $\mathcal{O}_{\boldsymbol{k}}$-lattice that is self-dual with respect to the hermitian form.

Attached to this data is a reductive algebraic group

$$G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W) \tag{1.1.1}$$

over $\mathbb{Q}$, defined as the subgroup on which the unitary similitude characters are equal, and a compact open subgroup $K \subset G(\mathbb{A}_f)$ depending on the above choice of self-dual lattices. As explained in § 2, there is an associated hermitian symmetric domain $\mathcal{D}$, and a Deligne-Mumford stack $\mathrm{Sh}(G,\mathcal{D})$ over $\boldsymbol{k}$ whose complex points are identified with the orbifold quotient

$$\mathrm{Sh}(G,\mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K.$$

This is the unitary Shimura variety of the title.

The stack $\mathrm{Sh}(G,\mathcal{D})$ can be interpreted as a moduli space of pairs (A_0, A) in which A_0 is an elliptic curve with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$, and A is a principally polarized abelian scheme of dimension n endowed with an $\mathcal{O}_{\boldsymbol{k}}$-action. The pair (A_0, A) is required to satisfy some additional conditions, which need not concern us in the introduction.

Using the moduli interpretation, one can construct an integral model of $\mathrm{Sh}(G,\mathcal{D})$ over $\mathcal{O}_{\boldsymbol{k}}$. In fact, following work of Pappas and Krämer, we explain in § 2.3 that there are two natural integral models related by a morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. Each integral model has a canonical toroidal compactification whose boundary is a disjoint union of smooth Cartier divisors, and the above morphism extends uniquely to a morphism

$$\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}} \tag{1.1.2}$$

of compactifications.

Each compactified integral model has its own desirable and undesirable properties. For example, $\mathcal{S}^*_{\mathrm{Kra}}$ is regular, while $\mathcal{S}^*_{\mathrm{Pap}}$ is not. On the other hand, every vertical (i.e., supported in nonzero characteristic) Weil divisor on $\mathcal{S}^*_{\mathrm{Pap}}$ has nonempty intersection with the boundary, while $\mathcal{S}^*_{\mathrm{Kra}}$ has certain *exceptional* divisors in characteristics $p \mid D$ that do not meet the boundary. An essential part of our method is to pass back and forth between these two models in order to exploit the best properties of each. For simplicity, we will state our main results in terms of the regular model $\mathcal{S}^*_{\mathrm{Kra}}$.

In § 2 we define a distinguished line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$, called the *line bundle of weight one modular forms*, and a family of Cartier divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$ indexed by integers $m > 0$. These special divisors were introduced in **[33, 34]**, and studied further in **[11, 23, 24]**. For the purposes of the introduction, we note only that one should regard the divisors as arising from embeddings of smaller unitary groups into G.

Denote by

$$\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \cong \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Kra}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

the Chow group of rational equivalence classes of divisors with $\mathbb{Q}$ coefficients. Each special divisor $\mathcal{Z}_{\mathrm{Kra}}(m)$ can be extended to a divisor on the toroidal compactification simply by taking its Zariski closure, denoted $\mathcal{Z}^*_{\mathrm{Kra}}(m)$. The *total special divisor* is defined as

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m) \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \tag{1.1.3}$$

where the boundary contribution is defined, as in (5.3.3), by

$$\mathcal{B}_{\mathrm{Kra}}(m) = \frac{m}{n-2} \sum_{\Phi} \#\{x \in L_0 : \langle x, x\rangle = m\} \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi).$$

The notation here is the following: The sum is over the equivalence classes of *proper cusp label representatives* Φ as defined in § 3.1. These index the connected components $\mathcal{S}^*_{\mathrm{Kra}}(\Phi) \subset \partial \mathcal{S}^*_{\mathrm{Kra}}$ of the boundary [(1)]. Inside the sum, $(L_0, \langle \cdot, \cdot \rangle)$ is a hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-2, 0)$, which depends on Φ.

The line bundle of modular forms $\boldsymbol{\omega}$ admits a canonical extension to the toroidal compactification, denoted the same way. For the sake of notational uniformity, we extend (1.1.3) to $m = 0$ by setting

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \boldsymbol{\omega}^{-1} + \mathrm{Exc} \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}). \tag{1.1.4}$$

Here Exc is the exceptional divisor of Theorem 2.3.4. It is a reduced effective divisor supported in characteristics $p \mid D$, disjoint from the boundary of the compactification. The following result appears in the text as Theorem 7.1.5.

Theorem A. — *Let* $\chi_{\boldsymbol{k}} : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ *be the Dirichlet character determined by* $\boldsymbol{k}/\mathbb{Q}$. *The formal generating series*

$$\sum_{m \geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]]$$

is modular of weight n, *level* $\Gamma_0(D)$, *and character* $\chi^n_{\boldsymbol{k}}$ *in the following sense: for every* $\mathbb{Q}$*-linear functional* $\alpha : \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}$, *the series*

$$\sum_{m \geq 0} \alpha(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)) \cdot q^m \in \mathbb{C}[[q]]$$

is the q*-expansion of a classical modular form of the indicated weight, level, and character.*

We can prove a stronger version of Theorem A. Denote by $\widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$ the Gillet-Soulé [20] arithmetic Chow group of rational equivalence classes of pairs $\widehat{\mathcal{Z}} = (\mathcal{Z}, \mathrm{Gr})$, where $\mathcal{Z}$ is a divisor on $\mathcal{S}^*_{\mathrm{Kra}}$ with rational coefficients, and Gr is a Green function

(1) After base change to $\mathbb{C}$, each $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ decomposes into h connected components, where h is the class number of $\boldsymbol{k}$.

for $\mathcal{Z}$. We allow the Green function to have additional log-log singularities along the boundary, as in the more general theory developed in [**13**]. See also [**8, 24**].

In § 7.3 we use the theory of regularized theta lifts to construct Green functions for the special divisors $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$, and hence obtain arithmetic divisors

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{Q}}(\mathcal{S}^{*}_{\mathrm{Kra}})$$

for $m > 0$. We also endow the line bundle $\boldsymbol{\omega}$ with a metric, and the resulting metrized line bundle $\widehat{\boldsymbol{\omega}}$ defines a class

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \widehat{\boldsymbol{\omega}}^{-1} + (\mathrm{Exc}, -\log(D)) \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{Q}}(\mathcal{S}^{*}_{\mathrm{Kra}}),$$

where the vertical divisor Exc has been endowed with the constant Green function $-\log(D)$. The following result is Theorem 7.3.1 in the text.

Theorem B. — *The formal generating series*

$$\widehat{\phi}(\tau) = \sum_{m \geq 0} \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^{m} \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{Q}}(\mathcal{S}^{*}_{\mathrm{Kra}})[[q]]$$

is modular of weight n, level $\Gamma_0(D)$, and character $\chi^{n}_{\boldsymbol{k}}$, where modularity is understood in the same sense as Theorem A.

Remark 1.1.1. — As this article was being revised for publication, Wei Zhang announced a proof of his *arithmetic fundamental lemma*, conjectured in [**52**]. Although the statement is a purely local result concerning intersections of cycles on unitary Rapoport-Zink spaces, Zhang's proof uses global calculations on unitary Shimura varieties, and makes essential use of the modularity result of Theorem B. See [**53**].

Remark 1.1.2. — Theorem B implies that the $\mathbb{Q}$-span of the classes $\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ is finite dimensional. See Remark 7.1.2.

Remark 1.1.3. — There is a second method of constructing Green functions for the special divisors, based on the methods of [**36**], which gives rise to a non-holomorphic variant of $\widehat{\phi}(\tau)$. It is a recent theorem of Ehlen-Sankaran [**16**] that Theorem B implies the modularity of this non-holomorphic generating series. See § 7.4.

One motivation for the modularity result of Theorem B is that it allows one to construct arithmetic theta lifts. If $g(\tau) \in S_n(\Gamma_0(D), \chi^{n}_{\boldsymbol{k}})$ is a classical scalar valued cusp form, we may form the Petersson inner product

$$\widehat{\theta}(g) \stackrel{\mathrm{def}}{=} \langle \widehat{\phi}, g \rangle_{\mathrm{Pet}} \in \widehat{\mathrm{Ch}}^{1}_{\mathbb{C}}(\mathcal{S}^{*}_{\mathrm{Kra}})$$

as in [**38**]. One expects, as in [*loc. cit.*], that the arithmetic intersection pairing of $\widehat{\theta}(g)$ against other cycle classes should be related to derivatives of L-functions, providing generalizations of the Gross-Zagier and Gross-Kohnen-Zagier theorems. Specific instances in which this expectation is fulfilled can be deduced from [**11, 23, 24**]. This will be explained in the companion paper [**10**].

As this paper is rather long, we explain in the next two subsections the main ideas that go into the proof of Theorem A. The proof of Theorem B is exactly the same, but one must keep track of Green functions.

1.2. Sketch of the proof, part I: the generic fiber. — In this subsection we sketch the proof of modularity only in the generic fiber. That is, the modularity of

$$\sum_{m\geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}})[[q]]. \tag{1.2.1}$$

The key to the proof is the study of *Borcherds products* [4, 5].

A Borcherds product is a meromorphic modular form on an orthogonal Shimura variety, whose construction depends on a choice of weakly holomorphic input form, typically of negative weight. In our case the input form is any

$$f(\tau) = \sum_{m\gg -\infty} c(m) q^m \in M^{!,\infty}_{2-n}(D, \chi^{n-2}_{\boldsymbol{k}}), \tag{1.2.2}$$

where the superscripts ! and ∞ indicate that the weakly holomorphic form $f(\tau)$ of weight $2-n$ and level $\Gamma_0(D)$ is allowed to have a pole at the cusp ∞, but must be holomorphic at all other cusps. We assume also that all $c(m)\in\mathbb{Z}$.

Our Shimura variety $\mathrm{Sh}(G,\mathcal{D})$ admits a natural map to an orthogonal Shimura variety. Indeed, the $\boldsymbol{k}$-vector space

$$V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

admits a natural hermitian form $\langle .,. \rangle$ of signature $(n-1,1)$, induced by the hermitian forms on W_0 and W. The natural action of G on V determines an exact sequence

$$1 \to \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \to G \to \mathrm{U}(V) \to 1 \tag{1.2.3}$$

of reductive groups over $\mathbb{Q}$.

We may also view V as a $\mathbb{Q}$-vector space endowed with the quadratic form $Q(x)=\langle x,x\rangle$ of signature $(2n-2,2)$, and so obtain a homomorphism $G\to\mathrm{SO}(V)$. This induces a map from $\mathrm{Sh}(G,\mathcal{D})$ to the Shimura variety associated with the group $\mathrm{SO}(V)$.

After possibly replacing f by a nonzero integer multiple, Borcherds constructs a meromorphic modular form on the orthogonal Shimura variety, which can be pulled back to a meromorphic modular form on $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. The result is a meromorphic section $\boldsymbol{\psi}(f)$ of $\boldsymbol{\omega}^k$, where the weight

$$k = \sum_{r\mid D} \gamma_r \cdot c_r(0) \in \mathbb{Z} \tag{1.2.4}$$

is the integer defined in § 5.3. The constant $\gamma_r = \prod_{p\mid r}\gamma_p$ is a 4^{th} root of unity (with $\gamma_1 = 1$) and $c_r(0)$ is the constant term of f at the cusp

$$\infty_r = \frac{r}{D} \in \Gamma_0(D)\backslash\mathbb{P}^1(\mathbb{Q}),$$

in the sense of Definition 4.1.1.

Initially, $\psi(f)$ is characterized by specifying $-\log\|\psi(f)\|$, where $\|\cdot\|$ is the Petersson norm on $\boldsymbol{\omega}^k$. In particular, $\psi(f)$ is only defined up to rescaling by a complex number of absolute value 1 on each connected component of $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. We prove that, after a suitable rescaling, $\psi(f)$ is the analytification of a rational section of the line bundle $\boldsymbol{\omega}^k$ on $\mathrm{Sh}(G,\mathcal{D})$. In other words, the Borcherds product is algebraic and defined over the reflex field $\boldsymbol{k}$. This allows us to view $\psi(f)$ as a rational section of $\boldsymbol{\omega}^k$ both on the integral model $\mathcal{S}_{\mathrm{Kra}}$, and on its toroidal compactification.

We compute the divisor of $\psi(f)$ on the generic fiber of the toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}}$, and find

$$\mathrm{div}(\psi(f))_{/\boldsymbol{k}} = \sum_{m>0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}}. \tag{1.2.5}$$

The calculation of the divisor on the interior $\mathcal{S}_{\mathrm{Kra}/\boldsymbol{k}}$ follows immediately from the corresponding calculations of Borcherds on the orthogonal Shimura variety. The multiplicities of the boundary components are computed using the results of [**32**], which describe the structure of the Fourier-Jacobi expansions of $\psi(f)$ along the various boundary components.

The equality of divisors (1.2.5) implies the relation

$$k\cdot\boldsymbol{\omega} = \sum_{m>0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}}$$

in the Chow group $\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}})$. The cusp $\infty_1 = 1/D$ is $\Gamma_0(D)$-equivalent to the usual cusp at ∞, and so $c_1(0)=c(0)$. Substituting the expression (1.2.4) for k into the left hand side and using (1.1.4) therefore yields the relation

$$\sum_{\substack{r\mid D\\ r>1}} \gamma_r c_r(0)\cdot\boldsymbol{\omega} = \sum_{m\geq 0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} \tag{1.2.6}$$

in $\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}})$. In § 4.2 we construct for each $r\mid D$ an Eisenstein series

$$E_r(\tau) = \sum_{m\geq 0} e_r(m)\cdot q^m \in M_n(D,\chi^n_{\boldsymbol{k}}),$$

which, by a simple residue calculation, satisfies

$$c_r(0) = -\sum_{m>0} c(-m)e_r(m).$$

Substituting this expression into (1.2.6) yields

$$0 = \sum_{m\geq 0} c(-m)\cdot\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} + \sum_{\substack{r\mid D\\ r>1}} \gamma_r e_r(m)\cdot\boldsymbol{\omega}\Big), \tag{1.2.7}$$

where we have also used the relation $e_r(0)=0$ for $r>1$.

We now invoke a variant of the modularity criterion of [**5**], which is our Theorem 4.2.3: if a formal q-expansion

$$\sum_{m\geq 0} d(m)q^m \in \mathbb{C}[[q]]$$

satisfies $0 = \sum_{m\geq 0} c(-m)d(m)$ for every input form (1.2.2), then it must be the q-expansion of a modular form of weight n, level $\Gamma_0(D)$, and character $\chi_{\boldsymbol{k}}^n$. It follows immediately from this and (1.2.7) that the formal q-expansion

$$\sum_{m\geq 0} \Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} + \sum_{\substack{r\mid D \\ r>1}} \gamma_r e_r(m) \cdot \boldsymbol{\omega} \Big) \cdot q^m$$

is modular in the sense of Theorem A. Rewriting this as

$$\sum_{m\geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}} \cdot q^m + \sum_{\substack{r\mid D \\ r>1}} \gamma_r E_r(\tau) \cdot \boldsymbol{\omega}$$

and using the modularity of each Eisenstein series $E_r(\tau)$, we deduce that (1.2.1) is modular.

1.3. Sketch of the proof, part II: vertical components. — In order to extend the arguments of § 1.2 to prove Theorem A, it is clear that one should attempt to compute the divisor of the Borcherds product $\boldsymbol{\psi}(f)$ on the integral model $\mathcal{S}^*_{\mathrm{Kra}}$ and hope for an expression similar to (1.2.5). Indeed, the bulk of this paper is devoted to precisely this problem.

The subtlety is that both $\mathrm{div}(\boldsymbol{\psi}(f))$ and $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ will turn out to have vertical components supported in characteristics dividing D. Even worse, in these bad characteristics the components of the exceptional divisor $\mathrm{Exc} \subset \mathcal{S}^*_{\mathrm{Kra}}$ do not intersect the boundary, and so the multiplicities of these components in the divisor of $\boldsymbol{\psi}(f)$ cannot be detected by examining its Fourier-Jacobi expansion.

This is where the second integral model $\mathcal{S}^*_{\mathrm{Pap}}$ plays an essential role. The morphism (1.1.2) sits in a cartesian diagram

$$\begin{array}{ccc} \mathrm{Exc} & \longrightarrow & \mathcal{S}^*_{\mathrm{Kra}} \\ \downarrow & & \downarrow \\ \mathrm{Sing} & \longrightarrow & \mathcal{S}^*_{\mathrm{Pap}}, \end{array}$$

where the *singular locus* $\mathrm{Sing} \subset \mathcal{S}^*_{\mathrm{Pap}}$ is the reduced closed substack of points at which the structure morphism $\mathcal{S}^*_{\mathrm{Pap}} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$ is not smooth. It is 0-dimensional and supported in characteristics dividing D. The right vertical arrow restricts to an isomorphism

$$\mathcal{S}^*_{\mathrm{Kra}} \smallsetminus \mathrm{Exc} \cong \mathcal{S}^*_{\mathrm{Pap}} \smallsetminus \mathrm{Sing}. \tag{1.3.1}$$

For each connected component $s \in \pi_0(\mathrm{Sing})$ the fiber

$$\mathrm{Exc}_s = \mathrm{Exc} \times_{\mathcal{S}^*_{\mathrm{Pap}}} s$$

is a smooth, irreducible, vertical Cartier divisor on $\mathcal{S}^*_{\mathrm{Kra}}$, and $\mathrm{Exc} = \bigsqcup_s \mathrm{Exc}_s$.

As the $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{S}^*_{\mathrm{Pap}}$ is proper and normal with normal fibers, every irreducible vertical divisor on it is the reduction, modulo some prime of $\mathcal{O}_{\boldsymbol{k}}$, of an entire connected (= irreducible) component. From this it follows that every vertical divisor meets the boundary. Thus one could hope to use (1.3.1) to view $\psi(f)$ as a rational section on $\mathcal{S}^*_{\mathrm{Pap}}$, compute its divisor there by examining Fourier-Jacobi expansions, and then pull that calculation back to $\mathcal{S}^*_{\mathrm{Kra}}$.

This is essentially what we do, but there is an added complication. The line bundle $\boldsymbol{\omega}$ on (1.3.1) does not extend to $\mathcal{S}^*_{\mathrm{Pap}}$, and similarly the divisor $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ on (1.3.1) cannot be extended across the singular locus to a Cartier divisor on $\mathcal{S}^*_{\mathrm{Pap}}$. However, if you square the line bundle and the divisors, they have much better behavior. This is the content of the following result, which is an amalgamation of Theorems 2.4.3, 2.5.3, 2.6.3, and 3.7.1 of the text.

Theorem C. — *There is a unique line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$ whose restriction to (1.3.1) is isomorphic to $\boldsymbol{\omega}^2$. Denoting by $\boldsymbol{\Omega}_{\mathrm{Kra}}$ its pullback to $\mathcal{S}^*_{\mathrm{Kra}}$, there is an isomorphism*

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc}).$$

Similarly, there is a unique Cartier divisor $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m)$ on $\mathcal{S}^*_{\mathrm{Pap}}$ whose restriction to (1.3.1) is equal to $2\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$. Its pullback $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ to $\mathcal{S}^*_{\mathrm{Kra}}$ satisfies

$$2\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x\rangle = m\} \cdot \mathrm{Exc}_s.$$

Here L_s is a positive definite self-dual hermitian lattice of rank n associated to the singular point s, and $\langle .,.\rangle$ is its hermitian form.

Setting $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \boldsymbol{\Omega}^{-1}_{\mathrm{Pap}}$, we obtain a formal generating series

$$\sum_{m \geq 0} \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})[[q]],$$

whose pullback via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$ is twice the generating series of Theorem A, up to an error term coming from the exceptional divisors. More precisely, Theorem C shows that the pullback is

$$2\sum_{m \geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m - \sum_{s \in \pi_0(\mathrm{Sing})} \vartheta_s(\tau) \cdot \mathrm{Exc}_s \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]],$$

where each $\vartheta_s(\tau)$ is the classical theta function whose coefficients count points in the positive definite hermitian lattice L_s.

Over (1.3.1) we have $\boldsymbol{\omega}^{2k} \cong \boldsymbol{\Omega}^k_{\mathrm{Pap}}$, which allows us to view $\psi(f)^2$ as a rational section of the line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$. We examine its Fourier-Jacobi expansions along the boundary components and are able to compute its divisor completely (it happens to include nontrivial vertical components). We then pull this calculation back to $\mathcal{S}^*_{\mathrm{Kra}}$, and find that $\psi(f)$, when viewed as a rational section of $\boldsymbol{\omega}^k$, has divisor

$$\begin{aligned}\operatorname{div}(\psi(f)) = &\sum_{m>0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) + \sum_{r\mid D}\gamma_r c_r(0)\cdot\Big(\frac{\mathrm{Exc}}{2}+\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}\Big)\\ &-\sum_{m>0}\frac{c(-m)}{2}\sum_{s\in\pi_0(\mathrm{Sing})}\#\{x\in L_s : \langle x,x\rangle = m\}\cdot \mathrm{Exc}_s\\ &-k\cdot\operatorname{div}(\delta),\end{aligned}$$

where $\delta\in\mathcal{O}_{\boldsymbol{k}}$ is a square root of $-D$, $\mathfrak{p}\subset\mathcal{O}_{\boldsymbol{k}}$ is the unique prime above $p\mid D$, and $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}$ is the mod $\mathfrak{p}$ fiber of $\mathcal{S}^*_{\mathrm{Kra}}$, viewed as a divisor. This is stated in the text as Theorem 5.3.3. Passing to the generic fiber recovers (1.2.5), as it must.

As in the argument leading to (1.2.7), this implies the relation

$$\begin{aligned}0 = &\sum_{m\geq 0} c(-m)\cdot\Bigg(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \frac{1}{2}\sum_{s\in\pi_0(\mathrm{Sing})}\#\{x\in L_s : \langle x,x\rangle = m\}\cdot\mathrm{Exc}_s\Bigg)\\ &+\sum_{m\geq 0} c(-m)\cdot\sum_{\substack{r\mid D\\ r>1}}\gamma_r e_r(m)\Bigg(\boldsymbol{\omega}-\frac{\mathrm{Exc}}{2}-\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}\Bigg)\end{aligned}$$

in the Chow group of $\mathcal{S}^*_{\mathrm{Kra}}$, and the modularity criterion implies that

$$\sum_{m\geq 0}\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)\cdot q^m - \frac{1}{2}\sum_{s\in\pi_0(\mathrm{Sing})}\vartheta_s(\tau)\cdot\mathrm{Exc}_s + \sum_{\substack{r\mid D\\ r>1}}\gamma_r E_r(\tau)\cdot\Bigg(\boldsymbol{\omega}-\frac{\mathrm{Exc}}{2}-\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}\Bigg)$$

is a modular form. As each theta series $\vartheta_s(\tau)$ and Eisenstein series $E_r(\tau)$ is modular, so is $\sum\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)\cdot q^m$. This completes the outline of the proof of Theorem A.

1.4. The structure of the paper. — We now briefly describe the contents of the various sections of the paper.

In § 2 we introduce the unitary Shimura variety associated to the group G of (1.1.1), and explain its realization as a moduli space of pairs (A_0, A) of abelian varieties with extra structure. We then review the integral models constructed by Pappas and Krämer, and the singular and exceptional loci of these models. These are related by a cartesian diagram

$$\begin{array}{ccc}\mathrm{Exc} & \longrightarrow & \mathcal{S}_{\mathrm{Kra}}\\ \downarrow & & \downarrow\\ \mathrm{Sing} & \longrightarrow & \mathcal{S}_{\mathrm{Pap}},\end{array}$$

where the vertical arrow on the right is an isomorphism outside of the 0-dimensional singular locus Sing. We also define the line bundle of modular forms $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$.

The first main result of § 2 is Theorem 2.4.3, which asserts the existence of a line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}_{\mathrm{Pap}}$ restricting to $\boldsymbol{\omega}^2$ over

$$\mathcal{S}_{\mathrm{Kra}} \setminus \mathrm{Exc} \cong \mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}.$$

We then define the Cartier divisor $\mathcal{Z}_{\mathrm{Kra}}(m)$ on $\mathcal{S}_{\mathrm{Kra}}$ and prove Theorem 2.5.3, which asserts the existence of a Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}}$ whose restriction to $\mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}$ coincides with $2\mathcal{Z}_{\mathrm{Kra}}(m)$. Up to error terms supported on the exceptional locus Exc, the pullbacks of Ω_{Pap} and $\mathcal{Y}_{\mathrm{Pap}}(m)$ to $\mathcal{S}_{\mathrm{Kra}}$ are therefore equal to $\boldsymbol{\omega}^2$ and $2\mathcal{Z}_{\mathrm{Kra}}(m)$, respectively. The error terms are computed in Theorem 2.6.3, which is the analogue of Theorem C for the noncompactified Shimura varieties.

In § 3 we describe the canonical toroidal compactifications $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$, and the structure of their formal completions along the boundary. In § 3.1 and § 3.2 we introduce the cusp labels Φ that index the boundary components, and their associated mixed Shimura varieties. In § 3.3 we construct smooth integral models $\mathcal{C}_\Phi$ of these mixed Shimura varieties, following the general recipes of the theory of arithmetic toroidal compactification, as moduli spaces of 1-motives. In § 3.4 we give a second moduli interpretation of these integral models. This is one of the key technical steps in our work, and allows us to compare Fourier-Jacobi expansions on our unitary Shimura varieties to Fourier-Jacobi expansions on orthogonal Shimura varieties. See the remarks at the beginning of § 3 for further discussion. In § 3.5 and § 3.6 we construct the line bundle of modular forms and the special divisors on the mixed Shimura varieties $\mathcal{C}_\Phi$. Theorem 3.7.1 describes the canonical toroidal compactifications $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ and their properties. In § 3.8 we describe the Fourier-Jacobi expansions of sections of $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$ in algebraic language, and in § 3.9 we explain how to express these Fourier-Jacobi coefficients in classical complex analytic coordinates.

In the short § 4 we introduce the weakly holomorphic modular forms that will be used as inputs for the construction of Borcherds products. We also state in Theorem 4.2.3 a variant of the modularity criterion of Borcherds.

In § 5 we consider the unitary Borcherds products associated to weakly holomorphic forms

$$f \in M^{!,\infty}_{2-n}(D, \chi^{n-2}_{\boldsymbol{k}}). \tag{1.4.1}$$

Ultimately, the integrality properties of the unitary Borcherds products will be deduced from an analysis of their Fourier-Jacobi expansions. These expansions involve certain products of Jacobi theta functions, and so, in § 5 we review facts about the arithmetic theory of Jacobi forms. For us, Jacobi forms will be sections of a suitable line bundle $\mathcal{J}_{k,m}$ on the universal elliptic curve living over the moduli stack (over $\mathbb{Z}$) of all elliptic curves. The key point is to have a precise description of the divisor of the canonical section

$$\Theta^{24} \in H^0(\mathcal{E}, \mathcal{J}_{0,12})$$

of Proposition 5.1.4. In § 5.2 we prove Borcherds quadratic identity, allowing us to relate $\mathcal{J}_{0,1}$ to a certain line bundle (determined by a Borcherds product) on the boundary component $\mathcal{B}_\Phi$ associated to a cusp label Φ.

After these technical preliminaries, we come to the statements of our main results about unitary Borcherds products. Theorem 5.3.1 asserts that, for each weakly holomorphic form (1.4.1) satisfying integrality conditions on the Fourier coefficients $c(m)$ with $m \leq 0$, there is a rational section $\boldsymbol{\psi}(f)$ of the line bundle $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$ with explicit divisor on the generic fiber and prescribed zeros and poles along each boundary component. Moreover, for each cusp label Φ, the leading Fourier-Jacobi coefficient of $\boldsymbol{\psi}(f)$ has an expression as a product of three factors, two of which, P^{vert}_{Φ} and P^{hor}_{Φ}, are constructed in terms of Θ^{24}. Theorem 5.3.3 gives the precise divisor of $\boldsymbol{\psi}(f)$ on $\mathcal{S}^*_{\mathrm{Kra}}$, and Theorem 5.3.4 gives an analogous formula on $\mathcal{S}^*_{\mathrm{Pap}}$. An essential ingredient in the calculation of these divisors is the calculation of the divisors of the factors P^{vert}_{Φ} and P^{hor}_{Φ}, which is done in § 5.4.

In § 6 we prove the main results stated in § 5.3. In § 6.1 we construct a vector valued form $\tilde{f}$ from (1.4.1), and give expressions for its Fourier coefficients in terms of those of f. The vector valued form $\tilde{f}$ defines a Borcherds product $\tilde{\boldsymbol{\psi}}(f)$ on the symmetric space $\tilde{\mathcal{D}}$ for the orthogonal group of the quadratic space (V, Q) and, in § 6.2, we define the unitary Borcherds product $\boldsymbol{\psi}(f)$ as its pullback to $\mathcal{D}$. In § 6.3 we determine the analytic Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$ at the cusp Φ by pulling back the product formula for $\tilde{\boldsymbol{\psi}}(f)$ computed in **[32]** along a one-dimensional boundary component of $\tilde{\mathcal{D}}$. In § 6.4 we show that the unitary Borcherds product constructed analytically arises from a rational section of $\boldsymbol{\omega}^k$ and that, after rescaling by a constant of absolute value 1, this section is defined over $\boldsymbol{k}$. This is Proposition 6.4.4. In § 6.5 we complete the proofs of Theorems 5.3.1, 5.3.3, and 5.3.4.

In § 7 we use the calculation of the divisors of Borcherds products to prove the modularity results discussed in detail earlier in the introduction.

In § 8 we provide some supplementary technical calculations.

1.5. The case $n = 2$. — Throughout the introduction we have assumed that $n \geq 3$, but one could ask if similar results hold for $n = 2$. This seems to be a delicate question.

The assumption that $n \geq 3$ guarantees that W contains an isotropic $\boldsymbol{k}$-line, which implies that $\mathrm{Sh}(G, \mathcal{D})$ has no compact (meaning proper over $\boldsymbol{k}$) components. When $n = 2$ the Shimura variety $\mathrm{Sh}(G, \mathcal{D})$ is essentially a union of classical modular curves (if W contains an isotropic $\boldsymbol{k}$-line) or of compact quaternionic Shimura curves (if W contains no isotropic $\boldsymbol{k}$-line).

When $n = 2$ one could still construct Borcherds products on $\mathrm{Sh}(G, \mathcal{D})$ as pullbacks from orthogonal Shimura varies, and use the results of **[26]** to prove that they are defined over the reflex field $\boldsymbol{k}$. Analyzing their divisors on the integral models $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ seems quite difficult. The compact case falls well outside the reach of our arguments, which rely in an essential way on the anaysis of Fourier-Jacobi expansions near the boundary of a toroidal compactification.

However, even in the noncompact $n = 2$ case there are some technical issues that we do not know how to resolve. Foremost among these is that when $n = 2$ the reduction of $\mathcal{S}_{\mathrm{Pap}}$ at a prime of $\mathcal{O}_{\boldsymbol{k}}$ above D is not normal, and so (as in the familiar case of modular curves) the reduction of an irreducible component need not remain irreducible. This causes the proof of Proposition 6.5.2 to break down in a serious way. In essence, we do not know how to exclude the possibility that constants κ_Φ appearing in Proposition 6.4.1 contribute some nontrivial error term to the divisor of the Borcherds product.

In § 2 and § 3 we assume $n \geq 2$, but from § 5 onwards we restrict to $n \geq 3$ (the integer n plays no role in the short § 4).

1.6. Thanks. — The results of this paper are the outcome of a long term project, begun initially in Bonn in June of 2013, and supported in a crucial way by three weeklong meetings at AIM, in Palo Alto (May of 2014) and San Jose (November of 2015 and 2016), as part of their AIM SQuaRE's program. The opportunity to spend these periods of intensely focused efforts on the problems involved was essential. We would like to thank the University of Bonn and AIM for their support.

1.7. Notation. — Throughout the paper, $\boldsymbol{k} \subset \mathbb{C}$ is a quadratic imaginary field of odd discriminant $\mathrm{disc}(\boldsymbol{k}) = -D$. Denote by $\delta = \sqrt{-D} \in \boldsymbol{k}$ the unique choice of square root with $\mathrm{Im}(\delta) > 0$, and by $\mathfrak{d} = \delta\mathcal{O}_{\boldsymbol{k}}$ the different of $\mathcal{O}_{\boldsymbol{k}}$.

Fix a $\pi \in \mathcal{O}_{\boldsymbol{k}}$ satisfying $\mathcal{O}_{\boldsymbol{k}} = \mathbb{Z} + \mathbb{Z}\pi$. If S is any $\mathcal{O}_{\boldsymbol{k}}$-scheme, define

$$\varepsilon_S = \pi \otimes 1 - 1 \otimes i_S(\overline{\pi}) \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S$$
$$\overline{\varepsilon}_S = \overline{\pi} \otimes 1 - 1 \otimes i_S(\overline{\pi}) \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S,$$

where $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$ is the structure map. The ideal sheaves generated by these sections are independent of the choice of π, and sit in exact sequences of free $\mathcal{O}_S$-modules

$$0 \to (\overline{\varepsilon}_S) \to \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{\alpha\otimes x \mapsto i_S(\alpha)x} \mathcal{O}_S \to 0$$

and

$$0 \to (\varepsilon_S) \to \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{\alpha\otimes x \mapsto i_S(\overline{\alpha})x} \mathcal{O}_S \to 0.$$

It is easy to see that $\varepsilon_S \cdot \overline{\varepsilon}_S = 0$, and that the images of (ε_S) and $(\overline{\varepsilon}_S)$ under

$$\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{\alpha\otimes x \mapsto i_S(\alpha)x} \mathcal{O}_S$$
$$\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{\alpha\otimes x \mapsto i_S(\overline{\alpha})x} \mathcal{O}_S,$$

respectively, are both equal to the sub-sheaf $\mathfrak{d}\mathcal{O}_S$. This defines isomorphisms of $\mathcal{O}_S$-modules

(1.7.1) $$(\varepsilon_S) \cong \mathfrak{d}\mathcal{O}_S \cong (\overline{\varepsilon}_S).$$

If N is an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S$-module then $N/\overline{\varepsilon}_S N$ is the maximal quotient of N on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$, and $N/\varepsilon_S N$ is the maximal quotient on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the complex conjugate of the structure morphism. If $D \in \mathcal{O}_S^\times$ then more is true: there is a decomposition

(1.7.2) $$N = \varepsilon_S N \oplus \overline{\varepsilon}_S N,$$

and the summands are the maximal submodules on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism and its conjugate, respectively. From this discussion it is clear that one should regard ε_S and $\overline{\varepsilon}_S$ as integral substitutes for the orthogonal idempotents in $\boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$. The $\mathcal{O}_{\boldsymbol{k}}$-scheme S will usually be clear from context, and we abbreviate ε_S and $\overline{\varepsilon}_S$ to ε and $\overline{\varepsilon}$.

Let $\boldsymbol{k}^{\mathrm{ab}} \subset \mathbb{C}$ be the maximal abelian extension of $\boldsymbol{k}$ in $\mathbb{C}$, and let

$$\mathrm{art} : \boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times \to \mathrm{Gal}(\boldsymbol{k}^{\mathrm{ab}}/\boldsymbol{k})$$

be the Artin map of class field theory, normalized as in [**43**, § 11]. As usual, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ is Deligne's torus.

For a prime $p \leq \infty$ we write $(a,b)_p$ for the Hilbert symbol of $a, b \in \mathbb{Q}_p^\times$. Recall that the *invariant* of a hermitian space V over $\boldsymbol{k}_p = \boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is defined by

$$\mathrm{inv}_p(V) = (\det V, -D)_p, \tag{1.7.3}$$

where $\det V$ is the determinant of the matrix of the hermitian form with respect to a $\boldsymbol{k}_p$-basis. If $p < \infty$ then V is determined up to isomorphism by its $\boldsymbol{k}_p$-rank and invariant. If $p = \infty$ then V is determined up to isomorphism by its signature (r, s), and its invariant is $\mathrm{inv}_\infty(V) = (-1)^s$.

The term *stack* always means *Deligne-Mumford stack.*

2. Unitary Shimura varieties

In this section we define a unitary Shimura variety $\mathrm{Sh}(G, \mathcal{D})$ over our quadratic imaginary field $\boldsymbol{k} \subset \mathbb{C}$ and describe its moduli interpretation. We then recall the work of Pappas and Krämer, which provides us with two integral models related by a surjection $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. This surjection becomes an isomorphism after restriction to $\mathcal{O}_{\boldsymbol{k}}[1/D]$. We define a line bundle of weight one modular forms $\boldsymbol{\omega}$ and a family of Cartier divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$, $m > 0$, on $\mathcal{S}_{\mathrm{Kra}}$,

The line bundle $\boldsymbol{\omega}$ and the divisors $\mathcal{Z}_{\mathrm{Kra}}(m)$ do not descend to $\mathcal{S}_{\mathrm{Pap}}$, and the main original material in § 2 is the construction of suitable substitutes on $\mathcal{S}_{\mathrm{Pap}}$. These substitutes consist of a line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ that agrees with $\boldsymbol{\omega}^2$ after restricting to $\mathcal{O}_{\boldsymbol{k}}[1/D]$, and Cartier divisors $\mathcal{Y}_{\mathrm{Pap}}(m)$ that agree with $2\mathcal{Z}_{\mathrm{Kra}}(m)$ after restricting to $\mathcal{O}_{\boldsymbol{k}}[1/D]$.

2.1. The Shimura variety. — Let W_0 and W be $\boldsymbol{k}$-vector spaces endowed with hermitian forms H_0 and H of signatures $(1,0)$ and $(n-1,1)$, respectively. We always assume that $n \geq 2$. Abbreviate

$$W(\mathbb{R}) = W \otimes_{\mathbb{Q}} \mathbb{R}, \quad W(\mathbb{C}) = W \otimes_{\mathbb{Q}} \mathbb{C}, \quad W(\mathbb{A}_f) = W \otimes_{\mathbb{Q}} \mathbb{A}_f,$$

and similarly for W_0. In particular, $W_0(\mathbb{R})$ and $W(\mathbb{R})$ are hermitian spaces over $\mathbb{C} = \boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{R}$.

We assume the existence of $\mathcal{O}_{\boldsymbol{k}}$-lattices $\mathfrak{a}_0 \subset W_0$ and $\mathfrak{a} \subset W$, self-dual with respect to the hermitian forms H_0 and H. As the inverse of $\delta = \sqrt{-D} \in \boldsymbol{k}$ generates the inverse different of $\boldsymbol{k}/\mathbb{Q}$, this is equivalent to self-duality with respect to the symplectic forms

$$\psi_0(w, w') = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} H_0(\delta^{-1}w, w'), \quad \psi(w, w') = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} H(\delta^{-1}w, w'). \tag{2.1.1}$$

This data will remain fixed throughout the paper.

As in (1.1.1), let $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ be the subgroup of pairs for which the similitude factors are equal. We denote by $\nu : G \to \mathbb{G}_m$ the common similitude character, and note that $\nu(G(\mathbb{R})) \subset \mathbb{R}^{>0}$.

Let $\mathcal{D}(W_0) = \{y_0\}$ be a one-point set, and define

$$\mathcal{D}(W) = \{\text{negative definite } \mathbb{C}\text{-planes } y \subset W(\mathbb{R})\}, \tag{2.1.2}$$

so that $G(\mathbb{R})$ acts on the connected hermitian domain

$$\mathcal{D} = \mathcal{D}(W_0) \times \mathcal{D}(W).$$

The lattices $\mathfrak{a}_0$ and $\mathfrak{a}$ determine a maximal compact open subgroup

$$K = \big\{ g \in G(\mathbb{A}_f) : g\widehat{\mathfrak{a}}_0 = \widehat{\mathfrak{a}}_0 \text{ and } g\widehat{\mathfrak{a}} = \widehat{\mathfrak{a}} \big\} \subset G(\mathbb{A}_f), \tag{2.1.3}$$

and the orbifold quotient

$$\mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \backslash \mathcal{D} \times G(\mathbb{A}_f)/K$$

is the space of complex points of a smooth $\boldsymbol{k}$-stack of dimension $n-1$, denoted $\mathrm{Sh}(G, \mathcal{D})$.

The symplectic forms (2.1.1) determine a $\boldsymbol{k}$-conjugate-linear isomorphism

$$\mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \xrightarrow{x \mapsto x^\vee} \mathrm{Hom}_{\boldsymbol{k}}(W, W_0), \tag{2.1.4}$$

characterized by $\psi(xw_0, w) = \psi_0(w_0, x^\vee w)$. The $\boldsymbol{k}$-vector space

$$V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

carries a hermitian form of signature $(n-1, 1)$ defined by

$$\langle x_1, x_2 \rangle = x_2^\vee \circ x_1 \in \mathrm{End}_{\boldsymbol{k}}(W_0) \cong \boldsymbol{k}. \tag{2.1.5}$$

The group G acts on V in a natural way, defining an exact sequence (1.2.3).

The hermitian form on V induces a quadratic form $Q(x) = \langle x, x \rangle$, with associated $\mathbb{Q}$-bilinear form

$$[x, y] = \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}} \langle x, y \rangle. \tag{2.1.6}$$

In particular, we obtain a representation $G \to \mathrm{SO}(V)$.

Proposition 2.1.1. — *The stack $\mathrm{Sh}(G, \mathcal{D})_{/\mathbb{C}}$ has $2^{1-o(D)}h^2$ connected components, where h is the class number of $\boldsymbol{k}$ and $o(D)$ is the number of prime divisors of D.*

Proof. — Each $g \in G(\mathbb{A}_f)$ determines $\mathcal{O}_{\boldsymbol{k}}$-lattices

$$g\mathfrak{a}_0 = W_0 \cap g\widehat{\mathfrak{a}}_0, \quad g\mathfrak{a} = W \cap g\widehat{\mathfrak{a}}.$$

The hermitian forms H_0 and H need not be $\mathcal{O}_{\boldsymbol{k}}$-valued on these lattices. However, if $\mathrm{rat}(\nu(g))$ denotes the unique positive rational number such that

$$\frac{\nu(g)}{\mathrm{rat}(\nu(g))} \in \widehat{\mathbb{Z}}^\times,$$

then the rescaled hermitian forms $\mathrm{rat}(\nu(g))^{-1}H_0$ and $\mathrm{rat}(\nu(g))^{-1}H$ make $g\mathfrak{a}_0$ and $g\mathfrak{a}$ into self-dual hermitian lattices.

As $\mathcal{D}$ is connected, the components of $\mathrm{Sh}(G,\mathcal{D})_{/\mathbb{C}}$ are in bijection with the set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$. The function $g \mapsto (g\mathfrak{a}_0, g\mathfrak{a})$ establishes a bijection from $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ to the set of isometry classes of pairs of self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices $(\mathfrak{a}_0', \mathfrak{a}')$ of signatures $(1,0)$ and $(n-1,1)$, respectively, for which the self-dual hermitian lattice $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0', \mathfrak{a}')$ lies in the same genus as $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \subset V$.

Using the fact that $\mathrm{SU}(V)$ satisfies strong approximation, one can show that there are exactly $2^{1-o(D)}h$ isometry classes in the genus of $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a})$, and each isometry class arises from exactly h isometry classes of pairs $(\mathfrak{a}_0', \mathfrak{a}')$. □

It will be useful at times to have other interpretations of the hermitian domain $\mathcal{D}$. The following remarks provide alternate points of view. Recalling the idempotents $\varepsilon, \overline{\varepsilon} \in \boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{C}$ of § 1.7, define isomorphisms of real vector spaces

$$\mathrm{pr}_{\varepsilon} : W(\mathbb{R}) \cong \varepsilon W(\mathbb{C}), \quad \mathrm{pr}_{\overline{\varepsilon}} : W(\mathbb{R}) \cong \overline{\varepsilon} W(\mathbb{C}) \tag{2.1.7}$$

as, respectively, the compositions

$$W(\mathbb{R}) \hookrightarrow W(\mathbb{C}) = \varepsilon W(\mathbb{C}) \oplus \overline{\varepsilon} W(\mathbb{C}) \xrightarrow{\text{proj.}} \varepsilon W(\mathbb{C})$$
$$W(\mathbb{R}) \hookrightarrow W(\mathbb{C}) = \varepsilon W(\mathbb{C}) \oplus \overline{\varepsilon} W(\mathbb{C}) \xrightarrow{\text{proj.}} \overline{\varepsilon} W(\mathbb{C}).$$

Remark 2.1.2. — Each pair $z = (y_0, y) \in \mathcal{D}$ determines a line $\mathrm{pr}_{\varepsilon}(y) \subset W(\mathbb{C})$, and hence a line

$$z = \mathrm{Hom}_{\mathbb{C}}(W_0(\mathbb{C})/\overline{\varepsilon} W_0(\mathbb{C}), \mathrm{pr}_{\varepsilon}(y)) \subset \varepsilon V(\mathbb{C}).$$

This construction identifies

$$\mathcal{D} \cong \big\{ z \in \varepsilon V(\mathbb{C}) : [z, \overline{z}] < 0 \big\} / \mathbb{C}^{\times} \subset \mathbb{P}(\varepsilon V(\mathbb{C}))$$

as an open subset of projective space.

Remark 2.1.3. — Define a Hodge structure

$$F^1 W_0(\mathbb{C}) = 0, \quad F^0 W_0(\mathbb{C}) = \overline{\varepsilon} W_0(\mathbb{C}), \quad F^{-1} W_0(\mathbb{C}) = W_0(\mathbb{C})$$

on $W_0(\mathbb{C})$, and identify the unique point $y_0 \in \mathcal{D}(W_0)$ with the corresponding morphism $\mathbb{S} \to \mathrm{GU}(W_0)_{\mathbb{R}}$. Every $y \in \mathcal{D}(W)$ defines a Hodge structure

$$F^1 W(\mathbb{C}) = 0, \quad F^0 W(\mathbb{C}) = \mathrm{pr}_{\varepsilon}(y) \oplus \mathrm{pr}_{\overline{\varepsilon}}(y^{\perp}), \quad F^{-1} W(\mathbb{C}) = W(\mathbb{C})$$

on $W(\mathbb{C})$. If we identify $y \in \mathcal{D}(W)$ with the corresponding morphism $\mathbb{S} \to \mathrm{GU}(W)_{\mathbb{R}}$, then for any point $z = (y_0, y) \in \mathcal{D}$ the product morphism

$$y_0 \times y : \mathbb{S} \to \mathrm{GU}(W_0)_{\mathbb{R}} \times \mathrm{GU}(W)_{\mathbb{R}}$$

takes values in $G_{\mathbb{R}}$. This realizes $\mathcal{D} \subset \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ as a $G(\mathbb{R})$-conjugacy class.

Remark 2.1.4. — In fact, the discussion above shows that $\mathrm{Sh}(G,\mathcal{D})$ admits a map to the Shimura variety defined the group $\mathrm{U}(V)$ together with the homomorphism

$$h_{\mathrm{Gross}} : \mathbb{S} \to \mathrm{U}(V)(\mathbb{R}), \qquad z \mapsto \mathrm{diag}(1,\ldots,1,\bar{z}/z).$$

Here we have chosen a basis for $V(\mathbb{R})$ for which the hermitian form has matrix $\mathrm{diag}(1_{n-1},-1)$. Note that, for analogous choices of bases for $W_0(\mathbb{R})$ and $W(\mathbb{R})$, the corresponding map is

$$h : \mathbb{S} \to G(\mathbb{R}), \qquad z \mapsto (z) \times \mathrm{diag}(z,\ldots,z,\bar{z}),$$

which, under composition with the homomorphism $G(\mathbb{R}) \to \mathrm{U}(V)(\mathbb{R})$, gives h_{Gross}. The existence of this map provides an answer to a question posed by Gross: how can one explicitly relate the Shimura variety defined by the unitary group $\mathrm{U}(V)$, as opposed to the Shimura variety defined by the similitude group $\mathrm{GU}(V)$, to a moduli space of abelian varieties? Our answer is that Gross's unitary Shimura variety is a quotient of our $\mathrm{Sh}(G,\mathcal{D})$, whose interpretation as a moduli space is explained in the next section.

2.2. Moduli interpretation. — We wish to interpret $\mathrm{Sh}(G,\mathcal{D})$ as a moduli space of pairs of abelian varieties with additional structure. First, we recall some generalities on abelian schemes.

For an abelian scheme $\pi : A \to S$ over an arbitrary base S, define the *first relative de Rham cohomology sheaf* $H^1_{\mathrm{dR}}(A) = \mathbb{R}^1\pi_*\Omega^\bullet_{A/S}$ as the relative hypercohomology of the de Rham complex $\Omega^\bullet_{A/S}$. The *relative de Rham homology*

$$H_1^{\mathrm{dR}}(A) = \underline{\mathrm{Hom}}(H^1_{\mathrm{dR}}(A),\mathcal{O}_S)$$

is a locally free $\mathcal{O}_S$-module of rank $2\cdot\dim(A)$, sitting in an exact sequence

$$0 \to F^0H_1^{\mathrm{dR}}(A) \to H_1^{\mathrm{dR}}(A) \to \mathrm{Lie}(A) \to 0.$$

Any polarization of A induces an $\mathcal{O}_S$-valued alternating pairing on $H_1^{\mathrm{dR}}(A)$, which in turn induces a pairing

$$F^0H_1^{\mathrm{dR}}(A) \otimes \mathrm{Lie}(A) \to \mathcal{O}_S. \tag{2.2.1}$$

If the polarization is principal then both pairings are perfect. When $S = \mathrm{Spec}(\mathbb{C})$, Betti homology satisfies $H_1(A(\mathbb{C}),\mathbb{C}) \cong H_1^{\mathrm{dR}}(A)$, and

$$A(\mathbb{C}) \cong H_1(A(\mathbb{C}),\mathbb{Z})\backslash H_1^{\mathrm{dR}}(A)/F^0H_1^{\mathrm{dR}}(A).$$

For any pair of nonnegative integers (s,t), define an algebraic stack $M_{(s,t)}$ over $\boldsymbol{k}$ as follows: for any $\boldsymbol{k}$-scheme S let $M_{(s,t)}(S)$ be the groupoid of triples (A,ι,ψ) in which

- $A \to S$ is an abelian scheme of relative dimension $s+t$,
- $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action such that the locally free summands

$$\mathrm{Lie}(A) = \varepsilon\mathrm{Lie}(A) \oplus \bar{\varepsilon}\mathrm{Lie}(A)$$

  of (1.7.2) have $\mathcal{O}_S$-ranks s and t, respectively,
- $\psi : A \to A^\vee$ is a principal polarization, such that the induced Rosati involution $\dagger$ on $\mathrm{End}^0(A)$ satisfies $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$.

We usually omit ι and ψ from the notation, and just write $A \in M_{(s,t)}(S)$.

Proposition 2.2.1. — *The Shimura variety* $\mathrm{Sh}(G,\mathcal{D})$ *is isomorphic to an open and closed substack*

$$\mathrm{Sh}(G,\mathcal{D}) \subset M_{(1,0)} \times_{\boldsymbol{k}} M_{(n-1,1)}. \tag{2.2.2}$$

More precisely, $\mathrm{Sh}(G,\mathcal{D})(S)$ *classifies, for any* $\boldsymbol{k}$*-scheme* S*, pairs*

$$(A_0, A) \in M_{(1,0)}(S) \times M_{(n-1,1)}(S) \tag{2.2.3}$$

for which there exists, at every geometric point $s \to S$*, an isomorphism of hermitian* $\mathcal{O}_{\boldsymbol{k},\ell}$*-modules*

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(T_\ell A_{0,s}, T_\ell A_s) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \otimes \mathbb{Z}_\ell \tag{2.2.4}$$

for every prime ℓ*. Here the hermitian form on the right hand side of* (2.2.4) *is the restriction of the hermitian form* (2.1.5) *on* $\mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \otimes \mathbb{Q}_\ell$*. The hermitian form on the left hand side is defined similarly, replacing the symplectic forms* (2.1.1) *on* W_0 *and* W *with the Weil pairings on the Tate modules* $T_\ell A_{0,s}$ *and* $T_\ell A_s$*.*

Proof. — As this is routine, we only describe the open and closed immersion on complex points. Fix a point

$$(z, g) \in \mathrm{Sh}(G,\mathcal{D})(\mathbb{C}).$$

The component g determines $\mathcal{O}_{\boldsymbol{k}}$-lattices $g\mathfrak{a}_0 \subset W_0$ and $g\mathfrak{a} \subset W$, which are self-dual with respect to the symplectic forms

$$\mathrm{rat}(\nu(g))^{-1}\psi_0 \quad \text{and} \quad \mathrm{rat}(\nu(g))^{-1}\psi$$

of (2.1.1), rescaled as in the proof of Proposition 2.1.1.

By Remark 2.1.3 the point $z \in \mathcal{D}$ determines Hodge structures on W_0 and W, and in this way (z, g) determines principally polarized complex abelian varieties

$$A_0(\mathbb{C}) = g\mathfrak{a}_0 \backslash W_0(\mathbb{C})/F^0(W_0)$$
$$A(\mathbb{C}) = g\mathfrak{a} \backslash W(\mathbb{C})/F^0(W),$$

with actions of $\mathcal{O}_{\boldsymbol{k}}$. One can easily check that the pair (A_0, A) determines a complex point of $M_{(1,0)} \times_{\boldsymbol{k}} M_{(n-1,1)}$, and this construction defines (2.2.2) on complex points. □

The following lemma will be needed in § 2.3 for the construction of integral models for $\mathrm{Sh}(G,\mathcal{D})$.

Lemma 2.2.2. — *Fix a* $\boldsymbol{k}$*-scheme* S*, a geometric point* $s \to S$*, a prime* p*, and a point* (2.2.3)*. If the relation* (2.2.4) *holds for all* $\ell \neq p$*, then it also holds for* $\ell = p$*.*

Proof. — As the stack $\mathrm{Sh}(G,\mathcal{D})$ is of finite type over $\boldsymbol{k}$, we may assume that $s = \mathrm{Spec}(\mathbb{C})$. The polarizations on A_0 and A induce symplectic forms on the first homology groups $H_1(A_{0,s}(\mathbb{C}),\mathbb{Z})$ and $H_1(A_s(\mathbb{C}),\mathbb{Z})$, and the construction (2.1.5) makes

$$L_{\mathrm{Be}}(A_{0,s},A_s)=\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}\big(H_1(A_{0,s}(\mathbb{C}),\mathbb{Z}),H_1(A_s(\mathbb{C}),\mathbb{Z})\big)$$

into a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(n-1,1)$, satisfying

$$L_{\mathrm{Be}}(A_{0,s},A_s)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell\cong\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(T_\ell A_{0,s},T_\ell A_s)$$

for all primes ℓ.

If the relation (2.2.4) holds for all primes $\ell\neq p$, then $L_{\mathrm{Be}}(A_{0,s},A_s)\otimes\mathbb{Q}$ and $\mathrm{Hom}_{\boldsymbol{k}}(W_0,W)$ are isomorphic as $\boldsymbol{k}$-hermitian spaces everywhere locally except at p, and so they are isomorphic at p as well. In particular, for every ℓ (including $\ell=p$) both sides of (2.2.4) are isomorphic to self-dual lattices in the hermitian space $\mathrm{Hom}_{\boldsymbol{k}}(W_0,W)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. By the results of Jacobowitz [**27**] all self-dual lattices in this local hermitian space are isomorphic (2), and so (2.2.4) holds for all ℓ. □

Remark 2.2.3. — For any positive integer m define

$$K(m)=\ker\big(K\to\mathrm{Aut}_{\mathcal{O}_{\boldsymbol{k}}}(\widehat{\mathfrak{a}}_0/m\widehat{\mathfrak{a}}_0)\times\mathrm{Aut}_{\mathcal{O}_{\boldsymbol{k}}}(\widehat{\mathfrak{a}}/m\widehat{\mathfrak{a}})\big).$$

For a $\boldsymbol{k}$-scheme S, a $K(m)$-*structure* on $(A_0,A)\in\mathrm{Sh}(G,\mathcal{D})(S)$ is a triple (α_0,α,ζ) in which $\zeta:\underline{\mu_m}\cong\underline{\mathbb{Z}/m\mathbb{Z}}$ is an isomorphism of S-group schemes, and

$$\alpha_0:A_0[m]\cong\underline{\widehat{\mathfrak{a}}_0/m\widehat{\mathfrak{a}}_0},\quad \alpha:A[m]\cong\underline{\widehat{\mathfrak{a}}/m\widehat{\mathfrak{a}}}$$

are $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphisms identifying the Weil pairings on $A_0[m]$ and $A[m]$ with the $\mathbb{Z}/m\mathbb{Z}$-valued symplectic forms on $\widehat{\mathfrak{a}}_0/m\widehat{\mathfrak{a}}_0$ and $\widehat{\mathfrak{a}}/m\widehat{\mathfrak{a}}$ deduced from the pairings (2.1.1). The Shimura variety $G(\mathbb{Q})\backslash\mathcal{D}\times G(\mathbb{A}_f)/K(m)$ admits a canonical model over $\boldsymbol{k}$, parametrizing $K(m)$-structures on points of $\mathrm{Sh}(G,\mathcal{D})$.

2.3. Integral models. — In this subsection we describe two integral models of $\mathrm{Sh}(G,\mathcal{D})$ over $\mathcal{O}_{\boldsymbol{k}}$, related by a morphism $\mathcal{S}_{\mathrm{Kra}}\to\mathcal{S}_{\mathrm{Pap}}$.

The first step is to construct an integral model of the moduli space $M_{(1,0)}$. More generally, we will construct an integral model of $M_{(s,0)}$ for any $s>0$. Define an $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{M}_{(s,0)}$ as the moduli space of triples (A,ι,ψ) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S such that

- $A\to S$ is an abelian scheme of relative dimension s,
- $\iota:\mathcal{O}_{\boldsymbol{k}}\to\mathrm{End}(A)$ is an action such $\overline{\varepsilon}\mathrm{Lie}(A)=0$, or, equivalently, such that the induced action of $\mathcal{O}_{\boldsymbol{k}}$ on the $\mathcal{O}_S$-module $\mathrm{Lie}(A)$ is through the structure map $i_S:\mathcal{O}_{\boldsymbol{k}}\to\mathcal{O}_S$,
- $\psi:A\to A^\vee$ is a principal polarization whose Rosati involution satisfies $\iota(\alpha)^\dagger=\iota(\overline{\alpha})$ for all $\alpha\in\mathcal{O}_{\boldsymbol{k}}$.

The stack $\mathcal{M}_{(s,0)}$ is smooth of relative dimension 0 over $\mathcal{O}_{\boldsymbol{k}}$ by [**24**, Proposition 2.1.2], and its generic fiber is the stack $M_{(s,0)}$ defined earlier.

Remark 2.3.1. — The stack $\mathcal{M}_{(n-2,0)}$ will play an important role in § 3. In the degenerate case $n=2$, we interpret this as $\mathcal{M}_{(0,0)}=\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$. The universal abelian scheme over it should be understood as the 0 group scheme.

(2) This uses our standing hypothesis that D is odd.

The question of integral models for $M_{(n-1,1)}$ is more subtle, but well-understood after work of Pappas and Krämer. The first integral model was defined by Pappas [**45**]. Let

$$\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

be the stack whose functor of points assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of triples (A, ι, ψ) in which

— $A \to S$ is an abelian scheme of relative dimension n,

— $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action satisfying the determinant condition

$$\det(T - \iota(\alpha) \mid \mathrm{Lie}(A)) = (T-\alpha)^{n-1}(T-\overline{\alpha}) \in \mathcal{O}_S[T]$$

for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$,

— $\psi : A \to A^\vee$ is a principal polarization whose Rosati involution satisfies $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$,

— viewing the elements ε_S and $\overline{\varepsilon}_S$ of § 1.7 as endomorphisms of $\mathrm{Lie}(A)$, the induced endomorphisms

$$\bigwedge^n \varepsilon_S : \bigwedge^n \mathrm{Lie}(A) \to \bigwedge^n \mathrm{Lie}(A)$$
$$\bigwedge^2 \overline{\varepsilon}_S : \bigwedge^2 \mathrm{Lie}(A) \to \bigwedge^2 \mathrm{Lie}(A)$$

are trivial (*Pappas's wedge condition*).

It is clear that the generic fiber of $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ is isomorphic to the moduli space $M_{(n-1,1)}$ defined earlier. Denote by

$$\mathrm{Sing}_{(n-1,1)} \subset \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$$

the singular locus: the reduced substack of points at which the structure morphism to $\mathcal{O}_{\boldsymbol{k}}$ is not smooth.

***Theorem 2.3.2* (Pappas).** — *The stack $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ is flat over $\mathcal{O}_{\boldsymbol{k}}$ of relative dimension $n-1$, and is Cohen-Macaulay and normal. Moreover:*

1. *For any prime $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$, the reduction $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)/\mathbb{F}_\mathfrak{p}}$ is Cohen-Macaulay. If $n > 2$ the reduction is geometrically normal.*
2. *The singular locus is a 0-dimensional stack, finite over $\mathcal{O}_{\boldsymbol{k}}$ and supported in characteristics dividing D. It is the reduced substack underlying the closed substack defined by $\delta \cdot \mathrm{Lie}(A) = 0$.*

Proof. — When $n > 2$ all of this is proved in [**45**] using the theory of local models, and it is straightforward to check that the arguments carry over (3) to the case $n = 2$. The only change is that if $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ lies above $p \mid D$, the stack $\mathcal{M}^{\mathrm{Pap}}_{(1,1)/\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}}$ is étale locally isomorphic to

$$\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}[x,y]/(xy-p)),$$

whose special fiber is not normal. □

(3) When $n = 2$, the $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ admits a canonical descent to $\mathbb{Z}$, and Pappas analyzes the structure of this descent. The descent is regular, but the regularity is destroyed by base change to $\mathcal{O}_{\boldsymbol{k}}$.

The stack $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ is not regular, but has a natural resolution of singularities. This leads us to our second integral model of $M_{(n-1,1)}$. As in the work of Krämer [**31**], define

$$\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

to be the stack whose functor of points assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of quadruples $(A, \iota, \psi, \mathcal{F}_A)$ in which

— $A \to S$ is an abelian scheme of relative dimension n,
— $\iota : \mathcal{O}_{\boldsymbol{k}} \to \mathrm{End}(A)$ is an action of $\mathcal{O}_{\boldsymbol{k}}$,
— $\psi : A \to A^\vee$ is a principal polarization satisfying $\iota(\alpha)^\dagger = \iota(\overline{\alpha})$ for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$,
— $\mathcal{F}_A \subset \mathrm{Lie}(A)$ is an $\mathcal{O}_{\boldsymbol{k}}$-stable $\mathcal{O}_S$-module local direct summand of rank $n-1$ satisfying *Krämer's condition*: $\mathcal{O}_{\boldsymbol{k}}$ acts on $\mathcal{F}_A$ via the structure map $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$, and acts on the line bundle $\mathrm{Lie}(A)/\mathcal{F}_A$ via the complex conjugate of the structure map.

There is a proper morphism

$$\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \to \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \tag{2.3.1}$$

defined by forgetting the subsheaf $\mathcal{F}_A$, and we define the *exceptional locus*

$$\mathrm{Exc}_{(n-1,1)} \subset \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \tag{2.3.2}$$

by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Exc}_{(n-1,1)} & \longrightarrow & \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \\ \downarrow & & \downarrow \\ \mathrm{Sing}_{(n-1,1)} & \longrightarrow & \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}. \end{array}$$

***Theorem 2.3.3* (Krämer).** — *The $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}$ is regular and flat with reduced fibers, and satisfies the following properties:*

1. *The exceptional locus* (2.3.2) *is a disjoint union of smooth Cartier divisors. Its fiber over a geometric point $s \to \mathrm{Sing}_{(n-1,1)}$ is isomorphic to the projective space $\mathbb{P}^{n-1}$ over $k(s)$.*
2. *The morphism* (2.3.1) *is proper and surjective, and restricts to an isomorphism*

$$\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \setminus \mathrm{Exc}_{(n-1,1)} \cong \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \setminus \mathrm{Sing}_{(n-1,1)}.$$

For an $\mathcal{O}_{\boldsymbol{k}}$-scheme S, the inverse of this isomorphism endows

$$A \in \big(\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)} \setminus \mathrm{Sing}_{(n-1,1)}\big)(S)$$

with the subsheaf $\mathcal{F}_A = \ker\big(\overline{\varepsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A)\big)$.

Proof. — When $n > 2$ all of this is proved in [**31**] using the theory of local models, and it is straightforward to check that nearly everything [(4)] carries over to the case $n = 2$. In particular, if $n = 2$ and $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ lies above $p \mid D$, the same arguments used in [*loc. cit.*] show that $\mathcal{M}^{\mathrm{Kra}}_{(1,1)/\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}}$ is étale locally isomorphic to the regular scheme

$$\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}[x, y]/(xy - \pi)),$$

for any uniformizer $\pi \in \mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$. □

Recalling (2.2.2), we define our first integral model

$$\mathcal{S}_{\mathrm{Pap}} \subset \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$$

as the Zariski closure of $\mathrm{Sh}(G, \mathcal{D})$ in the fiber product on the right, which, like all fiber products below, is taken over over $\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$. Using Lemma 2.2.2, one can show that it is characterized as the open and closed substack whose functor of points assigns to any $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of pairs

$$(A_0, A) \in \mathcal{M}_{(1,0)}(S) \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}(S)$$

such that, at any geometric point $s \to S$, the relation (2.2.4) holds for all primes $\ell \neq \mathrm{char}(k(s))$.

Our second integral model of $\mathrm{Sh}(G, \mathcal{D})$ is defined as the cartesian product

$$\begin{array}{ccc} \mathcal{S}_{\mathrm{Kra}} & \longrightarrow & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathrm{Pap}} & \longrightarrow & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}. \end{array}$$

The *singular locus* $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$ and *exceptional locus* $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ are defined by the cartesian squares

$$\begin{array}{ccc} \mathrm{Exc} & \longrightarrow & \mathcal{S}_{\mathrm{Kra}} \\ \downarrow & & \downarrow \\ \mathrm{Sing} & \longrightarrow & \mathcal{S}_{\mathrm{Pap}} \\ \downarrow & & \downarrow \\ \mathcal{M}_{(1,0)} \times \mathrm{Sing}_{(n-1,1)} & \longrightarrow & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}. \end{array}$$

(4) When $n > 2$, the statement of [**31**, Theorem 4.4] asserts that the special fiber of the local model of $\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}$ is the union of two smooth and geometrically irreducible varieties of dimension $n - 1$, whose intersection is smooth and geometrically irreducible of dimension $n - 2$. When $n = 2$, the structure of the local model is slightly different: its geometric special fiber is a union $X_1 \cup X_2 \cup X_3$ of three irreducible varieties, each isomorphic to $\mathbb{P}^1$, intersecting in such a way that $X_1 \cap X_2$ and $X_2 \cap X_3$ are distinct reduced points. The difference between the two cases occurs because the scheme $\mathcal{Q}$ defined in the proof of [**31**, Theorem 4.4], which parametrizes isotropic lines in a quadratic space of dimension n over a finite field, is geometrically irreducible only when $n > 2$.

Both loci are proper over $\mathcal{O}_{\boldsymbol{k}}$, and supported in characteristics dividing D.

***Theorem 2.3.4* (Pappas, Krämer).** — *The $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{S}_{\mathrm{Kra}}$ is regular and flat with reduced fibers. The $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{S}_{\mathrm{Pap}}$ is Cohen-Macaulay and normal, with Cohen-Macaulay fibers. Furthermore:*

1. *If $n > 2$, the geometric fibers of $\mathcal{S}_{\mathrm{Pap}}$ are normal.*
2. *The exceptional locus $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ is a disjoint union of smooth Cartier divisors. The singular locus $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$ is a reduced closed stack of dimension 0, supported in characteristics dividing D.*
3. *The fiber of Exc over a geometric point $s \to \mathrm{Sing}$ is isomorphic to the projective space $\mathbb{P}^{n-1}$ over $k(s)$.*
4. *The morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ is surjective, and restricts to an isomorphism*

$$\mathcal{S}_{\mathrm{Kra}} \setminus \mathrm{Exc} \cong \mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}. \tag{2.3.3}$$

For an $\mathcal{O}_{\boldsymbol{k}}$-scheme S, the inverse of this isomorphism endows

$$(A_0, A) \in \big(\mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}\big)(S)$$

with the subsheaf $\mathcal{F}_A = \ker\big(\overline{\varepsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A)\big)$.

Proof. — All of this follows from Theorems 2.3.2 and 2.3.3, along with the fact that $\mathcal{M}_{(1,0)} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$ is finite étale. □

Remark 2.3.5. — Let (A_0, A) be the universal pair over $\mathcal{S}_{\mathrm{Pap}}$. The vector bundle $H_1^{\mathrm{dR}}(A_0)$ is locally free of rank one over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}_{\mathrm{Pap}}}$ and, by definition of the moduli problem defining $\mathcal{S}_{\mathrm{Pap}}$, its quotient $\mathrm{Lie}(A_0)$ is annihilated by $\overline{\varepsilon}$. From this it is not hard to see that

$$F^0 H_1^{\mathrm{dR}}(A_0) = \overline{\varepsilon} H_1^{\mathrm{dR}}(A_0).$$

2.4. The line bundle of modular forms. — We now construct a line bundle of modular forms $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$, and consider the subtle question of whether or not it descends to $\mathcal{S}_{\mathrm{Pap}}$. The short answer is that it doesn't, but a more complete answer can be found in Theorems 2.4.3 and 2.6.3.

By Remark 2.1.3, every point $z \in \mathcal{D}$ determines Hodge structures on W_0 and W of weight -1, and hence a Hodge structure of weight 0 on $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$. Consider the holomorphic line bundle $\boldsymbol{\omega}^{\mathrm{an}}$ on $\mathcal{D}$ whose fiber at z is the complex line $\boldsymbol{\omega}^{\mathrm{an}}_z = F^1 V(\mathbb{C})$ determined by this Hodge structure.

Remark 2.4.1. — It is useful to interpret $\boldsymbol{\omega}^{\mathrm{an}}$ in the notation of Remark 2.1.2. The fiber of $\boldsymbol{\omega}^{\mathrm{an}}$ at $z = (y_0, y)$ is the line

$$\boldsymbol{\omega}^{\mathrm{an}}_z = \mathrm{Hom}_{\mathbb{C}}(W_0(\mathbb{C})/\overline{\varepsilon} W_0(\mathbb{C}), \mathrm{pr}_{\varepsilon}(y)) \subset \varepsilon V(\mathbb{C}), \tag{2.4.1}$$

and hence $\boldsymbol{\omega}^{\mathrm{an}}$ is simply the restriction of the tautological bundle via the inclusion

$$\mathcal{D} \cong \big\{ w \in \varepsilon V(\mathbb{C}) : [w, \overline{w}] < 0 \big\} / \mathbb{C}^\times \subset \mathbb{P}(\varepsilon V(\mathbb{C})).$$

There is a natural action of $G(\mathbb{R})$ on the total space of $\boldsymbol{\omega}^{\mathrm{an}}$, lifting the natural action on $\mathcal{D}$, and so $\boldsymbol{\omega}^{\mathrm{an}}$ descends to a line bundle on the complex orbifold $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. This descent is algebraic, has a canonical model over the reflex field, and extends in a natural way to the integral model $\mathcal{S}_{\mathrm{Kra}}$, as we now explain.

Let (A_0, A) be the universal object over $\mathcal{S}_{\mathrm{Kra}}$, let $\mathcal{F}_A \subset \mathrm{Lie}(A)$ be the universal subsheaf of Krämer's moduli problem, and let

$$\mathcal{F}_A^{\perp} \subset F^0 H_1^{\mathrm{dR}}(A)$$

be the orthogonal to $\mathcal{F}_A$ under the pairing (2.2.1). It is a rank one $\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}}}$-module local direct summand on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{S}_{\mathrm{Kra}}}$. Define the *line bundle of weight one modular forms* on $\mathcal{S}_{\mathrm{Kra}}$ by

$$\boldsymbol{\omega} = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \mathcal{F}_A^{\perp}),$$

or, equivalently, $\boldsymbol{\omega}^{-1} = \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A)/\mathcal{F}_A$.

Proposition 2.4.2. — *The line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$ just defined restricts to the already defined $\boldsymbol{\omega}^{\mathrm{an}}$ in the complex fiber. Moreover, on the complement of the exceptional locus* $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ *we have*

$$\boldsymbol{\omega} = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \varepsilon F^0 H_1^{\mathrm{dR}}(A)).$$

Proof. — The equality $\mathcal{F}_A^{\perp} = \varepsilon F^0 H_1^{\mathrm{dR}}(A)$ on the complement of Exc follows from the characterization

$$\mathcal{F}_A = \ker(\overline{\varepsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A))$$

of Theorem 2.3.4, and all of the claims follow easily from this and examination of the proof of Proposition 2.2.1. □

The line bundle $\boldsymbol{\omega}$ does not descend to $\mathcal{S}_{\mathrm{Pap}}$, but it is closely related to another line bundle that does. This is the content of the following theorem, whose proof will occupy the remainder of § 2.4. The result will be strengthened in Theorem 2.6.3.

Theorem 2.4.3. — *There is a unique line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}_{\mathrm{Pap}}$ whose restriction to the nonsingular locus* (2.3.3) *is isomorphic to $\boldsymbol{\omega}^2$. We denote by $\boldsymbol{\Omega}_{\mathrm{Kra}}$ its pullback via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$.*

Proof. — Let (A_0, A) be the universal object over $\mathcal{S}_{\mathrm{Pap}}$, and recall the short exact sequence

$$0 \to F^0 H_1^{\mathrm{dR}}(A) \to H_1^{\mathrm{dR}}(A) \xrightarrow{q} \mathrm{Lie}(A) \to 0$$

of vector bundles on $\mathcal{S}_{\mathrm{Pap}}$. As $H_1^{\mathrm{dR}}(A)$ is a locally free $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}_{\mathrm{Pap}}}$-module of rank n, the quotient $H_1^{\mathrm{dR}}(A)/\overline{\varepsilon} H_1^{\mathrm{dR}}(A)$ is a rank n vector bundle.

Define a line bundle

$$\mathcal{P}_{\mathrm{Pap}} = \underline{\mathrm{Hom}}\Big(\bigwedge\nolimits^n H_1^{\mathrm{dR}}(A)/\overline{\varepsilon} H_1^{\mathrm{dR}}(A), \bigwedge\nolimits^n \mathrm{Lie}(A) \Big)$$

on $\mathcal{S}_{\mathrm{Pap}}$, and denote by $\mathcal{P}_{\mathrm{Kra}}$ its pullback via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. Let

$$\psi : H_1^{\mathrm{dR}}(A) \otimes H_1^{\mathrm{dR}}(A) \to \mathcal{O}_{\mathcal{S}_{\mathrm{Pap}}}$$

be the alternating pairing induced by the principal polarization on A. If a and b are local sections of $H_1^{\mathrm{dR}}(A)$, define a local section $P_{a\otimes b}$ of $\mathcal{P}_{\mathrm{Pap}}$ by

$$P_{a\otimes b}(e_1\wedge\cdots\wedge e_n)=\sum_{k=1}^{n}(-1)^{k+1}\cdot\psi(\overline{\varepsilon}a,e_k)\cdot q(\overline{\varepsilon}b)\wedge\underbrace{q(e_1)\wedge\cdots\wedge q(e_n)}_{\text{omit } q(e_k)}.$$

Remark 2.4.4. — To see that $P_{a\otimes b}$ is well-defined, one must check that modifying any e_k by a section of $\overline{\varepsilon}H_1^{\mathrm{dR}}(A)$ leaves the right hand side unchanged. This is an easy consequence of the vanishing of

$$\bigwedge^2\overline{\varepsilon}:\bigwedge^2\mathrm{Lie}(A)\to\bigwedge^2\mathrm{Lie}(A)$$

imposed in the moduli problem defining $\mathcal{S}_{\mathrm{Pap}}$.

Lemma 2.4.5. — *The morphism*

$$P:H_1^{\mathrm{dR}}(A)\otimes H_1^{\mathrm{dR}}(A)\to\mathcal{P}_{\mathrm{Pap}} \tag{2.4.2}$$

defined by $a\otimes b\mapsto P_{a\otimes b}$ *factors through a morphism*

$$P:\mathrm{Lie}(A)\otimes\mathrm{Lie}(A)\to\mathcal{P}_{\mathrm{Pap}}.$$

After pullback to $\mathcal{S}_{\mathrm{Kra}}$ *there is a further factorization*

$$P:\mathrm{Lie}(A)/\mathcal{F}_A\otimes\mathrm{Lie}(A)/\mathcal{F}_A\to\mathcal{P}_{\mathrm{Kra}}, \tag{2.4.3}$$

and this map becomes an isomorphism after restriction to $\mathcal{S}_{\mathrm{Kra}}\setminus\mathrm{Exc}$.

Proof. — Let a and b be local sections of $H_1^{\mathrm{dR}}(A)$.

Assume first that a is contained in $F^0H_1^{\mathrm{dR}}(A)$. As $F^0H_1^{\mathrm{dR}}(A)$ is isotropic under the pairing ψ, $P_{a\otimes b}$ factors through a map

$$\bigwedge^n\mathrm{Lie}(A)/\overline{\varepsilon}\mathrm{Lie}(A)\to\bigwedge^n\mathrm{Lie}(A).$$

In the generic fiber of $\mathcal{S}_{\mathrm{Pap}}$, the sheaf $\mathrm{Lie}(A)/\overline{\varepsilon}\mathrm{Lie}(A)$ is a vector bundle of rank $n-1$. This proves that $P_{a\otimes b}$ is trivial over the generic fiber. As $P_{a\otimes b}$ is a morphism of vector bundles on a flat $\mathcal{O}_{\boldsymbol{k}}$-stack, we deduce that $P_{a\otimes b}=0$ identically on $\mathcal{S}_{\mathrm{Pap}}$.

If instead b is contained in $F^0H_1^{\mathrm{dR}}(A)$ then $q(\overline{\varepsilon}b)=0$, and again $P_{a\otimes b}=0$. These calculations prove that P factors through $\mathrm{Lie}(A)\otimes\mathrm{Lie}(A)$.

Now pullback to $\mathcal{S}_{\mathrm{Kra}}$. We need to check that $P_{a\otimes b}$ vanishes if either of a or b lies in $\mathcal{F}_A$. Once again it suffices to check this in the generic fiber, where it is clear from

$$\mathcal{F}_A=\ker(\overline{\varepsilon}:\mathrm{Lie}(A)\to\mathrm{Lie}(A)). \tag{2.4.4}$$

Over $\mathcal{S}_{\mathrm{Kra}}$ we now have a factorization (2.4.3), and it only remains to check that its restriction to (2.3.3) is an isomorphism. For this, it suffices to verify that (2.4.3) is surjective on the fiber at any geometric point

$$s=\mathrm{Spec}(\mathbb{F})\to\mathcal{S}_{\mathrm{Kra}}\setminus\mathrm{Exc}.$$

First suppose that $\mathrm{char}(\mathbb{F})$ is prime to D. In this case $\varepsilon, \overline{\varepsilon} \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}$ are (up to scaling by $\mathbb{F}^{\times}$) orthogonal idempotents, $\mathcal{F}_{A_s} = \varepsilon \mathrm{Lie}(A_s)$, and we may choose an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}$-basis $e_1, \ldots, e_n \in H_1^{\mathrm{dR}}(A_s)$ in such a way that

$$\varepsilon e_1, \overline{\varepsilon} e_2, \ldots, \overline{\varepsilon} e_n \in F^0 H_1^{\mathrm{dR}}(A_s)$$

and

$$q(\overline{\varepsilon} e_1), q(\varepsilon e_2), \ldots, q(\varepsilon e_n) \in \mathrm{Lie}(A_s)$$

are $\mathbb{F}$-bases. This implies that

$$P_{e_1 \otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \psi(\overline{\varepsilon} e_1, \varepsilon e_1) \cdot q(\overline{\varepsilon} e_1) \wedge q(\varepsilon e_2) \wedge \cdots \wedge q(\varepsilon e_n) \neq 0,$$

and so

$$P_{e_1 \otimes e_1} \in \mathrm{Hom}\big(\bigwedge\nolimits^n H_1^{\mathrm{dR}}(A_s) / \overline{\varepsilon} H_1^{\mathrm{dR}}(A_s), \bigwedge\nolimits^n \mathrm{Lie}(A_s) \big)$$

is a generator. Thus P is surjective in the fiber at z.

Now suppose that $\mathrm{char}(\mathbb{F})$ divides D. In this case there is an isomorphism

$$\mathbb{F}[x]/(x^2) \xrightarrow{x \mapsto \varepsilon = \overline{\varepsilon}} \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}.$$

By Theorem 2.3.4 the relation (2.4.4) holds in an étale neighborhood of s, and it follows that we may choose an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathbb{F}$-basis $e_1, \ldots, e_n \in H_1^{\mathrm{dR}}(A_s)$ in such a way that

$$e_2, \varepsilon e_2, \varepsilon e_3, \ldots, \varepsilon e_n \in F^0 H_1^{\mathrm{dR}}(A_s)$$

and

$$q(e_1), q(\varepsilon e_1), q(e_3) \ldots, q(e_n) \in \mathrm{Lie}(A_s)$$

are $\mathbb{F}$-bases. This implies that

$$P_{e_1 \otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \psi(\varepsilon e_1, e_2) \cdot q(\varepsilon e_1) \wedge q(e_1) \wedge q(e_3) \wedge \cdots \wedge q(e_n) \neq 0,$$

and so, as above, P is surjective in the fiber at z. □

We now complete the proof of Theorem 2.4.3. To prove the existence part of the claim, we define $\boldsymbol{\Omega}_{\mathrm{Pap}}$ by

$$\boldsymbol{\Omega}_{\mathrm{Pap}}^{-1} = \mathrm{Lie}(A_0)^{\otimes 2} \otimes \mathcal{P}_{\mathrm{Pap}},$$

and let $\boldsymbol{\Omega}_{\mathrm{Kra}}$ be its pullback via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$. Tensoring both sides of (2.4.3) with $\mathrm{Lie}(A_0)^{\otimes 2}$ defines a morphism

$$\boldsymbol{\omega}^{-2} \to \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1},$$

whose restriction to $\mathcal{S}_{\mathrm{Kra}} \setminus \mathrm{Exc}$ is an isomorphism. In particular $\boldsymbol{\omega}^2$ and $\boldsymbol{\Omega}_{\mathrm{Pap}}$ are isomorphic over (2.3.3).

The uniqueness of $\boldsymbol{\Omega}_{\mathrm{Pap}}$ is clear: as $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$ is a codimension ≥ 2 closed substack of a normal stack, any line bundle on the complement of Sing admits at most one extension to all of $\mathcal{S}_{\mathrm{Pap}}$. □

2.5. Special divisors. — Suppose S is a connected $\mathcal{O}_{\boldsymbol{k}}$-scheme, and

$$(A_0, A) \in \mathcal{S}_{\mathrm{Pap}}(S).$$

Imitating the construction of (2.1.5), there is a positive definite hermitian form on $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ defined by

$$\langle x_1, x_2 \rangle = x_2^\vee \circ x_1 \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A_0) \cong \mathcal{O}_{\boldsymbol{k}}, \tag{2.5.1}$$

where

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A) \xrightarrow{x \mapsto x^\vee} \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A, A_0)$$

is the $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear isomorphism induced by the principal polarizations on A_0 and A.

For any positive $m \in \mathbb{Z}$, define the $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{Z}_{\mathrm{Pap}}(m)$ as the moduli stack assigning to a connected $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of triples (A_0, A, x), where

— $(A_0, A) \in \mathcal{S}_{\mathrm{Pap}}(S)$,

— $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfies $\langle x, x \rangle = m$.

Define a stack $\mathcal{Z}_{\mathrm{Kra}}(m)$ in exactly the same way, but replacing $\mathcal{S}_{\mathrm{Pap}}$ by $\mathcal{S}_{\mathrm{Kra}}$. Thus we obtain a cartesian diagram

$$\begin{array}{ccc} \mathcal{Z}_{\mathrm{Kra}}(m) & \longrightarrow & \mathcal{S}_{\mathrm{Kra}} \\ \downarrow & & \downarrow \\ \mathcal{Z}_{\mathrm{Pap}}(m) & \longrightarrow & \mathcal{S}_{\mathrm{Pap}}, \end{array}$$

in which the horizontal arrows are relatively representable, finite, and unramified.

Each $\mathcal{Z}_{\mathrm{Kra}}(m)$ is, étale locally on $\mathcal{S}_{\mathrm{Kra}}$, a disjoint union of Cartier divisors. More precisely, around any geometric point of $\mathcal{S}_{\mathrm{Kra}}$ one can find an étale neighborhood U with the property that the morphism $\mathcal{Z}_{\mathrm{Kra}}(m)_U \to U$ restricts to a closed immersion on every connected component $Z \subset \mathcal{Z}_{\mathrm{Kra}}(m)_U$, and $Z \subset U$ is defined locally by one equation; this is [**24**, Proposition 3.2.3], but a cleaner argument (working on the Rapoport-Zink space corresponding to $\mathcal{S}_{\mathrm{Kra}}$) can be found in [**25**, Proposition 4.3]. Summing over all connected components Z allows us to view $\mathcal{Z}_{\mathrm{Kra}}(m)_U$ as a Cartier divisor on U, and gluing as U varies over an étale cover defines a Cartier divisor on $\mathcal{S}_{\mathrm{Kra}}$, which we again denote by $\mathcal{Z}_{\mathrm{Kra}}(m)$.

Remark 2.5.1. — It follows from (2.3.3) and the paragraph above that $\mathcal{Z}_{\mathrm{Pap}}(m)$ is locally defined by one equation away from the singular locus, and so defines a Cartier divisor on $\mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}$. This Cartier divisor does not extend to all of $\mathcal{S}_{\mathrm{Pap}}$.

Remark 2.5.2. — We can make the specal divisors more explicit in the complex fiber, as in [**34**, Proposition 3.5] or [**23**, § 3.8]. Recall from § 2.1 that the $\mathbb{Q}$-vector space $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ carries a quadratic form. Using the description

$$\mathcal{D} \cong \big\{ z \in \varepsilon V(\mathbb{C}) : [z, \overline{z}] < 0 \big\} / \mathbb{C}^\times \subset \mathbb{P}(\varepsilon V(\mathbb{C}))$$

of Remark 2.1.2, every $x \in V$ with $Q(x) > 0$ determines an analytic divisor

$$\mathcal{D}(x) = \{ z \in \mathcal{D} : [z, x] = 0 \}.$$

A choice of $g \in G(\mathbb{A}_f)$ determines a connected component

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash \mathcal{D} \xrightarrow{z \mapsto (z,g)} G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K \cong \mathcal{S}_{\mathrm{Kra}}(\mathbb{C}),$$

and if we set

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, g\mathfrak{a}) \subset V$$

the restriction of $\mathcal{Z}_{\mathrm{Kra}}(m)(\mathbb{C}) \to \mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ to this component is

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash \bigsqcup_{\substack{x \in L \\ Q(x)=m}} \mathcal{D}(x) \to (G(\mathbb{Q}) \cap gKg^{-1})\backslash \mathcal{D}.$$

The following theorem, whose proof will occupy the remainder of § 2.5, shows that $\mathcal{Z}_{\mathrm{Kra}}(m)$ is closely related to another Cartier divisor on $\mathcal{S}_{\mathrm{Kra}}$ that descends to $\mathcal{S}_{\mathrm{Pap}}$. This result will be strengthened in Theorem 2.6.3.

Theorem 2.5.3. — *For every $m > 0$ there is a unique Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}}$ whose restriction to $\mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}$ agrees with $2\mathcal{Z}_{\mathrm{Pap}}(m)$. In particular its pullback $\mathcal{Y}_{\mathrm{Kra}}(m)$ via $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ agrees with $2\mathcal{Z}_{\mathrm{Kra}}(m)$ over $\mathcal{S}_{\mathrm{Kra}} \setminus \mathrm{Exc}$.*

Proof. — The map $\mathcal{Z}_{\mathrm{Pap}}(m) \to \mathcal{S}_{\mathrm{Pap}}$ is finite, unramified, and relatively representable. It follows that every geometric point of $\mathcal{S}_{\mathrm{Pap}}$ admits an étale neighborhood $U \to \mathcal{S}_{\mathrm{Pap}}$ such that U is a scheme, and the morphism

$$\mathcal{Z}_{\mathrm{Pap}}(m)_U \to U$$

restricts to a closed immersion on every connected component

$$Z \subset \mathcal{Z}_{\mathrm{Pap}}(m)_U.$$

We will construct a Cartier divisor on any such U, and then glue them together as U varies over an étale cover to obtain the divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$.

Fix Z as above, let $\mathcal{I} \subset \mathcal{O}_U$ be its ideal sheaf, and let Z' be the closed subscheme of U defined by the ideal sheaf $\mathcal{I}^2$. Thus we have closed immersions

$$Z \subset Z' \subset U,$$

the first of which is a square-zero thickening.

By the very definition of $\mathcal{Z}_{\mathrm{Pap}}(m)$, along Z there is a universal $\mathcal{O}_{\boldsymbol{k}}$-linear map $x : A_{0Z} \to A_Z$. This map does not extend to a map $A_{0Z'} \to A_{Z'}$, however, by deformation theory [**40**, Chapter 2.1.6] the induced $\mathcal{O}_{\boldsymbol{k}}$-linear morphism of vector bundles

$$x : H_1^{\mathrm{dR}}(A_{0Z}) \to H_1^{\mathrm{dR}}(A_Z)$$

admits a canonical extension to

$$x' : H_1^{\mathrm{dR}}(A_{0Z'}) \to H_1^{\mathrm{dR}}(A_{Z'}). \tag{2.5.2}$$

Recalling the morphism (2.4.2), define $Y \subset Z'$ as the largest closed subscheme over which the composition

$$H_1^{\mathrm{dR}}(A_{0Z'}) \otimes H_1^{\mathrm{dR}}(A_{0Z'}) \xrightarrow{x' \otimes x'} H_1^{\mathrm{dR}}(A_{Z'}) \otimes H_1^{\mathrm{dR}}(A_{Z'}) \xrightarrow{P} \mathcal{P}_{\mathrm{Pap}}|_{Z'} \tag{2.5.3}$$

vanishes.

Lemma 2.5.4. — *If* $U \to \mathcal{S}_{\mathrm{Pap}}$ *factors through* $\mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}$, *then* $Y = Z'$.

Proof. — Lemma 2.4.5 provides us with a commutative diagram

$$\begin{array}{ccccc} H_1^{\mathrm{dR}}(A_{0Z'})^{\otimes 2} & \xrightarrow{x'\otimes x'} & H_1^{\mathrm{dR}}(A_{Z'})^{\otimes 2} & \xrightarrow{q\otimes q} & \big(\mathrm{Lie}(A_{Z'})/\mathcal{F}_{A_{Z'}}\big)^{\otimes 2} \\ & \searrow^{(2.5.3)} & & & \downarrow \cong \\ & & & & \mathcal{P}_{\mathrm{Pap}}|_{Z'}, \end{array}$$

where

$$\mathcal{F}_{A_{Z'}} = \ker(\overline{\varepsilon} : \mathrm{Lie}(A_{Z'}) \to \mathrm{Lie}(A_{Z'}))$$

as in Theorem 2.3.4.

By deformation theory, $Z \subset Z'$ is characterized as the largest closed subscheme over which (2.5.2) respects the Hodge filtrations. Using Remark 2.3.5, it is easily seen that $Z \subset Z'$ can also be characterized as the largest closed subscheme over which

$$H_1(A_{0Z'}) \xrightarrow{q\circ x'} \mathrm{Lie}(A_{Z'})/\mathcal{F}_{A_{Z'}}$$

vanishes identically. As $Z \subset Z'$ is a square zero thickening, it follows first that the horizontal composition in the above diagram vanishes identically, and then that (2.5.3) vanishes identically. In other words $Y = Z'$. □

Lemma 2.5.5. — *The closed subscheme* $Y \subset U$ *is defined locally by one equation.*

Proof. — Fix a closed point $y \in Y$ of characteristic p, let $\mathcal{O}_{U,y}$ be the local ring of U at y, and let $\mathfrak{m} \subset \mathcal{O}_{U,y}$ be the maximal ideal. For a fixed $k > 0$, let

$$\boldsymbol{U} = \mathrm{Spec}(\mathcal{O}_{U,y}/\mathfrak{m}^k) \subset U$$

be the k-th order infinitesimal neighborhood of y in U. The point of passing to the infinitesimal neighborhood is that p is nilpotent in $\mathcal{O}_{\boldsymbol{U}}$, and so we may apply Grothendieck-Messing deformation theory.

By construction we have closed immersions

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow & & \\ Z & \longrightarrow & Z' & \longrightarrow & U. \end{array}$$

Applying the fiber product $\times_U \boldsymbol{U}$ throughout the diagram, we obtain closed immersions

$$\begin{array}{ccccc} & & \boldsymbol{Y} & & \\ & & \downarrow & & \\ \boldsymbol{Z} & \longrightarrow & \boldsymbol{Z}' & \longrightarrow & \boldsymbol{U} \end{array}$$

of Artinian schemes. As k is arbitrary, it suffices to prove that $\boldsymbol{Y} \subset \boldsymbol{U}$ is defined by one equation.

First suppose that $p \nmid D$. In this case $\boldsymbol{U} \to U \to \mathcal{S}_{\mathrm{Pap}}$ factors through the nonsingular locus (2.3.3). It follows from Remark 2.5.1 that $\boldsymbol{Z} \subset \boldsymbol{U}$ is defined by one equation, and $\boldsymbol{Z}'$ is defined by the square of that equation. By Lemma 2.5.4, $\boldsymbol{Y} \subset \boldsymbol{U}$ is also defined by one equation.

For the remainder of the proof we assume that $p \mid D$. In particular $p > 2$. Consider the closed subscheme $Z'' \hookrightarrow U$ with ideal sheaf $\mathcal{I}^3$, so that we have closed immersions $Z \subset Z' \subset Z'' \subset U$. Taking the fiber product with $\boldsymbol{U}$, the above diagram extends to

$$\begin{array}{ccccccc} & & \boldsymbol{Y} & & & & \\ & & \downarrow & & & & \\ \boldsymbol{Z} & \longrightarrow & \boldsymbol{Z}' & \longrightarrow & \boldsymbol{Z}'' & \longrightarrow & \boldsymbol{U}. \end{array}$$

As $p > 2$, the cube zero thickening $\boldsymbol{Z} \subset \boldsymbol{Z}''$ admits divided powers extending the trivial divided powers on $\boldsymbol{Z} \subset \boldsymbol{Z}'$. Therefore, by Grothendieck-Messing theory, the restriction of (2.5.2) to

$$x' : H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}'}) \to H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}'})$$

admits a canonical extension to

$$x'' : H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \to H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}).$$

Define $\boldsymbol{Y}' \subset \boldsymbol{Z}''$ as the largest closed subscheme over which

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \xrightarrow{x''\otimes x''} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \xrightarrow{P} \mathcal{P}_{\mathrm{Pap}}|_{\boldsymbol{Z}''} \tag{2.5.4}$$

vanishes identically, so that there are closed immersions

$$\begin{array}{ccccccc} & & \boldsymbol{Y} & \longrightarrow & \boldsymbol{Y}' & & \\ & & \downarrow & & \downarrow & & \\ \boldsymbol{Z} & \longrightarrow & \boldsymbol{Z}' & \longrightarrow & \boldsymbol{Z}'' & \longrightarrow & \boldsymbol{U}. \end{array}$$

We pause the proof of Lemma 2.5.5 for a sub-lemma.

Lemma 2.5.6. — *We have* $\boldsymbol{Y} = \boldsymbol{Y}'$.

Proof. — As in the proof of Lemma 2.5.4, we may characterize $\boldsymbol{Z} \subset \boldsymbol{Z}''$ as the largest closed subscheme along which x'' respects the Hodge filtrations. Equivalently, by Remark 2.3.5, $\boldsymbol{Z} \subset \boldsymbol{Z}''$ is the largest closed subscheme over which the composition

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \xrightarrow{x''\circ\overline{\varepsilon}} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \xrightarrow{q} \mathrm{Lie}(A_{\boldsymbol{Z}''})$$

vanishes identically. This implies that $\boldsymbol{Z}' \subset \boldsymbol{Z}''$ is the largest closed subscheme over which

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})^{\otimes 2} \xrightarrow{(x''\circ\overline{\varepsilon})^{\otimes 2}} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''})^{\otimes 2} \xrightarrow{q^{\otimes 2}} \mathrm{Lie}(A_{\boldsymbol{Z}''})^{\otimes 2} \tag{2.5.5}$$

vanishes identically.

It follows directly from the definitions that $\boldsymbol{Y} = \boldsymbol{Y}' \cap \boldsymbol{Z}'$, and hence it suffices to show that $\boldsymbol{Y}' \subset \boldsymbol{Z}'$. In other words, it suffices to show that the vanishing of (2.5.4) implies the vanishing of (2.5.5).

For local sections a and b of $H_1(A_{\boldsymbol{Z}''})$, define

$$Q_{a\otimes b} : F^0 H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes \bigwedge^{n-1} \mathrm{Lie}(A_{\boldsymbol{Z}''}) \to \bigwedge^{n} \mathrm{Lie}(A_{\boldsymbol{Z}''})$$

by

$$Q_{a\otimes b}(e_1 \otimes q(e_2) \wedge \cdots \wedge q(e_n)) = \psi(a, e_1) \cdot q(b) \wedge q(e_2) \wedge \cdots \wedge q(e_n).$$

It is clear that $Q_{a\otimes b}$ depends only on the images of a and b in $\mathrm{Lie}(A_{\boldsymbol{Z}''})$, and that this construction defines an isomorphism

$$\mathrm{Lie}(A_{\boldsymbol{Z}''})^{\otimes 2} \xrightarrow{Q} \underline{\mathrm{Hom}}\Big(F^0 H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes \bigwedge^{n-1} \mathrm{Lie}(A_{\boldsymbol{Z}''}), \bigwedge^{n} \mathrm{Lie}(A_{\boldsymbol{Z}''})\Big). \tag{2.5.6}$$

It is related to the map

$$\mathrm{Lie}(A_{\boldsymbol{Z}''})^{\otimes 2} \xrightarrow{P} \underline{\mathrm{Hom}}\Big(\bigwedge^{n} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''})/\overline{\varepsilon} H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}), \bigwedge^{n} \mathrm{Lie}(A_{\boldsymbol{Z}''})\Big)$$

of Lemma 2.4.5 by

$$P_{a\otimes b}(e_1 \wedge \cdots \wedge e_n) = Q_{\overline{\varepsilon} a\otimes \overline{\varepsilon} b}(e_1 \otimes q(e_2) \wedge \cdots \wedge q(e_n))$$

for any local section $e_1 \otimes e_2 \otimes \cdots \otimes e_n$ of

$$F^0 H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}) \otimes \cdots \otimes H_1^{\mathrm{dR}}(A_{\boldsymbol{Z}''}).$$

Putting everything together, if (2.5.4) vanishes, then $P_{x''(a_0)\otimes x''(b_0)} = 0$ for all local sections a_0 and b_0 of $H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})$. Therefore

$$Q_{x''(\overline{\varepsilon} a_0)\otimes x''(\overline{\varepsilon} b_0)} = 0$$

for all local sections a_0 and b_0, which implies, as (2.5.6) is an isomorphism, that (2.5.5) vanishes. This proves that $\boldsymbol{Y}' \subset \boldsymbol{Z}'$, and hence $\boldsymbol{Y} = \boldsymbol{Y}'$. □

Returning to the proof of Lemma 2.5.5, the map (2.5.4), whose vanishing defines $\boldsymbol{Y}' \subset \boldsymbol{Z}''$, factors through a morphism of line bundles

$$H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})/\varepsilon H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \otimes H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''})/\varepsilon H_1^{\mathrm{dR}}(A_{0\boldsymbol{Z}''}) \to \mathcal{P}_{\mathrm{Pap}}|_{\boldsymbol{Z}''},$$

and hence $\boldsymbol{Y} = \boldsymbol{Y}'$ is defined inside of $\boldsymbol{Z}''$ locally by one equation. In other words, if we denote by $\boldsymbol{\mathcal{I}} \subset \mathcal{O}_{\boldsymbol{U}}$ and $\boldsymbol{\mathcal{J}} \subset \mathcal{O}_{\boldsymbol{U}}$ the ideal sheaves of $\boldsymbol{Z} \subset \boldsymbol{U}$ and $\boldsymbol{Y} \subset \boldsymbol{U}$, respectively, then $\boldsymbol{\mathcal{I}}^3$ is the ideal sheaf of $\boldsymbol{Z}'' \subset \boldsymbol{U}$, and

$$\boldsymbol{\mathcal{J}} = (f) + \boldsymbol{\mathcal{I}}^3$$

for some $f \in \mathcal{O}_{\boldsymbol{U}}$. But $\boldsymbol{Y} \subset \boldsymbol{Z}'$ implies that $\boldsymbol{\mathcal{I}}^2 \subset \boldsymbol{\mathcal{J}}$, and hence $\boldsymbol{\mathcal{I}}^3 \subset \boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{J}}$. It follows that the image of f under the composition

$$\boldsymbol{\mathcal{J}}/\boldsymbol{\mathcal{I}}^3 \to \boldsymbol{\mathcal{J}}/\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{J}} \to \boldsymbol{\mathcal{J}}/\mathfrak{m}\boldsymbol{\mathcal{J}}$$

is an $\mathcal{O}_{\boldsymbol{U}}$-module generator, and $\boldsymbol{\mathcal{J}}$ is principal by Nakayama's lemma. □

At last we can complete the proof of Theorem 2.5.3. For each connected component $Z \subset \mathcal{Z}_{\mathrm{Pap}}(m)_U$ we have now defined a closed subscheme $Y \subset Z'$. By Lemma 2.5.5 it is an effective Cartier divisor, and summing these Cartier divisors as Z varies over all connected components yields an effective Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)_U$ on U. Letting U vary over an étale cover and applying étale descent defines an effective Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}}$.

The Cartier divisor $\mathcal{Y}_{\mathrm{Pap}}(m)$ just defined agrees with $2\mathcal{Z}_{\mathrm{Pap}}(m)$ on $\mathcal{S}_{\mathrm{Pap}} \setminus \mathrm{Sing}$. This is clear from Lemma 2.5.4 and the definition of $\mathcal{Y}_{\mathrm{Pap}}(m)$. The uniqueness claim follows from the normality of $\mathcal{S}_{\mathrm{Pap}}$, exactly as in the proof of Theorem 2.4.3. □

2.6. Pullbacks of Cartier divisors. — After Theorem 2.4.3 we have two line bundles $\boldsymbol{\Omega}_{\mathrm{Kra}}$ and $\boldsymbol{\omega}^2$ on $\mathcal{S}_{\mathrm{Kra}}$, which agree over the complement of the exceptional locus Exc. We wish to pin down more precisely the relation between them.

Similarly, after Theorem 2.5.3 we have Cartier divisors $\mathcal{Y}_{\mathrm{Kra}}(m)$ and $2\mathcal{Z}_{\mathrm{Kra}}(m)$. These agree on the complement of Exc, and again we wish to pin down more precisely the relation between them.

Denote by $\pi_0(\mathrm{Sing})$ the set of connected components of the singular locus $\mathrm{Sing} \subset \mathcal{S}_{\mathrm{Pap}}$. For each $s \in \pi_0(\mathrm{Sing})$ there is a corresponding irreducible effective Cartier divisor

$$\mathrm{Exc}_s = \mathrm{Exc} \times_{\mathcal{S}_{\mathrm{Pap}}} s \hookrightarrow \mathcal{S}_{\mathrm{Kra}}$$

supported in a single characteristic dividing D. These satisfy

$$\mathrm{Exc} = \bigsqcup_{s \in \pi_0(\mathrm{Sing})} \mathrm{Exc}_s.$$

Remark 2.6.1. — As Sing is a reduced 0-dimensional stack of finite type over $\mathcal{O}_{\boldsymbol{k}}/\mathfrak{d}$, each $s \in \pi_0(\mathrm{Sing})$ can be realized as the stack quotient

$$s \cong G_s \backslash \mathrm{Spec}(\mathbb{F}_s)$$

for a finite field $\mathbb{F}_s$ of characteristic $p \mid D$ acted on by a finite group G_s.

Fix a geometric point $\mathrm{Spec}(\mathbb{F}) \to s$, and set $p = \mathrm{char}(\mathbb{F})$. By mild abuse of notation this geometric point will again be denoted simply by s. It determines a pair

$$(A_{0,s}, A_s) \in \mathcal{S}_{\mathrm{Pap}}(\mathbb{F}), \tag{2.6.1}$$

and hence a positive definite hermitian $\mathcal{O}_{\boldsymbol{k}}$-module

$$L_s = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0,s}, A_s)$$

as in (2.5.1). This hermitian lattice depends only on $s \in \pi_0(\mathrm{Sing})$, not on the choice of geometric point above it.

Proposition 2.6.2. — *For each $s \in \pi_0(\mathrm{Sing})$ the abelian varieties A_{0s} and A_s are supersingular, and there is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism of p-divisible groups*

$$A_s[p^\infty] \cong \underbrace{A_{0s}[p^\infty] \times \cdots \times A_{0s}[p^\infty]}_{n \text{ times}} \tag{2.6.2}$$

identifying the polarization on the left with the product polarization on the right. Moreover, the hermitian $\mathcal{O}_{\boldsymbol{k}}$-module L_s is self-dual of rank n.

Proof. — Certainly A_{0s} is supersingular, as p is ramified in $\mathcal{O}_{\boldsymbol{k}} \subset \mathrm{End}(A_{0s})$.

Denote by $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ be the unique prime above p. Let $W = W(\mathbb{F})$ be the Witt ring of $\mathbb{F}$, and let $\mathrm{Fr} \in \mathrm{Aut}(W)$ be the unique continuous lift of the p-power Frobenius on $\mathbb{F}$. Let $\mathbb{D}(W)$ denote the covariant Dieudonné module of A_s, endowed with its operators F and V satisfying $FV = p = VF$. The Dieudonné module is free of rank n over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$, and the short exact sequence

$$0 \to F^0 H_1^{\mathrm{dR}}(A_s) \to H_1^{\mathrm{dR}}(A_s) \to \mathrm{Lie}(A_s) \to 0$$

of $\mathbb{F}$-modules is identified with

$$0 \to V\mathbb{D}(W)/p\mathbb{D}(W) \to \mathbb{D}(W)/p\mathbb{D}(W) \to \mathbb{D}(W)/V\mathbb{D}(W) \to 0.$$

As D is odd, the element $\delta \in \mathcal{O}_{\boldsymbol{k}}$ fixed in § 1.7 satisfies $\mathrm{ord}_{\mathfrak{p}}(\delta) = 1$. This implies that

$$\delta \cdot \mathbb{D}(W) = V\mathbb{D}(W).$$

Indeed, by Theorem 2.3.2 the Lie algebra $\mathrm{Lie}(A_s)$ is annihilated by δ, and hence $\delta \cdot \mathbb{D}(W) \subset V\mathbb{D}(W)$. Equality holds as

$$\dim_{\mathbb{F}}\big(\mathbb{D}(W)/\delta \cdot \mathbb{D}(W)\big) = n = \dim_{\mathbb{F}}\big(\mathbb{D}(W)/V\mathbb{D}(W)\big).$$

Denote by $N \subset \mathbb{D}(W)$ the set of fixed points of the Fr-semilinear bijection

$$V^{-1} \circ \delta : \mathbb{D}(W) \to \mathbb{D}(W).$$

It is a free $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-module of rank n endowed with an isomorphism

$$\mathbb{D}(W) \cong N \otimes_{\mathbb{Z}_p} W$$

identifying $V = \delta \otimes \mathrm{Fr}^{-1}$. Moreover, the alternating form ψ on $\mathbb{D}(W)$ induced by the polarization on A_s has the form

$$\psi(n_1 \otimes w_1, n_2 \otimes w_2) = w_1 w_2 \cdot \mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\left(\frac{h(n_1, n_2)}{\delta}\right)$$

for a perfect hermitian pairing $h : N \times N \to \mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$. By diagonalizing this hermitian form, we obtain an orthogonal decomposition of N into rank one hermitian $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-modules, and tensoring this decomposition with W yields a decomoposition of $\mathbb{D}(W)$ as a direct sum of principally polarized Dieudonné modules, each of height 2 and slope 1/2. This corresponds to a decomposition (2.6.2) on the level of p-divisible groups.

In particular, A_s is supersingular, and hence is isogenous to n copies of A_{0s}. Using the Noether-Skolem theorem, this isogeny may be chosen to be $\mathcal{O}_{\boldsymbol{k}}$-linear. It follows first that L_s has $\mathcal{O}_{\boldsymbol{k}}$-rank n, and then that the natural map

$$L_s \otimes_{\mathbb{Z}} \mathbb{Z}_q \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}[q^\infty], A_s[q^\infty])$$

is an isomorphism of hermitian $\mathcal{O}_{\boldsymbol{k},q}$-modules for every rational prime q. It is easy to see, using (2.6.2) when $q = p$, that the hermitian module on the right is self-dual, and hence the same is true for $L_s \otimes_{\mathbb{Z}} \mathbb{Z}_q$. □

The remainder of § 2.6 is devoted to proving the following result.

Theorem 2.6.3. — *There is an isomorphism*

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc})$$

of line bundles on $\mathcal{S}_{\mathrm{Kra}}$, as well as an equality

$$2\mathcal{Z}_{\mathrm{Kra}}(m) = \mathcal{Y}_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x\rangle = m\} \cdot \mathrm{Exc}_s$$

of Cartier divisors.

Proof. — Recall from the proof of Theorem 2.4.3 the morphism

$$\begin{array}{ccc} \boldsymbol{\omega}^{-2} & & \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1} \\ \| & & \| \\ \mathrm{Lie}(A_0)^{\otimes 2} \otimes (\mathrm{Lie}(A)/\mathcal{F}_A))^{\otimes 2} & \xrightarrow{(2.4.3)} & \mathrm{Lie}(A_0)^{\otimes 2} \otimes \mathcal{P}_{\mathrm{Kra}}, \end{array}$$

whose restriction to $\mathcal{S}_{\mathrm{Kra}} \setminus \mathrm{Exc}$ is an isomorphism. If we view this morphism as a global section

$$\sigma \in H^0(\mathcal{S}_{\mathrm{Kra}}, \boldsymbol{\omega}^2 \otimes \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1}), \tag{2.6.3}$$

then

$$\mathrm{div}(\sigma) = \sum_{s \in \pi_0(\mathrm{Sing})} \ell_s(0) \cdot \mathrm{Exc}_s \tag{2.6.4}$$

for some integers $\ell_s(0) \geq 0$, and hence

$$\boldsymbol{\omega}^2 \otimes \boldsymbol{\Omega}_{\mathrm{Kra}}^{-1} \cong \bigotimes_{s \in \pi_0(\mathrm{Sing})} \mathcal{O}(\mathrm{Exc}_s)^{\otimes \ell_s(0)}. \tag{2.6.5}$$

We must show that each $\ell_s(0) = 1$.

Similarly, suppose $m > 0$. It follows from Theorem 2.5.3 that

$$2\mathcal{Z}_{\mathrm{Kra}}(m) = \mathcal{Y}_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \ell_s(m) \cdot \mathrm{Exc}_s \tag{2.6.6}$$

for some integers $\ell_s(m)$. Moreover, it is clear from the construction of $\mathcal{Y}_{\mathrm{Kra}}(m)$ that $2\mathcal{Z}_{\mathrm{Kra}}(m) - \mathcal{Y}_{\mathrm{Kra}}(m)$ is effective, and so $\ell_s(m) \geq 0$. We must show that

$$\ell_s(m) = \#\{x \in L_s : \langle x, x\rangle = m\}.$$

Fix $s \in \pi_0(\mathrm{Sing})$, and let $\mathrm{Spec}(\mathbb{F}) \to s$, $p = \mathrm{char}(\mathbb{F})$, and $(A_{0s}, A_s) \in \mathcal{S}_{\mathrm{Pap}}(\mathbb{F})$ be as in (2.6.1). Let $W = W(\mathbb{F})$ be the Witt ring of $\mathbb{F}$, and set $\mathcal{W} = \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$. It is a complete discrete valuation ring of absolute ramification degree 2. Fix a uniformizer $\varpi \in \mathcal{W}$. As p is odd, the quotient map

$$\mathcal{W} \to \mathcal{W}/\varpi\mathcal{W} = \mathbb{F}$$

admits canonical divided powers.

Denote by $\mathbb{D}_0$ and $\mathbb{D}$ the Grothendieck-Messing crystals of A_{0s} and A_s, respectively. Evaluation of the crystals [(5)] along the divided power thickening $\mathcal{W} \to \mathbb{F}$ yields free $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-modules $\mathbb{D}_0(\mathcal{W})$ and $\mathbb{D}(\mathcal{W})$ endowed with alternating $\mathcal{W}$-bilinear forms ψ_0 and ψ, and $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphisms

$$\mathbb{D}_0(\mathcal{W})/\varpi\mathbb{D}_0(\mathcal{W}) \cong \mathbb{D}_0(\mathbb{F}) \cong H_1^{\mathrm{dR}}(A_{0s})$$

and

$$\mathbb{D}(\mathcal{W})/\varpi\mathbb{D}(\mathcal{W}) \cong \mathbb{D}(\mathbb{F}) \cong H_1^{\mathrm{dR}}(A_s).$$

The W-modules $\mathbb{D}_0(W)$ and $\mathbb{D}(W)$ are canonically identified with the covariant Dieudonné modules of A_{0s} and A_s, respectively. The operators F and V on these Dieudonné modules induce operators, denoted the same way, on

$$\mathbb{D}_0(\mathcal{W}) \cong \mathbb{D}_0(W) \otimes_W \mathcal{W}, \quad \mathbb{D}(\mathcal{W}) \cong \mathbb{D}(W) \otimes_W \mathcal{W}.$$

For any elements $y_1, \ldots, y_k$ in an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-module, let $\langle y_1, \ldots, y_k \rangle$ be the $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-submodule generated by them. Recall from § 1.7 the elements

$$\varepsilon, \overline{\varepsilon} \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}.$$

Lemma 2.6.4. — *There is an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-basis $e_0 \in \mathbb{D}_0(\mathcal{W})$ such that*

$$F\,\mathbb{D}_0(\mathcal{W}) \stackrel{\mathrm{def}}{=} \langle \overline{\varepsilon} e_0 \rangle \subset \mathbb{D}_0(\mathcal{W})$$

is a totally isotropic $\mathcal{W}$-module direct summand lifting the Hodge filtration on $\mathbb{D}_0(\mathbb{F})$, and such that $Ve_0 = \delta e_0$.

Similarly, there is an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-basis $e_1, \ldots, e_n \in \mathbb{D}(\mathcal{W})$ such that

$$F\,\mathbb{D}(\mathcal{W}) \stackrel{\mathrm{def}}{=} \langle \varepsilon e_1, \overline{\varepsilon} e_2, \ldots, \overline{\varepsilon} e_n \rangle \subset \mathbb{D}(\mathcal{W})$$

is a totally isotropic $\mathcal{W}$-module direct summand lifting the Hodge filtration on $\mathbb{D}(\mathbb{F})$. This basis may be chosen so that $Ve_{k+1} = \delta e_k$, where the indices are understood in $\mathbb{Z}/n\mathbb{Z}$, and also so that

$$\psi\big(\langle e_i \rangle, \langle e_j \rangle\big) = \begin{cases} \mathcal{W} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — As in the proof of Proposition 2.6.2, we may identify

$$\mathbb{D}_0(W) \cong N_0 \otimes_{\mathbb{Z}_p} W$$

for some free $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-module N_0 of rank 1, in such a way that $V = \delta \otimes \mathrm{Fr}^{-1}$, and the alternating form on $\mathbb{D}_0(W)$ arises as the W-bilinear extension of an alternating form ψ_0 on N_0. Any $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$-generator $e_0 \in N_0$ determines a generator of the $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}} \otimes_{\mathbb{Z}_p} \mathcal{W}$-module

$$\mathbb{D}_0(\mathcal{W}) \cong N_0 \otimes_{\mathbb{Z}_p} \mathcal{W},$$

(5) If $p = 3$, the divided powers on $\mathcal{W} \to \mathbb{F}$ are not nilpotent, and so we cannot evaluate the usual Grothendieck-Messing crystals on this thickening. However, Proposition 2.6.2 implies that the p-divisible groups of A_{0s} and A_s are formal, and Zink's theory of displays **[54]** can be used as a substitute.

which, using Remark 2.3.5 has the desired properties.

Now set $N = N_0 \oplus \cdots \oplus N_0$ (n copies), so that, by Proposition 2.6.2, there is an isomorphism

$$\mathbb{D}(W) \cong N \otimes_{\mathbb{Z}_p} W$$

identifying $V = \delta \otimes \mathrm{Fr}^{-1}$, and the alternating bilinear form on $\mathbb{D}(W)$ arises from an alternating form ψ on N. Let $\mathbb{Z}_{p^n} \subset W$ be the ring of integers in the unique unramified degree n extension of $\mathbb{Q}_p$, and fix an action

$$\iota : \mathbb{Z}_{p^n} \to \mathrm{End}_{\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}}(N)$$

in such a way that $\psi(\iota(\alpha)x, y) = \psi(x, \iota(\alpha)y)$ for all $\alpha \in \mathbb{Z}_{p^n}$.

There is an induced decomposition

$$\mathbb{D}(W) \cong \bigoplus_{k \in \mathbb{Z}/n\mathbb{Z}} \mathbb{D}(W)_k,$$

where

$$\mathbb{D}(W)_k = \{e \in \mathbb{D}(W) : \forall \alpha \in \mathbb{Z}_{p^n},\ \iota(\alpha) \cdot e = \mathrm{Fr}^k(\alpha) \cdot e\}$$

is free of rank one over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$. Now pick any $\mathbb{Z}_{p^n}$-module generator $e \in N$, view it as an element of $\mathbb{D}(W)$, and let $e_k \in \mathbb{D}(W)_k$ be its projection to the k^{th} summand. This gives an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} W$-basis $e_1, \ldots, e_n \in \mathbb{D}(W)$, which determines an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$-basis of $\mathbb{D}(\mathcal{W})$ with the required properties. □

By the Serre-Tate theorem and Grothendieck-Messing theory, the lifts of the Hodge filtrations specified in Lemma 2.6.4 determine a lift

$$(\tilde{A}_{0s}, \tilde{A}_s) \in \mathcal{S}_{\mathrm{Pap}}(\mathcal{W}) \tag{2.6.7}$$

of the pair (A_{0s}, A_s). These come with canonical identifications

$$H_1^{\mathrm{dR}}(\tilde{A}_{0s}) \cong \mathbb{D}_0(\mathcal{W}), \quad H_1^{\mathrm{dR}}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W}),$$

under which the Hodge filtrations correspond to the filtrations chosen in Lemma 2.6.4. In particular, the Lie algebra of $\tilde{A}_s$ is

$$\mathrm{Lie}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W})/F\,\mathbb{D}(\mathcal{W}) = \langle e_1, e_2, \ldots, e_n\rangle / \langle \varepsilon e_1, \overline{\varepsilon} e_2, \ldots, \overline{\varepsilon} e_n\rangle.$$

The $\mathcal{W}$-module direct summand

$$\mathcal{F}_{\tilde{A}_s} = \langle e_2, \ldots, e_n\rangle / \langle \overline{\varepsilon} e_2, \ldots, \overline{\varepsilon} e_n\rangle$$

satisfies Krämer's condition (§ 2.3), and so determines a lift of (2.6.7) to

$$(\tilde{A}_{0s}, \tilde{A}_s) \in \mathcal{S}_{\mathrm{Kra}}(\mathcal{W}).$$

To summarize: starting from a geometric point $\mathrm{Spec}(\mathbb{F}) \to s$, we have used Lemma 2.6.4 to construct a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}(\mathbb{F}) & \longrightarrow & \mathrm{Exc}_s & \longrightarrow & s \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{W}) & \longrightarrow & \mathcal{S}_{\mathrm{Kra}} & \longrightarrow & \mathcal{S}_{\mathrm{Pap}}. \end{array} \tag{2.6.8}$$

Lemma 2.6.5. — *The pullback of the map* (2.4.3) *via* $\mathrm{Spec}(\mathcal{W}) \to \mathcal{S}_{\mathrm{Kra}}$ *vanishes identically along the closed subscheme* $\mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W})$, *but not along* $\mathrm{Spec}(\mathcal{W}/\varpi^2\mathcal{W})$.

Proof. — The $\mathcal{W}$-submodule of

$$\mathrm{Lie}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W})/\langle \varepsilon e_1, \overline{\varepsilon} e_2, \dots, \overline{\varepsilon} e_n \rangle \tag{2.6.9}$$

generated by e_1 is $\mathcal{O}_{\boldsymbol{k}}$-stable. The action of $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$ on this $\mathcal{W}$-line is via

$$\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W} \xrightarrow{\alpha\otimes x \mapsto i_{\mathcal{W}}(\overline{\alpha})x} \mathcal{W}$$

(where $i_{\mathcal{W}} : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{W}$ is the inclusion), and this map sends $\overline{\varepsilon}$ to a uniformizer of $\mathcal{W}$; see § 1.7. Thus the quotient map $q : \mathbb{D}(\mathcal{W}) \to \mathrm{Lie}(\tilde{A}_s)$ satisfies $q(\overline{\varepsilon} e_1) = \varpi q(e_1)$ up to multiplication by an element of $\mathcal{W}^\times$. It follows that

$$P_{e_1\otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \varpi \cdot \psi(\overline{\varepsilon} e_1, e_1) \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n)$$

up to scaling by $\mathcal{W}^\times$.

We claim that $\psi(\overline{\varepsilon} e_1, e_1) \in \mathcal{W}^\times$. Indeed, as $q(e_1)$ generates a $\mathcal{W}$-module direct summand of (2.6.9), there is some

$$x \in F\,\mathbb{D}(\mathcal{W}) = \langle \varepsilon e_1, \overline{\varepsilon} e_2, \dots, \overline{\varepsilon} e_n \rangle \subset \mathbb{D}(\mathcal{W}),$$

such that $\psi(x, e_1) \in \mathcal{W}^\times$. We chose our basis in Lemma 2.6.4 in such a way that $\psi(\overline{\varepsilon} e_i, e_1) = 0$ for $i > 1$. It follows that $\psi(\varepsilon e_1, e_1)$ is a unit, and hence the same is true for $\psi(\overline{\varepsilon} e_1, e_1) = \psi(e_1, \varepsilon e_1) = -\psi(\varepsilon e_1, e_1)$.

We have now proved that

$$P_{e_1\otimes e_1}(e_1 \wedge \cdots \wedge e_n) = \varpi \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n)$$

up to scaling by $\mathcal{W}^\times$, from which it follows that

$$P_{e_1\otimes e_1}(e_1 \wedge \cdots \wedge e_n) \in \bigwedge^n \mathrm{Lie}(\tilde{A}_s)$$

is divisible by ϖ, but not by ϖ^2.

The quotient

$$H_1^{\mathrm{dR}}(\tilde{A}_s)/\overline{\varepsilon} H_1^{\mathrm{dR}}(\tilde{A}_s) \cong \mathbb{D}(\mathcal{W})/\langle \overline{\varepsilon} e_1, \dots, \overline{\varepsilon} e_n \rangle$$

is generated as a $\mathcal{W}$-module by $e_1, \dots, e_n$. From the calculation of the previous paragraph, it now follows that $P_{e_1\otimes e_1} \in \mathcal{P}_{\mathrm{Kra}}|_{\mathrm{Spec}(\mathcal{W})}$ is divisible by ϖ but not by ϖ^2. The quotient

$$\mathrm{Lie}(\tilde{A}_s)/\mathcal{F}_{\tilde{A}_s} \cong \mathbb{D}(\mathcal{W})/\langle \varepsilon e_1, e_2, \dots, e_n \rangle$$

is generated as a $\mathcal{W}$-module by the image of e_1, and we at last deduce that

$$P \in \underline{\mathrm{Hom}}\big((\mathrm{Lie}(A)/\mathcal{F}_A)^{\otimes 2}, \mathcal{P}_{\mathrm{Kra}}\big)|_{\mathrm{Spec}(\mathcal{W})}$$

is divisible by ϖ but not by ϖ^2. □

Recall the global section σ of (2.6.3). It follows immediately from Lemma 2.6.5 that its pullback via $\mathrm{Spec}(\mathcal{W}) \to \mathcal{S}_{\mathrm{Kra}}$ has divisor $\mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W})$, and hence

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{div}(\sigma) = \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W}).$$

Comparison with (2.6.4) proves both that $\ell_s(0) = 1$, and that

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Exc}_s = \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W}). \tag{2.6.10}$$

Recalling (2.6.5), this completes the proof that

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc}).$$

It remains to prove the second claim of Theorem 2.6.3. Given any $x \in L_s = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}, A_s)$, denote by $k(x)$ the largest integer such that x lifts to a morphism

$$\tilde{A}_{0s} \otimes_{\mathcal{W}} \mathcal{W}/(\varpi^{k(x)}) \to \tilde{A}_s \otimes_{\mathcal{W}} \mathcal{W}/(\varpi^{k(x)}).$$

Lemma 2.6.6. — *As Cartier divisors on* $\mathrm{Spec}(\mathcal{W})$, *we have*

$$\mathcal{Z}_{\mathrm{Kra}}(m) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Spec}(\mathcal{W}) = \sum_{\substack{x \in L_s \\ \langle x,x\rangle = m}} \mathrm{Spec}(\mathcal{W}/\varpi^{k(x)}\mathcal{W}).$$

Proof. — Each $x \in L_s$ with $\langle x, x\rangle = m$ determines a geometric point

$$(A_{0z}, A_z, x) \in \mathcal{Z}_{\mathrm{Kra}}(m)(\mathbb{F}) \tag{2.6.11}$$

and surjective morphisms

$$\begin{array}{ccc} & \mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x} & \\ \swarrow & & \searrow \\ \mathcal{O}_{\mathcal{Z}_{\mathrm{Kra}}(m),x} & & \mathcal{W}, \end{array}$$

where $\mathcal{O}_{\mathcal{Z}_{\mathrm{Kra}}(m),x}$ is the étale local ring at (2.6.11), $\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x}$ is the étale local ring at the point below it, and the arrow on the right is induced by the map $\mathrm{Spec}(\mathcal{W}) \to \mathcal{S}_{\mathrm{Kra}}$ of (2.6.8). There is an induced isomorphism of $\mathcal{W}$-schemes

$$\mathcal{O}_{\mathcal{Z}_{\mathrm{Kra}}(m),x} \otimes_{\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x}} \mathcal{W} \cong \mathcal{W}/(\varpi^{k(x)})$$

and the claim follows by summing over x. □

Lemma 2.6.7. — *As Cartier divisors on* $\mathrm{Spec}(\mathcal{W})$, *we have*

$$\mathcal{Y}_{\mathrm{Kra}}(m) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Spec}(\mathcal{W}) = \sum_{\substack{x \in L_s \\ \langle x,x\rangle = m}} \mathrm{Spec}(\mathcal{W}/\varpi^{2k(x)-1}\mathcal{W}).$$

Proof. — Each $x \in L_s = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}, A_s)$ with $\langle x, x\rangle = m$ induces a morphism of crystals $\mathbb{D}_0 \to \mathbb{D}$, and hence a map

$$\mathbb{D}_0(\mathcal{W}) \xrightarrow{x} \mathbb{D}(\mathcal{W})$$

respecting the F and V operators. By Grothendieck-Messing deformation theory, the integer $k(x)$ is characterized as the largest integer such that the composition

$$\begin{array}{ccccccc}
F^0 H_1^{\mathrm{dR}}(\tilde{A}_{0s}) & \xrightarrow{\subset} & H_1^{\mathrm{dR}}(\tilde{A}_{0s}) & \xrightarrow{x} & H_1^{\mathrm{dR}}(\tilde{A}_s) & \xrightarrow{q} & \mathrm{Lie}(\tilde{A}_s) \\
\| & & \| & & \| & & \| \\
\overline{\varepsilon}\mathbb{D}_0(\mathcal{W}) & \xrightarrow{\subset} & \mathbb{D}_0(\mathcal{W}) & \xrightarrow{x} & \mathbb{D}(\mathcal{W}) & \longrightarrow & \frac{\mathbb{D}(\mathcal{W})}{\langle \varepsilon e_1, \overline{\varepsilon} e_2, \ldots, \overline{\varepsilon} e_n \rangle}
\end{array}$$

vanishes modulo $\varpi^{k(x)}$. In other words the composition

$$H_1^{\mathrm{dR}}(\tilde{A}_{0s}) \xrightarrow{x \circ \overline{\varepsilon}} H_1^{\mathrm{dR}}(\tilde{A}_s) \xrightarrow{q} \mathrm{Lie}(\tilde{A}_s)$$

vanishes modulo $\varpi^{k(x)}$, but not modulo $\varpi^{k(x)+1}$.

Using the bases of Lemma 2.6.4, we expand

$$x(e_0) = a_1 e_1 + \cdots + a_n e_n$$

with $a_1, \ldots, a_n \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$. The condition that x respects V implies that $a_1 = \cdots = a_n$. Let us call this common value a, so that

$$q(x(\overline{\varepsilon} e_0)) = \overline{\varepsilon} \cdot q(a e_1 + \cdots + a e_n) = a\overline{\varepsilon} \cdot q(e_1)$$

in $\mathrm{Lie}(\tilde{A}_s)$. By the previous paragraph, this element is divisible by $\varpi^{k(x)}$ but not by $\varpi^{k(x)+1}$, and so

$$q(a\overline{\varepsilon} e_1) = \varpi^{k(x)} q(e_1) \tag{2.6.12}$$

up to scaling by $\mathcal{W}^\times$.

On the other hand, the submodule of $\mathrm{Lie}(\tilde{A}_s)$ generated by $q(e_1)$ is isomorphic to $(\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W})/\langle \varepsilon \rangle \cong \mathcal{W}$, and $\overline{\varepsilon}$ acts on this quotient by a uniformizer in $\mathcal{W}$. Thus

$$\overline{\varepsilon} q(e_1) = \varpi q(e_1) \tag{2.6.13}$$

up to scaling by $\mathcal{W}^\times$.

Combining (2.6.12) and (2.6.13) shows that, up to scaling by $\mathcal{W}^\times$,

$$a\overline{\varepsilon} = \varpi^{k(x)-1}\overline{\varepsilon}$$

in the quotient $(\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W})/\langle \varepsilon \rangle$. By the injectivity of the quotient map $\langle \overline{\varepsilon} \rangle \to (\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W})/\langle \varepsilon \rangle$, this same equality holds in $\langle \overline{\varepsilon} \rangle \subset \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{W}$. Using this and (2.6.12), we compute

$$\begin{aligned}
P_{x(e_0)\otimes x(e_0)}(e_1 \wedge \cdots \wedge e_n) &= \psi(a\overline{\varepsilon} e_1, e_1) \cdot q(a\overline{\varepsilon} e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n) \\
&= \varpi^{2k(x)-1} \cdot \psi(\overline{\varepsilon} e_1, e_1) \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n) \\
&= \varpi^{2k(x)-1} \cdot q(e_1) \wedge q(e_2) \wedge \cdots \wedge q(e_n),
\end{aligned}$$

up to scaling by $\mathcal{W}^\times$. Here, as in the proof of Lemma 2.6.5, we have used $\psi(\overline{\varepsilon} e_1, e_1) \in \mathcal{W}^\times$.

This calculation shows that the composition

$$H_1^{\mathrm{dR}}(\tilde{A}_{0s})^{\otimes 2} \xrightarrow{x\otimes x} H_1^{\mathrm{dR}}(\tilde{A}_s)^{\otimes 2} \xrightarrow{P} \mathcal{P}|_{\mathrm{Spec}(\mathcal{W})}$$

vanishes modulo $\varpi^{2k(x)-1}$, but not modulo $\varpi^{2k(x)}$, and the remainder of the proof is the same as that of Lemma 2.6.6: comparing with the definition of $\mathcal{Y}_{\mathrm{Kra}}(m)$, see especially (2.5.3), shows that

$$\mathcal{O}_{\mathcal{Y}_{\mathrm{Kra}}(m),x} \otimes_{\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}},x}} \mathcal{W} \cong \mathcal{W}/(\varpi^{2k(x)-1}),$$

and summing over all x proves the claim. □

Combining Lemmas 2.6.6 and 2.6.7 shows that

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \big(2\mathcal{Z}_{\mathrm{Kra}}(m) - \mathcal{Y}_{\mathrm{Kra}}(m)\big) = \sum_{\substack{x\in L_s \\ \langle x,x\rangle = m}} \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W})$$

as Cartier divisors on $\mathrm{Spec}(\mathcal{W})$. We know from (2.6.10) that

$$\mathrm{Spec}(\mathcal{W}) \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Exc}_t = \begin{cases} \mathrm{Spec}(\mathcal{W}/\varpi\mathcal{W}) & \text{if } t = s, \\ 0 & \text{if } t \neq s \end{cases}$$

and comparison with (2.6.6) shows that

$$\ell_s(m) = \#\{x \in L_s : \langle x, x\rangle = m\},$$

completing the proof of Theorem 2.6.3. □

3. Toroidal compactification

In this section we describe canonical toroidal compactifications

$$\begin{array}{ccc} \mathcal{S}_{\mathrm{Kra}} & \longrightarrow & \mathcal{S}^*_{\mathrm{Kra}} \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathrm{Pap}} & \longrightarrow & \mathcal{S}^*_{\mathrm{Pap}} \end{array}$$

and the structure of their formal completions along the boundary. Using this description, we define Fourier-Jacobi expansions of modular forms.

The existence of toroidal compactifications with reasonable properties is not a new result. In fact the proof of Theorem 3.7.1, which asserts the existence of good compactifications of $\mathcal{S}_{\mathrm{Pap}}$ and $\mathcal{S}_{\mathrm{Kra}}$, simply refers to [**24**]. Of course [*loc. cit.*] is itself a very modest addition to the established literature [**17, 40, 41, 49**]. Because of this, the reader is perhaps owed a few words of explanation as to why § 3 is so long.

It is well-known that the boundary charts used to construct toroidal compactifications of PEL-type Shimura varieties are themselves moduli spaces of 1-motives (or, what is nearly the same thing, degeneration data in the sense of [**17**]). This moduli interpretation is explained in § 3.3.

It is a special feature of our particular Shimura variety $\mathrm{Sh}(G,\mathcal{D})$ that the boundary charts have a second, very different, moduli interpretation. This second moduli interpretation is explained in § 3.4. In some sense, the main result of § 3 is not Theorem 3.7.1 at all, but rather Proposition 3.4.4, which proves the equivalence of the two moduli problems.

The point is that our goal is to eventually study the integrality and rationality properties of Fourier-Jacobi expansions of Borcherds products on the integral models of $\mathrm{Sh}(G,\mathcal{D})$. A complex analytic description of these Fourier-Jacobi expansions can be deduced from [**32**], but it is not a priori clear how to deduce integrality and rationality properties from these purely complex analytic formulas.

To do so, we will exploit the fact that the formulas of [**32**] express the Fourier-Jacobi coefficients in terms of the classical Jacobi theta function. The Jacobi theta function can be viewed as a section of a line bundle on the universal elliptic curve fibered over the modular curve, and when interpreted in this way it has known integrality and rationality properties (this is explained in § 5.1).

By converting the moduli interpretation of the boundary charts from 1-motives to an interpretation that makes explicit reference to the universal elliptic curve and the line bundles that live over it, the integrality and rationality properties of the Fourier-Jacobi coefficients can be deduced, ultimately, from those of the classical Jacobi theta function.

3.1. Cusp label representatives. — Recall that W_0 and W are $\boldsymbol{k}$-hermitian spaces of signatures $(1,0)$ and $(n-1,1)$, respectively, with $n \geq 2$. Tautologically, the subgroup

$$G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$$

acts on both W_0 and W. If $J \subset W$ is an isotropic $\boldsymbol{k}$-line, its stabilizer $P = \mathrm{Stab}_G(J)$ in G is a parabolic subgroup. This establishes a bijection between isotropic $\boldsymbol{k}$-lines in W and proper parabolic subgroups of G. If $n > 2$ then such isotropic $\boldsymbol{k}$-lines always exist.

Definition 3.1.1. — A *cusp label representative* for $(G,\mathcal{D})$ is a pair $\Phi = (P,g)$ in which $g \in G(\mathbb{A}_f)$ and $P \subset G$ is a parabolic subgroup. If $P = \mathrm{Stab}_G(J)$ for an isotropic $\boldsymbol{k}$-line $J \subset W$, we call Φ a *proper cusp label representative.* If $P = G$ we call Φ an *improper cusp label representative.*

For each cusp label representative $\Phi = (P,g)$ there is a distinguished normal subgroup $Q_\Phi \lhd P$. If $P = G$ we simply take $Q_\Phi = G$. If $P = \mathrm{Stab}_G(J)$ for an isotropic $\boldsymbol{k}$-line $J \subset W$ then, following the recipe of [**47**, § 4.7], we define Q_Φ as the fiber product

$$\text{(3.1.1)} \qquad \begin{array}{ccc} Q_\Phi & \xrightarrow{\quad\nu_\Phi\quad} & \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \\ \big\downarrow & & \big\downarrow {\scriptstyle a\mapsto(a,\mathrm{Nm}(a),a,\mathrm{id})} \\ P & \longrightarrow & \mathrm{GU}(W_0)\times\mathrm{GL}(J)\times\mathrm{GU}(J^\perp/J)\times\mathrm{GL}(W/J^\perp). \end{array}$$

The morphism $G \to \mathrm{GU}(W)$ restricts to an injection $Q_\Phi \hookrightarrow \mathrm{GU}(W)$, as the action of Q_Φ on $J^\perp/J$ determines its action on W_0.

Let $K \subset G(\mathbb{A}_f)$ be the compact open subgroup (2.1.3). Any cusp label representative $\Phi = (P, g)$ determines compact open subgroups

$$K_\Phi = gKg^{-1} \cap Q_\Phi(\mathbb{A}_f), \quad \tilde{K}_\Phi = gKg^{-1} \cap P(\mathbb{A}_f),$$

and a finite group

$$\Delta_\Phi = \big(P(\mathbb{Q}) \cap Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi\big)/Q_\Phi(\mathbb{Q}). \tag{3.1.2}$$

Definition 3.1.2. — Two cusp label representatives $\Phi = (P, g)$ and $\Phi' = (P', g')$ are *K-equivalent* if there exist $\gamma \in G(\mathbb{Q})$, $h \in Q_\Phi(\mathbb{A}_f)$, and $k \in K$ such that

$$(P', g') = (\gamma P \gamma^{-1}, \gamma h g k).$$

One may easily verify that this is an equivalence relation. Obviously, there is a unique K-equivalence class of improper cusp label representatives.

From now through § 3.6, we fix a proper cusp label representative $\Phi = (P, g)$, with $P \subset G$ the stabilizer of an isotropic $\boldsymbol{k}$-line $J \subset W$. There is an induced weight filtration $\mathrm{wt}_i W \subset W$ defined by

$$\begin{array}{ccccccc} 0 & \subset & J & \subset & J^\perp & \subset & W \\ \| & & \| & & \| & & \| \\ \mathrm{wt}_{-3}W & \subset & \mathrm{wt}_{-2}W & \subset & \mathrm{wt}_{-1}W & \subset & \mathrm{wt}_0 W \end{array}$$

and an induced weight filtration on $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ defined by

$$\begin{array}{ccccccc} \mathrm{Hom}_{\boldsymbol{k}}(W_0, 0) & \subset & \mathrm{Hom}_{\boldsymbol{k}}(W_0, J) & \subset & \mathrm{Hom}_{\boldsymbol{k}}(W_0, J^\perp) & \subset & \mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \\ \| & & \| & & \| & & \| \\ \mathrm{wt}_{-2}V & \subset & \mathrm{wt}_{-1}V & \subset & \mathrm{wt}_0 V & \subset & \mathrm{wt}_1 V. \end{array}$$

It is easy to see that $\mathrm{wt}_{-1}V$ is an isotropic $\boldsymbol{k}$-line, whose orthogonal with respect to (2.1.5) is $\mathrm{wt}_0 V$. Denote by $\mathrm{gr}_i W = \mathrm{wt}_i W/\mathrm{wt}_{i-1}W$ the graded pieces, and similarly for V.

The $\mathcal{O}_{\boldsymbol{k}}$-lattice $g\mathfrak{a} \subset W$ determines an $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$\mathrm{gr}_i(g\mathfrak{a}) = \big(g\mathfrak{a} \cap \mathrm{wt}_i W\big)/\big(g\mathfrak{a} \cap \mathrm{wt}_{i-1}W\big) \subset \mathrm{gr}_i W.$$

The middle graded piece $\mathrm{gr}_{-1}(g\mathfrak{a})$ is endowed with a positive definite self-dual hermitian form, inherited from the self-dual hermitian form on $g\mathfrak{a}$ appearing in the proof of Proposition 2.1.1. The outer graded pieces

$$\mathfrak{m} = \mathrm{gr}_{-2}(g\mathfrak{a}), \quad \mathfrak{n} = \mathrm{gr}_0(g\mathfrak{a}) \tag{3.1.3}$$

are projective rank one $\mathcal{O}_{\boldsymbol{k}}$-modules [6], endowed with a perfect $\mathbb{Z}$-bilinear pairing $\mathfrak{m} \otimes_{\mathbb{Z}} \mathfrak{n} \to \mathbb{Z}$ inherited from the perfect symplectic form on $g\mathfrak{a}$ appearing in the proof of Proposition 2.2.1.

Remark 3.1.3. — The isometry class of $g\mathfrak{a}$ as a hermitian lattice is determined by the isomorphism classes of $\mathfrak{m}$ and $\mathfrak{n}$ as $\mathcal{O}_{\boldsymbol{k}}$-modules and the isometry class of $\mathrm{gr}_{-1}(g\mathfrak{a})$ as a hermitian lattice. This follows from the proof of [**24**, Proposition 2.6.3], which shows that one can find a splitting [7]

$$g\mathfrak{a} \cong \mathrm{gr}_{-2}(g\mathfrak{a}) \oplus \mathrm{gr}_{-1}(g\mathfrak{a}) \oplus \mathrm{gr}_0(g\mathfrak{a}),$$

in such a way that the outer summands are totally isotropic, and each is orthogonal to the middle summand.

Exactly as in (2.1.4), there is a $\boldsymbol{k}$-conjugate linear isomorphism

$$\mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_{-1}W) \xrightarrow{x \mapsto x^\vee} \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{gr}_{-1}W, W_0).$$

If we define

$$\begin{aligned} L_0 &= \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-1}(g\mathfrak{a})) \\ \Lambda_0 &= \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathrm{gr}_{-1}(g\mathfrak{a}), g\mathfrak{a}_0), \end{aligned} \tag{3.1.4}$$

then $x \mapsto x^\vee$ restricts to an $\mathcal{O}_{\boldsymbol{k}}$-conjugate linear isomorphism $L_0 \cong \Lambda_0$. These are, in a natural way, positive definite self-dual hermitian lattices. For $x_1, x_2 \in L_0$ the hermitian form on L_0 is defined, as in (2.1.5), by

$$\langle x_1, x_2 \rangle = x_1^\vee \circ x_2 \in \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0) \cong \mathcal{O}_{\boldsymbol{k}},$$

while the hermitian form on Λ_0 is defined by

$$\langle x_2^\vee, x_1^\vee \rangle = \langle x_1, x_2 \rangle.$$

Lemma 3.1.4. — *Two proper cusp label representatives Φ and Φ' are K-equivalent if and only if $\Lambda_0 \cong \Lambda_0'$ as hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules and $\mathfrak{n} \cong \mathfrak{n}'$ as $\mathcal{O}_{\boldsymbol{k}}$-modules. Moreover, the finite group* (3.1.2) *satisfies*

$$\Delta_\Phi \cong \mathrm{U}(\Lambda_0) \times \mathrm{GL}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}). \tag{3.1.5}$$

Proof. — The first claim is an elementary exercise, left to the reader. For the second claim we only define the isomorphism (3.1.5), and again leave the details to the reader. The group $P(\mathbb{Q})$ acts on both W_0 and W, preserving their weight filtrations, and so acts on both the hermitian space $\mathrm{Hom}_{\boldsymbol{k}}(\mathrm{gr}_{-1}W, W_0)$ and the $\boldsymbol{k}$-vector space $\mathrm{gr}_0 W$. The subgroup $P(\mathbb{Q}) \cap Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi$ preserves the lattices

$$\Lambda_0 \subset \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{gr}_{-1}W, W_0)$$

and $\mathfrak{n} \subset \mathrm{gr}_0 W$, inducing (3.1.5). □

[6] In fact $\mathfrak{m} \cong \mathfrak{n}$ as $\mathcal{O}_{\boldsymbol{k}}$-modules, but identifying them can only lead to confusion.
[7] This uses our standing assumption that $\boldsymbol{k}$ has odd discriminant.

3.2. Mixed Shimura varieties. — The subgroup $Q_\Phi(\mathbb{R}) \subset G(\mathbb{R})$ acts on

$$\mathcal{D}_\Phi(W) = \{\boldsymbol{k}\text{-stable } \mathbb{R}\text{-planes } y \subset W(\mathbb{R}) : W(\mathbb{R}) = J^\perp(\mathbb{R}) \oplus y\},$$

and so also acts on

$$\mathcal{D}_\Phi = \mathcal{D}(W_0) \times \mathcal{D}_\Phi(W).$$

The hermitian domain of (2.1.2) satisifies $\mathcal{D}(W) \subset \mathcal{D}_\Phi(W)$, and hence there is a canonical $Q_\Phi(\mathbb{R})$-equivariant inclusion $\mathcal{D} \subset \mathcal{D}_\Phi$.

The mixed Shimura variety

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) = Q_\Phi(\mathbb{Q})\backslash \mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi \tag{3.2.1}$$

admits a canonical model $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)$ over $\boldsymbol{k}$ by the general results of [**47**]. By rewriting the double quotient as

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \cong Q_\Phi(\mathbb{Q})\backslash \mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi/\tilde{K}_\Phi,$$

we see that (3.2.1) admits an action of the finite group Δ_Φ of (3.1.2), induced by the action of $P(\mathbb{Q}) \cap Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi$ on both factors of $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)\tilde{K}_\Phi$. This action descends to an action on the canonical model.

Proposition 3.2.1. — *The morphism* ν_Φ *of* (3.1.1) *induces a surjection*

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \xrightarrow{(z,h)\mapsto \nu_\Phi(h)} \boldsymbol{k}^\times\backslash \widehat{\boldsymbol{k}}^\times/\widehat{\mathcal{O}}_{\boldsymbol{k}}^\times$$

with connected fibers. This map is Δ_Φ*-equivariant, where* Δ_Φ *acts trivially on the target. In particular, the number of connected components of* (3.2.1) *is equal to the class number of* $\boldsymbol{k}$*, and the same is true of its orbifold quotient by the action of* Δ_Φ*.*

Proof. — The space $\mathcal{D}_\Phi$ is connected, and the kernel of $\nu_\Phi : Q_\Phi \to \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m$ is unipotent (so satisfies strong approximation). Therefore

$$\pi_0\big(\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})\big) \cong Q_\Phi(\mathbb{Q})\backslash Q_\Phi(\mathbb{A}_f)/K_\Phi \cong \boldsymbol{k}^\times\backslash \widehat{\boldsymbol{k}}^\times/\nu_\Phi(K_\Phi),$$

and an easy calculation shows that $\nu_\Phi(K_\Phi) = \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times$. □

It will be useful to have other interpretations of $\mathcal{D}_\Phi$.

Remark 3.2.2. — Any point $y \in \mathcal{D}_\Phi(W)$ determines a mixed Hodge structure on W whose weight filtration $\mathrm{wt}_i W \subset W$ was defined above, and whose Hodge filtration is defined exactly as in Remark 2.1.3. As in [**46**, p. 64] or [**47**, Proposition 1.2] there is an induced bigrading $W(\mathbb{C}) = \bigoplus W^{(p,q)}$, and this bigrading is induced by a morphism $\mathbb{S}_\mathbb{C} \to \mathrm{GU}(W)_\mathbb{C}$ taking values in the stabilizer of $J(\mathbb{C})$. The product of this morphism with the morphism $\mathbb{S}_\mathbb{C} \to \mathrm{GU}(W_0)_\mathbb{C}$ of Remark 2.1.3 defines a map $z : \mathbb{S}_\mathbb{C} \to Q_{\Phi\mathbb{C}}$, and this realizes $\mathcal{D}_\Phi \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}})$.

Remark 3.2.3. — Imitating the construction of Remark 2.1.2 identifies

$$\mathcal{D}_\Phi \cong \big\{w \in \varepsilon V(\mathbb{C}) : V(\mathbb{C}) = \mathrm{wt}_0 V(\mathbb{C}) \oplus \mathbb{C}w \oplus \mathbb{C}\overline{w}\big\}/\mathbb{C}^\times \subset \mathbb{P}(\varepsilon V(\mathbb{C}))$$

as an open subset of projective space.

3.3. The first moduli interpretation. — Using the pair $(\Lambda_0, \mathfrak{n})$ defined in § 3.1, we now construct a smooth integral model of the mixed Shimura variety (3.2.1). Following the general recipes of the theory of arithmetic toroidal compactifications, as in [**17, 24, 42, 40**], this integral model will be defined as the top layer of a tower of morphisms

$$\mathcal{C}_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi \to \operatorname{Spec}(\mathcal{O}_{\boldsymbol{k}}),$$

smooth of relative dimensions 1, $n-2$, and 0, respectively.

Recall from § 2.3 the smooth $\mathcal{O}_{\boldsymbol{k}}$-stack

$$\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-2,0)} \to \operatorname{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

of relative dimension 0 parametrizing certain pairs (A_0, B) of polarized abelian schemes over S with $\mathcal{O}_{\boldsymbol{k}}$-actions. The étale sheaf $\underline{\operatorname{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B, A_0)$ on S is locally constant; this is a consequence of [**11**, Theorem 5.1].

Define $\mathcal{A}_\Phi$ as the moduli space of triples (A_0, B, ϱ) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S, in which (A_0, B) is an S-point of $\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-2,0)}$, and

$$\varrho : \underline{\Lambda}_0 \cong \underline{\operatorname{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B, A_0)$$

is an isomorphism of étale sheaves of hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules.

Define $\mathcal{B}_\Phi$ as the moduli space of quadruples (A_0, B, ϱ, c) over $\mathcal{O}_{\boldsymbol{k}}$-schemes S, in which (A_0, B, ϱ) is an S-point of $\mathcal{A}_\Phi$, and $c : \mathfrak{n} \to B$ is an $\mathcal{O}_{\boldsymbol{k}}$-linear homomorphism of group schemes over S. In other words, if (A_0, B, ϱ) is the universal object over $\mathcal{A}_\Phi$, then

$$\mathcal{B}_\Phi = \underline{\operatorname{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B).$$

Suppose we fix $\mu, \nu \in \mathfrak{n}$. For any scheme U and any morphism $U \to \mathcal{B}_\Phi$, there is a corresponding quadruple (A_0, B, ϱ, c) over U. Evaluating the morphism of U-group schemes $c : \mathfrak{n} \to B$ at μ and ν determines U-points $c(\mu), c(\nu) \in B(U)$, and hence determines a morphism of U-schemes

$$U \xrightarrow{c(\mu)\times c(\nu)} B \times B \cong B \times B^\vee.$$

Denote by $\mathcal{L}(\mu, \nu)_U$ the pullback of the Poincaré bundle via this morphism. As U varies, these line bundles are obtained as the pullback of a single line bundle $\mathcal{L}(\mu, \nu)$ on $\mathcal{B}_\Phi$.

It follows from standard bilinearity properties of the Poincaré bundle that $\mathcal{L}(\mu, \nu)$ depends, up to canonical isomorphism, only on the image of $\mu \otimes \nu$ in

$$\operatorname{Sym}_\Phi = \operatorname{Sym}^2_{\mathbb{Z}}(\mathfrak{n}) / \big\langle (x\mu) \otimes \nu - \mu \otimes (\overline{x}\nu) : x \in \mathcal{O}_{\boldsymbol{k}},\ \mu, \nu \in \mathfrak{n} \big\rangle.$$

Thus we may associate to every $\chi \in \operatorname{Sym}_\Phi$ a line bundle $\mathcal{L}(\chi)$ on $\mathcal{B}_\Phi$, and there are canonical isomorphisms

$$\mathcal{L}(\chi) \otimes \mathcal{L}(\chi') \cong \mathcal{L}(\chi + \chi').$$

Our assumption that D is odd implies that Sym_Φ is a free $\mathbb{Z}$-module of rank one. Moreover, there is positive cone in $\operatorname{Sym}_\Phi \otimes_{\mathbb{Z}} \mathbb{R}$ uniquely determined by the condition

$\mu\otimes\mu \geq 0$ for all $\mu \in \mathfrak{n}$. Thus all of the line bundles $\mathcal{L}(\chi)$ are powers of the distinguished line bundle

$$\mathcal{L}_\Phi = \mathcal{L}(\chi_0) \tag{3.3.1}$$

determined by the unique positive generator $\chi_0 \in \mathrm{Sym}_\Phi$.

At last, define $\mathcal{B}_\Phi$-stacks

$$\mathcal{C}_\Phi = \underline{\mathrm{Iso}}(\mathcal{L}_\Phi, \mathcal{O}_{\mathcal{B}_\Phi}), \quad \mathcal{C}_\Phi^* = \underline{\mathrm{Hom}}(\mathcal{L}_\Phi, \mathcal{O}_{\mathcal{B}_\Phi}).$$

In other words, $\mathcal{C}_\Phi^*$ is the total space of the line bundle $\mathcal{L}_\Phi^{-1}$, and $\mathcal{C}_\Phi$ is the complement of the zero section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}_\Phi^*$. In slightly fancier language,

$$\mathcal{C}_\Phi = \underline{\mathrm{Spec}}_{\mathcal{B}_\Phi}\Big(\bigoplus_{\ell\in\mathbb{Z}} \mathcal{L}_\Phi^\ell\Big), \quad \mathcal{C}_\Phi^* = \underline{\mathrm{Spec}}_{\mathcal{B}_\Phi}\Big(\bigoplus_{\ell\geq 0} \mathcal{L}_\Phi^\ell\Big),$$

and the zero section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}_\Phi^*$ is defined by the ideal sheaf $\bigoplus_{\ell>0} \mathcal{L}_\Phi^\ell$.

Remark 3.3.1. — When $n = 2$ the situation is a bit degenerate. In this case

$$\mathcal{B}_\Phi = \mathcal{A}_\Phi = \mathcal{M}_{(1,0)},$$

$\mathcal{L}_\Phi$ is the trivial bundle, and $\mathcal{C}_\Phi \to \mathcal{B}_\Phi$ is the trivial $\mathbb{G}_m$-torsor.

Remark 3.3.2. — Using the isomorphism of Lemma 3.1.4, the group Δ_Φ acts on $\mathcal{B}_\Phi$ via

$$(u,t)\bullet(A_0, B, \varrho, c) = (A_0, B, \varrho\circ u^{-1}, c\circ t^{-1}),$$

for $(u,t) \in \mathrm{U}(\Lambda_0) \times \mathrm{GL}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n})$. The line bundle $\mathcal{L}_\Phi$ is invariant under Δ_Φ, and hence the action of Δ_Φ lifts to both $\mathcal{C}_\Phi$ and $\mathcal{C}_\Phi^*$.

Proposition 3.3.3. — *There is a Δ_Φ-equivariant isomorphism*

$$\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi) \cong \mathcal{C}_{\Phi/\boldsymbol{k}}.$$

Proof. — This is a special case of the general fact that mixed Shimura varieties appearing at the boundary of PEL Shimura varieties are themselves moduli spaces of 1-motives endowed with polarizations, endomorphisms, and level structure. The core of this is Deligne's theorem [**14**, § 10] that the category of 1-motives over $\mathbb{C}$ is equivalent to the category of integral mixed Hodge structures of types $(-1,-1)$, $(-1,0)$, $(0,-1)$, $(0,0)$. See [**42**], where this is explained for Siegel modular varieties, and also [**12**]. A good introduction to 1-motives is [**2**].

To make this a bit more explicit in our case, denote by $\mathcal{X}_\Phi$ the $\mathcal{O}_{\boldsymbol{k}}$-stack whose functor of points assigns to an $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid $\mathcal{X}_\Phi(S)$ of principally polarized 1-motives A consisting of diagrams

$$\begin{array}{ccccccccc} & & & & \mathfrak{n} & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathfrak{m}\otimes_{\mathbb{Z}}\mathbb{G}_m & \longrightarrow & \mathbb{B} & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

in which $B \in \mathcal{M}_{(n-2,0)}(S)$, $\mathbb{B}$ is an extension of B by the rank two torus $\mathfrak{m} \otimes_{\mathbb{Z}} \mathbb{G}_m$ in the category of group schemes with $\mathcal{O}_{\boldsymbol{k}}$-action, and the arrows are morphisms of fppf sheaves of $\mathcal{O}_{\boldsymbol{k}}$-modules.

To explain what it means to have a principal polarization of such a 1-motive A, set $\mathfrak{m}^\vee = \mathrm{Hom}(\mathfrak{m}, \mathbb{Z})$ and $\mathfrak{n}^\vee = \mathrm{Hom}(\mathfrak{n}, \mathbb{Z})$, and recall from [**14**, § 10] that A has a dual 1-motive $A^\vee$ consisting of a diagram

$$\begin{array}{ccccccccc} & & & & \mathfrak{m}^\vee & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & \mathfrak{n}^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m & \longrightarrow & \mathbb{B}^\vee & \longrightarrow & B^\vee & \longrightarrow & 0. \end{array}$$

A principal polarization is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism $\mathbb{B} \cong \mathbb{B}^\vee$ compatible with the given polarization $B \cong B^\vee$, and with the isomorphisms $\mathfrak{m} \cong \mathfrak{n}^\vee$ and $\mathfrak{n} \cong \mathfrak{m}^\vee$ determined by the perfect pairing $\mathfrak{m} \otimes_{\mathbb{Z}} \mathfrak{n} \to \mathbb{Z}$ defined after (3.1.3).

Using the "description plus symétrique" of 1-motives [**14**, (10.2.12)], the $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{C}_\Phi$ defined above can be identified with the moduli space whose S-points are triples (A_0, A, ϱ) in which

— $(A_0, A) \in \mathcal{M}_{(1,0)}(S) \times \mathcal{X}_\Phi(S)$,

— $\varrho : \underline{\Lambda}_0 \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B, A_0)$ is an isomorphism of étale sheaves of hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules, where $B \in \mathcal{M}_{(n-2,0)}(S)$ is the abelian scheme part of A.

To verify that $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)$ has the same functor of points, one uses Remark 3.2.2 to interpret $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$ as a moduli space of mixed Hodge structures on W_0 and W, and uses the theorem of Deligne cited above to interpret these mixed Hodge structures as 1-motives. This defines an isomorphism $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \cong \mathcal{C}_\Phi(\mathbb{C})$. The proof that it descends to the reflex field is identical to the proof for Siegel mixed Shimura varieties [**42**].

We remark in passing that any triple (A_0, A, ϱ) as above automatically satisfies (2.2.4) for every prime ℓ. Indeed, both sides of (2.2.4) are now endowed with weight filtrations, analogous to the weight filtration on $\mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ defined in § 3.1. The isomorphism ϱ induces an isomorphism (as hermitian $\mathcal{O}_{\boldsymbol{k},\ell}$-lattices) between the gr_0 pieces on either side. The gr_{-1} and gr_1 pieces have no structure other then projective $\mathcal{O}_{\boldsymbol{k},\ell}$-modules of rank 1, so are isomorphic. These isomorphisms of graded pieces imply the existence of an isomorphism (2.2.4), exactly as in Remark 3.1.3. □

3.4. The second moduli interpretation. — In order to make explicit calculations, it will be useful to interpret the moduli spaces

$$\mathcal{C}_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$$

in a different way.

Suppose $E \to S$ is an elliptic curve over any base scheme, and denote by $\mathcal{P}_E$ the Poincaré bundle on

$$E \times_S E \cong E \times_S E^\vee.$$

If U is any S-scheme and $a, b \in E(U)$, we obtain an $\mathcal{O}_U$-module $\mathcal{P}_E(a,b)$ by pulling back the Poincare bundle via

$$U \xrightarrow{(a,b)} E \times_S E \cong E \times_S E^\vee.$$

The notation is intended to remind the reader of the bilinearity properties of the Poincaré bundle, as expressed by canonical $\mathcal{O}_U$-module isomorphisms

$$\begin{aligned} \mathcal{P}_E(a+b,c) &\cong \mathcal{P}_E(a,c) \otimes \mathcal{P}_E(b,c) \\ \mathcal{P}_E(a,b+c) &\cong \mathcal{P}_E(a,b) \otimes \mathcal{P}_E(a,c) \\ \mathcal{P}_E(a,b) &\cong \mathcal{P}_E(b,a), \end{aligned} \tag{3.4.1}$$

along with $\mathcal{P}_E(e,b) \cong \mathcal{O}_U \cong \mathcal{P}_E(a,e)$. Here $e \in E(U)$ is the zero section.

Let $E \to \mathcal{M}_{(1,0)}$ be the universal elliptic curve with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$. Its Poincaré bundle satisfies, for all $\alpha \in \mathcal{O}_{\boldsymbol{k}}$, the additional relation $\mathcal{P}_E(\alpha a, b) \cong \mathcal{P}_E(a, \overline{\alpha} b)$.

Recall the positive definite self-dual hermitian lattice L_0 of (3.1.4). Using Serre's tensor construction, we define an abelian scheme

$$E \otimes L_0 = E \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \tag{3.4.2}$$

over $\mathcal{M}_{(1,0)}$. As explained in detail in **[1]**, the principal polarization on E and the hermitian form on L_0 can be combined to define a principal polarization on $E \otimes L_0$, and we denote by $\mathcal{P}_{E\otimes L_0}$ the Poincaré bundle on

$$(E \otimes L_0) \times_{\mathcal{M}_{(1,0)}} (E \otimes L_0) \cong (E \otimes L_0) \times_{\mathcal{M}_{(1,0)}} (E \otimes L_0)^\vee.$$

The Poincaré bundle $\mathcal{P}_{E\otimes L_0}$ can be expressed in terms of $\mathcal{P}_E$. If U is a scheme, a morphism

$$U \to (E \otimes L_0) \times_{\mathcal{M}_{(1,0)}} (E \otimes L_0)$$

is given by a pair of U-valued points

$$c = \sum s_i \otimes x_i \in E(U) \otimes L_0, \quad c' = \sum s'_j \otimes x'_j \in E(U) \otimes L_0,$$

and the pullback of $\mathcal{P}_{E\otimes L_0}$ to U is

$$\mathcal{P}_{E\otimes L_0}(c,c') = \bigotimes_{i,j} \mathcal{P}_E(\langle x_i, x'_j \rangle s_i, s'_j).$$

Define $\mathcal{Q}_{E\otimes L_0}$ to be the line bundle on $E \otimes L_0$ whose restriction to the U-valued point $c = \sum s_i \otimes x_i$ is

$$\mathcal{Q}_{E\otimes L_0}(c) = \bigotimes_{i<j} \mathcal{P}_E(\langle x_i, x_j \rangle s_i, s_j) \otimes \bigotimes_i \mathcal{P}_E(\gamma \langle x_i, x_i \rangle s_i, s_i), \tag{3.4.3}$$

where

$$\gamma = \frac{1+\delta}{2} \in \mathcal{O}_{\boldsymbol{k}}.$$

It is related to $\mathcal{P}_{E\otimes L_0}$ by canonical isomorphisms

$$\begin{aligned}\mathcal{P}_{E\otimes L_0}(a,b) &\cong \mathcal{Q}_{E\otimes L_0}(a+b)\otimes \mathcal{Q}_{E\otimes L_0}(a)^{-1}\otimes \mathcal{Q}_{E\otimes L_0}(b)^{-1}\\ \mathcal{P}_{E\otimes L_0}(a,a) &\cong \mathcal{Q}_{E\otimes L_0}(a)^{\otimes 2}.\end{aligned} \tag{3.4.4}$$

for all U-valued points $a,b\in E(U)\otimes L_0$.

Remark 3.4.1. — As in the constructions of [**40**, § 1.3.2] or [**44**, § 6.2], the line bundle $\mathcal{Q}_{E\otimes L_0}$ determines a morphism $E\otimes L_0\to (E\otimes L_0)^\vee$. The relations (3.4.4) amount to saying that this morphism is the principal polarization constructed in [**1**].

Remark 3.4.2. — The line bundle $\mathcal{P}_{E\otimes L_0}(\delta a,a)$ is canonically trivial. This follows by comparing

$$\mathcal{P}_{E\otimes L_0}(\gamma a,a)^{\otimes 2}\cong \mathcal{P}_{E\otimes L_0}(a,a)\otimes \mathcal{P}_{E\otimes L_0}(\delta a,a)$$

with

$$\mathcal{P}_{E\otimes L_0}(\gamma a,a)^{\otimes 2}\cong \mathcal{P}_{E\otimes L_0}(\gamma a,a)\otimes \mathcal{P}_{E\otimes L_0}(\overline{\gamma} a,a)\cong \mathcal{P}_{E\otimes L_0}(a,a).$$

Remark 3.4.3. — In the slightly degenerate case of $n=2$, $E\otimes L_0$ is the trivial group scheme over $\mathcal{M}_{(1,0)}$, and $\mathcal{P}_{E\otimes L_0}$ is the trivial bundle on $\mathcal{M}_{(1,0)}$.

Proposition 3.4.4. — *As above, let $E\to\mathcal{M}_{(1,0)}$ be the universal object. There are canonical isomorphisms*

$$\begin{array}{ccccc}\mathcal{C}_\Phi & \longrightarrow & \mathcal{B}_\Phi & \longrightarrow & \mathcal{A}_\Phi\\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong\\ \underline{\mathrm{Iso}}(\mathcal{Q}_{E\otimes L_0},\mathcal{O}_{E\otimes L_0}) & \longrightarrow & E\otimes L_0 & \longrightarrow & \mathcal{M}_{(1,0)},\end{array}$$

and the middle vertical arrow identifies $\mathcal{L}_\Phi\cong\mathcal{Q}_{E\otimes L_0}$.

Proof. — Define a morphism $\mathcal{A}_\Phi\to\mathcal{M}_{(1,0)}$ by sending a triple (A_0,B,ϱ) to the CM elliptic curve

$$E=\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n},A_0). \tag{3.4.5}$$

To show that this map is an isomorphism we will construct the inverse.

If S is any $\mathcal{O}_{\boldsymbol{k}}$-scheme and $E\in\mathcal{M}_{(1,0)}(S)$, we may define $(A_0,B,\varrho)\in\mathcal{A}_\Phi(S)$ by setting

$$A_0=E\otimes_{\mathcal{O}_{\boldsymbol{k}}}\mathfrak{n},\quad B=\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0,A_0),$$

and taking for $\varrho:\underline{\Lambda}_0\cong\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(B,A_0)$ the tautological isomorphism. The principal polarization on B is defined using the $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism

$$A_0\otimes_{\mathcal{O}_{\boldsymbol{k}}}L_0\xrightarrow{a\otimes x\mapsto\langle\,\cdot\,,x^\vee\rangle a}\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0,A_0)$$

and the principal polarization on $A_0\otimes_{\mathcal{O}_{\boldsymbol{k}}}L_0$ constructed in [**1**], exactly as in the discussion following (3.4.2). The construction $E\mapsto(A_0,B,\varrho)$ is inverse to the above morphism $\mathcal{A}_\Phi\to\mathcal{M}_{(1,0)}$.

Now identify $\mathcal{A}_\Phi \cong \mathcal{M}_{(1,0)}$ using the above isomorphism, and denote by (A_0, B, ϱ) and E the universal objects on the source and target. They are related by canonical isomorphisms

(3.4.6)
$$\begin{array}{ccc} & & \mathcal{B}_\Phi = \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B) \\ & \overset{\cong}{\nearrow} & \\ \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \Lambda_0, A_0) & & \\ & \overset{\cong}{\searrow} & \\ & & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, E). \end{array}$$

Combining this with the $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism

$$E \otimes L_0 \xrightarrow{a \otimes x \mapsto \langle\, . \,, x^\vee \rangle a} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, E)$$

defines $\mathcal{B}_\Phi \cong E \otimes L_0$. All that remains is to prove that this isomorphism identifies $\mathcal{L}_\Phi$ with $\mathcal{Q}_{E \otimes L_0}$, which amounts to carefully keeping track of the relations between the three Poincaré bundles $\mathcal{P}_B$, $\mathcal{P}_E$, and $\mathcal{P}_{A_0}$.

Any fractional ideal $\mathfrak{b} \subset \boldsymbol{k}$ admits a unique positive definite self-dual hermitian form, given explicitly by $\langle b_1, b_2 \rangle = b_1 \overline{b}_2 / \mathrm{N}(\mathfrak{b})$. It follows that any rank one projective $\mathcal{O}_{\boldsymbol{k}}$-module admits a unique positive definite self-dual hermitian form. For the $\mathcal{O}_{\boldsymbol{k}}$-module $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, \mathcal{O}_{\boldsymbol{k}})$, this hermitian form is

$$\langle \ell_1, \ell_2 \rangle = \ell_1(\mu)\overline{\ell_2(\nu)} + \ell_1(\nu)\overline{\ell_2(\mu)},$$

where $\mu \otimes \nu = \chi_0 \in \mathrm{Sym}_\Phi$ is the positive generator appearing in (3.3.1).

The relation (3.4.5) implies a relation between the line bundles $\mathcal{P}_E$ and $\mathcal{P}_{A_0}$. If U is any $\mathcal{A}_\Phi$-scheme and we are given points

$$s, s' \in E(U) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_{0U})$$

of the form $s = \ell(\cdot)a$ and $s' = \ell'(\cdot)a'$ with $\ell, \ell' \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, \mathcal{O}_{\boldsymbol{k}})$ and $a, a' \in A_0(U)$, then

$$\begin{aligned} \mathcal{P}_E(s, s') &\cong \mathcal{P}_{A_0}\big(\langle \ell, \ell' \rangle a, a'\big) \\ \mathcal{P}_E(\gamma s, s) &\cong \mathcal{P}_{A_0}\big(\ell(\mu)a, \ell(\nu)a\big). \end{aligned}$$

Similarly, the isomorphism $B \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_0)$ implies a relation between $\mathcal{P}_B$ and $\mathcal{P}_{A_0}$. If U is an S-scheme, a morphism $U \to B \times_{\mathcal{A}_\Phi} B$ is given by a pair of points

$$b, b' \in B(U) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_{0U})$$

of the form $b = \langle ., \lambda \rangle a$ and $b' = \langle ., \lambda' \rangle a'$ with $\lambda, \lambda' \in \Lambda_0$ and $a, a' \in A_0(U)$. The pullback of $\mathcal{P}_B$ to U is the line bundle

$$\mathcal{P}_B(b, b') = \mathcal{P}_{A_0}(a, \langle \lambda, \lambda' \rangle a').$$

Using the isomorphisms (3.4.6), a point $c \in \mathcal{B}_\Phi(U)$ admits three different interpretations. In one of them, c has the form

$$c = \sum \ell_i(\cdot)\langle ., \lambda_i\rangle a_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \Lambda, A_{0U}).$$

By setting

$$\begin{aligned} b_i &= \langle ., \lambda_i\rangle a_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_{0U}) = B(U) \\ s_i &= \ell_i(\cdot)a_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_{0U}) = E(U), \end{aligned}$$

we find the other two interpretations

$$\begin{aligned} c &= \sum \ell_i(\cdot)b_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B_U) \\ c &= \sum \langle ., \lambda_i\rangle s_i \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, E_U). \end{aligned}$$

The above relations between $\mathcal{P}_B$, $\mathcal{P}_E$, and $\mathcal{P}_{A_0}$ imply

$$\begin{aligned} \mathcal{P}_B(c(\mu), c(\nu)) &\cong \bigotimes_{i,j} \mathcal{P}_B(\ell_i(\mu)b_i, \ell_j(\nu)b_j) \\ &\cong \bigotimes_{i,j} \mathcal{P}_{A_0}(\ell_i(\mu)a_i, \langle\lambda_i, \lambda_j\rangle\ell_j(\nu)a_j) \\ &\cong \bigotimes_{i<j} \mathcal{P}_{A_0}(\langle\ell_i, \ell_j\rangle a_i, \langle\lambda_i, \lambda_j\rangle a_j) \otimes \bigotimes_{i} \mathcal{P}_{A_0}(\ell_i(\mu)a_i, \ell_i(\nu)\langle\lambda_i, \lambda_i\rangle a_i) \\ &\cong \bigotimes_{i<j} \mathcal{P}_E(s_i, \langle\lambda_i, \lambda_j\rangle s_j) \otimes \bigotimes_{i} \mathcal{P}_E(\gamma s_i, \langle\lambda_i, \lambda_i\rangle s_i). \end{aligned}$$

Now write $\lambda_i = x_i^\vee$ with $x_i \in L_0$, and use the relation

$$\mathcal{P}_E(s_i, \langle\lambda_i, \lambda_j\rangle s_j) = \mathcal{P}_E(\langle\lambda_j, \lambda_i\rangle s_i, s_j) = \mathcal{P}_E(\langle x_i, x_j\rangle s_i, s_j)$$

to obtain an isomorphism $\mathcal{P}_B(c(\mu), c(\nu)) \cong \mathcal{Q}_{E\otimes L_0}(c)$. The line bundle on the left is precisely the pullback of $\mathcal{L}_\Phi$ via c, and letting c vary we obtain an isomorphism $\mathcal{L}_\Phi \cong \mathcal{Q}_{E\otimes L_0}$. □

3.5. The line bundle of modular forms. — We now define a line bundle of weight one modular forms on our mixed Shimura variety, analogous to the one on the pure Shimura variety defined in § 2.4.

The holomorphic line bundle $\boldsymbol{\omega}^{\mathrm{an}}$ on $\mathcal{D}$ defined in § 2.4 admits a canonical extension to

$$\mathcal{D}_\Phi = \mathcal{D}(W_0) \times \mathcal{D}_\Phi(W),$$

which we denote by $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$. Indeed, recalling that $\mathcal{D}(W_0) = \{y_0\}$ is a one-point set, an element $z \in \mathcal{D}_\Phi$ is represented by a pair (y_0, y) in which y is a $\boldsymbol{k}$-stable $\mathbb{R}$-plane in $W(\mathbb{R})$ such that $W(\mathbb{R}) = J^\perp(\mathbb{R}) \oplus y$. The fiber of $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$ at z is the line

$$\mathrm{Hom}_{\mathbb{C}}(W_0(\mathbb{C})/\overline{\varepsilon}W_0(\mathbb{C}), \mathrm{pr}_\varepsilon(y)) \subset \varepsilon V(\mathbb{C}),$$

exactly as in Remark 2.1.2 and (2.4.1).

If we embed $\mathcal{D}_\Phi$ into projective space over $\varepsilon V(\mathbb{C})$ as in Remark 3.2.3, then $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$ is simply the restriction of the tautological bundle. There is an obvious action of $Q_\Phi(\mathbb{R})$ on the total space of $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$, lifting the natural action on $\mathcal{D}_\Phi$, and so $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$ determines a holomorphic line bundle on the complex orbifold $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$.

As in § 2.4, the holomorphic line bundle $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$ is algebraic and descends to the canonical model $\mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)$. In fact, it admits a canonical extension to the integral model $\mathcal{C}_\Phi$, as we now explain.

Recalling the $\mathcal{O}_{\boldsymbol{k}}$-modules $\mathfrak{m}$ and $\mathfrak{n}$ of (3.1.3), define rank two vector bundles on $\mathcal{A}_\Phi$ by

$$\mathfrak{M} = \mathfrak{m} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}, \quad \mathfrak{N} = \mathfrak{n} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}.$$

Each is locally free of rank one over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}$, and the perfect pairing between $\mathfrak{m}$ and $\mathfrak{n}$ defined after (3.1.3) induces a perfect bilinear pairing $\mathfrak{M} \otimes \mathfrak{N} \to \mathcal{O}_{\mathcal{A}_\Phi}$. Using the almost idempotents $\varepsilon, \overline{\varepsilon} \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}$ of § 1.7, there is an induced isomorphism of line bundles

$$(\mathfrak{M}/\varepsilon\mathfrak{M}) \otimes (\varepsilon\mathfrak{N}) \cong \mathcal{O}_{\mathcal{A}_\Phi}.$$

Recalling that $\mathcal{A}_\Phi$ carries over it a universal triple (A_0, B, ϱ), in which A_0 is an elliptic curve with $\mathcal{O}_{\boldsymbol{k}}$-action, we now define a line bundle on $\mathcal{A}_\Phi$ by

$$\boldsymbol{\omega}_\Phi = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \varepsilon\mathfrak{N}),$$

or, equivalently,

$$\boldsymbol{\omega}_\Phi^{-1} = \mathrm{Lie}(A_0) \otimes_{\mathcal{O}_{\mathcal{A}_\Phi}} \mathfrak{M}/\varepsilon\mathfrak{M}.$$

Denote in the same way its pullback to any step in the tower

$$\mathcal{C}_\Phi^* \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi.$$

The above definition of $\boldsymbol{\omega}_\Phi$ is a bit unmotivated, and so we explain why $\boldsymbol{\omega}_\Phi$ is analogous to the line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$ defined in § 2.4. Recall from the proof of Proposition 3.3.3 that $\mathcal{C}_\Phi$ carries over it a universal 1-motive A. This 1-motive has a de Rham realization $H_1^{\mathrm{dR}}(A)$, defined as the Lie algebra of the universal vector extension of A, as in [**14**, (10.1.7)]. It is a rank $2n$-vector bundle on $\mathcal{C}_\Phi$, locally free of rank n over $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{C}_\Phi}$, and sits in a diagram of vector bundles

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & F^0H_1^{\mathrm{dR}}(B) & & & & \mathfrak{M} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & F^0H_1^{\mathrm{dR}}(A) & \longrightarrow & H_1^{\mathrm{dR}}(A) & \longrightarrow & \mathrm{Lie}(A) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathfrak{N} & & & & \mathrm{Lie}(B) & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0, & &
\end{array}$$

with exact rows and columns. The polarization on A induces a perfect symplectic form on $H_1^{\mathrm{dR}}(A)$. This induces a perfect pairing

$$F^0 H_1^{\mathrm{dR}}(A) \otimes \mathrm{Lie}(A) \to \mathcal{O}_{\mathcal{C}_\Phi} \tag{3.5.1}$$

as in (2.2.1), which is compatible (in the obvious sense) with the pairings

$$F^0 H_1^{\mathrm{dR}}(B) \otimes \mathrm{Lie}(B) \to \mathcal{O}_{\mathcal{C}_\Phi}$$

and $\mathfrak{N} \otimes \mathfrak{M} \to \mathcal{O}_{\mathcal{C}_\Phi}$ that we already have.

The signature condition on B implies that $\varepsilon F^0 H_1^{\mathrm{dR}}(B) = 0$ and $\bar{\varepsilon}\mathrm{Lie}(B) = 0$. Using this, and arguing as in [**24**, Lemma 2.3.6], it is not difficult to see that

$$\mathcal{F}_A = \ker(\bar{\varepsilon} : \mathrm{Lie}(A) \to \mathrm{Lie}(A))$$

is the unique codimension one local direct summand of $\mathrm{Lie}(A)$ satisfying Kramer's condition as in § 2.3, and that its orthogonal under the pairing (3.5.1) is $\mathcal{F}_A^\perp = \varepsilon F^0 H_1^{\mathrm{dR}}(A)$. Moreover, the natural maps

$$\mathfrak{M}/\varepsilon\mathfrak{M} \to \mathrm{Lie}(A)/\mathcal{F}_A, \quad \mathcal{F}_A^\perp \to \varepsilon\mathfrak{N}$$

are isomorphisms. These latter isomorphisms allow us to identify

$$\boldsymbol{\omega}_\Phi = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \mathcal{F}_A^\perp), \quad \boldsymbol{\omega}_\Phi^{-1} = \mathrm{Lie}(A_0) \otimes \mathrm{Lie}(A)/\mathcal{F}_A$$

in perfect analogy with § 2.4.

Proposition 3.5.1. — *The isomorphism*

$$\mathcal{C}_\Phi(\mathbb{C}) \cong \mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$$

of Proposition 3.3.3 identifies the analytification of $\boldsymbol{\omega}_\Phi$ with the already defined $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$. Moreover, the isomorphism $\mathcal{A}_\Phi \cong \mathcal{M}_{(1,0)}$ of Proposition 3.4.4 identifies

$$\boldsymbol{\omega}_\Phi \cong \mathfrak{d} \cdot \mathrm{Lie}(E)^{-1} \subset \mathrm{Lie}(E)^{-1}$$

where $\mathfrak{d} = \delta\mathcal{O}_{\boldsymbol{k}}$ is the different of $\mathcal{O}_{\boldsymbol{k}}$, and $E \to \mathcal{M}_{(1,0)}$ is the universal elliptic curve with CM by $\mathcal{O}_{\boldsymbol{k}}$.

Proof. — Any point $z = (y_0, y) \in \mathcal{D}_\Phi$ determines, by Remarks 2.1.3 and 3.2.2, a pure Hodge structure on W_0 and a mixed Hodge structure on W, these induce a mixed Hodge structure on $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$, and the fiber of $\boldsymbol{\omega}_\Phi^{\mathrm{an}}$ at z is

$$\boldsymbol{\omega}_{\Phi,z}^{\mathrm{an}} = F^1 V(\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(W_0(\mathbb{C})/\bar{\varepsilon}W_0(\mathbb{C}), \varepsilon F^0 W(\mathbb{C})).$$

On the other hand, we have just seen that

$$\boldsymbol{\omega}_\Phi = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \mathcal{F}_A^\perp) = \underline{\mathrm{Hom}}(\mathrm{Lie}(A_0), \varepsilon F^0 H_1^{\mathrm{dR}}(A)).$$

With these identifications, the proof of the first claim amounts to carefully tracing through the construction of the isomorphism of Proposition 3.3.3.

For the second claim, the isomorphism $A_0 \cong E \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{n}$ induces a canonical isomorphism

$$\mathrm{Lie}(A_0) \cong \mathrm{Lie}(E) \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{n} \cong \mathrm{Lie}(E) \otimes \mathfrak{N}/\bar{\varepsilon}\mathfrak{N},$$

where we have used the fact that $\mathfrak{n} \otimes_{\mathcal{O}_k} \mathcal{O}_{\mathcal{A}_\Phi} = \mathfrak{N}/\bar{\varepsilon}\mathfrak{N}$ is the largest quotient of $\mathfrak{N}$ on which $\mathcal{O}_{\boldsymbol{k}}$ acts via the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_{\mathcal{A}_\Phi}$. Thus

$$\begin{aligned}\boldsymbol{\omega}_\Phi &= \underline{\mathrm{Hom}}(\mathrm{Lie}(A), \varepsilon\mathfrak{N}) \\ &\cong \underline{\mathrm{Hom}}(\mathrm{Lie}(E) \otimes \mathfrak{N}/\bar{\varepsilon}\mathfrak{N}, \varepsilon\mathfrak{N}) \\ &\cong \mathrm{Lie}(E)^{-1} \otimes_{\mathcal{O}_{\mathcal{A}_\Phi}} \underline{\mathrm{Hom}}(\mathfrak{N}/\bar{\varepsilon}\mathfrak{N}, \varepsilon\mathfrak{N}).\end{aligned}$$

Now recall the ideal sheaf $(\varepsilon) \subset \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{A}_\Phi}$ of § 1.7. There are canonical isomorphisms of line bundles

$$\mathfrak{d}\mathcal{O}_{\mathcal{A}_\Phi} \cong (\varepsilon) \cong \underline{\mathrm{Hom}}(\mathfrak{N}/\bar{\varepsilon}\mathfrak{N}, \varepsilon\mathfrak{N}),$$

where the first is (1.7.1) and the second is the tautological isomorphism sending ε to the multiplication-by-ε map $\mathfrak{N}/\bar{\varepsilon}\mathfrak{N} \to \varepsilon\mathfrak{N}$. These constructions determine the desired isomorphism

$$\boldsymbol{\omega}_\Phi \cong \mathrm{Lie}(E)^{-1} \otimes_{\mathcal{O}_{\mathcal{A}_\Phi}} \mathfrak{d}\mathcal{O}_{\mathcal{A}_\Phi}. \qquad \square$$

3.6. Special divisors. — Let $\mathcal{Y}_0(D)$ be the moduli stack over $\mathcal{O}_{\boldsymbol{k}}$ parametrizing cyclic D-isogenies of elliptic curves over $\mathcal{O}_{\boldsymbol{k}}$-schemes, and let $\mathcal{E} \to \mathcal{E}'$ be the universal object. See [**28**, Chapter 3] for the definitions.

Let (A_0, B, ϱ, c) be the universal object over $\mathcal{B}_\Phi$. Recalling the $\mathcal{O}_{\boldsymbol{k}}$-conjugate linear isomorphism $L_0 \cong \Lambda_0$ defined after (3.1.4), each $x \in L_0$ defines a morphism

$$\mathfrak{n} \xrightarrow{c} B \xrightarrow{\varrho(x^\vee)} A_0$$

of sheaves of $\mathcal{O}_{\boldsymbol{k}}$-modules on $\mathcal{B}_\Phi$. Define $\mathcal{Z}_\Phi(x) \subset \mathcal{B}_\Phi$ as the largest closed substack over which this morphism is trivial. We will see in a moment that this closed substack is defined locally by one equation. For any $m > 0$ define a stack over $\mathcal{B}_\Phi$ by

(3.6.1)
$$\mathcal{Z}_\Phi(m) = \bigsqcup_{\substack{x \in L_0 \\ \langle x, x\rangle = m}} \mathcal{Z}_\Phi(x).$$

We also view $\mathcal{Z}_\Phi(m)$ as a divisor on $\mathcal{B}_\Phi$, and denote in the same way the pullback of this divisor via $\mathcal{C}_\Phi^* \to \mathcal{B}_\Phi$.

Remark 3.6.1. — In the slightly degenerate case $n = 2$ we have $L_0 = 0$, and every special divisor $\mathcal{Z}_\Phi(m)$ is empty.

We will now reformulate the definition of $\mathcal{Z}_\Phi(x)$ in terms of the moduli problem of § 3.4. Recalling the isomorphisms of Proposition 3.4.4, every $x \in L_0$ determines a commutative diagram

$$\begin{array}{ccccccc} \mathcal{B}_\Phi & \xrightarrow{\cong} & E \otimes L_0 & \xrightarrow{\langle \cdot, x\rangle} & E & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}_\Phi & \xrightarrow{\cong} & \mathcal{M}_{(1,0)} & = & \mathcal{M}_{(1,0)} & \longrightarrow & \mathcal{Y}_0(D), \end{array}$$

where $\mathcal{M}_{(1,0)} \to \mathcal{Y}_0(D)$ sends E to the cyclic D-isogeny

$$E \to E \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathfrak{d}^{-1},$$

and the rightmost square is cartesian. The upper and lower horizontal compositions are denoted j_x and j, giving the diagram

$$\begin{array}{ccc} \mathcal{B}_\Phi & \xrightarrow{j_x} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathcal{A}_\Phi & \xrightarrow{j} & \mathcal{Y}_0(D). \end{array} \tag{3.6.2}$$

Proposition 3.6.2. — *For any nonzero $x \in L_0$, the closed substack $\mathcal{Z}_\Phi(x) \subset \mathcal{B}_\Phi$ is equal to the pullback of the zero section along j_x. It is an effective Cartier divisor, flat over $\mathcal{A}_\Phi$. In particular, as $\mathcal{A}_\Phi$ is flat over $\mathcal{O}_{\boldsymbol{k}}$, so is each divisor $\mathcal{Z}_\Phi(x)$.*

Proof. — Recall the isomorphisms

$$E \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_0), \quad B \cong \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_0)$$

from the proof of Proposition 3.4.4. If we identify $A_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \cong B$ using

$$A_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 \xrightarrow{a\otimes x \mapsto \langle \cdot, x^\vee \rangle a} \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_0, A_0) \cong B,$$

we obtain a commutative diagram of $\mathcal{A}_\Phi$-stacks

$$\begin{array}{ccccc} E \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0 & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_0 \otimes_{\mathcal{O}_{\boldsymbol{k}}} L_0) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, B) = \mathcal{B}_\Phi \\ \downarrow{\scriptstyle \langle \cdot, x \rangle} & & & & \downarrow{\scriptstyle \varrho(x^\vee)} \\ E & & \longrightarrow & & \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, A_0), \end{array}$$

in which all horizontal arrows are isomorphisms. The first claim follows immediately.

The remaining claims now follow from the cartesian diagram

$$\begin{array}{ccccc} \mathcal{Z}_\Phi(x) & & \longrightarrow & & \mathcal{M}_{(1,0)} \\ \downarrow & & & & \downarrow{\scriptstyle e} \\ \mathcal{B}_\Phi & \xrightarrow{\cong} & E \otimes L_0 & \xrightarrow{\langle \cdot, x \rangle} & E. \end{array}$$

The zero section $e : \mathcal{M}_{(1,0)} \hookrightarrow E$ is locally defined by a single nonzero equation [**28**, Lemma 1.2.2], and so the same is true of its pullback $\mathcal{Z}_\Phi(x) \hookrightarrow \mathcal{B}_\Phi$. Composition along the bottom row is flat by [**44**, Lemma 6.12], and hence so is the top horizontal arrow. ☐

Remark 3.6.3. — For those who prefer the language of 1-motives: As in the proof of Proposition 3.3.3, there is a universal triple (A_0, A, ϱ) over $\mathcal{C}_\Phi$ in which A_0 is an elliptic curve with $\mathcal{O}_{\boldsymbol{k}}$-action and A is a principally polarized 1-motive with $\mathcal{O}_{\boldsymbol{k}}$-action. The functor of points of $\mathcal{Z}_\Phi(m)$ assigns to any scheme $S \to \mathcal{C}_\Phi$ the set

$$\mathcal{Z}_\Phi(m)(S) = \{x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0,S}, A_S) : \langle x, x \rangle = m\},$$

where the positive definite hermitian form $\langle .,. \rangle$ is defined as in (2.5.1). Thus our special divisors are the exact analogues of the special divisors on $\mathcal{S}_{\mathrm{Kra}}$ defined in § 2.5.

3.7. The toroidal compactification. — We describe the canonical toroidal compactification of the integral models $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ of § 2.3.

Theorem 3.7.1. — *Let $\mathcal{S}_{\Box}$ denote either $\mathcal{S}_{\mathrm{Kra}}$ or $\mathcal{S}_{\mathrm{Pap}}$. There is a canonical toroidal compactification $\mathcal{S}_{\Box} \hookrightarrow \mathcal{S}^*_{\Box}$, flat over $\mathcal{O}_{\boldsymbol{k}}$ of relative dimension $n-1$. It admits a stratification*

$$\mathcal{S}^*_{\Box} = \bigsqcup_{\Phi} \mathcal{S}^*_{\Box}(\Phi)$$

as a disjoint union of locally closed substacks, indexed by the K-equivalence classes of cusp label representatives (defined in § 3.1).

1. *The $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{S}^*_{\mathrm{Kra}}$ is regular.*
2. *The $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{S}^*_{\mathrm{Pap}}$ is Cohen-Macaulay and normal, with Cohen-Macaulay fibers. If $n > 2$ its fibers are geometrically normal.*
3. *The open dense substack $\mathcal{S}_{\Box} \subset \mathcal{S}^*_{\Box}$ is the stratum indexed by the unique equivalence class of improper cusp label representatives. Its complement*

$$\partial \mathcal{S}^*_{\Box} = \bigsqcup_{\Phi \text{ proper}} \mathcal{S}^*_{\Box}(\Phi)$$

   is a smooth divisor, flat over $\mathcal{O}_{\boldsymbol{k}}$.

4. *For each proper Φ the stratum $\mathcal{S}^*_{\Box}(\Phi)$ is closed. All components of $\mathcal{S}^*_{\Box}(\Phi)_{/\mathbb{C}}$ are defined over the Hilbert class field $\boldsymbol{k}^{\mathrm{Hilb}}$, and they are permuted simply transitively by $\mathrm{Gal}(\boldsymbol{k}^{\mathrm{Hilb}}/\boldsymbol{k})$. Moreover, there is a canonical identification of $\mathcal{O}_{\boldsymbol{k}}$-stacks*

$$\begin{array}{ccc} \Delta_\Phi \backslash \mathcal{B}_\Phi & = & \mathcal{S}^*_{\Box}(\Phi) \\ \downarrow & & \downarrow \\ \Delta_\Phi \backslash \mathcal{C}^*_\Phi & & \mathcal{S}^*_{\Box} \end{array}$$

   such that the two stacks in the bottom row become isomorphic after completion along their common closed substack in the top row. In other words, there is a canonical isomorphism of formal stacks

$$\Delta_\Phi \backslash (\mathcal{C}^*_\Phi)^{\wedge}_{\mathcal{B}_\Phi} \cong (\mathcal{S}^*_{\Box})^{\wedge}_{\mathcal{S}^*_{\Box}(\Phi)}. \tag{3.7.1}$$

   The morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$ extends uniquely to a stratum preserving morphism of toroidal compactifications. This extension restricts to an isomorphism

$$\mathcal{S}^*_{\mathrm{Kra}} \setminus \mathrm{Exc} \cong \mathcal{S}^*_{\mathrm{Pap}} \setminus \mathrm{Sing}, \tag{3.7.2}$$

   compatible with (3.7.1) *for any proper Φ.*

The line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}$ defined in § 2.4 admits a unique extension (denoted the same way) to the toroidal compactification in such a way that (3.7.1) *identifies it with the line bundle $\boldsymbol{\omega}_\Phi$ on $\mathcal{C}^*_\Phi$. A similar statement holds for $\boldsymbol{\Omega}_{\mathrm{Kra}}$, and these two extensions are related by*

$$\boldsymbol{\omega}^2 \cong \boldsymbol{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc}).$$

The line bundle $\boldsymbol{\Omega}_{\mathrm{Pap}}$ on $\mathcal{S}_{\mathrm{Pap}}$ defined in § 2.4 admits a unique extension (denoted the same way) to the toroidal compactification, in such a way that (3.7.1) *identifies it with $\boldsymbol{\omega}^2_\Phi$.*

For any $m > 0$, define $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ as the Zariski closure of $\mathcal{Z}_{\mathrm{Kra}}(m)$ in $\mathcal{S}^*_{\mathrm{Kra}}$. The isomorphism (3.7.1) *identifies it with the Cartier divisor $\mathcal{Z}_\Phi(m)$ on $\mathcal{C}^*_\Phi$.*

For any $m > 0$, define $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ as the Zariski closure of $\mathcal{Y}_{\mathrm{Pap}}(m)$ in $\mathcal{S}^*_{\mathrm{Pap}}$. The isomorphism (3.7.1) *identifies it with $2\mathcal{Z}_\Phi(m)$. Moreover, the pullback of $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ to $\mathcal{S}^*_{\mathrm{Kra}}$, denoted $\mathcal{Y}^*_{\mathrm{Kra}}(m)$, satisfies*

$$2\mathcal{Z}^*_{\mathrm{Kra}}(m) = \mathcal{Y}^*_{\mathrm{Kra}}(m) + \sum_{s \in \pi_0(\mathrm{Sing})} \#\{x \in L_s : \langle x, x\rangle = m\} \cdot \mathrm{Exc}_s.$$

Proof. — Briefly, in [**24**, § 2] one finds the construction of a canonical toroidal compactification

$$\mathcal{M}^{\square}_{(n-1,1)} \hookrightarrow \mathcal{M}^{\square,*}_{(n-1,1)}.$$

Using the open and closed immersion

$$\mathcal{S}_\square \hookrightarrow \mathcal{M}_{(1,0)} \times \mathcal{M}^{\square}_{(n-1,1)},$$

the toroidal compactification $\mathcal{S}^*_\square$ is defined as the Zariski closure of $\mathcal{S}_\square$ in $\mathcal{M}_{(1,0)} \times \mathcal{M}^{\square,*}_{(n-1,1)}$. All of the claims follow by examination of the construction of the compactification, along with Theorem 2.6.3. □

Remark 3.7.2. — If W is anisotropic, so that $(G, \mathcal{D})$ has no proper cusp label representatives, the only new information in the theorem is that $\mathcal{S}_{\mathrm{Pap}}$ and $\mathcal{S}_{\mathrm{Kra}}$ are already proper over $\mathcal{O}_{\boldsymbol{k}}$, so that

$$\mathcal{S}_{\mathrm{Pap}} = \mathcal{S}^*_{\mathrm{Pap}}, \qquad \mathcal{S}_{\mathrm{Kra}} = \mathcal{S}^*_{\mathrm{Kra}}.$$

Corollary 3.7.3. — *Assume that $n > 2$. The Cartier divisor $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ on $\mathcal{S}^*_{\mathrm{Pap}}$ is $\mathcal{O}_{\boldsymbol{k}}$-flat, as is the restriction of $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ to $\mathcal{S}^*_{\mathrm{Kra}} \setminus \mathrm{Exc}$.*

Proof. — Fix a prime $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$, and let $\mathbb{F}_\mathfrak{p}$ be its residue field. To prove the first claim, it suffices to show that the support of the Cartier divisor $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ contains no irreducible components of the reduction $\mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_\mathfrak{p}}$.

By way of contradiction, suppose $\mathcal{E}_\mathfrak{p} \subset \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_\mathfrak{p}}$ is an irreducible component contained in $\mathcal{Y}^*_{\mathrm{Pap}}(m)$, and let $\mathcal{E} \subset \mathcal{S}^*_{\mathrm{Pap}}$ be the connected component containing it.

Properness of $\mathcal{S}^*_{\mathrm{Pap}}$ over $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$ implies that the reduction $\mathcal{E}_{/\mathbb{F}_\mathfrak{p}}$ is connected [**18**, Corollary 8.2.18]. The reduction $\mathcal{E}_{/\mathbb{F}_\mathfrak{p}}$ is normal by Theorem 3.7.1 and our assumption that $n > 2$, and hence is irreducible. Thus

$$\mathcal{E}_\mathfrak{p} = \mathcal{E}_{/\mathbb{F}_\mathfrak{p}}.$$

Our assumption that $n > 2$ also guarantees that W contains a nonzero isotropic vector, from which it follows that the boundary

$$\partial\mathcal{C} = \mathcal{C} \cap \partial\mathcal{S}^*_{\mathrm{Pap}}$$

is nonempty (one can check this in the complex fiber).

Proposition 3.6.2 implies that $\mathcal{Z}_\Phi(m)$ is $\mathcal{O}_{\boldsymbol{k}}$-flat for every proper cusp label representative Φ, and so it follows from Theorem 3.7.1 that $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ is $\mathcal{O}_{\boldsymbol{k}}$-flat when restricted to some étale neighborhood $U \to \mathcal{C}$ of $\partial\mathcal{C}$. On the other hand, the closed immersion

$$U_{/\mathbb{F}_\mathfrak{p}} \cong \mathcal{C}_\mathfrak{p} \times_{\mathcal{S}^*_{\mathrm{Pap}}} U \to \mathcal{Y}^*_{\mathrm{Pap}}(m) \times_{\mathcal{S}^*_{\mathrm{Pap}}} U$$

shows that the divisor $\mathcal{Y}^*_{\mathrm{Pap}}(m)|_U \to U$ contains the special fiber $U_{/\mathbb{F}_\mathfrak{p}}$, so is not $\mathcal{O}_{\boldsymbol{k}}$-flat. This contradiction completes the proof that $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ is flat.

As the isomorphism (3.7.2) identifies $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ with $2\mathcal{Z}^*_{\mathrm{Kra}}(m)$, it follows that the restriction of $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ to the complement of Exc is also flat. □

3.8. Fourier-Jacobi expansions. — We now define Fourier-Jacobi expansions of sections of the line bundle $\boldsymbol{\omega}^k$ of weight k modular forms on $\mathcal{S}^*_{\mathrm{Kra}}$.

Fix a proper cusp label representative $\Phi = (P, g)$. Suppose ψ is a rational function on $\mathcal{S}^*_{\mathrm{Kra}}$, regular on an open neighborhood of the closed stratum $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$. Using the isomorphism (3.7.1) we obtain a formal function, again denoted ψ, on the formal completion

$$(\mathcal{C}^*_\Phi)^\wedge_{\mathcal{B}_\Phi} = \underline{\mathrm{Spf}}_{\mathcal{B}_\Phi}\Big(\prod_{\ell \geq 0} \mathcal{L}^\ell_\Phi\Big).$$

Tautologically, there is a formal Fourier-Jacobi expansion

$$\psi = \sum_{\ell \geq 0} \mathrm{FJ}_\ell(\psi) \cdot q^\ell \tag{3.8.1}$$

with coefficients $\mathrm{FJ}_\ell(\psi) \in H^0(\mathcal{B}_\Phi, \mathcal{L}^\ell_\Phi)$. In the same way, any rational section ψ of $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$, regular on an open neighborhood of $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$, admits a Fourier-Jacobi expansion (3.8.1), but now with coefficients

$$\mathrm{FJ}_\ell(\psi) \in H^0(\mathcal{B}_\Phi, \boldsymbol{\omega}^k_\Phi \otimes \mathcal{L}^\ell_\Phi).$$

Remark 3.8.1. — Let $\pi : \mathcal{C}^*_\Phi \to \mathcal{B}_\Phi$ be the natural map. The formal symbol q can be understood as follows. As $\mathcal{C}^*_\Phi$ is the total space of the line bundle $\mathcal{L}^{-1}_\Phi$, there is a tautological section

$$q \in H^0(\mathcal{C}^*_\Phi, \pi^*\mathcal{L}^{-1}_\Phi),$$

whose divisor is the zero section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}^*_\Phi$. Any $\mathrm{FJ}_\ell \in H^0(\mathcal{B}_\Phi, \mathcal{L}^\ell_\Phi)$ pulls back to a section of $\pi^*\mathcal{L}^\ell_\Phi$, and so defines a function $\mathrm{FJ}_\ell \cdot q^\ell$ on $\mathcal{C}^*_\Phi$.

3.9. Explicit coordinates. — Once again, let $\Phi = (P, g)$ be a proper cusp label representative. The algebraic theory of § 3.8 realizes the Fourier-Jacobi coefficients of

$$\psi \in H^0(\mathcal{S}^*_{\mathrm{Kra}}, \boldsymbol{\omega}^k) \tag{3.9.1}$$

as sections of line bundles on the stack

$$\mathcal{B}_\Phi \cong E \otimes L_0.$$

Here $E \to \mathcal{M}_{(1,0)}$ is the universal CM elliptic curve, the tensor product is over $\mathcal{O}_{\boldsymbol{k}}$, and we are using the isomorphism of Proposition 3.4.4. Our goal is to relate this to the classical analytic theory of Fourier-Jacobi expansions by choosing explicit complex coordinates, so as to identify each coefficient $\mathrm{FJ}_\ell(\psi)$ with a holomorphic function on a complex vector space satisfying a particular transformation law.

The point of this discussion is to allow us, eventually, to show that the Fourier-Jacobi coefficients of Borcherds products, expressed in the classical way as holomorphic functions satisfying certain transformation laws, have algebraic meaning. More precisely, the following discussion will be used to deduce the algebraic statement of Proposition 6.4.1 from the analytic statement of Proposition 6.3.1.

Consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) & \xrightarrow{\cong} & \mathcal{C}_\Phi(\mathbb{C}) & \longrightarrow & \mathcal{B}_\Phi(\mathbb{C}) & \longrightarrow & \mathcal{A}_\Phi(\mathbb{C}) \\ \downarrow & & & & & & \downarrow \cong \\ \boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times / \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times & & & \xrightarrow[a \mapsto E^{(a)}]{} & & & \mathcal{M}_{(1,0)}(\mathbb{C}). \end{array}$$

Here the isomorphisms are those of Propositions 3.3.3 and 3.4.4, and the vertical arrow on the left is the surjection of Proposition 3.2.1. The bottom horizontal arrow is defined as the unique function making the diagram commute. It is a bijection, and is given explicitly by the following recipe: recalling the $\mathcal{O}_{\boldsymbol{k}}$-module $\mathfrak{n}$ of (3.1.3), each $a \in \widehat{\boldsymbol{k}}^\times$ determines a projective $\mathcal{O}_{\boldsymbol{k}}$-module

$$\mathfrak{b} = a \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n}, g\mathfrak{a}_0)$$

of rank one, and the elliptic curve $E^{(a)}$ has complex points

$$E^{(a)}(\mathbb{C}) = \mathfrak{b} \backslash (\mathfrak{b} \otimes_{\mathcal{O}_{\boldsymbol{k}}} \mathbb{C}). \tag{3.9.2}$$

For each $a \in \widehat{\boldsymbol{k}}^\times$ there is a cartesian diagram

$$\begin{array}{ccc} E^{(a)} \otimes L_0 & \longrightarrow & E \otimes L_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{E^{(a)}} & \mathcal{M}_{(1,0)}. \end{array}$$

Now suppose we have a section ψ as in (3.9.1). Using the isomorphisms $\mathcal{B}_\Phi \cong E \otimes L_0$ and $\boldsymbol{\omega}_\Phi \cong \mathfrak{d} \cdot \mathrm{Lie}(E)^{-1}$ of Propositions 3.4.4 and 3.5.1, we view its Fourier-Jacobi coefficients

$$\mathrm{FJ}_\ell(\psi) \in H^0(\mathcal{B}_\Phi, \boldsymbol{\omega}_\Phi^k \otimes \mathcal{L}_\Phi^\ell)$$

as sections

$$\mathrm{FJ}_\ell(\psi) \in H^0\big(E \otimes L_0, \mathfrak{d}^k \cdot \mathrm{Lie}(E)^{-k} \otimes \mathcal{Q}_{E\otimes L_0}^\ell\big),$$

which we pull back along the top map in the above diagram to obtain a section

$$\mathrm{FJ}_\ell^{(a)}(\psi) \in H^0\big(E^{(a)} \otimes L_0, \mathrm{Lie}(E^{(a)})^{-k} \otimes \mathcal{Q}_{E^{(a)}\otimes L_0}^\ell\big). \tag{3.9.3}$$

Remark 3.9.1. — Recalling that $\mathfrak{d} = \delta\mathcal{O}_{\boldsymbol{k}}$ is the different of $\boldsymbol{k}$, we are using the inclusion $\mathfrak{d}^k \subset \boldsymbol{k} \subset \mathbb{C}$ to identify

$$\mathfrak{d}^k \cdot \mathrm{Lie}(E^{(a)})^{-k} \cong \mathrm{Lie}(E^{(a)})^{-k}.$$

In particular, this isomorphism is *not* multiplication by δ^{-k}.

The explicit coordinates we will use to express (3.9.3) as a holomorphic function arise from a choice of Witt decomposition of the hermitian space $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$. The following lemma will allow us to choose this decomposition in a particularly nice way.

Lemma 3.9.2. — *The homomorphism ν_Φ of* (3.1.1) *admits a section*

$$Q_\Phi \underset{\nu_\Phi}{\overset{s}{\rightleftarrows}} \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m.$$

This section may be chosen so that $s(\widehat{\mathcal{O}}_{\boldsymbol{k}}^\times) \subset K_\Phi$, and such a choice determines a decomposition

$$\bigsqcup_{a \in \boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times / \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times} (Q_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1}) \backslash \mathcal{D}_\Phi \cong \mathrm{Sh}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}), \tag{3.9.4}$$

where the isomorphism is $z \mapsto (z, s(a))$ on the copy of $\mathcal{D}_\Phi$ indexed by a.

Proof. — Fix an isomorphism of hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules

$$g\mathfrak{a}_0 \oplus g\mathfrak{a} \cong g\mathfrak{a}_0 \oplus \mathrm{gr}_{-2}(ga) \oplus \mathrm{gr}_{-1}(ga) \oplus \mathrm{gr}_0(ga)$$

as in Remark 3.1.3. After tensoring with $\mathbb{Q}$, we let $\boldsymbol{k}^\times$ act on the right hand side by $a \mapsto (a, \mathrm{Nm}(a), a, 1)$. This defines a morphism $\boldsymbol{k}^\times \to G(\mathbb{Q})$, which, using (3.1.1), is easily seen to take values in the subgroup $Q_\Phi(\mathbb{Q})$. This defines the desired section s, and the decomposition (3.9.4) is immediate from Proposition 3.2.1. □

Fix a section s as in Lemma 3.9.2. Recall from § 3.1 the weight filtration $\mathrm{wt}_i V \subset V$ whose graded pieces

$$\begin{aligned} \mathrm{gr}_{-1}V &= \mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_{-2}W) \\ \mathrm{gr}_0 V &= \mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_{-1}W) \\ \mathrm{gr}_1 V &= \mathrm{Hom}_{\boldsymbol{k}}(W_0, \mathrm{gr}_0 W) \end{aligned}$$

have $\boldsymbol{k}$-dimensions 1, $n-2$, and 1, respectively. Recalling (3.1.1), which describes the action of Q_Φ on the graded pieces of V, the section s determines a splitting $V = V_{-1} \oplus V_0 \oplus V_1$ of the weight filtration by

$$\begin{aligned} V_{-1} &= \{v \in V : \forall\, a \in \boldsymbol{k}^\times, s(a)v = \overline{a}v\} \\ V_0 &= \{v \in V : \forall\, a \in \boldsymbol{k}^\times, s(a)v = v\} \\ V_1 &= \{v \in V : \forall\, a \in \boldsymbol{k}^\times, s(a)v = a^{-1}v\}. \end{aligned}$$

The summands V_{-1} and V_1 are isotropic $\boldsymbol{k}$-lines, and V_0 is the orthogonal complement of $V_{-1} + V_1$ with respect to the hermitian form on V. In particular, the restriction of the hermitian form to $V_0 \subset V$ is positive definite.

Fix an $a \in \widehat{\boldsymbol{k}}^\times$ and define an $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(s(a)g\mathfrak{a}_0, s(a)g\mathfrak{a}) \subset V.$$

Using the assumption $s(\widehat{\mathcal{O}}_{\boldsymbol{k}}^\times) \subset K_\Phi$, we obtain a decomposition

$$L = L_{-1} \oplus L_0 \oplus L_1$$

with $L_i = L \cap V_i$. The images of the lattices L_i in the graded pieces $\mathrm{gr}_i V$ are given by

$$\begin{aligned} L_{-1} &= \overline{a} \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-2}(g\mathfrak{a})) \\ L_0 &= \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_{-1}(g\mathfrak{a})) \\ L_1 &= a^{-1} \cdot \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, \mathrm{gr}_0(g\mathfrak{a})). \end{aligned}$$

In particular, L_0 is independent of a and agrees with (3.1.4).

Choose a $\mathbb{Z}$-basis $\mathrm{e}_{-1}, \mathrm{f}_{-1} \in L_{-1}$, and let $\mathrm{e}_1, \mathrm{f}_1 \in \mathfrak{d}^{-1}L_1$ be the dual basis with respect to the (perfect) $\mathbb{Z}$-bilinear pairing

$$[\,.\,,\,.\,] : L_{-1} \times \mathfrak{d}^{-1}L_1 \to \mathbb{Z},$$

obtained by restricting (2.1.6). This basis may be chosen so that

$$\begin{aligned} L_{-1} &= \mathbb{Z}\mathrm{e}_{-1} + \mathbb{Z}\mathrm{f}_{-1} & \mathfrak{d}^{-1}L_{-1} &= \mathbb{Z}\mathrm{e}_{-1} + D^{-1}\mathbb{Z}\mathrm{f}_{-1}, \\ L_1 &= \mathbb{Z}\mathrm{e}_1 + D\mathbb{Z}\mathrm{f}_1 & \mathfrak{d}^{-1}L_1 &= \mathbb{Z}\mathrm{e}_1 + \mathbb{Z}\mathrm{f}_1. \end{aligned} \tag{3.9.5}$$

As $\varepsilon V_1(\mathbb{C}) \subset V_1(\mathbb{C})$ is a line, there is a unique $\tau \in \mathbb{C}$ satisfying

$$\tau\mathrm{e}_1 + \mathrm{f}_1 \in \varepsilon V_1(\mathbb{C}). \tag{3.9.6}$$

After possibly replacing both e_1 and e_{-1} by their negatives, we may assume that $\mathrm{Im}(\tau) > 0$.

Proposition 3.9.3. — *The $\mathbb{Z}$-lattice $\mathfrak{b} = \mathbb{Z}\tau + \mathbb{Z}$ is contained in $\boldsymbol{k}$, and is a fractional $\mathcal{O}_{\boldsymbol{k}}$-ideal. The elliptic curve*

$$E^{(a)}(\mathbb{C}) = \mathfrak{b}\backslash\mathbb{C} \tag{3.9.7}$$

is isomorphic to (3.9.2), *and there is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism of complex abelian varieties*

$$E^{(a)}(\mathbb{C}) \otimes L_0 \cong \mathfrak{b}L_0\backslash V_0(\mathbb{R}). \tag{3.9.8}$$

Under this isomorphism the inverse of the line bundle (3.4.3) *has the form*

$$\mathcal{Q}^{-1}_{E^{(a)}(\mathbb{C})\otimes L_0} \cong \mathfrak{b}L_0\backslash(V_0(\mathbb{R})\times\mathbb{C}), \tag{3.9.9}$$

where the action of $y_0 \in \mathfrak{b}L_0$ *on* $V_0(\mathbb{R})\times\mathbb{C}$ *is*

$$y_0\cdot(w_0,q) = \big(w_0+\varepsilon y_0, q\cdot e^{\pi i\frac{\langle y_0,y_0\rangle}{\mathrm{N}(\mathfrak{b})}} e^{-\pi\frac{\langle w_0,y_0\rangle}{\mathrm{Im}(\tau)}-\pi\frac{\langle y_0,y_0\rangle}{2\mathrm{Im}(\tau)}}\big).$$

Proof. — Consider the $\mathbb{Q}$-linear map

$$V_{-1} \xrightarrow{\alpha \mathrm{e}_{-1}+\beta \mathrm{f}_{-1}\mapsto \alpha\tau+\beta} \mathbb{C}. \tag{3.9.10}$$

Its $\mathbb{C}$-linear extension $V_{-1}(\mathbb{C})\to\mathbb{C}$ kills the vector $\mathrm{e}_{-1}-\tau\mathrm{f}_{-1}\in\varepsilon V_{-1}(\mathbb{C})$, and hence factors through an isomorphism $V_{-1}(\mathbb{C})/\varepsilon V_{-1}(\mathbb{C})\cong\mathbb{C}$. This implies that (3.9.10) is $\boldsymbol{k}$-conjugate linear. As this map identifies $L_{-1}\cong\mathfrak{b}$, we find that the $\mathbb{Z}$-lattice $\mathfrak{b}\subset\mathbb{C}$ is $\mathcal{O}_{\boldsymbol{k}}$-stable. From $1\in\mathfrak{b}$ we then deduce that $\mathfrak{b}\subset\boldsymbol{k}$, and is a fractional $\mathcal{O}_{\boldsymbol{k}}$-ideal. Moreover, we have just shown that

$$L_{-1} \xrightarrow{\alpha \mathrm{e}_{-1}+\beta \mathrm{f}_{-1}\mapsto \alpha\tau+\beta} \mathfrak{b} \tag{3.9.11}$$

is an $\mathcal{O}_{\boldsymbol{k}}$-conjugate linear isomorphism.

Exactly as in (2.1.4), the self-dual hermitian forms on $g\mathfrak{a}_0$ and $g\mathfrak{a}$ induce an $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0,\mathrm{gr}_{-2}(g\mathfrak{a}))\cong\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathrm{gr}_0(g\mathfrak{a}),g\mathfrak{a}_0),$$

and hence determine an $\mathcal{O}_{\boldsymbol{k}}$-conjugate-linear isomorphism

$$\begin{aligned} L_{-1} &= \overline{a}\cdot\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0,\mathrm{gr}_{-2}(g\mathfrak{a}))\\ &\cong a\cdot\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathrm{gr}_0(g\mathfrak{a}),g\mathfrak{a}_0)\\ &= a\cdot\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n},g\mathfrak{a}_0).\end{aligned}$$

The composition

$$a\cdot\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{n},g\mathfrak{a}_0)\cong L_{-1}\xrightarrow{(3.9.11)}\mathfrak{b}$$

is an $\mathcal{O}_{\boldsymbol{k}}$-linear isomorphism, which identifies the fractional ideal $\mathfrak{b}$ with the projective $\mathcal{O}_{\boldsymbol{k}}$-module used in the definition of (3.9.2). In particular it identifies the elliptic curves (3.9.2) and (3.9.7), and also identifies

$$E^{(a)}(\mathbb{C})\otimes L_0 = (\mathfrak{b}\backslash\mathbb{C})\otimes L_0\cong(\mathfrak{b}\otimes L_0)\backslash(\mathbb{C}\otimes L_0).$$

Here, and throughout the remainder of the proof, all tensor products are over $\mathcal{O}_{\boldsymbol{k}}$. Identifying $\mathbb{C}\otimes L_0\cong V_0(\mathbb{R})$ proves (3.9.8).

It remains to explain the isomorphism (3.9.9). First consider the Poincaré bundle on the product

$$E^{(a)}(\mathbb{C})\times E^{(a)}(\mathbb{C})\cong(\mathfrak{b}\times\mathfrak{b})\backslash(\mathbb{C}\times\mathbb{C}).$$

Using classical formulas, the space of this line bundle can be identified with the quotient

$$\mathcal{P}_{E^{(a)}(\mathbb{C})} = (\mathfrak{b}\times\mathfrak{b})\backslash(\mathbb{C}\times\mathbb{C}\times\mathbb{C}),$$

where the action is given by

$$(b_1, b_2) \cdot (z_1, z_2, q) = \Big(z_1 + b_1, z_2 + b_2, q \cdot e^{\pi H_\tau(z_1,b_2) + \pi H_\tau(z_2,b_1) + \pi H_\tau(b_1,b_2)}\Big),$$

and we have set $H_\tau(w,z) = w\overline{z}/\mathrm{Im}(\tau)$ for complex numbers w and z.

Directly from the definition, the line bundle (3.4.3) on

$$E^{(a)}(\mathbb{C}) \otimes L_0 \cong (\mathfrak{b} \otimes L_0)\backslash(\mathbb{C} \otimes L_0)$$

is given by

$$\mathcal{Q}_{E^{(a)}(\mathbb{C})\otimes L_0} \cong (\mathfrak{b} \otimes L_0)\backslash\big((\mathbb{C} \otimes L_0) \times \mathbb{C}\big),$$

where the action of $\mathfrak{b} \otimes L_0$ on $(\mathbb{C} \otimes L_0) \times \mathbb{C}$ is given as follows: Choose any set $x_1, \ldots, x_n \in L_0$ of $\mathcal{O}_{\boldsymbol{k}}$-module generators, and extend the $\mathcal{O}_{\boldsymbol{k}}$-hermitian form on L_0 to a $\mathbb{C}$-hermitian form on $\mathbb{C} \otimes L_0$. If

$$y_0 = \sum_i b_i \otimes x_i \in \mathfrak{b} \otimes L_0$$

and

$$w_0 = \sum_i z_i \otimes x_i \in \mathbb{C} \otimes L_0$$

then

$$y_0 \cdot (w_0, q) = (w_0 + y_0, q \cdot e^{\pi X + \pi Y}),$$

where the factors X and Y are

$$\begin{aligned} X &= \sum_{i<j} \Big(H_\tau(\langle x_i, x_j\rangle z_i, b_j) + H_\tau(z_j, \langle x_i, x_j\rangle b_i) + H_\tau(\langle x_i, x_j\rangle b_i, b_j)\Big) \\ &= \frac{1}{\mathrm{Im}(\tau)} \sum_{i\neq j} \langle z_i \otimes x_i, b_j \otimes x_j\rangle + \frac{1}{\mathrm{Im}(\tau)} \sum_{i<j} \langle b_i \otimes x_i, b_j \otimes x_j\rangle \end{aligned}$$

and, recalling $\gamma = (1+\delta)/2$,

$$\begin{aligned} Y &= \sum_i \Big(H_\tau(\gamma\langle x_i, x_i\rangle z_i, b_i) + H_\tau(z_i, \gamma\langle x_i, x_i\rangle b_i) + H_\tau(\gamma\langle x_i, x_i\rangle b_i, b_i)\Big) \\ &= \frac{1}{\mathrm{Im}(\tau)} \sum_i \langle z_i \otimes x_i, b_i \otimes x_i\rangle + \frac{1}{\mathrm{Im}(\tau)} \sum_i \gamma\langle b_i \otimes x_i, b_i \otimes x_i\rangle. \end{aligned}$$

For elements $y_1, y_2 \in \mathfrak{b} \otimes L_0$, we abbreviate

$$\alpha(y_1, y_1) = \frac{\langle y_1, y_2\rangle}{\delta \mathrm{N}(\mathfrak{b})} - \frac{\langle y_2, y_1\rangle}{\delta \mathrm{N}(\mathfrak{b})} \in \mathbb{Z}.$$

Using $2i\mathrm{Im}(\tau) = \delta\mathrm{N}(\mathfrak{b})$, some elementary calculations show that

$$\begin{aligned} &\pi X + \pi Y - \frac{\pi\langle w_0, y_0\rangle}{\mathrm{Im}(\tau)} \\ &\quad = \frac{2\pi i}{\delta\mathrm{N}(\mathfrak{b})} \sum_{i<j} \langle b_i \otimes x_i, b_j \otimes x_j\rangle + \frac{2\pi i}{\delta\mathrm{N}(\mathfrak{b})} \sum_i \langle \gamma b_i \otimes x_i, b_i \otimes x_i\rangle \end{aligned}$$

$$\begin{aligned} = \quad & \frac{\pi}{2\mathrm{Im}(\tau)} \sum_{i,j} \langle b_i \otimes x_i, b_j \otimes x_j \rangle - \frac{\pi i}{\mathrm{N}(\mathfrak{b})} \sum_{i,j} \langle b_i \otimes x_i, b_j \otimes x_j \rangle \\ & + 2\pi i \sum_{i<j} \alpha(\gamma b_i \otimes x_i, b_j \otimes x_j) + \frac{2\pi i}{\mathrm{N}(\mathfrak{b})} \sum_{i} \langle b_i \otimes x_i, b_i \otimes x_i \rangle. \end{aligned}$$

All terms in the final line lie in $2\pi i\mathbb{Z}$, and so

$$e^{\pi X + \pi Y} = e^{\frac{\pi \langle w_0, y_0 \rangle}{\mathrm{Im}(\tau)}} e^{\frac{\pi \langle y_0, y_0 \rangle}{2\mathrm{Im}(\tau)}} e^{-\frac{\pi i \langle y_0, y_0 \rangle}{\mathrm{N}(\mathfrak{b})}}.$$

The relation (3.9.9) follows immediately. □

Proposition 3.9.3 allows us to express Fourier-Jacobi coefficients explicitly as functions on $V_0(\mathbb{R})$ satisfying certain transformation laws. Suppose we start with a global section

$$\psi \in H^0\big(\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}, \boldsymbol{\omega}^k\big). \tag{3.9.12}$$

For each $a \in \widehat{\boldsymbol{k}}^\times$ and $\ell \geq 0$ we have the algebraically defined Fourier-Jacobi coefficient

$$\mathrm{FJ}^{(a)}_\ell(\psi) \in H^0\big(E^{(a)} \otimes L_0, \mathcal{Q}^\ell_{E^{(a)} \otimes L_0}\big) \tag{3.9.13}$$

of (3.9.3), where we have trivialized $\mathrm{Lie}(E^{(a)})$ using (3.9.7). The isomorphism (3.9.9) now identifies (3.9.13) with a function on $V_0(\mathbb{R})$ satisfying the transformation law

$$\mathrm{FJ}^{(a)}_\ell(\psi)(w_0 + y_0) = \mathrm{FJ}^{(a)}_\ell(\psi)(w_0) \cdot e^{i\pi\ell \frac{\langle y_0, y_0 \rangle}{\mathrm{N}(\mathfrak{b})}} e^{\pi\ell \frac{\langle w_0, y_0 \rangle}{\mathrm{Im}(\tau)} + \pi\ell \frac{\langle y_0, y_0 \rangle}{2\mathrm{Im}(\tau)}} \tag{3.9.14}$$

for all $y_0 \in \mathfrak{b}L_0$.

Remark 3.9.4. — If we use the isomorphism $\mathrm{pr}_\varepsilon : V_0(\mathbb{R}) \cong \varepsilon V_0(\mathbb{C})$ of (2.1.7) to view (3.9.13) as a function of $w_0 \in \varepsilon V_0(\mathbb{C})$, the transformation law can be expressed in terms of the $\mathbb{C}$-bilinear form $[.,.]$ as

$$\mathrm{FJ}^{(a)}_\ell(\psi)(w_0 + \mathrm{pr}_\varepsilon(y_0)) = \mathrm{FJ}^{(a)}_\ell(\psi)(w_0) \cdot e^{i\pi\ell \frac{Q(y_0)}{\mathrm{N}(\mathfrak{b})}} e^{\pi\ell \frac{[w_0, y_0]}{\mathrm{Im}(\tau)} + \pi\ell \frac{Q(y_0)}{2\mathrm{Im}(\tau)}}$$

for all $y_0 \in \mathfrak{b}L_0$. This uses the (slightly confusing) commutativity of

$$\begin{array}{ccccc} V_0(\mathbb{R}) & \xrightarrow{\mathrm{pr}_\varepsilon} & \varepsilon V_0(\mathbb{C}) & \xrightarrow{\subset} & V_0(\mathbb{C}) \\ \Big\downarrow{\scriptstyle \langle \cdot, y_0 \rangle} & & & & \Big\downarrow{\scriptstyle [\cdot, y_0]} \\ \boldsymbol{k} \otimes_{\mathbb{Q}} \mathbb{R} & = & = & = & \mathbb{C}. \end{array}$$

In order to give another interpretation of our explicit coordinates, let $N_\Phi \subset Q_\Phi$ be the unipotent radical, and let $U_\Phi \subset N_\Phi$ be its center. The unipotent radical may be characterized as the kernel of the morphism ν_Φ of (3.1.1), or, equivalently, as the largest subgroup acting trivially on all graded pieces $\mathrm{gr}_i V$.

Proposition 3.9.5. — *There is a commutative diagram*

$$\begin{array}{ccc} (U_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1})\backslash \mathcal{D}_\Phi & \xrightarrow{z\mapsto(w_0,q)} & \varepsilon V_0(\mathbb{C}) \times \mathbb{C}^\times \\ \downarrow & & \downarrow \\ (N_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1})\backslash \mathcal{D}_\Phi & \longrightarrow & \mathfrak{b}L_0\backslash(\varepsilon V_0(\mathbb{C}) \times \mathbb{C}^\times), \end{array} \tag{3.9.15}$$

in which the horizontal arrows are holomorphic isomorphisms, and the action of $\mathfrak{b}L_0$ *on*

$$\varepsilon V_0(\mathbb{C}) \times \mathbb{C}^\times \cong V_0(\mathbb{R}) \times \mathbb{C}^\times$$

is the same as in Proposition 3.9.3.

Proof. — Recall from Remark 3.2.3 the isomorphism

$$\mathcal{D}_\Phi \cong \big\{ w \in \varepsilon V(\mathbb{C}) : \varepsilon V(\mathbb{C}) = \varepsilon V_{-1}(\mathbb{C}) \oplus \varepsilon V_0(\mathbb{C}) \oplus \mathbb{C}w \big\}/\mathbb{C}^\times.$$

As $\varepsilon V(\mathbb{C})$ is totally isotropic with respect to $[.,.]$, a simple calculation shows that every line $w \in \mathcal{D}_\Phi$ has a unique representative of the form

$$-\xi(\mathrm{e}_{-1} - \tau \mathrm{f}_{-1}) + w_0 + (\tau \mathrm{e}_1 + \mathrm{f}_1) \in \varepsilon V_{-1}(\mathbb{C}) \oplus \varepsilon V_0(\mathbb{C}) \oplus \varepsilon V_1(\mathbb{C})$$

with $\xi \in \mathbb{C}$ and $w_0 \in \varepsilon V_0(\mathbb{C}) = V_0(\mathbb{R})$. These coordinates define an isomorphism of complex manifolds

$$\mathcal{D}_\Phi \xrightarrow{w\mapsto(w_0,\xi)} \varepsilon V_0(\mathbb{C}) \times \mathbb{C}. \tag{3.9.16}$$

The action of G on V restricts to a faithful action of N_Φ, allowing us to express elements of $N_\Phi(\mathbb{Q})$ as matrices

$$n(\phi, \phi^*, u) = \begin{pmatrix} 1 & \phi^* & u + \frac{1}{2}\phi^* \circ \phi \\ & 1 & \phi \\ & & 1 \end{pmatrix} \in N_\Phi(\mathbb{Q})$$

for maps

$$\phi \in \mathrm{Hom}_{\boldsymbol{k}}(V_1, V_0), \quad \phi^* \in \mathrm{Hom}_{\boldsymbol{k}}(V_0, V_{-1}), \quad u \in \mathrm{Hom}_{\boldsymbol{k}}(V_1, V_{-1})$$

satisfying the relations

$$0 = \langle \phi(x_1), y_0 \rangle + \langle x_1, \phi^*(y_0) \rangle$$
$$0 = \langle u(x_1), y_1 \rangle + \langle x_1, u(y_1) \rangle$$

for $x_i, y_i \in V_i$. The subgroup $U_\Phi(\mathbb{Q})$ is defined by $\phi = 0 = \phi^*$.

The group $U_\Phi(\mathbb{Q}) \cap s(a)K_\Phi s(a)^{-1}$ is cyclic, and generated by the element $n(0,0,u)$ defined by

$$u(x_1) = \frac{\langle x_1, a \rangle}{[L_{-1} : \mathcal{O}_{\boldsymbol{k}} a]} \cdot \delta a$$

for any $a \in L_{-1}$. In terms of the bilinear form, this can be rewritten as

$$u(x_1) = -[x_1, \mathrm{f}_{-1}]\mathrm{e}_{-1} + [x_1, \mathrm{e}_{-1}]\mathrm{f}_{-1}.$$

In the coordinates of (3.9.16), the action of $n(0,0,u)$ on $\mathcal{D}_\Phi$ becomes

$$(w_0,\xi)\mapsto(w_0,\xi+1),$$

and setting $q=e^{2\pi i\xi}$ defines the top horizontal isomorphism in (3.9.15).

Let $\overline{V}_{-1}=V_{-1}$ with its conjugate action of $\boldsymbol{k}$. There are group isomorphisms

$$N_\Phi(\mathbb{Q})/U_\Phi(\mathbb{Q})\cong\overline{V}_{-1}\otimes_{\boldsymbol{k}}V_0\cong V_0. \tag{3.9.17}$$

The first sends

$$n(\phi,\phi^*,u)\mapsto y_{-1}\otimes y_0,$$

where y_{-1} and y_0 are defined by the relation $\phi(x_1)=\langle x_1,y_{-1}\rangle\cdot y_0$, and the second sends

$$(\alpha\mathrm{e}_{-1}+\beta\mathrm{f}_{-1})\otimes y_0\mapsto(\alpha\tau+\beta)y_0.$$

Compare with (3.9.11).

A slightly tedious calculation shows that (3.9.17) identifies

$$(N_\Phi(\mathbb{Q})\cap s(a)K_\Phi s(a)^{-1})/(U_\Phi(\mathbb{Q})\cap s(a)K_\Phi s(a)^{-1})\cong\mathfrak{b}L_0,$$

defining the bottom horizontal arrow in (3.9.15), and that the resulting action of $\mathfrak{b}L_0$ on $\varepsilon V_0(\mathbb{C})\times\mathbb{C}^\times$ agrees with the one defined in Proposition 3.9.3. We leave this to the reader. □

Any section (3.9.12) may now be pulled back via

$$(N_\Phi(\mathbb{Q})\cap s(a)K_\Phi s(a)^{-1})\backslash\mathcal{D}\xrightarrow{z\mapsto(z,s(a)g)}\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$$

to define a holomorphic section of $(\boldsymbol{\omega}^{\mathrm{an}})^k$, the k^{th} power of the tautological bundle on

$$\mathcal{D}\cong\big\{w\in\varepsilon V(\mathbb{C}):[w,\overline{w}]<0\big\}/\mathbb{C}^\times.$$

The tautological bundle admits a natural $N_\Phi(\mathbb{R})$-equivariant trivialization: any line $w\in\mathcal{D}$ must satisfy $[w,\mathrm{f}_{-1}]\neq0$, yielding an isomorphism

$$[\,\cdot\,,\mathrm{f}_{-1}]:\boldsymbol{\omega}^{\mathrm{an}}\cong\mathcal{O}_{\mathcal{D}}.$$

This trivialization allows us to identify ψ with a holomorphic function on $\mathcal{D}\subset\mathcal{D}_\Phi$, which then has an *analytic* Fourier-Jacobi expansion

$$\psi=\sum_\ell\mathrm{FJ}_\ell^{(a)}(\psi)(w_0)\cdot q^\ell \tag{3.9.18}$$

defined using the coordinates of Proposition 3.9.5. The fact that the coefficients here agree with (3.9.13) is a special case of the main results of [**39**], which compare algebraic and analytic Fourier-Jacobi coefficients on general PEL-type Shimura varieties.

4. Classical modular forms

Throughout § 4 we let D be any odd squarefree positive integer, and abbreviate $\Gamma=\Gamma_0(D)$. Let k be any positive integer.

4.1. Weakly holomorphic forms. — The positive divisors of D are in bijection with the cusps of the complex modular curve $X_0(D)(\mathbb{C})$, by sending $r \mid D$ to

$$\infty_r = \frac{r}{D} \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q}).$$

Note that $r = 1$ corresponds to the usual cusp at infinity, and so we sometimes abbreviate $\infty = \infty_1$.

Fix a positive divisor $r \mid D$, set $s = D/r$ and choose

$$R_r = \begin{pmatrix} \alpha & \beta \\ s\gamma & r\delta \end{pmatrix} \in \Gamma_0(s)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$. The corresponding Atkin-Lehner operator is defined by the matrix

$$W_r = \begin{pmatrix} r\alpha & \beta \\ D\gamma & r\delta \end{pmatrix} = R_r \begin{pmatrix} r & \\ & 1 \end{pmatrix}.$$

The matrix W_r normalizes Γ, and so acts on the cusps of $X_0(D)(\mathbb{C})$. This action satisfies $W_r \cdot \infty = \infty_r$.

Let χ be a quadratic Dirichlet character modulo D, and let

$$\chi = \chi_r \cdot \chi_s$$

be the unique factorization as a product of quadratic Dirichlet characters χ_r and χ_s modulo r and s, respectively. Write

$$M_k(D, \chi) \subset M_k^!(D, \chi)$$

for the spaces of holomorphic modular forms and weakly holomorphic modular forms of weight k, level Γ, and character χ. We assume that $\chi(-1) = (-1)^k$, since otherwise $M_k^!(D, \chi) = 0$.

Denote by $\mathrm{GL}_2^+(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$ the subgroup of elements with positive determinant. It acts on functions on the upper half plane by the usual weight k slash operator

$$(f \mid_k \gamma)(\tau) = \det(\gamma)^{k/2}(c\tau + d)^{-k} f(\gamma\tau), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

and $f \mapsto f \mid_k W_r$ defines an endomorphism of $M_k^!(D, \chi)$ satisfying

$$f \mid_k W_r^2 = \chi_r(-1)\chi_s(r) \cdot f.$$

In particular, W_r is an involution when χ is trivial.

Any weakly holomorphic modular form

$$f(\tau) = \sum_{m \gg -\infty} c(m) \cdot q^m \in M_k^!(D, \chi)$$

determines another weakly holomorphic modular form

$$\chi_r(\beta)\chi_s(\alpha) \cdot (f \mid_k W_r) \in M_k^!(D, \chi),$$

which is easily seen to be independent of the choice of parameters $\alpha, \beta, \gamma, \delta$ in the definition of W_r. This second modular form has a q-expansion at ∞, denoted

$$\chi_r(\beta)\chi_s(\alpha) \cdot (f \mid_k W_r) = \sum_{m \gg -\infty} c_r(m) \cdot q^m. \tag{4.1.1}$$

Definition 4.1.1. — We call (4.1.1) the *q-expansion of f at ∞_r*. Of special interest is $c_r(0)$, the *constant term of f at ∞_r*.

Remark 4.1.2. — If χ is nontrivial, the coefficients of (4.1.1) need not lie in the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f.

4.2. Eisenstein series and the modularity criterion. — Fix an integer $k \geq 2$. Denote by

$$M^{!,\infty}_{2-k}(D, \chi) \subset M^!_{2-k}(D, \chi)$$

the subspace of weakly holomorphic forms that are holomorphic outside the cusp ∞, and by

$$M^{\infty}_k(D, \chi) \subset M_k(D, \chi)$$

the subspace of holomorphic modular forms that vanish at all cusps different from ∞.

If $k > 2$ there is a decomposition

$$M^{\infty}_k(D, \chi) = \mathbb{C}E \oplus S_k(D, \chi),$$

where E is the Eisenstein series

$$E = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi(d) \cdot (1 \mid_k \gamma) \in M_k(D, \chi).$$

Here $\Gamma_\infty \subset \Gamma$ is the stabilizer of $\infty \in \mathbb{P}^1(\mathbb{Q})$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

We also define the (normalized) Eisenstein series for the cusp ∞_r by

$$E_r = \chi_r(-\beta)\chi_s(\alpha r) \cdot (E \mid_k W_r) \in M_k(D, \chi).$$

It is independent of the choice of the parameters in W_r, and we denote by

$$E_r(\tau) = \sum_{m \geq 0} e_r(m) \cdot q^m$$

its q-expansion at ∞.

Remark 4.2.1. — Our notation for the q-expansion of E_r is slightly at odds with (4.1.1), as the q-expansion of E at ∞_r is not $\sum e_r(m)q^m$. Instead, the q-expansion of E at ∞_r is $\chi_r(-1)\chi_s(r)\sum e_r(m)q^m$, while the q-expansion of E_r at ∞_r is $\sum e_1(m)q^m$. In any case, what matters most is that

$$\text{constant term of } E_r \text{ at } \infty_s = \begin{cases} 1 & \text{if } s = r, \\ 0 & \text{otherwise.} \end{cases}$$

The constant terms of weakly holomorphic modular forms in $M^{!,\infty}_{2-k}(D, \chi)$ can be computed using the above Eisenstein series.

Proposition 4.2.2. — *Assume $k>2$. Suppose $r\mid D$ and*

$$f(\tau)=\sum_{m\gg-\infty} c(m)\cdot q^m \in M^{!,\infty}_{2-k}(D,\chi).$$

The constant term of f at the cusp ∞_r, in the sense of Definition 4.1.1, satisfies

$$c_r(0)+\sum_{m>0} c(-m)e_r(m)=0.$$

Proof. — The meromorphic differential form $f(\tau)E_r(\tau)\,d\tau$ on $X_0(D)(\mathbb{C})$ is holomorphic away from the cusps ∞ and ∞_r. Summing its residues at these cusps gives the desired equality. □

***Theorem 4.2.3* (Modularity criterion).** — *Suppose $k\geq 2$. For a formal power series*

$$\sum_{m\geq 0} d(m)q^m \in \mathbb{C}[[q]], \tag{4.2.1}$$

the following are equivalent.

1. *The relation $\sum_{m\geq 0} c(-m)d(m)=0$ holds for every weakly holomorphic form*
$$\sum_{m\gg-\infty} c(m)\cdot q^m \in M^{!,\infty}_{2-k}(D,\chi).$$
2. *The formal power series (4.2.1) is the q-expansion of a modular form in $M^{\infty}_k(D,\chi)$.*

Proof. — As we assume $k\geq 2$, that the map sending a weakly holomorphic modular form f to its principal part at ∞ identifies

$$M^{!,\infty}_{2-k}(D,\chi)\subset \mathbb{C}[q^{-1}].$$

On the other hand, the map sending a holomorphic modular form to its q-expansion identifies

$$M^{\infty}_k(D,\chi)\subset \mathbb{C}[[q]].$$

A slight variant of the modularity criterion of [**5**, Theorem 3.1] shows that each subspace is the exact annihilator of the other under the bilinear pairing $\mathbb{C}[q^{-1}]\otimes\mathbb{C}[[q]]\longrightarrow\mathbb{C}$ sending $P\otimes g$ to the constant term of $P\cdot g$. The claim follows. □

5. Unitary Borcherds products

The goal of § 5 is to state Theorems 5.3.1, 5.3.3, and 5.3.4, which assert the existence of Borcherds products on $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ having prescribed divisors and prescribed leading Fourier-Jacobi coefficients. These theorems are the technical core of this work, and their proofs will occupy all of § 6.

We assume $n\geq 3$ throughout § 5.

5.1. Jacobi forms. — In this section we recall some of the rudiments of the arithmetic theory of Jacobi forms. A more systematic treatment can be found in the work of Kramer [**29, 30**].

Let $\mathcal{Y}$ be the moduli stack over $\mathbb{Z}$ classifying elliptic curves, and let $\pi : \mathcal{E} \to \mathcal{Y}$ be the universal elliptic curve. Abbreviate $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and let $\mathfrak{H}$ be the complex upper half-plane. The groups Γ and $\mathbb{Z}^2$ each act on $\mathfrak{H} \times \mathbb{C}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right), \qquad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot (\tau, z) = (\tau, z + \alpha\tau + \beta),$$

and this defines an action of the semi-direct product $\Gamma^* = \Gamma \ltimes \mathbb{Z}^2$. We identify the commutative diagrams (of complex orbifolds)

$$(5.1.1) \qquad \begin{array}{ccc} \Gamma\backslash(\mathfrak{H} \times \mathbb{C}) & & \\ \downarrow & \searrow & \\ \Gamma^*\backslash(\mathfrak{H} \times \mathbb{C}) & \longrightarrow & \Gamma\backslash\mathfrak{H} \end{array} \qquad \begin{array}{ccc} \mathrm{Lie}(\mathcal{E}(\mathbb{C})) & & \\ {\scriptstyle \exp}\downarrow & \searrow & \\ \mathcal{E}(\mathbb{C}) & \longrightarrow & \mathcal{Y}(\mathbb{C}) \end{array}$$

by sending $(\tau, z) \in \mathfrak{H} \times \mathbb{C}$ to the vector z in the Lie algebra of $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$.

Define a line bundle $\mathcal{O}(e)$ on $\mathcal{E}$ as the inverse ideal sheaf of the zero section $e : \mathcal{Y} \to \mathcal{E}$. The Lie algebra $\mathrm{Lie}(\mathcal{E})$ is (by definition) $e^*\mathcal{O}(e)$, and $\boldsymbol{\omega}_{\mathcal{Y}} = \mathrm{Lie}(\mathcal{E})^{-1}$ is the usual line bundle of weight one modular forms on $\mathcal{Y}$ (see Remark 5.1.3 below). In particular, the line bundle

$$\mathcal{Q} = \mathcal{O}(e) \otimes \pi^*\boldsymbol{\omega}_{\mathcal{Y}}$$

on $\mathcal{E}$ is canonically trivialized along the zero section. By the constructions of [**40**, § 1.3.2] and [**44**, § 6.2], this line bundle induces a homomorphism

$$(5.1.2) \qquad \mathcal{E} \to \mathcal{E}^\vee,$$

which is none other than the unique principal polarization of $\mathcal{E}$ (one can verify this fiber-by-fiber over geometric points of $\mathcal{Y}$, reducing the claim to standard properties of elliptic curves over fields). Denote by $\mathcal{P}$ the pullback of the Poincaré bundle via

$$\mathcal{E} \times_{\mathcal{Y}} \mathcal{E} \cong \mathcal{E} \times_{\mathcal{Y}} \mathcal{E}^\vee.$$

For a scheme U and points $a, b \in \mathcal{E}(U)$, denote by $\mathcal{Q}(a)$ the pullback of $\mathcal{Q}$ via $a : U \to \mathcal{E}$, and by $\mathcal{P}(a, b)$ the pullback of $\mathcal{P}$ via $(a, b) : U \to \mathcal{E} \times_{\mathcal{Y}} \mathcal{E}$. There are canonical isomorphisms

$$\mathcal{P}(a, b) \cong \mathcal{Q}(a + b) \otimes \mathcal{Q}(a)^{-1} \otimes \mathcal{Q}(b)^{-1}$$

and

$$\mathcal{P}(a, a) \cong \mathcal{Q}(a) \otimes \mathcal{Q}(a).$$

Given the way that (5.1.2) is constructed from $\mathcal{Q}$, the first isomorphism is essentially a tautology. The second is a consequence of the isomorphisms

$$\mathcal{Q}(2a) \cong \mathcal{Q}(a)^{\otimes 3} \otimes \mathcal{Q}(-a) \cong \mathcal{Q}(a)^{\otimes 4},$$

which follow from the theorem of the cube [**17**, Theorem I.1.3] and the invariance of $\mathcal{Q}$ under pullback by $[-1] : \mathcal{E} \to \mathcal{E}$, respectively.

Definition 5.1.1. — The diagonal restriction

$$\mathcal{J}_{0,1} = (\mathrm{diag})^* \mathcal{P} \cong \mathcal{Q}^2$$

is the line bundle of *Jacobi forms of weight* 0 *and index* 1 on $\mathcal{E}$. More generally,

$$\mathcal{J}_{k,m} = \mathcal{J}_{0,1}^m \otimes \pi^* \boldsymbol{\omega}_{\mathcal{Y}}^k$$

is the line bundle of *Jacobi forms of weight k and index m* on $\mathcal{E}$.

The isomorphism of the following proposition is presumably well-known. We include the proof in order to make explicit the normalization of the isomorphism (see Remark 5.1.3 below, for example).

Proposition 5.1.2. — *Let $p : \mathfrak{H} \times \mathbb{C} \to \mathcal{E}(\mathbb{C})$ be the quotient map. The holomorphic line bundle $\mathcal{J}_{k,m}^{\mathrm{an}}$ on $\mathcal{E}(\mathbb{C})$ is isomorphic to the holomorphic line bundle whose sections over an open set $\mathscr{U} \subset \mathcal{E}(\mathbb{C})$ are holomorphic functions $F(\tau, z)$ on $p^{-1}(\mathscr{U})$ satisfying the transformation laws*

$$F\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = F(\tau, z) \cdot (c\tau + d)^k \cdot e^{2\pi i m c z^2/(c\tau + d)}$$

and

$$F(\tau, z + \alpha\tau + \beta) = F(\tau, z) \cdot e^{-2\pi i m(\alpha^2\tau + 2\alpha z)}. \tag{5.1.3}$$

Proof. — Let $J_{k,m}$ be the holomorphic line bundle on $\mathcal{E}(\mathbb{C})$ defined by the above transformation laws.

By identifying the diagrams (5.1.1), a function f, defined on a Γ-invariant open subset of $\mathfrak{H}$ and satisfying the transformation law

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \cdot (c\tau + d)^{-1}$$

of a weight -1 modular form, defines a section $\tau \mapsto (\tau, f(\tau))$ of the line bundle

$$\Gamma \backslash (\mathfrak{H} \times \mathbb{C}) \cong \mathrm{Lie}(\mathcal{E}(\mathbb{C})) \cong (\boldsymbol{\omega}_{\mathcal{Y}}^{\mathrm{an}})^{-1}$$

on $\Gamma \backslash \mathfrak{H}$. This determines an isomorphism $J_{1,0} \cong \mathcal{J}_{1,0}^{\mathrm{an}}$. It now suffices to construct an isomorphism $J_{0,1} \cong \mathcal{J}_{0,1}^{\mathrm{an}}$, and then take tensor products.

Fix $\tau \in \mathfrak{H}$, set $E_\tau = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, and restrict both $\mathcal{J}_{0,1}^{\mathrm{an}}$ and $J_{0,1}$ to line bundles on $E_\tau \subset \mathcal{E}(\mathbb{C})$. The imaginary part of the hermitian form

$$H_\tau(z_1, z_2) = \frac{z_1 \overline{z_2}}{\mathrm{Im}(\tau)}$$

on $\mathbb{C}$ restricts to a Riemann form on $\mathbb{Z}\tau + \mathbb{Z}$. Using classical formulas for the Poincaré bundle on complex abelian varieties, as found in the proof of [**3**, Theorem 2.5.1], the restriction of $\mathcal{J}_{0,1}^{\mathrm{an}}$ to the fiber E_τ is isomorphic to the holomorphic line bundle determined by the Appell-Humbert data $2H_\tau$ and the trivial character $\mathbb{Z}\tau + \mathbb{Z} \to \mathbb{C}^\times$. The sections of this holomorphic line bundle are, by definition, holomorphic functions g_τ on $\mathbb{C}$ satisfying the transformation law

$$g_\tau(z+\ell) = g_\tau(z)\cdot e^{2\pi H_\tau(z,\ell)+\pi H_\tau(\ell,\ell)}$$

for all $\ell \in \mathbb{Z}\tau + \mathbb{Z}$. If we set

$$F(\tau,z) = g_\tau(z)\cdot e^{-\pi H_\tau(z,\overline{z})},$$

this transformation law becomes (5.1.3).

The above shows that $\mathcal{J}_{0,1}^{\mathrm{an}}$ and $J_{0,1}$ are isomorphic when restricted to the fiber over any point of $\mathcal{Y}(\mathbb{C})$, but such an isomorphism is only determined up to scaling by $\mathbb{C}^\times$. To pin down the scalars, and to get an isomorphism over the total space, use the fact that both $\mathcal{J}_{0,1}^{\mathrm{an}}$ and $J_{0,1}$ come (by construction) with canonical trivializations along the zero section. By the Seesaw Theorem [**3**, Appendix A], there is a unique isomorphism $\mathcal{J}_{0,1}^{\mathrm{an}} \cong J_{0,1}$ compatible with these trivializations. □

Remark 5.1.3. — The proof of Proposition 5.1.2 identifies a classical modular form $f(\tau) = \sum c(m)q^m$ of weight k and level Γ with a holomorphic section of $(\boldsymbol{\omega}_{\mathcal{Y}}^{\mathrm{an}})^k$, again denoted f, satisfying an additional growth condition at the cusp. Under our identification, the q-expansion principle takes the following form: if $R \subset \mathbb{C}$ is any subring, then f is the analytification of a global section $f \in H^0(\mathcal{Y}_{/R}, \boldsymbol{\omega}^k_{\mathcal{Y}/R})$ if and only if $c(m) \in (2\pi i)^k R$ for all m.

For $\tau \in \mathfrak{H}$ and $z \in \mathbb{C}$, we denote by

$$\vartheta_1(\tau,z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{1}{2}\right)^2\tau + 2\pi i\left(n+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}$$

the classical Jacobi theta function, and by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2n\pi i\tau})$$

Dedekind's eta function. Set

$$\Theta(\tau,z) \stackrel{\mathrm{def}}{=} i\frac{\vartheta_1(\tau,z)}{\eta(\tau)} = q^{1/12}(\zeta^{1/2}-\zeta^{-1/2})\prod_{n=1}^{\infty}(1-\zeta q^n)(1-\zeta^{-1}q^n),$$

where $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z}$.

Proposition 5.1.4. — *The Jacobi form Θ^{24} defines a global section*

$$\Theta^{24} \in H^0(\mathcal{E}, \mathcal{J}_{0,12})$$

with divisor $24e$, while $(2\pi i\eta^2)^{12}$ determines a nowhere vanishing section

$$(2\pi i\eta^2)^{12} \in H^0(\mathcal{Y}, \boldsymbol{\omega}^{12}_{\mathcal{Y}}).$$

Proof. — It is a classical fact that $(2\pi i \eta^2)^{12}$ is a nowhere vanishing modular form of weight 12. Noting Remark 5.1.3, the q-expansion principle shows that it descends to a section on $\mathcal{Y}_{/\mathbb{Q}}$, and thus may be viewed as a rational section on $\mathcal{Y}$. Another application of the q-expansion principle shows that its divisor has no vertical components. Thus its divisor is trivial.

Classical formulas show that Θ^{24} defines a holomorphic section of $\mathcal{J}^{\mathrm{an}}_{0,12}$ with divisor $24e$, and so the problem is to show that Θ^{24} is defined over $\mathbb{Q}$, and extends to a section on the integral model with the stated divisor. One could presumably deduce this from the q-expansion principle for Jacobi forms as in [**29, 30**]. We instead borrow an argument from [**51**, § 1.2], which requires only the more elementary q-expansion principle for *functions* on $\mathcal{E}$.

Let d be any positive integer. The bilinear relations (3.4.1) imply that the line bundle $\mathcal{J}_{0,1}^{d^2} \otimes [d]^* \mathcal{J}_{0,1}^{-1}$ on $\mathcal{E}$ is canonically trivial, and so

$$\theta_d^{24} = \Theta^{24d^2} \otimes [d]^* \Theta^{-24}$$

defines a meromorphic function on $\mathcal{E}(\mathbb{C})$. The crucial point is that θ_d^{24} is actually a rational function defined over $\mathbb{Q}$, and extends to a rational function on the integral model $\mathcal{E}$ with divisor

$$\operatorname{div}(\theta_d^{24}) = 24\big(d^2 \mathcal{E}[1] - \mathcal{E}[d]\big). \tag{5.1.4}$$

As in [**51**, p. 387], this follows by computing the divisor first in the complex fiber, then using the explicit formula

$$\theta_d^{24}(\tau, z) = q^{2(d^2-1)} \zeta^{-12d(d-1)} \left(\prod_{n \geq 0} \frac{(1-q^n\zeta)^{d^2}}{1-q^n\zeta^d} \prod_{n>0} \frac{(1-q^n\zeta^{-1})^{d^2}}{1-q^n\zeta^{-d}} \right)^{24}$$

and the q-expansion principle on $\mathcal{E}$ to see that the divisor has no vertical components.

The line bundle $\boldsymbol{\omega}_{\mathcal{Y}}^{12}$ is trivial, and hence there are isomorphisms

$$\mathcal{J}_{0,12} \cong \mathcal{Q}^{24} \cong \mathcal{O}(e)^{24} \otimes \pi^* \boldsymbol{\omega}_{\mathcal{Y}}^{12} \cong \mathcal{O}(e)^{24}.$$

Thus there is *some* $\tilde{\Theta}^{24} \in H^0(\mathcal{E}, \mathcal{J}_{0,12})$ with divisor $24e$, and the rational function

$$\tilde{\theta}_d^{24} = \tilde{\Theta}^{24d^2} \otimes [d]^* \tilde{\Theta}^{-24}$$

on $\mathcal{E}$ also has divisor (5.1.4).

Consider the meromorphic function $\rho = \Theta^{24}/\tilde{\Theta}^{24}$ on $\mathcal{E}(\mathbb{C})$. By computing the divisor in the complex fiber, we see that ρ is a nowhere vanishing holomorphic function, and hence is constant. But this implies that

$$\rho^{d^2-1} = \theta_d^{24}/\tilde{\theta}_d^{24}.$$

By what was said above, the right hand side is (the analytification of) a nowhere vanishing function on $\mathcal{E}$. This implies that $\rho^{d^2-1} = \pm 1$, and the only way this can hold for all $d > 1$ is if $\rho = \pm 1$. □

Now consider the tower of stacks

$$\mathcal{Y}_1(D) \to \mathcal{Y}_0(D) \to \mathcal{Y}$$

over Spec($\mathbb{Z}$) parametrizing elliptic curves with Drinfeld $\Gamma_1(D)$-level structure, $\Gamma_0(D)$-level structure, and no level structure, respectively. See [**28**, Chapter 3] or [**15**] for the definitions. We denote by $\mathcal{E}$ the universal elliptic curve over any one of these bases, and view the line bundle of Jacobi forms $\mathcal{J}_{0,12}$ as a line bundle on any one of the three universal elliptic curves. Similarly, we view the Jacobi forms Θ^{24} and $(2\pi i\eta^2)^{12}$ of Proposition 5.1.4 as being defined over any one of these bases.

The following lemma will be needed in § 5.3.

Lemma 5.1.5. — *Let $Q : \mathcal{Y}_1(D) \to \mathcal{E}$ be the universal D-torsion point. For any $r \mid D$ the line bundle*

$$\bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z}\\ b\neq 0\\ rb=0}} (bQ)^*\mathcal{J}_{0,12} \tag{5.1.5}$$

on $\mathcal{Y}_1(D)$ is canonically trivial, and its section

$$F_r^{24} = \bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z}\\ b\neq 0\\ rb=0}} (bQ)^*\Theta^{24}$$

admits a canonical descent, denoted the same way, to a section of the trivial bundle on $\mathcal{Y}_0(D)$.

Proof. — If $x_1, \dots, x_r$ are integers representing the r-torsion subgroup of $\mathbb{Z}/D\mathbb{Z}$, then $6\sum x_i^2 \equiv 0 \pmod{D}$. The bilinear relations (3.4.1) therefore provide a canonical isomorphism

$$\bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z}\\ b\neq 0\\ rb=0}} \mathcal{P}(bQ,bQ)^{\otimes 12} \cong \bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z}\\ b\neq 0\\ rb=0}} \mathcal{P}(Q,12b^2Q) \cong \mathcal{P}(Q,e) \cong \mathcal{O}_{\mathcal{Y}_1(D)}$$

of line bundles on $\mathcal{Y}_1(D)$. This is the desired trivialization of (5.1.5). The section F_r^{24} is obviously invariant under the action of the diamond operators on $\mathcal{Y}_1(D)$, and so descends to $\mathcal{Y}_0(D)$. □

5.2. Borcherds' quadratic identity. — For the remainder of § 5 we denote by $\chi_{\boldsymbol{k}} : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$ the Dirichlet character determined by the extension $\boldsymbol{k}/\mathbb{Q}$, abbreviate

$$\chi = \chi_{\boldsymbol{k}}^{n-2}, \tag{5.2.1}$$

and fix a weakly holomorphic form

$$f(\tau) = \sum_{m\gg-\infty} c(m)q^m \in M^{!,\infty}_{2-n}(D,\chi) \tag{5.2.2}$$

with $c(m) \in \mathbb{Z}$ for all $m \leq 0$.

For a proper cusp label representative Φ as in Definition 3.1.1, recall the self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice L_0 of signature $(n-2,0)$ defined by (3.1.4). The hermitian form on L_0 determines a quadratic form $Q(x)=\langle x,x\rangle$, with associated $\mathbb{Z}$-bilinear form $[x_1,x_2]=\mathrm{Tr}_{\boldsymbol{k}/\mathbb{Q}}\langle x_1,x_2\rangle$ of signature $(2n-4,0)$.

The modularity criterion of Theorem 4.2.3 implies the following identity of quadratic forms on $L_0\otimes\mathbb{R}$.

***Proposition 5.2.1* (Borcherds' quadratic identity).** — *For all $u\in L_0\otimes\mathbb{R}$,*

$$\sum_{x\in L_0} c(-Q(x))\cdot[u,x]^2=\frac{[u,u]}{2n-4}\sum_{x\in L_0} c(-Q(x))\cdot[x,x].$$

Proof. — The homogeneous polynomial

$$P(u,v)=[u,v]^2-\frac{[u,u]\cdot[v,v]}{2n-4}$$

on $L_0\otimes\mathbb{R}$ is harmonic in both variables u and v. For any fixed $u\in L_0\otimes\mathbb{R}$ there is a corresponding theta series

$$\theta(\tau,u,P)=\sum_{x\in L_0} P(u,x)\cdot q^{Q(x)}\in S_n(D,\chi).$$

The modularity criterion of Theorem 4.2.3 therefore shows that

$$\sum_{m>0} c(-m)\sum_{\substack{x\in L_0\\ Q(x)=m}}\left([u,x]^2-\frac{[u,u]\cdot[x,x]}{2n-4}\right)=0$$

for all $u\in L_0\otimes\mathbb{R}$. This implies the assertion. □

Recall from (3.6.2) that every $x\in L_0$ determines a diagram

(5.2.3)
$$\begin{array}{ccc}\mathcal{B}_\Phi & \xrightarrow{j_x} & \mathcal{E}\\ \downarrow & & \downarrow\\ \mathcal{A}_\Phi & \xrightarrow{j} & \mathcal{Y}_0(D),\end{array}$$

where, changing notation slightly from § 5.1, $\mathcal{Y}_0(D)$ is now the open modular curve over $\mathcal{O}_{\boldsymbol{k}}$. Recall also that $\mathcal{B}_\Phi$ carries a distinguished line bundle $\mathcal{L}_\Phi$ defined by (3.3.1), used to define the Fourier-Jacobi expansions of (3.8.1). We will use Borcherds' quadratic identity to relate the line bundle $\mathcal{L}_\Phi$ to the line bundle $\mathcal{J}_{0,1}$ of Jacobi forms on $\mathcal{E}$.

Proposition 5.2.2. — *The rational number*

(5.2.4)
$$\mathrm{mult}_\Phi(f)=\sum_{m>0}\frac{m\cdot c(-m)}{n-2}\cdot\#\{x\in L_0:Q(x)=m\}$$

lies in $\mathbb{Z}$, and there is a canonical isomorphism

$$\mathcal{L}_\Phi^{2\cdot\mathrm{mult}_\Phi(f)} \cong \bigotimes_{m>0} \bigotimes_{\substack{x\in L_0 \\ Q(x)=m}} j_x^*\mathcal{J}_{0,1}^{c(-m)}$$

of line bundles on $\mathcal{B}_\Phi$.

Proof. — Proposition 5.2.1 implies the equality of hermitian forms

$$\begin{aligned}\sum_{x\in L_0} c(-Q(x))\cdot\langle u,x\rangle\cdot\langle x,v\rangle &= \frac{\langle u,v\rangle}{2n-4}\sum_{x\in L_0} c(-Q(x))\cdot[x,x] \\ &= \langle u,v\rangle\cdot\mathrm{mult}_\Phi(f)\end{aligned}$$

for all $u,v\in L_0$. As L_0 is self-dual, we may choose u and v so that $\langle u,v\rangle = 1$, and the integrality of $\mathrm{mult}_\Phi(f)$ follows from the integrality of $c(-m)$.

Set $E = \mathcal{E}\times_{\mathcal{Y}_0(D)}\mathcal{A}_\Phi$, and use Proposition 3.4.4 to identify $\mathcal{B}_\Phi \cong E\otimes L_0$. The pullback of the line bundle

$$\bigotimes_{m>0} \bigotimes_{\substack{x\in L_0 \\ Q(x)=m}} j_x^*\mathcal{J}_{0,1}^{\otimes c(-m)} \cong \bigotimes_{x\in L_0} j_x^*\mathcal{J}_{0,1}^{\otimes c(-Q(x))}$$

via any T-valued point $a=\sum t_i\otimes y_i\in E(T)\otimes L_0$ is, in the notation of § 3.4,

$$\begin{aligned}\bigotimes_{x\in L_0}\mathcal{P}_E\Big(\sum_i\langle y_i,x\rangle t_i,\sum_j\langle y_j,x\rangle t_j\Big)^{\otimes c(-Q(x))} &\cong \bigotimes_{i,j}\bigotimes_{x\in L_0}\mathcal{P}_E\big(c(-Q(x))\cdot\langle y_i,x\rangle\cdot\langle x,y_j\rangle\cdot t_i,t_j\big) \\ &\cong \bigotimes_{i,j}\mathcal{P}_E\big(\langle y_i,y_j\rangle\cdot t_i,t_j\big)^{\otimes\mathrm{mult}_\Phi(f)} \\ &\cong \mathcal{P}_{E\otimes L_0}(a,a)^{\otimes\mathrm{mult}_\Phi(f)} \\ &\cong \mathcal{Q}_{E\otimes L_0}(a)^{\otimes 2\cdot\mathrm{mult}_\Phi(f)}.\end{aligned}$$

This, along with the isomorphism $\mathcal{Q}_{E\otimes L_0}\cong\mathcal{L}_\Phi$ of Proposition 3.4.4, proves that

$$\mathcal{L}_\Phi^{2\cdot\mathrm{mult}_\Phi(f)} \cong \mathcal{Q}_{E\otimes L_0}^{2\cdot\mathrm{mult}_\Phi(f)} \cong \bigotimes_{m>0} \bigotimes_{\substack{x\in L_0 \\ Q(x)=m}} j_x^*\mathcal{J}_{0,1}^{c(-m)}. \qquad \square$$

5.3. The unitary Borcherds product. — We now state our main results on Borcherds products.

For a prime p dividing D define

$$\gamma_p = \varepsilon_p^{-n}\cdot(D,p)_p^n\cdot\mathrm{inv}_p(V_p)\in\{\pm1,\pm i\}, \tag{5.3.1}$$

where $\mathrm{inv}_p(V_p)$ is the invariant of $V_p=\mathrm{Hom}_{\boldsymbol{k}}(W_0,W)\otimes_{\mathbb{Q}}\mathbb{Q}_p$ in the sense of (1.7.3), and

$$\varepsilon_p = \begin{cases} 1 & \text{if } p\equiv 1 \pmod 4 \\ i & \text{if } p\equiv 3 \pmod 4. \end{cases}$$

It is equal to the local Weil index of the Weil representation of $\mathrm{SL}_2(\mathbb{Z}_p)$ on $S_{L_p} \subset S(V_p)$, where V_p is viewed as a quadratic space as in (2.1.6). This is explained in more detail in § 8.1. For any r dividing D we define

$$\gamma_r = \prod_{p \mid r} \gamma_p. \tag{5.3.2}$$

Let $c_r(0)$ denote the constant term of f at the cusp ∞_r, as in Definition 4.1.1, and define

$$k = \sum_{r \mid D} \gamma_r \cdot c_r(0).$$

We will see later in Corollary 6.1.4 that all $\gamma_r \cdot c_r(0) \in \mathbb{Q}$.

For every $m > 0$ define a divisor

$$\mathcal{B}_{\mathrm{Kra}}(m) = \frac{m}{n-2} \sum_{\Phi} \#\{x \in L_0 : \langle x, x \rangle = m\} \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi) \tag{5.3.3}$$

with rational coefficients on $\mathcal{S}^*_{\mathrm{Kra}}$. Here the sum is over all K-equivalence classes of proper cusp label representatives Φ in the sense of § 3.2, L_0 is the hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-2, 0)$ defined by (3.1.4), and $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ is the boundary divisor of Theorem 3.7.1. It follows immediately from the definition (5.2.4) that

$$\sum_{m>0} c(-m) \cdot \mathcal{B}_{\mathrm{Kra}}(m) = \sum_{\Phi} \mathrm{mult}_{\Phi}(f) \cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi).$$

For $m > 0$ define the *total special divisor*

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m),$$

where $\mathcal{Z}^*_{\mathrm{Kra}}(m)$ is the special divisor defined on the open Shimura variety in § 2.5, and extended to the toroidal compactification in Theorem 3.7.1.

The following theorems assert the existence of Borcherds products on $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ having prescribed divisors and prescribed leading Fourier-Jacobi coefficients. Their proofs will occupy all of § 6.

Theorem 5.3.1. — *After possibly replacing the form f of (5.2.2) by a positive integer multiple, there is a rational section $\boldsymbol{\psi}(f)$ of the line bundle $\boldsymbol{\omega}^k$ on $\mathcal{S}^*_{\mathrm{Kra}}$ with the following properties.*

1. *In the generic fiber, the divisor of $\boldsymbol{\psi}(f)$ is*

$$\mathrm{div}(\boldsymbol{\psi}(f))_{/\boldsymbol{k}} = \sum_{m>0} c(-m) \cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)_{/\boldsymbol{k}}.$$

2. *For every proper cusp label representative Φ, the Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$, in the sense of (3.8.1), along the boundary divisor*

$$\Delta_{\Phi} \backslash \mathcal{B}_{\Phi} \cong \mathcal{S}^*_{\mathrm{Kra}}(\Phi)$$

has the form

$$\boldsymbol{\psi}(f) = q^{\mathrm{mult}_{\Phi}(f)} \sum_{\ell \geq 0} \boldsymbol{\psi}_{\ell} \cdot q^{\ell},$$

where ψ_ℓ is a rational section of $\boldsymbol{\omega}_\Phi^k \otimes \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)+\ell}$ over $\mathcal{B}_\Phi$.

3. *For any Φ as above, the leading coefficient ψ_0 admits a factorization*

$$\psi_0 = P_\Phi^\eta \otimes P_\Phi^{\mathrm{hor}} \otimes P_\Phi^{\mathrm{vert}},$$

where the three terms on the right are defined as follows.

   (a) *Proposition 3.5.1 provides us with an isomorphism*

$$\mathfrak{d}^{-1}\boldsymbol{\omega}_\Phi \cong j^*\boldsymbol{\omega}_\mathcal{Y}$$

   of line bundles on $\mathcal{A}_\Phi$, where $j : \mathcal{A}_\Phi \to \mathcal{Y}_0(D)$ is the morphism of (5.2.3), and $\boldsymbol{\omega}_\mathcal{Y} = \mathrm{Lie}(\mathcal{E})^{-1}$ is the pullback via $\mathcal{Y}_0(D) \to \mathcal{Y}$ of the line bundle of weight one modular forms. Pulling back the modular form $(2\pi i\eta^2)^{12}$ of Proposition 5.1.4 defines a nowhere vanishing section

$$j^*(2\pi i\eta^2)^k \in H^0(\mathcal{A}_\Phi, \mathfrak{d}^{-k}\boldsymbol{\omega}_\Phi^k).$$

   Using the canonical inclusion $\boldsymbol{\omega}_\Phi \subset \mathfrak{d}^{-1}\boldsymbol{\omega}_\Phi$, define

$$P_\Phi^\eta = j^*(2\pi i\eta^2)^k,$$

   but viewed as a rational section of $\boldsymbol{\omega}_\Phi^k$ over $\mathcal{A}_\Phi$. Denote in the same way its pullback to $\mathcal{B}_\Phi$.

   (b) *Recalling the function*

$$F_r^{24} = \bigotimes_{\substack{b\in\mathbb{Z}/D\mathbb{Z}\\ b\neq 0\\ rb=0}} (bQ)^*\Theta^{24}$$

   on $\mathcal{Y}_0(D)$ of Lemma 5.1.5, define a rational function

$$P_\Phi^{\mathrm{vert}} = \bigotimes_{\substack{r\mid D\\ r>1}} j^*F_r^{\gamma_r c_r(0)}$$

   on $\mathcal{A}_\Phi$, and again pull back to $\mathcal{B}_\Phi$.

   (c) *Using Proposition 5.2.2, define a rational section*

$$P_\Phi^{\mathrm{hor}} = \bigotimes_{m>0} \bigotimes_{\substack{x\in L_0\\ \langle x,x\rangle = m}} j_x^*\Theta^{c(-m)}$$

   of the line bundle $\mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ on $\mathcal{B}_\Phi$.

These properties determine $\psi(f)$ uniquely.

Remark 5.3.2. — In replacing f by a positive integer multiple, we are tacitly assuming that the constants $\gamma_r c_r(0)$ and $c(-m)$ are integer multiples of 24 for all $r \mid D$ and all $m > 0$. This is necessary in order to guarantee $k \in 12\mathbb{Z}$, and to make sense of the three factors $(2\pi i\eta_\Phi^2)^k$, P_Φ^{hor}, and P_Φ^{vert}.

In fact, we can strengthen Theorem 5.3.1 by computing precisely the divisor of $\psi(f)$ on the integral model $\mathcal{S}_{\mathrm{Kra}}^*$.

Theorem 5.3.3. — *The rational section $\psi(f)$ of $\boldsymbol{\omega}^k$ has divisor*

$$\begin{aligned}\operatorname{div}(\psi(f)) &= \sum_{m>0} c(-m)\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \\ &\quad + k\cdot\left(\frac{\mathrm{Exc}}{2} - \operatorname{div}(\delta)\right) + \sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}} \\ &\quad - \sum_{m>0}\frac{c(-m)}{2}\sum_{s\in\pi_0(\mathrm{Sing})}\#\{x\in L_s : \langle x,x\rangle = m\}\cdot\mathrm{Exc}_s,\end{aligned}$$

where $\mathfrak{p}\subset\mathcal{O}_{\boldsymbol{k}}$ is the unique prime above p, L_s is the self-dual Hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice defined in § 2.6, and $\mathrm{Exc}_s\subset\mathrm{Exc}$ is the fiber over the component $s\in\pi_0(\mathrm{Sing})$. Recall that $\delta=\sqrt{-D}\in\boldsymbol{k}$.

It is possible to give a statement analogous to Theorem 5.3.3 for the integral model $\mathcal{S}^*_{\mathrm{Pap}}$. To do this we first define, exactly as in (5.3.3), a Cartier divisor

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) = \mathcal{Y}^*_{\mathrm{Pap}}(m) + 2\mathcal{B}_{\mathrm{Pap}}(m)$$

with rational coefficients on $\mathcal{S}^*_{\mathrm{Pap}}$. Here $\mathcal{Y}^*_{\mathrm{Pap}}(m)$ is the Cartier divisor of Theorem § 3.7.1, and

$$\mathcal{B}_{\mathrm{Pap}}(m) = \frac{m}{n-2}\sum_{\Phi}\#\{x\in L_0 : \langle x,x\rangle = m\}\cdot\mathcal{S}^*_{\mathrm{Pap}}(\Phi).$$

It is clear from Theorem 3.7.1 that

$$2\cdot\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Kra}}(m) + \sum_{s\in\pi_0(\mathrm{Sing})}\#\{x\in L_s : \langle x,x\rangle = m\}\cdot\mathrm{Exc}_s, \tag{5.3.4}$$

where $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ denotes the pullback of $\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m)$ via $\mathcal{S}^*_{\mathrm{Kra}}\to\mathcal{S}^*_{\mathrm{Pap}}$.

The isomorphism

$$\boldsymbol{\omega}^2\cong\boldsymbol{\Omega}_{\mathrm{Kra}}\otimes\mathcal{O}(\mathrm{Exc})$$

of Theorem 3.7.1 identifies $\boldsymbol{\omega}^{2k}\cong\boldsymbol{\Omega}^k_{\mathrm{Kra}}$ in the generic fiber of $\mathcal{S}^*_{\mathrm{Kra}}$, allowing us to view $\psi(f)^2$ as a rational section of $\boldsymbol{\Omega}^k_{\mathrm{Kra}}$. As $\mathcal{S}^*_{\mathrm{Kra}}\to\mathcal{S}^*_{\mathrm{Pap}}$ is an isomorphism in the generic fiber, this section descends to a rational section of the line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$.

Theorem 5.3.4. — *When viewed as a rational section of $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$, the Borcherds product $\psi(f)^2$ has divisor*

$$\operatorname{div}(\psi(f)^2) = \sum_{m>0} c(-m)\cdot\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) - 2k\cdot\operatorname{div}(\delta) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_\mathfrak{p}}.$$

These three theorems will be proved simultaneously in § 6. Briefly, we will map our unitary Shimura variety $\mathrm{Sh}(G,\mathcal{D})$ to an orthogonal Shimura variety, where a meromorphic Borcherds product is already known to exist. If we pull back this Borcherds product to $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$, the leading coefficient in its analytic Fourier-Jacobi expansion is known from [**32**], up to multiplication by some unknown constants of absolute value 1.

By converting this analytic Fourier-Jacobi expansion into algebraic language, we will deduce the existence of a Borcherds product $\boldsymbol{\psi}(f)$ satisfying all of the properties stated in Theorem 5.3.1, up to some unknown constants in the leading Fourier-Jacobi coefficient. These unknown constants are the κ_Φ's appearing in Proposition 6.4.1. We then rescale the Borcherds product to make many $\kappa_\Phi = 1$ simultaneously.

After such a rescaling, the divisor of $\boldsymbol{\psi}(f)^2$ on $\mathcal{S}^*_{\mathrm{Pap}}$ can be computed from the Fourier-Jacobi expansions, and agrees with the divisor written in Theorem 5.3.4. Pulling back that divisor calculation via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$, and using Theorem 2.6.3, yields the divisor of Theorem 5.3.3.

Using the above divisor calculations, we prove that all κ_Φ are roots of unity. Thus, after replacing f by a multiple, which replaces $\boldsymbol{\psi}(f)$ by a power, we can force all $\kappa_\Phi = 1$, completing the proofs.

5.4. A divisor calculation at the boundary. — Let Φ be a proper cusp label representative for $(G,\mathcal{D})$. The following proposition is a key ingredient in the proofs of Theorems 5.3.1, 5.3.3, and 5.3.4.

Proposition 5.4.1. — *The rational sections P^η_Φ, P^{hor}_Φ, and P^{vert}_Φ of the line bundles $\boldsymbol{\omega}^k_\Phi$, $\mathcal{L}^{\mathrm{mult}_\Phi(f)}_\Phi$, and $\mathcal{O}_{\mathcal{B}_\Phi}$, respectively, have divisors*

$$\begin{aligned}\mathrm{div}(P^\eta_\Phi) &= -k\cdot\mathrm{div}(\delta)\\ \mathrm{div}(P^{\mathrm{hor}}_\Phi) &= \sum_{m>0} c(-m)\mathcal{Z}_\Phi(m)\\ \mathrm{div}(P^{\mathrm{vert}}_\Phi) &= \sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{B}_{\Phi/\mathbb{F}_\mathfrak{p}}.\end{aligned}$$

In particular, the divisor of P^{hor}_Φ is purely horizontal (Proposition 3.6.2), while the divisors of P^η_Φ and P^{vert}_Φ are purely vertical.

Proof. — By Proposition 5.1.4 the section

$$j^*(2\pi i\eta^2)^k \in H^0(\mathcal{A}_\Phi, \mathfrak{d}^{-k}\boldsymbol{\omega}^k_\Phi) \cong H^0(\mathcal{Y}_0(D), \boldsymbol{\omega}^k_\mathcal{Y})$$

has trivial divisor. When we use the inclusion $\boldsymbol{\omega}_\Phi \subset \mathfrak{d}^{-1}\boldsymbol{\omega}_\Phi$ to view it instead as a rational section P^η_Φ of $\boldsymbol{\omega}^k_\Phi$, its divisor becomes $\mathrm{div}(\delta^{-k})$. This proves the first equality.

To prove the remaining two equalities, let $\mathcal{E} \to \mathcal{Y}_0(D)$ be the universal elliptic curve, and denote by $e : \mathcal{Y}_0(D) \to \mathcal{E}$ the 0-section. It is an effective Cartier divisor on $\mathcal{E}$.

Directly from the definition of P^{hor}_Φ we have the equality

$$\mathrm{div}(P^{\mathrm{hor}}_\Phi) = \sum_{m>0}\frac{c(-m)}{24}\sum_{\substack{x\in L_0\\ \langle x,x\rangle = m}}\mathrm{div}(j^*_x\Theta^{24}).$$

Combining Proposition 5.1.4 with (3.6.1) shows that

$$\sum_{\substack{x\in L_0\\ \langle x,x\rangle=m}} \mathrm{div}(j_x^*\Theta^{24}) = \sum_{\substack{x\in L_0\\ \langle x,x\rangle=m}} 24j_x^*(e) = \sum_{\substack{x\in L_0\\ \langle x,x\rangle=m}} 24\mathcal{Z}_\Phi(x) = 24\mathcal{Z}_\Phi(m),$$

and the first equality follows immediately.

Recall the morphism $j : \mathcal{A}_\Phi \to \mathcal{Y}_0(D)$ of § 3.6. For the second equality it suffices to prove that the function F_r^{24} on $\mathcal{Y}_0(D)$ defined in Lemma 5.1.5 satisfies

$$\mathrm{div}(j^*F_r^{24}) = 24\sum_{p|r}\mathcal{A}_{\Phi/\mathbb{F}_\mathfrak{p}}. \tag{5.4.1}$$

Let $C\subset\mathcal{E}$ be the universal cyclic subgroup scheme of order D. For each $s\mid D$ denote by $C[s]\subset C$ the s-torsion subgroup, and by $C[s]^\times\subset C[s]$ the closed *subscheme of generators*. This is defined as follows. Noting that

$$C[s]=\prod_{p|s}C[p],$$

we define

$$C[s]^\times=\prod_{p|s}C[p]^\times,$$

where $C[p]^\times$ denotes the closed subscheme of generators of $C[p]$ as in [**21**, § 3.3]. Note that $C[p]^\times$ coincides with the subscheme of points of exact order p Z (see [**21**, Remark 3.3.2]) which allows the comparison with the formulation of the moduli problem in [**28**, Chapter 3]. Here and in the sequel, we are using [**21**, § 3.3] as a convenient summary of Oort-Tate theory (see also [**19**]) and of facts from [**28**] and [**15**].

There is an equality of Cartier divisors

$$\frac{1}{24}\mathrm{div}(F_r^{24}) = \big(C[r]-e\big)\times_{\mathcal{E},e}\mathcal{Y}_0(D) = \sum_{\substack{s|r\\ s\neq 1}}\big(C[s]^\times\times_{\mathcal{E},e}\mathcal{Y}_0(D)\big)$$

on $\mathcal{Y}_0(D)$. Indeed, one can check this after pullback to $\mathcal{Y}_1(D)$, where it is clear from Proposition 5.1.4, which asserts that the divisor of the section Θ^{24} appearing in the definition of F_r^{24} is equal to $24e$. If s is divisible by two distinct primes then

$$\big(C[s]^\times\times_{\mathcal{E},e}\mathcal{Y}_0(D)\big)=0,$$

and hence

$$\mathrm{div}(F_r^{24}) = 24\sum_{p|r}\big(C[p]^\times\times_{\mathcal{E},e}\mathcal{Y}_0(D)\big).$$

Now pull back this equality of Cartier divisors by j. Recall that j is defined as the composition

$$\mathcal{A}_\Phi\cong\mathcal{M}_{(1,0)}\xrightarrow{i}\mathcal{Y}_0(D),$$

where the isomorphism is the one provided by Proposition 3.4.4, and the arrow labeled i endows the universal CM elliptic curve $E \to \mathcal{M}_{(1,0)}$ with its cyclic subgroup scheme $E[\delta]$. Thus

$$i^*\mathrm{div}(F_r^{24}) = 24 \sum_{p|r} \big(E[\mathfrak{p}]^\times \times_{E,e} \mathcal{M}_{(1,0)}\big), \tag{5.4.2}$$

where $\mathfrak{p}$ denotes the unique prime ideal in $\mathcal{O}_{\boldsymbol{k}}$ over p.

Fix a geometric point $z : \mathrm{Spec}(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}) \to \mathcal{M}_{(1,0)}$, and view z also as a geometric point of E or $\mathcal{E}$ using

$$\mathcal{M}_{(1,0)} \xrightarrow{e} E \xrightarrow{i} \mathcal{E}.$$

Let $\mathcal{O}_{E,z}$ and $\mathcal{O}_{\mathcal{E},z}$ denote the completed étale local rings of E and $\mathcal{E}$ at z.

There is an isomorphism

$$\mathcal{O}_{\mathcal{E},z} \cong W[[X,Y,Z]]/(XY - w_p)$$

for some uniformizer w_p in the Witt ring $W = W(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})$. Compare with [**21**, Theorem 3.3.1]. Under this isomorphism the 0-section of $\mathcal{E}$ is defined by the equation $Z = 0$, and the divisor $C[p]^\times$ is defined by $Z^{p-1} - X = 0$. Moreover, noting that the completed étale local ring of $\mathcal{M}_{(1,0)}$ at z can be identified with $\mathcal{O}_{\boldsymbol{k}} \otimes W$, the natural map $\mathcal{O}_{\mathcal{E},z} \to \mathcal{O}_{E,z}$ is identified with the quotient map

$$W[[X,Y,Z]]/(XY - w_p) \to W[[X,Y,Z]]/(XY - w_p, X - uY)$$

for some $u \in W^\times$.

Under these identifications, the closed immersion

$$E[\mathfrak{p}]^\times \times_{E,e} \mathcal{M}_{(1,0)} \hookrightarrow \mathcal{M}_{(1,0)}$$

corresponds, on the level of completed local rings, to the quotient map

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{M}_{(1,0)},z} & = & W[[X,Y,Z]]/(XY - w_p, X - uY, Z) \\ & & \downarrow \\ \mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}} & = & W[[X,Y,Z]]/(XY - w_p, X - uY, Z, Z^{p-1} - X). \end{array}$$

This implies that

$$E[\mathfrak{p}]^\times \times_{E,e} \mathcal{M}_{(1,0)} = \mathcal{M}_{(1,0)/\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}}.$$

The equality (5.4.1) is clear from this and (5.4.2). □

6. Calculation of the Borcherds product divisor

In this section we prove Theorems 5.3.1, 5.3.3, and 5.3.4. We assume throughout that $n \geq 3$.

Throughout § 6 we keep f as in (5.2.2), and again assume that $c(-m) \in \mathbb{Z}$ for all $m \geq 0$. Recall that $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$ is endowed with the hermitian form $\langle x, y\rangle$ of

(2.1.5), as well as the $\mathbb{Q}$-bilinear form $[x, y]$ of (2.1.6). The associated quadratic form is

$$Q(x) = \langle x, x\rangle = \frac{[x, x]}{2}.$$

6.1. Vector-valued modular forms. — Let $L \subset V$ be any $\mathcal{O}_{\boldsymbol{k}}$-lattice, self-dual with respect to the hermitian form. The dual lattice of L with respect to the bilinear form $[.,.]$ is $L' = \mathfrak{d}^{-1}L$.

Let ω be the restriction to $\mathrm{SL}_2(\mathbb{Z})$ of the Weil representation of $\mathrm{SL}_2(\widehat{\mathbb{Q}})$ (associated with the standard additive character of $\mathbb{A}/\mathbb{Q}$) on the Schwartz-Bruhat functions on $L \otimes_{\mathbb{Z}} \mathbb{A}_f$. The restriction of ω to $\mathrm{SL}_2(\mathbb{Z})$ preserves the subspace $S_L = \mathbb{C}[L'/L]$ of Schwartz-Bruhat functions that are supported on $\widehat{L}'$ and invariant under translations by $\widehat{L}$. We obtain a representation

$$\omega_L : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(S_L).$$

For $\mu \in L'/L$, we denote by $\phi_\mu \in S_L$ the characteristic function of μ.

Remark 6.1.1. — The conjugate representation $\overline{\omega}_L$ on S_L, defined by

$$\overline{\omega}_L(\gamma)(\phi) = \overline{\omega_L(\gamma)(\overline{\phi})}$$

for $\phi \in S_L$, is the representation denoted ρ_L in **[4, 7, 9]**.

Recall the scalar valued modular form

$$f(\tau) = \sum_{m \gg -\infty} c(m) \cdot q^m \in M^{!,\infty}_{2-n}(D, \chi)$$

of (5.2.2), and continue to assume that $c(m) \in \mathbb{Z}$ for all $m \leq 0$. We will convert f into a $\mathbb{C}[L'/L]$-valued modular form $\tilde{f}$, to be used as input for Borcherds' construction of meromorphic modular forms on orthogonal Shimura varieties. The restriction of ω_L to $\Gamma_0(D)$ acts on the line $\mathbb{C} \cdot \phi_0$ via the character $\chi = \chi_{\boldsymbol{k}}^{n-2}$, and hence the induced function

$$\tilde{f}(\tau) = \sum_{\gamma \in \Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z})} (f\mid_{2-n} \gamma)(\tau) \cdot \omega_L(\gamma)^{-1}\phi_0 \tag{6.1.1}$$

is an S_L-valued weakly holomorphic modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2-n$ with representation ω_L. Its Fourier expansion is denoted

$$\tilde{f}(\tau) = \sum_{m \gg -\infty} \tilde{c}(m) \cdot q^m, \tag{6.1.2}$$

and we denote by $\tilde{c}(m, \mu)$ the value of $\tilde{c}(m) \in S_L$ at a coset $\mu \in L'/L$.

For any $r \mid D$ let $\gamma_r \in \{\pm 1, \pm i\}$ be as in (5.3.2), and let $c_r(m)$ be the m^{th} Fourier coefficient of f at the cusp ∞_r as in (4.1.1). For any $\mu \in L'/L$ define $r_\mu \mid D$ by

$$r_\mu = \prod_{\mu_p \neq 0} p, \tag{6.1.3}$$

where $\mu_p \in L'_p/L_p$ is the p-component of μ.

Proposition 6.1.2. — *For all $m \in \mathbb{Q}$ the coefficients $\tilde{c}(m) \in S_L$ satisfy*

$$\tilde{c}(m,\mu) = \begin{cases} \sum_{r_\mu | r | D} \gamma_r \cdot c_r(mr) & \text{if } m \equiv -Q(\mu) \pmod{\mathbb{Z}}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $m < 0$ we have

$$\tilde{c}(m,\mu) = \begin{cases} c(m) & \text{if } \mu = 0, \\ 0 & \text{if } \mu \neq 0, \end{cases}$$

and the constant term of $\tilde{f}$ is given by

$$\tilde{c}(0,\mu) = \sum_{r_\mu | r | D} \gamma_r \cdot c_r(0).$$

Proof. — The first formula is a special case of results of Scheithauer [**50**, Section 5]. For the reader's benefit we provide a direct proof in § 8.2.

The formula for the $m = 0$ coefficient is immediate from the general formula. So is the formula for $m < 0$, using the fact that the singularities of f are supported at the cusp at ∞. □

Remark 6.1.3. — The first formula of Proposition 6.1.2 actually also holds for f in the larger space $M^{!}_{2-n}(D,\chi)$.

Corollary 6.1.4. — *The coefficients $c(m)$ and $\tilde{c}(m)$ satisfy the following:*

1. *The $c(m)$ are rational for all m.*
2. *The $\tilde{c}(m,\mu)$ are rational for all m and μ, and are integral if $m < 0$.*
3. *For all $r \mid D$ we have $\gamma_r \cdot c_r(0) \in \mathbb{Q}$. In particular*

$$\tilde{c}(0,0) = \sum_{r|D} \gamma_r \cdot c_r(0) \in \mathbb{Q}.$$

Proof. — For the first claim, fix any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. The form $f^\sigma - f \in M^{!,\infty}_{2-n}$ is holomorphic at all cusps other than ∞, and vanishes at the cusp ∞ by the assumption that as $c(m) \in \mathbb{Z}$ for $m \leq 0$. Hence $f^\sigma - f$ is a holomorphic modular form of weight $2 - n < 0$, and therefore vanishes identically. It follows that $c(m) \in \mathbb{Q}$ for all m.

Now consider the second claim. In view of the Proposition 6.1.2 the coefficients $\tilde{c}(m,\mu)$ of $\tilde{f}$ with $m < 0$ are integers. Hence, for any $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, the function $\tilde{f}^\sigma - \tilde{f}$ is a holomorphic modular form of weight $2 - n < 0$, which is therefore identically 0. Therefore $\tilde{f}$ has rational Fourier coefficients.

The third claim follows from the second claim and the formula for the constant term of $\tilde{f}$ given in Proposition 6.1.2. □

6.2. Construction of the Borcherds product. — We now construct the Borcherds product $\boldsymbol{\psi}(f)$ of Theorem 5.3.1 as the pullback of a Borcherds product on the orthogonal Shimura variety defined by the quadratic space (V, Q). Useful references here include [**4, 7, 37, 22**].

After Corollary 6.1.4 we may replace f by a positive integer multiple in order to assume that $c(-m) \in 24\mathbb{Z}$ for all $m \geq 0$, and that $\gamma_r c_r(0) \in 24\mathbb{Z}$ for all $r \mid D$. In particular the rational number

$$k = \tilde{c}(0,0)$$

of Corollary 6.1.4 is an integer. Compare with Remark 5.3.2.

Define a hermitian domain

$$\tilde{\mathcal{D}} = \{w \in V(\mathbb{C}) : [w,w] = 0,\ [w,\overline{w}] < 0\}/\mathbb{C}^\times. \tag{6.2.1}$$

Let $\tilde{\boldsymbol{\omega}}^{\mathrm{an}}$ be the tautological bundle on $\tilde{\mathcal{D}}$, whose fiber at w is the line $\mathbb{C}w \subset V(\mathbb{C})$. The group of real points of $\mathrm{SO}(V)$ acts on (6.2.1), and this action lifts to an action on $\tilde{\boldsymbol{\omega}}^{\mathrm{an}}$.

As in Remark 2.1.2, any point $z \in \mathcal{D}$ determines a line $\mathbb{C}w \subset \varepsilon V(\mathbb{C})$. This construction defines a closed immersion

$$\mathcal{D} \hookrightarrow \tilde{\mathcal{D}}, \tag{6.2.2}$$

under which $\tilde{\boldsymbol{\omega}}^{\mathrm{an}}$ pulls back to the line bundle $\boldsymbol{\omega}^{\mathrm{an}}$ of § 2.4. The hermitian domain $\tilde{\mathcal{D}}$ has two connected components. Let $\tilde{\mathcal{D}}^+ \subset \tilde{\mathcal{D}}$ be the connected component containing $\mathcal{D}$.

For a fixed $g \in G(\mathbb{A}_f)$, we apply the constructions of § 6.1 to the input form f and the self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, g\mathfrak{a}) \subset V.$$

The result is a vector-valued modular form $\tilde{f}$ of weight $2-n$ and representation $\omega_L : \mathrm{SL}_2(\mathbb{Z}) \to S_L$. The form $\tilde{f}$ determines a Borcherds product $\Psi(\tilde{f})$ on $\tilde{\mathcal{D}}^+$; see [**4**, Theorem 13.3] and Theorem 7.2.4. For us it is more convenient to use the rescaled Borcherds product

$$\tilde{\boldsymbol{\psi}}_g(f) = (2\pi i)^{\tilde{c}(0,0)}\Psi(2\tilde{f}) \tag{6.2.3}$$

determined by $2\tilde{f}$. It is a meromorphic section of $(\tilde{\boldsymbol{\omega}}^{\mathrm{an}})^k$.

The subgroup $\mathrm{SO}(L)^+ \subset \mathrm{SO}(L)$ of elements preserving the component $\tilde{\mathcal{D}}^+$ acts on $\tilde{\boldsymbol{\psi}}_g(f)$ through a finite order character [**6**]. Replacing f by mf has the effect of replacing $\tilde{\boldsymbol{\psi}}_g(f)$ by $\tilde{\boldsymbol{\psi}}_g(f)^m$, and so after replacing f by a multiple we assume that $\tilde{\boldsymbol{\psi}}_g(f)$ is invariant under this action.

Denote by $\boldsymbol{\psi}_g(f)$ the pullback of $\tilde{\boldsymbol{\psi}}_g(f)$ via the map

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash\mathcal{D} \to \mathrm{SO}(L)^+\backslash\tilde{\mathcal{D}}^+$$

induced by (6.2.2). It is a meromorphic section of $(\boldsymbol{\omega}^{\mathrm{an}})^k$ on the connected component

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash\mathcal{D} \xrightarrow{z \mapsto (z,g)} \mathrm{Sh}(G,\mathcal{D})(\mathbb{C}).$$

By repeating the construction for all $g \in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$, we obtain a meromorphic section $\boldsymbol{\psi}(f)$ of the line bundle $(\boldsymbol{\omega}^{\mathrm{an}})^k$ on

$$\mathrm{Sh}(G,\mathcal{D})(\mathbb{C}) \cong \mathcal{S}_{\mathrm{Kra}}(\mathbb{C}).$$

After rescaling on every connected component by a complex constant of absolute value 1, this will be the section whose existence is asserted in Theorem 5.3.1.

Proposition 6.2.1. — *The divisor of* $\boldsymbol{\psi}(f)$ *is*

$$\mathrm{div}(\boldsymbol{\psi}(f)) = \sum_{m>0} c(-m) \cdot \mathcal{Z}_{\mathrm{Kra}}(m)(\mathbb{C}).$$

Proof. — The divisor of $\tilde{\boldsymbol{\psi}}_g(f)$ on $\tilde{\mathcal{D}}^+$ was computed by Borcherds in terms of the Fourier coefficients $\tilde{c}(-m)$ of $\tilde{f}$, and from this it is easy to obtain a formula for the divisor of $\boldsymbol{\psi}_g(f)$ on $\mathcal{D}$. See [**7**, Theorem 3.22] and [**22**, Theorem 8.1] for the details. The claim therefore follows by using Proposition 6.1.2 to rewrite this formula in terms of the $c(-m)$, and comparing with the explicit description of $\mathcal{Z}_{\mathrm{Kra}}(m)(\mathbb{C})$ stated in Remark 2.5.2. □

6.3. Analytic Fourier-Jacobi coefficients. — We return to the notation of § 3.9. Thus $\Phi = (P, g)$ is a proper cusp label representative for $(G, \mathcal{D})$, we have chosen

$$s : \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \to Q_\Phi$$

as in Lemma 3.9.2, and have fixed $a \in \widehat{\boldsymbol{k}}^\times$. This data determines a lattice

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(s(a)g\mathfrak{a}_0, s(a)g\mathfrak{a}),$$

and Witt decompositions

$$V = V_{-1} \oplus V_0 \oplus V_1, \quad L = L_{-1} \oplus L_0 \oplus L_1.$$

Choose bases $\mathrm{e}_{-1}, \mathrm{f}_{-1} \in L_{-1}$ and $\mathrm{e}_1, \mathrm{f}_1 \in L_1$ as in § 3.9.

Imitating the construction of (3.9.16) yields a commutative diagram

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{(6.2.2)} & \tilde{\mathcal{D}}^+ \\
{\scriptstyle w\mapsto(w_0,\xi)}\downarrow & & \downarrow{\scriptstyle w\mapsto(\tau,w_0,\xi)} \\
\varepsilon V_0(\mathbb{C}) \times \mathbb{C} & \longrightarrow & \mathfrak{H} \times V_0(\mathbb{C}) \times \mathbb{C}
\end{array}$$

in which the vertical arrows are open immersions, and the horizontal arrows are closed immersions. The vertical arrow on the right is defined as follows: Any $w \in \tilde{\mathcal{D}}$ pairs nontrivially with the isotropic vector f_{-1}, and so may be scaled so that $[w, \mathrm{f}_{-1}] = 1$. This allows us to identify

$$\tilde{\mathcal{D}} = \{w \in V(\mathbb{C}) : [w,w] = 0,\ [w,\overline{w}] < 0,\ [w,\mathrm{f}_{-1}] = 1\}.$$

Using this model, any $w \in \tilde{\mathcal{D}}^+$ has the form

$$w = -\xi\mathrm{e}_{-1} + (\tau\xi - Q(w_0))\mathrm{f}_{-1} + w_0 + \tau\mathrm{e}_1 + \mathrm{f}_1$$

with $\tau \in \mathfrak{H}$, $w_0 \in V_0(\mathbb{C})$, and $\xi \in \mathbb{C}$. The bottom horizontal arrow is $(w_0, \xi) \mapsto (\tau, w_0, \xi)$, where τ is determined by the relation (3.9.6).

The construction above singles out a nowhere vanishing section of $\tilde{\boldsymbol{\omega}}^{\mathrm{an}}$, whose value at an isotropic line $\mathbb{C}w$ is the unique vector in that line with $[w, \mathrm{f}_{-1}] = 1$. As in the discussion leading to (3.9.18), we obtain a trivialization

$$[\,\cdot\,, \mathrm{f}_{-1}] : \tilde{\boldsymbol{\omega}}^{\mathrm{an}} \cong \mathcal{O}_{\tilde{\mathcal{D}}^+}.$$

Now consider the Borcherds product $\tilde{\psi}_{s(a)g}(f)$ on $\tilde{\mathcal{D}}^+$ determined by the lattice L above (that is, replace g by $s(a)g$ throughout § 6.2). It is a meromorphic section of $(\tilde{\boldsymbol{\omega}}^{\mathrm{an}})^k$, and we use the trivialization above to identify it with a meromorphic function. In a neighborhood of the rational boundary component associated to the isotropic plane $V_{-1} \subset V$, this meromorphic function has a product expansion.

Proposition 6.3.1 **([32]).** — *There are positive constants A and B with the following property: For all points $w \in \tilde{\mathcal{D}}^+$ satisfying*

$$\mathrm{Im}(\xi) - \frac{Q(\mathrm{Im}(w_0))}{\mathrm{Im}(\tau)} > A\,\mathrm{Im}(\tau) + \frac{B}{\mathrm{Im}(\tau)},$$

there is a factorization

$$\tilde{\psi}_{s(a)g}(f) = \kappa \cdot (2\pi i)^k \cdot \eta^{2k}(\tau) \cdot e^{2\pi i I \xi} \cdot P_0(\tau) \cdot P_1(\tau, w_0) \cdot P_2(\tau, w_0, \xi)$$

in which $\kappa \in \mathbb{C}^\times$ has absolute value 1, η is the Dedekind η-function, and

$$I = \frac{1}{12} \sum_{b \in \mathbb{Z}/D\mathbb{Z}} \tilde{c}\left(0, -\frac{b}{D}\mathrm{f}_{-1}\right) - 2 \sum_{m>0} \sum_{x \in L_0} c(-m) \cdot \sigma_1(m - Q(x)).$$

The factors P_0 and P_1 are defined by

$$P_0(\tau) = \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0}} \Theta\left(\tau, \frac{b}{D}\right)^{\tilde{c}(0, \frac{b}{D}\mathrm{f}_{-1})}$$

and

$$P_1(\tau, w_0) = \prod_{m>0} \prod_{\substack{x \in L_0 \\ Q(x)=m}} \Theta\big(\tau, [w_0, x]\big)^{c(-m)}.$$

The remaining factor is

$$P_2(\tau, w_0, \xi) = \prod_{\substack{x \in \delta^{-1} L_0 \\ a \in \mathbb{Z} \\ b \in \mathbb{Z}/D\mathbb{Z} \\ c \in \mathbb{Z}_{>0}}} \Big(1 - e^{2\pi i c \xi} e^{2\pi i a \tau} e^{2\pi i b/D} e^{-2\pi i [x, w_0]}\Big)^{2 \cdot \tilde{c}(ac - Q(x), \mu)},$$

where $\mu = -a\mathrm{e}_{-1} - \frac{b}{D}\mathrm{f}_{-1} + x + c\mathrm{e}_1 \in \delta^{-1}L/L$.

Proof. — This is just a restatement of [**32**, Corollary 2.3], with some simplifications arising from the fact that the vector-valued form $\tilde{f}$ used to define the Borcherds product is induced from a scalar-valued form via (6.1.1).

A more detailed description of how these expressions arise from the general formulas in [**32**] is given in the appendix. □

If we pull back the formula for the Borcherds product $\tilde{\boldsymbol{\psi}}_{s(a)g}(f)$ found in Proposition 6.3.1 via the closed immersion (6.2.2), we obtain a formula for the Borcherds product $\boldsymbol{\psi}_{s(a)g}(f)$ on the connected component

$$(G(\mathbb{Q}) \cap s(a)gKg^{-1}s(a)^{-1})\backslash \mathcal{D} \xrightarrow{z\mapsto(z,s(a)g)} \mathrm{Sh}(G,\mathcal{D})(\mathbb{C}),$$

from which we can read off the leading analytic Fourier-Jacobi coefficient.

Corollary 6.3.2. — *The analytic Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$, in the sense of* (3.9.18)*, has the form*

$$\boldsymbol{\psi}_{s(a)g}(f) = \sum_{\ell \geq I} \mathrm{FJ}_{\ell}^{(a)}(\boldsymbol{\psi}(f))(w_0) \cdot q^{\ell},$$

where I is the integer of Proposition 6.3.1. The leading coefficient $\mathrm{FJ}_I^{(a)}(\boldsymbol{\psi}(f))$, viewed as a function on $V_0(\mathbb{R})$ as in the discussion leading to (3.9.14)*, is given by*

$$\mathrm{FJ}_I^{(a)}(\boldsymbol{\psi}(f))(w_0) = \kappa \cdot (2\pi i)^k \cdot \eta(\tau)^{2k} \cdot P_0(\tau) \cdot P_1(\tau, w_0), \tag{6.3.1}$$

where $\tau \in \mathfrak{H}$ is determined by (3.9.6),

$$P_0(\tau) = \prod_{r|D} \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b\neq 0 \\ rb=0}} \Theta\left(\tau, \frac{b}{D}\right)^{\gamma_r c_r(0)}$$

and

$$P_1(\tau, w_0) = \prod_{m>0} \prod_{\substack{x\in L_0 \\ Q(x)=m}} \Theta\big(\tau, \langle w_0, x\rangle\big)^{c(-m)}.$$

The constant $\kappa \in \mathbb{C}$, which depends on both Φ and a, has absolute value 1.

Proof. — Using Proposition 6.3.1, the pullback of $\tilde{\boldsymbol{\psi}}_{s(a)g}(f)$ via (6.2.2) factors as a product

$$\boldsymbol{\psi}_{s(a)g}(f) = \kappa \cdot (2\pi i)^k \cdot \eta^{2k}(\tau) \cdot e^{2\pi i \xi I} \cdot P_0(\tau) P_1(\tau, w_0) P_2(\tau, w_0, \xi),$$

where $\xi \in \mathbb{C}^{\times}$ and $w_0 \in V(\mathbb{R}) \cong \varepsilon V(\mathbb{C})$. The parameter $\tau \in \mathfrak{H}$ is now fixed, determined by (3.9.6). The equality

$$\prod_{\substack{b\in\mathbb{Z}/D\mathbb{Z} \\ b\neq 0}} \Theta\left(\tau, \frac{b}{D}\right)^{\tilde{c}(0, \frac{b}{D}\mathfrak{f}_{-1})} = \prod_{r|D} \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b\neq 0 \\ rb=0}} \Theta\left(\tau, \frac{b}{D}\right)^{\gamma_r c_r(0)}$$

follows from Proposition 6.1.2, and allows us to rewrite P_0 in the stated form. To rewrite the factor P_1 in terms of $\langle .,. \rangle$ instead of $[.,.]$, use the commutative diagram of Remark 3.9.4. Finally, as $\mathrm{Im}(\xi) \to \infty$, so $q = e^{2\pi i \xi} \to 0$, the factor P_2 converges to 1. This P_2 does not contribute to the leading Fourier-Jacobi coefficient. □

Proposition 6.3.3. — *The integer I defined in Proposition 6.3.1 is equal to the integer* $\mathrm{mult}_\Phi(f)$ *defined by* (5.2.4), *and the product* (6.3.1) *satisfies the transformation law* (3.9.14) *with* $\ell = \mathrm{mult}_\Phi(f)$.

Proof. — The Fourier-Jacobi coefficient $\mathrm{FJ}_I^{(a)}(\boldsymbol{\psi}(f))$ appearing on the left hand side of (6.3.1) is, by definition, a section of the line bundle $\mathcal{Q}^I_{E^{(a)}\otimes L}$ on $E^{(a)} \otimes L$. When viewed as a function of the variable $w_0 \in V_0(\mathbb{R})$ using our explicit coordinates, it therefore satisfies the transformation law (3.9.14) with $\ell = I$.

Now consider the right hand side of (6.3.1), and recall that τ is fixed, determined by (3.9.6). In our explicit coordinates the function $\Theta(\tau, \langle w_0, x\rangle)^{24}$ of $w_0 \in V_0(\mathbb{R})$ is identified with a section of the line bundle $j_x^*\mathcal{J}_{0,12}$ on $E^{(a)} \otimes L$; this is clear from the definition of j_x in (3.6.2), and Proposition 5.1.4. Thus $P_1(\tau, w_0)$, and hence the entire right hand side of (6.3.1), defines a section of the line bundle

$$\bigotimes_{m>0} \bigotimes_{\substack{x\in L_0\\ Q(x)=m}} j_x^*\mathcal{J}_{0,1}^{c(-m)/2} \cong \mathcal{L}_\Phi^{2\cdot \mathrm{mult}_\Phi(f/2)} \cong \mathcal{Q}^{\mathrm{mult}_\Phi(f)}_{E^{(a)}\otimes L},$$

where the isomorphisms are those of Proposition 5.2.2 and Proposition 3.4.4. This implies that the right hand side of (6.3.1) satisfies the transformation law (3.9.14) with $\ell = \mathrm{mult}_\Phi(f)$.

A function on $V_0(\mathbb{R})$ cannot satisfy the transformation law (3.9.14) for two different values of ℓ, and hence $I = \mathrm{mult}_\Phi(f)$. Note that we are using here the standing hypothesis $n > 2$; if $n = 2$ then $V_0(\mathbb{R}) = 0$, and the transformation law (3.9.14) is vacuous.

For a more direct proof of the proposition, see § 8.4. □

6.4. Algebraization and descent. — The following weak form of Theorem 5.3.1 shows that $\boldsymbol{\psi}(f)$ is algebraic, and provides an algebraic interpretation of its leading Fourier-Jacobi coefficients.

Proposition 6.4.1. — *The meromorphic section $\boldsymbol{\psi}(f)$ is the analytification of a rational section of the line bundle $\boldsymbol{\omega}^k$ on $\mathcal{S}_{\mathrm{Kra}/\mathbb{C}}$. This rational section satisfies the following properties:*

1. *When viewed as a rational section over the toroidal compactification,*
$$\mathrm{div}(\boldsymbol{\psi}(f)) = \sum_{m>0} c(-m)\cdot \mathcal{Z}^*_{\mathrm{Kra}}(m)_{/\mathbb{C}} + \sum_{\Phi} \mathrm{mult}_\Phi(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)_{/\mathbb{C}}.$$
2. *For every proper cusp label representative Φ, the Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$ along $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)_{/\mathbb{C}}$, in the sense of § 3.8, has the form*
$$\boldsymbol{\psi}(f) = q^{\mathrm{mult}_\Phi(f)} \sum_{\ell\geq 0} \boldsymbol{\psi}_\ell \cdot q^\ell.$$
3. *The leading coefficient $\boldsymbol{\psi}_0$, a rational section of $\boldsymbol{\omega}_\Phi^k \otimes \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ over $\mathcal{B}_{\Phi/\mathbb{C}}$, factors as*
$$\boldsymbol{\psi}_0 = \kappa_\Phi \otimes P_\Phi^\eta \otimes P_\Phi^{\mathrm{hor}} \otimes P_\Phi^{\mathrm{vert}}$$

for a unique section

$$\kappa_\Phi \in H^0(\mathcal{A}_{\Phi/\mathbb{C}}, \mathcal{O}^\times_{\mathcal{A}_\Phi/\mathbb{C}}).$$

This section satisfies $|\kappa_\Phi(z)| = 1$ *at every complex point* $z \in \mathcal{A}_\Phi(\mathbb{C})$*. (The other factors appearing on the right hand side were defined in Theorem 5.3.1.)*

Proof. — Using Corollary 6.3.2 and Proposition 6.3.3, one sees that $\boldsymbol{\psi}(f)$ extends to a meromorphic section of $\boldsymbol{\omega}^k$ over the toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$, vanishing to order $I = \mathrm{mult}_\Phi(f)$ along the closed stratum

$$\mathcal{S}^*_{\mathrm{Kra}}(\Phi)_{/\mathbb{C}} \subset \mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$$

indexed by a proper cusp label representative Φ.

The calculation of the divisor of $\boldsymbol{\psi}(f)$ over the open Shimura variety $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ is Proposition 6.2.1. The algebraicity claim now follows from GAGA (using the fact that the divisor is already known to be algebraic), proving all parts of the first claim. The second and third claims are just a translation of Corollary 6.3.2 into the algebraic language of Theorem 5.3.1, using the explicit coordinates of § 3.9 and the change of notation $(2\pi i\eta^2)^k = P^\eta_\Phi$, $P_0 = P^{\mathrm{vert}}_\Phi$ and $P_1 = P^{\mathrm{hor}}_\Phi$. □

We now prove that $\boldsymbol{\psi}(f)$, after minor rescaling, descends to $\boldsymbol{k}$. This can be deduced from the analogous statement about Borcherds products on orthogonal Shimura varieties proved in [**26**], but in the unitary case there is a much more elementary proof. This will require the following two lemmas.

Lemma 6.4.2. — *The geometric components of* $\mathrm{Sh}(G, \mathcal{D})$ *are defined over the Hilbert class field* $\boldsymbol{k}^{\mathrm{Hilb}}$ *of* $\boldsymbol{k}$*, and each such component has trivial stabilizer in* $\mathrm{Gal}(\boldsymbol{k}^{\mathrm{Hilb}}/\boldsymbol{k})$*.*

Proof. — One could prove this using Deligne's reciprocity law for connected components of Shimura varieties [**43**, § 13], but it also follows easily from the theory of toroidal compactification.

Our assumption that $n > 2$ guarantees that every connected component of $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$ contains some connected component of the boundary. It is a part [(8)] of Theorem 3.7.1 that all such boundary components are defined over the Hilbert class field, and it follows that the same is true for components of $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$. The same is therefore true for the components of the interior

$$\mathcal{S}_{\mathrm{Kra}/\mathbb{C}} \cong \mathrm{Sh}(G, \mathcal{D})_{/\mathbb{C}}.$$

The claim about stabilizers follows from the open and closed immersion

$$\mathrm{Sh}(G, \mathcal{D}) \subset M_{(1,0)} \times_{\boldsymbol{k}} M_{(n-1,1)}$$

of (2.2.2), along with the classical fact (from the theory of complex multiplication of elliptic curves) that the geometric components of $M_{(1,0)}$ form a simply transitive $\mathrm{Gal}(\boldsymbol{k}^{\mathrm{Hilb}}/\boldsymbol{k})$-set. □

(8) This particular part of Theorem 3.7.1 follows from the reciprocity law for the boundary components of $\mathcal{M}^{\mathrm{Pap}}_{(n-1,1)}$ proved in [**24**, Proposition 2.6.2].

The lemma allows us to choose a set of connected components

$$\{X_i\} \subset \pi_0\big(\mathrm{Sh}(G, \mathcal{D})_{/\boldsymbol{k}^{\mathrm{Hilb}}}\big)$$

in such a way that

$$\mathrm{Sh}(G, \mathcal{D})_{/\boldsymbol{k}^{\mathrm{Hilb}}} = \bigsqcup_i \bigsqcup_{\sigma \in \mathrm{Gal}(\boldsymbol{k}^{\mathrm{Hilb}}/\boldsymbol{k})} \sigma(X_i).$$

For each index i, pick $g_i \in G(\mathbb{A}_f)$ in such a way that $X_i(\mathbb{C})$ is equal to the image of

$$(G(\mathbb{Q}) \cap g_i K g_i^{-1})\backslash \mathcal{D} \xrightarrow{z \mapsto (z, g_i)} \mathrm{Sh}(G, \mathcal{D})(\mathbb{C}).$$

Choose an isotropic $\boldsymbol{k}$-line $J \subset W$, let $P \subset G$ be its stabilizer, and define a proper cusp label representative $\Phi_i = (P, g_i)$. The above choices pick out one boundary component on every component of the toroidal compactification, as the following lemma demonstrates.

Lemma 6.4.3. — *The natural maps*

$$\begin{array}{ccccccc}
 & & & & \bigsqcup_i \mathcal{S}^*_{\mathrm{Kra}}(\Phi_i) & \longrightarrow & \mathcal{S}^*_{\mathrm{Kra}} \\
 & & & \nearrow & \downarrow \cong & & \downarrow \\
\bigsqcup_i \mathcal{A}_{\Phi_i} & \longleftarrow & \bigsqcup_i \mathcal{B}_{\Phi_i} & & & & \\
 & & & \searrow & & & \\
 & & & & \bigsqcup_i \mathcal{S}^*_{\mathrm{Pap}}(\Phi_i) & \longrightarrow & \mathcal{S}^*_{\mathrm{Pap}}
\end{array}$$

induce bijections on connected components. The same is true after base change to $\boldsymbol{k}$ *or* $\mathbb{C}$.

Proof. — Let $X_i^* \subset \mathcal{S}^*_{\mathrm{Pap}}(\mathbb{C})$ be the closure of X_i. By examining the complex analytic construction of the toroidal compactification **[24, 39, 47]**, one sees that some component of the divisor $\mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C})$ lies on X_i^*.

Recall from Theorem 3.7.1 that the components of $\mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C})$ are defined over $\boldsymbol{k}^{\mathrm{Hilb}}$, and that the action of $\mathrm{Gal}(\boldsymbol{k}^{\mathrm{Hilb}}/\boldsymbol{k})$ is simply transitive. It follows immediately that

$$\mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C}) \subset \bigsqcup_{\sigma \in \mathrm{Gal}(\boldsymbol{k}^{\mathrm{Hilb}}/\boldsymbol{k})} \sigma(X_i^*),$$

and the inclusion induces a bijection on components. By Proposition 3.2.1 and the isomorphism of Proposition 3.3.3, the quotient map

$$\mathcal{C}_{\Phi}(\mathbb{C}) \to \Delta_{\Phi_i}\backslash \mathcal{C}_{\Phi_i}(\mathbb{C})$$

induces a bijection on connected components, and both maps $\mathcal{C}_\Phi \to \mathcal{B}_\Phi \to \mathcal{A}_\Phi$ have geometrically connected fibers (the first is a $\mathbb{G}_m$-torsor, and the second is an abelian scheme). We deduce that all maps in

$$\mathcal{A}_{\Phi_i}(\mathbb{C}) \leftarrow \mathcal{B}_{\Phi_i}(\mathbb{C}) \to \Delta_{\Phi_i}\backslash \mathcal{B}_{\Phi_i}(\mathbb{C}) \cong \mathcal{S}^*_{\mathrm{Kra}}(\Phi_i)(\mathbb{C}) \cong \mathcal{S}^*_{\mathrm{Pap}}(\Phi_i)(\mathbb{C})$$

induce bijections on connected components.

The above proves the claim over $\mathbb{C}$, and the claim over $\boldsymbol{k}$ follows formally from this. The claim for integral models follows from the claim in the generic fiber, using the fact that all integral models in question are normal and flat over $\mathcal{O}_{\boldsymbol{k}}$. □

Proposition 6.4.4. — *After possibly rescaling by a constant of absolute value* 1 *on every connected component of $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$, the Borcherds product $\boldsymbol{\psi}(f)$ is defined over $\boldsymbol{k}$, and the sections of Proposition 6.4.1 satisfy*

$$\kappa_\Phi \in H^0(\mathcal{A}_{\Phi/\boldsymbol{k}}, \mathcal{O}^\times_{\mathcal{A}_\Phi/\boldsymbol{k}})$$

for all proper cusp label representatives Φ. Furthermore, we may arrange that $\kappa_{\Phi_i} = 1$ for all i.

Proof. — Lemma 6.4.3 establishes a bijection between the connected components of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$ and the finite set $\bigsqcup_i \mathcal{A}_{\Phi_i}(\mathbb{C})$. On the component indexed by $z \in \mathcal{A}_{\Phi_i}(\mathbb{C})$, rescale $\boldsymbol{\psi}(f)$ by $\kappa_{\Phi_i}(z)^{-1}$. For this rescaled $\boldsymbol{\psi}(f)$ we have $\kappa_{\Phi_i} = 1$ for all i.

Suppose $\sigma \in \mathrm{Aut}(\mathbb{C}/\boldsymbol{k})$. The first claim of Proposition 6.4.1 implies that the divisor of $\boldsymbol{\psi}(f)$, when computed on the compactification $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$, is defined over $\boldsymbol{k}$. Therefore $\sigma(\boldsymbol{\psi}(f))/\boldsymbol{\psi}(f)$ has trivial divisor, and so is constant on every connected component.

By the third claim of Proposition 6.4.1, the leading coefficient in the Fourier-Jacobi expansion of $\boldsymbol{\psi}(f)$ along the boundary stratum $\mathcal{S}^*_{\mathrm{Kra}}(\Phi_i)$ is

$$\boldsymbol{\psi}_0 = P^\eta_{\Phi_i} \otimes P^{\mathrm{hor}}_{\Phi_i} \otimes P^{\mathrm{vert}}_{\Phi_i},$$

which is defined over $\boldsymbol{k}$. From this it follows that $\sigma(\boldsymbol{\psi}(f))/\boldsymbol{\psi}(f)$ is identically equal to 1 on every connected component of $\mathcal{S}^*_{\mathrm{Kra}/\mathbb{C}}$ meeting this boundary stratum. Varying i and using Lemma 6.4.3 shows that $\sigma(\boldsymbol{\psi}(f)) = \boldsymbol{\psi}(f)$.

This proves that $\boldsymbol{\psi}(f)$ is defined over $\boldsymbol{k}$, hence so are all of its Fourier-Jacobi coefficients along *all* boundary strata $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$. Appealing again to the calculation of the leading Fourier-Jacobi coefficient of Proposition 6.4.1, we deduce finally that κ_Φ is defined over $\boldsymbol{k}$ for all Φ. □

6.5. Calculation of the divisor, and completion of the proof. — The Borcherds product $\boldsymbol{\psi}(f)$ on $\mathcal{S}^*_{\mathrm{Kra}/\boldsymbol{k}}$ of Proposition 6.4.4 may be viewed as a rational section of $\boldsymbol{\omega}^k$ on the integral model $\mathcal{S}^*_{\mathrm{Kra}}$.

Let Φ be any proper cusp label representative. Combining Propositions 6.4.1 and 6.4.4 shows that the leading Fourier-Jacobi coefficient of $\boldsymbol{\psi}(f)$ along the boundary divisor $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ is

$$\boldsymbol{\psi}_0 = \kappa_\Phi \otimes P^\eta_\Phi \otimes P^{\mathrm{hor}}_\Phi \otimes P^{\mathrm{vert}}_\Phi. \tag{6.5.1}$$

Recall that this is a rational section of $\boldsymbol{\omega}^k_\Phi \otimes \mathcal{L}_\Phi^{\mathrm{mult}_\Phi(f)}$ on $\mathcal{B}_\Phi$. Here, by mild abuse of notation, we are viewing κ_Φ as a rational function on $\mathcal{A}_\Phi$, and denoting in the same way its pullback to any step in the tower

$$\mathcal{C}^*_\Phi \xrightarrow{\pi} \mathcal{B}_\Phi \to \mathcal{A}_\Phi.$$

Lemma 6.5.1. — *Recall that π has a canonical section $\mathcal{B}_\Phi \hookrightarrow \mathcal{C}_\Phi^*$, realizing $\mathcal{B}_\Phi$ as a divisor on $\mathcal{C}_\Phi^*$. If we use the isomorphism* (3.7.1) *to view $\boldsymbol{\psi}(f)$ as a rational section of the line bundle $\boldsymbol{\omega}_\Phi^k$ on the formal completion $(\mathcal{C}_\Phi^*)^\wedge_{\mathcal{B}_\Phi}$, its divisor satisfies*

$$\begin{aligned}\operatorname{div}(\boldsymbol{\psi}(f)) &= \operatorname{div}(\delta^{-k}\kappa_\Phi) + \operatorname{mult}_\Phi(f)\cdot\mathcal{B}_\Phi \\ &\quad + \sum_{m>0} c(-m)\mathcal{Z}_\Phi(m) + \sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\pi^*(\mathcal{B}_{\Phi/\mathbb{F}_\mathfrak{p}}).\end{aligned}$$

Proof. — The key step is to prove that the divisor of $\boldsymbol{\psi}(f)$ can be computed from the divisor of its leading Fourier-Jacobi coefficient $\boldsymbol{\psi}_0$ by the formula

$$\operatorname{div}(\boldsymbol{\psi}(f)) = \pi^*\operatorname{div}(\boldsymbol{\psi}_0) + \operatorname{mult}_\Phi(f)\cdot\mathcal{B}_\Phi. \tag{6.5.2}$$

Recalling the tautological section q with divisor $\mathcal{B}_\Phi$ from Remark 3.8.1, consider the rational section

$$R = q^{-\operatorname{mult}_\Phi(f)}\cdot\boldsymbol{\psi}(f) = \sum_{i\geq 0}\boldsymbol{\psi}_i\cdot q^i$$

of $\boldsymbol{\omega}_\Phi^k\otimes\pi^*\mathcal{L}_\Phi^{\operatorname{mult}_\Phi(f)}$ on the formal completion $(\mathcal{C}_\Phi^*)^\wedge_{\mathcal{B}_\Phi}$.

We claim that $\operatorname{div}(R) = \pi^*\Delta$ for *some* divisor Δ on $\mathcal{B}_\Phi$. Indeed, whatever $\operatorname{div}(R)$ is, it may decomposed as a sum of horizontal and vertical components. We know from Theorem 3.7.1 and Proposition 6.4.1 that the horizontal part is a linear combination of the divisors $\mathcal{Z}_\Phi(m)$ on $\mathcal{C}_\Phi^*$ defined by (3.6.1); these divisors are, by construction, pullbacks of divisors on $\mathcal{B}_\Phi$. On the other hand, the morphism $\mathcal{C}_\Phi^*\to\mathcal{B}_\Phi$ is the total space of a line bundle, and hence is smooth with connected fibers. Thus *every* vertical divisor on $\mathcal{C}_\Phi^*$, and in particular the vertical part of $\operatorname{div}(R)$, is the pullback of some divisor on $\mathcal{B}_\Phi$.

Denoting by $i:\mathcal{B}_\Phi\hookrightarrow\mathcal{C}_\Phi^*$ the zero section, we compute

$$\Delta = i^*\pi^*\Delta = i^*\operatorname{div}(R) = \operatorname{div}(i^*R) = \operatorname{div}(\boldsymbol{\psi}_0).$$

Pulling back by π proves that $\operatorname{div}(R) = \pi^*\operatorname{div}(\boldsymbol{\psi}_0)$, and (6.5.2) follows.

We now compute the divisor of $\boldsymbol{\psi}_0$ on $\mathcal{B}_\Phi$ using (6.5.1). The divisors of P_Φ^η, P_Φ^{hor}, and P_Φ^{vert} were computed in Proposition 5.4.1, which shows that on $\mathcal{B}_\Phi$ we have the equality

$$\operatorname{div}(\boldsymbol{\psi}_0) = \operatorname{div}(\delta^{-k}\kappa_\Phi) + \sum_{m>0} c(-m)\mathcal{Z}_\Phi(m) + \sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{B}_{\Phi/\mathbb{F}_\mathfrak{p}}.$$

Combining this with (6.5.2) completes the proof. □

Proposition 6.5.2. — *When viewed as a rational section of $\boldsymbol{\omega}^k$ on $\mathcal{S}_{\mathrm{Kra}}^*$, the Borcherds product $\boldsymbol{\psi}(f)$ has divisor*

$$\begin{aligned}\operatorname{div}(\boldsymbol{\psi}(f)) &= \sum_{m>0} c(-m)\cdot\mathcal{Z}_{\mathrm{Kra}}^*(m) + \sum_\Phi \operatorname{mult}_\Phi(f)\cdot\mathcal{S}_{\mathrm{Kra}}^*(\Phi) \\ &\quad + \operatorname{div}(\delta^{-k}) + \sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}^*\end{aligned}\tag{6.5.3}$$

up to a linear combination of irreducible components of the exceptional divisor $\mathrm{Exc} \subset \mathcal{S}^*_{\mathrm{Kra}}$. *Moreover, each section* κ_Φ *of Proposition 6.4.4 has finite multiplicative order, and extends to a section* $\kappa_\Phi \in H^0(\mathcal{A}_\Phi, \mathcal{O}^\times_{\mathcal{A}_\Phi})$.

Proof. — Recall from Lemma 6.4.3 that the natural maps

$$\begin{array}{ccccc} \bigsqcup_i \mathcal{B}_{\Phi_i} & \longrightarrow & \bigsqcup_i \mathcal{S}^*_{\mathrm{Pap}}(\Phi_i) & \longrightarrow & \mathcal{S}^*_{\mathrm{Pap}} \\ \downarrow & & & & \\ \bigsqcup_i \mathcal{A}_{\Phi_i} & & & & \end{array}$$

induce bijections on connected components, as well as on connected components of the generic fibers.

All stacks in the diagram are proper over $\mathcal{O}_{\boldsymbol{k}}$, and have normal fibers. (For $\mathcal{S}^*_{\mathrm{Pap}}$ this follows from Theorem 3.7.1 and our assumption that $n > 2$. The other stacks appearing in the diagram are smooth over $\mathcal{O}_{\boldsymbol{k}}$.) It follows from this and [**18**, Corollary 8.2.18] that all arrows in the diagram induce bijections between the irreducible (= connected) components modulo any prime $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$.

Deleting the (0-dimensional) singular locus $\mathrm{Sing} \subset \mathcal{S}^*_{\mathrm{Pap}}$ does not change the irreducible components of $\mathcal{S}^*_{\mathrm{Pap}}$ or its fibers, and so if we define

$$\mathcal{U} \stackrel{\mathrm{def}}{=} \mathcal{S}^*_{\mathrm{Pap}} \smallsetminus \mathrm{Sing} \cong \mathcal{S}^*_{\mathrm{Kra}} \smallsetminus \mathrm{Exc},$$

then the natural maps

$$\begin{array}{ccccc} \bigsqcup_i \mathcal{B}_{\Phi_i} & \longrightarrow & \bigsqcup_i \mathcal{S}^*_{\mathrm{Pap}}(\Phi_i) & \longrightarrow & \mathcal{U} \\ \downarrow & & & & \\ \bigsqcup_i \mathcal{A}_{\Phi_i} & & & & \end{array}$$

induce bijections on irreducible components, as well as on irreducible components modulo any prime $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$.

Suppose Φ is any proper cusp label representative, and let $\mathcal{U}_\Phi \subset \mathcal{U}$ be the union of all irreducible components that meet $\mathcal{S}^*_{\mathrm{Pap}}(\Phi)$. If we interpret $\mathrm{div}(\kappa_\Phi)$ as a divisor on $\mathcal{U}$ using the bijection

$$\{\text{vertical divisors on } \mathcal{A}_\Phi\} \cong \{\text{vertical divisors on } \mathcal{U}_\Phi\},$$

then the equality of divisors (6.5.3) holds after pullback to $\mathcal{U}_\Phi$, up to the error term $\mathrm{div}(\kappa_\Phi)$. Indeed, this equality holds in the generic fiber of $\mathcal{U}_\Phi$ by Proposition 6.4.1, and it holds over an open neighborhood of $\mathcal{S}^*_{\mathrm{Pap}}(\Phi)$ by Lemma 6.5.1 and the isomorphism of formal completions (3.7.1). As the union of the generic fiber with this open neighborhood is an open substack whose complement has codimension ≥ 2, the stated equality holds over all of $\mathcal{U}_\Phi$.

Letting Φ vary over the Φ_i and using $\kappa_{\Phi_i} = 1$, we see from the paragraph above that (6.5.3) holds over $\bigsqcup_i \mathcal{U}_{\Phi_i} = \mathcal{U}$. With this in hand, we may reverse the argument to see that the error term $\mathrm{div}(\kappa_\Phi)$ vanishes for every Φ. It follows that κ_Φ extends to a global section of $\mathcal{O}^\times_{\mathcal{A}_\Phi}$.

It only remains to show that each κ_Φ has finite order. Choose a finite extension $L/\boldsymbol{k}$ large enough that every elliptic curve over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$ admits a model over L with everywhere good reduction. Choosing such models determines a faithfully flat morphism

$$\bigsqcup \mathrm{Spec}(\mathcal{O}_L) \to \mathcal{M}_{(1,0)} \cong \mathcal{A}_\Phi,$$

and the pullback of κ_Φ is represented by a tuple of units $(x_\ell) \in \prod \mathcal{O}_L^\times$. Each x_ℓ has absolute value 1 at every complex embedding of L (this follows from the final claim of Proposition 6.4.1), and is therefore a root of unity. This implies that κ_Φ has finite order. □

Proof of Theorem 5.3.1. — Start with a weakly holomorphic form (5.2.2). As in § 6.2, after possibly replacing f by a positive integer multiple, we obtain a Borcherds product $\boldsymbol{\psi}(f)$. This is a meromorphic section of $(\boldsymbol{\omega}^{\mathrm{an}})^k$. By Proposition 6.4.1 it is algebraic, and by Proposition 6.4.4 it may be rescaled by a constant of absolute value 1 on each connected component in such a way that it descends to $\boldsymbol{k}$.

Now view $\boldsymbol{\psi}(f)$ as a rational section of $\boldsymbol{\omega}^k$ over $\mathcal{S}^*_{\mathrm{Kra}}$. By Proposition 6.5.2 we may replace f by a further positive integer multiple, and replace $\boldsymbol{\psi}(f)$ by a corresponding tensor power, in order to make all $\kappa_\Phi = 1$. Having trivialized the κ_Φ, the existence part of Theorem 5.3.1 now follows from Proposition 6.4.1. For uniqueness, suppose $\boldsymbol{\psi}'(f)$ also satisfies the conditions of that theorem. The quotient of the two Borcherds products is a rational function with trivial divisor, which is therefore constant on every connected component of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$. As the leading Fourier-Jacobi coefficients of $\boldsymbol{\psi}'(f)$ and $\boldsymbol{\psi}(f)$ are equal along every boundary stratum, those constants are all equal to 1. □

Proof of Theorem 5.3.4. — As in the statement of the theorem, we now view $\boldsymbol{\psi}(f)^2$ as a rational section of the line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$ on $\mathcal{S}^*_{\mathrm{Pap}}$. Combining Proposition 6.5.2 with the isomorphism

$$\mathcal{S}^*_{\mathrm{Kra}} \setminus \mathrm{Exc} \cong \mathcal{S}^*_{\mathrm{Pap}} \setminus \mathrm{Sing},$$

of (3.7.2), and recalling from Theorem 3.7.1 that this isomorphism identifies

$$\boldsymbol{\omega}^{2k} \cong \boldsymbol{\Omega}^k_{\mathrm{Kra}} \cong \boldsymbol{\Omega}^k_{\mathrm{Pap}},$$

we deduce the equality

$$\begin{aligned} \mathrm{div}(\boldsymbol{\psi}(f)^2) = \sum_{m>0} c(-m) \cdot \mathcal{Y}^*_{\mathrm{Pap}}(m) + 2 \sum_{\Phi} \mathrm{mult}_\Phi(f) \cdot \mathcal{S}^*_{\mathrm{Pap}}(\Phi) \\ + \mathrm{div}(\delta^{-2k}) + 2 \sum_{r \mid D} \gamma_r c_r(0) \sum_{p \mid r} \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_p} \end{aligned} \tag{6.5.4}$$

of Cartier divisors on $\mathcal{S}^*_{\mathrm{Pap}} \setminus \mathrm{Sing}$. As $\mathcal{S}^*_{\mathrm{Pap}}$ is normal and Sing lies in codimension ≥ 2, this same equality must hold on the entirety of $\mathcal{S}^*_{\mathrm{Pap}}$. □

Proof of Theorem 5.3.3. — If we pull back via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$ and view $\boldsymbol{\psi}(f)^2$ as a rational section of the line bundle

$$\boldsymbol{\Omega}^k_{\mathrm{Kra}} \cong \boldsymbol{\omega}^{2k} \otimes \mathcal{O}(\mathrm{Exc})^{-k},$$

the equality (6.5.4) on $\mathcal{S}^*_{\mathrm{Pap}}$ pulls back to

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)^2) &= \sum_{m>0} c(-m)\cdot \mathcal{Y}^*_{\mathrm{Kra}}(m) + 2\sum_{\Phi} \mathrm{mult}_\Phi(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)\\ &\quad + \mathrm{div}(\delta^{-2k}) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}.\end{aligned}$$

Theorem 2.6.3 allows us to rewrite this as

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)^2) &= 2\sum_{m>0} c(-m)\cdot \mathcal{Z}^*_{\mathrm{Kra}}(m) + 2\sum_{\Phi} \mathrm{mult}_\Phi(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)\\ &\quad + \mathrm{div}(\delta^{-2k}) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}\\ &\quad - \sum_{m>0} c(-m)\sum_{s\in\pi_0(\mathrm{Sing})} \#\{x\in L_s : \langle x,x\rangle = m\}\cdot \mathrm{Exc}_s.\end{aligned}$$

If we instead view $\boldsymbol{\psi}(f)^2$ as a rational section of $\boldsymbol{\omega}^{2k}$, this becomes

$$\begin{aligned}\mathrm{div}(\boldsymbol{\psi}(f)^2) &= 2\sum_{m>0} c(-m)\cdot \mathcal{Z}^*_{\mathrm{Kra}}(m) + 2\sum_{\Phi} \mathrm{mult}_\Phi(f)\cdot \mathcal{S}^*_{\mathrm{Kra}}(\Phi)\\ &\quad + \mathrm{div}(\delta^{-2k}) + 2\sum_{r\mid D}\gamma_r c_r(0)\sum_{p\mid r}\mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_\mathfrak{p}}\\ &\quad - \sum_{m>0} c(-m)\sum_{s\in\pi_0(\mathrm{Sing})} \#\{x\in L_s : \langle x,x\rangle = m\}\cdot \mathrm{Exc}_s\\ &\quad + k\cdot \mathrm{Exc}\end{aligned}$$

as desired. □

7. Modularity of the generating series

Now armed with the modularity criterion of Theorem 4.2.3 and the arithmetic theory of Borcherds products provided by Theorems 5.3.1, 5.3.3, and 5.3.4, we prove our main results: the modularity of generating series of divisors on the integral models $\mathcal{S}^*_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Pap}}$ of the unitary Shimura variety $\mathrm{Sh}(G,\mathcal{D})$. The strategy follows that of [5], which proves modularity of the generating series of divisors on the complex fiber of an orthogonal Shimura variety.

Throughout § 7 we assume $n \geq 3$.

7.1. The modularity theorems. — Denote by

$$\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}) \cong \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Kra}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

the Chow group of rational equivalence classes of Cartier divisors on $\mathcal{S}^*_{\mathrm{Kra}}$ with $\mathbb{Q}$ coefficients, and similarly for $\mathcal{S}^*_{\mathrm{Pap}}$. There is a natural pullback map

$$\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}}) \to \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

Let $\chi = \chi^n_{\boldsymbol{k}}$ be the quadratic Dirichlet character (5.2.1).

Definition 7.1.1. — If V is any $\mathbb{Q}$-vector space, a formal q-expansion

$$\sum_{m\geq 0} d(m) \cdot q^m \in V[[q]] \tag{7.1.1}$$

is *modular of level D, weight n, and character χ* if for any $\mathbb{Q}$-linear map $\alpha : V \to \mathbb{C}$ the q-expansion

$$\sum_{m\geq 0} \alpha(d(m)) \cdot q^m \in \mathbb{C}[[q]]$$

is the q-expansion of an element of $M_n(D,\chi)$.

Remark 7.1.2. — If (7.1.1) is modular then its coefficients $d(m)$ span a subspace of V of dimension $\leq \dim M_n(D,\chi)$. We leave the proof as an exercise for the reader.

We also define the notion of the constant term of (7.1.1) at a cusp ∞_r, generalizing Definition 4.1.1.

Definition 7.1.3. — Suppose a formal q-expansion $g \in V[[q]]$ is modular of level D, weight n, and character χ. For any $r \mid D$, a vector $v \in V(\mathbb{C})$ is said to be the *constant term of g at the cusp ∞_r* if, for every linear functional $\alpha : V(\mathbb{C}) \to \mathbb{C}$, $\alpha(v)$ is the constant term of $\alpha(g)$ at the cusp ∞_r in the sense of Definition 4.1.1.

For $m > 0$ we have defined in § 5.3 effective Cartier divisors

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \hookrightarrow \mathcal{S}^*_{\mathrm{Pap}}, \quad \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \hookrightarrow \mathcal{S}^*_{\mathrm{Kra}}$$

related by (5.3.4). We have defined in § 3.7 line bundles

$$\boldsymbol{\Omega}_{\mathrm{Pap}} \in \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Pap}}), \quad \boldsymbol{\omega} \in \mathrm{Pic}(\mathcal{S}^*_{\mathrm{Kra}})$$

extending the line bundles on the open integral models defined in § 2.4. For notational uniformity, we define

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \boldsymbol{\Omega}^{-1}_{\mathrm{Pap}}, \quad \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \boldsymbol{\omega}^{-1} \otimes \mathcal{O}(\mathrm{Exc}).$$

Theorem 7.1.4. — *The formal q-expansion*

$$\sum_{m\geq 0} \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})[[q]],$$

is a modular form of level D, weight n, and character χ. For any $r \mid D$, its constant term at the cusp ∞_r is

$$\gamma_r \cdot \Big(\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\sum_{p\mid r} \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_{\mathfrak{p}}}\Big) \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Here $\gamma_r \in \{\pm 1, \pm i\}$ is defined by (5.3.2), *$\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ is the unique prime above $p \mid r$, and $\mathbb{F}_{\mathfrak{p}}$ is its residue field.*

Proof. — Let f be a weakly holomorphic form as in (5.2.2), and assume again that $c(m) \in \mathbb{Z}$ for all $m \le 0$. The space $M^{!,\infty}_{2-n}(D,\chi)$ is spanned by such forms. The Borcherds product $\boldsymbol{\psi}(f)$ of Theorem 5.3.1 is a rational section of the line bundle

$$\boldsymbol{\omega}^k = \bigotimes_{r\mid D} \boldsymbol{\omega}^{\gamma_r c_r(0)},$$

on $\mathcal{S}^*_{\mathrm{Kra}}$. If we view $\boldsymbol{\psi}(f)^2$ as a rational section of the line bundle

$$\boldsymbol{\Omega}^k_{\mathrm{Pap}} \cong \bigotimes_{r\mid D} \boldsymbol{\Omega}^{\gamma_r c_r(0)}_{\mathrm{Pap}}$$

on $\mathcal{S}^*_{\mathrm{Pap}}$, exactly as in Theorem 5.3.4, then

$$\mathrm{div}(\boldsymbol{\psi}(f)^2) = -\sum_{r\mid D} \gamma_r c_r(0) \cdot \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0)$$

holds in the Chow group of $\mathcal{S}^*_{\mathrm{Pap}}$. Comparing this with the calculation of the divisor of $\boldsymbol{\psi}(f)^2$ found in Theorem 5.3.4 shows that

$$0 = \sum_{m\ge 0} c(-m) \cdot \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) + \sum_{\substack{r\mid D\\ r>1}} \gamma_r c_r(0) \cdot (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\mathcal{V}_r), \tag{7.1.2}$$

where we abbreviate $\mathcal{V}_r = \sum_{p\mid r} \mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_{\mathfrak{p}}}$.

For each $r \mid D$ we have defined in § 4.2 an Eisenstein series

$$E_r(\tau) = \sum_{m\ge 0} e_r(m) \cdot q^m \in M_n(D,\chi),$$

and Proposition 4.2.2 allows us to rewrite the above equality as

$$0 = \sum_{m\ge 0} c(-m) \cdot \Big[\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) - \sum_{\substack{r\mid D\\ r>1}} \gamma_r e_r(m) \cdot (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\mathcal{V}_r)\Big].$$

Note that we have used $e_r(0) = 0$ for $r > 1$, a consequence of Remark 4.2.1.

The modularity criterion of Theorem 4.2.3 now shows that

$$\sum_{m\ge 0} \mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m - \sum_{\substack{r\mid D\\ r>1}} \gamma_r E_r \cdot (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\mathcal{V}_r)$$

is a modular form of level D, weight n, and character χ, whose constant term vanishes at every cusp different from ∞.

The theorem now follows from the modularity of each E_r, together with the description of their constant terms found in Remark 4.2.1. □

Theorem 7.1.5. — *The formal q-expansion*

$$\sum_{m\geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)\cdot q^m \in \mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]],$$

is a modular form of level D, weight n, and character χ.

Proof. — Recall from Theorems 2.6.3 and 3.7.1 that pullback via $\mathcal{S}^*_{\mathrm{Kra}} \to \mathcal{S}^*_{\mathrm{Pap}}$ sends

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \mapsto 2\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \sum_{s\in\pi_0(\mathrm{Sing})} \#\{x\in L_s : \langle x,x\rangle = m\}\cdot \mathrm{Exc}_s$$

for all $m>0$. This relation also holds for $m=0$, as those same theorems show that

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \boldsymbol{\Omega}^{-1}_{\mathrm{Pap}} \mapsto \boldsymbol{\omega}^{-2}\otimes \mathcal{O}(\mathrm{Exc}) = 2\cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \mathrm{Exc}.$$

Pulling back the relation (7.1.2) shows that

$$\begin{aligned} 0 = \sum_{m\geq 0} c(-m)\cdot\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \sum_{s\in\pi_0(\mathrm{Sing})} \frac{\#\{x\in L_s : \langle x,x\rangle = m\}}{2}\cdot \mathrm{Exc}_s\Big)\\ + \sum_{\substack{r\mid D\\ r>1}} \gamma_r c_r(0)\cdot\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \frac{1}{2}\cdot\mathrm{Exc} + \mathcal{V}_r\Big) \end{aligned}$$

in $\mathrm{Ch}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$ for any input form (5.2.2), where we now abbreviate

$$\mathcal{V}_r = \sum_{p\mid r} \mathcal{S}^*_{\mathrm{Kra}/\mathbb{F}_{\mathfrak{p}}}.$$

Using Proposition 4.2.2 we rewrite this as

$$\begin{aligned} 0 = \sum_{m\geq 0} c(-m)\cdot\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) - \sum_{s\in\pi_0(\mathrm{Sing})} \frac{\#\{x\in L_s : \langle x,x\rangle = m\}}{2}\cdot \mathrm{Exc}_s\Big)\\ - \sum_{m\geq 0} c(-m)\sum_{\substack{r\mid D\\ r>1}} \gamma_r e_r(m)\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \frac{1}{2}\cdot\mathrm{Exc} + \mathcal{V}_r\Big), \end{aligned}$$

where we have again used the fact that $e_r(0)=0$ for $r>1$.

The modularity criterion of Theorem 4.2.3 now implies the modularity of

$$\sum_{m\geq 0} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)\cdot q^m - \frac{1}{2}\sum_{s\in\pi_0(\mathrm{Sing})} \vartheta_s(\tau)\cdot \mathrm{Exc}_s - \sum_{\substack{r\mid D\\ r>1}} \gamma_r E_r(\tau)\cdot\Big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(0) - \frac{1}{2}\cdot\mathrm{Exc} + \mathcal{V}_r\Big).$$

The theorem follows from the modularity of the Eisenstein series $E_r(\tau)$ and the theta series

$$\vartheta_s(\tau) = \sum_{x\in L_s} q^{\langle x,x\rangle} \in M_n(D,\chi).$$ □

7.2. Green functions. — Here we construct Green functions for special divisors on $\mathcal{S}^*_{\mathrm{Kra}}$ as regularized theta lifts of harmonic Maass forms.

Recall from Section 2 the isomorphism of complex orbifolds

$$\mathcal{S}_{\mathrm{Kra}}(\mathbb{C}) \cong \mathrm{Sh}(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{D} \times G(\mathbb{A}_f)/K.$$

We use the uniformization on the right hand side and the regularized theta lift to construct Green functions for the special divisors

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \mathcal{Z}^*_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m)$$

on $\mathcal{S}^*_{\mathrm{Kra}}$. The construction is a variant of the ones in [**9**] and [**11**], adapted to our situation.

We now recall some of the basic notions of the theory of harmonic Maass forms, as in [**9**, Section 3]. Let $H^{\infty}_{2-n}(D,\chi)$ denote the space of harmonic Maass forms f of weight $2-n$ for $\Gamma_0(D)$ with character χ such that

— f is bounded at all cusps of $\Gamma_0(D)$ different from the cusp ∞,

— f has polynomial growth at ∞, in sense that there is a

$$P_f = \sum_{m<0} c^+(m) q^m \in \mathbb{C}[q^{-1}]$$

such that $f - P_f$ is bounded as q goes to 0.

A harmonic Maass form $f \in H^{\infty}_{2-n}(D,\chi)$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\substack{m\in\mathbb{Z}\\ m\gg-\infty}} c^+(m) q^m + \sum_{\substack{m\in\mathbb{Z}\\ m<0}} c^-(m)\cdot\Gamma\big(n-1, 4\pi|m|\,\mathrm{Im}(\tau)\big)\cdot q^m, \tag{7.2.1}$$

where

$$\Gamma(s,x) = \int_x^{\infty} e^{-t} t^{s-1} dt$$

is the incomplete gamma function. The first summand on the right hand side of (7.2.1) is denoted by f^+ and is called the *holomorphic part* of f, the second summand is denoted by f^- and is called the *non-holomorphic part*.

If $f \in H^{\infty}_{2-n}(D,\chi)$ then (6.1.1) defines an S_L-valued harmonic Maass form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2-n$ with representation ω_L. Proposition 6.1.2 extends to such lifts of harmonic Maass forms, giving the same formulas for the coefficients $\tilde{c}^+(m,\mu)$ of the holomorphic part $\tilde{f}^+$ of $\tilde{f}$. In particular, if $m<0$ we have

$$\tilde{c}^+(m,\mu) = \begin{cases} c^+(m) & \text{if } \mu = 0,\\ 0 & \text{if } \mu \neq 0,\end{cases} \tag{7.2.2}$$

and the constant term of $\tilde{f}$ is given by

$$\tilde{c}^+(0,\mu) = \sum_{r_\mu | r | D} \gamma_r \cdot c^+_r(0).$$

The formula of Proposition 4.2.2 for the contant terms $c^+_r(0)$ of f at the other cusps also extends.

As before, we consider the hermitian self-dual $\mathcal{O}_{\boldsymbol{k}}$-lattice $L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a})$ in $V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$. The dual lattice of L with respect to the bilinear form $[.,.]$ is $L' = \mathfrak{d}^{-1}L$. Let

$$S_L \subset S(V(\mathbb{A}_f))$$

be the space of Schwartz-Bruhat functions that are supported on $\widehat{L}'$ and invariant under translations by $\widehat{L}$.

Recall from Remark 2.1.2 that we may identify

$$\mathcal{D} \cong \{w \in \varepsilon V(\mathbb{C}) : [w, \overline{w}] < 0\}/\mathbb{C}^\times,$$

and also

$$\mathcal{D} \cong \{\text{negative definite } \boldsymbol{k}\text{-stable } \mathbb{R}\text{-planes } z \subset V(\mathbb{R})\}.$$

For any $x \in V$ and $z \in \mathcal{D}$, let x_z be the orthogonal projection of x to the plane $z \subset V(\mathbb{R})$, and let $x_{z^\perp}$ be the orthogonal projection to $z^\perp$.

For $(\tau, z, g) \in \mathfrak{H} \times \mathcal{D} \times G(\mathbb{A}_f)$ and $\varphi \in S_L$, we define a theta function

$$\theta(\tau, z, g, \varphi) = \sum_{x \in V} \varphi(g^{-1}x) \cdot \varphi_\infty(\tau, z, x),$$

where the Schwartz function at ∞,

$$\varphi_\infty(\tau, z, x) = v \cdot e^{2\pi i Q(x_{z^\perp})\tau + 2\pi i Q(x_z)\bar{\tau}},$$

is the usual Gaussian involving the majorant associated to z. We may view θ as a function $\mathfrak{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to S_L^\vee$. As a function in (z, g) it is invariant under the left action of $G(\mathbb{Q})$. Under the right action of K it satisfies the transformation law

$$\theta(\tau, z, gk, \varphi) = \theta(\tau, z, g, \omega_L(k)\varphi), \quad k \in K,$$

where ω_L denotes the action of K on S_L by the Weil representation and $v = \mathrm{Im}(\tau)$. In the variable $\tau \in \mathfrak{H}$ it transforms as a $S_L^\vee$-valued modular form of weight $n-2$ for $\mathrm{SL}_2(\mathbb{Z})$.

Fix an $f \in H^\infty_{2-n}(D, \chi)$ with Fourier expansion as in (7.2.1), and assume that $c^+(m) \in \mathbb{Z}$ for $m \le 0$. We associate to f the divisors

$$\mathcal{Z}_{\mathrm{Kra}}(f) = \sum_{m>0} c^+(-m) \cdot \mathcal{Z}_{\mathrm{Kra}}(m)$$
$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(f) = \sum_{m>0} c^+(-m) \cdot \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$$

on $\mathcal{S}_{\mathrm{Kra}}$ and $\mathcal{S}^*_{\mathrm{Kra}}$, respectively. As the actions of $\mathrm{SL}_2(\mathbb{Z})$ and K via the Weil representation commute, the associated S_L-valued harmonic Maass form $\tilde{f}$ is invariant under K. Hence the natural pairing $S_L \times S_L^\vee \to \mathbb{C}$ gives rise to a scalar valued function $(\tilde{f}(\tau), \theta(\tau, z, g))$ in the variables $(\tau, z, g) \in \mathfrak{H} \times \mathcal{D} \times G(\mathbb{A}_f)$, which is invariant under the right action of K and the left action of $G(\mathbb{Q})$. Hence it descends to a function on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \times \mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$.

We define the *regularized theta lift of f* as

$$\Theta^{\mathrm{reg}}(z,g,f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}}^{\mathrm{reg}} \big(\tilde{f}(\tau), \theta(\tau,z,g)\big)\, \frac{du\, dv}{v^2}.$$

Here the regularization of the integral is defined as in [**4, 9, 11**]. We extend the incomplete Gamma function

$$\Gamma(0,t) = \int_t^\infty e^{-v}\,\frac{dv}{v} \tag{7.2.3}$$

to a function on $\mathbb{R}_{\geq 0}$ by setting

$$\widetilde{\Gamma}(0,t) = \begin{cases} \Gamma(0,t) & \text{if } t>0, \\ 0 & \text{if } t=0. \end{cases}$$

Theorem 7.2.1. — *The regularized theta lift $\Theta^{\mathrm{reg}}(z,g,f)$ defines a smooth function on $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C}) \setminus \mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$. For $g \in G(\mathbb{A}_f)$ and $z_0 \in \mathcal{D}$, there exists a neighborhood $U \subset \mathcal{D}$ of z_0 such that*

$$\Theta^{\mathrm{reg}}(z,g,f) - \sum_{\substack{x\in gL \\ x\perp z_0}} c^+(-\langle x,x\rangle)\cdot \widetilde{\Gamma}\big(0, 4\pi|\langle x_z, x_z\rangle|\big)$$

is a smooth function on U.

Proof. — Note that the sum over $x \in gL \cap z_0^\perp$ is finite. Since $\mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$ decomposes into a finite disjoint union of connected components of the form

$$(G(\mathbb{Q}) \cap gKg^{-1})\backslash \mathcal{D},$$

where $g \in G(\mathbb{A}_f)$, it suffices to consider the restriction of $\Theta^{\mathrm{reg}}(f)$ to these components.

On such a component, $\Theta^{\mathrm{reg}}(z,g,f)$ is the regularized theta lift considered in [**11**, Section 4] of the vector valued form $\tilde{f}$ for the lattice

$$gL = g\widehat{L} \cap V = \mathrm{Hom}_{\mathcal{O}_k}(g\mathfrak{a}_0, g\mathfrak{a}) \subset V,$$

and hence the assertion follows from (7.2.2) and [**11**, Theorem 4.1]. □

Remark 7.2.2. — Let $\Delta_{\mathcal{D}}$ denote the $\mathrm{U}(V)(\mathbb{R})$-invariant Laplacian on $\mathcal{D}$. There exists a non-zero real constant c (which only depends on the normalization of $\Delta_{\mathcal{D}}$ and which is independent of f), such that

$$\Delta_{\mathcal{D}}\Theta^{\mathrm{reg}}(z,g,f) = c\cdot \deg \mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$$

on the complement of the divisor $\mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$.

Using the fact that

$$\Gamma(0,t) = -\log(t) + \Gamma'(1) + o(t)$$

as $t \to 0$, Theorem 7.2.1 implies that $\Theta^{\mathrm{reg}}(f)$ is a (sub-harmonic) logarithmic Green function for the divisor $\mathcal{Z}_{\mathrm{Kra}}(f)(\mathbb{C})$ on the non-compactified Shimura variety $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$. These properties, together with an integrability condition, characterize it uniquely up to addition of a locally constant function [**11**, Theorem 4.6]. The following result describes the behavior of $\Theta^{\mathrm{reg}}(f)$ on the toroidal compactification.

Theorem 7.2.3. — *On $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$, the function $\Theta^{\mathrm{reg}}(f)$ is a logarithmic Green function for the divisor $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(f)(\mathbb{C})$ with possible additional log-log singularities along the boundary in the sense of* [**13**].

Proof. — As in the proof of Theorem 7.2.1 we reduce this to showing that $\Theta^{\mathrm{reg}}(f)$ has the correct growth along the boundary of the connected components of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$. Then it is a direct consequence of [**11**, Theorem 4.10] and [**11**, Corollary 4.12]. □

Recall that $\boldsymbol{\omega}^{\mathrm{an}}$ is the tautological bundle on

$$\mathcal{D} \cong \big\{ w \in \varepsilon V(\mathbb{C}) : [w, \overline{w}] < 0 \big\} / \mathbb{C}^\times .$$

We define the Petersson metric $\|\cdot\|$ on $\boldsymbol{\omega}^{\mathrm{an}}$ by

$$\|w\|^2 = -\frac{[w,\overline{w}]}{4\pi e^\gamma},$$

where $\gamma = -\Gamma'(1)$ denotes Euler's constant. This choice of metric on $\boldsymbol{\omega}^{\mathrm{an}}$ induces a metric on the line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ defined in § 2.4, which extends to a metric over $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$ with log-log singularities along the boundary [**11**, Proposition 6.3]. We obtain a hermitian line bundle on $\mathcal{S}^*_{\mathrm{Kra}}$, denoted

$$\widehat{\boldsymbol{\omega}} = (\boldsymbol{\omega}, \|\cdot\|).$$

If f is actually weakly holomorphic, that is, if it belongs to $M^{!,\infty}_{2-n}(D,\chi)$, then $\Theta^{\mathrm{reg}}(f)$ is given by the logarithm of a Borcherds product. More precisely, we have the following theorem, which follows immediately from [**4**, Theorem 13.3] and our construction of $\psi(f)$ as the pullback of a Borcherds product, renormalized by (6.2.3), on an orthogonal Shimura variety.

Theorem 7.2.4. — *Let $f \in M^{!,\infty}_{2-n}(D,\chi)$ be as in* (5.2.2). *The Borcherds product $\psi(f)$ of Theorem 5.3.1 satisfies*

$$\Theta^{\mathrm{reg}}(f) = -\log \|\psi(f)\|^2 .$$

7.3. Generating series of arithmetic special divisors. — We can now define arithmetic special divisors on $\mathcal{S}^*_{\mathrm{Kra}}$, and prove a modularity result for the corresponding generating series in the codimension one arithmetic Chow group. This result extends Theorem 7.1.5.

Recall our hypothesis that $n > 2$, and let m be a positive integer. As in [**9**, Proposition 3.11], or using Poincaré series, it can be shown that there exists a unique $f_m \in H^\infty_{2-n}(D,\chi)$ whose Fourier expansion at the cusp ∞ has the form

$$f_m = q^{-m} + O(1)$$

as $q \to 0$. According to Theorem 7.2.3, its regularized theta lift $\Theta^{\mathrm{reg}}(f_m)$ is a logarithmic Green function for $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$.

Denote by $\widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$ the arithmetic Chow group [20] of rational equivalence classes of arithmetic divisors with $\mathbb{Q}$-coefficients. We allow the Green functions of our arithmetic divisors to have possible additional log-log error terms along the boundary of $\mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$, as in the theory of [13]. For $m>0$ define an arithmetic special divisor

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = (\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m), \Theta^{\mathrm{reg}}(f_m)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})$$

on $\mathcal{S}^*_{\mathrm{Kra}}$, and for $m=0$ set

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \widehat{\boldsymbol{\omega}}^{-1} + (\mathrm{Exc}, -\log(D)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

In the theory of arithmetic Chow groups one usually works on a regular scheme such as $\mathcal{S}^*_{\mathrm{Kra}}$. However, the codimension one arithmetic Chow group of $\mathcal{S}^*_{\mathrm{Pap}}$ makes perfect sense: one only needs to specify that it consists of rational equivalence classes of *Cartier* divisors on $\mathcal{S}^*_{\mathrm{Pap}}$ endowed with a Green function.

With this in mind one can use the equality

$$\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m)(\mathbb{C}) = 2\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)(\mathbb{C})$$

in the complex fiber $\mathcal{S}^*_{\mathrm{Pap}}(\mathbb{C}) = \mathcal{S}^*_{\mathrm{Kra}}(\mathbb{C})$ to define arithmetic divisors

$$\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) = (\mathcal{Y}^{\mathrm{tot}}_{\mathrm{Pap}}(m), 2\Theta^{\mathrm{reg}}(f_m)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})$$

for $m>0$. For $m=0$ we define

$$\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(0) = \widehat{\boldsymbol{\Omega}}^{-1} + (0, -2\log(D)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}}),$$

where the metric on $\boldsymbol{\Omega}$ is induced from that on $\boldsymbol{\omega}$, again using $\boldsymbol{\Omega} \cong \boldsymbol{\omega}^2$ in the complex fiber.

Theorem 7.3.1. — *The formal q-expansions*

$$\widehat{\phi}(\tau) = \sum_{m\geq 0} \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]] \tag{7.3.1}$$

and

$$\sum_{m\geq 0} \widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})[[q]]$$

are modular forms of level D, weight n, and character χ.

Proof. — For any input form $f \in M^{!,\infty}_{2-n}(D,\chi)$ as in (5.2.2), the relation in the Chow group given by the Borcherds product $\psi(f)$ is compatible with the Green functions, in the sense that

$$-\log\|\psi(f)\|^2 = \sum_{m>0} c(-m)\cdot \Theta^{\mathrm{reg}}(f_m).$$

Indeed, this directly follows from $f = \sum_{m>0} c(-m) f_m$ and Theorem 7.2.4.

This observation allows us to simply repeat the argument of Theorems 7.1.4 and 7.1.5 on the level of arithmetic Chow groups. Viewing $\boldsymbol{\psi}(f)^2$ as a rational section of the metrized line bundle $\boldsymbol{\Omega}^k_{\mathrm{Pap}}$, the arithmetic divisor

$$\widehat{\mathrm{div}}(\boldsymbol{\psi}(f)^2) \stackrel{\mathrm{def}}{=} \big(\mathrm{div}(\boldsymbol{\psi}(f)^2), -2\log\|\boldsymbol{\psi}(f)\|^2\big) \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Pap}})$$

satisfies both

$$\widehat{\mathrm{div}}(\boldsymbol{\psi}(f)^2) = \widehat{\boldsymbol{\Omega}}^k_{\mathrm{Pap}} = -2k\cdot(0,\log(D)) - \sum_{r|D}\gamma_r c_r(0)\cdot\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(0) \tag{7.3.2}$$

and, recalling $\delta = \sqrt{-D}\in \boldsymbol{k}$,

$$\begin{aligned}\widehat{\mathrm{div}}(\boldsymbol{\psi}(f)^2) &= \sum_{m>0} c(-m)\cdot\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) - 2k\cdot(\mathrm{div}(\delta),0) + 2\sum_{r|D}\gamma_r c_r(0)\cdot\widehat{\mathcal{V}}_r\\ &= \sum_{m>0} c(-m)\cdot\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) - 2k\cdot(0,\log(D)) + 2\sum_{r|D}\gamma_r c_r(0)\cdot\widehat{\mathcal{V}}_r,\end{aligned} \tag{7.3.3}$$

where $\widehat{\mathcal{V}}_r$ is the the vertical divisor $\mathcal{V}_r = \sum_{p|r}\mathcal{S}^*_{\mathrm{Pap}/\mathbb{F}_\mathfrak{p}}$ endowed with the trivial Green function. Note that in the second equality we have used the relation

$$0 = \widehat{\mathrm{div}}(\delta) = (\mathrm{div}(\delta), -\log|\delta^2|) = (\mathrm{div}(\delta),0) - (0,\log(D))$$

in the arithmetic Chow group. Combining (7.3.2) and (7.3.3), we deduce that

$$0 = \sum_{m\geq 0} c(-m)\cdot\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(m) + \sum_{\substack{r|D\\ r>1}}\gamma_r c_r(0)\left(\widehat{\mathcal{Y}}^{\mathrm{tot}}_{\mathrm{Pap}}(0) + 2\cdot\widehat{\mathcal{V}}_r\right).$$

With this relation in hand, both proofs go through verbatim. □

7.4. Non-holomorphic generating series of special divisors. — In this subsection we discuss a non-holomorphic variant of the generating series (7.3.1), which is obtained by endowing the special divisors with other Green functions, namely with those constructed in [**23, 24**] following the method of [**36**]. By combining Theorem 7.3.1 with a recent result of Ehlen and Sankaran [**16**], we show that the non-holomorphic generating series is also modular.

For every $m\in\mathbb{Z}$ and $v\in\mathbb{R}_{>0}$ define a divisor

$$\mathcal{B}_{\mathrm{Kra}}(m,v) = \frac{1}{4\pi v}\sum_{\Phi}\#\{x\in L_0 :\ \langle x,x\rangle = m\}\cdot\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$$

with real coefficients on $\mathcal{S}^*_{\mathrm{Kra}}$. Here the sum is over all K-equivalence classes of proper cusp label representatives Φ in the sense of § 3.2, L_0 is the hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-2,0)$ defined by (3.1.4), and $\mathcal{S}^*_{\mathrm{Kra}}(\Phi)$ is the boundary divisor of Theorem 3.7.1. Note that $\mathcal{B}_{\mathrm{Kra}}(m,v)$ is trivial for all $m<0$. We define classes in $\mathrm{Ch}^1_{\mathbb{R}}(\mathcal{S}^*_{\mathrm{Kra}})$, depending on the parameter v, by

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v) = \begin{cases} \mathcal{Z}^{*}_{\mathrm{Kra}}(m) + \mathcal{B}_{\mathrm{Kra}}(m,v) & \text{if } m \neq 0 \\ \boldsymbol{\omega}^{-1} + \mathrm{Exc} + \mathcal{B}_{\mathrm{Kra}}(0,v) & \text{if } m = 0. \end{cases}$$

Following [**23, 24, 36**], Green functions for these divisors can be constructed as follows. For $x \in V(\mathbb{R})$ and $z \in \mathcal{D}$ we put

$$R(x,z) = -2Q(x_z).$$

Recalling the incomplete Gamma function (7.2.3), for $m \in \mathbb{Z}$ and

$$(v,z,g) \in \mathbb{R}_{>0} \times \mathcal{D} \times G(\mathbb{A}_f)$$

we define a Green function

$$\Xi(m,v,z,g) = \sum_{\substack{x \in V \setminus \{0\} \\ Q(x) = m}} \chi_{\widehat{L}}(g^{-1}x) \cdot \Gamma(0, 2\pi v R(x,z)), \tag{7.4.1}$$

where $\chi_{\widehat{L}} \in S_L$ denotes the characteristic function of $\widehat{L}$. As a function of the variable (z,g), (7.4.1) is invariant under the left action of $G(\mathbb{Q})$ and under the right action of K, and so descends to a function on $\mathbb{R}_{>0} \times \mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$. It was proved in [**24**, Theorem 3.4.7] that $\Xi(m,v)$ is a logarithmic Green function for $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v)$ when $m \neq 0$. When $m = 0$ it is a logarithmic Green function for $\mathcal{B}_{\mathrm{Kra}}(0,v)$.

Consequently, we obtain arithmetic special divisors in $\widehat{\mathrm{Ch}}^1_{\mathbb{R}}(\mathcal{S}^*_{\mathrm{Kra}})$ depending on the parameter v by putting

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v) = \begin{cases} (\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v), \Xi(m,v)) & \text{if } m \neq 0 \\ \widehat{\boldsymbol{\omega}}^{-1} + (\mathcal{B}_{\mathrm{Kra}}(0,v), \Xi(0,v)) + (\mathrm{Exc}, -\log(Dv)) & \text{if } m = 0. \end{cases}$$

Note that for $m < 0$ these divisors are supported in the archimedian fiber.

Theorem 7.4.1. — *The formal q-expansion*

$$\widehat{\phi}_{\text{non-hol}}(\tau) = \sum_{m \in \mathbb{Z}} \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m,v) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{R}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]],$$

is a non-holomorphic modular form of level D, weight n, and character χ. Here $q = e^{2\pi i \tau}$ and $v = \mathrm{Im}(\tau)$.

Proof. — Theorem 4.13 of [**16**] states that the difference

$$\widehat{\phi}_{\text{non-hol}}(\tau) - \widehat{\phi}(\tau) \tag{7.4.2}$$

is a non-holomorphic modular form of level D, weight n, and character χ, valued in $\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}})$. Hence the assertion follows from Theorem 7.3.1. □

The meaning of modularity in Theorem 7.4.1 is to be understood as in [**16**, Definition 4.11]. In our situation it reduces to the statement that there is a smooth function $s(\tau,z,g)$ on $\mathfrak{H} \times \mathrm{Sh}(G,\mathcal{D})(\mathbb{C})$ with the following properties:

1. in (z, g) the function $s(\tau, z, g)$ has at worst log-log-singularities at the boundary of $\mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$ (in particular it is a Green function for the trivial divisor);
2. $s(\tau, z, g)$ transforms in τ as a non-holomorphic modular form of level D, weight n, and character χ;
3. the difference $\widehat{\phi}_{\text{non-hol}}(\tau) - s(\tau, z, g)$ belongs to the space
$$M_n(D,\chi) \otimes_{\mathbb{C}} \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \oplus (R_{n-2} M_{n-2}(D,\chi)) \otimes_{\mathbb{C}} \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}),$$
where R_{n-2} denotes the Maass raising operator as in Section 8.4.

8. Appendix: some technical calculations

We collect some technical arguments and calculations. Strictly speaking, none of these are essential to the proofs in the body of the text. We explain the connection between the fourth roots of unity γ_p defined by (5.3.1) and the local Weil indices appearing in the theory of the Weil representation, provide alternative proofs of Propositions 6.1.2 and 6.3.3, and explain in greater detail how Proposition 6.3.1 is deduced from the formulas of [**32**].

8.1. Local Weil indices. — In this subsection, we explain how the quantity γ_p defined in (5.3.1) is related to the local Weil representation.

Let $L \subset V$ be as in § 6.1, and recall that $S_L = \mathbb{C}[L'/L]$ is identified with a subspace of $S(V(\mathbb{A}_f))$ by sending $\mu \in L'/L$ to the characteristic function ϕ_μ of $\mu + \widehat{L} \subset V(\mathbb{A}_f)$.

As $\dim_{\mathbb{Q}} V = 2n$ and D is odd, the representation ω_L of $\mathrm{SL}_2(\mathbb{Z})$ on S_L is the pullback via
$$\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \prod_{p \mid D} \mathrm{SL}_2(\mathbb{Z}_p)$$
of the representation
$$\omega_L = \bigotimes_{p \mid D} \omega_p,$$
where $\omega_p = \omega_{L_p}$ is the Weil representation of $\mathrm{SL}_2(\mathbb{Z}_p)$ on $S_{L_p} \subset S(V_p)$. These Weil representations are defined using the standard global additive character $\psi = \otimes_p \psi_p$, which is trivial on $\widehat{\mathbb{Z}}$ and on $\mathbb{Q}$ and whose restriction to $\mathbb{R} \subset \mathbb{A}$ is given by $\psi(x) = \exp(2\pi i x)$. Recall that, for $a \in \mathbb{Q}_p^\times$ and $b \in \mathbb{Q}_p$,
$$\omega_p(n(b))\phi(x) = \psi_p(bQ(x)) \cdot \phi(x)$$
$$\omega_p(m(a))\phi(x) = \chi^n_{\boldsymbol{k},p}(a) \cdot |a|^n_p \cdot \phi(ax)$$
$$\omega_p(w)\phi(x) = \gamma_p \int_{V_p} \psi_p(-[x,y]) \cdot \phi(y)\, dy, \quad w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix},$$
where $\gamma_p = \gamma_p(L)$ is the Weil index of the quadratic space V_p with respect to ψ_p and $\chi_{\boldsymbol{k},p}$ is the quadratic character of $\mathbb{Q}_p^\times$ corresponding to $\boldsymbol{k}_p$. Note that dy is the self-dual measure with respect to the pairing $\psi_p([x,y])$.

Lemma 8.1.1. — *The Weil representation ω_p satisfies the following properties.*

1. *When restricted to the subspace $S_{L_p} \subset S(V_p)$, the action of $\gamma \in \mathrm{SL}_2(\mathbb{Z}_p)$ depends only on the image of γ in $\mathrm{SL}_2(\mathbb{F}_p)$.*
2. *The Weil index is given by*

$$\gamma_p = \varepsilon_p^{-n} \cdot (D,p)_p^n \cdot \mathrm{inv}_p(V_p)$$

where $(a,b)_p$ is the Hilbert symbol for $\mathbb{Q}_p$ and $\mathrm{inv}_p(V_p)$ is the invariant of V_p in the sense of (1.7.3).

Proof. — (i) It suffices to check this on the generators. We omit this.
(ii) We can choose an $O_{\boldsymbol{k},p}$-basis for L_p such that the matrix for the hermitian form is $\mathrm{diag}(a_1,\dots,a_n)$, with $a_j \in \mathbb{Z}_p^\times$. The matrix for the bilinear form $[x,y] = \mathrm{Tr}_{K_p/\mathbb{Q}_p}(\langle x,y\rangle)$ is then $\mathrm{diag}(2a_1,\dots,2a_n,2Da_1,\dots,2Da_n)$. Then, according to the formula for β_V in [**35**, p. 379], we have

$$\gamma_p^{-1} = \gamma_{\mathbb{Q}_p}(\frac{1}{2}\cdot\psi_p\circ V) = \prod_{j=1}^n \gamma_{\mathbb{Q}_p}(a_j\psi_p)\cdot\gamma_{\mathbb{Q}_p}(Da_j\psi_p),$$

where we note that, in the notation there, $x(w) = 1$, and $j = j(w) = 1$. Next by Proposition A.11 of the appendix to [**48**], for any $\alpha \in \mathbb{Z}_p^\times$, we have $\gamma_{\mathbb{Q}_p}(\alpha\psi_p) = 1$ and

$$\gamma_{\mathbb{Q}_p}(\alpha p\psi_p) = \left(\frac{-\alpha}{p}\right)\cdot\varepsilon_p = (-\alpha,p)_p\cdot\varepsilon_p.$$

Here note that if $\eta = \alpha p\psi_p$, then the resulting character $\bar\eta$ of $\mathbb{F}_p$ is given by

$$\bar\eta(\bar a) = \psi_p(p^{-1}a) = e(-p^{-1}a).$$

and $\gamma_{\mathbb{F}_p}(\bar\eta) = \left(\frac{-1}{p}\right)\cdot\varepsilon_p$. Thus

$$\gamma_p = \varepsilon_p^{-n}\cdot(-D/p,p)_p^n\cdot(\det(V),p)_p,$$

as claimed. □

8.2. A direct proof of Proposition 6.1.2. — The proof of Proposition 6.1.2, which expresses the Fourier coefficients of the vector valued form $\tilde f$ in terms of those of the scalar valued form $f \in M^!_{2-n}(D,\chi)$, appealed to the more general results of [**50**]. In some respects, it is easier to prove Proposition 6.1.2 from scratch than it is to extract it from [*loc. cit.*]. This is what we do here.

Recall that $\tilde f$ is defined from f by the induction procedure of (6.1.1), and that the coefficients $\tilde c(m,\mu)$ in its Fourier expansion (6.1.2) are indexed by $m \in \mathbb{Q}$ and $\mu \in L'/L$. Recall that, for $r \mid D$, $rs = D$,

$$W_r = \begin{pmatrix} r\alpha & \beta \\ D\gamma & r\delta\end{pmatrix} = R_r\begin{pmatrix} r & \\ & 1\end{pmatrix},\quad R_r = \begin{pmatrix}\alpha & \beta\\ s\gamma & r\delta\end{pmatrix}\in\Gamma_0(s).$$

Note that

$$\Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z})/\Gamma(D) \simeq \prod_{p|D} B_p\backslash \mathrm{SL}_2(\mathbb{F}_p), \tag{8.2.1}$$

so this set has order $\prod_{p|D}(p+1)$. A set of coset representatives is given by

$$\bigsqcup_{\substack{r|D \\ c\,(\mathrm{mod}\,r)}} R_r \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}.$$

Now, using (4.1.1), we have

$$\begin{aligned}\left(f\big|_{2-n} R_r \begin{pmatrix} 1 & c \\ & 1 \end{pmatrix}\right)(\tau) &= \left(f\big|_{2-n} W_r \begin{pmatrix} r^{-1} & r^{-1}c \\ & 1 \end{pmatrix}\right)(\tau) \\ &= \chi_r(\beta)\chi_s(\alpha) \sum_{m \gg -\infty} r^{\frac{n}{2}-1} c_r(m) \cdot e^{\frac{2\pi i m(\tau+c)}{r}}.\end{aligned} \tag{8.2.2}$$

On the other hand, the image of the inverse of our coset representative on the right side of (8.2.1) has components

$$\begin{cases} \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & -\beta \\ -s\gamma & \alpha \end{pmatrix} & \text{if } p \mid r \\ \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix}\begin{pmatrix} r\delta & -\beta \\ 0 & \alpha \end{pmatrix} & \text{if } p \mid s. \end{cases}$$

Note that $r\alpha\delta - s\beta\gamma = 1$. Then, as elements of $\mathrm{SL}_2(\mathbb{F}_p)$, we have

$$\begin{cases} \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix}\begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \alpha\beta \\ & 1 \end{pmatrix} & \text{if } p \mid r \\ \begin{pmatrix} 1 & -c \\ & 1 \end{pmatrix}\begin{pmatrix} \alpha^{-1} & \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & -\alpha\beta \\ & 1 \end{pmatrix} & \text{if } p \mid s. \end{cases}$$

The element on the second line just multiplies $\phi_{0,p}$ by $\chi_p(\alpha)$. For the element on the first line, the factor on the right fixes ϕ_0 and

$$\omega_p\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\right)\phi_0 = \gamma_p\, p^{-\frac{n}{2}} \sum_{\mu \in L'_p/L_p} \phi_\mu.$$

Thus, the element on the first line carries $\phi_{0,p}$ to

$$\chi_p(\beta)\gamma_p\, p^{-\frac{n}{2}} \sum_{\mu \in L'_p/L_p} \psi_p(-c\,Q(\mu))\,\phi_\mu.$$

Recall from (6.1.3) that for $\mu \in L'/L$, r_μ is the product of the primes $p \mid D$ such that $\mu_p \neq 0$. Thus

$$\omega_L\left(R_r\begin{pmatrix}1 & c\\ & 1\end{pmatrix}\right)^{-1}\phi_0 = \chi_s(\alpha)\chi_r(\beta)\,\gamma_r\,r^{-\frac{n}{2}}\sum_{\substack{\mu\in L'/L\\ r_\mu\mid r}} e^{2\pi i c Q(\mu)}\phi_\mu. \tag{8.2.3}$$

Taking the product of (8.2.2) and (8.2.3) and summing on c and on r, we obtain

$$\begin{aligned}\sum_{r\mid D}\gamma_r\cdot r^{-1}\sum_{c\,(\mathrm{mod}\,r)}\sum_{\substack{\mu\in L'/L\\ r_\mu\mid r}} e^{2\pi i c Q(\mu)}\phi_\mu\sum_{m\gg-\infty} c_r(m)e^{\frac{2\pi i m(\tau+c)}{r}}\\
=\sum_{r\mid D}\gamma_r\sum_{\substack{\mu\in L'/L\\ r_\mu\mid r}}\phi_\mu\sum_{\substack{m\gg-\infty\\ \frac{m}{r}+Q(\mu)\in\mathbb{Z}}} c_r(m)\,q^{\frac{m}{r}}\\
=\sum_{\substack{m\in\mathbb{Q}\\ m\gg-\infty}}\sum_{\substack{\mu\in L'/L\\ m+Q(\mu)\in\mathbb{Z}}}\sum_{\substack{r\\ r_\mu\mid r\mid D}}\gamma_r c_r(mr)\phi_\mu\,q^m.\end{aligned}$$

This gives the claimed general expression for $\tilde{c}(m,\mu)$ and completes the proof of Proposition 6.1.2.

8.3. A more detailed proof of Proposition 6.3.1. — In this section, we explain in more detail how to obtain the product formula of Proposition 6.3.1 from the general formula given in [**32**].

For our weakly holomorphic S_L-valued modular form $\tilde{f}$ of weight $2-n$, with Fourier expansion given by (6.1.2), the corresponding meromorphic Borcherds product $\Psi(\tilde{f})$ on $\tilde{\mathcal{D}}^+$ has a product formula [**32**, Corollary 2.3] in a neighborhood of the 1-dimensional boundary component associated to L_{-1}. It is given as a product of 4 factors, labeled (a), (b), (c) and (d). We note that, in our present case, there is a basic simplification in factor (b) due to the restriction on the support of the Fourier coefficients of $\tilde{f}$. More precisely, for $m>0$, $\tilde{c}(-m,\mu)=0$ for $\mu\notin L$, and $\tilde{c}(-m,0)=c(-m)$. In particular, if $x\in L'$ with $[x,\mathrm{e}_{-1}]=[x,\mathrm{f}_{-1}]=0$, then $Q(x)=Q(x_0)$, where x_0 is the $(L_0)_{\mathbb{Q}}$ component of x. If $x_0\neq 0$, then $Q(x)>0$, and $\tilde{c}(-Q(x),\mu)=0$ for $\mu\notin L$. The factors for $\Psi(\tilde{f})$ are then given by:

(a)

$$\prod_{\substack{x\in L'\\ [x,\mathrm{f}_{-1}]=0\\ [x,\mathrm{e}_{-1}]>0\\ \mathrm{mod}\ L\cap\mathbb{Q}\mathrm{f}_{-1}}}\left(1-e^{-2\pi i[x,w]}\right)^{\tilde{c}(-Q(x),x)}.$$

(b)

$$P_1(w_0,\tau_1)\stackrel{\mathrm{def}}{=}\prod_{\substack{x\in L_0\\ [x,W_0]>0}}\left(\frac{\vartheta_1(-[x,w],\tau_1)}{\eta(\tau_1)}\right)^{c(-Q(x))},$$

where W_0 is a Weyl chamber in $V_0(\mathbb{R})$, as in [**32**, § 2].

(c)

$$P_0(\tau_1) \overset{\text{def}}{=} \prod_{\substack{x \in \mathfrak{d}^{-1}L_{-1}/L_{-1} \\ x \neq 0}} \left(\frac{\vartheta_1(-[x,w],\tau_1)}{\eta(\tau_1)} \, e^{\pi i [x,w]\cdot[x,\mathrm{e}_1]} \right)^{\tilde{c}(0,x)/2}$$

(d) and

$$\kappa\, \eta(\tau_1)^{\tilde{c}(0,0)}\, q_2^{I_0},$$

where κ is a scalar of absolute value 1, and

$$I_0 = -\sum_m \sum_{\substack{x \in L' \cap (L_{-1})^\perp \\ \text{mod } L_{-1}}} \tilde{c}(-m,x)\, \sigma_1(m - Q(x)).$$

The factors given in Proposition 6.3.1 are for the form

$$\tilde{\psi}_g(f) \overset{\text{def}}{=} (2\pi i)^{\tilde{c}(0,0)}\, \Psi(2\tilde{f}).$$

The quantity q_2 in [**32**] is our $e(\xi)$, and τ_1 there is our τ.

Recall from (3.9.5) that $\mathfrak{d}^{-1}L_{-1} = \mathbb{Z}\mathrm{e}_{-1} + D^{-1}\mathbb{Z}\mathrm{f}_{-1}$, so that, in factor (c), the product runs over vectors $D^{-1}b\,\mathrm{f}_{-1}$, with $b \pmod D$ nonzero. For these vectors $[x,\mathrm{e}_1] = 0$. In the formula for I, x runs over vectors of the form

$$x = -\frac{b}{D}\mathrm{f}_{-1} + x_0,$$

with $x_0 \in \mathfrak{d}^{-1}L_0$. But, again, if $x_0 \neq 0$, $Q(x) = Q(x_0) > 0$ and $\tilde{c}(-Q(x),x) = 0$ unless $b = 0$, and so the sum in that term runs over $x_0 \in L_0$ $x_0 \neq 0$ and over $-\frac{b}{D}\mathrm{f}_{-1}$'s.

Thus the factors for $\tilde{\psi}_g(f)$ are given by:

(a)

$$\prod_{\substack{x \in L' \\ [x,\mathrm{f}_{-1}]=0 \\ [x,\mathrm{e}_{-1}]>0 \\ \text{mod } L \cap \mathbb{Q}\mathrm{f}_{-1}}} \left(1 - e^{-2\pi i [x,w]}\right)^{2\,\tilde{c}(-Q(x),x)},$$

(b)

$$P_1(w_0,\tau_1) \overset{\text{def}}{=} \prod_{\substack{x_0 \in L_0 \\ x_0 \neq 0}} \left(\frac{\vartheta_1(-[x_0,w],\tau_1)}{\eta(\tau_1)} \right)^{c(-Q(x_0))},$$

(c)

$$P_0(\tau_1) \overset{\text{def}}{=} \prod_{\substack{b \in \mathbb{Z}/D\mathbb{Z} \\ b \neq 0}} \left(\frac{\vartheta_1(-[x,w],\tau_1)}{\eta(\tau_1)} \right)^{\tilde{c}(0,\frac{b}{D}f_{-1})},$$

(d) and, setting $k = \tilde{c}(0,0)$,

$$\kappa^2\, (2\pi i\, \eta^2(\tau))^k\, q_2^{2I_0},$$

where κ is a scalar of absolute value 1, and

$$I_0 = -2 \sum_{m>0} \sum_{x_0 \in L_0} c(-m)\, \sigma_1(m - Q(x_0)) + \frac{1}{12} \sum_{b \in \mathbb{Z}/D\mathbb{Z}} \tilde{c}(0, \frac{b}{D}\mathrm{f}_{-1}).$$

Here note that for $\tilde{\psi}_g(f) = (2\pi i)^{\tilde{c}(0,0)} \Psi(2\tilde{f})$ we have multiplied the previous expression by 2.

Finally recall

$$w = -\xi \mathrm{e}_{-1} + (\tau\xi - Q(w_0))\mathrm{f}_{-1} + w_0 + \tau \mathrm{e}_1 + f_1.$$

If $[x, f_{-1}] = 0$, then x has the form

$$x = -a\mathrm{e}_{-1} - \frac{b}{D}\mathrm{f}_{-1} + x_0 + c\mathrm{e}_1,$$

so that

$$[x, w] = -c\,\xi + [x_0, w_0] - a\tau - \frac{b}{D},$$

and

$$Q(x) = -ac + Q(x_0).$$

Using these values, the formulas given in Proposition 6.3.1 follow immediately.

8.4. A direct proof of Proposition 6.3.3. — Here we give a direct proof of Proposition 6.3.3, which does not rely on Corollary 6.3.2. We begin by recalling some general facts about derivatives of modular forms.

We let $q\frac{d}{dq}$ be the Ramanujan theta operator on q-series. Recall that the image under $q\frac{d}{dq}$ of a holomorphic modular form g of weight k is in general not a modular form. However, the function

$$D(g) = q\frac{dg}{dq} - \frac{k}{12} g E_2 \tag{8.4.1}$$

is a holomorphic modular form of weight $k+2$ (see [**11**, § 4.2]). Here

$$E_2(\tau) = -24 \sum_{m \geq 0} \sigma_1(m) q^m$$

denotes the non-modular Eisenstein series of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. In particular $\sigma_1(0) = -\frac{1}{24}$. We extend σ_1 to rational arguments by putting $\sigma_1(r) = 0$ if $r \notin \mathbb{Z}_{\geq 0}$. If $R_k = 2i\frac{\partial}{\partial \tau} + \frac{k}{v}$ denotes the Maass raising operator, and

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi v}$$

is the non-holomorphic (but modular) Eisenstein series of weight 2, we also have

$$D(g) = -\frac{1}{4\pi} R_k(g) - \frac{k}{12} g E_2^*.$$

Proposition 8.4.1. — *Let* $f \in M^{!,\infty}_{2-n}(D,\chi)$ *as in* (5.2.2)*. The integer*

$$I = \frac{1}{12} \sum_{\alpha \in \mathfrak{d}^{-1}L_{-1}/L_{-1}} \tilde{c}(0,\alpha) - 2\sum_{m>0} c(-m) \sum_{x\in L_0} \sigma_1(m-Q(x)).$$

defined in Proposition 6.3.1 is equal to the integer

$$\mathrm{mult}_\Phi(f) = \frac{1}{n-2} \sum_{x\in L_0} c(-Q(x))Q(x)$$

defined by (5.2.4)*.*

Proof. — Consider the $S_{L_0}^\vee$-valued theta function

$$\Theta_0(\tau) = \sum_{x\in L_0'} q^{Q(x)} \chi^\vee_{x+L_0} \in M_{n-2}(\omega^\vee_{L_0}).$$

Applying the above construction (8.4.1) to Θ_0 we obtain an $S_{L_0}^\vee$-valued modular form

$$D(\Theta_0) = \sum_{x\in L_0'} Q(x)q^{Q(x)}\chi^\vee_{x+L_0} - \frac{n-2}{12}\Theta_0 E_2 \in M_n(\omega^\vee_{L_0})$$

of weight n. For its Fourier coefficients we have

$$D(\Theta_0) = \sum_{\nu\in L_0'/L_0} \sum_{m\geq 0} b(m,\nu) q^m \chi^\vee_\nu$$

$$b(m,\nu) = \sum_{\substack{x\in \nu+L_0 \\ Q(x)=m}} Q(x) + 2(n-2) \sum_{x\in\nu+L_0} \sigma_1(m-Q(x)).$$

As in [**11**, (4.8)], an S_L-valued modular form F induces an S_{L_0}-valued form F_{L_0}. If we denote by F_μ the components of F with respect to the standard basis (χ_μ) of S_L, we have

(8.4.2)
$$F_{L_0,\nu} = \sum_{\alpha\in\mathfrak{d}^{-1}L_{-1}/L_{-1}} F_{\nu+\alpha}$$

for $\nu \in L_0'/L_0$.

Let $\tilde{f} \in M^!_{2-n}(\omega_L)$ be the S_L-valued form corresponding to f, as in (6.1.1). Using (8.4.2) we obtain

$$\tilde{f}_{L_0} \in M^!_{2-n}(\omega_{L_0})$$

with Fourier expansion

$$\tilde{f}_{L_0} = \sum_{\nu,m} \sum_{\alpha\in\delta^{-1}I/I} \tilde{c}(m,\nu+\alpha) q^m \chi_{\nu+L_0}.$$

We consider the natural pairing between the S_{L_0}-valued modular form $\tilde{f}_{L_0}$ of weight $2-n$ and the $S^\vee_{L_0}$-valued modular form $D(\Theta_0)$ of weight n,

$$(\tilde{f}_{L_0}, D(\Theta_0)) \in M^!_2(\mathrm{SL}_2(\mathbb{Z})).$$

By the residue theorem, the constant term of the q-expansion vanishes, and so

$$\sum_{m\geq 0}\sum_{\substack{\nu\in L_0'/L_0\\ \alpha\in\delta^{-1}I/I}}\tilde{c}(-m,\nu+\alpha)b(m,\nu)=0. \tag{8.4.3}$$

We split this up in the sum over $m>0$ and the contribution from $m=0$. Employing Proposition 6.1.2, we obtain that the sum over $m>0$ is equal to

$$\sum_{m>0}c(-m)b(m,0).$$

For the contribution of $m=0$ we notice

$$b(0,\nu)=\begin{cases}-\frac{n-2}{12}, & \nu=0\in L_0'/L_0,\\ 0, & \nu\neq 0.\end{cases}$$

Hence this part is equal to

$$-\frac{n-2}{12}\sum_{\alpha\in\mathfrak{d}^{-1}L_{-1}/L_{-1}}\tilde{c}(0,\alpha).$$

Inserting the two contributions into (8.4.3), we obtain

$$\begin{aligned}0&=\sum_{m>0}c(-m)b(m,0)-\frac{n-2}{12}\sum_{\alpha\in\mathfrak{d}^{-1}L_{-1}/L_{-1}}\tilde{c}(0,\alpha)\\&=\sum_{m>0}c(-m)\Big(\sum_{\substack{x\in L_0\\ Q(x)=m}}Q(x)+2(n-2)\sum_{x\in L_0}\sigma_1(m-Q(x))\Big)\\&\quad-\frac{n-2}{12}\sum_{\alpha\in\mathfrak{d}^{-1}L_{-1}/L_{-1}}\tilde{c}(0,\alpha)\\&=\sum_{x\in L_0}c(-Q(x))Q(x)+2(n-2)\sum_{m>0}c(-m)\sum_{x\in L_0}\sigma_1(m-Q(x))\\&\quad-\frac{n-2}{12}\sum_{\alpha\in\mathfrak{d}^{-1}L_{-1}/L_{-1}}\tilde{c}(0,\alpha)\\&=(n-2)\mathrm{mult}_\Phi(f)-(n-2)I.\end{aligned}$$

This concludes the proof of the proposition. □

Now we verify directly the other claim of Proposition 6.3.3: the function

$$P_1(\tau,w_0)=\prod_{m>0}\prod_{\substack{x\in L_0\\ Q(x)=m}}\Theta\big(\tau,\langle w_0,x\rangle\big)^{c(-m)}$$

satisfies the transformation law (3.9.14) with respect to the translation action of $\mathfrak{b}L_0$ on the variable w_0.

First recall that, for a, $b\in\mathbb{Z}$,

$$\Theta(\tau,z+a\tau+b)=\exp\big(-\pi i a^2\tau-2\pi i a z+\pi i(b-a)\big)\cdot\Theta(\tau,z).$$

If we write $\alpha = a\tau + b$ and $\tau = u + iv$, then

$$a = \frac{\operatorname{Im}(\alpha)}{v} = \frac{\alpha - \bar{\alpha}}{2iv}, \qquad b = \operatorname{Re}(\alpha) - \frac{u}{v}\operatorname{Im}(\alpha).$$

Thus

$$\frac{1}{2}a^2\tau + az + \frac{1}{2}(a-b) = \frac{1}{4iv}(\alpha-\bar{\alpha})\alpha + \frac{1}{2iv}(\alpha - \bar{\alpha})z + \frac{1}{2}(a-b-ab).$$

For z and w in $\mathbb{C}$, write

$$R(z,w) = R_\tau(z,w) = B_\tau(z,w) - H_\tau(z,w) = \frac{1}{v}z(w-\bar{w}).$$

Then

$$\frac{1}{4v}(\alpha-\bar{\alpha})\alpha + \frac{1}{2v}(\alpha-\bar{\alpha})z = \frac{1}{2}R(z,\alpha) + \frac{1}{4}R(\alpha,\alpha),$$

and we can write

$$\Theta(\tau, z+\alpha) = \exp(-\pi R(z,\alpha) - \frac{\pi}{2}R(\alpha,\alpha)) \cdot \exp(\pi i(a-b-ab))^{-1}\,\Theta(\tau,z).$$

We will consider the contribution of the $\frac{1}{2}(a-b-ab)$ term separately.

For $\beta \in V_0$, we have $\langle w_0 + \beta, x\rangle = \langle w_0, x\rangle + \langle \beta, x\rangle$. Suppose that for all $x \in L_0$, we have $\langle \beta, x\rangle = a\tau + b$ for a and b in $\mathbb{Z}$. Writing $\mathfrak{b} = \mathbb{Z} + \mathbb{Z}\tau$, this is precisely the condition that $\beta \in \mathfrak{b}\,L_0$. Then we obtain a factor

$$\exp\left(-\pi\sum_{m>0}\sum_{\substack{x\in L_0\\ Q(x)=m}} c(-m)\left[R\big(\langle w_0,x\rangle,\langle\beta,x\rangle\big) + \frac{R\big(\langle\beta,x\rangle,\langle\beta,x\rangle\big)}{2}\right]\right).$$

Expanding the sum and using the hermitian version of Borcherds' quadratic identity from the proof of Proposition 5.2.2, we have

$$\begin{aligned}
&\sum_{x\in L_0}\frac{c(-Q(x))}{v}\left[\langle w_0,x\rangle\langle\beta,x\rangle - \langle w_0,x\rangle\langle x,\beta\rangle + \frac{\langle\beta,x\rangle\langle\beta,x\rangle}{2} - \frac{\langle\beta,x\rangle\langle x,\beta\rangle}{2}\right]\\
&= -\frac{1}{v}\left(\langle w_0,\beta\rangle + \frac{1}{2}\langle\beta,\beta\rangle\right)\cdot\frac{1}{2n-4}\cdot\sum_{x\in L_0}c(-Q(x))\,[x,x]\\
&= -\frac{1}{v}\left(\langle w_0,\beta\rangle + \frac{1}{2}\langle\beta,\beta\rangle\right)\cdot\operatorname{mult}_\Phi(f).
\end{aligned}$$

Thus, using $I = \operatorname{mult}_\Phi(f)$, we have a contribution of

$$\exp\Big(\frac{\pi\langle w_0,\beta\rangle}{v} + \frac{\pi\langle\beta,\beta\rangle}{2v}\Big)^I$$

to the transformation law.

Next we consider the quantity

$$a-b-ab=\frac{\mathrm{Im}(\alpha)}{v}-\mathrm{Re}(\alpha)-\frac{u\,\mathrm{Im}(\alpha)}{v}-\frac{\mathrm{Im}(\alpha)}{v}\Big(\mathrm{Re}(\alpha)-\frac{u\,\mathrm{Im}(\alpha)}{v}\Big)$$
$$=\frac{\alpha-\bar\alpha}{2iv}-\frac{(\alpha+\bar\alpha)}{2}-\frac{u(\alpha-\bar\alpha)}{2iv}-\frac{\alpha-\bar\alpha}{2iv}\Big(\frac{(\alpha+\bar\alpha)}{2}-\frac{u(\alpha-\bar\alpha)}{2iv}\Big).$$

This will contribute $\exp(-\pi i A)$, where A is defined as the sum

$$\sum_{x\neq 0} c(-Q(x))\Big[\frac{\alpha-\bar\alpha}{2iv}-\frac{\alpha+\bar\alpha}{2}-\frac{u(\alpha-\bar\alpha)}{2iv}-\frac{\alpha-\bar\alpha}{2iv}\Big(\frac{(\alpha+\bar\alpha)}{2}-\frac{u(\alpha-\bar\alpha)}{2iv}\Big)\Big],$$

where $\alpha=\langle\beta,x\rangle$. Since x and $-x$ both occur in the sum, the linear terms vanish and

$$A=\sum_{x\neq 0} c(-Q(x))\Big[-\frac{\alpha-\bar\alpha}{2iv}\Big(\frac{(\alpha+\bar\alpha)}{2}-\frac{u(\alpha-\bar\alpha)}{2iv}\Big)\Big].$$

Using the hermitian version of Borcherds quadratic identity, as in the proof of Proposition 5.2.2, we obtain

$$A=\frac{uI}{2v^2}\cdot\langle\beta,\beta\rangle.$$

Thus we have

$$P_1(\tau,w_0+\beta)=P_1(\tau,w_0)\cdot\exp\Big(\frac{\pi}{v}\langle w_0,\beta\rangle+\frac{\pi}{2v}\langle\beta,\beta\rangle\Big)^I\cdot\exp\Big(\frac{-2\pi i u\langle\beta,\beta\rangle}{4v^2}\Big)^I.$$

Finally, we recall the conjugate linear isomorphism $L_{-1}\cong\mathfrak{b}$ of (3.9.11) defined by $e_{-1}\mapsto\tau$ and $f_{-1}\mapsto 1$. As

$$\mathfrak{d}^{-1}L_{-1}=\mathbb{Z}e_{-1}+D^{-1}\mathbb{Z}f_{-1},$$

we have $-\delta^{-1}\tau=a\tau+D^{-1}b$ for some $a,b\in\mathbb{Z}$, and hence

$$\tau=-D^{-1}b(a+\delta^{-1})^{-1}.$$

This gives $u/v=a\,D^{\frac12}$. Also, using

$$\delta e_{-1}=-Dae_{-1}-b\,f_{-1},$$

we have

$$\frac12(1+\delta)\,e_{-1}=\frac12(1-Da)\,e_{-1}-\frac12 b\,f_{-1}\in\mathbb{Z}e_{-1}+\mathbb{Z}f_{-1}=L_{-1}.$$

Thus a is odd and b is even. Recall that $\mathrm{N}(\mathfrak{b})=2v/\sqrt{D}$. Thus

$$\frac{u}{4v^2}=\frac{aD^{\frac12}}{2\mathrm{N}(\mathfrak{b})D^{\frac12}},$$

and, since $\langle\beta,\beta\rangle\in\mathrm{N}(\mathfrak{b})$, we have

$$\exp\Big(-\frac{2\pi i u\langle\beta,\beta\rangle}{4v^2}\Big)=\exp\Big(-\frac{\pi i\langle\beta,\beta\rangle}{\mathrm{N}(\mathfrak{b})}\Big)=\pm1.$$

The transformation law is then

$$P_1(\tau, w_0 + \beta) = \exp\left(\frac{\pi}{v}\langle w_0, \beta\rangle + \frac{\pi}{2v}\langle \beta, \beta\rangle - i\pi\,\frac{\langle \beta, \beta\rangle}{\mathrm{N}(\mathfrak{b})}\right)^{I} \cdot P_1(\tau, w_0),$$

as claimed in Proposition 6.3.3.

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MODULARITY OF GENERATING SERIES OF DIVISORS ON UNITARY SHIMURA VARIETIES II: ARITHMETIC APPLICATIONS

by

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Abstract. — We prove two formulas in the style of the Gross-Zagier theorem, relating derivatives of L-functions to arithmetic intersection pairings on a unitary Shimura variety. We also prove a special case of Colmez's conjecture on the Faltings heights of abelian varieties with complex multiplication. These results are derived from the authors' earlier results on the modularity of generating series of divisors on unitary Shimura varieties.

Résumé **(Modularité des séries génératrices de diviseurs sur les variétés de Shimura unitaires II: applications arithmétiques)**

Nous prouvons deux formules dans le style du théorème de Gross-Zagier, reliant les dérivées des fonctions L aux accouplements d'intersection arithmétique sur une variété de Shimura unitaire. Nous prouvons également un cas particulier de la conjecture de Colmez sur les hauteurs de Faltings des variétés abéliennes à multiplication complexe. Ces résultats sont déduits des résultats antérieurs des auteurs sur la modularité des séries génératrices de diviseurs sur les variétés de Shimura unitaires.

1. Introduction

Fix an integer $n \geq 3$, and a quadratic imaginary field $\boldsymbol{k} \subset \mathbb{C}$ of odd discriminant $\mathrm{disc}(\boldsymbol{k}) = -D$. Let $\chi_{\boldsymbol{k}} : \mathbb{A}^\times \to \{\pm 1\}$ be the associated quadratic character, let $\mathfrak{d}_{\boldsymbol{k}} \subset \mathcal{O}_{\boldsymbol{k}}$ denote the different of $\boldsymbol{k}$, let $h_{\boldsymbol{k}}$ be the class number of $\boldsymbol{k}$, and let $w_{\boldsymbol{k}}$ be the number of roots of unity in $\boldsymbol{k}$.

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By a *hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice* we mean a projective $\mathcal{O}_{\boldsymbol{k}}$-module of finite rank endowed with a nondegenerate hermitian form.

1.1. Arithmetic theta lifts. — Suppose we are given a pair $(\mathfrak{a}_0, \mathfrak{a})$ in which
— $\mathfrak{a}_0$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(1,0)$,
— $\mathfrak{a}$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(n-1,1)$.
This pair determines hermitian $\boldsymbol{k}$-spaces $W_0 = \mathfrak{a}_{0\mathbb{Q}}$ and $W = \mathfrak{a}_{\mathbb{Q}}$.

From this data we constructed in [**6**] a smooth Deligne-Mumford stack $\mathrm{Sh}(G, \mathcal{D})$ of dimension $n-1$ over $\boldsymbol{k}$ with complex points

$$\mathrm{Sh}(G,\mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K.$$

The reductive group $G \subset \mathrm{GU}(W_0) \times \mathrm{GU}(W)$ is the largest subgroup on which the two similitude characters agree, and $K \subset G(\mathbb{A}_f)$ is the largest subgroup stabilizing the $\widehat{\mathbb{Z}}$-lattices $\widehat{\mathfrak{a}}_0 \subset W_0(\mathbb{A}_f)$ and $\widehat{\mathfrak{a}} \subset W(\mathbb{A}_f)$.

We also defined in [**6**, §2.3] an integral model

$$\mathcal{S}_{\mathrm{Kra}} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \tag{1.1.1}$$

of $\mathrm{Sh}(G,\mathcal{D})$. It is regular and flat over $\mathcal{O}_{\boldsymbol{k}}$, and admits a canonical toroidal compactification $\mathcal{S}_{\mathrm{Kra}} \hookrightarrow \mathcal{S}^*_{\mathrm{Kra}}$ whose boundary is a smooth divisor.

The main result of [**6**] is the construction of a formal generating series of arithmetic divisors

$$\widehat{\phi}(\tau) = \sum_{m\geq 0} \widehat{\mathcal{Z}}^{\mathrm{total}}_{\mathrm{Kra}}(m) \cdot q^m \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}})[[q]] \tag{1.1.2}$$

valued in the Gillet-Soulé codimension one arithmetic Chow group with rational coefficients, extended to allow log-log Green functions at the boundary as in [**10, 4**], and the proof that this generating series is modular of weight n, level $\Gamma_0(D)$, and character $\chi^n_{\boldsymbol{k}}$. The modularity result implies that the coefficients span a finite-dimensional subspace of the arithmetic Chow group [**6**, Remark 7.1.2].

After passing to the arithmetic Chow group with complex coefficients, for any classical modular form

$$g \in S_n(\Gamma_0(D), \chi^n_{\boldsymbol{k}})$$

we may form the Petersson inner product

$$\langle \widehat{\phi}, g\rangle_{\mathrm{Pet}} = \int_{\Gamma_0(D)\backslash\mathcal{H}} \overline{g(\tau)} \cdot \widehat{\phi}(\tau)\, \frac{du\, dv}{v^{2-n}}$$

where $\tau = u + iv$. As in [**24**], define the *arithmetic theta lift*

$$\widehat{\theta}(g) = \langle \widehat{\phi}, g\rangle_{\mathrm{Pet}} \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}). \tag{1.1.3}$$

Armed with the construction of the arithmetic theta lift (1.1.3), we are now able to complete the program of [**18, 19, 7**] to prove Gross-Zagier style formulas relating arithmetic intersections to derivatives of L-functions.

The Shimura variety $\mathcal{S}^*_{\mathrm{Kra}}$ carries different families of codimension $n-1$ cycles constructed from complex multiplication points, and our results show that the arithmetic intersections of these families with arithmetic lifts are related to central derivatives of L-functions.

1.2. Central derivatives and small CM points. — In §2 we construct an étale and proper Deligne-Mumford stack $\mathcal{Y}_{\mathrm{sm}}$ over $\mathcal{O}_{\boldsymbol{k}}$, along with a morphism

$$\mathcal{Y}_{\mathrm{sm}} \to \mathcal{S}^*_{\mathrm{Kra}}.$$

This is the *small CM cycle*. Intersecting arithmetic divisors against $\mathcal{Y}_{\mathrm{sm}}$ defines a linear functional

$$[-:\mathcal{Y}_{\mathrm{sm}}] : \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C},$$

and our first main result computes the image of the arithmetic theta lift (1.1.3) under this linear functional.

The statement involves the convolution L-function $L(\tilde{g}, \theta_\Lambda, s)$ of two modular forms

$$\tilde{g} \in S_n(\overline{\omega}_L), \qquad \theta_\Lambda \in M_{n-1}(\omega_\Lambda^\vee)$$

valued in finite-dimensional representations of $\mathrm{SL}_2(\mathbb{Z})$. We refer the reader to §2.3 for the precise definitions. Here we note only that $\tilde{g}$ is the image of g under an induction map

$$S_n(\Gamma_0(D), \chi^n_{\boldsymbol{k}}) \to S_n(\overline{\omega}_L) \tag{1.2.1}$$

from scalar-valued forms to vector-valued forms, that θ_Λ is the theta function attached to a quadratic space Λ over $\mathbb{Z}$ of signature $(2n-2, 0)$, and that the L-function $L(\tilde{g}, \theta_\Lambda, s)$ vanishes at its center of symmetry $s = 0$.

Theorem A. — *The arithmetic theta lift* (1.1.3) *satisfies*

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{sm}}] = -\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) \cdot \frac{d}{ds} L(\tilde{g}, \theta_\Lambda, s)\big|_{s=0}.$$

Here we have defined

$$\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) = \sum_{y \in \mathcal{Y}_{\mathrm{sm}}(\mathbb{C})} \frac{1}{|\mathrm{Aut}(y)|},$$

where the sum is over the finitely many isomorphism classes of the groupoid of complex points of $\mathcal{Y}_{\mathrm{sm}}$, viewed as an $\mathcal{O}_{\boldsymbol{k}}$-stack.

The proof is given in §2, by combining the modularity result of [**6**] with the main result of [7]. In §3 we provide alternative formulations of Theorem A that involve the usual convolution L-function of scalar-valued modular forms, as opposed to the vector-valued forms $\tilde{g}$ and θ_Λ. See especially Theorem 3.4.1.

1.3. Central derivatives and big CM points. — Fix a totally real field F of degree n, and define a CM field

$$E = \boldsymbol{k} \otimes_{\mathbb{Q}} F.$$

Let $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ be a CM type of signature $(n-1, 1)$, in the sense that there is a unique $\varphi^{\mathrm{sp}} \in \Phi$, called the *special embedding*, whose restriction to $\boldsymbol{k}$ agrees with the complex conjugate of the inclusion $\boldsymbol{k} \subset \mathbb{C}$. The reflex field of the pair (E, Φ) is

$$E_\Phi = \varphi^{\mathrm{sp}}(E) \subset \mathbb{C},$$

and we denote by $\mathcal{O}_\Phi \subset E_\Phi$ its ring of integers.

We define in §4.2 an étale and proper Deligne-Mumford stack $\mathcal{Y}_{\mathrm{big}}$ over $\mathcal{O}_\Phi$, along with a morphism of $\mathcal{O}_{\boldsymbol{k}}$-stacks

$$\mathcal{Y}_{\mathrm{big}} \to \mathcal{S}^*_{\mathrm{Kra}}.$$

This is the *big CM cycle*. Here we view $\mathcal{Y}_{\mathrm{big}}$ as an $\mathcal{O}_{\boldsymbol{k}}$-stack using the inclusion $\mathcal{O}_{\boldsymbol{k}} \subset \mathcal{O}_\Phi$ of subrings of $\mathbb{C}$ (which is the complex conjugate of the special embedding $\varphi^{\mathrm{sp}} : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_\Phi$). Intersecting arithmetic divisors against $\mathcal{Y}_{\mathrm{big}}$ defines a linear functional

$$[- : \mathcal{Y}_{\mathrm{big}}] : \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}.$$

Our second main result relates the image of the arithmetic theta lift (1.1.3) under this linear functional to the central derivative of a generalized L-function defined as the Petersson inner product $\langle E(s), \tilde{g}\rangle_{\mathrm{Pet}}$. The modular form $\tilde{g}(\tau)$ is, once again, the image of $g(\tau)$ under the induction map (1.2.1). The modular form $E(\tau, s)$ is defined as the restriction via the diagonal embedding $\mathcal{H} \to \mathcal{H}^n$ of a weight one Hilbert modular Eisenstein series valued in the space of the contragredient representation $\omega_L^\vee$. See §4.3 for details.

Theorem B. — *Assume that the discriminants of $\boldsymbol{k}/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime. The arithmetic theta lift* (1.1.3) *satisfies*

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{big}}] = \frac{-1}{n} \cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) \cdot \frac{d}{ds} \langle E(s), \tilde{g}\rangle_{\mathrm{Pet}}\big|_{s=0}.$$

Here we have defined

$$\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) = \sum_{y \in \mathcal{Y}_{\mathrm{big}}(\mathbb{C})} \frac{1}{|\mathrm{Aut}(y)|},$$

where the sum is over the finitely many isomorphism classes of the groupoid of complex points of $\mathcal{Y}_{\mathrm{big}}$, viewed as an $\mathcal{O}_{\boldsymbol{k}}$-stack.

The proof is given in §4, by combining the modularity result of [**6**] with the intersection calculations of [**8, 18, 19**].

1.4. Colmez's conjecture. — Suppose E is a CM field with maximal totally real subfield F. Let D_E and D_F be the absolute discriminants of E and F, set $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, and define the completed L-function

$$\Lambda(s,\chi_E) = \left|\frac{D_E}{D_F}\right|^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s+1)^{[F:\mathbb{Q}]} L(s,\chi_E)$$

of the character $\chi_E : \mathbb{A}_F^\times \to \{\pm 1\}$ determined by E/F. It satisfies the functional equation $\Lambda(1-s,\chi_E) = \Lambda(s,\chi_E)$, and

$$\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)} = \frac{L'(0,\chi_E)}{L(0,\chi_E)} + \frac{1}{2}\log\left|\frac{D_E}{D_F}\right| - \frac{[F:\mathbb{Q}]}{2}\log(4\pi e^\gamma),$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.

Suppose A is an abelian variety over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_E$ and CM type Φ. In particular A is defined over the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. It is a theorem of Colmez [**12**] that the Faltings height

$$h^{\mathrm{Falt}}_{(E,\Phi)} = h^{\mathrm{Falt}}(A)$$

depends only on the pair (E,Φ), and not on A itself. Moreover, Colmez gave a conjectural formula for this Faltings height in terms of logarithmic derivatives of Artin L-functions. In the special case where $E = \boldsymbol{k}$, Colmez's conjecture reduces to the well-known Chowla-Selberg formula

$$h^{\mathrm{Falt}}_{\boldsymbol{k}} = -\frac{1}{2}\cdot\frac{\Lambda'(0,\chi_{\boldsymbol{k}})}{\Lambda(0,\chi_{\boldsymbol{k}})} - \frac{1}{4}\cdot\log(16\pi^3 e^\gamma), \tag{1.4.1}$$

where we omit the CM type $\{\mathrm{id}\} \subset \mathrm{Hom}(\boldsymbol{k},\mathbb{C})$ from the notation.

Now suppose we are in the special case of §1.3, where

$$E = \boldsymbol{k} \otimes_{\mathbb{Q}} F$$

and $\Phi \subset \mathrm{Hom}(E,\mathbb{C})$ has signature $(n-1,1)$. In this case, Colmez's conjecture simplifies to the equality of the following theorem.

Theorem C **([29])**. — *For a pair (E,Φ) as above,*

$$h^{\mathrm{Falt}}_{(E,\Phi)} = -\frac{2}{n}\cdot\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)} + \frac{4-n}{2}\cdot\frac{\Lambda'(0,\chi_{\boldsymbol{k}})}{\Lambda(0,\chi_{\boldsymbol{k}})} - \frac{n}{4}\cdot\log(16\pi^3 e^\gamma).$$

In [**6**, §2.4] we defined the line bundle of weight one modular forms $\boldsymbol{\omega}$ on $\mathcal{S}^*_{\mathrm{Kra}}$. It was endowed it with a hermitian metric in [**6**, §7.2], and the resulting metrized line bundle determines a class

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Ch}}^1_{\mathbb{Q}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

The constant term of (1.1.2) is

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = -\widehat{\boldsymbol{\omega}} + (\mathrm{Exc}, -\log(D)) \tag{1.4.2}$$

where Exc is the *exceptional locus* of $\mathcal{S}^*_{\mathrm{Kra}}$ appearing in [**6**, Theorem 2.3.4]. It is a smooth effective Cartier divisor supported in characteristics dividing D, and we view it as an arithmetic divisor by endowing it with the constant Green function $-\log(D)$ in the complex fiber.

Theorem D. — *The metrized line bundle $\widehat{\boldsymbol{\omega}}$ satisfies*

$$[\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{big}}] = \frac{-2}{n} \cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) \cdot \frac{\Lambda'(0, \chi_E)}{\Lambda(0, \chi_E)}.$$

Theorem C is proved in [**29**] as a consequence of the average version of Colmez's conjecture [**2, 30, 20**]. Note that the proof in [**29**] does not require our standing hypothesis that $\mathrm{disc}(\boldsymbol{k})$ is odd. Of course the assumption that $\mathrm{disc}(\boldsymbol{k})$ is odd is still needed for Theorem D, as it is only under these hypotheses that we have even defined the integral model $\mathcal{S}^*_{\mathrm{Kra}}$ and its line bundle of weight one modular forms.

In §5 we will show that Theorems C and D are equivalent. One can interpret this in one of two ways. As Theorem C is already known, this equivalence proves Theorem D. On the other hand, in §4.5 will give an independent proof of Theorem D under the additional assumption that the discriminants of $\boldsymbol{k}$ and F are odd and relatively prime. In this way we obtain a new proof of Theorem C under these extra hypotheses.

1.5. The case $n = 2$. — Throughout the introduction we have assumed that $n \geq 3$, and the reader might wonder how much of what we have written extends to the case $n = 2$.

As explained in [**6**, §1.6], when $n = 2$ the proof of the modularity of (1.1.2) breaks down because there is no known integral model of $\mathrm{Sh}(G, \mathcal{D})$ whose reduction at the primes of $\mathcal{O}_{\boldsymbol{k}}$ dividing D is normal. The existence of such a model when $n > 2$ is used in [*loc. cit.*] to compute the vertical components of divisors of Borcherds products.

When $n = 2$, the Shimura variety $\mathrm{Sh}(G, \mathcal{D})$ is essentially a union of modular curves (if the $\boldsymbol{k}$-hermitian space W admits an isotropic line) or compact quaternionic Shimura curves (if W is anisotropic). In either case the analogues of Theorems A and B are close in spirit to the Gross-Zagier theorem [**15**] and its generalizations [**31**]. In particular, the statement of Theorems A is quite parallel to the key result Theorem 6.1 in [**15**, Section 1.6]. If we interchange in the computation of $[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{sm}}]$ the order of taking the Petersson inner product and the height pairing, this quantity is very analogous to the left hand side of Theorem 6.1 in [**15**]. Both quantities are expressed as central derivatives of a Rankin convolution L-function of g and a binary theta function which is determined by the CM cycle in question. If g is a newform, then $\widehat{\theta}(g)$ should lie in a g-isotypical component and the height pairing in our Theorem A should be proportional to the height of the g-isotypical component of (a twist of) $\mathcal{Y}_{\mathrm{sm}}$. It would be interesting to make such a comparison precise. However, note that there are substantial differences as well. While we work with unitary Shimura varieties and CM points whose discriminants are equal to the level, Gross and Zagier work with GL_2 Shimura varieties and CM points whose discriminants are coprime to the level.

Theorem C is true as stated when $n = 2$, and is proved in [**29**]. Indeed, Colmez's conjecture is known for all quartic CM fields. If the quartic CM field is Galois over $\mathbb{Q}$, then the Galois group is abelian and Colmez's conjecture is known by work of Colmez

[12] and Obus [25]. In the non-Galois case the CM types form a single $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$-orbit; as Colmez's conjecture is constant on such orbits, the full Colmez conjecture follows from the average case proved in [2] and [30].

Theorem D is also true as stated when $n = 2$. Indeed, when we prove the equivalence of Theorems C and D in §5 we only assume $n \geq 2$.

1.6. Thanks. — The results of this paper are the outcome of a long term project, begun initially in Bonn in June of 2013, and supported in a crucial way by three weeklong meetings at AIM, in Palo Alto (May of 2014) and San Jose (November of 2015 and 2016), as part of their AIM SQuaRE's program. The opportunity to spend these periods of intensely focused efforts on the problems involved was essential. We would like to thank the University of Bonn and AIM for their support.

2. Small CM cycles and derivatives of L-functions

In this section we combine the results of [6] and [7] to prove Theorem A. Although we will restrict to $n \geq 3$ in §2.5, we allow $n \geq 2$ until that point.

2.1. A Shimura variety of dimension zero. — Define a rank three torus T_{sm} over $\mathbb{Q}$ as the fiber product

$$\begin{array}{ccc} T_{\mathrm{sm}} & \longrightarrow & \mathbb{G}_m \\ \big\downarrow & & \big\downarrow \scriptstyle{\mathrm{diag.}} \\ \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m & \xrightarrow[\mathrm{Nm}\times\mathrm{Nm}]{} & \mathbb{G}_m \times \mathbb{G}_m. \end{array}$$

Its group of $\mathbb{Q}$-points is

$$T_{\mathrm{sm}}(\mathbb{Q}) \cong \{(x, y) \in \boldsymbol{k}^\times \times \boldsymbol{k}^\times : x\overline{x} = y\overline{y}\}.$$

The fixed embedding $\boldsymbol{k} \subset \mathbb{C}$ identifies Deligne's torus $\mathbb{S}$ with the real algebraic group $(\mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m)_\mathbb{R}$, and the diagonal inclusion

$$\mathbb{S} \hookrightarrow (\mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m)_\mathbb{R} \times (\mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m)_\mathbb{R}$$

factors through a morphism $h_{\mathrm{sm}} : \mathbb{S} \to T_{\mathrm{sm},\mathbb{R}}$. The pair $(T_{\mathrm{sm}}, \{h_{\mathrm{sm}}\})$ is a Shimura datum, which, along with the compact open subgroup

$$K_{\mathrm{sm}} = T_{\mathrm{sm}}(\mathbb{A}_f) \cap (\widehat{\mathcal{O}}_{\boldsymbol{k}}^\times \times \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times),$$

determines a 0-dimensional $\boldsymbol{k}$-stack $\mathrm{Sh}(T_{\mathrm{sm}})$ with complex points

$$\mathrm{Sh}(T_{\mathrm{sm}})(\mathbb{C}) = T_{\mathrm{sm}}(\mathbb{Q})\backslash\{h_{\mathrm{sm}}\} \times T_{\mathrm{sm}}(\mathbb{A}_f)/K_{\mathrm{sm}}.$$

2.2. The small CM cycle. — The Shimura variety just constructed has a moduli interpretation, which allows us to construct an integral model. The interpretation we have in mind requires first choosing a triple $(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b})$ in which

— $\mathfrak{a}_0$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(1,0)$,
— $\mathfrak{a}_1$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(0,1)$,
— $\mathfrak{b}$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(n-1,0)$.

The hermitian forms on $\mathfrak{a}_0$ and $\mathfrak{b}$ induce a hermitian form of signature $(n-1,0)$ on the projective $\mathcal{O}_{\boldsymbol{k}}$-module

$$\Lambda = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{b}),$$

as explained in [**7**, §2.1] or [**6**, (2.1.5)].

Recall from [**7**, §3.1] or [**6**, §2.3] the $\mathcal{O}_{\boldsymbol{k}}$-stacks $\mathcal{M}_{(p,0)}$ and $\mathcal{M}_{(0,p)}$. Both parametrize abelian schemes $A \to S$ of relative dimension $p \geq 1$ over $\mathcal{O}_{\boldsymbol{k}}$-schemes, endowed with principal polarizations and $\mathcal{O}_{\boldsymbol{k}}$-actions. For the first moduli problem we impose the signature $(p,0)$ condition that $\mathcal{O}_{\boldsymbol{k}}$ acts on the $\mathcal{O}_S$-module $\mathrm{Lie}(A)$ via the structure morphism $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$. For the second we impose the signature $(0,p)$ condition that the action is by the complex conjugate of the structure morphism. Both of these stacks are étale and proper over $\mathcal{O}_{\boldsymbol{k}}$ by [**19**, Proposition 2.1.2].

Remark 2.2.1. — The generic fibers of $\mathcal{M}_{(1,0)}$ and $\mathcal{M}_{(0,1)}$ are the Shimura varieties associated to $\mathfrak{a}_{0\mathbb{Q}}$ and $\mathfrak{a}_{1\mathbb{Q}}$, while the generic fiber of $\mathcal{M}_{(n-1,0)}$ contains the Shimura variety associated to $\mathfrak{b}_{\mathbb{Q}}$ as an open and closed substack. For more precise information, see [**23**, Proposition 2.13] and the lemma that precedes it.

Denote by $\widetilde{\mathcal{Y}}_{\mathrm{sm}}$ the functor that associates to every $\mathcal{O}_{\boldsymbol{k}}$-scheme S the groupoid of quadruples (A_0, A_1, B, η) in which

$$(A_0, A_1, B) \in \mathcal{M}_{(1,0)}(S) \times \mathcal{M}_{(0,1)}(S) \times \mathcal{M}_{(n-1,0)}(S), \tag{2.2.1}$$

and

$$\eta : \underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B) \cong \underline{\Lambda} \tag{2.2.2}$$

is an isomorphism of étale sheaves of hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules, where the hermitian form on the left hand side is defined as in [**6**, (2.5.1)]. We impose the further condition that for every geometric point $s \to S$, and every prime $\ell \neq \mathrm{char}(s)$, there is an isomorphism of hermitian $\mathcal{O}_{\boldsymbol{k},\ell}$-lattices

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}[\ell^\infty], A_{1s}[\ell^\infty]) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}_1) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell. \tag{2.2.3}$$

Lemma 2.2.2. — *If*

$$s \to \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(0,1)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-1,0)}$$

is a geometric point of characteristic 0 *such that* (2.2.3) *holds for all primes* ℓ *except possibly one, then it holds for the remaining prime as well.*

Proof. — The proof is identical to [**6**, Lemma 2.2.2]. □

Proposition 2.2.3. — *The functor $\widetilde{\mathcal{Y}}_{\mathrm{sm}}$ is represented by a Deligne-Mumford stack, étale and proper over $\mathcal{O}_{\boldsymbol{k}}$, and there is a canonical isomorphism of $\boldsymbol{k}$-stacks*

$$\mathrm{Sh}(T_{\mathrm{sm}}) \cong \widetilde{\mathcal{Y}}_{\mathrm{sm}/\boldsymbol{k}}. \tag{2.2.4}$$

Proof. — For any $\mathcal{O}_{\boldsymbol{k}}$-scheme S, let $\mathcal{N}(S)$ be the groupoid of triples (2.2.1) satisfying (2.2.3) for every geometric point $s \to S$ and every prime $\ell \neq \mathrm{char}(s)$. In other words, the definition is the same as $\widetilde{\mathcal{Y}}_{\mathrm{sm}}$ except that we omit the datum (2.2.2) from the moduli problem.

We interrupt the proof of Proposition 2.2.3 for a lemma.

Lemma 2.2.4. — *The functor $\mathcal{N}$ is represented by an open and closed substack*

$$\mathcal{N} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(0,1)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-1,0)}.$$

Proof. — This is [**7**, Proposition 5.2]. As the proof there is left to the reader, we indicate the idea. Let

$$\mathcal{B} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(0,1)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{M}_{(n-1,0)}$$

be one connected component, and suppose there is a geometric point $s \to \mathcal{B}$ of characteristic p such that (2.2.3) holds for all $\ell \neq p$. The geometric fibers of the ℓ-adic sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(A_0[\ell^\infty], A_1[\ell^\infty])$ on

$$\mathcal{B}_{(p)} = \mathcal{B} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}_{(p)})$$

are all isomorphic, and therefore (2.2.3) holds for all geometric points $s \to \mathcal{B}_{(p)}$ and all $\ell \neq p$. In particular, using Lemma 2.2.2, if $s \to \mathcal{B}$ is a geometric point of characteristic 0, then (2.2.3) holds for every prime ℓ. Having proved this, one can reverse the argument to see that (2.2.3) holds for *every* geometric point $s \to \mathcal{B}$ and every $\ell \neq \mathrm{char}(s)$. Thus if the condition (2.2.3) holds at one geometric point, it holds at all geometric points on the same connected component. □

We now return to the proof of Proposition 2.2.3. As noted above, the stacks $\mathcal{M}_{(p,0)}$ and $\mathcal{M}_{(0,p)}$ are étale and proper over $\mathcal{O}_{\boldsymbol{k}}$, and hence the same is true of $\mathcal{N}$.

Let (A_0, A_1, B) be the universal object over $\mathcal{N}$. Combining [**7**, Theorem 5.1] and [**17**, Corollary 6.9], the étale sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B)$ is represented by a Deligne-Mumford stack whose connected components are finite étale over $\mathcal{N}$. Fixing a geometric point $s \to \mathcal{N}$, we obtain a representation of $\pi_1^{et}(\mathcal{N}, s)$ on a finitely generated $\mathcal{O}_{\boldsymbol{k}}$-module $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_{0s}, B_s)$, and the kernel of this representation cuts out a finite étale cover $\mathcal{N}' \to \mathcal{N}$ over which the sheaf $\underline{\mathrm{Hom}}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B)$ becomes constant.

It is now easy to see that the functor $\widetilde{\mathcal{Y}}_{\mathrm{sm}}$ is represented by the disjoint union of finitely many copies of the maximal open and closed substack of $\mathcal{N}'$ over which there exists an isomorphism (2.2.2).

It remains to construct the isomorphism (2.2.4). The natural actions of $\mathcal{O}_{\boldsymbol{k}}$ on $\mathfrak{a}_0$ and $\mathfrak{b}$, along with the *complex conjugate* of the natural action of $\mathcal{O}_{\boldsymbol{k}}$ on $\mathfrak{a}_1$, determine a morphism of reductive groups

$$\mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \xrightarrow{(w,z)\mapsto(w,\overline{z},z)} \mathrm{GU}(\mathfrak{a}_{0\mathbb{Q}}) \times \mathrm{GU}(\mathfrak{a}_{1\mathbb{Q}}) \times \mathrm{GU}(\mathfrak{b}_{\mathbb{Q}}).$$

Restricting this morphism to the subtorus T_{sm} defines a morphism

$$\mathbb{S} \xrightarrow{h_{\mathrm{sm}}} T_{\mathrm{sm},\mathbb{R}} \to \mathrm{GU}(\mathfrak{a}_{0\mathbb{R}}) \times \mathrm{GU}(\mathfrak{a}_{1\mathbb{R}}) \times \mathrm{GU}(\mathfrak{b}_{\mathbb{R}}),$$

endowing the real vector spaces $\mathfrak{a}_{0\mathbb{R}}$, $\mathfrak{a}_{1\mathbb{R}}$, and $\mathfrak{b}_{\mathbb{R}}$ with complex structures.

The isomorphism (2.2.4) on complex points sends a pair

$$(h_{\mathrm{sm}}, g) \in \mathrm{Sh}(T_{\mathrm{sm}})(\mathbb{C})$$

to the quadruple (A_0, A_1, B, η) defined by

$$A_0(\mathbb{C}) = \mathfrak{a}_{0\mathbb{R}}/g\mathfrak{a}_0, \quad A_1(\mathbb{C}) = \mathfrak{a}_{1\mathbb{R}}/g\mathfrak{a}_1, \quad B(\mathbb{C}) = \mathfrak{b}_{\mathbb{R}}/g\mathfrak{b},$$

endowed with their natural $\mathcal{O}_{\boldsymbol{k}}$-actions and polarizations as in the proof of [**6**, Proposition 2.2.1]. The datum η is the canonical identification

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, B) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(g\mathfrak{a}_0, g\mathfrak{b}) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{b}) = \Lambda.$$

It follows from the theory of canonical models that this isomorphism on complex points descends to an isomorphism of $\boldsymbol{k}$-stacks, completing the proof of Proposition 2.2.3. □

The finite group $\mathrm{Aut}(\Lambda)$ of automorphisms of the hermitian lattice Λ acts on $\widetilde{\mathcal{Y}}_{\mathrm{sm}}$ by

$$\gamma * (A_0, A_1, B, \eta) = (A_0, A_1, B, \gamma \circ \eta),$$

allowing us to form the stack quotient $\mathcal{Y}_{\mathrm{sm}} = \mathrm{Aut}(\Lambda)\backslash\widetilde{\mathcal{Y}}_{\mathrm{sm}}$. The forgetful map

$$\widetilde{\mathcal{Y}}_{\mathrm{sm}} \to \mathcal{M}_{(1,0)} \times \mathcal{M}_{(0,1)} \times \mathcal{M}_{(n-1,0)}$$

(all fiber products over $\mathcal{O}_{\boldsymbol{k}}$) factors through an open and closed immersion

$$\mathcal{Y}_{\mathrm{sm}} \to \mathcal{M}_{(1,0)} \times \mathcal{M}_{(0,1)} \times \mathcal{M}_{(n-1,0)}$$

whose image is the open and closed substack $\mathcal{N}$ of Lemma 2.2.4.

The triple $(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b})$ determines a pair $(\mathfrak{a}_0, \mathfrak{a})$ as in the introduction, simply by setting $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{b}$. This data determines a unitary Shimura variety with integral model $\mathcal{S}_{\mathrm{Kra}}$ as in (1.1.1), and there is a commutative diagram

$$\begin{array}{ccc}
\mathcal{Y}_{\mathrm{sm}} & \longrightarrow & \mathcal{M}_{(1,0)} \times \mathcal{M}_{(0,1)} \times \mathcal{M}_{(n-1,0)} \\
{\scriptstyle \pi}\downarrow & & \downarrow \\
\mathcal{S}_{\mathrm{Kra}} & \xrightarrow{\subset} & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}.
\end{array}$$

The vertical arrow on the right sends

$$(A_0, A_1, B) \mapsto (A_0, A_1 \times B),$$

and the arrow π is defined by the commutativity of the diagram.

Remark 2.2.5. — In order for $A_1 \times B$ to define a point of $\mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}$, we must endow its Lie algebra with a codimension one subsheaf

$$\mathcal{F}_{A_1 \times B} \subset \mathrm{Lie}(A_1 \times B)$$

satisfying Krämer's condition [**6**, §2.3]. We choose $\mathcal{F}_{A_1 \times B} = \mathrm{Lie}(B)$.

Definition 2.2.6. — Composing the morphism π in the diagram above with the inclusion of $\mathcal{S}_{\mathrm{Kra}}$ into its toroidal compactification, we obtain a morphism of $\mathcal{O}_{\boldsymbol{k}}$-stacks

$$\pi : \mathcal{Y}_{\mathrm{sm}} \to \mathcal{S}^*_{\mathrm{Kra}}$$

called the *small CM cycle*.

As in [**19**, Definition 3.1.8], there is a linear functional

$$\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}$$

called *arithmetic degree along* $\mathcal{Y}_{\mathrm{sm}}$ and denoted $\widehat{\mathcal{Z}} \mapsto [\widehat{\mathcal{Z}} : \mathcal{Y}_{\mathrm{sm}}]$, defined as the composition

$$\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \xrightarrow{\pi^*} \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) \xrightarrow{\widehat{\deg}} \mathbb{C}.$$

The first arrow is pullback of arithmetic divisors. The second arrow (*arithmetic degree*) is normalized as follows: An irreducible divisor $\mathcal{Z} \subset \mathcal{Y}_{\mathrm{sm}}$ is necessarily supported in finitely many nonzero characteristics, and hence any $\mathbb{C}$-valued function $\mathrm{Gr}(\mathcal{Z}, .)$ on the finite set $\mathcal{Y}_{\mathrm{sm}}(\mathbb{C})$ defines a Green function for it. The arithmetic degree of the arithmetic divisor

$$(\mathcal{Z}, \mathrm{Gr}(\mathcal{Z}, .)) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}})$$

is defined to be

$$\widehat{\deg}(\mathcal{Z}, \mathrm{Gr}(\mathcal{Z}, .)) = \sum_{\mathfrak{q} \subset \mathcal{O}_{\boldsymbol{k}}} \sum_{z \in \mathcal{Z}(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{q}})} \frac{\log(\mathrm{N}(\mathfrak{q}))}{\#\mathrm{Aut}_{\mathcal{X}}(z)} + \sum_{z \in \mathcal{Y}_{\mathrm{sm}}(\mathbb{C})} \frac{\mathrm{Gr}(\mathcal{Z}, z)}{\#\mathrm{Aut}_{\mathcal{Y}_{\mathrm{sm}}(\mathbb{C})}(z)},$$

where $\mathbb{F}^{\mathrm{alg}}_{\mathfrak{q}}$ is an algebraic closure of $\mathcal{O}_{\boldsymbol{k}}/\mathfrak{q}$, and $\mathrm{N}(\mathfrak{q}) = \#(\mathcal{O}_{\boldsymbol{k}}/\mathfrak{q})$.

Remark 2.2.7. — The above definition of arithmetic degree does not include a factor of $1/2$ in front of the archimedean contribution, seemingly in disagreement with the usual definition (see [**13**, §3.4.3] for example). In fact there is no disagreement. Our convention is that $\mathcal{Y}_{\mathrm{sm}}(\mathbb{C})$ means the complex points of $\mathcal{Y}_{\mathrm{sm}}(\mathbb{C})$ as a $\boldsymbol{k}$-stack, whereas in the usual definition it would be regarded as a $\mathbb{Q}$-stack. Thus the usual definition includes a sum over twice as many complex points, but with a $1/2$ in front.

Remark 2.2.8. — The small CM cycle arises from a morphism of Shimura varieties. Indeed, there is a morphism of Shimura data $(T_{\mathrm{sm}}, \{h_{\mathrm{sm}}\}) \to (G, \mathcal{D})$, and the induced morphism of Shimura varieties sits in a commutative diagram

$$\begin{array}{ccc} \mathrm{Sh}(T_{\mathrm{sm}}) & \longrightarrow & \mathrm{Sh}(G, \mathcal{D}) \\ \cong\downarrow & & \downarrow\cong \\ \widetilde{\mathcal{Y}}_{\mathrm{sm}/\boldsymbol{k}} \longrightarrow \mathcal{Y}_{\mathrm{sm}/\boldsymbol{k}} & \xrightarrow{\pi} & \mathcal{S}_{\mathrm{Kra}/\boldsymbol{k}}. \end{array}$$

Proposition 2.2.9. — *The degree* $\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}})$ *of Theorem A satisfies*

$$\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) = (h_{\boldsymbol{k}}/w_{\boldsymbol{k}})^2 \cdot \frac{2^{1-o(D)}}{|\mathrm{Aut}(\Lambda)|},$$

where $o(D)$ is the number of distinct prime divisors of D.

Proof. — This is an elementary calculation. Briefly, the groupoid $\mathcal{Y}_{\mathrm{sm}}(\mathbb{C})$ has $2^{1-o(D)}h_{\boldsymbol{k}}^2$ isomorphism classes of points, and each point has the same automorphism group $\mathcal{O}_{\boldsymbol{k}}^\times \times \mathcal{O}_{\boldsymbol{k}}^\times \times U(\Lambda)$. □

Recall from (1.4.2) that the constant term of (1.1.2) is

$$\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = -\widehat{\boldsymbol{\omega}} + (\mathrm{Exc}, -\log(D)),$$

where $\widehat{\boldsymbol{\omega}}$ is the metrized line bundle of weight one modular forms. The exceptional locus $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ was defined in [**6**, §2.3]. It is a reduced effective Cartier divisor supported in characteristics dividing D, and can be characterized as follows. The integral model $\mathcal{S}_{\mathrm{Kra}}$ carries over it an abelian scheme $A \to \mathcal{S}_{\mathrm{Kra}}$ of relative dimension n endowed with an action of $\mathcal{O}_{\boldsymbol{k}}$. This abelian scheme is obtained by pulling back the universal object from the second factor of the fiber product in (1.1.1). If we let $\delta \in \mathcal{O}_{\boldsymbol{k}}$ be a fixed square root of $-D$, then Exc is the reduced stack underlying closed substack of $\mathcal{S}_{\mathrm{Kra}}$ defined by $\delta \cdot \mathrm{Lie}(A) = 0$.

Proposition 2.2.10. — *The constant term* (1.4.2) *satisfies*

$$[\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) : \mathcal{Y}_{\mathrm{sm}}] = -[\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{sm}}] = 2\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) \cdot \frac{\Lambda'(0,\chi_{\boldsymbol{k}})}{\Lambda(0,\chi_{\boldsymbol{k}})}.$$

Proof. — The second equality was proved in the course of proving [**7**, Theorem 6.4]. We note that the argument uses the Chowla-Selberg formula (1.4.1) in an essential way.

The first equality is equivalent to

$$[(\mathrm{Exc}, -\log(D)) : \mathcal{Y}_{\mathrm{sm}}] = 0,$$

and so it suffices to prove

$$[(0, \log(D)) : \mathcal{Y}_{\mathrm{sm}}] = \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) \cdot \log(D) = [(\mathrm{Exc}, 0) : \mathcal{Y}_{\mathrm{sm}}]. \tag{2.2.5}$$

The first equality in (2.2.5) is obvious from the definitions. To prove the second equality, we first prove

$$\mathcal{Y}_{\mathrm{sm}} \times_{\mathcal{S}_{\mathrm{Kra}}} \mathrm{Exc} = \mathcal{Y}_{\mathrm{sm}} \times_{\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})} \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}/\mathfrak{d}_{\boldsymbol{k}}). \tag{2.2.6}$$

As the exceptional locus $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ is reduced and supported in characteristics dividing D, it satisfies

$$\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}} \times_{\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})} \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}/\mathfrak{d}_{\boldsymbol{k}}).$$

This implies the inclusion $\subset$ in (2.2.6). As $\mathcal{Y}_{\mathrm{sm}}$ is étale over $\mathcal{O}_{\boldsymbol{k}}$, the right hand side of (2.2.6) is reduced, and hence so is the left hand side. To prove that equality holds in (2.2.6), it now suffices to check the inclusion $\supset$ on the level of geometric points.

As above, let $\delta \in \mathcal{O}_{\boldsymbol{k}}$ be a square root of $-D$. Suppose $p \mid D$ is a prime, $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$ is the unique prime above it, and $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$ is an algebraic closure of its residue field. Suppose we have a point $y \in \mathcal{Y}_{\mathrm{sm}}(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})$ corresponding to a triple (A_0, A_1, B) over $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$. As $\delta = 0$

in $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$, the signature conditions imply that the endomorphism $\delta \in \mathcal{O}_{\boldsymbol{k}}$ kills the Lie algebras of A_0, A_1, and B. In particular δ kills the Lie algebra of $A_1 \times B$, which is the pullback via

$$\pi : \mathcal{Y}_{\mathrm{sm}} \to \mathcal{S}_{\mathrm{Kra}}$$

of the universal $A \to \mathcal{S}_{\mathrm{Kra}}$. Using the characterization of Exc recalled above, we find that that $\pi(y) \in \mathrm{Exc}$. This proves (2.2.6).

The equality (2.2.6), and the fact that both sides of that equality are reduced, implies that

$$[(\mathrm{Exc}, 0) : \mathcal{Y}_{\mathrm{sm}}] = \sum_{p \mid D} \log(p) \sum_{y \in \mathcal{Y}_{\mathrm{sm}}(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})} \frac{1}{|\mathrm{Aut}(y)|}.$$

On the other hand, the étaleness of $\mathcal{Y}_{\mathrm{sm}} \to \mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$ implies that the right hand side is equal to

$$\sum_{p \mid D} \log(p) \sum_{y \in \mathcal{Y}_{\mathrm{sm}}(\mathbb{C})} \frac{1}{|\mathrm{Aut}(y)|} = \log(D) \cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}),$$

completing the proof of the second equality in (2.2.5). □

2.3. The convolution L-function. — Recall that we have defined a hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice $\Lambda = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{b})$ of signature $(n-1, 0)$. We also define hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices

$$L_0 = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}_1), \quad L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}),$$

of signature $(1,0)$ and $(n-1,1)$, so that $L \cong L_0 \oplus \Lambda$.

The hermitian form $\langle .,. \rangle : L \times L \to \mathcal{O}_{\boldsymbol{k}}$ determines a $\mathbb{Z}$-valued quadratic form $Q(x) = \langle x, x \rangle$ on L, and we denote in the same way its restrictions to L_0 and Λ. The dual lattice of L with respect to the $\mathbb{Z}$-bilinear form

$$[x_1, x_2] = Q(x_1 + x_2) - Q(x_1) - Q(x_2) \tag{2.3.1}$$

is $L' = \mathfrak{d}_{\boldsymbol{k}}^{-1} L$.

As in [7, §2.2] we denote by $S_L = \mathbb{C}[L'/L]$ the space of complex-valued functions on L'/L, and by $\omega_L : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_L)$ the Weil representation. There is a complex conjugate representation $\overline{\omega}_L$ on S_L defined by

$$\overline{\omega}_L(\gamma)\phi = \overline{\omega_L(\gamma)\overline{\phi}}.$$

Suppose we begin with a classical scalar-valued cusp form

$$g(\tau) = \sum_{m > 0} c(m) q^m \in S_n(\Gamma_0(D), \chi_{\boldsymbol{k}}^n).$$

Such a form determines a vector-valued form

$$\tilde{g}(\tau) = \sum_{\gamma \in \Gamma_0(D) \backslash \mathrm{SL}_2(\mathbb{Z})} (g|_n \gamma)(\tau) \cdot \overline{\omega_L(\gamma^{-1})\phi_0} \in S_n(\overline{\omega}_L), \tag{2.3.2}$$

where $\phi_0 \in S_L$ is the characteristic function of the trivial coset. This construction defines the induction map (1.2.1). The form $\tilde{g}(\tau)$ has a q-expansion

$$\tilde{g}(\tau) = \sum_{m>0} \tilde{c}(m) q^m$$

with coefficients $\tilde{c}(m) \in S_L$.

There is a similar Weil representation $\omega_\Lambda : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_\Lambda)$, and for every $m \in \mathbb{Q}$ we define a linear functional $R_\Lambda(m) \in S_\Lambda^\vee$ by

$$R_\Lambda(m)(\phi) = \sum_{\substack{x \in \Lambda' \\ \langle x,x \rangle = m}} \phi(x)$$

where $\phi \in S_\Lambda$ and $\langle .,. \rangle : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \boldsymbol{k}$ is the $\mathbb{Q}$-linear extension of the hermitian form on Λ. The theta series

$$\theta_\Lambda(\tau) = \sum_{m \in \mathbb{Q}} R_\Lambda(m) q^m \in M_{n-1}(\omega_\Lambda^\vee)$$

is a modular form valued in the contragredient representation $S_\Lambda^\vee$.

As in [**7**, §5.3] or [**9**, §4.4], we define the *Rankin-Selberg convolution L-function*

$$L(\tilde{g}, \theta_\Lambda, s) = \Gamma\left(\frac{s}{2} + n - 1\right) \sum_{m \geq 0} \frac{\{\overline{\tilde{c}(m)}, R_\Lambda(m)\}}{(4\pi m)^{\frac{s}{2}+n-1}}. \tag{2.3.3}$$

Here $\{.,.\} : S_L \times S_L^\vee \to \mathbb{C}$ is the tautological pairing. The inclusion

$$\Lambda'/\Lambda \to L'/L$$

induces a linear map $S_L \to S_\Lambda$ by restriction of functions, and we use the dual $S_\Lambda^\vee \to S_L^\vee$ to view $R_\Lambda(m)$ as an element of $S_L^\vee$.

Remark 2.3.1. — The convolution L-function satisfies a functional equation in $s \mapsto -s$, forcing $L(\tilde{g}, \theta_\Lambda, 0) = 0$.

Remark 2.3.2. — In this generality, neither the cusp form g nor the theta series θ_Λ is a Hecke eigenform. Thus the convolution L-function (2.3.3) cannot be expected to have an Euler product expansion.

2.4. A preliminary central derivative formula. — We now recall the main result of [**7**], and explain the connection between the cycles and Shimura varieties here and in that work.

Define hermitian $\widehat{\mathcal{O}}_{\boldsymbol{k}}$-lattices

$$\mathbb{L}_{0,f} = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}_1) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}, \quad \mathbb{L}_f = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}},$$

and let $\mathbb{L}_{0,\infty}$ and $\mathbb{L}_\infty$ be $\boldsymbol{k}_{\mathbb{R}}$-hermitian spaces of signatures $(1,0)$ and $(n,0)$, respectively. In the terminology of [**7**, §2.1], the pairs

$$\mathbb{L}_0 = (\mathbb{L}_{0,\infty}, \mathbb{L}_{0,f}), \quad \mathbb{L} = (\mathbb{L}_\infty, \mathbb{L}_f)$$

are *incoherent hermitian* $(\boldsymbol{k}_{\mathbb{R}}, \widehat{\mathcal{O}}_{\boldsymbol{k}})$*-modules.* Our small CM cycle is related to the cycle of [7, §5.1] by

$$\begin{array}{ccc} \mathcal{Y}_{\mathrm{sm}} & \longrightarrow & \mathcal{S}_{\mathrm{Kra}} \\ \| & & \| \\ \mathcal{Y}_{(\mathbb{L}_0,\Lambda)} & \longrightarrow & \mathcal{M}_{\mathbb{L}}, \end{array}$$

and the metrized line bundle $\widehat{\boldsymbol{\omega}}^{-1}$ of [6] agrees with the metrized cotautological bundle $\widehat{\mathcal{T}}_{\mathbb{L}}$ of [7].

Let Δ be the automorphism group of the finite abelian group L'/L endowed with the quadratic form $L'/L \to \mathbb{Q}/\mathbb{Z}$ obtained by reduction of $Q : L \to \mathbb{Z}$. The tautological action of Δ on $S_L = \mathbb{C}[L'/L]$ commutes with the Weil representation ω_L, and hence Δ acts on all spaces of modular forms valued in the representation ω_L.

Let $H_{2-n}(\omega_L)$ be the space of harmonic Maass forms of [7, §2.2]. Every $f \in H_{2-n}(\omega_L)$ has a *holomorphic part*

$$f^+(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c_f^+(m) \cdot q^m,$$

which is a formal q-expansion with coefficients in S_L. Let $c_f^+(0,0)$ be the value of $c_f^+(0) \in S_L$ at the trivial coset.

As in [5] or [9, §3.1], there is a Δ-equivariant, surjective, conjugate linear differential operator

$$\xi : H_{2-n}(\omega_L) \to S_n(\overline{\omega}_L),$$

and the construction of [7, (4.15)] defines a linear functional

$$\widehat{\mathcal{Z}} : H_{2-n}(\omega_L)^{\Delta} \to \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}). \tag{2.4.1}$$

These are related by the main result of [7], which we now state.

Theorem 2.4.1 ([7]). — *The equality*

$$[\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathrm{sm}}] - c_f^+(0,0) \cdot [\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{sm}}] = -\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) \cdot L'(\xi(f), \theta_\Lambda, 0)$$

holds for any Δ*-invariant* $f \in H_{2-n}(\omega_L)$.

2.5. The proof of Theorem A. — Throughout §2.5 we assume $n \geq 3$. Under this assumption the linear functional (2.4.1) is closely related to the coefficients of the generating series (1.1.2). Indeed, If m is a positive integer, [7, Lemma 3.10] shows that there is a unique

$$f_m \in H_{2-n}(\omega_L)^{\Delta}$$

with holomorphic part

$$f_m^+(\tau) = \phi_0 \cdot q^{-m} + O(1), \tag{2.5.1}$$

where $\phi_0 \in S_L$ is the characteristic function of the trivial coset. Applying the above linear functional to this form recovers the m-th coefficient

$$\widehat{\mathcal{Z}}^{\text{tot}}_{\text{Kra}}(m) = \widehat{\mathcal{Z}}(f_m)$$

of the generating series (1.1.2).

The following proposition explains the connection between the linear functional (2.4.1) and the arithmetic theta lift (1.1.3).

Proposition 2.5.1. — *For every $g \in S_n(\Gamma_0(D), \chi_{\boldsymbol{k}}^n)$ there is a Δ-invariant form $f \in H_{2-n}(\omega_L)$ such that*

$$\widehat{\theta}(g) = \widehat{\mathcal{Z}}(f) + c_f^+(0,0) \cdot \widehat{\mathcal{Z}}^{\text{tot}}_{\text{Kra}}(0), \tag{2.5.2}$$

and such that $\xi(f)$ is equal to the form $\tilde{g} \in S_n(\overline{\omega}_L)$ defined by (2.3.2). *Moreover, we may choose f to be a linear combination of the forms f_m characterized by* (2.5.1).

Proof. — Consider the space $H^{\infty}_{2-n}(\Gamma_0(D), \chi_{\boldsymbol{k}}^n)$ of harmonic Maass forms of [**6**, §7.2]. The constructions of [**5**] provide us with a surjective conjugate linear differential operator

$$\xi : H^{\infty}_{2-n}(\Gamma_0(D), \chi_{\boldsymbol{k}}^n) \to S_n(\Gamma_0(D), \chi_{\boldsymbol{k}}^n),$$

and we choose an $f_0 \in H^{\infty}_{2-n}(\Gamma_0(D), \chi_{\boldsymbol{k}}^n)$ such that $\xi(f_0) = g$. It is easily seen that f_0 may be chosen to vanish at all cusps of $\Gamma_0(D)$ different from ∞. This can, for instance, be attained by adding a suitable weakly holomorphic form in the space $M^{!,\infty}_{2-n}(\Gamma_0(D), \chi_{\boldsymbol{k}}^n)$ of [**6**, §4.2]. The Fourier expansion of the holomorphic part of f_0 is denoted

$$f_0^+(\tau) = \sum_{m \in \mathbb{Q}} c_0^+(m) q^m.$$

As in (2.3.2), the form f_0 determines an S_L-valued harmonic Maass form

$$f(\tau) = \sum_{\gamma \in \Gamma_0(D) \backslash \mathrm{SL}_2(\mathbb{Z})} (f_0|_{2-n}\gamma)(\tau) \cdot \omega_L(\gamma^{-1})\phi_0 \in H_{2-n}(\omega_L)^{\Delta}.$$

As the ξ-operator is equivariant for the action of $\mathrm{SL}_2(\mathbb{Z})$, we have $\xi(f) = \tilde{g}$. According to [**6**, Proposition 6.1.2], which holds analogously for harmonic Maass forms, the coefficients of the holomorphic part f^+ satisfy

$$c_f^+(m,\mu) = \begin{cases} c_0^+(m) & \text{if } \mu = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for all $m \leq 0$. This equality implies that

$$f = \sum_{m > 0} c_0^+(-m) f_m,$$

where $f_m \in H_{2-n}(\omega_L)^{\Delta}$ is the harmonic form characterized by (2.5.1). Indeed, the difference between the two forms is a harmonic form h whose holomorphic part $\sum_{m \geq 0} c_h^+(m) q^m$ has no principal part. It follows from [**5**, Theorem 3.6] that such a harmonic form is actually holomorphic, and therefore vanishes because the weight is negative.

The above decomposition of f as a linear combination of the f_m's implies that

$$\widehat{\mathcal{Z}}(f) = \sum_{m>0} c_0^+(-m) \cdot \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}),$$

and consequently

$$\begin{aligned} \widehat{\theta}(g) &= \langle \widehat{\phi}, \xi(f_0) \rangle_{\mathrm{Pet}} \\ &= \{f_0, \widehat{\phi}\} \\ &= \sum_{m\geq 0} c_0^+(-m) \cdot \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \\ &= \widehat{\mathcal{Z}}(f) + c_f^+(0,0) \cdot \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0). \end{aligned}$$

Here, in the second line, we have used the bilinear pairing

$$\{.,.\} : H^{\infty}_{2-n}(\Gamma_0(D), \chi^n_{\boldsymbol{k}}) \times M_n(\Gamma_0(D), \chi^n_{\boldsymbol{k}}) \to \mathbb{C}$$

analogous to [**5**, Proposition 3.5], and the fact that f_0 vanishes at all cusps different from ∞. □

Remark 2.5.2. — It is incorrectly claimed in [**7**, §1.3] that (2.5.2) holds for *every* form f with $\xi(f) = \tilde{g}$.

The following is stated in the introduction as Theorem A.

Theorem 2.5.3. — *If $g \in S_n(\Gamma_0(D), \chi^n_{\boldsymbol{k}})$ and $\tilde{g} \in S_n(\overline{\omega}_L)$ are related by* (2.3.2), *then*

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{sm}}] = -\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) \cdot L'(\tilde{g}, \theta_\Lambda, 0).$$

Proof. — Choosing f as in Proposition 2.5.1, and using the first equality of Proposition 2.2.10, yields

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{sm}}] = [\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathrm{sm}}] - c_f^+(0,0) \cdot [\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{sm}}].$$

Thus the claim follows from Theorem 2.4.1. □

3. Further results on the convolution L-function

In this section we specialize to the case where $g \in S_n(\Gamma_0(D), \chi^n_{\boldsymbol{k}})$ is a new eigenform, and express the convolution L-function (2.3.3) associated to the vector valued cusp form $\tilde{g}$ in terms of the usual L-function associated to g.

This allows us, in Theorem 3.4.1 below, to rewrite Theorem A of the introduction in a way that avoids vector-valued modular forms. When n is even, it also allows us to formulate a version of Theorem A in which the L-function has an Euler product.

We assume $n \geq 2$ until we reach §3.4, at which point we restrict to $n \geq 3$.

3.1. Atkin-Lehner operators. — Recall that $\chi_{\boldsymbol{k}}$ is the idele class character associated to the quadratic field $\boldsymbol{k}$. If we view $\chi_{\boldsymbol{k}}$ as a Dirichlet character modulo D, then any factorization $D = Q_1Q_2$ induces a factorization

$$\chi_{\boldsymbol{k}} = \chi_{Q_1}\chi_{Q_2}$$

where $\chi_{Q_i} : (\mathbb{Z}/Q_i\mathbb{Z})^\times \to \mathbb{C}^\times$ is a quadratic Dirichlet character.

Fix a normalized cuspidal new eigenform

$$g(\tau) = \sum_{m>0} c(m)q^m \in S_n(\Gamma_0(D), \chi_{\boldsymbol{k}}^n).$$

As in [**6**, Section 4.1], for each positive divisor $Q \mid D$, fix a matrix

$$R_Q = \begin{pmatrix} \alpha & \beta \\ \frac{D}{Q}\gamma & Q\delta \end{pmatrix} \in \Gamma_0(D/Q)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, and define the Atkin-Lehner operator

$$W_Q = \begin{pmatrix} Q\alpha & \beta \\ D\gamma & Q\delta \end{pmatrix} = R_Q \begin{pmatrix} Q & \\ & 1 \end{pmatrix}.$$

The cusp form

$$\begin{aligned} g_Q(\tau) &= \chi_Q^n(\beta)\chi_{D/Q}^n(\alpha) \cdot g|_n W_Q \\ &= \sum_{m>0} c_Q(m)q^m, \end{aligned}$$

is then independent of the choice of α, β, γ, δ.

Let $\varepsilon_Q(g)$ be the fourth root of unity

$$\varepsilon_Q(g) = \prod_{\substack{q \mid Q \\ q \text{ prime}}} \chi_Q^n(Q/q) \cdot \lambda_q,$$

where

$$\lambda_q = \overline{c(q)} \cdot \begin{cases} -q^{1-\frac{n}{2}} & \text{if } n \equiv 0 \pmod 2 \\ \delta_q q^{\frac{1-n}{2}} & \text{if } n \equiv 1 \pmod 2, \end{cases}$$

and δ_q is defined by

(3.1.1)
$$\delta_q = \begin{cases} 1 & \text{if } q \equiv 1 \pmod 4 \\ i & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

According to [**3**, Theorem 2], we have

$$\begin{aligned} c_Q(m) &= \varepsilon_Q(g)\chi_Q^n(m)c(m) && \text{if } (m,Q) = 1, \\ c_Q(m) &= \varepsilon_Q(g)\chi_{D/Q}^n(m)\overline{c(m)} && \text{if } (m, D/Q) = 1, \\ c_Q(m_1m_2) &= \varepsilon_Q(g)^{-1}c_Q(m_1)c_Q(m_2) && \text{if } (m_1,m_2) = 1. \end{aligned}$$

Remark 3.1.1. — If n is even, then the Fourier coefficients of g are totally real. It follows that $g_Q = \varepsilon_Q(g)g$ for every divisor $Q \mid D$. Furthermore,

$$\varepsilon_Q(g) = \prod_{q|Q} \big(- q^{1-\frac{n}{2}} c(q)\big) = \pm 1.$$

3.2. Twisting theta functions. — Let $(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{b})$ be a triple of self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices of signatures $(1,0)$, $(0,1)$, and $(n-1,0)$, as in §2.2, and recall that from this data we constructed hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{b}, \qquad L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \tag{3.2.1}$$

of signature $(n-1,1)$. We also define

$$L_1 = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}_1), \qquad \Lambda = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{b}), \tag{3.2.2}$$

so that $L = L_1 \oplus \Lambda$.

Let $\mathrm{GU}(\Lambda)$ be the unitary similitude group associated with Λ, viewed as an algebraic group over $\mathbb{Z}$. For any $\mathbb{Z}$-algebra R its R-valued points are given by

$$\mathrm{GU}(\Lambda)(R) = \{h \in \mathrm{GL}_{\mathcal{O}_{\boldsymbol{k}}}(\Lambda_R) : \langle hx, hy\rangle = \nu(h)\langle x, y\rangle \ \forall x, y \in \Lambda_R\},$$

where $\nu(h) \in R^\times$ denotes the similitude factor of h. Note the relation

$$\mathrm{Nm}_{\boldsymbol{k}/\mathbb{Q}}(\det(h)) = \nu(h)^{n-1}. \tag{3.2.3}$$

For $h \in \mathrm{GU}(\Lambda)(\mathbb{R})$ the similitude factor $\nu(h)$ belongs to $\mathbb{R}_{>0}$.

As Λ is positive definite, the set

$$X_\Lambda = \mathrm{GU}(\Lambda)(\mathbb{Q})\backslash \mathrm{GU}(\Lambda)(\mathbb{A}_f)/\mathrm{GU}(\Lambda)(\widehat{\mathbb{Z}})$$

is finite. Denoting by

$$\mathrm{CL}(\boldsymbol{k}) = \boldsymbol{k}^\times \backslash \widehat{\boldsymbol{k}}^\times / \widehat{\mathcal{O}}_{\boldsymbol{k}}^\times$$

the ideal class group of $\boldsymbol{k}$, the natural map $\mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \to \mathrm{GU}(\Lambda)$ to the center induces an action

$$\mathrm{CL}(\boldsymbol{k}) \times X_\Lambda \longrightarrow X_\Lambda. \tag{3.2.4}$$

As in the proof of [**6**, Proposition 2.1.1], any $h \in \mathrm{GU}(\Lambda)(\mathbb{A}_f)$ determines an $\mathcal{O}_{\boldsymbol{k}}$-lattice

$$\Lambda_h = \Lambda_{\mathbb{Q}} \cap h\widehat{\Lambda}.$$

This lattice is not self-dual under the hermitian form $\langle -, - \rangle$ on $\Lambda_{\mathbb{Q}}$. However, there is a unique positive rational number $\mathrm{rat}(\nu(h))$ such that

$$\frac{\nu(h)}{\mathrm{rat}(\nu(h))} \in \widehat{\mathbb{Z}}^\times,$$

and the lattice Λ_h is self-dual under the rescaled hermitian form

$$\langle x, y\rangle_h = \frac{1}{\mathrm{rat}(\nu(h))} \cdot \langle x, y\rangle.$$

If $h \in \mathrm{GU}(\Lambda)(\widehat{\mathbb{Z}})$ then $\Lambda_h = \Lambda$. If $h \in \mathrm{GU}(\Lambda)(\mathbb{Q})$, then $\Lambda_h \cong \Lambda$ as hermitian $\mathcal{O}_{\boldsymbol{k}}$-modules. Hence $h \mapsto \Lambda_h$ defines a function from X_Λ to the set of isometry classes of self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-module of signature $(n-1, 0)$.

Similarly, for any $h \in \mathrm{GU}(\Lambda)(\mathbb{A}_f)$ we define a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(0,1)$ by endowing

$$L_{1,h} = L_{1\mathbb{Q}} \cap \det(h)\widehat{L}_1$$

with the hermitian form

$$\langle x, y\rangle_h = \frac{1}{\mathrm{rat}(\nu(h))^{n-1}} \cdot \langle x, y\rangle.$$

The assignment $h \mapsto L_{1,h}$ defines a map from X_Λ to the set of isometry classes of self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices of signature $(0,1)$.

Lemma 3.2.1. — *For any $h \in \mathrm{GU}(\Lambda)(\mathbb{A}_f)$ the hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice*

$$L_h = L_{1,h} \oplus \Lambda_h$$

is isomorphic everywhere locally to L. Moreover, L_h and L become isomorphism after tensoring with $\mathbb{Q}$.

Proof. — Let p be a prime. As in [**6**, §1.8], a $\boldsymbol{k}_p$-hermitian space is determined by its dimension and invariant. The relations

$$\det(\Lambda_h \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{rat}(\nu(h))^{1-n} \cdot \det(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}),$$
$$\det(L_{1,h} \otimes_{\mathbb{Z}} \mathbb{Q}) = \mathrm{rat}(\nu(h))^{1-n} \cdot \det(L_1 \otimes_{\mathbb{Z}} \mathbb{Q}),$$

combined with (3.2.3), imply that $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L_h \otimes_{\mathbb{Z}} \mathbb{Q}$ have the same invariant everywhere locally. As they both have signature $(n-1,1)$, they are isomorphic everywhere locally, and hence isomorphic globally.

A result of Jacobowitz [**22**] shows that any two self-dual lattices in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic everywhere locally, and hence it follows from the previous paragraph that L and L_h are isomorphic everywhere locally. □

Define a linear map

$$M_{n-1}(\omega_\Lambda^\vee) \to M_{n-1}(\Gamma_0(D), \chi_{\boldsymbol{k}}^{n-1})$$

from $S_\Lambda^\vee$-valued modular forms to scalar-valued modular forms by evaluation at the characteristic function $\phi_0 \in S_\Lambda$ of the trivial coset $0 \in \Lambda'/\Lambda$. This map takes the vector valued theta series $\theta_\Lambda \in M_{n-1}(\omega_\Lambda^\vee)$ of §2.3 to the scalar valued theta series

$$\theta_\Lambda^{\mathrm{sc}}(\tau) = \sum_{m \in \mathbb{Z}_{\geq 0}} R_\Lambda^{\mathrm{sc}}(m) \cdot q^m,$$

where $R_\Lambda^{\mathrm{sc}}(m)$ is the number of ways to represent m by Λ.

Let η be an algebraic automorphic form for $\mathrm{GU}(\Lambda)$ which is trivial at ∞ and right $\mathrm{GU}_\Lambda(\widehat{\mathbb{Z}})$-invariant. In other words, a function

$$\eta : X_\Lambda \longrightarrow \mathbb{C}.$$

Throughout we assume that under the action (3.2.4) the function η transforms with a character $\chi_\eta : \mathrm{CL}(\boldsymbol{k}) \to \mathbb{C}^\times$, that is,

$$\eta(\alpha h) = \chi_\eta(\alpha)\eta(h). \tag{3.2.5}$$

We associate a theta function to η by setting

$$\theta^{\mathrm{sc}}_{\eta,\Lambda} = \sum_{h\in X_\Lambda} \frac{\eta(h)}{|\mathrm{Aut}(\Lambda_h)|} \cdot \theta^{\mathrm{sc}}_{\Lambda_h} \in M_{n-1}(\Gamma_0(D), \chi_{\boldsymbol{k}}^{n-1}).$$

This form is cuspidal when the character χ_η is non-trivial. We denote its Fourier expansion by

$$\theta^{\mathrm{sc}}_{\eta,\Lambda}(\tau) = \sum_{m\geq 0} R^{\mathrm{sc}}_{\eta,\Lambda}(m) \cdot q^m.$$

Similarly, we may define

$$\theta_{\eta,\Lambda}(\tau) = \sum_{h\in X_\Lambda} \frac{\eta(h)}{|\mathrm{Aut}(\Lambda_h)|} \cdot \theta_{\Lambda_h}(\tau),$$

but this is only a formal sum: as h varies the forms θ_{Λ_h} take values in the varying spaces $S^\vee_{\Lambda_h}$.

Lemma 3.2.1 allows us to identify $S_L \cong S_{L_h}$, and hence make sense of the L-function $L(\tilde{g}, \theta_{\Lambda_h}, s)$ as in (2.3.3). In the next subsection we will compare

$$L(\tilde{g}, \theta_{\eta,\Lambda}, s) = \sum_{h\in X_\Lambda} \frac{\eta(h)}{|\mathrm{Aut}(\Lambda_h)|} \cdot L(\tilde{g}, \theta_{\Lambda_h}, s) \tag{3.2.6}$$

to the usual convolution L-function

$$L(g, \theta^{\mathrm{sc}}_{\eta,\Lambda}, s) = \Gamma\Big(\frac{s}{2} + n - 1\Big) \sum_{m=1}^{\infty} \frac{\overline{c(m)} R^{\mathrm{sc}}_{\eta,\Lambda}(m)}{(4\pi m)^{\frac{s}{2}+n-1}} \tag{3.2.7}$$

of the scalar-valued forms g and $\theta^{\mathrm{sc}}_{\eta,\Lambda}$.

3.3. Rankin-Selberg L-functions for scalar and vector valued forms. — In this subsection we prove a precise relation between (3.2.6) and (3.2.7). First, we give an explicit formula for the Fourier coefficients $a(m,\mu)$ of $\tilde{g}$ in terms of those of g analogous to [**6**, Proposition 6.1.2].

For a prime p dividing D define

$$\gamma_p = \delta_p^{-n} \cdot (D,p)_p^n \cdot \mathrm{inv}_p(V_p) \in \{\pm 1, \pm i\}, \tag{3.3.1}$$

where $\mathrm{inv}_p(V_p)$ is the invariant of $V_p = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ in the sense of [**6**, (1.8.3)] and $\delta_p \in \{1, i\}$ is as before. It is equal to the local Weil index of the Weil representation of $\mathrm{SL}_2(\mathbb{Z}_p)$ on $S_{L_p} \subset S(V_p)$, where V_p is viewed as a quadratic space by taking the trace of the hermitian form. This is explained in more detail in [**6**, Section 8.1]. For any Q dividing D we define

$$\gamma_Q = \prod_{q\mid Q} \gamma_q. \tag{3.3.2}$$

Remark 3.3.1. — If n is even and $p \mid D$, then (3.3.1) simplifies to

$$\gamma_p = \left(\frac{-1}{p}\right)^{n/2} \mathrm{inv}_p(V_p).$$

For any $\mu \in L'/L$ define $Q_\mu \mid D$ by

$$Q_\mu = \prod_{\substack{p \mid D \\ \mu_p \neq 0}} p,$$

where μ_p is the image of μ in L'_p/L_p. Let $\phi_\mu \in S_L$ be the characteristic function of μ.

Proposition 3.3.2. — *For all $m \in \mathbb{Q}$ the coefficients $\tilde{a}(m) \in S_L$ of $\tilde{g}$ satisfy*

$$\tilde{a}(m,\mu) = \begin{cases} \sum_{Q_\mu \mid Q \mid D} Q^{1-n} \overline{\gamma_Q} \cdot c_Q(mQ) & \text{if } m \equiv -Q(\mu) \pmod{\mathbb{Z}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — The first formula is a special case of results of Scheithauer [**26**, Section 5]. It can also be proved in the same way as Proposition 6.1.2 of [**6**]. The complex conjugation over γ_Q arises because of the fact that $\tilde{g}$ transforms with the complex conjugate representation $\overline{\omega}_L$. The additional factor Q^{1-n} is due to the fact that we work here in weight n. □

Proposition 3.3.3. — *The convolution L-function* (2.3.3) *satisfies*

$$L(\tilde{g}, \theta_\Lambda, s) = \sum_{Q \mid D} Q^{\frac{s}{2}} \gamma_Q \cdot L(g_Q, \theta^{\mathrm{sc}}_{\Lambda_{\mathfrak{q}}}, s),$$

where $\mathfrak{q} \in \widehat{\boldsymbol{k}}^\times$ is such that $\mathfrak{q}^2 \widehat{\mathcal{O}}^\times_{\boldsymbol{k}} = Q \widehat{\mathcal{O}}^\times_{\boldsymbol{k}}$. Moreover, for any $\eta : X_\Lambda \to \mathbb{C}$ satisfying (3.2.5) *the L-functions* (3.2.6) *and* (3.2.7) *are related by*

$$L(\tilde{g}, \theta_{\eta,\Lambda}, s) = \sum_{Q \mid D} Q^{\frac{s}{2}} \gamma_Q \cdot \chi_\eta(\mathfrak{q}^{-1}) L(g_Q, \theta^{\mathrm{sc}}_{\eta,\Lambda}, s).$$

Proof. — Proposition 3.3.2 implies

$$\begin{aligned} \frac{L(\tilde{g}, \theta_\Lambda, s)}{\Gamma(\frac{s}{2}+n-1)} &= \sum_{\mu \in \Lambda'/\Lambda} \sum_{m \in \mathbb{Q}_{>0}} \sum_{Q_\mu \mid Q \mid D} Q^{1-n} \gamma_Q \cdot \frac{\overline{c_Q(mQ)} R_\Lambda(m, \phi_\mu)}{(4\pi m)^{\frac{s}{2}+n-1}} \\ &= \sum_{Q \mid D} Q^{1-n} \gamma_Q \sum_{m \in \frac{1}{Q}\mathbb{Z}_{>0}} \frac{\overline{c_Q(mQ)}}{(4\pi m)^{\frac{s}{2}+n-1}} \sum_{\substack{\mu \in \Lambda'/\Lambda \\ Q_\mu \mid Q}} R_\Lambda(m, \phi_\mu) \\ &= \sum_{Q \mid D} Q^{\frac{s}{2}} \gamma_Q \sum_{m \in \mathbb{Z}_{>0}} \frac{\overline{c_Q(m)}}{(4\pi m)^{\frac{s}{2}+n-1}} \sum_{\substack{\mu \in \Lambda'/\Lambda \\ Q_\mu \mid Q}} R_\Lambda(m/Q, \phi_\mu). \end{aligned}$$

The first claim now follows from the relation

$$\sum_{\substack{\mu\in\Lambda'/\Lambda\\ Q_\mu|Q}} R_\Lambda(m/Q,\mu) = R_{\Lambda_{\mathfrak{q}^{-1}}}(m,0) = R_{\Lambda_{\mathfrak{q}}}(m,0).$$

For the second claim, if we replace Λ by Λ_h and L_1 by $L_{1,h}$ for $h\in X_\Lambda$, then L and γ_Q remain unchanged. The above calculations therefore imply that

$$\begin{aligned} L(\tilde{g},\theta_{\eta,\Lambda},s) &= \sum_{Q|D}\gamma_Q Q^{\frac{s}{2}}\sum_{h\in X_\Lambda}\frac{\eta(h)}{|\mathrm{Aut}(\Lambda_h)|}L(g_Q,\theta^{\mathrm{sc}}_{\Lambda_{\mathfrak{q}h}},s)\\ &= \sum_{Q|D}\gamma_Q Q^{\frac{s}{2}}\sum_{h\in X_\Lambda}\frac{\eta(\mathfrak{q}^{-1}h)}{|\mathrm{Aut}(\Lambda_h)|}L(g_Q,\theta^{\mathrm{sc}}_{\Lambda_h},s)\\ &= \sum_{Q|D}\gamma_Q Q^{\frac{s}{2}}\cdot\chi_\eta(\mathfrak{q}^{-1})L(g_Q,\theta^{\mathrm{sc}}_{\eta,\Lambda},s), \end{aligned}$$

where we have used (3.2.5) and the fact that $|\mathrm{Aut}(\Lambda_h)| = |\mathrm{Aut}(\Lambda_{\mathfrak{q}h})|$. □

Corollary 3.3.4. — *If n is even, then*

$$L(\tilde{g},\theta_{\eta,\Lambda},s) = L(g,\theta^{\mathrm{sc}}_{\eta,\Lambda},s)\cdot\prod_{p|D}\big(1+\chi_\eta(\mathfrak{p}^{-1})\varepsilon_p(g)\gamma_p p^{\frac{s}{2}}\big).$$

Proof. — This is immediate from Proposition 3.3.3 and Remark 3.1.1. □

3.4. Small CM cycles and derivatives of L-functions, revisited. — Now we are ready to state a variant of Theorem A using only scalar valued modular forms. Assume $n\geq 3$.

Every $h\in X_\Lambda$ determines a codimension $n-1$ cycle

$$\mathcal{Y}_{\mathrm{sm},h}\to\mathcal{S}^*_{\mathrm{Kra}} \tag{3.4.1}$$

as follows. From the triple $(\mathfrak{a}_0,\mathfrak{a}_1,\mathfrak{b})$ fixed in §3.2 and the hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices $L_h = L_{1,h}\oplus\Lambda_h$ of Lemma 3.2.1, we denote by $\mathfrak{a}_{1,h}$ and $\mathfrak{b}_h$ the unique hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattices satisfying

$$L_{1,h}\cong\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0,\mathfrak{a}_{1,h}),\quad \Lambda_h\cong\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0,\mathfrak{b}_h),$$

and set $\mathfrak{a}_h=\mathfrak{a}_{1,h}\oplus\mathfrak{b}_h$ so that $L_h\cong\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0,\mathfrak{a}_h)$. Compare with (3.2.1) and (3.2.2).

Repeating the construction of the small CM cycle $\mathcal{Y}_{\mathrm{sm}}$ with the triple $(\mathfrak{a}_0,\mathfrak{a}_1,\mathfrak{b})$ replaced by $(\mathfrak{a}_0,\mathfrak{a}_{1,h},\mathfrak{b}_h)$ results in a proper étale $\mathcal{O}_{\boldsymbol{k}}$-stack $\mathcal{Y}_{\mathrm{sm},h}$. Repeating the construction of the Shimura variety $\mathcal{S}_{\mathrm{Kra}}$ with the triple $(\mathfrak{a}_0,\mathfrak{a})$ replaced by $(\mathfrak{a}_0,\mathfrak{a}_h)$ results in a new Shimura variety $\mathcal{S}_{\mathrm{Kra},h}$, along with a finite and unramified morphism

$$\mathcal{Y}_{\mathrm{sm},h}\to\mathcal{S}_{\mathrm{Kra},h}.$$

It follows from Lemma 3.2.1 that $\mathfrak{a}$ and $\mathfrak{a}_h$ are isomorphic everywhere locally, and examination of the moduli problem defining $\mathcal{S}_{\mathrm{Kra}}$ in [**6**, §2.3] shows that $\mathcal{S}_{\mathrm{Kra}}$ depends only the everywhere local data determined by the pair $(\mathfrak{a}_0,\mathfrak{a})$, and not on the actual global $\mathcal{O}_{\boldsymbol{k}}$-hermitian lattices. Therefore, there is a canonical morphism of $\mathcal{O}_{\boldsymbol{k}}$-stacks

$$\mathcal{Y}_{\mathrm{sm},h}\to\mathcal{S}_{\mathrm{Kra},h}\cong\mathcal{S}_{\mathrm{Kra}}$$

in which the isomorphism is simply the identity functor on the moduli problems. In the end, this amounts to simply repeating the construction of $\mathcal{Y}_{\mathrm{sm}} \to \mathcal{S}_{\mathrm{Kra}}$ from Definition 2.2.6 word-for-word, but replacing Λ by Λ_h everywhere. This defines the desired cycle (3.4.1).

Each algebraic automorphic form $\eta : X_\Lambda \to \mathbb{C}$ satisfying (3.2.5) now determines a cycle

$$\eta\mathcal{Y}_{\mathrm{sm}} = \sum_{h \in X_\Lambda} \eta(h) \cdot \mathcal{Y}_{\mathrm{sm},h}$$

on $\mathcal{S}^*_{\mathrm{Kra}}$ with complex coefficients, and a corresponding linear functional

$$[- : \eta\mathcal{Y}_{\mathrm{sm}}] : \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \mathbb{C}.$$

Theorem 3.4.1. — *The arithmetic theta lift* (1.1.3) *satisfies*

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{sm}}] = -\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm}}) \cdot \frac{d}{ds}\Big[\sum_{Q|D} Q^{\frac{s}{2}}\gamma_Q L(g_Q, \theta^{\mathrm{sc}}_{\Lambda_{\mathfrak{q}}}, s)\Big]\big|_{s=0},$$

where $\mathfrak{q} \in \widehat{\boldsymbol{k}}^\times$ *is such that* $\mathfrak{q}^2\widehat{\mathcal{O}}^\times_{\boldsymbol{k}} = Q\widehat{\mathcal{O}}^\times_{\boldsymbol{k}}$. *Moreover, if n is even and* $\eta : X_\Lambda \to \mathbb{C}$ *satisfies* (3.2.5), *then*

$$[\widehat{\theta}(g) : \eta\mathcal{Y}_{\mathrm{sm}}] = -2^{1-o(d_{\boldsymbol{k}})}\,(h_{\boldsymbol{k}}/w_{\boldsymbol{k}})^2 \cdot \frac{d}{ds}\Big[L(g, \theta^{\mathrm{sc}}_{\eta,\Lambda}, s) \cdot \prod_{p|D}\big(1 + \chi_\eta(\mathfrak{p}^{-1})\varepsilon_p(g)\gamma_p p^{\frac{s}{2}}\big)\Big]\big|_{s=0},$$

where $\mathfrak{p} \in \widehat{\boldsymbol{k}}^\times$ *such that* $\mathfrak{p}^2\widehat{\mathcal{O}}^\times_{\boldsymbol{k}} = p\widehat{\mathcal{O}}^\times_{\boldsymbol{k}}$. *Note that in the first formula the sum is over all positive divisors $Q \mid D$, while in the second the product is over the prime divisors $p \mid D$.*

Proof. — The first assertion follows from Theorem A and Proposition 3.3.3.

For the second assertion, applying Theorem A to

$$\mathcal{Y}_{\mathrm{sm},h} \to \mathcal{S}^*_{\mathrm{Kra},h} \cong \mathcal{S}^*_{\mathrm{Kra}}$$

yields

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{sm},h}] = -\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{sm},h}) \cdot \frac{d}{ds} L(\tilde{g}, \theta_{\Lambda_h}, s)\big|_{s=0}.$$

Combining this with Proposition 2.2.9 yields

$$[\widehat{\theta}(g) : \eta\mathcal{Y}_{\mathrm{sm}}] = -2^{1-o(d_k)}\,(h_{\boldsymbol{k}}/w_{\boldsymbol{k}})^2 \cdot \frac{d}{ds} L(\tilde{g}, \theta_{\eta,\Lambda}, s)\big|_{s=0},$$

and an application of Corollary 3.3.4 completes the proof. □

Remark 3.4.2. — Since the L-function (3.2.6) vanishes at $s = 0$, the same must be true for the expressions in brackets on the right hand sides of the equalities of the above theorem. In particular, when n is even, then either $L(g, \theta^{\mathrm{sc}}_{\eta,\Lambda}, s)$ or at least one of the factors

$$1 + \chi_\eta(\mathfrak{p}^{-1})\varepsilon_p(g)\gamma_p p^{\frac{s}{2}}$$

(for a prime $p \mid D$) vanishes at $s = 0$. If we pick the newform g such that the latter local factors are nonvanishing, then $L(g, \theta^{\mathrm{sc}}_{\eta,\Lambda}, 0) = 0$ and we obtain

$$[\widehat{\theta}(g) : \eta\mathcal{Y}_{\mathrm{sm}}] = -2^{1-o(d_{\boldsymbol{k}})} \frac{h_{\boldsymbol{k}}^2}{w_{\boldsymbol{k}}^2} \cdot \prod_{p \mid D} \big(1 + \chi_\eta(\mathfrak{p}^{-1})\varepsilon_p(g)\gamma_p\big) \cdot L'(g, \theta^{\mathrm{sc}}_{\eta,\Lambda}, 0).$$

4. Big CM cycles and derivatives of L-functions

In this section we prove Theorem B by combining results of [**6**] and [**18, 19, 8**]. We asume $n \geq 2$ until §4.4, at which point we restrict to $n \geq 3$.

4.1. A Shimura variety of dimension zero. — Let F be a totally real field of degree n, and define a CM field $E = \boldsymbol{k} \otimes_{\mathbb{Q}} F$. Define a rank $n+2$ torus T_{big} over $\mathbb{Q}$ as the fiber product

$$\begin{array}{ccc} T_{\mathrm{big}} & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow {\scriptstyle \mathrm{diag.}} \\ \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m & \xrightarrow[\mathrm{Nm}\times\mathrm{Nm}]{} & \mathbb{G}_m \times \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m. \end{array}$$

Its group of $\mathbb{Q}$-points is

$$T_{\mathrm{big}}(\mathbb{Q}) \cong \{(x, y) \in \boldsymbol{k}^\times \times E^\times : x\overline{x} = y\overline{y}\}.$$

Remark 4.1.1. — There is an isomorphism

$$T_{\mathrm{big}}(\mathbb{Q}) \cong \boldsymbol{k}^\times \times \ker(\mathrm{Nm} : E^\times \to F^\times)$$

defined by $(x, y) \mapsto (x, x^{-1}y)$. It is clear that this arises from an isomorphism

$$T_{\mathrm{big}} \cong \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \times \ker\big(\mathrm{Nm} : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m\big).$$

As in the discussion preceding Theorem B, let $\Phi \subset \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{C})$ be a CM type of signature $(n-1, 1)$, let

$$\varphi^{\mathrm{sp}} : E \to \mathbb{C}$$

be its special element, and let $\mathcal{O}_\Phi$ be the ring of integers of $E_\Phi = \varphi^{\mathrm{sp}}(E)$.

The CM type Φ determines an isomorphism $\mathbb{C}^n \cong E_{\mathbb{R}}$, and hence an embedding $\mathbb{C}^\times \to E_{\mathbb{R}}^\times$ arising from a morphism of real algebraic groups $\mathbb{S} \to (\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{R}}$. This induces a morphism

$$\mathbb{S} \to (\mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{R}} \times (\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{R}},$$

which factors through a morphism

$$h_{\mathrm{big}} : \mathbb{S} \to T_{\mathrm{big},\mathbb{R}}.$$

The pair $(T_{\mathrm{big}}, \{h_{\mathrm{big}}\})$ is a Shimura datum, which, along with the compact open subgroup

$$K_{\mathrm{big}} = T_{\mathrm{big}}(\mathbb{A}_f) \cap (\widehat{\mathcal{O}}_{\boldsymbol{k}}^\times \times \widehat{\mathcal{O}}_E^\times),$$

determines a 0-dimensional E_Φ-stack $\mathrm{Sh}(T_{\mathrm{big}})$ with complex points

$$\mathrm{Sh}(T_{\mathrm{big}})(\mathbb{C}) = T_{\mathrm{big}}(\mathbb{Q})\backslash\{h_{\mathrm{big}}\} \times T_{\mathrm{big}}(\mathbb{A}_f)/K_{\mathrm{big}}.$$

4.2. The big CM cycle. — The Shimura variety just constructed has a moduli interpretation, which we will use to construct an integral model. The interpretation we have in mind requires first choosing a triple $(\mathfrak{a}_0, \mathfrak{a}, i_E)$ in which

— $\mathfrak{a}_0$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(1,0)$,
— $\mathfrak{a}$ is a self-dual hermitian $\mathcal{O}_{\boldsymbol{k}}$-lattice of signature $(n-1,1)$,
— $i_E : \mathcal{O}_E \to \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a})$ is an action extending the action of $\mathcal{O}_{\boldsymbol{k}}$.

Denoting by $H : \mathfrak{a} \times \mathfrak{a} \to \mathcal{O}_{\boldsymbol{k}}$ the hermitian form, we require further that

$$H(i_E(x)a, b) = H(a, i_E(\overline{x})b)$$

for all $x \in \mathcal{O}_E$ and $a, b \in \mathfrak{a}$, and that in the decomposition

$$\mathfrak{a}_{\mathbb{R}} \cong \bigoplus_{\varphi_F : F \to \mathbb{R}} \mathfrak{a} \otimes_{\mathcal{O}_F, \varphi_F} \mathbb{R}$$

the summand indexed by $\varphi_F = \varphi^{\mathrm{sp}}|_F$ is negative definite (which, by the signature condition, implies that the other summands are positive definite).

Remark 4.2.1. — In general such a triple need not exist. In the applications will assume that the discriminants of $\boldsymbol{k}/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime, and in this case one can construct such a triple using the argument of [**18**, Proposition 3.1.6].

We now define a moduli space of abelian varieties with complex multiplication by $\mathcal{O}_E$ and type Φ, as in [**18**, §3.1]. Denote by $\mathcal{CM}_\Phi$ the functor that associates to every $\mathcal{O}_\Phi$-scheme S the groupoid of triples (A, ι, ψ) in which

— $A \to S$ is an abelian scheme of dimenension n,
— $\iota : \mathcal{O}_E \to \mathrm{End}(A)$ is an $\mathcal{O}_E$-action,
— $\psi : A \to A^\vee$ is a principal polarization such that

$$\iota(x)^\vee \circ \psi = \psi \circ \iota(\overline{x})$$

for all $x \in \mathcal{O}_E$.

We also impose the Φ-*determinant condition* that every $x \in \mathcal{O}_E$ acts on $\mathrm{Lie}(A)$ with characteristic polynomial equal to the image of

$$\prod_{\varphi \in \Phi} (T - \varphi(x)) \in \mathcal{O}_\Phi[T]$$

in $\mathcal{O}_S[T]$. We usually abbreviate $A \in \mathcal{CM}_\Phi(S)$, and suppress the data ι and ψ from the notation. By [**18**, Proposition 3.1.2], the functor $\mathcal{CM}_\Phi$ is represented by a Deligne-Mumford stack, proper and étale over $\mathcal{O}_\Phi$.

Remark 4.2.2. — The Φ-determinant condition defined above agrees with that of [**18**, §3.1]. As in [**16**, Proposition 2.1.3], this is a consequence of Amitsur's formula, which can be found in [**1**, Theorem A] or [**11**, Lemma 1.12].

Define an open and closed substack

$$\mathcal{Y}_{\mathrm{big}} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_{\Phi}$$

as the union of connected components $\mathcal{B} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_{\Phi}$ satisfying the following property: for every complex point $y = (A_0, A) \in \mathcal{B}(\mathbb{C})$, and for all primes ℓ, there is an $\mathcal{O}_E$-linear isomorphism of hermitian $\mathcal{O}_{\boldsymbol{k},\ell}$-lattices

$$\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k},\ell}}(A_0[\ell^\infty], A[\ell^\infty]) \cong \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a}) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell. \tag{4.2.1}$$

Remark 4.2.3. — To verify that a connected component $\mathcal{B} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_{\Phi}$ is contained in $\mathcal{Y}_{\mathrm{big}}$, it suffices to check that (4.2.1) holds for one complex point $y \in \mathcal{B}(\mathbb{C})$. This is a consequence of the main theorem of complex multiplication and the fact that the points of $\mathcal{B}(\mathbb{C})$ form a single $\mathrm{Aut}(\mathbb{C}/E_\Phi)$-orbit.

Proposition 4.2.4. — *There is a canonical isomorphism of E_Φ-stacks*

$$\mathrm{Sh}(T_{\mathrm{big}}) \cong \mathcal{Y}_{\mathrm{big}/E_\Phi}.$$

Proof. — The natural actions of $\mathcal{O}_{\boldsymbol{k}}$ and $\mathcal{O}_E$ on $\mathfrak{a}_0$ and $\mathfrak{a}$ determine an action of the subtorus

$$T_{\mathrm{big}} \subset \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$$

on $\mathfrak{a}_{0\mathbb{Q}}$ and $\mathfrak{a}_{\mathbb{Q}}$, and the morphism $h_{\mathrm{big}} : \mathbb{S} \to T_{\mathrm{sm},\mathbb{R}}$ endows each of the real vector spaces $\mathfrak{a}_{0\mathbb{R}}$ and $\mathfrak{a}_{\mathbb{R}}$ with a complex structure.

The desired isomorphism on complex points sends

$$(h_{\mathrm{big}}, g) \in \mathrm{Sh}(T_{\mathrm{sm}})(\mathbb{C})$$

to the pair (A_0, A) defined by

$$A_0(\mathbb{C}) = \mathfrak{a}_{0\mathbb{R}}/g\mathfrak{a}_0, \quad A(\mathbb{C}) = \mathfrak{a}_{\mathbb{R}}/g\mathfrak{a}.$$

The elliptic curve A_0 is endowed with its natural $\mathcal{O}_{\boldsymbol{k}}$-action and its unique principal ploarization. The abelian variety A is endowed with its natural $\mathcal{O}_E$-action, and the polarization induced by the symplectic form determined by its $\mathcal{O}_{\boldsymbol{k}}$-hermitian form, as in the proof of [**6**, Proposition 2.2.1].

It follows from the theory of canonical models that this isomorphism on complex points descends to an isomorphism of E_Φ-stacks. □

The triple $(\mathfrak{a}_0, \mathfrak{a}, i_E)$ determines a pair $(\mathfrak{a}_0, \mathfrak{a})$ as in the introduction, which determines a unitary Shimura variety with integral model $\mathcal{S}_{\mathrm{Kra}}$ as in (1.1.1). Recalling that $\mathcal{O}_{\boldsymbol{k}} \subset \mathcal{O}_\Phi$ as subrings of $\mathbb{C}$, we now view both $\mathcal{Y}_{\mathrm{big}}$ and $\mathcal{CM}_\Phi$ as $\mathcal{O}_{\boldsymbol{k}}$-stacks. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{Y}_{\mathrm{big}} & \longrightarrow & \mathcal{M}_{(1,0)} \times \mathcal{CM}_\Phi \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ \mathcal{S}_{\mathrm{Kra}} & \longrightarrow & \mathcal{M}_{(1,0)} \times \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)} \end{array}$$

(all fiber products are over $\mathcal{O}_{\boldsymbol{k}}$), in which the vertical arrow on the right is the identity on the first factor and "forget complex multiplication" on the second. The arrow π is defined by the commutativity of the diagram.

Remark 4.2.5. — In order to define the morphism

$$\mathcal{CM}_\Phi \to \mathcal{M}^{\mathrm{Kra}}_{(n-1,1)}$$

in the diagram above, we must endow a point $A \in \mathcal{CM}_\Phi(S)$ with a subsheaf $\mathcal{F}_A \subset \mathrm{Lie}(A)$ satisfying Krämer's condition [**6**, §2.3]. Using the morphism

$$\mathcal{O}_E \xrightarrow{\varphi^{\mathrm{sp}}} \mathcal{O}_\Phi \to \mathcal{O}_S,$$

denote by $J_{\varphi^{\mathrm{sp}}} \subset \mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_S$ the kernel of

$$\mathcal{O}_E \otimes_{\mathbb{Z}} \mathcal{O}_S \xrightarrow{x\otimes y \mapsto \varphi^{\mathrm{sp}}(x)\cdot y} \mathcal{O}_S.$$

According to [**19**, Lemma 4.1.2], the subsheaf $\mathcal{F}_A = J_{\varphi^{\mathrm{sp}}}\mathrm{Lie}(A)$ has the desired properties.

Definition 4.2.6. — Composing the morphism π in the diagram above with the inclusion of $\mathcal{S}_{\mathrm{Kra}}$ into its toroidal compactification, we obtain a morphism of $\mathcal{O}_{\boldsymbol{k}}$-stacks

$$\pi : \mathcal{Y}_{\mathrm{big}} \to \mathcal{S}^*_{\mathrm{Kra}},$$

called the *big CM cycle.*

Exactly as in §2.2, the *arithmetic degree along* $\mathcal{Y}_{\mathrm{big}}$ is the composition

$$\widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}) \xrightarrow{\pi^*} \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) \xrightarrow{\widehat{\deg}} \mathbb{C}.$$

We denote this linear functional by $\widehat{\mathcal{Z}} \mapsto [\widehat{\mathcal{Z}} : \mathcal{Y}_{\mathrm{big}}]$.

Remark 4.2.7. — The big CM cycle arises from a morphism of Shimura varieties. Indeed, there is a morphism of Shimura data $(T_{\mathrm{big}}, \{h_{\mathrm{big}}\}) \to (G, \mathcal{D})$, and the induced morphism of Shimura varieties sits in a commutative diagram of E_Φ-stacks

$$\begin{array}{ccc} \mathrm{Sh}(T_{\mathrm{big}}) & \longrightarrow & \mathrm{Sh}(G,\mathcal{D})_{/E_\Phi} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathcal{Y}_{\mathrm{big}/E_\Phi} & \xrightarrow{\pi} & \mathcal{S}_{\mathrm{Kra}/E_\Phi}. \end{array}$$

Proposition 4.2.8. — *The degree* $\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})$ *of Theorem B satisfies*

$$\frac{1}{n} \cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) = \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \cdot \frac{\Lambda(0,\chi_E)}{2^{r-1}},$$

where r is the number of places of F that ramify in E (including all archimedean places).

Proof. — It is clear from Proposition 4.2.4 that

$$\frac{1}{n}\cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) = \sum_{y\in \mathrm{Sh}(T_{\mathrm{big}})(\mathbb{C})} \frac{1}{|\mathrm{Aut}(y)|} = \frac{|T_{\mathrm{big}}(\mathbb{Q})\backslash T_{\mathrm{big}}(\mathbb{A}_f)/K_{\mathrm{big}}|}{|T_{\mathrm{big}}(\mathbb{Q})\cap K_{\mathrm{big}}|}.$$

Note that when we defined the degree on the left we counted the complex points of $\mathcal{Y}_{\mathrm{big}}$ viewed as an $\mathcal{O}_{\boldsymbol{k}}$-stack, whereas in the middle expression we are viewing $\mathrm{Sh}(T_{\mathrm{big}})$ as an E_Φ-stack. This is the reason for the correction factor of $n=[E_\Phi:\boldsymbol{k}]$ on the left.

Let $E'\subset E^\times$ be the kernel of the norm map $\mathrm{Nm}:E^\times\to F^\times$, and define

$$\widehat{E}'\subset \widehat{E}^\times,\quad \widehat{\mathcal{O}}'_E\subset \widehat{\mathcal{O}}_E^\times$$

similarly. Note that $\mu(E)=E'\cap\widehat{\mathcal{O}}'_E$ is the group of roots of unity in E, whose order we denote by w_E. Using the isomorphism $T_{\mathrm{big}}(\mathbb{Q})\cong \boldsymbol{k}^\times\times E'$ of Remark 4.1.1, we find

$$\frac{|T_{\mathrm{big}}(\mathbb{Q})\backslash T_{\mathrm{big}}(\mathbb{A}_f)/K_{\mathrm{big}}|}{|T_{\mathrm{big}}(\mathbb{Q})\cap K_{\mathrm{big}}|} = \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}}\cdot\frac{|E'\backslash \widehat{E}'/\widehat{\mathcal{O}}'_E|}{w_E}. \tag{4.2.2}$$

Denote by C_F and C_E the ideal class groups of E and F, and by $\tilde F$ and $\tilde E$ their Hilbert class fields. As E/F is ramified at all archimedean places, $\tilde F\cap E=F$, and the natural map

$$\mathrm{Gal}(\tilde E/E)\to \mathrm{Gal}(\tilde F/F)$$

is surjective. Hence, by class field theory, the norm

$$\mathrm{Nm}:C_E\to C_F$$

is surjective. Denote its kernel by B, so that we have a short exact sequence

$$1\to B\to C_E\xrightarrow{\mathrm{Nm}} C_F\to 1.$$

Define a group

$$\tilde B = E^\times\backslash\left\{(\mathfrak{B},\beta):\begin{array}{c}\mathfrak{B}\subset E \text{ is a fractional } \mathcal{O}_E\text{-ideal,}\\ \beta\in F^\times,\ \text{and } \mathrm{Nm}(\mathfrak{B})=\beta\mathcal{O}_F\end{array}\right\},$$

where the action of $E^\times$ is by $\alpha\cdot(\mathfrak{B},\beta)=(\alpha\mathfrak{B},\alpha\overline{\alpha}\beta)$. There is an evident short exact sequence

$$1\to \mathrm{Nm}(\mathcal{O}_E^\times)\backslash\mathcal{O}_F^\times\xrightarrow{\beta\mapsto(\mathcal{O}_E,\beta)}\tilde B\to B\to 1.$$

Lemma 4.2.9. — *We have* $[\mathcal{O}_E^\times:\mathrm{Nm}(\mathcal{O}_E^\times)]=2^{n-1}w_E$.

Proof. — Let $Q=[\mathcal{O}_E^\times:\mu(E)\mathcal{O}_F^\times]$. If $Q=1$ then

$$[\mathrm{Nm}(\mathcal{O}_E^\times):\mathcal{O}_F^{\times,2}]=1\quad\text{and}\quad[\mathcal{O}_E^\times:\mathcal{O}_F^\times]=\frac{1}{2}\cdot w_E,$$

and so

$$[\mathcal{O}_F^\times:\mathrm{Nm}(\mathcal{O}_E^\times)]=[\mathcal{O}_F^\times:\mathcal{O}_F^{\times,2}]=2^n=\frac{2^{n-1}w_E}{[\mathcal{O}_E^\times:\mathcal{O}_F^\times]},$$

where the middle equality follows from Dirichlet's unit theorem.

If $Q > 1$ then [**27**, Theorem 4.12] and its proof show that $Q = 2$, and that the image of the map $\phi : \mathcal{O}_E^\times \to \mathcal{O}_E^\times$ defined by $\phi(x) = x/\overline{x}$ is the index two subgroup $\phi(\mathcal{O}_E^\times) = \mu(E)^2 \subset \mu(E)$. From this it follows easily that

$$[\mathrm{Nm}(\mathcal{O}_E^\times) : \mathcal{O}_F^{\times,2}] = 2 \quad \text{and} \quad [\mathcal{O}_E^\times : \mathcal{O}_F^\times] = w_E,$$

and so

$$[\mathcal{O}_F^\times : \mathrm{Nm}(\mathcal{O}_E^\times)] = \frac{1}{2} \cdot [\mathcal{O}_F^\times : \mathcal{O}_F^{\times,2}] = 2^{n-1} = \frac{2^{n-1} w_E}{[\mathcal{O}_E^\times : \mathcal{O}_F^\times]}. \qquad \square$$

Combining the information we have so far gives

$$|\tilde{B}| = [\mathcal{O}_F^\times : \mathrm{Nm}(\mathcal{O}_E^\times)] \cdot |B| = \frac{2^{n-1} w_E}{[\mathcal{O}_E^\times : \mathcal{O}_F^\times]} \cdot \frac{|C_E|}{|C_F|} = w_E \cdot \Lambda(0, \chi_E), \tag{4.2.3}$$

where the final equality is a consequence of Dirichlet's class number formula.

Lemma 4.2.10. — *There is an exact sequence*

$$1 \to E' \backslash \widehat{E}' / \widehat{\mathcal{O}}'_E \to \tilde{B} \to \{\pm 1\}^r \to \{\pm 1\} \to 1.$$

Proof. — Every $x \in \widehat{E}'$ determines a fractional $\mathcal{O}_E$-ideal $\mathfrak{B} = x\mathcal{O}_E$ with $\mathrm{Nm}(\mathfrak{B}) = \mathcal{O}_F$, and the rule $x \mapsto (\mathfrak{B}, 1)$ is easily seen to define an injection

$$E' \backslash \widehat{E}' / \widehat{\mathcal{O}}'_E \to \tilde{B}. \tag{4.2.4}$$

Given a $(\mathfrak{B}, \beta) \in \tilde{B}$, consider the elements $\chi_{E,v}(\beta) \in \{\pm 1\}$ as v runs over all places of F. If v is split in E then certainly $\chi_{E,v}(\beta) = 1$. If v is inert in E then $\mathrm{Nm}(\mathfrak{B}) = \beta\mathcal{O}_F$ implies that $\chi_{E,v}(\beta) = 1$. As the product over all v of $\chi_{E,v}(\beta)$ is equal to 1, we see that sending $(\mathfrak{B}, \beta)$ to the tuple of $\chi_{E,v}(\beta)$ with v ramified in E defines a homomorphism

$$\tilde{B} \to \ker\big(\{\pm 1\}^r \xrightarrow{\text{product}} \{\pm 1\}\big). \tag{4.2.5}$$

To see that (4.2.5) is surjective, fix a tuple $(\varepsilon_v)_v \in \{\pm 1\}^r$ indexed by the places of F ramified in E, and assume that $\prod_v \varepsilon_v = 1$. Let $b \in \mathbb{A}_F^\times$ be any idele satisfying:
— If v is ramified in E then $\chi_{E,v}(b_v) = \varepsilon_v$.
— If v is a finite place of F then $b_v \in \mathcal{O}_{F,v}^\times$.
The second condition implies that $\chi_{E,v}(b_v) = 1$ whenever v is unramified in E, and hence

$$\chi_E(b) = \prod_v \varepsilon_v = 1.$$

Thus b lies in the kernel of the reciprocity map

$$\mathbb{A}_F^\times \to F^\times \backslash \mathbb{A}_F^\times / \mathrm{Nm}(\mathbb{A}_E^\times) \cong \mathrm{Gal}(E/F),$$

and so can be factored as $b = \beta^{-1} x\overline{x}$ for some $\beta \in F^\times$ and $x \in \mathbb{A}_E^\times$. Setting $\mathfrak{B} = x\mathcal{O}_E$, the pair $(\mathfrak{B}, \beta) \in \tilde{B}$ maps to $(\varepsilon_v)_v$ under (4.2.5).

It only remains to show that the image of (4.2.4) is equal to the kernel of (4.2.5). It is clear from the definitions that the composition

$$E' \backslash \widehat{E}' / \widehat{\mathcal{O}}'_E \to \tilde{B} \to \{\pm 1\}^r$$

is trivial, proving one inclusion. For the other inclusion, suppose $(\mathfrak{B},\beta)\in\tilde{B}$ lies in the kernel of (4.2.5). We have already seen that this implies that $\beta\in F^\times$ satisfies $\chi_{E,v}(\beta)=1$ for every place v of F, and so β is a norm from E everywhere locally. By the Hasse-Minkowski theorem, β is a norm globally, say $\beta=\alpha\overline{\alpha}$ with $\alpha\in E^\times$. In the group $\tilde{B}$, we therefore have the relation

$$(\mathfrak{B},\beta)=\alpha^{-1}(\mathfrak{B},\beta)=(\mathfrak{A},1)$$

for a fractional $\mathcal{O}_E$-ideal $\mathfrak{A}=\alpha^{-1}\mathfrak{B}$ satisfying $\mathrm{Nm}(\mathfrak{A})=\mathcal{O}_F$. Any such $\mathfrak{A}$ has the form $\mathfrak{A}=x\mathcal{O}_E$ for some $x\in\widehat{E}'$, proving that $(\mathfrak{B},\beta)$ lies in the image of (4.2.4). □

Combining the lemma with (4.2.3) gives

$$\frac{|E'\backslash\widehat{E}'/\widehat{\mathcal{O}}'_E|}{w_E}=\frac{|\tilde{B}|}{2^{r-1}w_E}=\frac{\Lambda(0,\chi_E)}{2^{r-1}},$$

and combining this with (4.2.2) completes the proof of Proposition 4.2.8. □

Proposition 4.2.11. — *Assume that the discriminants of $\boldsymbol{k}$ and F are relatively prime. The constant term* (1.4.2) *satisfies*

$$[\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0):\mathcal{Y}_{\mathrm{big}}]=-[\widehat{\boldsymbol{\omega}}:\mathcal{Y}_{\mathrm{big}}].$$

Proof. — The stated equality is equivalent to

$$[(\mathrm{Exc},-\log(D)):\mathcal{Y}_{\mathrm{big}}]=0,$$

and so it suffices to prove

$$[(0,\log(D)):\mathcal{Y}_{\mathrm{big}}]=\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})\cdot\log(D)=[(\mathrm{Exc},0):\mathcal{Y}_{\mathrm{big}}].$$

The first equality is clear from the definitions. To prove the second equality, we first argue that

$$\mathcal{Y}_{\mathrm{big}}\times_{\mathcal{S}_{\mathrm{Kra}}}\mathrm{Exc}=\mathcal{Y}_{\mathrm{big}}\times_{\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})}\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}/\mathfrak{d}_{\boldsymbol{k}}),\tag{4.2.6}$$

as in the proof of Proposition 2.2.10.

The inclusion $\subset$ of (4.2.6) is again clear from

$$\mathrm{Exc}\subset\mathcal{S}_{\mathrm{Kra}}\times_{\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})}\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}}/\mathfrak{d}_{\boldsymbol{k}}).$$

Recall that $\mathcal{Y}_{\mathrm{big}}\to\mathrm{Spec}(\mathcal{O}_\Phi)$ is étale. Our hypothesis on the discriminants of $\boldsymbol{k}$ and F implies that $\mathrm{Spec}(\mathcal{O}_\Phi)\to\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$ is étale at all primes dividing $\mathfrak{d}_{\boldsymbol{k}}$, and hence the same is true for $\mathcal{Y}_{\mathrm{big}}\to\mathrm{Spec}(\mathcal{O}_{\boldsymbol{k}})$. This implies that the right hand side of (4.2.6) is reduced, and hence so is the left hand side. To prove equality in (4.2.6), it therefore suffices to prove the inclusion $\supset$ on the level of geometric points.

Suppose $\mathfrak{p}\mid\mathfrak{d}_{\boldsymbol{k}}$ is prime, and let $\mathbb{F}^{\mathrm{alg}}_{\mathfrak{p}}$ be an algebraic closure of its residue field. Suppose that $y\in\mathcal{Y}_{\mathrm{big}}(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{p}})$ corresponds to the pair (A_0,A), so that $A\in\mathcal{CM}_\Phi(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{p}})$. Let W be the completed étale local ring of the geometric point

$$\mathrm{Spec}(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{p}})\xrightarrow{y}\mathcal{Y}_{\mathrm{big}}\to\mathrm{Spec}(\mathcal{O}_\Phi).$$

More concretely, W is the completion of the maximal unramified extension of $\mathcal{O}_{\boldsymbol{k},\mathfrak{p}}$, equipped with an injective ring homomorphism $\mathcal{O}_\Phi\to W$. Let $\mathbb{C}_{\mathfrak{p}}$ be the completion of an algebraic closure of the fraction field of W, and fix an isomorphism of E_Φ-algebras $\mathbb{C} \cong \mathbb{C}_\mathfrak{p}$.

For every $\varphi \in \Phi$ the induced map $\mathcal{O}_E \to \mathbb{C} \cong \mathbb{C}_\mathfrak{p}$ takes values in the subring W, and the induced map

$$\mathcal{O}_E \otimes_\mathbb{Z} W \to \prod_{\varphi \in \Phi} W$$

is surjective (by our hypothesis that $\boldsymbol{k}$ and F have relatively prime discriminants). Denote its kernel by $J_\Phi \subset \mathcal{O}_E \otimes_\mathbb{Z} W$, and define an $\mathcal{O}_E \otimes_\mathbb{Z} W$-module

$$\mathrm{Lie}_\Phi = (\mathcal{O}_E \otimes_\mathbb{Z} W)/J_\Phi \cong \prod_{\varphi \in \Phi} W.$$

As in the proof of [**19**, Lemma 4.1.2], there is an isomorphism of $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{F}_\mathfrak{p}^{\mathrm{alg}}$-modules

$$\mathrm{Lie}(A) \cong \mathrm{Lie}_\Phi \otimes_W \mathbb{F}_\mathfrak{p}^{\mathrm{alg}} \cong \prod_{\varphi \in \Phi} \mathbb{F}_\mathfrak{p}^{\mathrm{alg}}.$$

Let $\delta \in \mathcal{O}_{\boldsymbol{k}}$ be a square root of $-D$. As the image of δ under

$$\mathcal{O}_E \xrightarrow{\varphi} W \to \mathbb{F}_\mathfrak{p}^{\mathrm{alg}}$$

is 0 for every $\varphi \in \Phi$, it follows from what was said above that δ annihilates $\mathrm{Lie}(A)$. Exactly as in the proof of Proposition 2.2.10, this implies that the image of y under $\mathcal{Y}_{\mathrm{big}} \to \mathcal{S}_{\mathrm{Kra}}$ lies on the exceptional divisor. This completes the proof of (4.2.6), and the remainder of the proof is exactly as in Proposition 2.2.10. □

4.3. A generalized L-function. — The action $i_E : \mathcal{O}_E \to \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a})$ makes

$$L = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(\mathfrak{a}_0, \mathfrak{a})$$

into a projective $\mathcal{O}_E$-module of rank one, and the $\mathcal{O}_{\boldsymbol{k}}$-hermitian form on L defined by [**6**, (2.1.5)] satisfies $\langle \alpha x_1, x_2 \rangle = \langle x_1, \overline{\alpha} x_2 \rangle$ for all $\alpha \in \mathcal{O}_E$ and $x_1, x_2 \in L$. It is a formal consequence of this that the E-vector space $\mathscr{V} = L \otimes_\mathbb{Z} \mathbb{Q}$ carries an E-hermitian form

$$\langle -, - \rangle_{\mathrm{big}} : \mathscr{V} \times \mathscr{V} \to E,$$

uniquely determined by the property

$$\langle x_1, x_2 \rangle = \mathrm{Tr}_{E/\boldsymbol{k}} \langle x_1, x_2 \rangle_{\mathrm{big}}.$$

This hermitian form has signature $(0,1)$ at $\varphi^{\mathrm{sp}}|_F$, and signature $(1,0)$ at all other archimedean places of F.

From the E-hermitian form we obtain an F-valued quadratic form $\mathscr{Q}(x) = \langle x, x \rangle_{\mathrm{big}}$ on $\mathscr{V}$ with signature $(0,2)$ at $\varphi^{\mathrm{sp}}|_F$, and signature $(2,0)$ at all other archimedean places of F. The $\mathbb{Q}$-quadratic form

$$Q(x) = \mathrm{Tr}_{F/\mathbb{Q}} \mathscr{Q}(x) \tag{4.3.1}$$

is $\mathbb{Z}$-valued on $L \subset \mathscr{V}$, and agrees with the quadratic form of §2.3. Let

$$\omega_L : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}_\mathbb{C}(S_L)$$

be the Weil representation on the space $S_L = \mathbb{C}[L'/L]$, where $L' = \mathfrak{d}_{\boldsymbol{k}}^{-1}L$ is the dual lattice of L relative to the $\mathbb{Z}$-bilinear form (2.3.1).

Write each $\vec{\tau} \in F_{\mathbb{C}}$ in the form $\vec{\tau} = \vec{u} + i\vec{v}$ with $\vec{u}, \vec{v} \in F_{\mathbb{R}}$, and set

$$\mathcal{H}_F = \{\vec{\tau} \in F_{\mathbb{C}} : \vec{v} \text{ is totally positive}\}.$$

Every Schwartz function $\phi \in S(\widehat{\mathscr{V}})$ determines an incoherent Hilbert modular Eisenstein series

$$E(\vec{\tau}, s, \phi) = \sum_{\alpha \in F} E_\alpha(\vec{v}, s, \phi) \cdot q^\alpha \tag{4.3.2}$$

on $\mathcal{H}_F$, as in [**8**, (4.4)] and [**2**, §6.1]. If we identify

$$S_L = \mathbb{C}[L'/L] \subset S(\widehat{\mathscr{V}})$$

as the space of $\widehat{L}$-invariant functions supported on $\widehat{L}'$, then (4.3.2) can be viewed as a function $E(\vec{\tau}, s)$ on $\mathcal{H}_F$ taking values in the complex dual $S_L^\vee$.

We quickly recall the construction of (4.3.2). If v is an arichmedean place of F, denote by $(\mathscr{C}_v, \mathscr{Q}_v)$ the unique positive definite rank 2 quadratic space over F_v. Set $\mathscr{C}_\infty = \prod_{v|\infty} \mathscr{C}_v$. The rank 2 quadratic space

$$\mathscr{C} = \mathscr{C}_\infty \times \widehat{\mathscr{V}}$$

over $\mathbb{A}_F$ is *incoherent*, in the sense that it is not the adelization of any F-quadratic space. In fact, $\mathscr{C}$ is isomorphic to $\mathscr{V}$ everywhere locally, except at the unique archimedean place $\varphi^{\mathrm{sp}}|_F$ at which $\mathscr{V}$ is negative definite.

Let $\psi_{\mathbb{Q}} : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the standard additive character, and define

$$\psi_F : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$$

by $\psi_F = \psi_{\mathbb{Q}} \circ \mathrm{Tr}_{F/\mathbb{Q}}$. Denote by $I(s, \chi_E)$ the degenerate principal series representation of $\mathrm{SL}_2(\mathbb{A}_F)$ induced from the character $\chi_E|\cdot|^s$ on the subgroup $B \subset \mathrm{SL}_2$ of upper triangular matrices. Thus $I(s, \chi_E)$ consists of all smooth functions $\Phi(g, s)$ on $\mathrm{SL}_2(\mathbb{A}_F)$ satisfying the transformation law

$$\Phi\left(\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} g, s\right) = \chi_E(a)|a|^{s+1}\Phi(g, s).$$

The Weil representation $\omega_{\mathscr{C}}$ determined by the character ψ_F defines an action of $\mathrm{SL}_2(\mathbb{A}_F)$ on $S(\mathscr{C})$, and for any Schwartz function

$$\phi_\infty \otimes \phi \in S(\mathscr{C}_\infty) \otimes S(\widehat{\mathscr{V}}) \cong S(\mathscr{C})$$

the function

$$\Phi(g, 0) = \omega_{\mathscr{C}}(g)(\phi_\infty \otimes \phi)(0) \tag{4.3.3}$$

lies in the induced representation $I(0, \chi_E)$. It extends uniquely to a standard section $\Phi(g, s)$ of $I(s, \chi_E)$, which determines an Eisenstein series

$$E(g, s, \phi_\infty \otimes \phi) = \sum_{\gamma \in B(F)\backslash \mathrm{SL}_2(F)} \Phi(\gamma g, s)$$

in the variable $g \in \mathrm{SL}_2(\mathbb{A}_F)$.

We always choose $\phi \in S_L \subset S(\mathscr{V})$, and take the archimedean component ϕ_∞ of our Schwartz function to be the Gaussian distribution

$$\phi_\infty^{\mathbf{1}} = \otimes \phi_v^{\mathbf{1}} \in \bigotimes_{v|\infty} S(\mathscr{C}_v)$$

defined by $\phi_v^{\mathbf{1}}(x) = e^{-2\pi \mathscr{Q}_v(x)}$, so that the resulting Eisenstein series

$$E(\vec{\tau}, s, \phi) = \frac{1}{\sqrt{\mathrm{Nm}(\vec{v})}} \cdot E(g_{\vec{\tau}}, s, \phi_\infty^{\mathbf{1}} \otimes \phi)$$

has parallel weight 1. Here

$$g_{\vec{\tau}} = \begin{pmatrix} 1 & \vec{u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\vec{v}} & \\ & 1/\sqrt{\vec{v}} \end{pmatrix} \in \mathrm{SL}_2(F_\mathbb{R})$$

and $\mathrm{Nm} : F_\mathbb{R}^\times \to \mathbb{R}^\times$ is the norm.

A choice of ordering of the embeddings $F \to \mathbb{R}$ fixes an isomorphism of $\mathcal{H}_F$ with the n-fold product of the complex upper half-plane with itself, and the diagonal inclusion $\mathcal{H} \hookrightarrow \mathcal{H}_F$ is independent of the choice of ordering. By restricting our Eisenstein series to the diagonal we obtain an $S_L^\vee$-valued function

$$E(\tau, s) = E(\vec{\tau}, s)|_{\mathcal{H}}$$

in the variable $\tau \in \mathcal{H}$, which transforms like a modular form of weight n and representation $\omega_L^\vee$ under the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Given a cusp form $\tilde{g} \in S_n(\overline{\omega}_L)$ valued in S_L, consider the Petersson inner product

$$\langle E(s), \tilde{g} \rangle_{\mathrm{Pet}} = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}} \left\{ \overline{\tilde{g}(\tau)}, E(\tau, s) \right\} \frac{du\, dv}{v^{2-n}}, \tag{4.3.4}$$

where $\{.,.\} : S_L \times S_L^\vee \to \mathbb{C}$ is the tautological pairing. This is an unnormalized version of the *generalized L-function*

$$\mathcal{L}(s, \tilde{g}) = \Lambda(s+1, \chi_E) \cdot \langle E(s), \tilde{g} \rangle_{\mathrm{Pet}}$$

of [**8**, (1.2)] or [**2**, §6.3].

Let $F_+ \subset F$ be the subset of totally positive elements. The Eisenstein series $E(\vec{\tau}, s)$ satisfies a functional equation in $s \mapsto -s$, forcing it to vanish at $s = 0$. As in [**8**, Proposition 4.6] and [**2**, §6.2], we can extract from the central derivative $E'(\vec{\tau}, 0)$ a formal q-expansion

$$a_F(0) + \sum_{\alpha \in F_+} a_F(\alpha) \cdot q^\alpha.$$

If $\alpha \in F_+$ then $E'_\alpha(\vec{v}, 0, \phi)$ is independent of $\vec{v}$, and we define $a_F(\alpha) \in S_L^\vee$ by

$$a_F(\alpha, \phi) = \Lambda(0, \chi_E) \cdot E'_\alpha(\vec{v}, 0, \phi).$$

We define $a_F(0) \in S_L^\vee$ by

$$a_F(0, \phi) = \Lambda(0, \chi) \cdot E'_0(\vec{v}, 0, \phi) - \Lambda(0, \chi_E) \cdot \phi(0) \log \mathrm{Nm}(\vec{v}).$$

Again, this is independent of $\vec{v}$.

Remark 4.3.1. — For notational simplicity, we often denote by $a_F(\alpha,\mu)$ the value of $a_F(\alpha): S_L \to \mathbb{C}$ at the characteristic function of a coset $\mu \in L'/L$.

For any nonzero $\alpha \in F$, define

$$\mathrm{Diff}(\mathscr{C},\alpha) = \{\text{places } v \text{ of } F : \mathscr{C}_v \text{ does not represent } \alpha\}.$$

This is a finite set of odd cardinality, and any $v \in \mathrm{Diff}(\mathscr{C},\alpha)$ is necessarily nonsplit in E. We are really only interested in this set when $\alpha \in F_+$. As $\mathscr{C}$ is positive definite at all archimedean places, for such α we have

$$\mathrm{Diff}(\mathscr{C},\alpha) = \{\text{primes } \mathfrak{p} \subset \mathcal{O}_F : \mathscr{V}_\mathfrak{p} \text{ does not represent } \alpha\}.$$

We will need explicit formulas for all $a_F(\alpha,\mu)$ with $\alpha \in F_+$, but only for the trivial coset $\mu = 0$. These are provided by the following proposition.

Proposition 4.3.2. — *Suppose $\alpha \in F_+$.*

1. *If* $|\mathrm{Diff}(\mathscr{C},\alpha)| > 1$ *then* $a_F(\alpha) = 0$.
2. *If* $\mathrm{Diff}(\mathscr{C},\alpha) = \{\mathfrak{p}\}$, *then*

$$a_F(\alpha,0) = -2^{r-1}\cdot \rho(\alpha\mathfrak{d}_F\mathfrak{p}^{-\varepsilon_\mathfrak{p}})\cdot \mathrm{ord}_\mathfrak{p}(\alpha\mathfrak{p}\mathfrak{d}_F)\cdot \log(\mathrm{N}(\mathfrak{p})),$$

where the notation is as follows: r is the number of places of F ramified in E (including all archimedean places), $\mathfrak{d}_F \subset \mathcal{O}_F$ is the different of F, and

$$\varepsilon_\mathfrak{p} = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ is inert in } E \\ 0 & \text{if } \mathfrak{p} \text{ is ramified in } E. \end{cases}$$

Moreover, for any fractional $\mathcal{O}_F$-ideal $\mathfrak{b} \subset F$ we have set

$$\rho(\mathfrak{b}) = |\{\text{ideals } \mathfrak{B} \subset \mathcal{O}_E : \mathfrak{B}\overline{\mathfrak{B}} = \mathfrak{b}\mathcal{O}_E\}|.$$

In particular, $\rho(\mathfrak{b}) = 0$ unless $\mathfrak{b} \subset \mathcal{O}_F$.

Proof. — Up to a change of notation, this is [**18**, Proposition 4.2.1], whose proof amounts to collecting together calculations of [**28**]. More general formulas can be found in [**2**, §7.1] and [**21**, §4.6]. □

Proposition 4.3.3. — *Assume that the discriminants of $\boldsymbol{k}$ and F are relatively prime. For any $\mu \in L'/L$ we have*

$$a_F(0,\mu) = \begin{cases} -2\Lambda'(0,\chi_E) & \text{if } \mu = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — Let $\Phi_\mu = \prod_\mathfrak{p} \Phi_{\mu,\mathfrak{p}}$ be the standard section of $I(s,\chi_E)$ determined by the characteristic function $\phi_\mu \in S_L \subset S(\mathscr{V})$ of $\mu \in L'/L$. According to [**2**, Proposition 6.2.3], we then have

$$a_F(0,\mu) = -2\phi_\mu(0)\Lambda'(0,\chi_E) - \Lambda(0,\chi_E)\cdot \frac{d}{ds}\Big(\prod_\mathfrak{p} M_\mathfrak{p}(s,\phi_\mu)\Big)\big|_{s=0}, \tag{4.3.5}$$

where the product is over all finite places $\mathfrak{p}$ of F, and the local factors on the right have the form

$$M_{\mathfrak{p}}(s,\phi_\mu) = c_{\mathfrak{p}} \cdot \frac{L_{\mathfrak{p}}(s+1,\chi_E)}{L_{\mathfrak{p}}(s,\chi_E)} \cdot W_{0,\mathfrak{p}}(s,\Phi_\mu) \tag{4.3.6}$$

for some constants $c_{\mathfrak{p}}$ independent of s. Here, setting

$$w = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad n(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right),$$

the function

$$W_{0,\mathfrak{p}}(s,\Phi_\mu) = \int_{F_{\mathfrak{p}}} \Phi_{\mu,\mathfrak{p}}\left(wn(b),s\right)\, db$$

is the value of the local Whittaker function $W_{0,\mathfrak{p}}(g,s,\Phi_\mu)$ at the identity in $\mathrm{SL}_2(F_{\mathfrak{p}})$. Our goal is to prove that $M_{\mathfrak{p}}(s,\phi_\mu)$ is independent of s, and hence both the particular value of $c_{\mathfrak{p}}$ and the choice of Haar measure on $F_{\mathfrak{p}}$ are irrelevant to us.

Fix a prime $\mathfrak{p} \subset \mathcal{O}_F$, and let p be the rational prime below it. We may identify $\mathscr{V}_{\mathfrak{p}} \cong E_{\mathfrak{p}}$ in such a way that $L_{\mathfrak{p}} \cong \mathcal{O}_{E,\mathfrak{p}}$, and so that the $F_{\mathfrak{p}}$-valued quadratic form $\mathscr{Q}$ on $\mathscr{V}_{\mathfrak{p}} \cong E_{\mathfrak{p}}$ becomes

$$\mathscr{Q}(x) = \beta x\overline{x}$$

for some $\beta \in F_{\mathfrak{p}}^\times$. If $\mathfrak{d}_F$ denotes the different of $F/\mathbb{Q}$, then

$$\beta\mathcal{O}_{F,\mathfrak{p}} = \mathfrak{d}_F^{-1}\mathcal{O}_{F,\mathfrak{p}}. \tag{4.3.7}$$

Indeed, let $\mathfrak{d}_E$ be the different of $E/\mathbb{Q}$. The lattice $L'_{\mathfrak{p}} = \mathfrak{d}_{\boldsymbol{k}}^{-1}\mathcal{O}_{E,\mathfrak{p}}$ is the dual lattice of $\mathcal{O}_{E,\mathfrak{p}}$ relative to the $\mathbb{Q}_p$-bilinear form $[x,y] = \mathrm{Tr}_{E_{\mathfrak{p}}/\mathbb{Q}_p}(\beta x\overline{y})$, which implies the first equality in

$$\beta^{-1}\mathcal{O}_{E,\mathfrak{p}} = \mathfrak{d}_E\mathfrak{d}_{\boldsymbol{k}}^{-1}\mathcal{O}_{E,\mathfrak{p}} = \mathfrak{d}_F\mathcal{O}_{E,\mathfrak{p}}.$$

The second equality is a consequence of our assumption that the discriminants of $\boldsymbol{k}$ and F are relatively prime.

If we endow $\mathscr{V}_{\mathfrak{p}} = E_{\mathfrak{p}}$ with the rescaled quadratic form

$$\mathscr{Q}^\sharp(x) \stackrel{\mathrm{def}}{=} \beta^{-1}\mathscr{Q}(x) = x\overline{x},$$

and define a new additive character

$$\psi^\sharp_{F,\mathfrak{p}}(x) \stackrel{\mathrm{def}}{=} \psi_{F,\mathfrak{p}}(\beta x)$$

(unramified by (4.3.7)), we obtain a new Weil representation

$$\omega^\sharp : \mathrm{SL}_2(F_{\mathfrak{p}}) \to \mathrm{Aut}(S(\mathscr{V}_{\mathfrak{p}})),$$

and hence, as in (4.3.3), a function

$$S(\mathscr{V}_{\mathfrak{p}}) \xrightarrow{\phi \mapsto \Phi^\sharp_{\mathfrak{p}}(s,g)} I_{\mathfrak{p}}(s,\chi_E)$$

defined by first setting $\Phi^\sharp_{\mathfrak{p}}(0,g) = \omega^\sharp(g)\phi(0)$, and then extending to a standard section.

The local Schwartz function $\phi_{\mu,\mathfrak{p}} \in S(\mathscr{V}_{\mathfrak{p}})$ now determines a standard section $\Phi^{\sharp}_{\mu,\mathfrak{p}}(g,s)$ of $I_{\mathfrak{p}}(s,\chi_E)$, and explicit formulas for the Weil representation, as in [**21**, (4.2.1)], show that

$$\int_{F_{\mathfrak{p}}} \Phi_{\mu,\mathfrak{p}}\left(wn(b),s\right)\, db = \int_{F_{\mathfrak{p}}} \Phi^{\sharp}_{\mu,\mathfrak{p}}\left(wn(b),s\right)\, db.$$

What our discussion shows is that there is no harm in rescaling the quadratic form on $\mathscr{V}_{\mathfrak{p}}$ to make $\beta = 1$, and simultaneously modifying the additive character $\psi_{F,\mathfrak{p}}$ to make it unramified.

After this rescaling, one can easily deduce explicit formulas for $W_{0,\mathfrak{p}}(s,\Phi_\mu)$ from the literature. Indeed, if the local component $\mu_{\mathfrak{p}} \in L'_{\mathfrak{p}}/L_{\mathfrak{p}}$ is zero, then the calculations found in [**28**, §2] imply that

$$W_{0,\mathfrak{p}}(s,\Phi_\mu) = \frac{L_{\mathfrak{p}}(s,\chi_E)}{L_{\mathfrak{p}}(s+1,\chi_E)}$$

up to scaling by a nonzero constant independent of s. If instead $\mu_{\mathfrak{p}} \neq 0$ then $\mathfrak{p}$ is ramified in E (and in particular $p > 2$), and it follows from the calculations found in the proof of [**21**, Proposition 4.6.4] that $W_{0,\mathfrak{p}}(s,\Phi_\mu) = 0$. In any case (4.3.6) is independent of s for every $\mathfrak{p}$, and so the derivative in (4.3.5) vanishes. □

4.4. A preliminary central derivative formula. — The entirety of §4.4 is devoted to proving Theorem 4.4.1, which a big CM analogue of Theorem 2.4.1. The proof will make essential use of the calculations of [**18, 19, 8**].

We assume $n \geq 3$ throughout §4.4. This allows us to make use of the distinguished harmonic forms

$$f_m \in H_{2-n}(\overline{\omega}_L)^{\Delta}$$

(for $m > 0$) characterized by (2.5.1).

Theorem 4.4.1. — *Assume that the discriminants of $\boldsymbol{k}/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime, and fix a positive integer m. If $f = f_m$ is the harmonic form above, and $\widehat{\mathcal{Z}}$ is the linear function* (2.4.1), *then*

$$\frac{n\cdot[\widehat{\mathcal{Z}}(f):\mathcal{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} + 2c_f^{+}(0,0)\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)} = -\frac{d}{ds}\langle E(s),\xi(f)\rangle_{\mathrm{Pet}}\big|_{s=0}.$$

For the form $f = f_m$ we have

$$\widehat{\mathcal{Z}}(f) = \widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = \left(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m),\Theta^{\mathrm{reg}}(f_m)\right) \in \widehat{\mathrm{Ch}}^{1}(\mathcal{S}^{*}_{\mathrm{Kra}}),$$

where the Green function $\Theta^{\mathrm{reg}}(f_m)$ for the divisor $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)$ is constructed in [**6**, §7] as a regularized theta lift. The arithmetic degree appearing in Theorem 4.4.1 decomposes as

(4.4.1)
$$[\widehat{\mathcal{Z}}(f):\mathcal{Y}_{\mathrm{big}}] = \sum_{\mathfrak{p}\subset\mathcal{O}_{\boldsymbol{k}}} \log(\mathrm{N}(\mathfrak{p})) \sum_{y\in(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)\cap\mathcal{Y}_{\mathrm{big}})(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{p}})} \frac{\mathrm{length}(\mathcal{O}_y)}{|\mathrm{Aut}(y)|} + \sum_{y\in\mathcal{Y}_{\mathrm{big}}(\mathbb{C})} \frac{\Theta^{\mathrm{reg}}(f_m)(y)}{|\mathrm{Aut}(y)|},$$

where $\mathbb{F}_{\mathfrak{p}} = \mathcal{O}_{\boldsymbol{k}}/\mathfrak{p}$, and $\mathcal{O}_y$ is the étale local ring of

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cap \mathcal{Y}_{\mathrm{big}} \stackrel{\mathrm{def}}{=} \mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \times_{\mathcal{S}^*_{\mathrm{Kra}}} \mathcal{Y}_{\mathrm{big}} \tag{4.4.2}$$

at y. The final summation is over all complex points of $\mathcal{Y}_{\mathrm{big}}$, viewed as an $\mathcal{O}_{\boldsymbol{k}}$-stack. We will see that the terms on the right hand side of (4.4.1) are intimately related to the Eisenstein series coefficients $a_F(\alpha)$ of §4.3.

We first study the structure of the stack-theoretic intersection (4.4.2). Suppose S is a connected $\mathcal{O}_\Phi$-scheme, and

$$(A_0, A) \in \big(\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi\big)(S)$$

is an S-point. The $\mathcal{O}_{\boldsymbol{k}}$-module $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ carries an $\mathcal{O}_{\boldsymbol{k}}$-hermitian form $\langle -, - \rangle$ defined by [**6**, (2.5.1)]. The construction of this hermitian form only uses the underlying point of $\mathcal{S}_{\mathrm{Kra}}$, and not the action $\mathcal{O}_E \to \mathrm{End}_{\mathcal{O}_{\boldsymbol{k}}}(A)$. As in [**19**, §3.2], the extra action of $\mathcal{O}_E$ makes $\mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ into a projective $\mathcal{O}_E$-module, and there is a totally positive definite E-hermitian form $\langle -, - \rangle_{\mathrm{big}}$ on

$$\mathscr{V}(A_0, A) = \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{4.4.3}$$

characterized by the relation

$$\langle x_1, x_2 \rangle = \mathrm{Tr}_{E/\boldsymbol{k}} \langle x_1, x_2 \rangle_{\mathrm{big}}.$$

for all $x_1, x_2 \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$.

Fix an $\alpha \in F_+$. Recalling that

$$\mathcal{Y}_{\mathrm{big}} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi \tag{4.4.4}$$

as an open and closed substack, for any $\mathcal{O}_\Phi$-scheme S let $\mathcal{Z}_{\mathrm{big}}(\alpha)(S)$ be the groupoid of triples (A_0, A, x), in which

— $(A_0, A) \in \mathcal{Y}_{\mathrm{big}}(S)$,

— $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfies $\langle x, x \rangle_{\mathrm{big}} = \alpha$.

This functor is represented by an $\mathcal{O}_\Phi$-stack $\mathcal{Z}_{\mathrm{big}}(\alpha)$, and the evident forgetful morphism

$$\mathcal{Z}_{\mathrm{big}}(\alpha) \to \mathcal{Y}_{\mathrm{big}}$$

is finite and unramified.

This construction is entirely analogous to the construction of the special divisors $\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \to \mathcal{S}_{\mathrm{Kra}}$ of [**6**]. In fact, directly from the definitions, if S is an $\mathcal{O}_\Phi$-scheme an S-point

$$(A_0, A, x) \in \big(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cap \mathcal{Y}_{\mathrm{big}}\big)(S)$$

consists of a pair $(A_0, A) \in \mathcal{Y}_{\mathrm{big}}(S)$ and an $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfying $m = \langle x, x \rangle$. From this it is clear that there is an isomorphism

$$\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \cap \mathcal{Y}_{\mathrm{big}} \cong \bigsqcup_{\substack{\alpha \in F_+ \\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha) = m}} \mathcal{Z}_{\mathrm{big}}(\alpha), \tag{4.4.5}$$

defined by sending the triple (A_0, A, x) to the same triple, but now viewed as an S-point of the stack $\mathcal{Z}_{\mathrm{big}}(\alpha)$ determined by $\alpha = \langle x, x \rangle_{\mathrm{big}}$.

Proposition 4.4.2. — *For each $\alpha \in F_+$ the stack $\mathcal{Z}_{\mathrm{big}}(\alpha)$ is either empty, or has dimension 0 and is supported at a single prime of $\mathcal{O}_\Phi$. Moreover,*

1. *If $|\mathrm{Diff}(\mathscr{C},\alpha)| > 1$ then $\mathcal{Z}_{\mathrm{big}}(\alpha) = \emptyset$.*
2. *Suppose that $\mathrm{Diff}(\mathscr{C},\alpha) = \{\mathfrak{p}\}$ for a single prime $\mathfrak{p} \subset \mathcal{O}_F$, let $\mathfrak{q} \subset \mathcal{O}_E$ be the unique prime above it, and denote by $\mathfrak{q}_\Phi \subset \mathcal{O}_\Phi$ the corresponding prime under the isomorphism $\varphi^{\mathrm{sp}} : E \cong E_\Phi$. Then $\mathcal{Z}_{\mathrm{big}}(\alpha)$ is supported at the prime $\mathfrak{q}_\Phi$, and satisfies*
$$\sum_{y \in \mathcal{Z}_{\mathrm{big}}(\alpha)(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{q}_\Phi})} \frac{1}{|\mathrm{Aut}(y)|} = \frac{h_{\boldsymbol{k}}}{w_{\boldsymbol{k}}} \cdot \rho(\alpha\mathfrak{d}_F\mathfrak{p}^{-\varepsilon_\mathfrak{p}}),$$
where $\mathbb{F}_{\mathfrak{q}_\Phi}$ is the residue field of $\mathfrak{q}_\Phi$, and $\varepsilon_\mathfrak{p}$ and ρ are as in Proposition 4.3.2. Moreover, the étale local rings at all geometric points
$$y \in \mathcal{Z}_{\mathrm{big}}(\alpha)(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{q}_\Phi})$$
have the same length
$$\mathrm{length}(\mathcal{O}_y) = \mathrm{ord}_\mathfrak{p}(\alpha\mathfrak{p}\mathfrak{d}_F) \cdot \begin{cases} 1/2 & \text{if } E_\mathfrak{q}/F_\mathfrak{p} \text{ is unramified} \\ 1 & \text{otherwise.} \end{cases}$$

Proof. — This is essentially contained in [**18**, §3]. In that work we studied the $\mathcal{O}_\Phi$-stack $\mathcal{Z}_\Phi(\alpha)$ classifying triples (A_0, A, x) exactly as in the definition of $\mathcal{Z}_{\mathrm{big}}(\alpha)$, except we allowed the pair (A_0, A) to be any point of $\mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi$ rather than a point of the substack (4.4.4). Thus we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{Z}_{\mathrm{big}}(\alpha) & \longrightarrow & \mathcal{Z}_\Phi(\alpha) \\ \downarrow & & \downarrow \\ \mathcal{Y}_{\mathrm{big}} & \longrightarrow & \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi. \end{array}$$

As the bottom horizontal arrow is an open and closed immersion, so is the top horizontal arrow. In other words, our $\mathcal{Z}_{\mathrm{big}}(\alpha)$ is a union of connected components of the stack $\mathcal{Z}_\Phi(\alpha)$ of [**18**].

Lemma 4.4.3. — *Each $\mathcal{Z}_\Phi(\alpha)$ has dimension 0. If y is a geometric point of $\mathcal{Z}_\Phi(\alpha)$ corresponding to a triple (A_0, A, x) over $k(y)$, then $k(y)$ has nonzero characteristic, A_0 and A are supersingular, and the E-hermitian space (4.4.3) has dimension one. Moreover, if $\mathfrak{p} \subset \mathcal{O}_F$ denotes the image of y under the composition*

(4.4.6)
$$\mathcal{Z}_\Phi(\alpha) \to \mathrm{Spec}(\mathcal{O}_\Phi) \cong \mathrm{Spec}(\mathcal{O}_E) \to \mathrm{Spec}(\mathcal{O}_F)$$

(the isomorphism is $\varphi^{\mathrm{sp}} : E \cong E_\Phi$), then $\mathfrak{p}$ is nonsplit in E, and the following are equivalent:

— *The geometric point y factors through the open and closed substack*
$$\mathcal{Z}_{\mathrm{big}}(\alpha) \subset \mathcal{Z}_\Phi(\alpha).$$

— *The E-hermitian space* (4.4.3) *is isomorphic to $\mathscr{V}$ everywhere locally except at $\mathfrak{p}$ and $\varphi^{\mathrm{sp}}|_F$.*

Proof. — This is an easy consequence of [**18**, Proposition 3.4.5] and [**18**, Proposition 3.5.2]. The only part that requires explanation is the final claim.

Fix a connected component

$$\mathcal{B} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi.$$

As in [**18**, §3.4], for each complex point $y = (A_0, A) \in \mathcal{B}(\mathbb{C})$ one can construct from the Betti realizations of A_0 and A an E-hermitian space

$$\mathscr{V}(\mathcal{B}) = \mathrm{Hom}_{\boldsymbol{k}}\big(H_1(A_0(\mathbb{C}), \mathbb{Q}), H_1(A(\mathbb{C}), \mathbb{Q})\big)$$

of dimension 1. This hermitian space has signature $(0,1)$ at $\varphi^{\mathrm{sp}}|_F$, and signature $(1,0)$ at all other archimedean places of F. Moreover, as in Remark 4.2.3, this hermitian space depends only on the connected component $\mathcal{B}$, and not on the particular complex point y. The open and closed substack

$$\mathcal{Y}_{\mathrm{big}} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi$$

can be characterized as the union of all components $\mathcal{B}$ for which $\mathscr{V}(\mathcal{B}) \cong \mathscr{V}$.

So, suppose we have a geometric point $y = (A_0, A, x)$ of $\mathcal{Z}_\Phi(\alpha)$, and denote by

$$\mathcal{B} \subset \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi$$

the connected component containing the underlying point $y = (A_0, A)$. The content of [**18**, Proposition 3.4.5] is that the hermitian space (4.4.3) is isomorphic to $\mathscr{V}(\mathcal{B})$ everywhere locally except at $\mathfrak{p}$ and $\varphi^{\mathrm{sp}}|_F$. From this we deduce the equivalence of the following statements:

— The geometric point $y \to \mathcal{Z}_\Phi(\alpha)$ factors through $\mathcal{Z}_{\mathrm{big}}(\alpha)$.
— The underlying point $y \to \mathcal{M}_{(1,0)} \times_{\mathcal{O}_{\boldsymbol{k}}} \mathcal{CM}_\Phi$ factors through $\mathcal{Y}_{\mathrm{big}}$.
— The hermitian spaces $\mathscr{V}(\mathcal{B})$ and $\mathscr{V}$ are isomorphic.
— The E-hermitian space (4.4.3) is isomorphic to $\mathscr{V}$ everywhere locally except at $\mathfrak{p}$ and $\varphi^{\mathrm{sp}}|_F$. □

Now suppose that $\mathcal{Z}_{\mathrm{big}}(\alpha)$ is nonempty. If we fix a geometric point $y = (A_0, A, x)$ as above, the vector $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfies $\langle x, x\rangle_{\mathrm{big}} = \alpha$, and hence (4.4.3) represents α. The above lemma now implies that $\mathscr{V}$ represents α everywhere locally except at $\mathfrak{p}$ and $\varphi^{\mathrm{sp}}|_F$, where $\mathfrak{p}$ is the image of y under (4.4.6). From this it follows first $\mathrm{Diff}(\mathscr{C}, \alpha) = \{\mathfrak{p}\}$, and then that all geometric points of $\mathcal{Z}_{\mathrm{big}}(\alpha)$ have the same image under (4.4.6), and lie above the same prime $\mathfrak{q}_\Phi \subset \mathcal{O}_\Phi$ characterized as in the statement of Proposition 4.4.2. In particular, if $|\mathrm{Diff}(\mathscr{C}, \alpha)| > 1$ then $\mathcal{Z}_{\mathrm{big}}(\alpha) = \emptyset$.

It remains to prove part (2) of the proposition. For this we need the following lemma.

Lemma 4.4.4. — *Assume that* $\mathrm{Diff}(\mathscr{C}, \alpha) = \{\mathfrak{p}\}$ *for some prime* $\mathfrak{p} \subset \mathcal{O}_F$, *and let* $\mathfrak{q} \subset \mathcal{O}_E$ *be the unique prime above it. The open and closed substack* $\mathcal{Z}_{\mathrm{big}}(\alpha) \subset \mathcal{Z}_\Phi(\alpha)$

is equal to the union of all connected components of $\mathcal{Z}_\Phi(\alpha)$ that are supported at the prime $\mathfrak{q}_\Phi$.

Proof. — We have already seen that every geometric point of $\mathcal{Z}_{\mathrm{big}}(\alpha)$ lies above the prime $\mathfrak{q}_\Phi$, and so it suffices to prove that every geometric point of $\mathcal{Z}_\Phi(\alpha)$ lying above the prime $\mathfrak{q}_\Phi$ factors through $\mathcal{Z}_{\mathrm{big}}(\alpha)$. Let $y \to \mathcal{Z}_\Phi(\alpha)$ be such a point.

If y corresponds to the triple (A_0, A, x), then $x \in \mathrm{Hom}_{\mathcal{O}_{\boldsymbol{k}}}(A_0, A)$ satisfies $\langle x, x\rangle_{\mathrm{big}} = \alpha$, and hence (4.4.3) represents α. But the assumption that $\mathrm{Diff}(\mathscr{C}, \alpha) = \{\mathfrak{p}\}$ implies that $\mathscr{V}$ represents α everywhere locally except at $\mathfrak{p}$ and $\varphi^{\mathrm{sp}}|_F$, and it follows from this that $\mathscr{V}$ and (4.4.3) are isomorphic locally everywhere except at $\mathfrak{p}$ and $\varphi^{\mathrm{sp}}|_F$. By the previous lemma, this implies that y factors through $\mathcal{Z}_{\mathrm{big}}(\alpha)$. □

With this last lemma in hand, all parts of (2) follow from the corresponding statements for $\mathcal{Z}_\Phi(\alpha)$ proved in [**18**, Theorem 3.5.3] and [**18**, Theorem 3.6.2]. □

Proposition 4.4.5. — *For every $\alpha \in F_+$ we have*

$$\sum_{\mathfrak{p}\subset\mathcal{O}_{\boldsymbol{k}}} \frac{n\cdot\log(\mathrm{N}(\mathfrak{p}))}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} \sum_{y\in\mathcal{Z}_{\mathrm{big}}(\alpha)(\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}})} \frac{\mathrm{length}(\mathcal{O}_y)}{|\mathrm{Aut}(y)|} = -\frac{a_F(\alpha,0)}{\Lambda(0,\chi_E)},$$

where the inner sum is over all $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$-points of $\mathcal{Z}_{\mathrm{big}}(\alpha)$, viewed as an $\mathcal{O}_{\boldsymbol{k}}$-stack.

Proof. — Combining Propositions 4.2.8, 4.3.2, and 4.4.2 shows that

$$\sum_{\mathfrak{q}_\Phi\subset\mathcal{O}_\Phi} \frac{n\cdot\log(\mathrm{N}(\mathfrak{q}_\Phi))}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} \sum_{y\in\mathcal{Z}_{\mathrm{big}}(\alpha)(\mathbb{F}_{\mathfrak{q}_\Phi}^{\mathrm{alg}})} \frac{\mathrm{length}(\mathcal{O}_y)}{|\mathrm{Aut}(y)|} = -\frac{a_F(\alpha,0)}{\Lambda(0,\chi_E)},$$

where the inner sum is over all $\mathbb{F}_{\mathfrak{q}_\Phi}^{\mathrm{alg}}$ points of $\mathcal{Z}_{\mathrm{big}}(\alpha)$, viewed as an $\mathcal{O}_\Phi$-stack. The claim follows by collecting together all primes $\mathfrak{q}_\Phi \subset \mathcal{O}_\Phi$ lying above a common prime $\mathfrak{p} \subset \mathcal{O}_{\boldsymbol{k}}$. □

Proposition 4.4.6. — *The regularized theta lift $\Theta^{\mathrm{reg}}(f_m)$ satisfies*

$$\begin{aligned}\frac{n}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} &\sum_{y\in\mathcal{Y}_{\mathrm{big}}(\mathbb{C})} \frac{\Theta^{\mathrm{reg}}(f_m)(y)}{|\mathrm{Aut}(y)|} \\ &= -\frac{d}{ds}\langle E(s), \xi(f_m)\rangle_{\mathrm{Pet}}\big|_{s=0} + \sum_{\substack{\alpha\in F_+\\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m}} \frac{a_F(\alpha,0)}{\Lambda(0,\chi_E)} - 2c_{f_m}^+(0,0)\cdot\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)}.\end{aligned}$$

Proof. — This is a special case of the main result of [**8**]. This requires some explanation, as that work deals with cycles on Shimura varieties of type GSpin, rather than the unitary Shimura varieties under current consideration.

Recall that we have an F-quadratic space $(\mathscr{V}, \mathscr{Q})$ of rank two, and a $\mathbb{Q}$-quadratic space (V, Q) whose underlying $\mathbb{Q}$-vector space

$$V = \mathrm{Hom}_{\boldsymbol{k}}(W_0, W)$$

is equal to $\mathscr{V}$, and whose quadratic form is (4.3.1). As in [**8**, §2] or [**2**, §5.3] this data determines a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & T_{\mathrm{GSpin}} & \longrightarrow & T_{\mathrm{SO}} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathrm{GSpin}(V) & \longrightarrow & \mathrm{SO}(V) & \longrightarrow & 1, \end{array}$$

with exact rows, of algebraic groups over $\mathbb{Q}$. The torus $T_{\mathrm{SO}} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SO}(\mathscr{V})$ has $\mathbb{Q}$-points

$$T_{\mathrm{SO}}(\mathbb{Q}) = \{y \in E^\times : y\overline{y} = 1\},$$

while the torus T_{GSpin} has $\mathbb{Q}$-points

$$T_{\mathrm{GSpin}}(\mathbb{Q}) = E^\times/\ker(\mathrm{Norm} : F^\times \to \mathbb{Q}^\times).$$

The map $T_{\mathrm{GSpin}} \to T_{\mathrm{SO}}$ is $x \mapsto x/\overline{x}$. To these groups one can associate morphisms of Shimura data

$$\begin{array}{ccc} (T_{\mathrm{GSpin}}, \{h_{\mathrm{GSpin}}\}) & \longrightarrow & (T_{\mathrm{SO}}, \{h_{\mathrm{SO}}\}) \\ \downarrow & & \downarrow \\ (\mathrm{GSpin}(V), \mathcal{D}_{\mathrm{GSpin}}) & \longrightarrow & (\mathrm{SO}(V), \mathcal{D}_{\mathrm{SO}}). \end{array}$$

In the top row both data have reflex field E_Φ. In the bottom row both data have reflex field $\mathbb{Q}$.

Let $K_{\mathrm{SO}} \subset \mathrm{SO}(V)(\mathbb{A}_f)$ be any compact open subgroup that stabilizes the lattice $L \subset V$, and fix any compact open subgroup $K_{\mathrm{GSpin}} \subset \mathrm{GSpin}(V)(\mathbb{A}_f)$ contained in the preimage of K_{SO}. The Shimura data in the bottom row, along with these compact open subgroups, determine Shimura varieties $M_{\mathrm{GSpin}} \to M_{\mathrm{SO}}$. These are $\mathbb{Q}$-stacks of dimension $2n-2$.

The Shimura data in the top row, along with the compact open subgroups $K_{\mathrm{GSpin}} \cap T_{\mathrm{GSpin}}(\mathbb{A}_f)$ and $K_{\mathrm{SO}} \cap T_{\mathrm{SO}}(\mathbb{A}_f)$, determine Shimura varieties $Y_{\mathrm{GSpin}} \to Y_{\mathrm{SO}}$. These are E_Φ-stacks of dimension 0, but we instead view them as stacks over $\mathrm{Spec}(\mathbb{Q})$, so that there is a commutative diagram

(4.4.7)
$$\begin{array}{ccc} Y_{\mathrm{GSpin}} & \longrightarrow & Y_{\mathrm{SO}} \\ \downarrow & & \downarrow \\ M_{\mathrm{GSpin}} & \longrightarrow & M_{\mathrm{SO}}. \end{array}$$

Assume that the compact open subgroup K_{SO} acts trivially on the quotient L'/L. For every form $f \in H_{2-n}(\overline{\omega}_L)$, one can find in [**8**, Theorem 3.2] the construction of a divisor $Z_{\mathrm{GSpin}}(f)$ on M_{GSpin}, along with a Green function $\Theta^{\mathrm{reg}}_{\mathrm{GSpin}}(f)$ for that divisor, constructed as a regularized theta lift. Up to change of notation, [**8**, Theorem 1.1]

asserts that
(4.4.8)

$$\frac{n}{\deg_{\mathbb{C}}(Y_{\mathrm{GSpin}})}\sum_{y\in Y_{\mathrm{GSpin}}(\mathbb{C})}\frac{\Theta^{\mathrm{reg}}_{\mathrm{GSpin}}(f,y)}{|\mathrm{Aut}(y)|} = -\frac{d}{ds}\langle E(s),\xi(f)\rangle_{\mathrm{Pet}}\big|_{s=0} + \sum_{\substack{m\geq 0\\ \mu\in L'/L}}\frac{a(m,\mu)c_f^+(-m,\mu)}{\Lambda(0,\chi_E)},$$

where the coefficients $a(m)\in S_L$ are defined by

$$a(m) = \sum_{\substack{\alpha\in F_+\\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m}} a_F(\alpha)$$

if $m>0$, and by $a(0)=a_F(0)$.

It is not difficult to see, directly from the constructions, that both the divisor $Z_{\mathrm{GSpin}}(f)$ and the Green function $\Theta^{\mathrm{reg}}_{\mathrm{GSpin}}(f)$ descend to the quotient M_{SO}. If we call these descents $Z_{\mathrm{SO}}(f)$ and $\Theta^{\mathrm{reg}}_{\mathrm{SO}}(f)$, it is a formal consequence of the commutativity of (4.4.7) that the equality (4.4.8) continues to hold if all subscripts GSpin are replaced by SO.

Moreover, suppose that our form $f\in H_{2-n}(\overline{\omega}_L)$ is invariant under the action of the finite group Δ of §2.4, as is true for the form f_m of (2.5.1). In this case one can see, directly from the definitions, that the divisor $Z_{\mathrm{SO}}(f)$ and the Green function $\Theta^{\mathrm{reg}}_{\mathrm{SO}}(f)$ descend to the orthogonal Shimura variety determined by the maximal compact open subgroup

$$K_{\mathrm{SO}} = \{g\in \mathrm{SO}(V)(\mathbb{A}_f) : gL = L\}.$$

From now on we fix this choice of K_{SO}.

Specializing (4.4.8) to the form $f=f_m$, and using the formula for $a(0)=a_F(0)$ found in Proposition 4.3.3, we obtain

$$\begin{aligned}\frac{n}{\deg_{\mathbb{C}}(Y_{\mathrm{SO}})}\sum_{y\in Y_{\mathrm{SO}}(\mathbb{C})}\frac{\Theta^{\mathrm{reg}}_{\mathrm{SO}}(f_m)(y)}{|\mathrm{Aut}(y)|} &= -\frac{d}{ds}\langle E(s),\xi(f_m)\rangle_{\mathrm{Pet}}\big|_{s=0}\\ &\quad + \frac{a(m,0)}{\Lambda(0,\chi_E)} - 2c^+_{f_m}(0,0)\cdot\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)}.\end{aligned}\tag{4.4.9}$$

As in [**6**, §2.1], our group $G\subset \mathrm{GU}(W_0)\times\mathrm{GU}(W)$ acts on V in a natural way, defining a homomorphism $G\to\mathrm{SO}(V)$. On the other hand, Remark 4.1.1 shows that $T_{\mathrm{big}}\cong \mathrm{Res}_{\boldsymbol{k}/\mathbb{Q}}\mathbb{G}_m\times T_{\mathrm{SO}}$, and projection to the second factor defines a morphism $T_{\mathrm{big}}\to T_{\mathrm{SO}}$. We obtain morphisms of Shimura data

$$\begin{array}{ccc}(T_{\mathrm{big}},\{h_{\mathrm{big}}\}) & \longrightarrow & (T_{\mathrm{SO}},\{h_{\mathrm{SO}}\})\\ \big\downarrow & & \big\downarrow\\ (G,\mathcal{D}) & \longrightarrow & (\mathrm{SO}(V),\mathcal{D}_{\mathrm{SO}}),\end{array}$$

which induce morphisms of $\boldsymbol{k}$-stacks

$$\begin{array}{ccc} \mathcal{Y}_{\mathrm{big}/\boldsymbol{k}} & \longrightarrow & Y_{\mathrm{SO}/\boldsymbol{k}} \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathrm{Kra}/\boldsymbol{k}} & \longrightarrow & M_{\mathrm{SO}/\boldsymbol{k}}. \end{array}$$

The Green function $\Theta^{\mathrm{reg}}(f_m)$ on $\mathcal{S}_{\mathrm{Kra}/\boldsymbol{k}}$ defined in [**6**, §7.2] is simply the pullback of the Green function $\Theta^{\mathrm{reg}}_{\mathrm{SO}}(f_m)$ via the bottom horizontal arrow. It follows easily that

$$\frac{n}{\deg_{\mathbb{C}}(Y_{\mathrm{SO}})}\sum_{y\in Y_{\mathrm{SO}}(\mathbb{C})}\frac{\Theta^{\mathrm{reg}}_{\mathrm{SO}}(f_m)(y)}{|\mathrm{Aut}(y)|}=\frac{n}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})}\sum_{y\in \mathcal{Y}_{\mathrm{big}}(\mathbb{C})}\frac{\Theta^{\mathrm{reg}}(f_m)(y)}{|\mathrm{Aut}(y)|},$$

and comparison with (4.4.9) completes the proof of Proposition 4.4.6. □

Proof of Theorem 4.4.1. — Combining the decomposition (4.4.5) with Proposition 4.4.5 shows that

$$\sum_{\mathfrak{p}\subset\mathcal{O}_{\boldsymbol{k}}}\frac{n\log(\mathrm{N}(\mathfrak{p}))}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})}\sum_{y\in(\mathcal{Z}^{\mathrm{tot}}_{\mathrm{Kra}}(m)\cap\mathcal{Y}_{\mathrm{big}})(\mathbb{F}^{\mathrm{alg}}_{\mathfrak{p}})}\frac{\mathrm{length}(\mathcal{O}_y)}{|\mathrm{Aut}(y)|}=\sum_{\substack{\alpha\in F_+\\ \mathrm{Tr}_{F/\mathbb{Q}}(\alpha)=m}}\frac{-a_F(\alpha,0)}{\Lambda(0,\chi_E)}.$$

Plugging this formula and the archimedean calculation of Proposition 4.4.6 into (4.4.1) leaves

$$\frac{n\cdot[\widehat{\mathcal{Z}}(f_m):\mathcal{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})}=-2c^+_{f_m}(0,0)\cdot\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)}-\frac{d}{ds}\langle E(s),\xi(f_m)\rangle_{\mathrm{Pet}}\big|_{s=0},$$

as desired. □

4.5. The proof of Theorem B. — We now use Theorem 4.4.1 to prove a special case of Theorem D, and then prove Theorem B. We assume $n\geq 3$.

Recall the differential operator

$$\xi : H_{2-n}(\omega_L)\to S_n(\overline{\omega}_L)$$

of §2.4. Its kernel is the subspace

$$M^!_{2-n}(\omega_L)\subset H_{2-n}(\omega_L)$$

of weakly holomorphic forms.

Lemma 4.5.1. — *In the notation of §2.4, there exists a Δ-invariant form $f\in M^!_{2-n}(\omega_L)$ such that $c^+_f(0,0)\neq 0$, and*

$$\widehat{\mathcal{Z}}(f)+c^+_f(0,0)\cdot\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0)=0.$$

Proof. — Denote by

$$S^{!,\infty}_{2-n}(\Gamma_0(D),\chi^n_{\boldsymbol{k}})\subset M^!_{2-n}(\Gamma_0(D),\chi^n_{\boldsymbol{k}})$$

the subspace of forms that vanish at all cusps other than ∞, and choose any form

$$f_0(\tau) = \sum_{\substack{m\in\mathbb{Z} \\ m\gg-\infty}} c_0(m)\cdot q^m \in S^{!,\infty}_{2-n}(\Gamma_0(D),\chi^n_{\boldsymbol{k}})$$

such that $c_0(0) \neq 0$. The existence of such a form can be proved as in [**4**, Lemma 4.11]. As in (2.3.2) there is an induced form

$$f(\tau) = \sum_{\gamma\in\Gamma_0(D)\backslash \mathrm{SL}_2(\mathbb{Z})} (f_0|_{2-n}\gamma)(\tau)\cdot\omega_L(\gamma^{-1})\phi_0 \in M^!_{2-n}(\omega_L)^\Delta,$$

which we claim has the desired properties.

Indeed, the proof of Proposition 2.5.1 shows that $c^+_f(0,0) = c_0(0)$, and that $f = \sum_{m>0} c_0(-m) f_m$. In particular,

$$\widehat{\mathcal{Z}}(f) = \sum_{m>0} c_0(-m)\cdot\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) \in \widehat{\mathrm{Ch}}^1_{\mathbb{C}}(\mathcal{S}^*_{\mathrm{Kra}}).$$

Given any modular form

$$g(\tau) = \sum_{m\geq 0} d(m)\cdot q^m \in M_n(D,\chi^n_{\boldsymbol{k}}),$$

summing the residues of the meromorphic form $f_0(\tau)g(\tau)d\tau$ on $X_0(D)(\mathbb{C})$ shows that

$$\sum_{m\geq 0} c_0(-m)\cdot d(m) = 0.$$

Thus the modularity of the generating series (1.1.2) implies the second equality in

$$\widehat{\mathcal{Z}}(f) + c_0(0)\cdot\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) = \sum_{m\geq 0} c_0(-m)\cdot\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(m) = 0. \tag{4.5.1}$$ □

We can now prove Theorem D under some additional hypotheses. These hypotheses will be removed in §5.

Theorem 4.5.2. — *If the discriminants of $\boldsymbol{k}/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime, then*

$$[\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{big}}] = \frac{-2}{n}\cdot\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})\cdot\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)}.$$

Proof. — If we choose f as in Lemma 4.5.1 then $\xi(f) = 0$, and so Theorem 4.4.1 simplifies to

$$-nc^+_f(0,0)\cdot\frac{[\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) : \mathcal{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} + 2c^+_f(0,0)\cdot\frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)} = 0.$$

An application of Proposition 4.2.11 completes the proof. □

The following is Theorem B in the introduction.

Theorem 4.5.3. — *Assume that the discriminants of $\boldsymbol{k}/\mathbb{Q}$ and $F/\mathbb{Q}$ are odd and relatively prime, and let $g \in S_n(\Gamma_0(D), \chi^n)$ and $\tilde{g} \in S_n(\overline{\omega}_L)$ be related by* (2.3.2). *The central derivative of the Petersson inner product* (4.3.4) *is related to the arithmetic theta lift* (1.1.3) *by*

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{big}}] = \frac{-1}{n} \cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) \cdot \frac{d}{ds} \langle E(s), \tilde{g} \rangle_{\mathrm{Pet}}\big|_{s=0}.$$

Proof. — If we choose f as in Proposition 2.5.1, then $\xi(f) = \tilde{g}$ and

$$[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{big}}] = [\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathrm{big}}] + c_f^+(0,0) \cdot [\widehat{\mathcal{Z}}^{\mathrm{tot}}_{\mathrm{Kra}}(0) : \mathcal{Y}_{\mathrm{big}}].$$

Proposition 4.2.11 and Theorem 4.5.2 allow us to rewrite this as

$$\begin{aligned}[\widehat{\theta}(g) : \mathcal{Y}_{\mathrm{big}}] &= [\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathrm{big}}] - c_f^+(0,0) \cdot [\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{big}}] \\ &= [\widehat{\mathcal{Z}}(f) : \mathcal{Y}_{\mathrm{big}}] + \frac{2}{n} \cdot c_f^+(0,0) \cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) \cdot \frac{\Lambda'(0,\chi_E)}{\Lambda(0,\chi_E)},\end{aligned}$$

and comparison with Theorem 4.4.1 completes the proof. □

5. Faltings heights of CM abelian varieties

In §5 we assume $n \geq 2$, and study Theorems C and D of the introduction. As in §1.3, let F be a totally real field of degree n, set

$$E = \boldsymbol{k} \otimes_{\mathbb{Q}} F,$$

and let $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ be a CM type of signature $(n-1, 1)$. We fix a triple $(\mathfrak{a}_0, \mathfrak{a}, i_E)$ as in §4.2.

5.1. Some metrized line bundles.

— By virtue of the inclusion (1.1.1), there is a universal pair (A_0, A) over $\mathcal{S}_{\mathrm{Kra}}$ consisting of an elliptic curve $\pi_0 : A_0 \to \mathcal{S}_{\mathrm{Kra}}$ and an abelian scheme $\pi : A \to \mathcal{S}_{\mathrm{Kra}}$ of dimension n.

Endowing the Lie algebras of A_0 and A with their Faltings (a.k.a. Hodge) metrics gives rise to metrized line bundles

$$\mathrm{Lie}(A_0) \in \widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}), \qquad \det(\mathrm{Lie}(A)) \in \widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}).$$

A vector η in the fiber

$$\det(\mathrm{Lie}(A_s))^{-1} \cong \bigwedge^n \mathrm{Fil}^1 H^1_{\mathrm{dR}}(A_s) \subset \bigwedge^n H^1_{\mathrm{dR}}(A_s)$$

at a complex point $s \in \mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ has norm

(5.1.1) $$\|\eta\|_s^2 = \Big| \int_{A_s(\mathbb{C})} \eta \wedge \overline{\eta} \Big|.$$

The metric on $\mathrm{Lie}(A_0)$ is defined similarly.

We now recall some notation from [**6**, §1.8]. Fix a $\pi \in \mathcal{O}_{\boldsymbol{k}}$ such that $\mathcal{O}_{\boldsymbol{k}} = \mathbb{Z} + \mathbb{Z}\pi$. If S is any $\mathcal{O}_{\boldsymbol{k}}$-scheme, define

$$\begin{aligned} \varepsilon_S &= \pi \otimes 1 - 1 \otimes i_S(\overline{\pi}) \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S \\ \overline{\varepsilon}_S &= \overline{\pi} \otimes 1 - 1 \otimes i_S(\overline{\pi}) \in \mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S, \end{aligned} \tag{5.1.2}$$

where $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$ is the structure map. We usually just write ε and $\overline{\varepsilon}$, when the scheme S is clear from context.

Remark 5.1.1. — If N is an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_S$-module then $N/\overline{\varepsilon}N$ is the maximal quotient of N on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism $i_S : \mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_S$, and $N/\varepsilon N$ is the maximal quotient on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the conjugate of the structure morphism. If $D \in \mathcal{O}_S^\times$ then

$$N = \varepsilon N \oplus \overline{\varepsilon} N,$$

and the summands are the maximal submodules on which $\mathcal{O}_{\boldsymbol{k}}$ acts through the structure morphism and its conjugate, respectively.

As in [**6**, §2.2], the relative de Rham homology $H_1^{\mathrm{dR}}(A)$ is a rank $2n$ vector bundle on $\mathcal{S}_{\mathrm{Kra}}$ endowed with an action of $\mathcal{O}_{\boldsymbol{k}}$ induced from that on A. In fact, it is locally free of rank n as an $\mathcal{O}_{\boldsymbol{k}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{S}_{\mathrm{Kra}}}$-module, and

$$\mathcal{V} = H_1^{\mathrm{dR}}(A)/\overline{\varepsilon} H_1^{\mathrm{dR}}(A)$$

is a rank n vector bundle. We make $\det(\mathcal{V})$ into a metrized line bundle by declaring that a local section η of its inverse

$$\det(\mathcal{V})^{-1} \cong \bigwedge^n \varepsilon H^1_{\mathrm{dR}}(A) \subset H^n_{\mathrm{dR}}(A)$$

has norm (5.1.1) at a complex point $s \in \mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$.

As the exceptional divisor $\mathrm{Exc} \subset \mathcal{S}_{\mathrm{Kra}}$ of [**6**, §2.3] is supported in characteristics dividing D, the line bundle $\mathcal{O}(\mathrm{Exc})$ is canonically trivial in the generic fiber. We endow it with the trivial metric. That is to say, the constant function 1, viewed as a section of $\mathcal{O}(\mathrm{Exc})$, has norm $\|1\|^2 = 1$.

Recall that the line bundle $\boldsymbol{\omega}$ of [**6**, §2.4] was endowed with a metric in [**6**, §7.2], defining

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}).$$

For any positive real number c, denote by

$$\mathcal{O}\langle c\rangle \in \widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}})$$

the trivial bundle $\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}}}$ endowed with the constant metric $\|1\|^2 = c$.

Proposition 5.1.2. — *There is an isomorphism*

$$\mathcal{O}\langle 8\pi^2 e^\gamma D^{-1}\rangle^{\otimes 2} \otimes \widehat{\boldsymbol{\omega}}^{\otimes 2} \otimes \det(\mathrm{Lie}(A)) \otimes \mathrm{Lie}(A_0)^{\otimes 2} \cong \mathcal{O}(\mathrm{Exc}) \otimes \det(\mathcal{V})$$

of metrized line bundles on $\mathcal{S}_{\mathrm{Kra}}$.

Proof. — In [**6**, §2.4] we defined a line bundle $\mathbf{\Omega}_{\mathrm{Kra}}$ on $\mathcal{S}_{\mathrm{Kra}}$ by

$$\mathbf{\Omega}_{\mathrm{Kra}} = \det(\mathrm{Lie}(A))^{-1} \otimes \mathrm{Lie}(A_0)^{\otimes -2} \otimes \det(\mathcal{V}),$$

and in [**6**, Theorem 2.6.3] we constructed an isomorphism

$$\boldsymbol{\omega}^{\otimes 2} \cong \mathbf{\Omega}_{\mathrm{Kra}} \otimes \mathcal{O}(\mathrm{Exc}).$$

This defines the desired isomorphism

$$\boldsymbol{\omega}^{\otimes 2} \otimes \det(\mathrm{Lie}(A)) \otimes \mathrm{Lie}(A_0)^{\otimes 2} \cong \mathcal{O}(\mathrm{Exc}) \otimes \det(\mathcal{V}) \tag{5.1.3}$$

on underlying line bundles, and it remains to compare the metrics.

In the complex fiber this can be made more explicit. At any complex point $s \in \mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ the Hodge short exact sequence admits a canonical splitting

$$H_1^{\mathrm{dR}}(A_s) = F^0(A_s) \oplus \mathrm{Lie}(A_s),$$

where $F^0(A_s) = \mathrm{Fil}^0 H_1^{\mathrm{dR}}(A_s)$ is the nontrivial step in the Hodge filtration. When combined with the decomposition of Remark 5.1.1 we obtain

$$H_1^{\mathrm{dR}}(A_s) = \underbrace{\varepsilon F^0(A_s)}_{1} \oplus \underbrace{\overline{\varepsilon} F^0(A_s)}_{n-1} \oplus \underbrace{\varepsilon \mathrm{Lie}(A_s)}_{n-1} \oplus \underbrace{\overline{\varepsilon} \mathrm{Lie}(A_s)}_{1},$$

where the subscripts indicate the dimensions as $\mathbb{C}$-vector spaces. There is a similar decomposition

$$H_1^{\mathrm{dR}}(A_{0s}) = \underbrace{\varepsilon F^0(A_{0s})}_{0} \oplus \underbrace{\overline{\varepsilon} F^0(A_{0s})}_{1} \oplus \underbrace{\varepsilon \mathrm{Lie}(A_{0s})}_{1} \oplus \underbrace{\overline{\varepsilon} \mathrm{Lie}(A_{0s})}_{0}.$$

Denote by

$$\psi : H_1^{\mathrm{dR}}(A_s) \times H_1^{\mathrm{dR}}(A_s) \to \mathbb{C} \tag{5.1.4}$$

the alternating pairing determined by the principal polarization on A_s. The two direct summands

$$\varepsilon F^0(A_s) \oplus \overline{\varepsilon} \mathrm{Lie}(A_s) \subset H_1^{\mathrm{dR}}(A_s)$$

are interchanged by complex conjugation. We endow both $\varepsilon F^0(A_s)$ and $\overline{\varepsilon}\mathrm{Lie}(A_s)$ with the metric

$$\|b\|_s^2 = \left| \frac{\psi(b, \overline{b})}{2\pi i} \right|, \tag{5.1.5}$$

so that the pairing

$$\psi : \varepsilon F^0(A_s) \otimes \overline{\varepsilon} \mathrm{Lie}(A_s) \to \mathcal{O}\langle 4\pi^2 \rangle_s^{-1} \tag{5.1.6}$$

is an isometry.

For $a, b \in \overline{\varepsilon}\mathrm{Lie}(A_s)$, define $p_{a\otimes b} : \varepsilon F^0(A_s) \to \overline{\varepsilon}\mathrm{Lie}(A_s)$ by

$$p_{a\otimes b}(e) = \psi(\overline{\varepsilon} a, e) \cdot \overline{\varepsilon} b = -D\psi(a, e) \cdot b. \tag{5.1.7}$$

The factor of $-D$ comes from the observation that $\overline{\varepsilon}$ acts on $\overline{\varepsilon}\mathrm{Lie}(A_s)$ as $\pm\sqrt{-D}$, where the sign depends on the choice of π used in (5.1.2).

We now define $P_{a\otimes b}$ by the commutativity of

$$\begin{array}{ccc} \det(\mathcal{V}_s) & \xrightarrow{P_{a\otimes b}} & \det(\mathrm{Lie}(A_s)) \\ \cong\downarrow & & \downarrow\cong \\ \varepsilon F^0(A_s)\otimes\det(\varepsilon\mathrm{Lie}(A_s)) & \xrightarrow[p_{a\otimes b}\otimes\mathrm{id}]{} & \overline{\varepsilon}\mathrm{Lie}(A_s)\otimes\det(\varepsilon\mathrm{Lie}(A_s)). \end{array} \tag{5.1.8}$$

This defines the isomorphism

$$(\overline{\varepsilon}\mathrm{Lie}(A_s))^{\otimes 2} \xrightarrow{P} \mathrm{Hom}\big(\det(\mathcal{V}_s), \det(\mathrm{Lie}(A_s))\big) \tag{5.1.9}$$

of [**6**, Lemma 2.4.5].

Lemma 5.1.3. — *The isomorphism* (5.1.9) *defines an isometry*

$$\det(\mathcal{V}_s) \cong \mathcal{O}\langle 2\pi D^{-1}\rangle_s^{\otimes 2} \otimes (\varepsilon F^0(A_s))^{\otimes 2} \otimes \det(\mathrm{Lie}(A_s)).$$

Proof. — Fix an isomorphism $\bigwedge^{2n} H_1(A_s(\mathbb{C}),\mathbb{Z}) \cong \mathbb{Z}$ and extend it to a $\mathbb{C}$-linear isomorphism

$$\mathrm{vol} : \bigwedge^{2n} H_1^{\mathrm{dR}}(A_s) \cong \mathbb{C}.$$

Under the de Rham comparison isomorphism $H_1(A_s(\mathbb{C}),\mathbb{C}) \cong H_1^{\mathrm{dR}}(A_s)$, the pairing (5.1.4) restricts to a perfect pairing

$$\psi : H_1(A_s(\mathbb{C}),\mathbb{Z}) \times H_1(A_s(\mathbb{C}),\mathbb{Z}) \to 2\pi i\mathbb{Z}.$$

It follows that there is a unique element $\Psi = \alpha\wedge\beta \in \bigwedge^2 H_1(A_s(\mathbb{C}),\mathbb{Z})$ such that

$$2\pi i \cdot \psi(a,b) = \psi(\alpha,a)\psi(\beta,b) - \psi(\alpha,b)\psi(\beta,a)$$

for all $a, b \in H_1(A_s(\mathbb{C}),\mathbb{Z})$. The map

$$\Big(\bigwedge^{n-1} H_1(A_s(\mathbb{C}),\mathbb{Z})\Big) \otimes \Big(\bigwedge^{n-1} H_1(A_s(\mathbb{C}),\mathbb{Z})\Big) \to \mathbb{Z}$$

defined by $a\otimes b \mapsto \mathrm{vol}(\Psi\wedge a\wedge b)$ is a perfect pairing of $\mathbb{Z}$-modules.

We now metrize the line

$$\det(\varepsilon\mathrm{Lie}(A_s)) \subset \bigwedge^{n-1} \varepsilon H_1^{\mathrm{dR}}(A_s)$$

by $\|\mu\|^2 = |\mathrm{vol}(\Psi\wedge\mu\wedge\overline{\mu})|$. With this definition, the vertical arrows in (5.1.8) are isometries.

Using (5.1.6) and (5.1.7), one sees that the map

$$p_{a\otimes b} \in \mathrm{Hom}(F^0(A_s), \overline{\varepsilon}\mathrm{Lie}(A_s))$$

satisfies $\|p_{a\otimes b}\| = 2\pi D\cdot\|a\otimes b\|$, and hence also $\|P_{a\otimes b}\| = 2\pi D\cdot\|a\otimes b\|$. This proves that the isomorphism P defines an isometry

$$\mathcal{O}\langle 2\pi D\rangle_s^{\otimes 2} \otimes (\overline{\varepsilon}\mathrm{Lie}(A_s))^{\otimes 2} \cong \mathrm{Hom}\big(\det(\mathcal{V}_s), \det(\mathrm{Lie}(A_s))\big).$$

The isomorphism (5.1.6) allows us to rewrite this as

$$\det(\mathcal{V}_s) \cong \mathcal{O}\langle 2\pi D^{-1}\rangle_s^{\otimes 2} \otimes (\varepsilon F^0(A_s))^{\otimes 2} \otimes \det(\mathrm{Lie}(A_s)). \qquad \square$$

The proof of [6, Proposition 2.4.2] gives an isomorphism

$$\boldsymbol{\omega}_s \cong \mathrm{Hom}(\mathrm{Lie}(A_{0s}), \varepsilon F^0(A_s)) \subset \varepsilon V_{\mathbb{C}}, \tag{5.1.10}$$

where

$$V = \mathrm{Hom}_{\boldsymbol{k}}\big(H_1(A_{0s}(\mathbb{C}), \mathbb{Q}), H_1(A_s(\mathbb{C}), \mathbb{Q})\big).$$

As in [6, §2.1], there is a $\mathbb{Q}$-bilinear form $[.,.] : V \times V \to \mathbb{Q}$ induced by the polarizations on A_{0s} and A_s. If we extend this to a $\mathbb{C}$-bilinear form on

$$V_{\mathbb{C}} = \mathrm{Hom}_{\boldsymbol{k}\otimes\mathbb{C}}\big(H_1^{\mathrm{dR}}(A_{0s}), H_1^{\mathrm{dR}}(A_s)\big)$$

then the metric on $\boldsymbol{\omega}_s$ is defined, as in [6, §7.2], by

$$\|x\|^2 = \frac{|[x, \overline{x}]|}{4\pi e^{\gamma}}$$

for any $x \in \mathrm{Hom}(\mathrm{Lie}(A_{0s}), \varepsilon F^0(A_s))$.

On the other hand, we have defined the Faltings metric on $\mathrm{Lie}(A_{0s})$, and defined a metric on $\varepsilon F^0(A_s)$ by (5.1.5). The following lemma shows that (5.1.10) respects the metrics, up to scaling by a factor of $4\pi e^{\gamma}$.

Lemma 5.1.4. — *The isomorphism* (5.1.10) *defines an isometry*

$$\mathcal{O}\langle 4\pi e^{\gamma}\rangle_s \otimes \widehat{\boldsymbol{\omega}}_s \cong \mathrm{Hom}(\mathrm{Lie}(A_{0s}), \varepsilon F^0(A_s)).$$

Proof. — The alternating form

$$\psi_0 : H_1^{\mathrm{dR}}(A_{0s}) \times H_1^{\mathrm{dR}}(A_{0s}) \to \mathbb{C}$$

analogous to (5.1.4) restricts to a perfect pairing

$$\psi_0 : H_1(A_{0s}(\mathbb{C}), \mathbb{Z}) \times H_1(A_{0s}(\mathbb{C}), \mathbb{Z}) \to 2\pi i\mathbb{Z},$$

and hence the Faltings metric on $\mathrm{Lie}(A_{0s}) = \varepsilon H_1^{\mathrm{dR}}(A_{0s})$ is

$$\|a\|^2 = (2\pi)^{-1}|\psi_0(a, \overline{a})|.$$

From the definition of the bilinear form on V, one can show that

$$[x, \overline{x}] \cdot \psi_0(a, \overline{a}) = \psi(xa, \overline{xa})$$

for all $x \in \varepsilon V_{\mathbb{C}}$. Comparing with the metric on $\varepsilon F^0(A_s)$ shows that

$$4\pi e^{\gamma} \cdot \|x\|^2 \cdot \|a\|^2 = (2\pi)^{-1} \cdot |\psi(xa, \overline{xa})| = \|xa\|^2,$$

for all $x \in \boldsymbol{\omega}_s$ and $a \in \mathrm{Lie}(A_{0s})$, as claimed. □

The two lemmas provide us with isometries

$$\begin{aligned}\det(\mathcal{V}_s) &\cong \mathcal{O}\langle 2\pi D^{-1}\rangle_s^{\otimes 2} \otimes (\varepsilon F^0(A_s))^{\otimes 2} \otimes \det(\mathrm{Lie}(A_s)) \\ &\cong \mathcal{O}\langle 8\pi^2 e^{\gamma} D^{-1}\rangle_s^{\otimes 2} \otimes \widehat{\boldsymbol{\omega}}_s^{\otimes 2} \otimes \mathrm{Lie}(A_{0s})^{\otimes 2} \det(\mathrm{Lie}(A_s))\end{aligned}$$

and the composition agrees with the isomorphism (5.1.3). This completes the proof of Proposition 5.1.2. □

Recall the big CM cycle $\pi : \mathcal{Y}_{\mathrm{big}} \to \mathcal{S}^*_{\mathrm{Kra}}$ of Definition 4.2.6. All of the metrized line bundles on $\mathcal{S}_{\mathrm{Kra}}$ appearing in Proposition 5.1.2 can be extended to the toroidal compactification $\mathcal{S}^*_{\mathrm{Kra}}$ (with possible log-singularities along the boundary) so as to define classes in the codimension one arithmetic Chow group. However, we don't actually need this. Indeed, we can define a homomorphism

$$[- : \mathcal{Y}_{\mathrm{big}}] : \widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}) \to \mathbb{R}$$

as the composition

$$\widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}) \xrightarrow{\pi^*} \widehat{\mathrm{Pic}}(\mathcal{Y}_{\mathrm{big}}) \cong \widehat{\mathrm{Ch}}^1(\mathcal{Y}_{\mathrm{big}}) \xrightarrow{\widehat{\deg}} \mathbb{R}.$$

As the big CM cycle does not meet the boundary of the toroidal compactification, the composition

$$\widehat{\mathrm{Ch}}^1(\mathcal{S}^*_{\mathrm{Kra}}) \cong \widehat{\mathrm{Pic}}(\mathcal{S}^*_{\mathrm{Kra}}) \to \widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}) \xrightarrow{[-:\mathcal{Y}_{\mathrm{big}}]} \mathbb{R}$$

agrees with the arithmetic degree along $\mathcal{Y}_{\mathrm{big}}$ of Definition 4.2.6.

Remark 5.1.5. — Directly from the definitions, and recalling Remark 2.2.7, the metrized line bundle $\mathcal{O}\langle c\rangle$ satisfies

$$[\mathcal{O}\langle c\rangle : \mathcal{Y}_{\mathrm{big}}] = \sum_{y\in\mathcal{Y}_{\mathrm{big}}(\mathbb{C})} -\log\|1\|^2 = -\log(c)\cdot \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}).$$

5.2. The Faltings height. — Recall from §4.2 the moduli stack $\mathcal{CM}_\Phi$ of abelian varieties over $\mathcal{O}_\Phi$-schemes with complex multiplication by $\mathcal{O}_E$ and CM type Φ.

Suppose $A \in \mathcal{CM}_\Phi(\mathbb{C})$. Choose a model of A over a number field $L\subset\mathbb{C}$ large enough that the Néron model $\pi : \mathcal{A} \to \mathrm{Spec}(\mathcal{O}_L)$ has everywhere good reduction. Pick a nonzero rational section s of the line bundle $\pi_*\Omega^{\dim(A)}_{\mathcal{A}/\mathcal{O}_L}$ on $\mathrm{Spec}(\mathcal{O}_L)$, and define

$$h^{\mathrm{Falt}}_\infty(A,s) = \frac{-1}{2[L:\mathbb{Q}]}\sum_{\sigma:L\to\mathbb{C}} \log\Big|\int_{\mathcal{A}^\sigma(\mathbb{C})} s^\sigma\wedge\overline{s^\sigma}\Big|,$$

and

$$h^{\mathrm{Falt}}_f(A,s) = \frac{1}{[L:\mathbb{Q}]}\sum_{\mathfrak{p}\subset\mathcal{O}_L} \mathrm{ord}_{\mathfrak{p}}(s)\cdot\log \mathrm{N}(\mathfrak{p}).$$

By a result of Colmez **[12]**, the *Faltings height*

$$h^{\mathrm{Falt}}_{(E,\Phi)} = h^{\mathrm{Falt}}_f(A,s) + h^{\mathrm{Falt}}_\infty(A,s)$$

depends only on the pair (E,Φ).

Proposition 5.2.1. — *The arithmetic degree of* $\mathrm{Lie}(A)$ *along* $\mathcal{Y}_{\mathrm{big}}$ *satisfies*

$$[\det(\mathrm{Lie}(A)) : \mathcal{Y}_{\mathrm{big}}] = -2\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})\cdot h^{\mathrm{Falt}}_{(E,\Phi)}.$$

Similarly, recalling the Faltings height $h^{\mathrm{Falt}}_{\boldsymbol{k}}$ *of* (1.4.1),

$$[\mathrm{Lie}(A_0) : \mathcal{Y}_{\mathrm{big}}] = -2\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})\cdot h^{\mathrm{Falt}}_{\boldsymbol{k}}.$$

Proof. — Suppose we are given a morphism $y : \mathrm{Spec}(\mathcal{O}_L) \to \mathcal{Y}_{\mathrm{big}}$ for some finite extension L/E_Φ. The restriction of A to $\mathcal{O}_L$ has complex multiplication by $\mathcal{O}_E$ and CM type Φ, and comparing the definition of the Faltings height with the definition of $\widehat{\deg}$ found in [19, §3.1], shows that the composition

$$\widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}) \xrightarrow{\pi^*} \widehat{\mathrm{Ch}}^1(\mathcal{Y}_{\mathrm{big}}) \xrightarrow{y^*} \widehat{\mathrm{Ch}}^1(\mathrm{Spec}(\mathcal{O}_L)) \xrightarrow{\widehat{\deg}} \mathbb{R}$$

sends $\mathrm{Lie}(A)^{-1}$ to $[L:\mathbb{Q}] \cdot h^{\mathrm{Falt}}_{(E,\Phi)}$.

We may choose L in such a way that the $\mathcal{O}_{\boldsymbol{k}}$-stack

$$\mathcal{Y}_{\mathrm{big}} \times_{\mathrm{Spec}(\mathcal{O}_\Phi)} \mathrm{Spec}(\mathcal{O}_L)$$

admits a finite étale cover by a disjoint union $Y_{\mathrm{big}} = \bigsqcup \mathrm{Spec}(\mathcal{O}_L)$ of, say, m copies of $\mathrm{Spec}(\mathcal{O}_L)$, and then

$$\frac{[\mathrm{Lie}(A) : \mathcal{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} = \frac{[\mathrm{Lie}(A) : Y_{\mathrm{big}}]}{\deg_{\mathbb{C}}(Y_{\mathrm{big}})} = -\frac{m[L:\mathbb{Q}] \cdot h^{\mathrm{Falt}}_{(E,\Phi)}}{m[L:\boldsymbol{k}]} = -2 \cdot h^{\mathrm{Falt}}_{(E,\Phi)}.$$

This proves the first equality, and the proof of the second is similar. □

5.3. Gross's trick. — The goal of §5.3 is to compute the degree of the metrized line bundle $\det(\mathcal{V})$ along the big CM cycle. The impatient reader may skip directly to Proposition 5.3.6 for the answer. However, the strategy of the calculation is simple enough that we can explain it in a few sentences.

It is an observation of Gross [14] that the metrized line bundle $\det(\mathcal{V})$ behaves, for all practical purposes, like the trivial bundle $\mathcal{O}_{\mathcal{S}_{\mathrm{Kra}}}$ endowed with the constant metric $\|1\|^2 = \exp(-c)$ for a certain period c. This is made more precise in Theorem 5.3.1 and Corollary 5.3.2 below. A priori, the constant c is something mysterious, but one can evaluate it by computing the degree of $\det(\mathcal{V})$ along *any* codimension $n-1$ cycle that one chooses. We choose a cycle along which the universal abelian scheme $A \to \mathcal{S}_{\mathrm{Kra}}$ degenerates to a product of CM elliptic curves. Using this, one can express the value of c in terms of the Faltings height $h^{\mathrm{Falt}}_{\boldsymbol{k}}$ appearing in (1.4.1). The degree of $\det(\mathcal{V})$ along $\mathcal{Y}_{\mathrm{big}}$ is readily computed from this.

To carry out this procedure, the first step is to construct a cover of $\mathcal{S}_{\mathrm{Kra}}(\mathbb{C})$ over which the line bundle $\det(\mathcal{V})$ can be trivialized analytically. Fix a positive integer m, let $K(m) \subset K$ be the compact open subgroup of [6, Remark 2.2.3], and consider the finite étale cover

$$\begin{array}{ccc} \mathrm{Sh}_{K(m)}(G,\mathcal{D})(\mathbb{C}) & = & G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K(m) \\ \downarrow & & \downarrow \\ \mathrm{Sh}(G,\mathcal{D})(\mathbb{C}) & = & G(\mathbb{Q})\backslash \mathcal{D} \times G(\mathbb{A}_f)/K. \end{array}$$

This cover has a moduli interpretation, exactly as with $\mathcal{S}_{\mathrm{Kra}}$ itself, but with additional level m structure. This allows us to construct a regular integral model $\mathcal{S}_{\mathrm{Kra}}(m)$ over $\mathcal{O}_{\boldsymbol{k}}[1/m]$ of $\mathrm{Sh}_{K(m)}(G,\mathcal{D})$, along with a finite étale morphism

$$\mathcal{S}_{\mathrm{Kra}}(m) \to \mathcal{S}_{\mathrm{Kra}/\mathcal{O}_{\boldsymbol{k}}[1/m]}.$$

We use the notation $\det(\mathcal{V})$ for both the metrized line bundle on $\mathcal{S}_{\mathrm{Kra}}$, and for its pullback to $\mathcal{S}_{\mathrm{Kra}}(m)$.

The following results extends a theorem of Gross [**14**, Theorem 1] to integral models.

Theorem 5.3.1. — *Suppose $m \geq 3$, let $\mathbb{Z}^{\mathrm{alg}} \subset \mathbb{C}$ be the subring of all algebraic integers, and fix a connected component*

$$\mathcal{C} \subset \mathcal{S}_{\mathrm{Kra}}(m)_{/\mathbb{Z}^{\mathrm{alg}}[1/m]}.$$

The line bundle $\det(\mathcal{V})$ *admits a nowhere vanishing section*

$$\eta \in H^0(\mathcal{C}, \det(\mathcal{V})).$$

Such a section is unique up to scaling by $\mathbb{Z}^{\mathrm{alg}}[1/m]^\times$, and its norm $\|\eta\|^2$ is constant on $\mathcal{C}(\mathbb{C})$.

Proof. — For some $g \in G(\mathbb{A}_f)$ we have a complex uniformization

$$\Gamma\backslash\mathcal{D} \xrightarrow{z\mapsto(z,g)} \mathcal{C}(\mathbb{C}) \subset \mathrm{Sh}_{K(m)}(G,\mathcal{D})(\mathbb{C}),$$

where $\Gamma = G(\mathbb{Q}) \cap gK(m)g^{-1}$, and under this uniformization the total space of the vector bundle $\det(\mathcal{V})$ is isomorphic to $\Gamma\backslash(\mathcal{D}\times\mathbb{C})$, where the action of Γ on $\mathbb{C}$ is via the composition

$$\Gamma \subset G(\mathbb{Q}) \to \mathrm{GL}(W) \xrightarrow{\det} \boldsymbol{k}^\times \subset \mathbb{C}^\times.$$

The compact open subgroup $K(m)$ is constructed in such a way that there is a $\mathcal{O}_{\boldsymbol{k}}$-lattice $g\mathfrak{a} \subset W(\boldsymbol{k})$ stabilized by Γ, and such that Γ acts trivially on $g\mathfrak{a}/mg\mathfrak{a}$. This implies that the above composition actually takes values in the subgroup

$$\{\zeta \in \mathcal{O}_{\boldsymbol{k}}^\times : \zeta \equiv 1 \pmod{m\mathcal{O}_{\boldsymbol{k}}}\},$$

which is trivial by our assumption that $m \geq 3$. In other words, the vector bundle $\det(\mathcal{V})$ becomes (non-canonically) trivial after restriction to $\mathcal{X}(\mathbb{C})$. In fact, the argument of [**14**, Theorem 1] shows that one can find a trivializing section η that is algebraic and defined over $\mathbb{Q}^{\mathrm{alg}} \subset \mathbb{C}$, and that such a section is unique up to scaling by $(\mathbb{Q}^{\mathrm{alg}})^\times$ and has constant norm $\|\eta\|^2$.

All that remains to show is that η may be chosen so that it extends to a nowhere vanishing section over $\mathbb{Z}^{\mathrm{alg}}[1/m]$. The key is to recall from [**6**, §2.3] that $\mathrm{Sh}(G,\mathcal{D})$ has a second integral model $\mathcal{S}_{\mathrm{Pap}}$ over $\mathcal{O}_{\boldsymbol{k}}$, which is normal with geometrically normal fibers. It is related to the first by a surjective morphism $\mathcal{S}_{\mathrm{Kra}} \to \mathcal{S}_{\mathrm{Pap}}$, which restricts to an isomorphism over $\mathcal{O}_{\boldsymbol{k}}[1/D]$. It has a moduli interpretation very similar to that of $\mathcal{S}_{\mathrm{Kra}}$, which allows us to do two things. First, there is a canonical descent of the vector bundle $\mathcal{V}$ to $\mathcal{S}_{\mathrm{Pap}}$, defined again by $\mathcal{V} = H_1^{\mathrm{dR}}(A)/\overline{\varepsilon}H_1^{\mathrm{dR}}(A)$, but where now (A_0, A) is the universal pair over $\mathcal{S}_{\mathrm{Pap}}$. Second, we can add level $K(m)$ structure to obtain a cartesian diagram

$$\begin{array}{ccc} \mathcal{S}_{\mathrm{Kra}}(m) & \longrightarrow & \mathcal{S}_{\mathrm{Kra}/\mathcal{O}_{\boldsymbol{k}}[1/m]} \\ \downarrow & & \downarrow \\ \mathcal{S}_{\mathrm{Pap}}(m) & \longrightarrow & \mathcal{S}_{\mathrm{Pap}/\mathcal{O}_{\boldsymbol{k}}[1/m]} \end{array}$$

of $\mathcal{O}_{\boldsymbol{k}}[1/m]$-stacks with étale horizontal arrows.

In particular, $\mathcal{S}_{\mathrm{Pap}}(m)$ is normal with geometrically normal fibers, from which it follows that the above diagram extends to

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{S}_{\mathrm{Kra}}(m)_{/\mathbb{Z}^{\mathrm{alg}}[1/m]} & \longrightarrow & \mathcal{S}_{\mathrm{Kra}/\mathbb{Z}^{\mathrm{alg}}[1/m]} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{S}_{\mathrm{Pap}}(m)_{/\mathbb{Z}^{\mathrm{alg}}[1/m]} & \longrightarrow & \mathcal{S}_{\mathrm{Pap}/\mathbb{Z}^{\mathrm{alg}}[1/m]} \end{array}$$

for some connected component $\mathcal{B} \subset \mathcal{S}_{\mathrm{Pap}}(m)_{/\mathbb{Z}^{\mathrm{alg}}[1/m]}$ with irreducible fibers.

Now fix a number field $L \subset \mathbb{C}$ containing $\boldsymbol{k}$ large enough that the section η and the components $\mathcal{C}$ and $\mathcal{B}$ are defined over $\mathcal{O}_L[1/m]$. Viewing η as a rational section of the line bundle $\det(\mathcal{V})$ on $\mathcal{B}$, its divisor is a finite sum of vertical fibers of $\mathcal{B}$, and so there is a fractional $\mathcal{O}_L[1/m]$-ideal $\mathfrak{b} \subset L$ such that

$$\mathrm{div}(\eta) = \sum_{\mathfrak{q} \mid \mathfrak{b}} \mathrm{ord}_{\mathfrak{q}}(\mathfrak{b}) \cdot \mathcal{B}_{\mathfrak{q}},$$

where $\mathcal{B}_{\mathfrak{q}}$ is the mod $\mathfrak{q}$ fiber of $\mathcal{Y}$. By enlarging L we may assume that $\mathfrak{b}$ is principal, and hence η can be rescaled by an element of $L^{\times}$ to have trivial divisor on $\mathcal{B}$. But then η also has trivial divisor on $\mathcal{C}$, as desired. □

Corollary 5.3.2. — *Let $\mathcal{A} \subset \mathcal{S}_{\mathrm{Kra}}$ be a connected component. There is a constant $c = c_{\mathcal{A}} \in \mathbb{R}$ with the following property: for any finite extension $L/\boldsymbol{k}$ and any morphism $\mathrm{Spec}(\mathcal{O}_L) \to \mathcal{A}$, the image of $\det(\mathcal{V})$ under*

$$\widehat{\mathrm{Pic}}(\mathcal{S}_{\mathrm{Kra}}) \to \widehat{\mathrm{Pic}}(\mathcal{A}) \to \widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathcal{O}_L)) \xrightarrow{\widehat{\deg}} \mathbb{R} \tag{5.3.1}$$

is equal to $c \cdot [L : \boldsymbol{k}]$.

Proof. — Fix an integer $m \geq 3$. The open and closed substack

$$\mathcal{A}(m) = \mathcal{A} \times_{\mathcal{S}_{\mathrm{Kra}}} \mathcal{S}_{\mathrm{Kra}}(m)$$

of $\mathcal{S}_{\mathrm{Kra}}(m)$, may be disconnected, so we fix one of its connected components $\mathcal{A}(m)^{\circ} \subset \mathcal{A}(m)$. This is an $\mathcal{O}_{\boldsymbol{k}}[1/m]$-stack, which may become disconnected after base change to $\mathbb{Z}^{\mathrm{alg}}[1/m]$. Fix one connected component

$$\mathcal{C} \subset \mathcal{A}(m)^{\circ}_{/\mathbb{Z}^{\mathrm{alg}}[1/m]}$$

and let $\eta \in H^0(\mathcal{C}, \det(\mathcal{V}))$ be a trivializing section as in Theorem 5.3.1.

Choose a finite Galois extension $M/\boldsymbol{k}$ contained in $\mathbb{C}$, large enough that $\mathcal{C}$ and η are defined over $\mathcal{O}_M[1/m]$. For each $\sigma \in \mathrm{Gal}(M/\boldsymbol{k})$ we obtain a trivializing section

$$\eta^\sigma \in H^0(\mathcal{C}^\sigma, \det(\mathcal{V})),$$

which, by Theorem 5.3.1, has constant norm $\|\eta^\sigma\|$.

Let $\mathbb{R}(m)$ be the quotient of $\mathbb{R}$ by the $\mathbb{Q}$-span of $\{\log(p) : p \mid m\}$, and define

$$c(m) = \frac{-1}{[M:\boldsymbol{k}]} \sum_{\sigma \in \mathrm{Gal}(M/\boldsymbol{k})} \log \|\eta^\sigma\|^2 \in \mathbb{R}(m).$$

This is independent of the choice of M, and also independent of η by the uniqueness claim of Theorem 5.3.1. Moreover, for any number field $L/\boldsymbol{k}$ and any morphism

$$\mathrm{Spec}(\mathcal{O}_L[1/m]) \to \mathcal{A}(m)^\circ,$$

the image of $\det(\mathcal{V})$ under

$$\widehat{\mathrm{Pic}}(\mathcal{A}(m)^\circ) \to \widehat{\mathrm{Pic}}(\mathrm{Spec}(\mathcal{O}_L[1/m])) \xrightarrow{\widehat{\deg}} \mathbb{R}(m)$$

is equal to $c(m) \cdot [L:\boldsymbol{k}]$.

Now suppose we are given some $\mathrm{Spec}(\mathcal{O}_L) \to \mathcal{A}$ as in the statement of the corollary. After possible enlarging L, this morphism admits a lift

$$\begin{array}{ccc} & & \mathcal{A}(m)^\circ \\ & \nearrow & \downarrow \\ \mathrm{Spec}(\mathcal{O}_L[1/m]) & \longrightarrow & \mathcal{A}_{/\mathcal{O}_{\boldsymbol{k}}[1/m]}, \end{array}$$

and from this it is easy to see that the image of $\det(\mathcal{V})$ under the composition of (5.3.1) with $\mathbb{R} \to \mathbb{R}(m)$ is equal to $c(m) \cdot [L:\boldsymbol{k}]$.

In particular, the image of $\det(\mathcal{V})$ under the composition of (5.3.1) with the diagonal embedding

$$\mathbb{R} \hookrightarrow \prod_{m \geq 3} \mathbb{R}(m)$$

is equal to the tuple of constants $c(m) \cdot [L:\mathbb{Q}]$. What this proves is that there is a unique $c \in \mathbb{R}$ whose image under the diagonal embedding is the tuple of constants $c(m)$, and that this is the c we seek. □

Proposition 5.3.3. — *The constant $c = c_{\mathcal{A}}$ of Corollary 5.3.2 is independent of $\mathcal{A}$, and is equal to*

$$c = (4-2n) h^{\mathrm{Falt}}_{\boldsymbol{k}} + \log(4\pi^2 D),$$

where $h^{\mathrm{Falt}}_{\boldsymbol{k}}$ is the Faltings height (1.4.1).

Proof. — Recall that we have fixed a triple $(\mathfrak{a}_0, \mathfrak{a}, i_E)$ as in §4.2. Fix a $g \in G(\mathbb{A}_f)$ in such a way that the map

$$\mathcal{D} \xrightarrow{z \mapsto (z,g)} \mathrm{Sh}(G, \mathcal{D})(\mathbb{C})$$

factors through $\mathcal{A}(\mathbb{C})$, and a decomposition of $\mathcal{O}_{\boldsymbol{k}}$-modules

$$g\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n,$$

in which each $\mathfrak{a}_i$ is projective of rank 1. Define elliptic curves over the complex numbers by

$$A_i(\mathbb{C}) = g\mathfrak{a}_i \backslash \mathfrak{a}_{i\mathbb{C}} / \overline{\varepsilon}\mathfrak{a}_{i\mathbb{C}}$$

for $0 \leq i < n$, and

$$A_n(\mathbb{C}) = g\mathfrak{a}_n \backslash \mathfrak{a}_{n\mathbb{C}} / \varepsilon\mathfrak{a}_{n\mathbb{C}}.$$

Endow the abelian variety $A = A_1 \times \cdots \times A_n$ with the diagonal action of $\mathcal{O}_{\boldsymbol{k}}$, and the principal polarization induced by the perfect symplectic form on $g\mathfrak{a}$, as in the proof of [**6**, Proposition 2.2.1]. The pair (A_0, A) then corresponds to a point $(z, g) \in \mathcal{A}(\mathbb{C})$.

As each A_i has complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$, we may choose a number field L containing $\boldsymbol{k}$ over which all of these elliptic curves are defined and have everywhere good reduction. If we denote again by $A_0, \ldots, A_n$ and A the Néron models over $\operatorname{Spec}(\mathcal{O}_L)$, the pair (A_0, A) determines a morphism

$$\operatorname{Spec}(\mathcal{O}_L) \to \mathcal{A} \subset \mathcal{S}_{\mathrm{Kra}}.$$

The pullback of $\mathcal{V}$ to $\operatorname{Spec}(\mathcal{O}_L)$ is the rank n vector bundle

$$\mathcal{V}|_{\operatorname{Spec}(\mathcal{O}_L)} \cong \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_n,$$

where $\mathcal{V}_i = H_1^{\mathrm{dR}}(A_i)/\overline{\varepsilon} H_1^{\mathrm{dR}}(A_i)$. We endow $\mathcal{V}_i^{-1} \cong \varepsilon H^1_{\mathrm{dR}}(A_i)$ with the metric (5.1.1), so that

$$\det(\mathcal{V})|_{\operatorname{Spec}(\mathcal{O}_L)} \cong \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_n$$

is an isomorphism of metrized line bundles.

The following two lemmas relate the images of $\mathcal{V}_1, \ldots, \mathcal{V}_n$ under the arithmetic degree

$$\widehat{\operatorname{Pic}}(\operatorname{Spec}(\mathcal{O}_L)) \xrightarrow{\widehat{\deg}} \mathbb{R} \tag{5.3.2}$$

to the Faltings height $h_{\boldsymbol{k}}^{\mathrm{Falt}}$.

Lemma 5.3.4. — *For* $1 \leq i < n$, *the arithmetic degree* (5.3.1180equation.5.3.12 *sends*

$$\mathcal{V}_i \mapsto -[L : \mathbb{Q}] \cdot h_{\boldsymbol{k}}^{\mathrm{Falt}}.$$

Proof. — The action of $\mathcal{O}_{\boldsymbol{k}}$ on $\operatorname{Lie}(A_i)$ is through the inclusion $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_L$, and hence, as in [**6**, Remark 2.3.5], the quotient map

$$H_1^{\mathrm{dR}}(A_i) \to \operatorname{Lie}(A_i)$$

descends to an isomorphism of line bundles $\mathcal{V}_i \cong \operatorname{Lie}(A_i)$. If we endow $\operatorname{Lie}(A_i)^{-1}$ with the Faltings metric (5.1.1) then this isomorphism respects the metrics, and the claim follows as in the proof of Proposition 5.2.1. □

Lemma 5.3.5. — *The arithmetic degree* (5.3.1180equation.5.3.12 *sends*

$$\mathcal{V}_n \mapsto [L : \mathbb{Q}] \cdot \big(h_{\boldsymbol{k}}^{\mathrm{Falt}} - \frac{1}{2}\log(4\pi^2 D)\big).$$

Proof. — The action of $\mathcal{O}_{\boldsymbol{k}}$ on $\mathrm{Lie}(A_i)$ is through the complex conjugate of the inclusion $\mathcal{O}_{\boldsymbol{k}} \to \mathcal{O}_L$, from which it follows that the Hodge short exact sequence takes the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^0(A_n) & \longrightarrow & H_1^{\mathrm{dR}}(A_n) & \longrightarrow & \mathrm{Lie}(A_n) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \varepsilon H_1^{\mathrm{dR}}(A_0) & \longrightarrow & H_1^{\mathrm{dR}}(A_n) & \longrightarrow & H_1^{\mathrm{dR}}(A_n)/\varepsilon H_1^{\mathrm{dR}}(A_n) & \longrightarrow & 0. \end{array}$$

In particular, the endomorphism ε on $H_1^{\mathrm{dR}}(A_n)$ descends to an isomorphism $\mathcal{V}_n \cong F^0(A_n)$.

Let

$$\psi_n : H_1^{\mathrm{dR}}(A_n) \otimes H_1^{\mathrm{dR}}(A_n) \to \mathcal{O}_L$$

be the perfect pairing induced by the principal polarization on A_n, and define a second pairing $\Psi(x, y) = \psi_n(\varepsilon x, y)$. It follows from the previous paragraph that this descends to a perfect pairing

$$\Psi : \mathcal{V}_n \otimes \mathrm{Lie}(A_n) \cong \mathcal{O}_L.$$

However, if we endow $\mathrm{Lie}(A_n)^{-1}$ with the Faltings metric (5.1.1), then this pairing is not a duality between metrized line bundles.

Instead, an argument as in the proof of Proposition 5.1.2 shows that

$$\Psi : \mathcal{V}_n \otimes \mathrm{Lie}(A_n) \cong \mathcal{O}_L \left\langle \frac{1}{2\pi\sqrt{D}} \right\rangle.$$

is an isomorphism of metrized line bundles. With this isomorphism in hand, the remainder of the proof is exactly as in the previous lemma. ☐

The two lemmas show that the image of $\det(\mathcal{V})$ under (5.3.1) is

$$\sum_{i=1}^{n} \widehat{\deg}(\mathcal{V}_i) = [L : \mathbb{Q}] \cdot \Big((2-n) \cdot h_{\boldsymbol{k}}^{\mathrm{Falt}} - \frac{1}{2} \log(4\pi^2 D) \Big)$$

as claimed. This completes the proof of Proposition 5.3.3. ☐

Proposition 5.3.6. — *The metrized line bundle* $\det(\mathcal{V})$ *satisfies*

$$[\det(\mathcal{V}) : \mathcal{Y}_{\mathrm{big}}] = \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}}) \cdot \Big((4-2n) h_{\boldsymbol{k}}^{\mathrm{Falt}} + \log(4\pi^2 D) \Big).$$

Proof. — As in the proof of Proposition 5.2.1, we may fix a finite extension L/E_Φ and a finite étale cover $Y_{\mathrm{big}} = \bigsqcup \mathrm{Spec}(\mathcal{O}_L)$ of the $\mathcal{O}_{\boldsymbol{k}}$-stack

$$\mathcal{Y}_{\mathrm{big}} \times_{\mathrm{Spec}(\mathcal{O}_\Phi)} \mathrm{Spec}(\mathcal{O}_L)$$

by, say, m copies of $\mathrm{Spec}(\mathcal{O}_L)$. Corollary 5.3.2 then implies

$$\frac{[\det(\mathcal{V}) : \mathcal{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} = \frac{[\det(\mathcal{V}) : \mathrm{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathrm{Y}_{\mathrm{big}})} = \frac{cm \cdot [L : \boldsymbol{k}]}{m \cdot [L : \boldsymbol{k}]} = c.$$

Appealing to the evaluation of the constant c found in Proposition 5.3.3 completes the proof. ☐

5.4. Theorems C and D. — We can now put everything together, and relate the arithmetic degree of $\widehat{\omega}$ along $\mathcal{Y}_{\mathrm{big}}$ to the Faltings height $h^{\mathrm{Falt}}_{(E,\Phi)}$.

Proposition 5.4.1. — *The metrized line bundle $\widehat{\boldsymbol{\omega}}$ satisfies*

$$\frac{[\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} = h^{\mathrm{Falt}}_{(E,\Phi)} + \frac{n-4}{2} \cdot \frac{\Lambda'(0,\chi_{\boldsymbol{k}})}{\Lambda(0,\chi_{\boldsymbol{k}})} + \frac{n}{4}\log(16\pi^3 e^{\gamma}).$$

Proof. — Proposition 5.1.2 shows that

$$2\cdot[\mathcal{O}\langle 8\pi^2 e^{\gamma} D^{-1}\rangle \otimes \widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{big}}] + [\det(\mathrm{Lie}(A)) : \mathcal{Y}_{\mathrm{big}}] + 2\cdot[\mathrm{Lie}(A_0) : \mathcal{Y}_{\mathrm{big}}] \\ = [\mathcal{O}(\mathrm{Exc}) : \mathcal{Y}_{\mathrm{big}}] + [\det(\mathcal{V}) : \mathcal{Y}_{\mathrm{big}}].$$

Proposition 5.2.1 and Remark 5.1.5 imply that the left hand side is equal to

$$2\cdot[\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{big}}] - 2\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})\cdot\Big(\log(8\pi^2 e^{\gamma} D^{-1}) + h^{\mathrm{Falt}}_{(E,\Phi)} + 2\cdot h^{\mathrm{Falt}}_{\boldsymbol{k}}\Big),$$

while Proposition 5.3.6 shows that the right hand side is equal to

$$2\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})\cdot\big((2-n)h^{\mathrm{Falt}}_{\boldsymbol{k}} + \log(2\pi D)\big).$$

Note that we have used here the equality

$$[\mathcal{O}(\mathrm{Exc}) : \mathcal{Y}_{\mathrm{big}}] = [(\mathrm{Exc}, 0) : \mathcal{Y}_{\mathrm{big}}] = \deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})\cdot\log(D).$$

from the proof of Proposition 4.2.11.

Combining these formulas yields

$$\frac{[\widehat{\boldsymbol{\omega}} : \mathcal{Y}_{\mathrm{big}}]}{\deg_{\mathbb{C}}(\mathcal{Y}_{\mathrm{big}})} = h^{\mathrm{Falt}}_{(E,\Phi)} + (4-n)h^{\mathrm{Falt}}_{\boldsymbol{k}} + \log(16\pi^3 e^{\gamma}),$$

and substituting the value (1.4.1) for $h^{\mathrm{Falt}}_{\boldsymbol{k}}$ completes the proof. □

It is clear from Proposition 5.4.1 that Theorems C and Theorem D are equivalent. As Theorem C is proved in [**29**], this completes the proof of Theorem D.

On the other hand, we proved Theorem D in §4.5 under the assumption that $n \geq 3$ and the discriminants of $\boldsymbol{k}$ and F are odd and relatively prime, and so this gives a new proof of Theorem C under these hypotheses.

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ARITHMETIC OF BORCHERDS PRODUCTS

by

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Abstract. — We compute the divisors of Borcherds products on integral models of orthogonal Shimura varieties. As an application, we obtain an integral version of a theorem of Borcherds on the modularity of a generating series of special divisors.

***Résumé* (Arithmétique des produits de Borcherds).** — Nous calculons les diviseurs des produits de Borcherds sur des modèles intégraux de variétés de Shimura orthogonales. Comme application, nous obtenons une version intégrale d'un théorème de Borcherds sur la modularité d'une série génératrice de diviseurs spéciaux.

1. Introduction

In the series of papers [**4, 5, 6**], Borcherds introduced a family of meromorphic modular forms on orthogonal Shimura varieties, whose zeroes and poles are prescribed linear combinations of special divisors arising from embeddings of smaller orthogonal Shimura varieties. These meromorphic modular forms are the Borcherds products of the title.

After work of Kisin [**31**] on integral models of general Hodge and abelian type Shimura varieties, the theory of integral models of orthogonal Shimura varieties and their special divisors was developed further in [**26, 27**] and [**39, 1, 2**].

The goal of this paper is to combine the above theories to compute the divisor of a Borcherds product on the integral model of an orthogonal Shimura variety. We show that such a divisor is given as a prescribed linear combination of special divisors, exactly as in the generic fiber.

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The first such results were obtained by Bruinier, Burgos Gil, and Kühn [**9**], who worked on Hilbert modular surfaces (a special type of signature $(2,2)$ orthogonal Shimura variety). Those results were later extended to more general orthogonal Shimura varieties by Hörmann [**26, 27**], but with some restrictions.

Our results extend Hörmann's, but with substantially weaker hypotheses. For example, our results include cases where the integral model is not smooth, cases where the divisors in question may have irreducible components supported in nonzero characteristics, and even cases where the Shimura variety is compact (so that one has no theory of q-expansions with which to analyze the arithmetic properties of Borcherds products).

1.1. Orthogonal Shimura varieties. — Given an integer $n \geq 1$ and a quadratic space (V, Q) over $\mathbb{Q}$ of signature $(n, 2)$, one can construct a Shimura datum $(G, \mathcal{D})$ with reflex field $\mathbb{Q}$.

The group $G = \mathrm{GSpin}(V)$ is a subgroup of the group of units in the Clifford algebra $C(V)$, and sits in a short exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathrm{SO}(V) \to 1.$$

The hermitian symmetric domain is

$$\mathcal{D} = \{z \in V_{\mathbb{C}} : [z, z] = 0,\ [z, \overline{z}] < 0\}/\mathbb{C}^\times \subset \mathbb{P}(V_{\mathbb{C}}),$$

where the bilinear form

$$[x, y] = Q(x + y) - Q(x) - Q(y) \tag{1.1.1}$$

on V has been extended $\mathbb{C}$-bilinearly to $V_{\mathbb{C}}$.

To define a Shimura variety, fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ on which the quadratic form is $\mathbb{Z}$-valued, and a compact open subgroup $K \subset G(\mathbb{A}_f)$ such that

$$K \subset G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times. \tag{1.1.2}$$

Here $C(V_{\widehat{\mathbb{Z}}})$ is the Clifford algebra of the $\widehat{\mathbb{Z}}$-quadratic space $V_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$. The canonical model of the complex orbifold

$$\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \backslash \big(\mathcal{D} \times G(\mathbb{A}_f)/K\big)$$

is a smooth n-dimensional Deligne-Mumford stack

$$\mathrm{Sh}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Q}).$$

As in work of Kudla [**32, 34**], our Shimura variety carries a family of effective Cartier divisors

$$Z(m, \mu) \to \mathrm{Sh}_K(G, \mathcal{D})$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$, and a metrized line bundle

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathrm{Sh}_K(G, \mathcal{D}))$$

of weight one modular forms. Under the complex uniformization of the Shimura variety, this line bundle pulls back to the tautological bundle on $\mathcal{D}$, with the metric defined by (4.2.3).

We say that $V_{\mathbb{Z}}$ is *maximal* if there is no proper superlattice in V on which Q takes integer values, and is *maximal at* p if the $\mathbb{Z}_p$-quadratic space $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$ has the analogous property. It is clear that $V_{\mathbb{Z}}$ is maximal at every prime not dividing the discriminant $[V_{\mathbb{Z}}^{\vee} : V_{\mathbb{Z}}]$.

Let Ω be a finite set of rational primes containing all primes at which $V_{\mathbb{Z}}$ is not maximal, and abbreviate

$$\mathbb{Z}_{\Omega} = \mathbb{Z}[1/p : p \in \Omega].$$

Assume that (1.1.2) factors as $K = \prod_p K_p$, in such a way that

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^{\times}$$

for every prime $p \notin \Omega$. For such K there is a flat and normal integral model

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_{\Omega})$$

of $\mathrm{Sh}_K(G, \mathcal{D})$. It is a Deligne-Mumford stack of finite type over $\mathbb{Z}_{\Omega}$, and is a scheme if K is sufficiently small. At any prime $p \notin \Omega$, it satisfies the following properties:

1. If the lattice $V_{\mathbb{Z}}$ is self-dual at a prime p (or even *almost self-dual* in the sense of Definition 6.1.1) then the restriction of the integral model to $\mathrm{Spec}(\mathbb{Z}_{(p)})$ is smooth.
2. If p is odd and p^2 does not divide the discriminant $[V_{\mathbb{Z}}^{\vee} : V_{\mathbb{Z}}]$, then the restriction of the integral model to $\mathrm{Spec}(\mathbb{Z}_{(p)})$ is regular.
3. If $n \geq 6$ then the reduction mod p is geometrically normal.

The integral model carries over it a metrized line bundle

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G, \mathcal{D}))$$

of weight one modular forms, extending the one already available in the generic fiber, and a family of effective Cartier divisors

$$\mathcal{Z}(m, \mu) \to \mathcal{S}_K(G, \mathcal{D})$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$.

Remark 1.1.1. — If $V_{\mathbb{Z}}$ is itself maximal, one can take $\Omega = \emptyset$, choose

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^{\times}$$

for the level subgroup, and obtain an integral model of $\mathrm{Sh}_K(G, \mathcal{D})$ over $\mathbb{Z}$.

1.2. Borcherds products. — In § 5.1, we recall the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}})$$

of the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$ on the $\mathbb{C}$-vector space

$$S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}].$$

Any weakly holomorphic form

$$f(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m\gg-\infty}} c(m)\cdot q^m \in M^{!}_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

valued in the complex-conjugate representation has Fourier coefficients

$$c(m) \in S_{V_{\mathbb{Z}}},$$

and we denote by $c(m,\mu)$ the value of $c(m)$ at the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Fix such an f, assume that f is *integral* in the sense that $c(m,\mu) \in \mathbb{Z}$ for all m and μ.

Using the theory of regularized theta lifts, Borcherds **[5]** constructs a Green function $\Theta^{\mathrm{reg}}(f)$ for the analytic divisor

$$\sum_{\substack{m>0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu)\cdot Z(m,\mu)(\mathbb{C}) \tag{1.2.1}$$

on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$, and shows (after possibly replacing f by a suitable multiple) that some power of $\boldsymbol{\omega}^{\mathrm{an}}$ admits a meromorphic section $\psi(f)$ satisfying

$$-2\log\|\psi(f)\| = \Theta^{\mathrm{reg}}(f). \tag{1.2.2}$$

This implies that the divisor of $\psi(f)$ is (1.2.1). These meromorphic sections are the *Borcherds products* of the title.

Our main result, stated in the text as Theorem 9.1.1, asserts that the Borcherds product $\psi(f)$ is algebraic, defined over $\mathbb{Q}$, and has the expected divisor when viewed as a rational section over the integral model.

Theorem A. — *After possibly replacing f by a positive integer multiple, there is a rational section $\psi(f)$ of the line bundle $\boldsymbol{\omega}^{c(0,0)}$ on $\mathcal{S}_K(G,\mathcal{D})$ whose norm under the metric* (4.2.3) *satisfies* (1.2.2)*, and whose divisor is*

$$\mathrm{div}(\psi(f)) = \sum_{\substack{m>0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu)\cdot \mathcal{Z}(m,\mu).$$

The unspecified positive integer by which one must multiply f can be made at least somewhat more explicit. For example, it depends only on the lattice $V_{\mathbb{Z}}$, and not on the form f. See the discussion of § 9.3.

As noted earlier, similar results can be found in the work of Hörmann **[26, 27]**. Hörmann only considers self-dual lattices, so that the corresponding integral model $\mathcal{S}_K(G,\mathcal{D})$ is smooth, and always assumes that the quadratic space V admits an isotropic line. This allows him to prove the flatness of $\mathrm{div}(\psi(f))$ by examining the

q-expansion of $\psi(f)$ at a cusp. As Hörmann's special divisors $\mathcal{Z}(m,\mu)$, unlike ours, are defined as the Zariski closures of their generic fibers, the equality of divisors stated in Theorem A is then a formal consequence of the same equality in the generic fiber.

In contrast, we can prove Theorem A even in cases where the divisors in question may not be flat, and in cases where V is anisotropic, so no theory of q-expansions is available.

The reader may be surprised to learn that even the descent of $\psi(f)$ to $\mathbb{Q}$ was not previously known in full generality. Indeed, there is a product formula for the Borcherds product giving its q-expansions at every cusp, and so one should be able to detect the field of definition of $\psi(f)$ from a suitable q-expansion principle.

If V is anisotropic then $\mathrm{Sh}_K(G,\mathcal{D})$ is proper over $\mathbb{Q}$, no theory of q-expansions exists, and the above strategy fails completely. But even when V is isotropic there is a serious technical obstruction to this argument. The product formula of Borcherds is not completely precise, in that the q-expansion of $\psi(f)$ at a given cusp is only specified up to multiplication by an unknown constant of absolute value 1, and there is no a priori relation between the different constants at different cusps. These constants are the $\kappa^{(a)}$ appearing in Proposition 5.4.2.

If $\mathrm{Sh}_K(G,\mathcal{D})$ admits (a toroidal compactification with) a cusp defined over $\mathbb{Q}$ there is no problem: simply rescale the Borcherds product by a constant of absolute value 1 to remove the mysterious constant at that cusp, and now $\psi(f)$ is defined over $\mathbb{Q}$. But if $\mathrm{Sh}_K(G,\mathcal{D})$ has no rational cusps, then to prove that $\psi(f)$ descends to $\mathbb{Q}$ one must compare the q-expansions of $\psi(f)$ at all points in a Galois orbit of cusps. One can rescale the Borcherds product to trivialize the constant at one cusp, but then one has no control over the constants at other cusps in the Galois orbit.

Using the q-expansion principle alone, is seems that the best one can prove is that $\psi(f)$ descends to the minimal field of definition of a cusp. Our strategy to improve on this is sketched in § 1.4 below.

Remark 1.2.1. — As in the statement and proof of [**26**, Theorem 10.4.12], there is an elementary argument using Hilbert's Theorem 90 that allows one to rescale the Borcherds product so that it descends to $\mathbb{Q}$, but in this argument one has no control over the scaling factor, and it need not have absolute value 1. In particular this rescaling may destroy the norm relation (1.2.2). Even worse, rescaling by such factors may introduce unwanted and unknown vertical components into the divisor of the Borcherds product on the integral model of the Shimura variety, and understanding what's happening on the integral model is the central concern of this work.

1.3. Modularity of generating series. — The family of special divisors determines a family of line bundles

$$\mathcal{Z}(m,\mu) \in \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. We extend the definition to $m = 0$ by setting

$$\mathcal{Z}(0,\mu) = \begin{cases} \boldsymbol{\omega}^{-1} & \text{if } \mu = 0 \\ \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} & \text{if } \mu \neq 0. \end{cases}$$

Exactly as in the work of Borcherds [**6**], Theorem A produces enough relations in the Picard group to prove the modularity of the generating series of these line bundles. Let

$$\phi_\mu \in S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}]$$

denote the characteristic function of the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$.

Theorem B. — *The formal q-expansion*

$$\sum_{\substack{m \geq 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu) \otimes \phi_\mu \cdot q^m$$

is a modular form valued in $\mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \otimes S_{V_{\mathbb{Z}}}$. *More precisely, we have*

$$\sum_{\substack{m \geq 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \alpha(\mathcal{Z}(m,\mu)) \cdot \phi_\mu \cdot q^m \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$$

for any $\mathbb{Z}$*-linear map* $\alpha : \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$.

Theorem B is stated in the text as Theorem 9.4.1. After endowing the special divisors with Green functions as in [**8**], we also prove a modularity result in the group of metrized line bundles. See Theorem 9.5.1.

1.4. Idea of the proof. — We first prove Theorem A assuming that $n \geq 6$, and that $V_{\mathbb{Z}}$ splits an integral hyperbolic plane. This assumption has three crucial consequences. First, it guarantees the existence of cusps of $\mathrm{Sh}_K(G,\mathcal{D})$ defined over $\mathbb{Q}$. Second, it guarantees that our integral model has geometrically normal fibers, so that we may use the results of [**40**] to fix a toroidal compactification in such a way that every irreducible component of every mod p fiber of $\mathcal{S}_K(G,\mathcal{D})$ meets a cusp. Finally, it guarantees the flatness of all special divisors $\mathcal{Z}(m,\mu)$.

As noted above, the existence of cusps over $\mathbb{Q}$ allows us to deduce the descent of $\psi(f)$ to $\mathbb{Q}$ using the q-expansion principle. Moreover, by examining the q-expansions of $\psi(f)$ at the cusps, one can show that its divisor is flat over $\mathbb{Z}_\Omega$, and the equality of divisors in Theorem A then follows from the known equality in the generic fiber.

Remark 1.4.1. — In fact, we prove that our divisors are flat over $\mathbb{Z}$ as soon as $n \geq 4$. When $n \in \{1,2,3\}$ the orthogonal Shimura varieties and their special divisors can be interpreted as a moduli space of abelian varieties with additional structure, as in the work of Kudla-Rapoport [**35, 36, 37**]. Already in the case of $n = 1$, Kudla and Rapoport [**37**] provide examples in which the special divisors are not flat.

To understand how to deduce the general case from the special case above, we first recall how Borcherds constructs $\psi(f)$ in the complex fiber. If V contains an isotropic line, the construction boils down to explicitly writing down its q-expansion as an infinite product. This gives the desired $\psi(f)$, along with the norm relation (1.2.2), on the region of convergence. The right hand side of (1.2.2) is a pluriharmonic function defined on the complement of the support of (1.2.1), and the meromorphic continuation of $\psi(f)$ follows more-or-less formally from this.

Suppose now that V is anisotropic. The idea of Borcherds is to fix isometric embeddings of V into two (very particular) quadratic spaces $V^{[1]}$ and $V^{[2]}$ of signature $(n+24,2)$. From this one can construct morphisms of orthogonal Shimura varieties

$$\begin{array}{ccc} & \mathrm{Sh}_K(G,\mathcal{D}) & \\ {}^{j^{[1]}}\swarrow & & \searrow^{j^{[2]}} \\ \mathrm{Sh}_{K^{[1]}}(G^{[1]},\mathcal{D}^{[1]}) & & \mathrm{Sh}_{K^{[2]}}(G^{[2]},\mathcal{D}^{[2]}). \end{array}$$

As both $V^{[1]}$ and $V^{[2]}$ contain isotropic lines, one already has Borcherds products on their associated Shimura varieties.

The next step should be to define

$$\psi(f) = \frac{(j^{[2]})^*\psi(f^{[2]})}{(j^{[1]})^*\psi(f^{[1]})} \tag{1.4.1}$$

for (very particular) weakly holomorphic forms $f^{[1]}$ and $f^{[2]}$. The problem is that the quotient on the right hand side is nearly always either $0/0$ or ∞/∞, and so doesn't really make sense.

Borcherds gets around this via an analytic construction on the level of hermitian domains. On the hermitian domain

$$\mathcal{D}^{[i]} = \{z \in V^{[i]}_{\mathbb{C}} : [z,z]=0,\ [z,\overline{z}]<0\}/\mathbb{C}^\times \subset \mathbb{P}(V^{[i]}_{\mathbb{C}}),$$

every irreducible component of every special divisor has the form

$$\mathcal{D}^{[i]}(x) = \{z \in \mathcal{D}^{[i]} : z \perp x\}$$

for some $x \in V^{[i]}$, and the dual of the tautological line bundle $\boldsymbol{\omega}_{\mathcal{D}^{[i]}}$ on $\mathcal{D}^{[i]}$ admits a canonical section

$$\mathrm{obst}^{\mathrm{an}}_x \in H^0(\mathcal{D}^{[i]}, \boldsymbol{\omega}^{-1}_{\mathcal{D}^{[i]}})$$

with zero locus $\mathcal{D}^{[i]}(x)$. See the discussion at the beginning of § 6.5.

Whenever there is an $x \in V^{[i]}$ such that $\mathcal{D} \subset \mathcal{D}^{[i]}(x)$, Borcherds multiplies $\psi(f^{[i]})$ by a suitable power of $\mathrm{obst}^{\mathrm{an}}_x$ in order to remove the component $\mathcal{D}^{[i]}(x)$ from $\mathrm{div}(\psi(f^{[i]})$. After modifying both $\psi(f^{[1]})$ and $\psi(f^{[2]})$ in this way, the quotient (1.4.1) is defined. This process is what Borcherds calls the *embedding trick* in **[5]**. As understood by Borcherds, the embedding trick is a purely analytic construction. The sections $\mathrm{obst}^{\mathrm{an}}_x$ over $\mathcal{D}^{[i]}$ do not descend to the Shimura varieties, and have no obvious algebraic properties. In particular, even if one knows that the $\psi(f^{[i]})$ are defined over $\mathbb{Q}$, it is not obvious that the renormalized quotient (1.4.1) is defined over $\mathbb{Q}$.

One of the main contributions of this paper is an algebraic analogue of the embedding trick, which works even on the level of integral models. This is based on the methods used to compute improper intersections in **[11, 1, 28]**. The idea is to use deformation theory to construct an analogue of the section $\mathrm{obst}_x^{\mathrm{an}}$, not over all of $\mathrm{Sh}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]})$, but only over the first order infinitesimal neighborhood of $\mathrm{Sh}_K(G, \mathcal{D})$ in $\mathrm{Sh}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]})$. This section is the *obstruction to deforming x* appearing in § 6.5.

With this algebraic analogue of the embedding trick in hand, we can make sense of the quotient (1.4.1), and compute the divisor of the left hand side in terms of the divisors of the numerator and denominator on the right. This allows us to deduce the general case of Theorem A from the special case explained above.

1.5. Organization of the paper. — Ultimately, all arithmetic information about Borcherds products comes from their q-expansions, and so we must make heavy use of the arithmetic theory of toroidal compactifications of Shimura varieties of **[47, 40]**. This theory requires introducing a substantial amount of notation just to state the main results. Also, because Borcherds products are rational sections of powers of the line bundle $\boldsymbol{\omega}$, we need the theory of automorphic vector bundles on toroidal compactifications. This theory is distributed across a series of papers of Harris **[24, 18, 19, 20]** and Harris-Zucker **[21, 22, 23]**.

Accordingly, before we even begin to talk about orthogonal Shimura varieties, we first recall in § 2 the main results on toroidal compactification from Pink's thesis **[47]**, and in § 3 the results of Harris and Harris-Zucker on automorphic vector bundles. All of this is in the generic fiber of fairly general Shimura varieties.

Beginning in § 4 we specialize to case of orthogonal Shimura varieties. We consider their toroidal compactifications, and give a purely algebraic definition of q-expansions of modular forms on them. In particular, we prove the q-expansion principle Proposition 4.6.3, which can be used to detect their fields of definition.

In § 5 we introduce Borcherds products and, when V admits an isotropic line, describe their q-expansions.

In § 6 we introduce integral models of orthogonal Shimura varieties over $\mathbb{Z}_{(p)}$, along with their line bundles of modular forms and special divisors. This material is drawn from **[39, 1, 2]**, although we work here in slightly more generality. The main new result in § 6 is the pullback formula of Proposition 6.6.3, which explains how the special divisors behave under pullback via embeddings of orthogonal Shimura varieties. This formula, whose proof is similar to calculations of improper intersections found in **[11, 1, 28]**, is essential to our algebraic version of the embedding trick.

In § 7 we prove some technical properties of the integral models over $\mathbb{Z}_{(p)}$. We show that the special divisors are flat when $n \geq 4$, and the integral model has geometrically normal fibers when $n \geq 6$. When $p \neq 2$ these results already appear in **[2]**. The methods here are similar, except that we appeal to the work of Ogus **[44]** instead of **[29]** (which excludes $p = 2$) to control the dimension of the supersingular locus.

In § 8 we extend the theory of toroidal compactifications and q-expansions to our integral models, making use of the general theory of toroidal compactifications of Hodge type Shimura varieties found in [**40**]. The culmination of the discussion is Corollary 8.2.4, which allows one to use q-expansions to detect the flatness of divisors of rational sections of $\boldsymbol{\omega}$ and its powers.

Finally, in § 9 we put everything together to prove Theorem A. The modularity result of Theorem B (and its extension to the group of metrized line bundles) follows immediately from Theorem A and the modularity criterion of Borcherds.

1.6. Notation and conventions. — For every $a \in \mathbb{A}_f^\times$ there is a unique factorization

$$a = \mathrm{rat}(a) \cdot \mathrm{unit}(a)$$

in which $\mathrm{rat}(a)$ is a positive rational number and $\mathrm{unit}(a) \in \widehat{\mathbb{Z}}^\times$.

Class field theory provides us with a reciprocity map

$$\mathrm{rec} : \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}),$$

which we normalize as follows. Let μ_∞ be the set of all roots of unity in $\mathbb{C}$, so that $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\mu_\infty)$ is the maximal abelian extension of $\mathbb{Q}$. The group $(\mathbb{Z}/M\mathbb{Z})^\times$ acts on the set of M-th roots of unity in the usual way, by letting $u \in (\mathbb{Z}/M\mathbb{Z})^\times$ act by $\zeta \mapsto \zeta^u$. Passing to the limit yields an action of $\widehat{\mathbb{Z}}^\times$ on μ_∞, and the reciprocity map is characterized by

$$\zeta^{\mathrm{rec}(a)} = \zeta^{\mathrm{unit}(a)}$$

for all $a \in \mathbb{A}_f^\times$ and $\zeta \in \mu_\infty$.

We follow the conventions of [**14**] and [**47**, Chapter 1] for Hodge structures and mixed Hodge structures. As usual, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m\mathbb{C}}$ is Deligne's torus, so that $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$, with complex conjugation acting by $(t_1, t_2) \mapsto (\bar{t}_2, \bar{t}_1)$. In particular, $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$ by $(t, \bar{t}) \mapsto t$. If V is a rational vector space endowed with a Hodge structure $\mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$, then $V^{(p,q)} \subset V_\mathbb{C}$ is the subspace on which $(t_1, t_2) \in \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}(\mathbb{C})$ acts via $t_1^{-p} t_2^{-q}$. There is a distinguished cocharacter

$$\mathrm{wt} : \mathbb{G}_{m\mathbb{R}} \to \mathbb{S}$$

defined on complex points by $t \mapsto (t^{-1}, t^{-1})$. The composition

$$\mathbb{G}_{m\mathbb{R}} \xrightarrow{\mathrm{wt}} \mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$$

encodes the weight grading on $V_\mathbb{R}$, in the sense that

$$\bigoplus_{p+q=k} V^{(p,q)} = \{v \in V_\mathbb{C} : \mathrm{wt}(z) \cdot v = z^k \cdot v, \, \forall z \in \mathbb{C}^\times\}.$$

Now suppose that V is endowed with a mixed Hodge structure. This consists of an increasing weight filtration $\mathrm{wt}_\bullet V$ on V, and a decreasing Hodge filtration $F^\bullet V_\mathbb{C}$ on $V_\mathbb{C}$, whose induced filtration on every graded piece

$$\mathrm{gr}_k(V) = \mathrm{wt}_k V / \mathrm{wt}_{k-1} V \tag{1.6.1}$$

is a pure Hodge structure of weight k. By [46, Lemma-Definition 3.4] there is a canonical bigrading $V_{\mathbb{C}} = \bigoplus V^{(p,q)}$ with the property that

$$\mathrm{wt}_k V_{\mathbb{C}} = \bigoplus_{p+q\leq k} V^{(p,q)}, \quad F^i V_{\mathbb{C}} = \bigoplus_{p\geq i} V^{(p,q)}.$$

This bigrading is induced by a morphism $\mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$.

2. Toroidal compactification

This section is a (relatively) short summary of Pink's thesis [47] on toroidal compactifications of canonical models of Shimura varieties. See also [26] and [21, 23]. We limit ourselves to what is needed in the sequel, and simplify the discussion somewhat by only dealing with those mixed Shimura varieties that appear at the boundaries of pure Shimura varieties.

2.1. Shimura varieties. — Throughout § 2 and § 3 we let $(G, \mathcal{D})$ be a (pure) Shimura datum in the sense of [47, § 2.1]. Thus G is a reductive group over $\mathbb{Q}$, and $\mathcal{D}$ is a $G(\mathbb{R})$-homogeneous space equipped with a finite-to-one $G(\mathbb{R})$-equivariant map

$$\mathtt{h} : \mathcal{D} \to \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$$

such that the pair $(G, \mathtt{h}(\mathcal{D}))$ satisfies Deligne's axioms [14, (2.1.1.1)-(2.1.1.3)]. We often abuse notation and confuse $z \in \mathcal{D}$ with its image $\mathtt{h}(z)$.

The *weight cocharacter*

$$w \stackrel{\text{def}}{=} \mathtt{h}(z) \circ \mathrm{wt} : \mathbb{G}_{m\mathbb{R}} \to G_{\mathbb{R}} \tag{2.1.1}$$

of $(G, \mathcal{D})$ is independent of $z \in \mathcal{D}$, and takes values in the center of $G_{\mathbb{R}}$.

Hypothesis 2.1.1. — Because it will simplify much of what follows, and because it is assumed throughout [23], we always assume that our Shimura datum $(G, \mathcal{D})$ satisfies:

1. The weight cocharacter (2.1.1) is defined over $\mathbb{Q}$.
2. The connected center of G is isogenous to the product of a $\mathbb{Q}$-split torus with a torus whose group of real points is compact.

Suppose $K \subset G(\mathbb{A}_f)$ is any compact open subgroup. The associated Shimura variety

$$\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash\big(\mathcal{D} \times G(\mathbb{A}_f)/K\big)$$

is a complex orbifold. Its canonical model $\mathrm{Sh}_K(G, \mathcal{D})$ is a Deligne-Mumford stack over the reflex field $E(G, \mathcal{D}) \subset \mathbb{C}$. If K is neat in the sense of [47, § 0.6], then $\mathrm{Sh}_K(G, \mathcal{D})$ is a quasi-projective scheme. By slight abose of notation, the image of a point $(z, g) \in \mathcal{D} \times G(\mathbb{A}_f)$ is again denoted

$$(z, g) \in \mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}).$$

Remark 2.1.2. — Let $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$ act on the two point set

$$\mathcal{H}_0 \stackrel{\text{def}}{=} \{2\pi\epsilon \in \mathbb{C} : \epsilon^2 = -1\}$$

via the unique continuous transitive action: positive real numbers act trivially, and negative real numbers swap the two points. If we define

$$\mathcal{H}_0 \to \mathrm{Hom}(\mathbb{S}, \mathbb{G}_{m\mathbb{R}})$$

by sending both points to the norm map $\mathbb{C}^\times \to \mathbb{R}^\times$, then $(\mathbb{G}_m, \mathcal{H}_0)$ is a Shimura datum in the sense of [**47**].

2.2. Mixed Shimura varieties. — Toroidal compactifications of Shimura varieties are obtained by gluing together certain mixed Shimura varieties, which we now define.

Recall from [**47**, Definition 4.5] the notion of an *admissible* parabolic subgroup $P \subset G$. If G^{ad} is simple, this just means that P is either a maximal proper parabolic subgroup, or is all of G. In general, it means that if we write $G^{ad} = G_1 \times \cdots \times G_s$ as a product of simple groups, then P is the preimage of a subgroup $P_1 \times \cdots \times P_s$, where each $P_i \subset G_i$ is an admissible parabolic.

Definition 2.2.1. — A *cusp label representative* $\Phi = (P, \mathcal{D}^\circ, h)$ for $(G, \mathcal{D})$ is a triple consisting of an admissible parabolic subgroup P, a connected component $\mathcal{D}^\circ \subset \mathcal{D}$, and an $h \in G(\mathbb{A}_f)$.

As in [**47**, § 4.11 and § 4.12], any cusp label representative $\Phi = (P, \mathcal{D}^\circ, h)$ determines a mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$, whose construction we now recall.

Let $W_\Phi \subset P$ be the unipotent radical, and let U_Φ be the center of W_Φ. According to [**47**, § 4.1] there is a distinguished central cocharacter $\lambda : \mathbb{G}_m \to P/W_\Phi$. The weight cocharacter $w : \mathbb{G}_m \to G$ is central, so takes values in P, and therefore determines a new central cocharacter

$$w \cdot \lambda^{-1} : \mathbb{G}_m \to P/W_\Phi. \tag{2.2.1}$$

Suppose $G \to \mathrm{GL}(N)$ is a faithful representation on a finite dimensional $\mathbb{Q}$-vector space. Each point $z \in \mathcal{D}$ determines a Hodge filtration $F^\bullet N_\mathbb{C}$ on N. Any lift of (2.2.1) to a cocharacter $\mathbb{G}_m \to P$ determines a grading $N = \bigoplus N^k$, and the associated *weight filtration*

$$\mathrm{wt}_\ell N = \bigoplus_{k \le \ell} N^k$$

is independent of the lift. The triple $(N, F^\bullet N_\mathbb{C}, \mathrm{wt}_\bullet N)$ is a mixed Hodge structure [**47**, § 4.12, Remark (i)], and the associated bigrading of $N_\mathbb{C}$ determines a morphism $\mathtt{h}_\Phi(z) \in \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$ independent of the choice of faithful representation N.

As in [**47**, § 4.7], define $Q_\Phi \subset P$ to be the smallest closed normal subgroup through which every such $\mathtt{h}_\Phi(z)$ factors. Thus we have normal subgroups

$$U_\Phi \lhd W_\Phi \lhd Q_\Phi \lhd P,$$

and a map

$$\mathtt{h}_\Phi : \mathcal{D} \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}}).$$

The cocharacter (2.2.1) takes values in Q_Φ/W_Φ, defining the *weight cocharacter*

$$w_\Phi : \mathbb{G}_m \to Q_\Phi/W_\Phi. \tag{2.2.2}$$

Remark 2.2.2. — Being an abelian unipotent group, $\mathrm{Lie}(U_\Phi) \cong U_\Phi$ has the structure of a $\mathbb{Q}$-vector space. By [**47**, Proposition 2.14], the conjugation action of Q_Φ on U_Φ is through a character

$$\nu_\Phi : Q_\Phi \to \mathbb{G}_m. \tag{2.2.3}$$

By [**47**, Proposition 4.15(a)], the map $\mathtt{h}_\Phi$ restricts to an open immersion on every connected component of $\mathcal{D}$, and so the diagonal map

$$\mathcal{D} \to \pi_0(\mathcal{D}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}})$$

is a $P(\mathbb{R})$-equivariant open immersion. The action of the subgroup $U_\Phi(\mathbb{R})$ on $\pi_0(\mathcal{D})$ is trivial, and we extend it to the trivial action of $U_\Phi(\mathbb{C})$ on $\pi_0(\mathcal{D})$. Now define

$$\mathcal{D}_\Phi = Q_\Phi(\mathbb{R})U_\Phi(\mathbb{C})\mathcal{D}^\circ \subset \pi_0(\mathcal{D}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}}).$$

Projection to the second factor defines a finite-to-one map

$$\mathtt{h}_\Phi : \mathcal{D}_\Phi \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}}),$$

and we usually abuse notation and confuse $z \in \mathcal{D}_\Phi$ with its image $\mathtt{h}_\Phi(z)$.

Having now defined the mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$, the compact open subgroup

$$K_\Phi \stackrel{\mathrm{def}}{=} hKh^{-1} \cap Q_\Phi(\mathbb{A}_f)$$

determines a mixed Shimura variety

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) = Q_\Phi(\mathbb{Q})\backslash\big(\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big), \tag{2.2.4}$$

which has a canonical model $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ over its reflex field. Note that the reflex field is again $E(G, \mathcal{D})$, by [**47**, Proposition 12.1]. The canonical model is a quasi-projective scheme if K (hence K_Φ) is neat.

Remark 2.2.3. — If we choose our cusp label representative to have the form $\Phi = (G, \mathcal{D}^\circ, h)$, then $(Q_\Phi, \mathcal{D}_\Phi) = (G, \mathcal{D})$ and

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) = \mathrm{Sh}_{hKh^{-1}}(G, \mathcal{D}) \cong \mathrm{Sh}_K(G, \mathcal{D}).$$

As a consequence, all of our statements about the mixed Shimura varieties $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ include the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$ as a special case.

2.3. The torsor structure. — Define $\bar{Q}_\Phi = Q_\Phi/U_\Phi$ and $\bar{\mathcal{D}}_\Phi = U_\Phi(\mathbb{C})\backslash\mathcal{D}_\Phi$. The pair

$$(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi) = (Q_\Phi, \mathcal{D}_\Phi)/U_\Phi$$

is the quotient mixed Shimura datum in the sense of [**47**, §2.9]. Let $\bar{K}_\Phi$ be the image of K_Φ under the quotient map $Q_\Phi(\mathbb{A}_f) \to \bar{Q}_\Phi(\mathbb{A}_f)$, so that we have a canonical morphism

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \to \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi), \tag{2.3.1}$$

where the target mixed Shimura variety is defined in the same way as (2.2.4).

Proposition 2.3.1. — *Define a $\mathbb{Z}$-lattice in $U_\Phi(\mathbb{Q})$ by $\Gamma_\Phi = K_\Phi \cap U_\Phi(\mathbb{Q})$. The morphism* (2.3.1) *is canonically a torsor for the relative torus*

$$T_\Phi \stackrel{\text{def}}{=} \Gamma_\Phi(-1) \otimes \mathbb{G}_m$$

with cocharacter group $\Gamma_\Phi(-1) = (2\pi i)^{-1}\Gamma_\Phi$.

Proof. — This is proved in [**47**, § 6.6]. In what follows we only want to make the torsor structure explicit on the level of complex points.

The character (2.2.3) factors through a character $\bar{\nu}_\Phi : \bar{Q}_\Phi \to \mathbb{G}_m$. A pair $(z, g) \in \mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)$ determines points

$$(z, g) \in \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}), \quad (\bar{z}, \bar{g}) \in \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)(\mathbb{C}),$$

and we define $\mathbf{T}_\Phi(\mathbb{C}) \to \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)(\mathbb{C})$ as the relative torus with fiber

$$U_\Phi(\mathbb{C})/(gK_\Phi g^{-1} \cap U_\Phi(\mathbb{Q})) = U_\Phi(\mathbb{C})/\mathrm{rat}(\bar{\nu}_\Phi(\bar{g})) \cdot \Gamma_\Phi \tag{2.3.2}$$

at $(\bar{z}, \bar{g})$. There is a natural action of $\mathbf{T}_\Phi(\mathbb{C})$ on (2.2.4) defined as follows: using the natural action of $U_\Phi(\mathbb{C})$ on $\mathcal{D}_\Phi$, a point u in the fiber (2.3.2) acts as $(z, g) \mapsto (uz, g)$.

It now suffices to construct an isomorphism

$$\mathbf{T}_\Phi(\mathbb{C}) \cong T_\Phi(\mathbb{C}) \times \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)(\mathbb{C}), \tag{2.3.3}$$

and this is essentially [**47**, § 3.16]. First choose a morphism

$$\bar{\mathcal{D}}_\Phi \xrightarrow{\bar{z} \mapsto 2\pi\epsilon(\bar{z})} \mathcal{H}_0 \tag{2.3.4}$$

in such a way that it, along with the character $\bar{\nu}_\Phi$, induces a morphism of mixed Shimura data $(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi) \to (\mathbb{G}_m, \mathcal{H}_0)$. Such a morphism always exists, by the remark of [**47**, § 6.8]. The fiber (2.3.2) is

$$U_\Phi(\mathbb{C})/\mathrm{rat}(\bar{\nu}_\Phi(\bar{g})) \cdot \Gamma_\Phi \xrightarrow{2\pi\epsilon(\bar{z})/\mathrm{rat}(\bar{\nu}_\Phi(\bar{g}))} U_\Phi(\mathbb{C})/\Gamma_\Phi(1),$$

and this identifies $\mathbf{T}_\Phi(\mathbb{C})$ fiber-by-fiber with the constant torus

$$U_\Phi(\mathbb{C})/\Gamma_\Phi(1) \cong \Gamma_\Phi \otimes \mathbb{C}/\mathbb{Z}(1) \cong \Gamma_\Phi \otimes \mathbb{C}^\times \xrightarrow{(-2\pi\epsilon^\circ)^{-1}} \Gamma_\Phi(-1) \otimes \mathbb{C}^\times. \tag{2.3.5}$$

Here $2\pi\epsilon^\circ$ is the image of $\mathcal{D}^\circ$ under $\mathcal{D}_\Phi \to \bar{\mathcal{D}}_\Phi \to \mathcal{H}_0$, and the minus sign is included so that (2.6.5) holds below; compare with the definition of the function "ord" in [**47**, § 5.8].

One can easily check that the trivialization (2.3.3) does not depend on the choice of (2.3.4). □

Remark 2.3.2. — Our $\mathbb{Z}$-lattice $\Gamma_\Phi \subset U_\Phi(\mathbb{Q})$ agrees with the seemingly more complicated lattice of [**47**, § 3.13], defined as the image of

$$\{(c,\gamma) \in Z_\Phi(\mathbb{Q})_0 \times U_\Phi(\mathbb{Q}) : c\gamma \in K_\Phi\} \xrightarrow{(c,\gamma)\mapsto\gamma} U_\Phi(\mathbb{Q}).$$

Here Z_Φ is the center of Q_Φ, and $Z_\Phi(\mathbb{Q})_0 \subset Z_\Phi(\mathbb{Q})$ is the largest subgroup acting trivially on $\mathcal{D}_\Phi$ (equivalently, acting trivially on $\pi_0(\mathcal{D}_\Phi)$). This follows from the final comments of [*loc. cit.*] and the simplifying Hypothesis 2.1.1, which implies that the connected center of Q_Φ/U_Φ is isogenous to the product of a $\mathbb{Q}$-split torus and a torus whose group of real points is compact (see the proof of [**47**, Corollary 4.10]).

Denoting by $\langle -,- \rangle : \Gamma_\Phi^\vee(1) \times \Gamma_\Phi(-1) \to \mathbb{Z}$ the tautological pairing, define an isomorphism

$$\Gamma_\Phi^\vee(1) \xrightarrow{\alpha\mapsto q_\alpha} \mathrm{Hom}(\Gamma_\Phi(-1)\otimes\mathbb{G}_m, \mathbb{G}_m) = \mathrm{Hom}(T_\Phi, \mathbb{G}_m)$$

by $q_\alpha(\beta\otimes z) = z^{\langle\alpha,\beta\rangle}$. This determines an isomorphism

$$T_\Phi \cong \mathrm{Spec}\Big(\mathbb{Q}[q_\alpha]_{\alpha\in\Gamma_\Phi^\vee(1)}\Big),$$

and hence, for any rational polyhedral cone [(1)] $\sigma \subset U_\Phi(\mathbb{R})(-1)$, a partial compactification

$$T_\Phi(\sigma) \stackrel{\mathrm{def}}{=} \mathrm{Spec}\Big(\mathbb{Q}[q_\alpha]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geq 0}}\Big). \tag{2.3.6}$$

More generally, the T_Φ-torsor structure on (2.3.1) determines, by the general theory of torus embeddings [**47**, § 5], a partial compactification

$$\begin{array}{ccc} \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) & \longrightarrow & \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) \\ \downarrow & \swarrow & \\ \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi,\bar{\mathcal{D}}_\Phi) & & \end{array} \tag{2.3.7}$$

with a stratification by locally closed substacks

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) = \bigsqcup_\tau Z^\tau_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) \tag{2.3.8}$$

indexed by the faces $\tau \subset \sigma$. The unique open stratum

$$Z^{\{0\}}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) = \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)$$

corresponds to $\tau = \{0\}$. The unique closed stratum corresponds to $\tau = \sigma$.

(1) By which we mean a *convex rational polyhedral cone* in the sense of [**47**, § 5.1]. In particular, each σ is a closed subset of the real vector space $U_\Phi(\mathbb{R})(-1)$.

2.4. Rational polyhedral cone decompositions. — Let $\Phi = (P, \mathcal{D}^\circ, h)$ be a cusp label representative for $(G, \mathcal{D})$, with associated mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$. We denote by $\mathcal{D}_\Phi^\circ = U_\Phi(\mathbb{C})\mathcal{D}^\circ$ the connected component of $\mathcal{D}_\Phi$ containing $\mathcal{D}^\circ$.

Define the *projection to the imaginary part* $c_\Phi : \mathcal{D}_\Phi \to U_\Phi(\mathbb{R})(-1)$ by

$$c_\Phi(z)^{-1} \cdot z \in \pi_0(\mathcal{D}) \times \mathrm{Hom}(\mathbb{S}, Q_{\Phi\mathbb{R}})$$

for every $z \in \mathcal{D}_\Phi$. By [**47**, Proposition 4.15] there is an open convex cone

$$C_\Phi \subset U_\Phi(\mathbb{R})(-1) \tag{2.4.1}$$

characterized by $\mathcal{D}^\circ = \{z \in \mathcal{D}_\Phi^\circ : c_\Phi(z) \in C_\Phi\}$.

Definition 2.4.1. — Suppose $\Phi = (P, \mathcal{D}^\circ, h)$ and $\Phi_1 = (P_1, \mathcal{D}_1^\circ, h_1)$ are cusp label representatives. A *K-morphism*

$$\Phi \xrightarrow{(\gamma, q)} \Phi_1 \tag{2.4.2}$$

is a pair $(\gamma, q) \in G(\mathbb{Q}) \times Q_{\Phi_1}(\mathbb{A}_f)$, such that

$$\gamma Q_\Phi \gamma^{-1} \subset Q_{\Phi_1}, \quad \gamma \mathcal{D}^\circ = \mathcal{D}_1^\circ, \quad \gamma h \in q h_1 K.$$

A K-morphism is a *K-isomorphism* if $\gamma Q_\Phi \gamma^{-1} = Q_{\Phi_1}$.

Remark 2.4.2. — The Baily-Borel compactification of $\mathrm{Sh}_K(G, \mathcal{D})$ admits a stratification by locally closed substacks, defined over the reflex field, whose strata are indexed by the K-isomorphism classes of cusp label representatives. Whenever there is a K-morphism $\Phi \to \Phi_1$, the stratum indexed by Φ is "deeper into the boundary" than the stratum indexed by Φ_1, in the sense that the Φ-stratum is contained in the closure of the Φ_1-stratum. The unique open stratum, which is just the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$, is indexed by the K-isomorphism class consisting of all cusp label representatives of the form $(G, \mathcal{D}^\circ, h)$ as $\mathcal{D}^\circ$ and h vary.

Suppose we have a K-morphism (2.4.2) of cusp label representatives. It follows from [**47**, Proposition 4.21] that $U_{\Phi_1} \subset \gamma U_\Phi \gamma^{-1}$, and the image of the open convex cone C_{Φ_1} under

$$U_{\Phi_1}(\mathbb{R})(-1) \xrightarrow{u \mapsto \gamma^{-1} u \gamma} U_\Phi(\mathbb{R})(-1) \tag{2.4.3}$$

lies in the closure of the open convex cone C_Φ. Define, as in [**47**, Definition-Proposition 4.22],

$$C_\Phi^* = \bigcup_{\Phi \to \Phi_1} \gamma^{-1} C_{\Phi_1} \gamma \subset U_\Phi(\mathbb{R})(-1),$$

where the union is over all K-morphisms with source Φ. This is a convex cone lying between C_Φ and its closure, but in general C_Φ^* is neither open nor closed. For every K-morphism $\Phi \to \Phi_1$ as above, the injection (2.4.3) identifies $C_{\Phi_1}^* \subset C_\Phi^*$.

Definition 2.4.3. — A *(rational polyhedral) partial cone decomposition* of C_Φ^* is a collection $\Sigma_\Phi = \{\sigma\}$ of rational polyhedral cones $\sigma \subset U_\Phi(\mathbb{R})(-1)$ such that

— each $\sigma \in \Sigma_\Phi$ satisfies $\sigma \subset C_\Phi^*$,

— every face of every $\sigma \in \Sigma_\Phi$ is again an element of Σ_Φ,
— the intersection of any $\sigma, \tau \in \Sigma_\Phi$ is a face of both σ and τ,
— $\{0\} \in \Sigma_\Phi$.

We say that Σ_Φ is *smooth* if it is smooth, in the sense of [**47**, § 5.2], with respect to the lattice $\Gamma_\Phi(-1) \subset U_\Phi(\mathbb{R})(-1)$. It is *complete* if

$$C_\Phi^* = \bigcup_{\sigma \in \Sigma_\Phi} \sigma.$$

Definition 2.4.4. — A *K-admissible (rational polyhedral) partial cone decomposition* $\Sigma = \{\Sigma_\Phi\}_\Phi$ for $(G, \mathcal{D})$ is a collection of partial cone decompositions Σ_Φ for C_Φ^*, one for every cusp label representative Φ, such that for any K-morphism $\Phi \to \Phi_1$, the induced inclusion $C_{\Phi_1}^* \subset C_\Phi^*$ identifies

$$\Sigma_{\Phi_1} = \{\sigma \in \Sigma_\Phi : \sigma \subset C_{\Phi_1}^*\}.$$

We say that Σ is *smooth* if every Σ_Φ is smooth, and *complete* if every Σ_Φ is complete.

Fix a K-admissible complete cone decomposition Σ of $(G, \mathcal{D})$.

Definition 2.4.5. — A *toroidal stratum representative* for $(G, \mathcal{D}, \Sigma)$ is a pair (Φ, σ) in which Φ is a cusp label representative and $\sigma \in \Sigma_\Phi$ is a rational polyhedral cone whose interior is contained in C_Φ. In other words, σ is not contained in any proper subset $C_{\Phi_1}^* \subsetneq C_\Phi^*$ determined by a K-morphism $\Phi \to \Phi_1$.

We now extend Definition 2.4.1 from cusp label representatives to toroidal stratum representatives.

Definition 2.4.6. — A *K-morphism* of toroidal stratum representatives

$$(\Phi, \sigma) \xrightarrow{(\gamma, q)} (\Phi_1, \sigma_1)$$

consists of a pair $(\gamma, q) \in G(\mathbb{Q}) \times Q_{\Phi_1}(\mathbb{A}_f)$ such that

$$\gamma Q_\Phi \gamma^{-1} \subset Q_{\Phi_1}, \quad \gamma \mathcal{D}^\circ = \mathcal{D}_1^\circ, \quad \gamma h \in q h_1 K,$$

and such that the injection (2.4.3) identifies σ_1 with a face of σ. Such a K-morphism is a *K-isomorphism* if $\gamma Q_\Phi \gamma^{-1} = Q_{\Phi_1}$ and $\gamma^{-1} \sigma_1 \gamma = \sigma$.

The set of K-isomorphism classes of toroidal stratum representatives will be denoted $\mathrm{Strat}_K(G, \mathcal{D}, \Sigma)$.

Definition 2.4.7. — We say that Σ is *finite* if $\#\mathrm{Strat}_K(G, \mathcal{D}, \Sigma) < \infty$.

Definition 2.4.8. — We say that Σ has the *no self-intersection property* if the following holds: whenever we are given toroidal stratum representatives (Φ, σ) and (Φ_1, σ_1), and two K-morphisms

$$(\Phi, \sigma) \rightrightarrows (\Phi_1, \sigma_1),$$

the two injections

$$U_{\Phi_1}(\mathbb{R})(-1) \rightrightarrows U_\Phi(\mathbb{R})(-1)$$

of (2.4.3) send σ_1 to the same face of σ.

The no self-intersection property is just a rewording of the condition of [**47**, § 7.12]. If Σ has the no self-intersection property then so does any refinement (in the sense of [**47**, § 5.1]).

Remark 2.4.9. — Any finite and K-admissible cone decomposition Σ for $(G,\mathcal{D})$ acquires the no self-intersection property after possibly replacing K by a smaller compact open subgroup [**47**, § 7.13]. Moreover, by examining the proof one can see that if K factors as $K = K_\ell K^\ell$ for some prime ℓ with $K_\ell \subset G(\mathbb{Q}_\ell)$ and $K^\ell \subset G(\mathbb{A}_f^\ell)$, then it suffices to shrink K_ℓ while holding K^ℓ fixed.

2.5. Functoriality of cone decompositions. — Suppose that we have an embedding $(G,\mathcal{D}) \to (G',\mathcal{D}')$ of Shimura data.

As explained in [**40**, (2.1.28)], every cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, g)$$

for $(G,\mathcal{D})$ determines a cusp label representative

$$\Phi' = (P', \mathcal{D}'^{,\circ}, g')$$

for $(G',\mathcal{D}')$. More precisely, we define $g' = g$, let $\mathcal{D}'^{,\circ} \subset \mathcal{D}'$ be the connected component containing $\mathcal{D}^\circ$, and let $P' \subset G'$ be the smallest admissible parabolic subgroup containing P. In particular,

$$Q_\Phi \subset Q_{\Phi'}, \quad U_\Phi \subset U_{\Phi'}, \quad C_\Phi \subset C_{\Phi'}.$$

If $K \subset G(\mathbb{A}_f)$ is a compact open subgroup contained in a compact open subgroup $K' \subset G'(\mathbb{A}_f)$, then every K-morphism

$$\Phi \xrightarrow{(\gamma,q)} \Phi_1$$

determines a K'-morphism

$$\Phi' \xrightarrow{(\gamma,q)} \Phi_1'.$$

Any K'-admissible rational cone decomposition Σ' for $(G',\mathcal{D}')$ pulls back to a K-admissible rational cone decomposition Σ for $(G,\mathcal{D})$, defined by

$$\Sigma_\Phi = \{\sigma' \cap C_\Phi^* : \sigma' \in \Sigma'_{\Phi'}\}$$

for every cusp label representative Φ of $(G,\mathcal{D})$. It is shown in [**20**, § 3.3] that Σ is finite whenever Σ' is so. It is also not hard to check that Σ has the no self-intersection property whenever Σ' does, and that it is complete when Σ' is so.

Given a cusp label representative Φ for $(G,\mathcal{D})$ and a $\sigma \in \Sigma_\Phi$, there is a unique rational polyhedral cone $\sigma' \in \Sigma'_{\Phi'}$ such that $\sigma \subset \sigma'$, but σ is not contained in any proper face of σ'. The assignment $(\Phi,\sigma) \mapsto (\Phi',\sigma')$ induces a function

$$\mathrm{Strat}_K(G,\mathcal{D},\Sigma) \to \mathrm{Strat}_{K'}(G',\mathcal{D}',\Sigma')$$

on K-isomorphism classes of toroidal stratum representatives.

2.6. Compactification of canonical models. — In this subsection we assume that $K \subset G(\mathbb{A}_f)$ is neat. Suppose Σ is a finite and K-admissible complete cone decomposition for $(G, \mathcal{D})$.

Remark 2.6.1. — A Σ with the above properties always exists, and may be refined, in the sense of [**47**, § 5.1], to make it smooth. This is the content of [**47**, Theorem 9.21].

The main result of [**47**, § 12] is the existence of a proper toroidal compactification

$$\mathrm{Sh}_K(G, \mathcal{D}) \hookrightarrow \mathrm{Sh}_K(G, \mathcal{D}, \Sigma),$$

in the category of algebraic spaces over $E(G, \mathcal{D})$, along with a stratification

$$\mathrm{Sh}_K(G, \mathcal{D}, \Sigma) = \bigsqcup_{(\Phi,\sigma) \in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)} Z_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma) \tag{2.6.1}$$

by locally closed subspaces indexed by the finite set $\mathrm{Strat}_K(G, \mathcal{D}, \Sigma)$ appearing in Definition 2.4.7. The stratum indexed by (Φ, σ) lies in the closure of the stratum indexed by (Φ_1, σ_1) if and only if there is a K-morphism of toroidal stratum representatives $(\Phi, \sigma) \to (\Phi_1, \sigma_1)$.

If Σ is smooth then so is the toroidal compacification.

After possibly shrinking K, we may assume that Σ has the no self-intersection property (see Remark 2.4.9). The no self-intersection property guarantees that the strata appearing in (2.6.1) have an especially simple shape. Fix one $(\Phi, \sigma) \in \mathrm{Strat}_K(G, \mathcal{D}, \Sigma)$ and write $\Phi = (P, \mathcal{D}^\circ, h)$. Pink shows that there is a canonical isomorphism

$$\begin{array}{ccc} Z^{\sigma}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) & \xrightarrow{\cong} & Z_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma) \\ \downarrow & & \downarrow \\ \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) & & \mathrm{Sh}_K(G, \mathcal{D}, \Sigma) \end{array} \tag{2.6.2}$$

such that the two algebraic spaces in the bottom row become isomorphic after formal completion along their common locally closed subspace in the top row. See [**47**, Corollary 7.17] and [**47**, Theorem 12.4].

In other words, if we abbreviate

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) = \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)^{\wedge}_{Z^{\sigma}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)}$$

for the formal completion of $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$ along its closed stratum, and abbreviate [(2)]

$$\widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma) = \mathrm{Sh}_K(G, \mathcal{D}, \Sigma)^{\wedge}_{Z_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)}$$

for the formal completion of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)$ along the locally closed stratum $Z_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma)$, there is an isomorphism of formal algebraic spaces

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma). \tag{2.6.3}$$

(2) In order to limit the already burdensome notation, we choose to suppress the dependence on (Φ, σ) of the left hand side. The meaning will always be clear from context.

Remark 2.6.2. — In [**47**] the isomorphism (2.6.3) is constructed after the left hand side is replace by its quotient by a finite group action. Thanks to Hypothesis 2.1.1 and the assumption that K is neat, the finite group in question is trivial. See [**48**, Lemma 1.7 and Remark 1.8].

We can make the above more explicit on the level of complex points. Suppose (Φ,σ) is a toroidal stratum representative with underlying cusp label representative $\Phi=(P,\mathcal{D}^\circ,h)$, and denote by $Q_\Phi(\mathbb{R})^\circ\subset Q_\Phi(\mathbb{R})$ the stabilizer of the connected component $\mathcal{D}^\circ\subset\mathcal{D}$. The complex manifold

$$\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)=Q_\Phi(\mathbb{Q})^\circ\backslash(\mathcal{D}^\circ\times Q_\Phi(\mathbb{A}_f)/K_\Phi)$$

sits in a diagram

$$\begin{array}{ccc} \mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) & \longrightarrow & \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}) \\ {\scriptstyle (z,g)\mapsto(z,gh)}\downarrow & & \\ \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}), & & \end{array} \tag{2.6.4}$$

in which the horizontal arrow is an open immersion, and the vertical arrow is a local isomorphism. This allows us to define a partial compactification

$$\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)\hookrightarrow\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)$$

as the interior of the closure of $\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)$ in $\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)(\mathbb{C})$.

Any K-morphism as in Definition 2.4.6 induces a morphism of complex manifolds

$$\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)\xrightarrow{(z,g)\mapsto(\gamma z,\gamma g\gamma^{-1}q)}\mathscr{U}_{K_{\Phi_1}}(Q_{\Phi_1},\mathcal{D}_{\Phi_1}),$$

which extends uniquely to

$$\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)\to\mathscr{U}_{K_{\Phi_1}}(Q_{\Phi_1},\mathcal{D}_{\Phi_1},\sigma_1).$$

Complex analytically, the toroidal compactification is defined as the quotient

$$\mathrm{Sh}_K(G,\mathcal{D},\Sigma)(\mathbb{C})=\Big(\bigsqcup_{(\Phi,\sigma)\in\mathrm{Strat}_K(G,\mathcal{D},\Sigma)}\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)\Big)\Big/\sim,$$

where $\sim$ is the equivalence relation generated by the graphs of all such morphisms.

By [**47**, § 6.13] the closed stratum appearing in (2.3.8) satisfies

$$Z^\sigma_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)(\mathbb{C})\subset\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma). \tag{2.6.5}$$

The morphisms in (2.6.4) extend continuously to morphisms

$$\begin{array}{ccc} \mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) & \longrightarrow & \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)(\mathbb{C}) \\ \downarrow & & \\ \mathrm{Sh}_K(G,\mathcal{D},\Sigma)(\mathbb{C}) & & \end{array} \tag{2.6.6}$$

in such a way that the vertical map identifies

$$Z^\sigma_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)(\mathbb{C})\cong Z^{(\Phi,\sigma)}_K(G,\mathcal{D},\Sigma)(\mathbb{C}).$$

This agrees with the analytification of the isomorphism (2.6.2).

Now pick any point $z \in Z^{\sigma}_{\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C})$. Let R be the completed local ring of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$ at z, and let R_Φ be the completed local ring of $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}}$ at z. Each completed local ring can be computed with respect to the étale or analytic topologies, and the results are canonically identified. Working in the analytic topology, the morphisms in (2.6.6) induce an isomorphism $R \cong R_\Phi$, as they identify both rings with the completed local ring of $\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$ at z. This analytic isomorphism agrees with the one induced by the algebraic isomorphism (2.6.3).

3. Automorphic vector bundles

Throughout § 3 we fix a Shimura datum $(G, \mathcal{D})$ satisfying Hypothesis 2.1.1, and a compact open subgroup $K \subset G(\mathbb{A}_f)$.

We recall the theory of automorphic vector bundles on the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$, on its toroidal compactification, and on the mixed Shimura varieties appearing along the boundary. The main reference is [**23**].

3.1. Holomorphic vector bundles. — Let $\Phi = (P, \mathcal{D}^\circ, h)$ be a cusp label representative for $(G, \mathcal{D})$. As in § 2, this determines a mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$ and a compact open subgroup $K_\Phi \subset Q_\Phi(\mathbb{A}_f)$.

Suppose we have a representation $Q_\Phi \to \mathrm{GL}(N)$ on a finite dimensional $\mathbb{Q}$-vector space. Given a point $z \in \mathcal{D}_\Phi$, its image under

$$\mathcal{D}_\Phi \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}})$$

determines a mixed Hodge structure $(N, F^\bullet N_\mathbb{C}, \mathrm{wt}_\bullet N)$. The weight filtration is independent of z, and is split by any lift $\mathbb{G}_m \to Q_\Phi$ of the weight cocharacter (2.2.2).

Denote by $(\boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}}, F^\bullet \boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}}, \mathrm{wt}_\bullet \boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}})$ the doubly filtered holomorphic vector bundle on $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi$ whose fiber at (z, g) is the vector space $N_\mathbb{C}$ endowed with the Hodge and weight filtrations determined by z. There is a natural action of $Q_\Phi(\mathbb{Q})$ on this doubly filtered vector bundle, covering the action on the base. By taking the quotient, we obtain a functor

$$N \mapsto (\boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}}, F^\bullet \boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}}, \mathrm{wt}_\bullet \boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}}) \tag{3.1.1}$$

from finite dimensional representations of Q_Φ to doubly filtered holomorphic vector bundles on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$. Ignoring the double filtration, this functor is simply

$$N \mapsto \boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}} = Q_\Phi(\mathbb{Q}) \backslash \big(\mathcal{D}_\Phi \times N_\mathbb{C} \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big). \tag{3.1.2}$$

Given a K_Φ-stable $\widehat{\mathbb{Z}}$-lattice $N_{\widehat{\mathbb{Z}}} \subset N \otimes \mathbb{A}_f$, we may define a $\mathbb{Z}$-lattice

$$gN_\mathbb{Z} = gN_{\widehat{\mathbb{Z}}} \cap N$$

for every $g \in Q_\Phi(\mathbb{A}_f)$, along with a weight filtration

$$\mathrm{wt}_\bullet(gN_\mathbb{Z}) = gN_{\widehat{\mathbb{Z}}} \cap \mathrm{wt}_\bullet N.$$

Denote by $(\boldsymbol{N}_{\mathrm{Be}}, \mathrm{wt}_\bullet \boldsymbol{N}_{\mathrm{Be}})$ the filtered $\mathbb{Z}$-local system on $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi$ whose fiber at (z,g) is $(gN_{\mathbb{Z}}, \mathrm{wt}_\bullet(gN_{\mathbb{Z}}))$. This local system has an obvious action of $Q_\Phi(\mathbb{Q})$, covering the action on the base. Passing to the quotient, we obtain a functor

$$N_{\widehat{\mathbb{Z}}} \mapsto (\boldsymbol{N}_{\mathrm{Be}}, \mathrm{wt}_\bullet \boldsymbol{N}_{\mathrm{Be}})$$

from K_Φ-stable $\widehat{\mathbb{Z}}$-lattices in $N \otimes \mathbb{A}_f$ to filtered $\mathbb{Z}$-local systems on (2.2.4).

By construction there is a canonical isomorphism

$$(\boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}}, \mathrm{wt}_\bullet \boldsymbol{N}^{\mathrm{an}}_{\mathrm{dR}}) \cong (\boldsymbol{N}_{\mathrm{Be}} \otimes \mathcal{O}^{\mathrm{an}}, \mathrm{wt}_\bullet \boldsymbol{N}_{\mathrm{Be}} \otimes \mathcal{O}^{\mathrm{an}}), \tag{3.1.3}$$

where $\mathcal{O}^{\mathrm{an}}$ denotes the structure sheaf on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$.

3.2. The Borel morphism. — Suppose $G \to \mathrm{GL}(N)$ is any faithful representation of G on a finite dimensional $\mathbb{Q}$-vector space. A point $z \in \mathcal{D}$ determines a Hodge structure $\mathbb{S} \to \mathrm{GL}(N_{\mathbb{R}})$ on N, and we denote by $F^\bullet N_{\mathbb{C}}$ the induced Hodge filtration. As in [**43**, § III.1] and [**23**, § 1], define the *compact dual*

$$\check{M}(G, \mathcal{D})(\mathbb{C}) = \left\{ \begin{array}{c} \text{descending filtrations on } N_{\mathbb{C}} \\ \text{that are } G(\mathbb{C})\text{-conjugate to } F^\bullet N_{\mathbb{C}} \end{array} \right\}. \tag{3.2.1}$$

By construction, there is a canonical $G(\mathbb{R})$-equivariant finite-to-one *Borel morphism*

$$\mathcal{D} \to \check{M}(G, \mathcal{D})(\mathbb{C})$$

sending a point of $\mathcal{D}$ to the induced Hodge filtration on $N_{\mathbb{C}}$. The compact dual is the space of complex points of a smooth projective variety $\check{M}(G, \mathcal{D})$ defined over the reflex field $E(G, \mathcal{D})$, and admitting an action of $G_{E(G,\mathcal{D})}$ inducing the natural action of $G(\mathbb{C})$ on complex points. It is independent of the choice of z, and of the choice of faithful representation N.

More generally, there is an analogue of (3.2.1) for the mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$, as in [**26**, Main Theorem 3.4.1] and [**27**, Main Theorem 2.5.12]. Let $Q_\Phi \to \mathrm{GL}(N)$ be a faithful representation on a finite dimensional $\mathbb{Q}$-vector space. Any point $z \in \mathcal{D}_\Phi$ then determines a mixed Hodge structure $(N, F^\bullet N_{\mathbb{C}}, \mathrm{wt}_\bullet N)$, and we define the *dual* of $(Q_\Phi, \mathcal{D}_\Phi)$ by

$$\check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) = \left\{ \begin{array}{c} \text{descending filtrations on } N_{\mathbb{C}} \\ \text{that are } Q_\Phi(\mathbb{C})\text{-conjugate to } F^\bullet N_{\mathbb{C}} \end{array} \right\}.$$

It is the space of complex points of an open $Q_{\Phi, E(G,\mathcal{D})}$-orbit

$$\check{M}(Q_\Phi, \mathcal{D}_\Phi) \subset \check{M}(G, \mathcal{D}),$$

independent of the choice of $z \in \mathcal{D}_\Phi$ and N. By construction, there is a $Q_\Phi(\mathbb{C})$-equivariant *Borel morphism*

$$\mathcal{D}_\Phi \to \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}).$$

3.3. The standard torsor. — We want to give a more algebraic interpretation of the functor (3.1.1).

Harris and Zucker [**23**, § 1] prove that the mixed Shimura variety (2.2.4) carries a *standard torsor*(3). This consists of a diagram of $E(G,\mathcal{D})$-stacks

$$
\begin{array}{ccc}
J_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) & \xrightarrow{b} & \check{M}(Q_\Phi,\mathcal{D}_\Phi) \\
\downarrow{\scriptstyle a} & & \\
\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi), & &
\end{array}
\tag{3.3.1}
$$

in which a is a relative Q_Φ-torsor, and b is Q_Φ-equivariant. See also the papers of Harris [**24, 18, 19**], Harris-Zucker [**21, 22**], and Milne [**42, 43**]. Complex analytically, the standard torsor is the complex orbifold

$$J_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}) = Q_\Phi(\mathbb{Q})\backslash\big(\mathcal{D}_\Phi \times Q_\Phi(\mathbb{C}) \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big),$$

with $Q_\Phi(\mathbb{C})$ acting by $s\cdot(z,t,g) = (z, ts^{-1}, g)$. The morphisms a and b are, respectively,

$$(z,t,g) \mapsto (z,g) \quad \text{and} \quad (z,t,g) \mapsto t^{-1}z.$$

Exactly as in [**23**], we can use the standard torsor to define models of the vector bundles (3.1.1) over the reflex field. First, we require a lemma.

Lemma 3.3.1. — *Suppose $\check{N} \to \check{M}(Q_\Phi,\mathcal{D}_\Phi)$ is a Q_Φ-equivariant vector bundle; that is, a finite rank vector bundle endowed with an action of $Q_{\Phi,E(G,\mathcal{D})}$ covering the action on the base. There are canonical Q_Φ-equivariant filtrations $\mathrm{wt}_\bullet \check{N}$ and $F^\bullet \check{N}$ on $\check{N}$, and the construction*

$$\check{N} \mapsto (\check{N}, F^\bullet\check{N}, \mathrm{wt}_\bullet\check{N})$$

is functorial in $\check{N}$.

Proof. — Fix a faithful representation $Q_\Phi \to \mathrm{GL}(H)$. Suppose we are given an étale neighborhood $U \to \check{M}(Q_\Phi,\mathcal{D}_\Phi)$ of some geometric point x of $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$. By the very definition of $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$, U determines a $Q_{\Phi U}$-stable filtration $F^\bullet H_U$ on $H_U = H \otimes \mathcal{O}_U$. After possibly shrinking U we may choose a cocharacter $\mu_x : \mathbb{G}_m \to Q_{\Phi U}$ splitting this filtration.

As $Q_{\Phi U}$ acts on $\check{N}_U$, the cocharacter μ_x determines a filtration $F^\bullet\check{N}_U$, which does not depend on the choice of splitting. Gluing over an étale cover of $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$ defines the desired filtration $F^\bullet\check{N}$. The definition of $\mathrm{wt}_\bullet\check{N}$ is similar, but easier: it is the filtration split by any lift $\mathbb{G}_m \to Q_\Phi$ of the weight cocharacter (2.2.2). □

Now suppose we have a representation $Q_\Phi \to \mathrm{GL}(N)$ on a finite dimensional $\mathbb{Q}$-vector space. Applying Lemma 3.3.1 to the constant Q_Φ-equivariant vector bundle

$$\check{N} = \check{M}(Q_\Phi,\mathcal{D}_\Phi) \times_{\mathrm{Spec}(E(G,\mathcal{D}))} N_{E(G,\mathcal{D})}$$

(3) A.k.a. *standard principal bundle.*

yields a Q_Φ-equivariant doubly filtered vector bundle $(\check{N}, F^\bullet\check{N}, \mathrm{wt}_\bullet\check{N})$ on $\check{M}(Q_\Phi, \mathcal{D}_\Phi)$. The construction

$$N \mapsto (\boldsymbol{N}_{\mathrm{dR}}, F^\bullet\boldsymbol{N}_{\mathrm{dR}}, \mathrm{wt}_\bullet\boldsymbol{N}_{\mathrm{dR}}) = Q_\Phi\backslash b^*(\check{N}, F^\bullet\check{N}, \mathrm{wt}_\bullet\check{N}) \tag{3.3.2}$$

defines a functor from representations of Q_Φ to doubly filtered vector bundles on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$. Passing to the complex fiber recovers the functor (3.1.1).

The following proposition extends the above functor to partial compactifications.

Proposition 3.3.2. — *For any rational polyhedral cone $\sigma \subset U_\Phi(\mathbb{R})(-1)$ there is a functor*

$$N \mapsto (\boldsymbol{N}_{\mathrm{dR}}, F^\bullet\boldsymbol{N}_{\mathrm{dR}}, \mathrm{wt}_\bullet\boldsymbol{N}_{\mathrm{dR}}),$$

extending (3.3.2)*, from representations of Q_Φ on finite dimensional $\mathbb{Q}$-vector spaces to doubly filtered vector bundles on* $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$*.*

Proof. — This is part of [**23**, Definition-Proposition 1.3.5]. Here we sketch a different argument.

Recall the T_Φ-torsor structure on (2.3.1). On complex points, this action was deduced from the natural left action of $U_\Phi(\mathbb{C})$ on $\mathcal{D}_\Phi$. Of course the group $U_\Phi(\mathbb{C})$ also acts on both factors of $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{C})$ on the left, and imitating the proof of Proposition 2.3.1 yields action of the relative torus $T_\Phi(\mathbb{C})$ on the standard torsor $J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$, covering the action on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$.

To see that the action is algebraic and defined over the reflex field, one can reduce, exactly as in the proof of [**23**, Proposition 1.2.4], to the case in which $(Q_\Phi, \mathcal{D}_\Phi)$ is either a pure Shimura datum, or is a mixed Shimura datum associated with a Siegel Shimura datum. The pure case is vacuous (the relative torus is trivial). The Siegel mixed Shimura varieties are moduli spaces of polarized 1-motives, and it is not difficult to give a moduli-theoretic interpretation of the torus action, along the lines of [**40**, § 2.2.8]. From this interpretation the descent to the reflex field is obvious.

In the diagram (3.3.1), the arrow a is T_Φ-equivariant, and the arrow b is constant on T_Φ-orbits. This is clear from the complex analytic description.

Taking the quotient of the standard torsor by this action, we obtain a diagram

$$\begin{array}{ccc} T_\Phi\backslash J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) & \xrightarrow{\quad b \quad} & \check{M}(Q_\Phi, \mathcal{D}_\Phi) \\ \downarrow a & & \\ \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi), & & \end{array}$$

in which a is a relative Q_Φ-torsor and b is Q_Φ-equivariant. Pulling back the quotient $T_\Phi\backslash J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ along the diagonal arrow in (2.3.7) defines the upper left entry in the diagram

$$\begin{array}{ccc} J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) & \xrightarrow{\quad b \quad} & \check{M}(Q_\Phi, \mathcal{D}_\Phi) \\ \Big\downarrow{\scriptstyle a} & & \\ \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) & & \end{array}$$

extending (3.3.1), in which a is a Q_Φ-torsor, and b is Q_Φ-equivariant. Now simply repeat the construction (3.3.2) to obtain the desired functor. □

Remark 3.3.3. — The proof actually shows more: because the standard torsor admits a canonical descent to $\mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)$, the same is true of all doubly filtered vector bundles (3.3.2). Compare with [**23**, (1.2.11)].

3.4. Automorphic vector bundles on toroidal compactifications. — Assume that K is neat, and that Σ is a finite K-admissible complete cone decomposition for $(G, \mathcal{D})$ having the no self-intersection property.

By results of Harris and Harris-Zucker, see especially [**23**], one can glue together the diagrams in the proof of Proposition 3.3.2 as (Φ, σ) varies in order to obtain a diagram

$$\begin{array}{ccc} J_K(G, \mathcal{D}, \Sigma) & \xrightarrow{\ b\ } & \check{M}(G, \mathcal{D}) \\ \Big\downarrow{\scriptstyle a} & & \\ \mathrm{Sh}_K(G, \mathcal{D}, \Sigma) & & \end{array} \tag{3.4.1}$$

in which a is a G-torsor and b is G-equivariant. This implies the following:

Theorem 3.4.1. — *There is a functor $N \mapsto (\boldsymbol{N}_{\mathrm{dR}}, F^\bullet \boldsymbol{N}_{\mathrm{dR}})$ from representations of G on finite dimensional $\mathbb{Q}$-vector spaces to filtered vector bundles on $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)$, compatible, in the obvious sense, with the isomorphism*

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)$$

of (2.6.3) *and the functor of Proposition 3.3.2, for every toroidal stratum representative*

$$(\Phi, \sigma) \in \mathrm{Strat}_K(G, \mathcal{D}, \Sigma).$$

In other words, there is an arithmetic theory of automorphic vector bundles on toroidal compactifications.

Remark 3.4.2. — Over the open Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$ there is also a weight filtration $\mathrm{wt}_\bullet \boldsymbol{N}_{\mathrm{dR}}$ on $\boldsymbol{N}_{\mathrm{dR}}$, but it is not compatible with the weight filtrations along the boundary. It is also not very interesting. On an irreducible representation N the (central) weight cocharacter $w : \mathbb{G}_m \to G$ acts through $z \mapsto z^k$ for some k, and the weight filtration has a unique nonzero graded piece $\mathrm{gr}_k \boldsymbol{N}_{\mathrm{dR}}$.

3.5. A simple Shimura variety. — Let $(\mathbb{G}_m, \mathcal{H}_0)$ be the Shimura datum of Remark 2.1.2. For any compact open subgroup $K \subset \mathbb{A}_f^\times$, we obtain a 0-dimensional Shimura variety

$$\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \backslash (\mathcal{H}_0 \times \mathbb{A}_f^\times / K), \tag{3.5.1}$$

with a canonical model $\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)$ over $\mathbb{Q}$.

The action of $\mathrm{Aut}(\mathbb{C})$ on its complex points satisfies

$$\tau \cdot (2\pi\epsilon, a) = (2\pi\epsilon, aa_\tau) \tag{3.5.2}$$

whenever $\tau \in \mathrm{Aut}(\mathbb{C})$ and $a_\tau \in \mathbb{A}_f^\times$ are related by $\tau|_{\mathbb{Q}^{\mathrm{ab}}} = \mathrm{rec}(a_\tau)$. This implies that

$$\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0) \cong \mathrm{Spec}(F),$$

where $F/\mathbb{Q}$ is the abelian extension characterized by

$$\mathrm{rec} : \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K \cong \mathrm{Gal}(F/\mathbb{Q}).$$

The following proposition shows that all automorphic vector bundles on (3.5.1) are canonically trivial. The particular trivializations will be essential in our later discussion of q-expansions. See especially Proposition 4.6.1.

Proposition 3.5.1. — *For any representation $\mathbb{G}_m \to \mathrm{GL}(N)$ there is a canonical isomorphism*

$$N \otimes \mathcal{O}_{\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)} \xrightarrow{n \otimes 1 \mapsto \boldsymbol{n}} \boldsymbol{N}_{\mathrm{dR}}$$

of vector bundles. If $\mathbb{G}_m$ acts on N through the character $z \mapsto z^k$, the global section $\boldsymbol{n} = n \otimes 1$ is given, in terms of the complex parametrization

$$\boldsymbol{N}_{\mathrm{dR}}^{\mathrm{an}} = \mathbb{Q}^\times \backslash (\mathcal{H}_0 \times N_{\mathbb{C}} \times \mathbb{A}_f^\times / K)$$

of (3.1.2), *by*

$$(2\pi\epsilon, a) \mapsto \left(2\pi\epsilon, \frac{\mathrm{rat}(a)^k}{(2\pi\epsilon)^k} \cdot n, a \right).$$

Proof. — First set $N = \mathbb{Q}$ with $\mathbb{G}_m$ acting via the identity character $z \mapsto z$, and set $N_{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Z}}$. Recalling (3.1.3), the quotient $\boldsymbol{N}_{\mathrm{Be}} \backslash \boldsymbol{N}_{\mathrm{dR}}^{\mathrm{an}}$ defines an analytic family of rank one tori over $\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$, whose relative Lie algebra is the line bundle

$$\mathrm{Lie}(\boldsymbol{N}_{\mathrm{Be}} \backslash \boldsymbol{N}_{\mathrm{dR}}^{\mathrm{an}}) = \boldsymbol{N}_{\mathrm{dR}}^{\mathrm{an}} = \mathbb{Q}^\times \backslash (\mathcal{H}_0 \times \mathbb{C} \times \mathbb{A}_f^\times / K).$$

Using this identification, we may identify the standard $\mathbb{C}^\times$-torsor

$$J_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \backslash \big(\mathcal{H}_0 \times \mathbb{C}^\times \times \mathbb{A}_f^\times / K\big) \tag{3.5.3}$$

with the $\mathbb{C}^\times$-torsor of trivializations of $\mathrm{Lie}(\boldsymbol{N}_{\mathrm{Be}} \backslash \boldsymbol{N}_{\mathrm{dR}}^{\mathrm{an}})$.

On the other hand, the isomorphisms

$$(N \cap aN_{\widehat{\mathbb{Z}}}) \backslash N_{\mathbb{C}} = (\mathbb{Q} \cap a\widehat{\mathbb{Z}}) \backslash \mathbb{C} \xrightarrow{2\pi\epsilon/\mathrm{rat}(a)} \mathbb{Z}(1) \backslash \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$$

identify $\boldsymbol{N}_{\mathrm{Be}}\backslash\boldsymbol{N}_{\mathrm{dR}}^{\mathrm{an}}$, fiber-by-fiber, with the constant torus $\mathbb{C}^\times$, and so identify (3.5.3) with the $\mathbb{C}^\times$-torsor of trivializations of $\mathrm{Lie}(\mathbb{C}^\times)$. The canonical model of (3.5.3) is now concretely realized as the $\mathbb{G}_m$-torsor

$$J_K(\mathbb{G}_m,\mathcal{H}_0)=\underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m),\mathcal{O}_{\mathrm{Sh}_K(\mathbb{G}_m,\mathcal{H}_0)}\big).$$

For any ring R, the Lie algebra of $\mathbb{G}_m=\mathrm{Spec}(R[q,q^{-1}])$ is canonically trivialized by the invariant derivation $q\cdot d/dq$. Thus the standard torsor admits a canonical section which, in terms of the uniformization (3.5.3), is

$$(2\pi\epsilon,a)\mapsto\left(2\pi\epsilon,\frac{\mathrm{rat}(a)}{2\pi\epsilon},a\right).$$

This section trivializes the standard torsor, and induces the desired trivialization of any automorphic vector bundle. □

Remark 3.5.2. — Let $\mathbb{G}_m$ act on N via $z\mapsto z^k$. What the above proof actually shows is that there are canonical isomorphisms

$$N\otimes\mathcal{O}_{\mathrm{Sh}_K(\mathbb{G}_m,\mathcal{H}_0)}\cong N\otimes\mathrm{Lie}(\mathbb{G}_m)^{\otimes k}\cong\boldsymbol{N}_{\mathrm{dR}}.$$

4. Orthogonal Shimura varieties

In § 4 we specialize the preceding theory to the case of Shimura varieties associated to the group of spinor similitudes of a quadratic space (V,Q) over $\mathbb{Q}$ of signature $(n,2)$ with $n\geq 1$. This will allow us to define q-expansions of modular forms on such Shimura varieties, and prove the q-expansion principle of Proposition 4.6.3.

4.1. The GSpin Shimura variety. — Let $G=\mathrm{GSpin}(V)$ as in [**39**]. This is a reductive group over $\mathbb{Q}$ sitting in an exact sequence

$$1\to\mathbb{G}_m\to G\to\mathrm{SO}(V)\to 1.$$

There is a distinguished character $\nu:G\to\mathbb{G}_m$, called the *spinor similitude*. Its kernel is the usual spin double cover of $\mathrm{SO}(V)$, and its restriction to $\mathbb{G}_m$ is $z\mapsto z^2$.

The group $G(\mathbb{R})$ acts on the hermitian domain

$$\mathcal{D}=\big\{z\in V_{\mathbb{C}}:[z,z]=0\text{ and }[z,\overline{z}]<0\big\}/\mathbb{C}^\times\subset\mathbb{P}(V_{\mathbb{C}}) \tag{4.1.1}$$

in the obvious way. This hermitian domain has two connected components, interchanged by the action of any $\gamma\in G(\mathbb{R})$ with $\nu(\gamma)<0$. The pair $(G,\mathcal{D})$ is the *GSpin Shimura datum.* Its reflex field is $\mathbb{Q}$.

By construction, G is a subgroup of the multiplicative group of the Clifford algebra $C(V)$. As such, G has two distinguished representations. One is the standard representation $G\to\mathrm{SO}(V)$, and the other is the faithful action on $H=C(V)$ defined by left multiplication in the Clifford algebra. These two representations are related by a G-equivariant injection

$$V\to\mathrm{End}_{\mathbb{Q}}(H) \tag{4.1.2}$$

defined by the left multiplication action of $V \subset C(V)$ on H. A point $z \in \mathcal{D}$ determines a Hodge structure on any representation of G. For the representations V and H, the induced Hodge filtrations are

$$F^2 V_{\mathbb{C}} = 0, \quad F^1 V_{\mathbb{C}} = \mathbb{C}z, \quad F^0 V_{\mathbb{C}} = (\mathbb{C}z)^{\perp}, \quad F^{-1} V_{\mathbb{C}} = V_{\mathbb{C}}, \tag{4.1.3}$$

and

$$F^1 H_{\mathbb{C}} = 0, \quad F^0 H_{\mathbb{C}} = zH_{\mathbb{C}}, \quad F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}. \tag{4.1.4}$$

Here we are using (4.1.2) to view $\mathbb{C}z \subset \mathrm{End}_{\mathbb{C}}(H_{\mathbb{C}})$.

In order to obtain a Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$ as in 1.1, we fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ on which Q is $\mathbb{Z}$-valued and assume that the compact open subgroup $K \subset G(\mathbb{A}_f)$ is chosen as in (1.1.2). According to [**39**, Lemma 2.6], any such K stabilizes both $V_{\widehat{\mathbb{Z}}}$ and its dual, and acts trivially on the discriminant group

$$V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \cong V_{\widehat{\mathbb{Z}}}^{\vee}/V_{\widehat{\mathbb{Z}}}. \tag{4.1.5}$$

4.2. The line bundle of modular forms. — Applying the functor of Proposition 3.3.2 to the standard representation $G \to \mathrm{SO}(V)$ yields a filtered vector bundle $(\boldsymbol{V}_{\mathrm{dR}}, F^{\bullet}\boldsymbol{V}_{\mathrm{dR}})$ on $\mathrm{Sh}_K(G, \mathcal{D})$. The filtration has the form

$$0 = F^2 \boldsymbol{V}_{\mathrm{dR}} \subset F^1 \boldsymbol{V}_{\mathrm{dR}} \subset F^0 \boldsymbol{V}_{\mathrm{dR}} \subset F^{-1} \boldsymbol{V}_{\mathrm{dR}} = \boldsymbol{V}_{\mathrm{dR}},$$

in which $F^1 \boldsymbol{V}_{\mathrm{dR}}$ is a line, isotropic with respect to the bilinear form

$$[-,-] : \boldsymbol{V}_{\mathrm{dR}} \otimes \boldsymbol{V}_{\mathrm{dR}} \to \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})} \tag{4.2.1}$$

induced by (1.1.1), and $F^0 \boldsymbol{V}_{\mathrm{dR}} = (F^1 \boldsymbol{V}_{\mathrm{dR}})^{\perp}$. These properties are clear from the complex analytic definition (3.1.1) of $\boldsymbol{V}_{\mathrm{dR}}^{\mathrm{an}}$, and the explicit description of the Hodge filtration (4.1.3). In particular, the filtration on $\boldsymbol{V}_{\mathrm{dR}}$ is completely determined by the isotropic line $F^1 \boldsymbol{V}_{\mathrm{dR}}$.

Definition 4.2.1. — The *line bundle of weight one modular forms* on $\mathrm{Sh}_K(G, \mathcal{D})$ is defined by

$$\boldsymbol{\omega} = F^1 \boldsymbol{V}_{\mathrm{dR}}.$$

For any $g \in G(\mathbb{A}_f)$, the pullback of $\boldsymbol{\omega}$ via the complex uniformization

$$\mathcal{D} \xrightarrow{z \mapsto (z,g)} \mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$$

is just the tautological bundle on the hermitian domain (4.1.1). In particular, the line bundle $\boldsymbol{\omega}$ carries a metric, inherited from the metric

$$\|z\|^2_{\mathrm{naive}} = -[z, \overline{z}] \tag{4.2.2}$$

on the tautological bundle. We will more often use the rescaled metric

$$\|z\|^2 = -\frac{[z, \overline{z}]}{4\pi e^{\gamma}} \tag{4.2.3}$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.

4.3. The Hodge embedding. — As above, let $H = C(V)$ viewed as a faithful 2^{n+2}-dimensional representation of $G \subset C(V)^\times$ via left multiplication. If we define a $\mathbb{Z}$-lattice in H by

$$H_{\mathbb{Z}} = C(V_{\mathbb{Z}}),$$

the inclusion (1.1.2) implies that $H_{\widehat{\mathbb{Z}}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is K-stable.

The discussion of § 3 provides a filtered vector bundle $(\boldsymbol{H}_{\mathrm{dR}}, F^\bullet \boldsymbol{H}_{\mathrm{dR}})$ on $\mathrm{Sh}_K(G, \mathcal{D})$, and a $\mathbb{Z}$-local system $\boldsymbol{H}_{\mathrm{Be}}$ over the complex fiber endowed with an isomorphism

$$\boldsymbol{H}_{\mathrm{Be}} \otimes \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})} \cong \boldsymbol{H}^{\mathrm{an}}_{\mathrm{dR}}.$$

The double quotient

$$A(\mathbb{C}) = \boldsymbol{H}_{\mathrm{Be}} \backslash \boldsymbol{H}^{\mathrm{an}}_{\mathrm{dR}} / F^0 \boldsymbol{H}^{\mathrm{an}}_{\mathrm{dR}} \tag{4.3.1}$$

defines an analytic family of complex tori over $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$. In fact, this arises from an abelian scheme over $\mathrm{Sh}_K(G, \mathcal{D})$, as we now explain.

As in [**1**, § 2.2], one may choose a symplectic form ψ on H such that the representation of G factors through $G^{\mathrm{Sg}} = \mathrm{GSp}(H)$, and induces a Hodge embedding

$$(G, \mathcal{D}) \to (G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})$$

into the Siegel Shimura datum determined by (H, ψ). Explicitly, choose any vectors $v, w \in V$ of negative length with $[v, w] = 0$ and set

$$\delta = vw \in C(V).$$

If we denote by $c \mapsto c^*$ the $\mathbb{Q}$-algebra involution on $C(V)$ fixing pointwise the subset $V \subset C(V)$, then $\delta^* = -\delta$. Denoting by $\mathrm{Trd} : C(V) \to \mathbb{Q}$ the reduced trace, the symplectic form

$$\psi(x, y) = \mathrm{Trd}(x \delta y^*)$$

has the desired properties.

As in (3.2.1), we may describe the compact dual $\check{M}(G, \mathcal{D})$ as a G-orbit of descending filtrations on the faithful representation H. It is more convenient to characterize the compact dual as the $\mathbb{Q}$-scheme with functor of points

$$\check{M}(G, \mathcal{D})(S) = \{\text{isotropic lines } z \subset V \otimes \mathcal{O}_S\},$$

where *line* means a locally free $\mathcal{O}_S$-module direct summand of rank one. In order to realize $\check{M}(G, \mathcal{D})$ as a space of filtrations on H, first define

$$\check{M}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})(S) = \{\text{Lagrangian subsheaves } F^0 \subset H \otimes \mathcal{O}_S\}$$

and then use (4.1.2) to define a closed immersion

$$\check{M}(G, \mathcal{D}) \to \check{M}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}),$$

sending the isotropic line $z \subset V$ to the Lagrangian $zH \subset H$.

By rescaling, we may assume that ψ is $\mathbb{Z}$-valued on $H_{\mathbb{Z}}$, and so the Hodge embedding defines a morphism from $\mathrm{Sh}_K(G, \mathcal{D})$ to a moduli stack of polarized abelian varieties of dimension 2^{n+1}. Pulling back the universal object defines the *Kuga-Satake abelian scheme*

$$\pi : A \to \mathrm{Sh}_K(G, \mathcal{D}).$$

The Kuga-Satake abelian scheme does not depend on the choice of ψ, but the polarization on it does. Passing to the complex analytic fiber recovers the family of complex tori defined by (4.3.1).

The first relative de Rham homology of A, with its Hodge filtration, is related to the vector bundle $\boldsymbol{H}_{\mathrm{dR}}$ by a canonical isomorphism of filtered vector bundles

$$\boldsymbol{H}_{\mathrm{dR}} \cong \underline{\mathrm{Hom}}\big(R^1\pi_*\Omega^\bullet_{A/\mathrm{Sh}_K(G,\mathcal{D})}, \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})}\big).$$

4.4. Cusp label representatives: isotropic lines. — We wish to make more explicit the structure of the mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$ associated to a cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h)$$

for $(G, \mathcal{D})$. See § 2.2 for the definitions.

The admissible parabolic $P \subset G$ is the stabilizer of a totally isotropic subspace $I \subset V$ with $\dim(I) \in \{0, 1, 2\}$. In this subsection we assume that $P \subset G$ is the stabilizer of an isotropic line $I \subset V$. The case of isotropic planes will be considered in § 4.5.

The P-stable weight filtration on V defined by

$$\mathrm{wt}_{-3}V = 0, \quad \mathrm{wt}_{-2}V = \mathrm{wt}_{-1}V = I, \quad \mathrm{wt}_0 V = \mathrm{wt}_1 V = I^\perp, \quad \mathrm{wt}_2 V = V,$$

and the Hodge filtration (4.1.3) determined by a point $z \in \mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_\mathbb{C} \xrightarrow{\mathtt{h}_\Phi(z)} P_\mathbb{C} \to \mathrm{SO}(V_\mathbb{C})$$

on V of type $\{(-1,-1), (0,0), (1,1)\}$.

Similarly,the P-stable weight filtration on H defined by

$$\mathrm{wt}_{-3}H = 0, \quad \mathrm{wt}_{-2}H = \mathrm{wt}_{-1}H = IH, \quad \mathrm{wt}_0 H = H,$$

and the Hodge filtration (4.1.4) determined by a point $z \in \mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_\mathbb{C} \xrightarrow{\mathtt{h}_\Phi(z)} P_\mathbb{C} \to \mathrm{GSp}(H_\mathbb{C})$$

on H of type $\{(-1,-1), (0,0)\}$. In the definition of the weight filtration we are using the inclusion $I \subset \mathrm{End}_\mathbb{Q}(H)$ determined by (4.1.2), and setting

$$IH = \mathrm{Span}_\mathbb{Q}\{\ell x : \ell \in I,\, x \in H\}.$$

The proof of the following lemma is left as an exercise to the reader.

Lemma 4.4.1. — *Recalling the notation* (1.6.1)*, the largest closed normal subgroup $Q_\Phi \subset P$ through which every such $\mathtt{h}_\Phi(z)$ factors is*

$$Q_\Phi = \ker\big(P \to \mathrm{GL}(\mathrm{gr}_0(H))\big).$$

The action $Q_\Phi \to \mathrm{SO}(V)$ is faithful, and is given on the graded pieces of $\mathrm{wt}_\bullet V$ by the commutative diagram

$$\begin{array}{ccc} Q_\Phi & \xrightarrow{\nu_\Phi} & \mathbb{G}_m \\ \downarrow & & \downarrow{\scriptstyle t\mapsto(t,1,t^{-1})} \\ P & \longrightarrow & \mathrm{GL}(I)\times\mathrm{SO}(I^\perp/I)\times\mathrm{GL}(V/I^\perp), \end{array} \tag{4.4.1}$$

in which ν_Φ is the restriction to Q_Φ of the spinor similitude on G. This agrees with the character (2.2.3). The groups U_Φ and W_Φ are

$$U_\Phi = W_\Phi = \ker(\nu_\Phi : Q_\Phi \to \mathbb{G}_m),$$

and there is an isomorphism of $\mathbb{Q}$-vector spaces

$$(I^\perp/I)\otimes I \cong U_\Phi(\mathbb{Q}) \tag{4.4.2}$$

sending $v\otimes\ell \in (I^\perp/I)\otimes I$ to the unipotent transformation of V defined by

$$x \mapsto x + [x,\ell]v - [x,v]\ell - Q(v)[x,\ell]\ell.$$

The dual of $(Q_\Phi,\mathcal{D}_\Phi)$ is the $\mathbb{Q}$-scheme with functor of points

$$\check{M}(Q_\Phi,\mathcal{D}_\Phi)(S) = \left\{ \begin{array}{c} \text{isotropic lines } z\subset V\otimes\mathcal{O}_S \text{ such that} \\ V\to V/I^\perp \\ \text{identifies } z\cong (V/I^\perp)\otimes\mathcal{O}_S \end{array} \right\}. \tag{4.4.3}$$

Every point

$$z\in\mathcal{D}_\Phi\subset\pi_0(\mathcal{D})\times\mathrm{Hom}(\mathbb{S}_\mathbb{C},Q_{\Phi\mathbb{C}})$$

determines a mixed Hodge structure on V of type $\{(-1,-1),(0,0),(1,1)\}$, and the Borel morphism

$$\mathcal{D}_\Phi \to \check{M}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C})$$

sends z to the isotropic line $F^1V_\mathbb{C}\subset V_\mathbb{C}$. This induces an isomorphism

$$\mathcal{D}_\Phi \cong \pi_0(\mathcal{D})\times\check{M}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}). \tag{4.4.4}$$

4.5. Cusp label representatives: isotropic planes. — In this subsection we fix a cusp label representative $\Phi=(P,\mathcal{D}^\circ,h)$ with $P\subset G$ the stabilizer of an isotropic plane $I\subset V$.

The P-stable weight filtrations on V defined by

$$\mathrm{wt}_{-2}V=0,\quad \mathrm{wt}_{-1}V=I,\quad \mathrm{wt}_0V=I^\perp,\quad \mathrm{wt}_1V=V,$$

and the Hodge filtration (4.1.3) determined by a point $z\in\mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_\mathbb{C}\xrightarrow{\mathrm{h}_\Phi(z)} P_\mathbb{C}\to\mathrm{SO}(V_\mathbb{C})$$

on V of type $\{(-1,0),(0,-1),(0,0),(1,0),(0,1)\}$.

Similarly, the P-stable weight filtration on H defined by

$$\mathrm{wt}_{-3}H=0,\quad \mathrm{wt}_{-2}H=I^2H,\quad \mathrm{wt}_{-1}H=IH,\quad \mathrm{wt}_0H=H,$$

and the Hodge filtration (4.1.4) determined by a point $z \in \mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_{\mathbb{C}} \xrightarrow{\mathtt{h}_{\Phi}(z)} P_{\mathbb{C}} \to \mathrm{GSp}(H_{\mathbb{C}})$$

on H of type $\{(-1,-1),(-1,0),(0,-1),(0,0)\}$. In the definition of the weight filtration we are using the inclusion $I \subset \mathrm{End}_{\mathbb{Q}}(H)$ determined by (4.1.2), and setting

$$\begin{aligned} IH &= \mathrm{Span}_{\mathbb{Q}}\{\ell x : \ell \in I,\, x \in H\} \\ I^2 H &= \mathrm{Span}_{\mathbb{Q}}\{\ell\ell' x : \ell, \ell' \in I,\, x \in H\}. \end{aligned}$$

The proof of the following lemma is left as an exercise to the reader.

Lemma 4.5.1. — *Recalling the notation* (1.6.1), *the largest closed normal subgroup $Q_\Phi \subset P$ through which every such $\mathtt{h}_\Phi(z)$ factors is*

$$Q_\Phi = \ker\big(P \to \mathrm{GL}(\mathrm{gr}_0(H))\big).$$

The natural action $Q_\Phi \to \mathrm{SO}(V)$ is faithful, and is trivial on the quotient $I^\perp/I$. The groups $U_\Phi \lhd W_\Phi \lhd Q_\Phi$ are

$$W_\Phi = \ker(Q_\Phi \to \mathrm{GL}(I)),$$

and

$$U_\Phi \cong \bigwedge^2 I,$$

where we identify $a \wedge b \in \bigwedge^2 I$ with the unipotent transformation of V defined by

$$x \mapsto x + [x,a]b - [x,b]a.$$

The dual of $(Q_\Phi, \mathcal{D}_\Phi)$ is the $\mathbb{Q}$-scheme with functor of points

$$\check{M}(Q_\Phi, \mathcal{D}_\Phi)(S) = \left\{ \begin{array}{c} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^\perp \text{ identifies } z \text{ with a rank one} \\ \text{local direct summand of } (V/I^\perp) \otimes \mathcal{O}_S \end{array} \right\}.$$

Every point $z \in \mathcal{D}_\Phi$ determines a mixed Hodge structure on V of type

$$\{(-1,0),(0,-1),(0,0),(0,1),(1,0)\},$$

and again the Borel morphism

$$\mathcal{D}_\Phi \to \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$$

sends $z \mapsto F^1 V_{\mathbb{C}}$. It identifies $\mathcal{D}_\Phi$ with the open subset

$$\mathcal{D}_\Phi = U_\Phi(\mathbb{C})\mathcal{D} \subset \pi_0(\mathcal{D}) \times \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}).$$

4.6. The q-expansion principle. — Now suppose the compact open subgroup K of (1.1.2) is neat, and small enough that there exists a finite K-admissible complete cone decomposition Σ for $(G, \mathcal{D})$ having the no self-intersection property. See § 2.4 for the definitions.

The results of Pink recalled in § 2.6 provide us with a toroidal compactification

$$\mathrm{Sh}_K(G, \mathcal{D}, \Sigma) = \bigsqcup_{(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)} Z_K^{(\Phi,\cdot\sigma)}(G, \mathcal{D}, \Sigma), \tag{4.6.1}$$

and the result of Harris-Zucker recalled as Theorem 3.4.1 gives a filtered vector bundle

$$0 = F^2\boldsymbol{V}_{\mathrm{dR}} \subset F^1\boldsymbol{V}_{\mathrm{dR}} \subset F^0\boldsymbol{V}_{\mathrm{dR}} \subset F^{-1}\boldsymbol{V}_{\mathrm{dR}} = \boldsymbol{V}_{\mathrm{dR}}$$

on the compactification, endowed with a quadratic form

$$[-,-] : \boldsymbol{V}_{\mathrm{dR}} \to \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D},\Sigma)}$$

induced by the bilinear form on V. Exactly as in 4.2, the *line bundle of weight one modular forms*

$$\boldsymbol{\omega} = F^1\boldsymbol{V}_{\mathrm{dR}}$$

is isotropic with respect to this bilinear form, and $F^0\boldsymbol{V}_{\mathrm{dR}} = (F^1\boldsymbol{V}_{\mathrm{dR}})^\perp$. These constructions extend those of § 4.2 from the open Shimura variety to its compactification.

In order to define q-expansions of sections of $\boldsymbol{\omega}^{\otimes k}$ on (4.6.1), we need to make some additional choices. The first choice is a boundary stratum

$$Z_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \subset \mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \tag{4.6.2}$$

indexed by a toroidal stratum representative (Φ, σ) in which the parabolic subgroup appearing in the underlying cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h)$$

is the stabilizer of an isotropic line $I \subset V$. The second choice is a nonzero vector $\ell \in I$, which will determine a trivialization of $\boldsymbol{\omega}$ in a formal neighborhood of the stratum (4.6.2).

As $\mathcal{D}$ has two connected components, there are exactly two continuous surjections $\nu : \mathcal{D} \to \mathcal{H}_0$. Fix one of them. It, along with the spinor similitude $\nu : G \to \mathbb{G}_m$, induces a morphism of Shimura data

$$(G, \mathcal{D}) \xrightarrow{\nu} (\mathbb{G}_m, \mathcal{H}_0).$$

Denote by $2\pi\epsilon^\circ \in \mathcal{H}_0$ the image of the component $\mathcal{D}^\circ$. There is a unique continuous extension of $\nu : \mathcal{D} \to \mathcal{H}_0$ to $\nu_\Phi : \mathcal{D}_\Phi \to \mathcal{H}_0$, and this determines a morphism of mixed Shimura data

$$(Q_\Phi, \mathcal{D}_\Phi) \xrightarrow{\nu_\Phi} (\mathbb{G}_m, \mathcal{H}_0), \tag{4.6.3}$$

where $\nu_\Phi : Q_\Phi \to \mathbb{G}_m$ is the character of (4.4.1).

Applying the functor of Proposition 3.3.2 to the Q_Φ-representations $I \subset V$ determines vector bundles $\boldsymbol{I}_{\mathrm{dR}} \subset \boldsymbol{V}_{\mathrm{dR}}$ on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$. The vector bundle $\boldsymbol{V}_{\mathrm{dR}}$ is endowed with a filtration and a symmetric bilinear pairing, exactly as in the discussion following (4.6.1), and restricting the bilinear pairing yields a homomorphism

$$[.,.] : \boldsymbol{I}_{\mathrm{dR}} \otimes \boldsymbol{\omega} \to \mathcal{O}_{\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)}. \tag{4.6.4}$$

The choice of nonzero vector $\ell \in I$ defines a section

$$\boldsymbol{\ell}^{\mathrm{an}} \in H^0\big(\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}), \boldsymbol{I}^{\mathrm{an}}_{\mathrm{dR}}\big)$$

of the line bundle

$$\boldsymbol{I}^{\mathrm{an}}_{\mathrm{dR}} = Q_\Phi(\mathbb{Q})\backslash\big(\mathcal{D}_\Phi \times I_{\mathbb{C}} \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big)$$

by sending

$$(z,g) \mapsto \left(z, \frac{\mathrm{rat}(\nu_\Phi(g))}{\nu_\Phi(z)} \cdot \ell, g\right).$$

Proposition 4.6.1. — *The holomorphic section $\boldsymbol{\ell}^{\mathrm{an}}$ extends uniquely to the partial compactification $\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)(\mathbb{C})$. This extension is algebraic and defined over $\mathbb{Q}$, and so arises from a unique global section*

$$\boldsymbol{\ell} \in H^0\big(\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma), \boldsymbol{I}_{\mathrm{dR}}\big). \tag{4.6.5}$$

Moreover, (4.6.4) *is an isomorphism, and induces an isomorphism*

$$\boldsymbol{\omega} \xrightarrow{\psi\mapsto[\boldsymbol{\ell},\psi]} \mathcal{O}_{\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)}.$$

Proof. — As the action of Q_Φ on I is via $\nu_\Phi : Q_\Phi/U_\Phi \to \mathbb{G}_m$, the discussion of § 3.5 (see especially Remark 3.5.2) identifies $\boldsymbol{I}_{\mathrm{dR}}$ with the pullback of the line bundle $I \otimes \mathrm{Lie}(\mathbb{G}_m) \cong I \otimes \mathcal{O}_{\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0)}$ via

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) \xrightarrow{(2.3.7)} \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi,\bar{\mathcal{D}}_\Phi) = \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0).$$

The section (4.6.5) is simply the pullback of the trivializing section

$$\ell \otimes 1 \in H^0\big(\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0), I \otimes \mathcal{O}_{\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0)}\big).$$

It now suffices to prove that (4.6.4) is an isomorphism. Recall from § 4.4 that $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$ has functor of points

$$\check{M}(Q_\Phi,\mathcal{D}_\Phi)(S) = \left\{ \begin{array}{c} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^\perp \text{ identifies } z \cong (V/I^\perp) \otimes \mathcal{O}_S \end{array} \right\}.$$

Let $\check{I}$ and $\check{V}$ be the (constant) Q_Φ-equivariant vector bundles on $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$ determined by the representations I and V. In the notation of Lemma 3.3.1, the line bundle $\check{\omega} = F^1\check{V}$ is the tautological bundle, and the bilinear form on V determines a Q_Φ-equivariant isomorphism

$$\check{I} \otimes \check{\omega} \to \check{V} \otimes \check{V} \xrightarrow{[-,-]} \mathcal{O}_{\check{M}(Q_\Phi,\mathcal{D}_\Phi)}.$$

By examining the construction of the functor in Proposition 3.3.2, the induced morphism (4.6.4) is also an isomorphism. □

It follows from the analysis of § 4.4 that the diagram (2.3.7) has the form

$$\begin{array}{ccc} \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) & \longrightarrow & \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \\ \downarrow{\scriptstyle \nu_\Phi} & \swarrow & \\ \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0), & & \end{array}$$

in which the arrow labeled ν_Φ is a torsor for the n-dimensional torus

$$T_\Phi = \mathrm{Spec}\Big(\mathbb{Q}[q_\alpha]_{\alpha \in \Gamma_\Phi^\vee(1)}\Big)$$

over the 0-dimensional base $\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0)$. To define q-expansions we will trivialize this torsor over an étale extension of the base, effectively putting coordinates on the mixed Shimura variety $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$.

Choose an auxiliary isotropic line $I_* \subset V$ with $[I, I_*] \neq 0$. This choice fixes a section

$$(Q_\Phi, \mathcal{D}_\Phi) \underset{\nu_\Phi}{\overset{s}{\rightleftarrows}} (\mathbb{G}_m, \mathcal{H}_0).$$

The underlying morphism of groups $s : \mathbb{G}_m \to Q_\Phi$ sends, for any $\mathbb{Q}$-algebra R, $a \in R^\times$ to the orthogonal transformation

$$s(a) \cdot x = \begin{cases} ax & \text{if } x \in I_R \\ a^{-1}x & \text{if } x \in I_{*,R} \\ x & \text{if } x \in (I \oplus I_*)^\perp_R. \end{cases} \tag{4.6.6}$$

To characterize $s : \mathcal{H}_0 \to \mathcal{D}_\Phi$, we first use (4.4.3) to view

$$I_{*\mathbb{C}} \in \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}).$$

Recalling the isomorphism (4.4.4), the preimage of $I_{*\mathbb{C}}$ under the projection

$$\mathcal{D}_\Phi \cong \pi_0(\mathcal{D}) \times \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \to \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$$

consists of two points, indexed by the two connected components of $\mathcal{D}$. The function $s : \mathcal{H}_0 \to \mathcal{D}_\Phi$ is defined by sending $2\pi\epsilon^\circ \in \mathcal{H}_0$ to the point indexed by $\mathcal{D}^\circ$, and the other element of $\mathcal{H}_0$ to the point indexed by the other connected component of $\mathcal{D}$.

The section s determines a Levi decomposition $Q_\Phi = \mathbb{G}_m \ltimes U_\Phi$. Choose a compact open subgroup $K_0 \subset \mathbb{G}_m(\mathbb{A}_f)$ small enough that its image under (4.6.6) is contained in K_Φ, and set

$$K_{\Phi 0} = K_0 \ltimes (U_\Phi(\mathbb{A}_f) \cap K_\Phi) \subset K_\Phi.$$

Our hypothesis that K is neat implies that $K_0 \subset K_{\Phi 0} \subset K_\Phi$ are also neat.

Proposition 4.6.2. — *Assume that the rational polyhedral cone $\sigma \subset U_\Phi(\mathbb{R})(-1)$ has (top) dimension n. The above choices determine a commutative diagram*

$$\begin{array}{ccc} \bigsqcup_{a\in\mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/K_0}\widehat{T}_\Phi(\sigma)_{/\mathbb{C}} & \xrightarrow{\;\cong\;} & \widehat{\mathrm{Sh}}_{K_{\Phi0}}(Q_\Phi,\mathcal{D}_\Phi,\sigma)_{/\mathbb{C}} \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Sh}}_K(G,\mathcal{D},\Sigma)_{/\mathbb{C}} & \xrightarrow{\;\cong\;} & \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)_{/\mathbb{C}} \end{array}$$

of formal algebraic spaces, in which the vertical arrows are formally étale surjections. Here

$$\widehat{T}_\Phi(\sigma) \stackrel{\mathrm{def}}{=} \mathrm{Spf}\Big(\mathbb{Q}[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geq 0}}\Big) \tag{4.6.7}$$

is the formal completion of (2.3.6) *along its closed stratum, the lower left corner is the formal completion of* $\mathrm{Sh}_K(G,\mathcal{D},\Sigma)_{/\mathbb{C}}$ *along the* 0*-dimensional stratum* (4.6.2)*, and the bottom isomorphism is* (2.6.3)*.*

Proof. — Consider the diagram

$$\begin{array}{ccccc} \mathrm{Sh}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)\times_{\mathrm{Spec}(\mathbb{Q})} T_\Phi & = & \mathrm{Sh}_{K_{\Phi0}}(Q_\Phi,\mathcal{D}_\Phi) & \longrightarrow & \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \\ & \searrow & {\scriptstyle\nu_\Phi}\downarrow \uparrow{\scriptstyle s} & & \downarrow{\scriptstyle\nu_\Phi} \\ & & \mathrm{Sh}_{K_0}(\mathbb{G}_m,\mathcal{H}_0) & \longrightarrow & \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0), \end{array}$$

in which the arrows labeled ν_Φ are the T_Φ-torsors of (2.3.1), and the isomorphism "=" is the trivialization induced by the section s.

There is a canonical bijection

$$\mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/K_0 \xrightarrow{\simeq} \mathrm{Sh}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)(\mathbb{C}) \tag{4.6.8}$$

defined by $a \mapsto [(2\pi\epsilon^\circ, a)]$. We remind the reader that $2\pi\epsilon^\circ \in \mathcal{H}_0$ was fixed in the discussion preceding (4.6.3).

Using this, the top row of the above diagram exhibits $\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)_{/\mathbb{C}}$ as an étale quotient

$$\bigsqcup_{a\in\mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/K_0} T_{\Phi/\mathbb{C}} \cong \mathrm{Sh}_{K_{\Phi0}}(Q_\Phi,\mathcal{D}_\Phi)_{/\mathbb{C}} \to \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)_{/\mathbb{C}}. \tag{4.6.9}$$

This morphism extends to partial compactifications, and formally completing along the closed stratum yields a formally étale morphism

$$\bigsqcup_{a\in\mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/K_0} \widehat{T}_\Phi(\sigma)_{/\mathbb{C}} \cong \widehat{\mathrm{Sh}}_{K_{\Phi0}}(Q_\Phi,\mathcal{D}_\Phi,\sigma)_{/\mathbb{C}} \to \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)_{/\mathbb{C}}.$$

This defines the top horizontal arrow and the right vertical arrow in the diagram. The vertical arrow on the left is defined by the commutativity of the diagram. □

Propositions 4.6.2 and 4.6.1 now give us a working theory of q-expansions along the 0-dimensional boundary stratum (4.6.2) determined by a top dimensional cone $\sigma \subset U_\Phi(\mathbb{R})(-1)$. Taking tensor powers in Proposition 4.6.1 determines an isomorphism

$$[\boldsymbol{\ell}^{\otimes k}, .] : \boldsymbol{\omega}^{\otimes k} \cong \mathcal{O}_{\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)},$$

and hence any global section

$$\psi \in H^0\big(\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}, \boldsymbol{\omega}^{\otimes k}\big)$$

determines a formal function $[\boldsymbol{\ell}^{\otimes k}, \psi]$ on

$$\widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \cong \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}}.$$

Now pull this formal function back via the formally étale surjection

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \widehat{T}_\Phi(\sigma)_{/\mathbb{C}} \to \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$$

of Proposition 4.6.2. By restricting the pullback to the copy of $\widehat{T}_\Phi(\sigma)_{/\mathbb{C}}$ indexed by a, we obtain a formal q-expansion (*a.k.a.* Fourier Jacobi expansion)

$$\mathrm{FJ}^{(a)}(\psi) = \sum_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geq 0}} \mathrm{FJ}^{(a)}_\alpha(\psi) \cdot q_\alpha \in \mathbb{C}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geq 0}}. \tag{4.6.10}$$

We emphasize that (4.6.10) depends on the choice of toroidal stratum representative (Φ, σ), as well as on the choices of $\nu : \mathcal{D} \to \mathcal{H}_0$, I_*, and $\ell \in I$. These will always be clear from context.

For each $\tau \in \mathrm{Aut}(\mathbb{C})$, denote by $a_\tau \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times$ the unique element with

$$\mathrm{rec}(a_\tau) = \tau|_{\mathbb{Q}^{\mathrm{ab}}}.$$

The following is our q-expansion principle; see also [**27**, Theorem 2.8.7].

***Proposition 4.6.3* (Rational q-expansion principle).** — *For any $a \in \mathbb{A}_f^\times$ and $\tau \in \mathrm{Aut}(\mathbb{C})$, the q-expansion coefficients of ψ and ψ^τ are related by*

$$\mathrm{FJ}^{(aa_\tau)}_\alpha(\psi^\tau) = \tau\big(\mathrm{FJ}^{(a)}_\alpha(\psi)\big).$$

Moreover, ψ is defined over a subfield $L \subset \mathbb{C}$ if and only if

$$\mathrm{FJ}^{(aa_\tau)}_\alpha(\psi) = \tau\big(\mathrm{FJ}^{(a)}_\alpha(\psi)\big)$$

for all $a \in \mathbb{A}_f^\times$, all $\tau \in \mathrm{Aut}(\mathbb{C}/L)$, and all $\alpha \in \Gamma_\Phi^\vee(1)$.

Proof. — The formal scheme $\widehat{T}_\Phi(\sigma)$ of (4.6.7) has a distinguished $\mathbb{Q}$-valued point defined by $q_\alpha = 0$ (*i.e.*, the unique point of the underlying reduced $\mathbb{Q}$-scheme), and so has a distinguished $\mathbb{C}$-valued point. Hence, using the morphism

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \widehat{T}_\Phi(\sigma)_{/\mathbb{C}} \to \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C})$$

of Proposition 4.6.2, each $a \in \mathbb{Q}^\times_{>0}\backslash \mathbb{A}_f^\times$ determines a distinguished point

$$\mathrm{cusp}^{(a)} \in \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}).$$

By examining the proof of Proposition 4.6.2, the reciprocity law (3.5.2) implies that

$$\mathrm{cusp}^{(aa_\tau)} = \tau(\mathrm{cusp}^{(a)})$$

for any $\tau \in \mathrm{Aut}(\mathbb{C})$, and the q-expansion (4.6.10) is, tautologically, the image of the formal function $[\boldsymbol{\ell}^{\otimes k}, \psi]$ in the completed local ring at $\mathrm{cusp}^{(a)}$. The first claim is now a consequence of the equality

$$[\boldsymbol{\ell}^{\otimes k}, \psi]^\tau = [\boldsymbol{\ell}^{\otimes k}, \psi^\tau]$$

of formal functions on

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma).$$

The second claim follows from the first, and the observation that two rational sections ψ_1 and ψ_2 are equal if and only if $\mathrm{FJ}^{(a)}(\psi_1) = \mathrm{FJ}^{(a)}(\psi_2)$ for all a. Indeed, to check that $\psi_1 = \psi_2$, it suffices to check this in a formal neighborhood of one point on each connected component of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$. Using strong approximation for the simply connected group

$$\mathrm{Spin}(V) = \ker\big(\nu : G \to \mathbb{G}_m\big),$$

one can show that the fibers of

$$\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) \to \mathrm{Sh}_{\nu(K)}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$$

are connected. This implies that the images of the points $\mathrm{cusp}^{(a)}$ under

$$\widehat{\mathrm{Sh}}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}) \to \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)(\mathbb{C})$$

hit every connected component of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)(\mathbb{C})$. □

5. Borcherds products

Once again, we work with a fixed $\mathbb{Q}$-quadratic space (V, Q) of signature $(n, 2)$ with $n \geq 1$, and denote by $(G, \mathcal{D})$ the associated GSpin Shimura datum of § 4.1. Fix a $\mathbb{Z}$-lattice $V_\mathbb{Z} \subset V$ on which the quadratic form is $\mathbb{Z}$-valued, and let K be as in (1.1.2). Recalling the notation of Remark 2.1.2, fix a choice of

$$2\pi i \in \mathcal{H}_0.$$

We recall the analytic theory of Borcherds products **[5, 8]** on $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$ using the adelic formulation as in **[33]**. Assuming that V contains an isotropic line, we express their product expansions in the algebraic language of § 4.6.

5.1. Weakly holomorphic forms. — Let $S(V_{\mathbb{A}_f})$ be the Schwartz space of locally constant $\mathbb{C}$-valued compactly supported functions on $V_{\mathbb{A}_f} = V \otimes \mathbb{A}_f$. For any $g \in G(\mathbb{A}_f)$ abbreviate

$$gV_{\mathbb{Z}} = gV_{\widehat{\mathbb{Z}}} \cap V.$$

Denote by $S_{V_{\mathbb{Z}}} \subset S(V_{\mathbb{A}_f})$ the finite dimensional subspace of functions invariant under $V_{\widehat{\mathbb{Z}}}$, and supported on its dual lattice; we often identify it with the space

$$S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}]$$

of functions on $V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. The metaplectic double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ acts via the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}}),$$

as in **[5, 8, 10]**. Define the complex conjugate representation by

$$\overline{\rho}_{V_{\mathbb{Z}}}(\gamma) \cdot \varphi = \overline{(\rho_{V_{\mathbb{Z}}}(\gamma) \cdot \overline{\varphi})},$$

for $\gamma \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ and $\varphi \in S_{V_{\mathbb{Z}}}$.

Remark 5.1.1. — There is a canonical basis

$$\{\phi_\mu : \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}\} \subset S_{V_{\mathbb{Z}}},$$

in which ϕ_μ is the characteristic function of $\mu + V_{\mathbb{Z}}$. This allows us to identify $S_{V_{\mathbb{Z}}}$ with its own $\mathbb{C}$-linear dual. Under this identification, the complex conjugate representation $\overline{\rho}_{V_{\mathbb{Z}}}$ agrees with with contragredient representation $\rho_{V_{\mathbb{Z}}}^{\vee}$. It also agrees with the representation denoted $\omega_{V_{\mathbb{Z}}}$ in **[1, 2]**.

Denote by $M^!_{1-n/2}(\overline{\rho}_{V_{\mathbb{Z}}})$ the space of weakly holomorphic forms for $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ of weight $1 - n/2$ and representation $\overline{\rho}_{V_{\mathbb{Z}}}$, as in **[5, 8, 10]**. In particular, any

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}) \tag{5.1.1}$$

is an $S_{V_{\mathbb{Z}}}$-valued holomorphic function on the complex upper half-plane $\mathcal{H}$. Each Fourier coefficient $c(m) \in S_{V_{\mathbb{Z}}}$ is determined by its values $c(m, \mu)$ at the various cosets $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Moreover, $c(m, \mu) \neq 0$ implies $m \equiv Q(\mu)$ modulo $\mathbb{Z}$.

Definition 5.1.2. — The weakly holomorphic form (5.1.1) is *integral* if

$$c(m, \mu) \in \mathbb{Z}$$

for all $m \in \mathbb{Q}$ and all $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$.

It is a theorem of McGraw **[41]** that the space of all forms (5.1.1) has a $\mathbb{C}$-basis of integral forms.

5.2. Borcherds products and regularized theta lifts. — We now recall the meromorphic Borcherds products of [**5, 8, 33**].

Write $\tau = u + iv \in \mathcal{H}$ for the variable on the complex upper half-plane. For each $\varphi \in S(V_{\mathbb{A}_f})$ there is a Siegel theta function

$$\vartheta(\tau, z, g; \varphi) : \mathcal{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to \mathbb{C},$$

as in [**33**, (1.37)], satisfying the transformation law

$$\vartheta(\tau, \gamma z, \gamma gh; \varphi) = \vartheta(\tau, z, g; \varphi \circ h^{-1})$$

for any $\gamma \in G(\mathbb{Q})$ and any $h \in G(\mathbb{A}_f)$. If we use the basis of Remark 5.1.1 to define

$$\vartheta(\tau, z, g) = \sum_{\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}} \vartheta(\tau, z, g, \phi_\mu) \cdot \phi_\mu,$$

we obtain a function

$$\vartheta(\tau, z, g) : \mathcal{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to S_{V_{\mathbb{Z}}},$$

which transforms in the variable τ like a modular form of weight $\frac{n}{2} - 1$ and representation $\rho_{V_{\mathbb{Z}}}$.

Given a weakly holomorphic form (5.1.1) one can regularize the divergent integral

$$\Theta^{\mathrm{reg}}(f)(z, g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\vartheta(\tau, z, g)\ \frac{du\,dv}{v^2} \tag{5.2.1}$$

as in [**5, 8, 33**]. Here we are using the map $S_{V_{\mathbb{Z}}} \otimes S_{V_{\mathbb{Z}}} \to \mathbb{C}$ defined by

$$\phi_\mu \otimes \phi_\nu \mapsto \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise} \end{cases}$$

to obtain an $\mathrm{SL}_2(\mathbb{Z})$-invariant scalar-valued integrand $f(\tau)\vartheta(\tau, z, g)$.

As the subgroup K acts trivially on the quotient (4.1.5), the subspace $S_{V_{\mathbb{Z}}} \subset S(V_{\mathbb{A}_f})$ is K-invariant. It follows that (5.2.1) satisfies

$$\Theta^{\mathrm{reg}}(f)(\gamma z, \gamma gk) = \Theta^{\mathrm{reg}}(f)(z, g)$$

for any $\gamma \in G(\mathbb{Q})$ and any $k \in K$. This allows us to view $\Theta^{\mathrm{reg}}(f)$ as a function on $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$, which we call the *regularized theta lift*. Our $\Theta^{\mathrm{reg}}(f)$ is usually denoted $\Phi(f)$ in the literature.

Remark 5.2.1. — The following fundamental theorem of Borcherds implies that the regularized theta lift is real analytic away from a prescribed divisor, with logarithmic singularities along that divisor. Remarkably, the regularization process gives $\Theta^{\mathrm{reg}}(f)$ a meaningful value at *every* point of $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$, including along the singular divisor. In the context of unitary Shimura varieties, this is [**11**, Theorem 4.1] and [**11**, Corollary 4.2], and the proof for orthogonal Shimura varieties is identical. In other words, $\Theta^{\mathrm{reg}}(f)$ is a well-defined (but discontinuous) function on all of $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$. Its values along the singular divisor will be made more explicit in § 9.2 when we use the embedding trick to complete the proof of Theorem 9.1.1.

***Theorem 5.2.2* (Borcherds).** — *Assume that f is integral. After multiplying f by any sufficiently divisible positive integer*[(4)]*, there is a meromorphic section $\Psi(f)$ of the analytic line bundle $(\boldsymbol{\omega}^{\mathrm{an}})^{\otimes c(0,0)/2}$ on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ such that, away from the support of $\mathrm{div}(\Psi(f))$, we have*

$$-4\log\|\Psi(f)\|_{\mathrm{naive}} = \Theta^{\mathrm{reg}}(f) + c(0,0)\log(\pi) + c(0,0)\Gamma'(1). \tag{5.2.2}$$

Here $\Gamma'(s)$ is the derivative of the usual Gamma function, and $\|-\|_{\mathrm{naive}}$ is the metric of (4.2.2).

Proof. — Choose a connected component $\mathcal{D}^\circ \subset \mathcal{D}$, let $G(\mathbb{R})^\circ \subset G(\mathbb{R})$ be its stabilizer (this is just the subgroup of elements on which the spinor similitude $\nu : G \to \mathbb{G}_m$ is positive) and define $G(\mathbb{Q})^\circ$ similarly. Denote by $\boldsymbol{\omega}_{\mathcal{D}^\circ}$ the restriction to $\mathcal{D}^\circ$ of the tautological line bundle on (4.1.1). It carries an action of $G(\mathbb{R})^\circ$ covering the action on the base, and a $G(\mathbb{R})^\circ$ invariant metric (4.2.2).

For any $g \in G(\mathbb{A}_f)$, denote by

$$\Theta^{\mathrm{reg}}_g(f)(z) \stackrel{\mathrm{def}}{=} \Theta^{\mathrm{reg}}(f)(z,g)$$

the restriction of the regularized theta lift to the connected component

$$(G(\mathbb{Q})^\circ \cap gKg^{-1})\backslash\mathcal{D}^\circ \xrightarrow{z\mapsto(z,g)} \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}). \tag{5.2.3}$$

Borcherds [5] proves the existence of a meromorphic section $\Psi_g(f)$ of $\boldsymbol{\omega}_{\mathcal{D}^\circ}^{\otimes c(0,0)/2}$ satisfying

$$-4\log\|\Psi_g(f)\|_{\mathrm{naive}} = \Theta^{\mathrm{reg}}_g(f) + c(0,0)\log(\pi) + c(0,0)\Gamma'(1). \tag{5.2.4}$$

Note that Borcherds does not work adelically. Instead, for every input form (5.1.1) he constructs a single meromorphic section $\Psi_{\mathrm{classical}}(f)$ over $\mathcal{D}^\circ$. However, $g \in G(\mathbb{A}_f)$ determines an isomorphism $V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}} \to gV_{\mathbb{Z}}^\vee/gV_{\mathbb{Z}}$, which induces an isomorphism

$$M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}) \xrightarrow{f\mapsto g\cdot f} M^!_{1-\frac{n}{2}}(\overline{\rho}_{gV_{\mathbb{Z}}}).$$

Replacing the pair $(V_{\mathbb{Z}}, f)$ by $(gV_{\mathbb{Z}}, gf)$ yields another meromorphic section $\Psi_{\mathrm{classical}}(gf)$ over $\mathcal{D}^\circ$, and

$$\Psi_g(f) = \Psi_{\mathrm{classical}}(gf).$$

The relation (5.2.4) determines $\Psi_g(f)$ up to scaling by a complex number of absolute value 1, and the linearity of $f \mapsto \Theta^{\mathrm{reg}}_g(f)$ implies the multiplicativity

$$\Psi_g(f_1+f_2) = \Psi_g(f_1)\otimes\Psi_g(f_2)$$

relation, up to the ambiguity just noted. As $\Theta^{\mathrm{reg}}_g(f)$ is invariant under translation by every $\gamma \in G(\mathbb{Q})^\circ \cap gKg^{-1}$, we must have

$$\gamma\cdot\Psi_g(f)(z) = \xi_g(\gamma)\cdot\Psi_g(f)(\gamma z)$$

for some unitary character

$$\xi_g : G(\mathbb{Q})^\circ \cap gKg^{-1} \to \mathbb{C}^\times. \tag{5.2.5}$$

(4) In particular, we may assume $c(0,0) \in 2\mathbb{Z}$.

The main result of [7] asserts that the character ξ_g is of finite order, and so we may replace f by a positive integer multiple in order to make it trivial. The section $\Psi_g(f)$ now descends to the quotient (5.2.3).

Repeating this procedure on every connected component of $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ yields a section $\Psi(f)$ satisfying (5.2.2). □

The meromorphic section $\Psi(f)$ of the theorem is what is usually called the *Borcherds product* (or Borcherds lift) of f. We will use the same terminology to refer to the meromorphic section

$$\psi(f) = (2\pi i)^{c(0,0)}\Psi(2f)$$

of $(\boldsymbol{\omega}^{\mathrm{an}})^{\otimes c(0,0)}$, which has better arithmetic properties. We will see in § 9 that, after rescaling by a constant of absolute value 1 on every connected component of $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$, the section $\psi(f)$ is algebraic and defined over the reflex field $\mathbb{Q}$.

Proposition 5.2.3. — *Assume that either $n \geq 3$, or that $n = 2$ and V has Witt index* 1*. The Borcherds product $\Psi(f)$ of Theorem 5.2.2, a priori a meromorphic section on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$, is the analytification of a rational section on $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$.*

Proof. — It suffices to prove this after shrinking K, so we may assume that K is neat and $\mathrm{Sh}_K(G,\mathcal{D})$ is a quasi-projective variety. The hypotheses on n imply that the boundary of the (normal and projective) Baily-Borel compacification

$$\mathrm{Sh}_K(G,\mathcal{D}) \hookrightarrow \mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{BB}}$$

lies in codimension ≥ 2.

Let D be the polar part of the divisor of $\Psi(f)$, so that D is an effective analytic divisor on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ with $\mathrm{div}(\Psi(f)) + D$ effective. The proof of Levi's generalization of Hartogs' theorem [**17**, § 9.5] shows that the topological closure of D in $\mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{BB}}(\mathbb{C})$ is again an analytic divisor. By Chow's theorem on the algebraicity of analytic divisors on projective varieties, this closure is algebraic, and so D itself was algebraic.

Now view $\Psi(f)$ as a holomorphic section of the analytification of the line bundle $\boldsymbol{\omega}^{\otimes c(0,0)/2} \otimes \mathcal{O}(D)$ on $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$. By Hartshorne's extension of GAGA [**25**, Theorem VI.2.1] this section is algebraic, as desired. □

5.3. The product expansion I. — As in the proof of Theorem 5.2.2, fix a connected component $\mathcal{D}^\circ \subset \mathcal{D}$ and an $h \in G(\mathbb{A}_f)$, and denote by $\Psi_h(f)$ the restriction of the Borcherds product to the connected component

$$(G(\mathbb{Q})^\circ \cap hKh^{-1})\backslash\mathcal{D}^\circ \xrightarrow{z\mapsto(z,h)} \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}).$$

In this subsection we recall the product expansion for $\Psi_h(f)$ due to Borcherds. Let $\boldsymbol{\omega}_{\mathcal{D}^\circ}$ be the restriction to $\mathcal{D}^\circ$ of the tautological bundle on (4.1.1).

Assume throughout § 5.3 that there exists an isotropic line $I \subset V$. Choose a second isotropic line $I_* \subset V$ with $[I, I_*] \neq 0$, but do this in a particular way: first choose a $\mathbb{Z}$-module generator

$$\ell \in I \cap hV_{\mathbb{Z}},$$

and then choose a $k \in hV_{\mathbb{Z}}^{\vee}$ such that $[\ell, k] = 1$. Now take I_* be the span of the isotropic vector

$$\ell_* = k - Q(k)\ell. \tag{5.3.1}$$

Obviously $[\ell, \ell_*] = 1$, but we need not have $\ell_* \in hV_{\mathbb{Z}}^{\vee}$.

Abbreviate $V_0 = I^{\perp}/I$. This is a $\mathbb{Q}$-vector space endowed with a quadratic form of signature $(n-1, 1)$, and a $\mathbb{Z}$-lattice

$$V_{0\mathbb{Z}} = (I^{\perp} \cap hV_{\mathbb{Z}})/(I \cap hV_{\mathbb{Z}}) \subset V_0. \tag{5.3.2}$$

Denote by

$$\mathrm{LightCone}(V_{0\mathbb{R}}) = \{w \in V_{0\mathbb{R}} : Q(w) < 0\}$$

the light cone in $V_{0\mathbb{R}}$. It is a disjoint union of two open convex cones. Every $v \in I_{\mathbb{C}}^{\perp}$ determines an isotropic vector

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in V_{\mathbb{C}},$$

depending only on the image $v \in V_{0\mathbb{C}}$. The resulting injection $V_{0\mathbb{C}} \to \mathbb{P}^1(V_{\mathbb{C}})$ restricts to an isomorphism

$$V_{0\mathbb{R}} + i \cdot \mathrm{LightCone}(V_{0\mathbb{R}}) \cong \mathcal{D},$$

and we let

$$\mathrm{LightCone}^{\circ}(V_{0\mathbb{R}}) \subset \mathrm{LightCone}(V_{0\mathbb{R}}) \tag{5.3.3}$$

be the connected component with

$$V_{0\mathbb{R}} + i \cdot \mathrm{LightCone}^{\circ}(V_{0\mathbb{R}}) \cong \mathcal{D}^{\circ}.$$

There is an action $\rho_{V_{0\mathbb{Z}}}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ on the finite dimensional $\mathbb{C}$-vector space $S_{V_{0\mathbb{Z}}}$, exactly as in § 5.1, and a weakly holomorphic modular form

$$f_0(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} \sum_{\lambda \in V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}} c_0(m, \lambda) \cdot q^m \in M^{!}_{1-\frac{n}{2}}(\overline{\rho}_{V_{0\mathbb{Z}}}),$$

whose coefficients are defined by

$$c_0(m, \lambda) = \sum_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} c(m, h^{-1}\mu).$$

Here we understand $h^{-1}\mu$ to mean the image of μ under the isomorphism $hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \to V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$ defined by multiplication by h^{-1}. The notation $\mu \sim \lambda$ requires explanation: denoting by

$$p : (I^{\perp} \cap hV_{\mathbb{Z}}^{\vee})/(I^{\perp} \cap hV_{\mathbb{Z}}) \to V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}$$

the natural map, $\mu \sim \lambda$ means that there is a

$$\tilde{\mu} \in I^{\perp} \cap (\mu + hV_{\mathbb{Z}}) \tag{5.3.4}$$

such that $p(\tilde{\mu}) = \lambda$.

Every vector $x \in V_0$ of positive length determines a hyperplane $x^{\perp} \subset V_{0\mathbb{R}}$. For each $m \in \mathbb{Q}_{>0}$ and $\lambda \in V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}$ define a formal sum of hyperplanes

$$H(m,\lambda) = \sum_{\substack{x \in \lambda + V_{0\mathbb{Z}} \\ Q(x) = m}} x^{\perp},$$

in $V_{0\mathbb{R}}$, and set

$$H(f_0) = \sum_{\substack{m \in \mathbb{Q}_{>0} \\ \lambda \in V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}}} c_0(-m,\lambda) \cdot H(m,\lambda).$$

Definition 5.3.1. — A *Weyl chamber* for f_0 is a connected component

$$\mathscr{W} \subset \mathrm{LightCone}^{\circ}(V_{0\mathbb{R}}) \setminus \mathrm{Support}(H(f_0)). \tag{5.3.5}$$

Let N be the positive integer determined by $N\mathbb{Z} = [hV_{\mathbb{Z}}, I \cap hV_{\mathbb{Z}}]$, and note that $\ell/N \in hV_{\mathbb{Z}}^{\vee}$. Set

$$A = \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x/N}\right)^{c(0, xh^{-1}\ell/N)}. \tag{5.3.6}$$

Tautologically, every fiber of $\boldsymbol{\omega}_{\mathcal{D}^{\circ}}$ is a line in $V_{\mathbb{C}}$, and each such fiber pairs nontrivially with the isotropic line $I_{\mathbb{C}}$. Using the nondegenerate pairing

$$[\,.,.\,] : I_{\mathbb{C}} \otimes \boldsymbol{\omega}_{\mathcal{D}^{\circ}} \to \mathcal{O}_{\mathcal{D}^{\circ}},$$

the Borcherds product $\Psi_h(f)$ and the isotropic vector $\ell \in I$ determine a meromorphic function $[\ell^{\otimes c(0,0)/2}, \Psi_h(f)]$ on $\mathcal{D}^{\circ}$. It is this function that Borcherds expresses as an infinite product.

***Theorem 5.3.2* (Borcherds [5, 8])**. — *For each Weyl chamber $\mathscr{W}$ there is a vector $\varrho \in V_0$ with the following property: For all*

$$v \in V_{0\mathbb{R}} + i \cdot \mathscr{W} \subset V_{0\mathbb{C}}$$

with $|Q(\mathrm{Im}(v))| \gg 0$, the value of $[\ell^{\otimes c(0,0)/2}, \Psi_h(f)]$ at the isotropic line

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^{\circ}$$

is given by the (convergent) infinite product

$$\kappa A \cdot e^{2\pi i[\varrho, v]} \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^{\vee} \\ [\lambda, \mathscr{W}] > 0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - \zeta_{\mu} \cdot e^{2\pi i[\lambda, v]}\right)^{c(-Q(\lambda), h^{-1}\mu)}$$

for some $\kappa \in \mathbb{C}$ of absolute value 1. Here, recalling the vector $k \in hV_{\mathbb{Z}}^{\vee}$ appearing in (5.3.1), we have set

$$\zeta_{\mu} = e^{2\pi i[\mu, k]}.$$

Remark 5.3.3. — The vector $\varrho \in V_0$ of the theorem is the *Weyl vector*. It is completely determined by the weakly holomorphic form f_0 and the choice of Weyl chamber $\mathscr{W}$.

5.4. The product expansion II. — We now connect the product expansion of Theorem 5.3.2 with the algebraic theory of q-expansions from § 4.6. Throughout § 5.4 we assume that K is neat.

The theory of § 4.6 applies to sections of the algebraic line bundle $\boldsymbol{\omega}_{/\mathbb{C}}$ on $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$ and at the moment we only know the algebraicity of Borcherds products in special cases (Proposition 5.2.3). Throughout § 5.4, we simply assume that our given Borcherds product $\Psi(f)$ is algebraic.

Begin by choosing a cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h)$$

for which P is the stabilizer of an isotropic line I. Let $\ell \in I \cap hV_{\mathbb{Z}}$ be a generator, let ℓ_* be as in (5.3.1), and let $I_* = \mathbb{Q}\ell_*$.

Recall from the discussion surrounding (4.6.6) that the choice of I_* determines morphisms of mixed Shimura data

$$(Q_\Phi, \mathcal{D}_\Phi) \xrightarrow[\nu_\Phi]{} (\mathbb{G}_m, \mathcal{H}_0), \qquad s : (\mathbb{G}_m, \mathcal{H}_0) \to (Q_\Phi, \mathcal{D}_\Phi)$$

where we specify that $\nu_\Phi : \mathcal{D}_\Phi \to \mathcal{H}_0$ sends the connected component $\mathcal{D}_\Phi^\circ \subset \mathcal{D}_\Phi$ containing $\mathcal{D}^\circ$ to the $2\pi i \in \mathcal{H}_0$ fixed at the beginning of § 5.

Set $V_0 = I^\perp/I$ as before. The connected component (5.3.3) was chosen in such a way that the isomorphism

$$V_{0\mathbb{C}} \xrightarrow{\otimes \ell} V_{0\mathbb{C}} \otimes I \xrightarrow{(4.4.2)} U_\Phi(\mathbb{C})$$

identifies

$$V_{0\mathbb{R}} + i \cdot \mathrm{LightCone}^\circ(V_{0\mathbb{R}}) \cong U_\Phi(\mathbb{R}) + C_\Phi,$$

where $C_\Phi \subset U_\Phi(\mathbb{R})(-1)$ is the open convex cone (2.4.1). Equivalently, the isomorphism

$$V_{0\mathbb{R}} \xrightarrow{\otimes (-2\pi i)^{-1}\ell} V_{0\mathbb{R}} \otimes I(-1) \cong U_\Phi(\mathbb{R})(-1) \tag{5.4.1}$$

(note the minus sign!) identifies $\mathrm{LightCone}^\circ(V_{0\mathbb{R}}) \cong C_\Phi$.

Lemma 5.4.1. — *Fix a Weyl chamber $\mathscr{W}$ as in (5.3.5). After possibly shrinking K, there exists a K-admissible, complete cone decomposition Σ of $(G, \mathcal{D})$ having the no self-intersection property, and such that the following holds: there is some top-dimensional rational polyhedral cone $\sigma \in \Sigma_\Phi$ whose interior is identified with an open subset of $\mathscr{W}$ under the above isomorphism*

$$C_\Phi \cong \mathrm{LightCone}^\circ(V_{0\mathbb{R}}).$$

Proof. — This is an elementary exercise. Using Remark 2.6.1, we first shrink K in order to find some K-admissible, complete cone decomposition Σ of $(G, \mathcal{D})$ having the no self-intersection property. We may furthermore choose Σ to be smooth, and applying barycentric subdivision [**47**, § 5.24] finitely many times yields a refinement of Σ with the desired properties. □

For the remainder of § 5.4 we assume that K, Σ, $\mathscr{W}$, and $\sigma \subset U_\Phi(\mathbb{R})(-1)$ are as in Lemma 5.4.1. As in § 4.6, the line bundle $\boldsymbol{\omega}$ on $\mathrm{Sh}_K(G, \mathcal{D})$ has a canonical extension to $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)$, and we view $\Psi(f)$ as a rational section over $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$.

The top-dimensional cone σ singles out a 0-dimensional stratum

$$Z_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma) \subset \mathrm{Sh}_K(G, \mathcal{D}, \Sigma)$$

as in § 2.6. Completing along this stratum, Proposition 4.6.2 provides us with a formally étale surjection

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \mathrm{Spf}\Big(\mathbb{C}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geq 0}} \Big) \to \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}},$$

where $K_0 \subset \mathbb{A}_f^\times$ is chosen small enough that the section (4.6.6) satisfies $s(K_0) \subset K_\Phi$. As in (4.6.10), the Borcherds product $\Psi(f)$ and the isotropic vector ℓ determine a rational formal function $[\boldsymbol{\ell}^{\otimes c(0,0)/2}, \Psi(f)]$ on the target, which pulls back to a rational formal function

$$\mathrm{FJ}^{(a)}(\Psi(f)) \in \mathrm{Frac}\Big(\mathbb{C}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geq 0}} \Big) \tag{5.4.2}$$

for every index a. The following proposition explains how this formal q-expansion varies with a.

Proposition 5.4.2. — *Let $F \subset \mathbb{C}$ be the abelian extension of $\mathbb{Q}$ determined by*

$$\mathrm{rec} : \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0 \cong \mathrm{Gal}(F/\mathbb{Q}).$$

The rational formal function (5.4.2) *has the form*

$$(2\pi i)^{c(0,0)/2} \cdot \mathrm{FJ}^{(a)}(\Psi(f)) = \kappa^{(a)} A^{\mathrm{rec}(a)} q_{\alpha(\varrho)} \cdot \mathrm{BP}(f)^{\mathrm{rec}(a)}. \tag{5.4.3}$$

Here $\kappa^{(a)} \in \mathbb{C}$ is some constant of absolute value 1, *and the power series*

$$\mathrm{BP}(f) \in \mathcal{O}_F[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geq 0}}$$

(<u>B</u>orcherds <u>P</u>roduct) is the infinite product

$$\mathrm{BP}(f) = \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^\vee \\ [\lambda, \mathscr{W}] > 0}} \prod_{\substack{\mu \in h V_\mathbb{Z}^\vee / h V_\mathbb{Z} \\ \mu \sim \lambda}} \Big(1 - \zeta_\mu \cdot q_{\alpha(\lambda)} \Big)^{c(-Q(\lambda), h^{-1}\mu)}.$$

The constant A and the roots of unity ζ_μ have the same meaning as in Theorem 5.3.2, and these constants lie in $\mathcal{O}_F$. The meaning of $q_{\alpha(\lambda)}$ is as follows: dualizing the isomorphism (5.4.1) *yields an isomorphism*

$$V_{0\mathbb{R}} \xrightarrow{\lambda\mapsto\alpha(\lambda)} U_\Phi(\mathbb{R})^\vee(1), \tag{5.4.4}$$

and the image of each $\lambda \in V_{0\mathbb{R}}$ appearing in the product satisfies

$$\alpha(\lambda) \in \Gamma_\Phi^\vee(1).$$

The condition $[\lambda, \mathscr{W}] > 0$ implies $\langle\alpha(\lambda), \sigma\rangle > 0$. Of course $q_{\alpha(\varrho)}$ has the same meaning, with $\varrho \in V_0$ the Weyl vector of Theorem 5.3.2. Again $\alpha(\varrho) \in \Gamma_\Phi^\vee(1)$, but need not satisfy the positivity condition with respect to σ.

Proof. — First we address the field of definition of the constants A and ζ_μ.

Lemma 5.4.3. — *The constant A of* (5.3.6) *lies in $\mathcal{O}_F$, and $\zeta_\mu \in \mathcal{O}_F$ for every μ appearing in the above product.*

Proof. — Suppose $a \in K_0$. It follows from the discussion preceeding (4.1.5) that $s(a) \in hKh^{-1}$ stabilizes the lattice $hV_\mathbb{Z}$, and acts trivially on the quotient $hV_\mathbb{Z}^\vee/hV_\mathbb{Z}$. In particular, $s(a)$ acts trivially on the vector $\ell/N \in hV_\mathbb{Z}^\vee/hV_\mathbb{Z}$. On the other hand, by its very definition (4.6.6) we know that $s(a)$ acts by a on this vector. It follows that $(a-1)\ell/N \in hV_\mathbb{Z}$, from which we deduce first $a - 1 \in N\widehat{\mathbb{Z}}$, and then $A^{\mathrm{rec}(a)} = A$.

Suppose $\mu \in hV_\mathbb{Z}^\vee/hV_\mathbb{Z}$ satisfies $\mu \sim \lambda$ for some $\lambda \in V_{0\mathbb{Z}}^\vee$. By (5.3.4) we may fix some $\tilde\mu \in I^\perp \cap (\mu + hV_\mathbb{Z})$. This allows us to compute, using (5.3.1),

$$\begin{aligned}\zeta_\mu^{\mathrm{rec}(a)} = e^{2\pi i[\tilde\mu, ak]} &= e^{2\pi i[\tilde\mu, a\ell_*]}e^{2\pi i Q(k)\cdot[\tilde\mu, a\ell]}\\ &= e^{2\pi i[\tilde\mu, s(a)^{-1}\ell_*]}e^{2\pi i Q(k)\cdot[\tilde\mu, s(a)\ell]}.\end{aligned}$$

As $[\tilde\mu, \ell] = 0$, we have $[\tilde\mu, s(a)\ell] = 0 = [\tilde\mu, s(a)^{-1}\ell]$. Thus

$$\zeta_\mu^{\mathrm{rec}(a)} = e^{2\pi i[\tilde\mu, s(a)^{-1}\ell_*]}e^{2\pi i Q(k)\cdot[\tilde\mu, s(a)^{-1}\ell]} = e^{2\pi i[\tilde\mu, s(a)^{-1}k]} = e^{2\pi i[s(a)\tilde\mu, k]}.$$

As above, $s(a)$ acts trivially on $hV_\mathbb{Z}^\vee/hV_\mathbb{Z}$, and we conclude that

$$\zeta_\mu^{\mathrm{rec}(a)} = e^{2\pi i[\tilde\mu, k]} = \zeta_\mu. \qquad \square$$

Suppose $a \in \mathbb{A}_f^\times$. The image of the discrete group

$$\Gamma_\Phi^{(a)} = s(a)K_\Phi s(a)^{-1} \cap Q_\Phi(\mathbb{Q})^\circ$$

under $\nu_\Phi : Q_\Phi \to \mathbb{G}_m$ is contained in $\widehat{\mathbb{Z}}^\times \cap \mathbb{Q}_{>0}^\times = \{1\}$, and hence $\Gamma_\Phi^{(a)}$ is contained in $\ker(\nu_\Phi) = U_\Phi$. Recalling that the conjugation action of Q_Φ on U_Φ is by ν_Φ, we find that

$$\Gamma_\Phi^{(a)} = \mathrm{rat}(\nu_\Phi(s(a))) \cdot \big(K_\Phi \cap U_\Phi(\mathbb{Q})\big) = \mathrm{rat}(a) \cdot \Gamma_\Phi$$

as lattices in $U_\Phi(\mathbb{Q})$.

Recalling (4.6.9) and (2.6.4), consider the following commutative diagram of complex analytic spaces

$$
\begin{array}{ccccc}
\bigsqcup_a \Gamma_\Phi^{(a)}\backslash \mathcal{D}^\circ & \dashrightarrow & \bigsqcup_a T_\Phi(\mathbb{C}) & = & \bigsqcup_a \Gamma_\Phi(-1)\otimes \mathbb{C}^\times \\
{\scriptstyle \cong}\big\downarrow{\scriptstyle z\mapsto (z,s(a))} & & \big\downarrow{\scriptstyle \cong} & & \\
\mathscr{U}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi) & \longrightarrow & \mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) & & \\
\big\downarrow & & \big\downarrow & & \\
\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) & \longrightarrow & \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) & & \\
\big\downarrow{\scriptstyle (z,g)\mapsto (z,gh)} & & & & \\
\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}), & & & &
\end{array}
\tag{5.4.5}
$$

in which all horizontal arrows are open immersions, all vertical arrows are local isomorphisms on the source, and the disjoint unions are over a set of coset representatives $a \in \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K_0$. The dotted arrow is, by definition, the unique open immersion making the upper left square commute.

Lemma 5.4.4. — *Fix a $\lambda \in V_{0\mathbb{R}}$ whose image under* (5.4.4) *satisfies*

$$\alpha(\lambda) \in \Gamma_\Phi^\vee(1),$$

and suppose

$$v \in V_{0\mathbb{R}} + i \cdot \mathrm{LightCone}^\circ(V_{0\mathbb{R}}).$$

If we restrict the character

$$q_{\alpha(\lambda)} : T_\Phi(\mathbb{C}) \to \mathbb{C}^\times$$

to a function $\Gamma_\Phi^{(a)}\backslash \mathcal{D}^\circ \to \mathbb{C}^\times$ via the open immersion in the top row of (5.4.5)*, its value at the isotropic vector*

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^\circ$$

is $e^{2\pi i[\lambda,v]/\mathrm{rat}(a)}$.

Proof. — The proof is a (rather tedious) exercise in tracing through the definitions. The dotted arrow in the diagram above is induced by the open immersion $\mathcal{D}^\circ \subset U_\Phi(\mathbb{C})\mathcal{D}^\circ = \mathcal{D}_\Phi^\circ$ and the isomorphisms

$$\bigsqcup_a \Gamma_\Phi^{(a)}\backslash \mathcal{D}_\Phi^\circ \cong \mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \cong \bigsqcup_a T_\Phi(\mathbb{C}). \tag{5.4.6}$$

The second isomorphism is the trivialization of the $T_\Phi(\mathbb{C})$-torsor

$$\mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \to \mathrm{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) \tag{5.4.7}$$

induced by the section $s : (\mathbb{G}_m, \mathcal{H}_0) \to (Q_\Phi, \mathcal{D}_\Phi)$, as in the proof of Proposition 4.6.2.

Tracing through the proof of Proposition 2.3.1, this isomorphism is obtained by combining the isomorphism

$$U_\Phi(\mathbb{C})/\Gamma_\Phi \cong \Gamma_\Phi(-1)\otimes \mathbb{C}/\mathbb{Z}(1) \xrightarrow{\mathrm{id}\otimes\exp} \Gamma_\Phi(-1)\otimes\mathbb{C}^\times = T_\Phi(\mathbb{C}) \tag{5.4.8}$$

with the isomorphism

$$U_\Phi(\mathbb{C})/\Gamma_\Phi \xrightarrow{-\mathrm{rat}(a)} U_\Phi(\mathbb{C})/\Gamma_\Phi^{(a)} \cong \Gamma_\Phi^{(a)}\backslash \mathcal{D}_\Phi^\circ \tag{5.4.9}$$

obtained by trivializing $\Gamma_\Phi^{(a)}\backslash\mathcal{D}_\Phi^\circ$ as a $U_\Phi(\mathbb{C})/\Gamma_\Phi^{(a)}$-torsor using the point $\ell_* \in \mathcal{D}_\Phi^\circ$. Note the minus sign in (5.4.9), which arises from the minus sign in the isomorphism (2.3.5) used to define the torsor structure on (5.4.7).

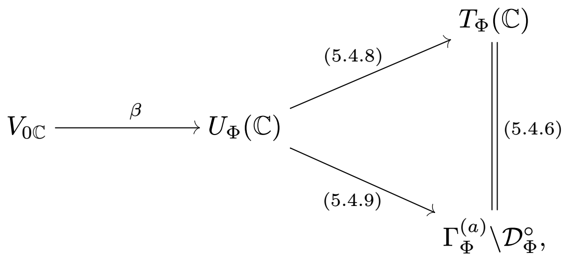
Denote by β the composition

$$V_{0\mathbb{R}} \xrightarrow{\otimes\ell} V_{0\mathbb{R}}\otimes I \xrightarrow{(4.4.2)} U_\Phi(\mathbb{R}).$$

It is related to $\alpha(\lambda)\in U_\Phi(\mathbb{R})^\vee(1)$ by

$$\langle\alpha(\lambda),\beta(v)\rangle = -2\pi i\cdot[\lambda,v],$$

for all $v\in V_{0\mathbb{R}}$. Extending β complex linearly yields a commutative diagram



and going all the way back to the definitions preceding (2.3.6), we find that the pullback of $q_{\alpha(\lambda)} : T_\Phi(\mathbb{C})\to\mathbb{C}^\times$ to a function on $V_{0\mathbb{C}}$ is given by

$$q_{\alpha(\lambda)}(v) = e^{-2\pi i[\lambda,v]}.$$

On the other hand, the composition along the bottom row sends

$$\frac{v}{-\mathrm{rat}(a)}\in V_{0\mathbb{C}}$$

to the point obtained by translating $\ell_*\in\mathcal{D}_\Phi^\circ$ by the vector $v\otimes\ell\in V_{0\mathbb{C}}\otimes I$, viewed as an element of $U_\Phi(\mathbb{C})$ using (4.4.2). This translate is

$$\ell_* + v - [\ell_*,v]\ell - Q(v)\ell \in \mathcal{D}_\Phi^\circ,$$

and hence the value of $q_{\alpha(\lambda)}$ at this point is

$$q_{\alpha(\lambda)}\left(\frac{v}{-\mathrm{rat}(a)}\right) = e^{2\pi i[\lambda,v]/\mathrm{rat}(a)}. \qquad\square$$

Lemma 5.4.5. — *Suppose $v \in V_{0\mathbb{R}} + i \cdot \mathscr{W}$ with $|Q(\mathrm{Im}(v))| \gg 0$. The value of the meromorphic function*

$$\mathrm{rat}(a)^{c(0,0)/2} \cdot [\ell^{c(0,0)/2}, \Psi_{s(a)h}(f)]$$

at the isotropic line $\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^\circ$ is

$$A_\Phi^{\mathrm{rec}(a)} \cdot e^{2\pi i[\varrho,v]/\mathrm{rat}(a)} \times \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^\vee \\ [\lambda,\mathscr{W}]>0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - \zeta_\mu^{\mathrm{rec}(a)} \cdot e^{2\pi i[\lambda,v]/\mathrm{rat}(a)}\right)^{c(-Q(\lambda),h^{-1}\mu)},$$

up to scaling by a complex number of absolute value 1.

Proof. — The proof amounts to carefully keeping track of how Theorem 5.3.2 changes when $\Psi_h(f)$ is replaced by $\Psi_{s(a)h}(f)$. The main source of confusion is that the vectors ℓ and ℓ_* appearing in Theorem 5.3.2 were chosen to have nice properties with respect to the lattice $hV_{\mathbb{Z}}$, and so we must first pick new isotropic vectors $\ell^{(a)}$ and $\ell_*^{(a)}$ having similarly nice properties with respect to $s(a)hV_{\mathbb{Z}}$.

Set $\ell^{(a)} = \mathrm{rat}(a)\ell$. This is a generator of

$$I \cap s(a)hV_{\mathbb{Z}} = \mathrm{rat}(a) \cdot (I \cap hV_{\mathbb{Z}}).$$

Now choose a $k^{(a)} \in s(a)hV_{\mathbb{Z}}^\vee$ such that $[\ell^{(a)}, k^{(a)}] = 1$, and let $I_*^{(a)} \subset V$ be the span of the isotropic vector

$$\ell_*^{(a)} = k^{(a)} - Q(k^{(a)})\ell^{(a)}.$$

Using the fact that Q_Φ acts trivially on the quotient $I^\perp/I$, it is easy to see that the lattice

$$V_{0\mathbb{Z}}^{(a)} = (I^\perp \cap s(a)hV_{\mathbb{Z}})/(I \cap s(a)hV_{\mathbb{Z}}) \subset I^\perp/I$$

is equal, as a subset of $I^\perp/I$, to the lattice $V_{0\mathbb{Z}}$ of (5.3.2). Thus replacing $hV_{\mathbb{Z}}$ by $s(a)hV_{\mathbb{Z}}$ has no effect on the construction of the modular form f_0, or on the formation of Weyl chambers or their corresponding Weyl vectors.

Similarly, as Q_Φ stabilizes I, the ideal $N\mathbb{Z} = [hV_{\mathbb{Z}}, I \cap hV_{\mathbb{Z}}]$ is unchanged if h is replaced by $s(a)h$. Replacing h by $s(a)h$ in the definition of A now determines a new constant

$$\begin{aligned} A^{(a)} &= \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x/N}\right)^{c(0, x \cdot h^{-1} s(a)^{-1} \ell^{(a)}/N)} \\ &= \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x/N}\right)^{c(0, x \cdot \mathrm{unit}(a)^{-1} h^{-1} \ell/N)} \\ &= \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x \cdot \mathrm{unit}(a)/N}\right)^{c(0, x h^{-1} \ell/N)} \\ &= A^{\mathrm{rec}(a)}. \end{aligned}$$

Citing Theorem 5.3.2 with h replaced by $s(a)h$ everywhere, and using the isomorphism

$$s(a)hV_{\mathbb{Z}}^{\vee}/s(a)hV_{\mathbb{Z}} \cong hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$$

induced by the action of $s(a)^{-1}$, we find that the value of

$$[(\ell^{(a)})^{c(0,0)/2}, \Psi_{s(a)h}(f)] = \mathrm{rat}(a)^{c(0,0)/2}[\ell^{c(0,0)/2}, \Psi_{s(a)h}(f)] \tag{5.4.10}$$

at the isotropic line

$$\ell_*^{(a)} + v - [\ell_*^{(a)}, v]\ell^{(a)} - Q(v)\ell^{(a)} \in \mathcal{D}^\circ$$

is given by the infinite product

$$A_\Phi^{(a)} \cdot e^{2\pi i[\varrho, v]} \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^\vee \\ [\lambda, \mathscr{W}]>0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - e^{2\pi i[s(a)\mu, k^{(a)}]} \cdot e^{2\pi i[\lambda, v]}\right)^{c(-Q(\lambda), h^{-1}\mu)}.$$

Now make a change of variables. If we set $v^{(a)} = \ell_* - \mathrm{rat}(a)\ell_*^{(a)} \in V_0$, we find that the value of (5.4.10) at the isotropic line

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell = \ell_*^{(a)} + \Big(\frac{v + v^{(a)}}{\mathrm{rat}(a)}\Big) - \Big[\ell_*^{(a)}, \Big(\frac{v + v^{(a)}}{\mathrm{rat}(a)}\Big)\Big]\ell^{(a)} - Q\Big(\frac{v + v^{(a)}}{\mathrm{rat}(a)}\Big)\ell^{(a)}$$

is

$$A_\Phi^{(a)} \cdot e^{2\pi i[\varrho, v+v^{(a)}]/\mathrm{rat}(a)} \times \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^\vee \\ [\lambda, \mathscr{W}]>0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} \left(1 - e^{2\pi i[s(a)\mu, k^{(a)}]} e^{2\pi i[\lambda, v+v^{(a)}]/\mathrm{rat}(a)}\right)^{c(-Q(\lambda), h^{-1}\mu)}.$$

Assuming that $\mu \sim \lambda$, we may lift $\lambda \in I^\perp/I$ to $\tilde{\mu} \in I^\perp \cap (\mu + hV_{\mathbb{Z}})$. As $s(a) \in Q_\Phi(\mathbb{A}_f)$ acts trivially on $(I^\perp/I) \otimes \mathbb{A}_f$, we have

$$[\lambda, v^{(a)}] = [\lambda, s(a)^{-1}v^{(a)}] = [\tilde{\mu}, s(a)^{-1}v^{(a)}].$$

Using (4.6.6) and the definition of $v^{(a)}$, we find

$$\mathrm{rat}(a)^{-1}s(a)^{-1}v^{(a)} = \mathrm{unit}(a)\ell_* - s(a)^{-1}\ell_*^{(a)}.$$

Combining these relations with $[\tilde{\mu}, \ell_*] = [\tilde{\mu}, k]$ and $[\tilde{\mu}, \ell_*^{(a)}] = [\tilde{\mu}, k^{(a)}]$ shows that

$$\frac{[\lambda, v^{(a)}]}{\mathrm{rat}(a)} = [\tilde{\mu}, \mathrm{unit}(a)k - s(a)^{-1}k^{(a)}].$$

As $\mathrm{unit}(a)k - s(a^{-1})k^{(a)} \in hV_{\widehat{\mathbb{Z}}}^\vee$ and $\tilde{\mu} - \mu \in hV_{\mathbb{Z}}$, we deduce the equality

$$\frac{[\lambda, v^{(a)}]}{\mathrm{rat}(a)} = [\mu, \mathrm{unit}(a)k - s(a)^{-1}k^{(a)}]$$

in $\mathbb{Q}/\mathbb{Z} \cong \widehat{\mathbb{Q}}/\widehat{\mathbb{Z}}$. Thus

$$\begin{aligned} e^{2\pi i[s(a)\mu, k^{(a)}]} e^{2\pi i[\lambda, v+v^{(a)}]/\mathrm{rat}(a)} &= e^{2\pi i[\mu, s(a)^{-1}k^{(a)}]} e^{2\pi i[\lambda, v+v^{(a)}]/\mathrm{rat}(a)} \\ &= e^{2\pi i[\mu, \mathrm{unit}(a)k]} e^{2\pi i[\lambda, v]/\mathrm{rat}(a)} \\ &= \zeta_\mu^{\mathrm{unit}(a)} \cdot e^{2\pi i[\lambda, v]/\mathrm{rat}(a)}. \end{aligned}$$

Finally, the equality

$$e^{2\pi i[\varrho, v+v^{(a)}]/\mathrm{rat}(a)} = e^{2\pi i[\varrho, v]/\mathrm{rat}(a)}$$

holds up to a root of unity, simply because $[\varrho, v^{(a)}] \in \mathbb{Q}$. □

Working on one connected component

$$\Gamma_\Phi^{(a)} \backslash \mathcal{D}^\circ \hookrightarrow \mathscr{U}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi),$$

the pullback of $\Psi(f)$ is $\Psi_{s(a)h}(f)$. The pullback of the section $\boldsymbol{\ell}^{\mathrm{an}}$ of the constant vector bundle $\boldsymbol{I}_{\mathrm{dR}}^{\mathrm{an}}$ determined by $I_\mathbb{C}$ is, by the definition preceding Proposition 4.6.1, the constant section determined by

$$\frac{\mathrm{rat}(a)}{2\pi i} \cdot \ell \in I_\mathbb{C}.$$

Thus on $\Gamma_\Phi^{(a)} \backslash \mathcal{D}^\circ$ we have the equality of meromorphic functions

$$(2\pi i)^{c(0,0)/2} \cdot [\boldsymbol{\ell}^{\otimes c(0,0)/2}, \Psi(f)] = \mathrm{rat}(a)^{c(0,0)/2} \cdot [\ell^{\otimes c(0,0)/2}, \Psi_{s(a)h}(f)].$$

Combining the two lemmas above, we see that the value of this meromorphic function at the isotropic line $\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^\circ$ is

$$A_\Phi^{\mathrm{rec}(a)} \cdot q_{\alpha(\varrho)} \cdot \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^\vee \\ [\lambda, \mathscr{W}] > 0}} \prod_{\substack{\mu \in hV_\mathbb{Z}^\vee / hV_\mathbb{Z} \\ \mu \sim \lambda}} \left(1 - \zeta_\mu^{\mathrm{rec}(a)} \cdot q_{\alpha(\lambda)}\right)^{c(-Q(\lambda), h^{-1}\mu)},$$

up to scaling by a complex number of absolute value 1. The stated q-expansion (5.4.3) follows from this.

It remains to prove the integrality conditions $\alpha(\lambda) \in \Gamma_\Phi^\vee(1)$. A priori, every $\alpha(\lambda)$ appearing in the product above (including $\lambda = \varrho$) lies in $U_\Phi(\mathbb{Q})^\vee(1)$. However, as the product itself is invariant under the action of

$$\Gamma_\Phi^{(a)} = \mathrm{rat}(a) \cdot \Gamma_\Phi \subset U_\Phi(\mathbb{Q})$$

on $\mathcal{D}^\circ$, the uniqueness of the q-expansion implies that only those terms

$$q_{\alpha(\lambda)} = e^{2\pi i[\lambda, v]/\mathrm{rat}(a)} \tag{5.4.11}$$

that are themselves invariant under $\Gamma_\Phi^{(a)}$ can appear. Pullback by the action of $u \in U_\Phi(\mathbb{Q})$ sends

$$q_{\alpha(\lambda)} \mapsto q_{\alpha(\lambda)} \cdot e^{\langle \alpha(\lambda), u \rangle / \mathrm{rat}(a)},$$

where $\langle -, - \rangle : U_\Phi(\mathbb{Q})^\vee(1) \otimes U_\Phi(\mathbb{Q}) \to \mathbb{Q}(1)$ is the tautological pairing, and it follows that the invariance of (5.4.11) under $\Gamma_\Phi^{(a)}$ is equivalent to the integrality condition $\alpha(\lambda) \in \Gamma_\Phi^\vee(1)$.

This completes the proof of Proposition 5.4.2. □

6. Integral models

As in § 5, we keep $V_{\mathbb{Z}} \subset V$ of signature $(n, 2)$ with $n \geq 1$. Fix a prime p at which $V_{\mathbb{Z}}$ is maximal in the sense of § 1.1, and assume that the compact open subgroup (1.1.2) factors as $K = K_p K^p$ with p-component

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times.$$

Under this assumption, we recall from [**1, 2**] the construction of an integral model

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_{(p)})$$

of $\mathrm{Sh}_K(G, \mathcal{D})$, and extensions to this model of the line bundle of weight one modular forms and the special divisors. In those references it is assumed that $V_{\mathbb{Z}}$ is maximal at *every* prime, but nearly everything extends verbatim to the more general case considered here. Indeed, one only has to be careful about the definitions of special divisors in § 6.4. Once the correct definitions are formulated the proofs of [*loc. cit.*] go through without significant change, and we simply give the appropriate citations without further comment.

The main new result is the pullback formula of Proposition 6.6.3, which describes how special divisors restrict under embeddings between orthogonal Shimura varieties of different dimension. This will be a crucial ingredient in our algebraic variant of the embedding trick of Borcherds.

6.1. Almost self-dual lattices. — The motivation for the following definition will become clear in § 6.3.

Definition 6.1.1. — We say that $V_{\mathbb{Z}_p}$ is *almost self-dual* if it has one of the following (mutually exclusive) properties:
— $V_{\mathbb{Z}_p}$ is self-dual;
— $p = 2$, $\dim_{\mathbb{Q}}(V)$ is odd, and $[V_{\mathbb{Z}_2}^\vee : V_{\mathbb{Z}_2}]$ is not divisible by 4.

Remark 6.1.2. — Almost self-duality is equivalent to the smoothness of the quadric over $\mathbb{Z}_p$ parameterizing isotropic lines in $V_{\mathbb{Z}_p}$. Here, an *isotropic line* in V_R for an $\mathbb{Z}_p$-algebra R is a local direct summand $I \subset V_R$ of rank 1 that is locally generated by an element $v \in I$ satisfying $Q(v) = 0$.

Recall from § 4.3 that G acts on the $\mathbb{Q}$-vector space $H = C(V)$, and that one may choose a $\mathbb{Z}$-valued symplectic form ψ on

$$H_{\mathbb{Z}} = C(V_{\mathbb{Z}})$$

in such a way that the action of G induces a Hodge embedding into the Siegel Shimura datum determined by (H, ψ). The following lemma will be used in § 8 to choose ψ in a particularly nice way.

Lemma 6.1.3. — *Assume that V has Witt index 2 (this is automatic if $n \geq 5$). If $V_{\mathbb{Z}_p}$ is almost self-dual, then we may choose ψ as above in such a way that $H_{\mathbb{Z}_p}$ is self-dual.*

Proof. — Choose any isotropic line $I \subset V$, and let $\ell \in I \cap V_{\mathbb{Z}}$ be a $\mathbb{Z}$-module generator. Let N be the positive integer defined by

$$N\mathbb{Z} = [V_{\mathbb{Z}}, \ell].$$

On the one hand, $\ell/N \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$ is isotropic under the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form induced by Q. On the other hand, maximality of $V_{\mathbb{Z}_p}$ implies that $V_{\mathbb{Z}_p}^{\vee}/V_{\mathbb{Z}_p}$ has no nonzero isotropic vectors. Thus $\ell/N \in V_{\mathbb{Z}_p}$, and so $p \nmid N$.

It follows that there is some $k \in V_{\mathbb{Z}}$ such that $p \nmid [k, \ell]$, and from this it is easy to see that there exists a vector $v \in \mathbb{Z}k + \mathbb{Z}\ell$ such that $Q(v)$ is negative and prime to p.

The $\mathbb{Q}$-span of $k, \ell \in V$ is a hyperbolic plane over $\mathbb{Q}$, and the $\mathbb{Z}_p$-span of $k, \ell \in V_{\mathbb{Z}_p}$ is an integral hyperbolic plane over $\mathbb{Z}_p$. It follows that the orthogonal complement

$$W = (\mathbb{Q}k + \mathbb{Q}\ell)^{\perp} \subset V$$

has Witt index 1, and that the $\mathbb{Z}$-lattice $W_{\mathbb{Z}} = W \cap V_{\mathbb{Z}}$ satisfies

$$V_{\mathbb{Z}_p} = \mathbb{Z}_p k \oplus \mathbb{Z}_p \ell \oplus W_{\mathbb{Z}_p}.$$

In particular $W_{\mathbb{Z}_p}$ is again maximal. Repeating the argument above with $V_{\mathbb{Z}}$ replaced by $W_{\mathbb{Z}}$, we find another vector $w \in V_{\mathbb{Z}}$ with $Q(w)$ negative and prime to p, and $[v, w] = 0$.

We have now constructed an element $\delta = vw \in C(V_{\mathbb{Z}})$ such that

$$\delta^2 = -Q(v)Q(w) \in \mathbb{Z}_{(p)}^{\times}.$$

Set $\psi(x, y) = \mathrm{Trd}(x\delta y^*)$, exactly as in § 4.3.

It remains to prove that $H_{\mathbb{Z}_p}$ is self-dual. We will use the decomposition

$$H_{\mathbb{Z}_p} = H_{\mathbb{Z}_p}^{+} \oplus H_{\mathbb{Z}_p}^{-}$$

induced by the decomposition $C(V_{\mathbb{Z}_p}) = C^+(V_{\mathbb{Z}_p}) \oplus C^-(V_{\mathbb{Z}_p})$ into even and odd parts. It is not hard to see that these direct summands of $H_{\mathbb{Z}_p}$ are orthogonal to each other under ψ, and so it suffices to prove the self-duality of each summand individually.

According to [**13**, § C.2], the almost self-duality of $V_{\mathbb{Z}_p}$ implies that the even Clifford algebra $C^+(V_{\mathbb{Z}_p})$ is an Azumaya algebra over its center, and this center is itself a finite étale $\mathbb{Z}_p$-algebra. Equivalently, $C^+(V_{\mathbb{Z}_p})$ is isomorphic étale locally on $\mathrm{Spec}(\mathbb{Z}_p)$ to a finite product of matrix algebras. It follows from this that

$$C^+(V_{\mathbb{Z}_p}) \otimes C^+(V_{\mathbb{Z}_p}) \xrightarrow{x \otimes y \mapsto \mathrm{Trd}(xy)} \mathbb{Z}_p$$

is a perfect bilinear pairing. The self-duality of $H_{\mathbb{Z}_p}^{+}$ under ψ follows easily from this. The self-duality of $H_{\mathbb{Z}_p}^{-}$ then follows using the isomorphism

$$H_{\mathbb{Z}_p}^{-} \cong H_{\mathbb{Z}_p}^{+}$$

given by right multiplication by the $v \in C(V_{\mathbb{Z}})$ chosen above, and the relation

$$\psi(xv, yv) = -Q(v) \cdot \psi(x, y)$$

for all $x, y \in H$. □

6.2. Isometric embeddings. — We will repeatedly find ourselves in the following situation. Suppose we have another quadratic space $(V^\diamond, Q^\diamond)$ of signature $(n^\diamond, 2)$, and an isometric embedding $V \hookrightarrow V^\diamond$. This induces a morphism of Clifford algebras $C(V) \to C(V^\diamond)$, which induces a morphism of GSpin Shimura data

$$(G, \mathcal{D}) \to (G^\diamond, \mathcal{D}^\diamond).$$

Just as we assume for (V, Q), suppose we are given a $\mathbb{Z}$-lattice $V^\diamond_{\mathbb{Z}} \subset V^\diamond$ on which $Q^\diamond$ is integer valued, and which is maximal at p. Let

$$K^\diamond = K^\diamond_p \cdot K^{\diamond,p} \subset G^\diamond(\mathbb{A}_f) \cap C(V^\diamond_{\widehat{\mathbb{Z}}})^\times$$

be a compact open subgroup with p-component

$$K^\diamond_p = G^\diamond(\mathbb{Q}_p) \cap C(V^\diamond_{\mathbb{Z}_p})^\times.$$

Assume that $V_{\mathbb{Z}} \subset V^\diamond_{\mathbb{Z}}$ and $K \subset K^\diamond$, so that we have a finite and unramified morphism

$$j : \mathrm{Sh}_K(G, \mathcal{D}) \to \mathrm{Sh}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond) \tag{6.2.1}$$

of canonical models. Our choices imply (using the assumption that $V_{\mathbb{Z}_p}$ is maximal) that $V_{\mathbb{Z}_p} = V_{\mathbb{Q}_p} \cap V^\diamond_{\mathbb{Z}_p}$ and $K_p = K^\diamond_p \cap G(\mathbb{Q}_p)$.

Lemma 6.2.1. — *It is possible to choose $(V^\diamond, Q^\diamond)$ and $V^\diamond_{\mathbb{Z}}$ as above in such a way that $V^\diamond_{\mathbb{Z}}$ is self-dual. Moreover, we can ensure that $V \subset V^\diamond$ has codimension at most 2 if n is even, and has codimension at most 3 if n is odd.*

Proof. — An exercise in the classification of quadratic spaces over $\mathbb{Q}$ shows that we may choose a positive definite quadratic space W in such a way that the orthogonal direct sum $V^\diamond = V \oplus W$ admits a self-dual lattice locally at every finite prime (for example, we may arrange for $V^\diamond$ to be a sum of hyperbolic planes locally at every finite prime). From Eichler's theorem that any two maximal lattices in a $\mathbb{Q}_p$-quadratic space are isometric [**16**, Theorem 8.8], it follows that any maximal lattice in $V^\diamond$ is self-dual. Enlarging $V_{\mathbb{Z}}$ to a maximal lattice $V^\diamond_{\mathbb{Z}} \subset V^\diamond$ proves the first claim.

A more careful analysis, once again using the classification of quadratic spaces, also yields the second claim. □

6.3. Definition of the integral model. — We now define our integral model of the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$.

Assume first that $V_{\mathbb{Z}_p}$ is almost self-dual. This implies, by [**13**, § C.4], that

$$\mathcal{G} = \mathrm{GSpin}(V_{\mathbb{Z}_{(p)}})$$

is a reductive group scheme over $\mathbb{Z}_{(p)}$, and hence that $K_p = \mathcal{G}(\mathbb{Z}_p)$ is a hyperspecial compact open subgroup of $G(\mathbb{Q}_p)$. Thus $\mathrm{Sh}_K(G, \mathcal{D})$ admits a canonical smooth integral model $\mathcal{S}_K(G, \mathcal{D})$ over $\mathbb{Z}_{(p)}$ by the results of Kisin [**31**] (and [**30**] if $p = 2$).

Remark 6.3.1. — The notion of almost self-duality does not appear anywhere in our main references [**39, 1, 2**] on integral models of $\mathrm{Sh}_K(G,\mathcal{D})$. This is due to an oversight on the authors' part: we did not realize that one could obtain smooth integral models even if $V_{\mathbb{Z}_p}$ fails to be self-dual.

According to [**30**, Proposition 3.7] there is a functor

$$N \mapsto (\boldsymbol{N}_{\mathrm{dR}}, F^{\bullet}\boldsymbol{N}_{\mathrm{dR}}) \tag{6.3.1}$$

from representations $\mathcal{G} \to \mathrm{GL}(N)$ on free $\mathbb{Z}_{(p)}$-modules of finite rank to filtered vectors bundles on $\mathcal{S}_K(G,\mathcal{D})$, restricting to the functor (3.3.2) in the generic fiber. [5]

Applying this functor to the representation $V_{\mathbb{Z}_{(p)}}$ yields a filtered vector bundle $(\boldsymbol{V}_{\mathrm{dR}}, F^{\bullet}\boldsymbol{V}_{\mathrm{dR}})$. Applying the functor to the representation $H_{\mathbb{Z}_{(p)}} = C(V_{\mathbb{Z}_{(p)}})$ yields a filtered vector bundle $(\boldsymbol{H}_{\mathrm{dR}}, F^{\bullet}\boldsymbol{H}_{\mathrm{dR}})$. The inclusion (4.1.2) restricts to $V_{\mathbb{Z}_{(p)}} \to \mathrm{End}(H_{\mathbb{Z}_{(p)}})$, which determines an injection

$$\boldsymbol{V}_{\mathrm{dR}} \to \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})$$

onto a local direct summand.

For any local section x of $\boldsymbol{V}_{\mathrm{dR}}$, the composition $x \circ x$ is a local section of the subsheaf $\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})$. This defines a quadratic form

$$\boldsymbol{Q} : \boldsymbol{V}_{\mathrm{dR}} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})},$$

with an associated bilinear form

$$[-,-] : \boldsymbol{V}_{\mathrm{dR}} \otimes \boldsymbol{V}_{\mathrm{dR}} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})}$$

related as in (1.1.1). The filtration on $\boldsymbol{V}_{\mathrm{dR}}$ has the form

$$0 = F^2\boldsymbol{V}_{\mathrm{dR}} \subset F^1\boldsymbol{V}_{\mathrm{dR}} \subset F^0\boldsymbol{V}_{\mathrm{dR}} \subset F^{-1}\boldsymbol{V}_{\mathrm{dR}} = \boldsymbol{V}_{\mathrm{dR}},$$

in which $F^1\boldsymbol{V}_{\mathrm{dR}}$ is an isotropic line, and $F^0\boldsymbol{V}_{\mathrm{dR}} = (F^1\boldsymbol{V}_{\mathrm{dR}})^{\perp}$. As in § 4.2, the *line bundle of weight one modular forms* on $\mathcal{S}_K(G,\mathcal{D})$ is

$$\boldsymbol{\omega} = F^1\boldsymbol{V}_{\mathrm{dR}}.$$

If $V_{\mathbb{Z}_p}$ is not almost self-dual then choose auxiliary data $(V^{\diamond}, Q^{\diamond})$ as in § 6.2 in such a way that $V^{\diamond}_{\mathbb{Z}_p}$ is almost self-dual. This determines a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_K(G,\mathcal{D}) & \longleftarrow & \mathrm{Sh}_K(G,\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}) & \longleftarrow & \mathrm{Sh}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}), \end{array} \tag{6.3.2}$$

in which the lower left corner is the canonical integral model of $\mathrm{Sh}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond})$, and the upper right corner is defined as its normalization in $\mathrm{Sh}_K(G,\mathcal{D})$, in the sense of [**2**, Definition 4.2.1]. By construction, $\mathcal{S}_K(G,\mathcal{D})$ is a normal Deligne-Mumford stack, flat and of finite type over $\mathbb{Z}_{(p)}$.

[5] There is also a weight filtration on $\boldsymbol{N}_{\mathrm{dR}}$, but, as noted in Remark 3.4.2, it is not very interesting over the pure Shimura variety.

Define the *line bundle of weight one modular forms* on $\mathcal{S}_K(G,\mathcal{D})$ by

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{\diamond}|_{\mathcal{S}_K(G,\mathcal{D})}, \tag{6.3.3}$$

where $\boldsymbol{\omega}^{\diamond}$ is the line bundle on $\mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond})$ constructed in the almost self-dual case above. The line bundle (6.3.3) extends the line bundle of the same name previously constructed on the generic fiber.

The following is [**2**, Proposition 4.4.1].

Proposition 6.3.2. — *The $\mathbb{Z}_{(p)}$-stack $\mathcal{S}_K(G,\mathcal{D})$ and the line bundle $\boldsymbol{\omega}$ are independent of the auxiliary choices of $(V^{\diamond},Q^{\diamond})$, $V_{\mathbb{Z}}^{\diamond}$, and $K^{\diamond}$ used in their construction, and the Kuga-Satake abelian scheme of § 4.3 extends uniquely to an abelian scheme $\mathcal{A}\to\mathcal{S}_K(G,\mathcal{D})$.*

The following is a restatement of the main result of [**39**].

Proposition 6.3.3. — *If $p>2$ and p^2 does not divide $[V_{\mathbb{Z}}^{\vee}:V_{\mathbb{Z}}]$, then $\mathcal{S}_K(G,\mathcal{D})$ is regular.*

Remark 6.3.4. — Our $\mathcal{S}_K(G,\mathcal{D})$ is not quite the same as the integral model of [**1**]. That integral model is obtained from $\mathcal{S}_K(G,\mathcal{D})$ by deleting certain closed substacks supported in characteristics p for which p^2 divides $[V_{\mathbb{Z}}^{\vee}:V_{\mathbb{Z}}]$. The point of deleting such substacks is that the vector bundle $\boldsymbol{V}_{\mathrm{dR}}$ on $\mathrm{Sh}_K(G,\mathcal{D})$ of § 4.2 then extends canonically to the remaining open substack. In the present work, as in [**2**], the only automorphic vector bundle required on $\mathcal{S}_K(G,\mathcal{D})$ is the line bundle of modular forms $\boldsymbol{\omega}$ just constructed; we have no need of an extension of $\boldsymbol{V}_{\mathrm{dR}}$ to $\mathcal{S}_K(G,\mathcal{D})$.

6.4. Special divisors. — For $m\in\mathbb{Q}_{>0}$ and $\mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$ there is a Cartier divisor $\mathcal{Z}(m,\mu)$ on $\mathcal{S}_K(G,\mathcal{D})$, defined in [**1, 2**] in the case where $V_{\mathbb{Z}}$ is maximal. As we now assume only the weaker hypothesis that $V_{\mathbb{Z}}$ is maximal at p, the definition requires minor adjustment.

We first define the divisors in the generic fiber, where they were originally constructed by Kudla [**32**]. Our construction is different, and has a more moduli-theoretic flavor.

By the theory of automorphic vector bundles described in § 3, the G-equivariant inclusion (4.1.2) determines an inclusion

$$\boldsymbol{V}_{\mathrm{dR}}\subset\underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}}) \tag{6.4.1}$$

of vector bundles on $\mathrm{Sh}_K(G,\mathcal{D})$, respecting the Hodge filtrations. Recall from § 4.3 that the filtered vector bundle $\boldsymbol{H}_{\mathrm{dR}}$ is canonically identified with the first relative de Rham homology of the Kuga-Satake abelian scheme

$$\pi: A\to\mathrm{Sh}_K(G,\mathcal{D}).$$

The compact open subgroup $K\subset G(\mathbb{A}_f)$ appears as a quotient of the étale fundamental group of $\mathrm{Sh}_K(G,\mathcal{D})$, and hence representations of K give rise to étale local systems. In particular, for any prime ℓ the $\mathbb{Z}_\ell$-lattice $H_{\mathbb{Z}_\ell}$ determines an étale sheaf of $\mathbb{Z}_\ell$-modules $\boldsymbol{H}_\ell$ on $\mathrm{Sh}_K(G,\mathcal{D})$. This is just the relative ℓ-adic Tate module

$$\boldsymbol{H}_\ell \cong \underline{\mathrm{Hom}}(R^1\pi_{et,*}\underline{\mathbb{Z}_\ell}, \underline{\mathbb{Z}_\ell})$$

of the Kuga-Satake abelian scheme.

As in the discussion preceding (4.1.5), K also acts on both

$$V_{\mathbb{Z}_\ell} = V_{\mathbb{Z}} \otimes \mathbb{Z}_\ell \qquad \text{and} \qquad V_{\mathbb{Z}_\ell}^\vee = V_{\mathbb{Z}}^\vee \otimes \mathbb{Z}_\ell,$$

and the induced action on the quotient $V_{\mathbb{Z}_\ell}^\vee / V_{\mathbb{Z}_\ell}$ is trivial. These representations of K determine étale sheaves of $\mathbb{Z}_\ell$-modules $\boldsymbol{V}_\ell \subset \boldsymbol{V}_\ell^\vee$, along with an inclusion of étale $\mathbb{Z}_\ell$-sheaves

$$\boldsymbol{V}_\ell \subset \underline{\mathrm{End}}(\boldsymbol{H}_\ell). \tag{6.4.2}$$

and a canonical trivialization $\boldsymbol{V}_\ell^\vee/\boldsymbol{V}_\ell \cong V_{\mathbb{Z}_\ell}^\vee/V_{\mathbb{Z}_\ell}$. In particular, each $\mu_\ell \in V_{\mathbb{Z}_\ell}^\vee/V_{\mathbb{Z}_\ell}$ determines a subsheaf of sets

$$\mu_\ell + \boldsymbol{V}_\ell \subset \boldsymbol{V}_\ell \otimes \mathbb{Q}_\ell. \tag{6.4.3}$$

Suppose we are given a $\mathbb{Q}$-scheme S and a morphism $S \to \mathrm{Sh}_K(G,\mathcal{D})$. Denote by $A_S \to S$ the pullback of the Kuga-Satake abelian scheme. A quasi-endomorphism [(6)] $x \in \mathrm{End}(A_S) \otimes \mathbb{Q}$ is *special* if

— its de Rham realization

$$x_{\mathrm{dR}} \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})|_S)$$

lies in the subsheaf $\boldsymbol{V}_{\mathrm{dR}}|_S$, and

— its ℓ-adic realization

$$x_\ell \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_\ell)|_S \otimes \mathbb{Q}_\ell)$$

lies in the subsheaf $\boldsymbol{V}_\ell|_S \otimes \mathbb{Q}_\ell$ for every prime ℓ.

The space of all special quasi-endomorphisms of A_S is a $\mathbb{Q}$-subspace

$$V(A_S) \subset \mathrm{End}(A_S) \otimes \mathbb{Q}.$$

Under the inclusion $V \subset \mathrm{End}(H)$, the quadratic form on V becomes $Q(x) = x \circ x$. Similarly, the square of any $x \in V(A_S)$ lies in $\mathbb{Q} \subset \mathrm{End}(A_S) \otimes \mathbb{Q}$, and $V(A_S)$ is endowed with the positive definite quadratic form $Q(x) = x \circ x$. For each $\mu \in V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}$, we now define

$$V_\mu(A_S) \subset V(A_S) \tag{6.4.4}$$

to be the set of all special quasi-endomorphisms whose ℓ-adic realization lies in the subsheaf (6.4.3) for every prime ℓ, and set

$$Z(m,\mu)(S) \stackrel{\mathrm{def}}{=} \{x \in V_\mu(A_S) : Q(x) = m\}.$$

---

(6) A quasi-endomorphism should really be defined as global section of the Zariski sheaf $\underline{\mathrm{End}}(A_S) \otimes \mathbb{Q}$ on S. If S is not of finite type over $\mathbb{Q}$, the space of such global sections can be strictly larger than $\mathrm{End}(A_S) \otimes \mathbb{Q}$. For simplicity of notation, we ignore this minor technical point.

We now explain how to extend this definition to the integral model.

First assume that $V_{\mathbb{Z}}$ is self-dual at p. As in the discussion following (6.3.1), the inclusion of vector bundles (6.4.1) has a canonical extension to the integral model $\mathcal{S}_K(G,\mathcal{D})$. Directly from the definitions, so does the inclusion of étale $\mathbb{Q}_\ell$-sheaves (6.4.2) for any $\ell \neq p$. As a substitute for p-adic étale cohomology, we use the inclusion

$$\boldsymbol{V}_{\mathrm{crys}} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{crys}}) \tag{6.4.5}$$

of locally free crystals on $\mathcal{S}_K(G,\mathcal{D})_{/\mathbb{F}_p}$ as in [**2**, Proposition 4.2.5]. There is a canonical isomorphism

$$\boldsymbol{H}_{\mathrm{crys}} \cong \underline{\mathrm{Hom}}(R^1\pi_{\mathrm{crys},*}\mathcal{O}^{\mathrm{crys}}_{\mathcal{A}_{\mathbb{F}_p}/\mathbb{Z}_p}, \mathcal{O}^{\mathrm{crys}}_{\mathcal{M}_{\mathbb{F}_p}/\mathbb{Z}_p})$$

between $\boldsymbol{H}_{\mathrm{crys}}$ and the first relative crystalline homology of the reduction of the Kuga-Satake abelian scheme $\pi : \mathcal{A} \to \mathcal{S}_K(G,\mathcal{D})$ of Proposition 6.3.2.

Still assuming that $V_{\mathbb{Z}}$ is self-dual at p, suppose we are given a $\mathbb{Z}_{(p)}$-scheme S and a morphism $S \to \mathcal{S}_K(G,\mathcal{D})$, and let $\mathcal{A}_S$ be the pullback of the Kuga-Satake abelian scheme. We call $x \in \mathrm{End}(A_S) \otimes \mathbb{Z}_{(p)}$ *special* if

— its de Rham realization

$$x_{\mathrm{dR}} \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})|_S)$$

lies in the subsheaf $\boldsymbol{V}_{\mathrm{dR}}|_S$,

— its ℓ-adic realization

$$x_\ell \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_\ell)|_S \otimes \mathbb{Q}_\ell)$$

lies in the subsheaf $\boldsymbol{V}_\ell|_S \otimes \mathbb{Q}_\ell$ for every prime $\ell \neq p$,

— its p-adic realization

$$x_p \in H^0(S_{\mathbb{Q}}, \underline{\mathrm{End}}(\boldsymbol{H}_p)|_{S_{\mathbb{Q}}})$$

over the generic fiber $S_{\mathbb{Q}}$ lies in the subsheaf $\boldsymbol{V}_p|_{S_{\mathbb{Q}}}$, and

— its crystalline realization

$$x_{\mathrm{crys}} \in H^0(S_{\mathbb{F}_p}, \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{crys}})|_{S_{\mathbb{F}_p}})$$

over the special fiber $S_{\mathbb{F}_p}$ lies in the subcrystal $\boldsymbol{V}_{\mathrm{crys}}|_{S_{\mathbb{F}_p}}$.

The space of all such $x \in \mathrm{End}(A_S) \otimes \mathbb{Z}_{(p)}$ is denoted

$$V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}} \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Z}_{(p)},$$

and tensoring with $\mathbb{Q}$ defines the subspace of all special quasi-endomorphisms

$$V(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}.$$

It endowed with a positive definite quadratic form $Q(x) = x \circ x$ exactly as above. For any $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$ we define

$$V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}} \tag{6.4.6}$$

as the subset of elements whose ℓ-adic realization lies in (6.4.3) for every prime $\ell \neq p$.

Now consider the general case in which $V_{\mathbb{Z}} \subset V$ is only assumed to be maximal at p. In this generality we still have the étale $\mathbb{Q}_\ell$-sheaves (6.4.2) for $\ell \neq p$. However, there is no adequate theory of automorphic vector bundles or crystals on $\mathcal{S}_K(G, \mathcal{D})$; compare with Remark 6.3.4. In particular, we have no adequate substitute for the sheaves in (6.4.5).

So that we may apply the results of [**1, 2**], enlarge $V_{\mathbb{Z}}$ to a lattice $V'_{\mathbb{Z}} \subset V$ that is maximal at every prime. This choice determines a second $\mathbb{Z}$-lattice $H'_{\mathbb{Z}} \subset H_{\mathbb{Q}}$, and hence a second Kuga-Satake abelian scheme

$$\mathcal{A}' \to \mathcal{S}_K(G, \mathcal{D})$$

endowed with an isogeny $\mathcal{A} \to \mathcal{A}'$ of degree prime to p. Choose a larger quadratic space $V^\diamond$ as in § 6.2 admitting a maximal lattice $V^\diamond_{\mathbb{Z}} \subset V^\diamond$ that is self-dual at p, and an isometric embedding $V'_{\mathbb{Z}} \to V^\diamond_{\mathbb{Z}}$.

By the very construction of the integral model, there is a finite morphism

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond).$$

According to [**1**, Proposition 2.5.1], the abelian schemes $\mathcal{A}'$ and

$$\mathcal{A}^\diamond \to \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond)$$

carry right actions of the integral Clifford algebras $C(V'_{\mathbb{Z}})$ and $C(V^\diamond_{\mathbb{Z}})$, respectively, and are related by a canonical isomorphism

$$\mathcal{A}' \otimes_{C(V'_{\mathbb{Z}})} C(V^\diamond_{\mathbb{Z}}) \cong \mathcal{A}^\diamond|_{\mathcal{S}_K(G,\mathcal{D})}. \tag{6.4.7}$$

Note that the Serre tensor construction on the left is defined because the maximality of $V'_{\mathbb{Z}}$ implies that $V'_{\mathbb{Z}} \subset V^\diamond_{\mathbb{Z}}$ as a $\mathbb{Z}$-module direct summand, which implies that the natural map $C(V'_{\mathbb{Z}}) \to C(V^\diamond_{\mathbb{Z}})$ makes $C(V^\diamond_{\mathbb{Z}})$ into a free $C(V'_{\mathbb{Z}})$-module.

Definition 6.4.1. — Suppose we are given a morphism $S \to \mathcal{S}_K(G, \mathcal{D})$. A quasi-endomorphism

$$x \in \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}$$

is *special* if the induced quasi-endomorphism of $\mathcal{A}'_S$ commutes with the action of $C(V'_{\mathbb{Z}})$, and its image under the map

$$\mathrm{End}_{C(V'_{\mathbb{Z}})}(\mathcal{A}'_S) \otimes \mathbb{Q} \to \mathrm{End}_{C(V^\diamond_{\mathbb{Z}})}(\mathcal{A}^\diamond_S) \otimes \mathbb{Q}$$

induced by (6.4.7) is a special quasi-endomorphism of $\mathcal{A}^\diamond_S$ (in the sense already defined for the self-dual-at-p lattice $V^\diamond_{\mathbb{Z}}$).

The following is [**2**, Proposition 4.3.4].

Proposition 6.4.2. — *If S is connected, then $x \in \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}$ is special if and only if the restriction $x_s \in \mathrm{End}(\mathcal{A}_s) \otimes \mathbb{Q}$ is special for some (equivalently, every) geometric point $s \to S$.*

Once again, the space of all special quasi-endomorphisms

$$V(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}$$

carries a positive definite quadratic form $Q(x) = x \circ x$. By construction it comes with an isometric embedding

$$V(\mathcal{A}_S) \subset V(\mathcal{A}_S^{\diamond}). \tag{6.4.8}$$

It remains to define a subset

$$V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S) \tag{6.4.9}$$

for each coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Let $\mu_\ell \in V_{\mathbb{Z}_\ell}^{\vee}/V_{\mathbb{Z}_\ell}$ be the ℓ-component. If $\ell \neq p$ let

$$V_{\mu_\ell}(\mathcal{A}_S) \subset V(\mathcal{A}_S)$$

be the subset of elements whose ℓ-adic realization lies in the subsheaf (6.4.3). To treat the p-part of μ, define

$$\Lambda = \{x \in V_{\mathbb{Z}}^{\diamond} : x \perp V_{\mathbb{Z}}\}.$$

The maximality of $V_{\mathbb{Z}}$ at p implies that $V_{\mathbb{Z}_p} \subset V_{\mathbb{Z}_p}^{\diamond}$ is a $\mathbb{Z}_p$-module direct summand. From this and the self-duality of $V_{\mathbb{Z}}^{\diamond}$ at p it is easy to see that the projections to the two factors in

$$V^{\diamond} = V \oplus \Lambda_{\mathbb{Q}}$$

induce bijections

$$V_{\mathbb{Z}_p}^{\vee}/V_{\mathbb{Z}_p} \cong (V_{\mathbb{Z}_p}^{\diamond})^{\vee}/V_{\mathbb{Z}_p}^{\diamond} \cong \Lambda_{\mathbb{Z}_p}^{\vee}/\Lambda_{\mathbb{Z}_p}. \tag{6.4.10}$$

The image of μ_p under this bijection is denoted $\bar{\mu}_p \in \Lambda_{\mathbb{Z}_p}^{\vee}/\Lambda_{\mathbb{Z}_p}$. As in [**1**, Proposition 2.5.1], there is a canonical isometric embedding

$$\Lambda \to V(\mathcal{A}_S^{\diamond})_{\mathbb{Z}_{(p)}}$$

whose image is orthogonal to that of (6.4.8). In fact, we have an orthogonal decmposition

$$V(\mathcal{A}_S^{\diamond}) = V(\mathcal{A}_S) \oplus \Lambda_{\mathbb{Q}},$$

which allows us to define

$$V_{\mu_p}(\mathcal{A}_S) = \{x \in V(\mathcal{A}_S) : x + \bar{\mu}_p \in V(\mathcal{A}_S^{\diamond})_{\mathbb{Z}_{(p)}}\}. \tag{6.4.11}$$

Finally, define (6.4.9) by

$$V_\mu(\mathcal{A}_S) = \bigcap_{\ell} V_{\mu_\ell}(\mathcal{A}_S).$$

This set is independent of the choice of auxiliary data $V_{\mathbb{Z}}' \subset V_{\mathbb{Z}}^{\diamond} \subset V^{\diamond}$ used in its definition, and agrees with the definition (6.4.4) if S is a $\mathbb{Q}$-scheme. See [**2**, Proposition 4.5.3].

The following is [**1**, Proposition 2.7.2].

Proposition 6.4.3. — *Given a positive $m \in \mathbb{Q}$ and a $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$, the functor sending an $\mathcal{S}_K(G,\mathcal{D})$-scheme S to*

$$\mathcal{Z}(m,\mu)(S) = \{x \in V_{\mu}(\mathcal{A}_S) : Q(x) = m\}$$

is represented by a finite, unramified, and relatively representable morphism of Deligne-Mumford stacks

$$\mathcal{Z}(m,\mu) \to \mathcal{S}_K(G,\mathcal{D}). \tag{6.4.12}$$

In the next subsection we will justify in what sense the morphisms (6.4.12), which are not even closed immersions, deserve the name *special divisors.*

We end this section by describing what the morphism (6.4.12) looks like in the complex fiber. For each $g \in G(\mathbb{A}_f)$, the pullback of (6.4.12) via the complex uniformization

$$\mathcal{D} \xrightarrow{z \mapsto (z,g)} \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$$

can be made explicit. Each $x \in V$ with $Q(x) > 0$ determines an analytic subset

$$\mathcal{D}(x) = \{z \in \mathcal{D} : [z,x] = 0\}$$

of the hermitian domain (4.1.1).

From the discussion of § 4.3, we see that the fiber of the Kuga-Satake abelian scheme at a point $z \in \mathcal{D}$ is the complex torus

$$A_z(\mathbb{C}) = gH_{\mathbb{Z}} \backslash H_{\mathbb{C}} / zH_{\mathbb{C}}.$$

The action of $x \in V \subset \mathrm{End}(H)$ by left multiplication in the Clifford algebra $C(V)$ defines a quasi-endomorphism of $A_z(\mathbb{C})$ if and only if it preserves the subspace $zH_{\mathbb{C}} \subset H_{\mathbb{C}}$, and a linear algebra exercise shows that this condition is equivalent to $z \in \mathcal{D}(x)$. Using this, one can check that the pullback of (6.4.12) via the above complex uniformization is

$$\bigsqcup_{\substack{x \in g\mu + gV_{\mathbb{Z}} \\ Q(x)=m}} \mathcal{D}(x) \to \mathcal{D}. \tag{6.4.13}$$

Here, by mild abuse of notation, $g\mu$ is the image of μ under the action-by-g isomorphism $V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \to gV_{\mathbb{Z}}^{\vee}/gV_{\mathbb{Z}}$

6.5. Deformation theory. — We need to explain the sense in which the morphism (6.4.12) merits the name *special divisor.* This is closely tied up with the deformation theory of special endomorphisms, which will also be needed in the proof of the pullback formula of Proposition 6.6.3 below.

It is enlightening to first consider the complex analytic situation of (6.4.13). Each subset $\mathcal{D}(x) \subset \mathcal{D}$ is not only an analytic divisor, but arises as the 0-locus of a canonical section

$$\mathrm{obst}_x^{\mathrm{an}} \in H^0(\mathcal{D}, \boldsymbol{\omega}_{\mathcal{D}}^{-1}) \tag{6.5.1}$$

of the inverse of the tautological bundle $\boldsymbol{\omega}_{\mathcal{D}}$ on (4.1.1). Indeed, recalling that the fiber of $\boldsymbol{\omega}_{\mathcal{D}}$ at $z \in \mathcal{D}$ is the isotropic line $\mathbb{C}z \subset V_{\mathbb{C}}$, we define (6.5.1) as the linear functional

$$\mathbb{C}z \xrightarrow{z \mapsto [z,x]} \mathbb{C}.$$

This is the *analytic obstruction to deforming* x.

Returning to the algebraic world, suppose

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{Z}(m,\mu) \\ \downarrow & & \downarrow \\ \widetilde{S} & \longrightarrow & \mathcal{S}_K(G,\mathcal{D}) \end{array} \tag{6.5.2}$$

is a commutative diagram of stacks in which $S \to \widetilde{S}$ is a closed immersion of schemes defined by an ideal sheaf $J \subset \mathcal{O}_{\widetilde{S}}$ with $J^2 = 0$. After pullback to S, the Kuga-Satake abelian scheme $\mathcal{A} \to \mathcal{S}_K(G,\mathcal{D})$ acquires a tautological special quasi-endomorphism

$$x \in V_\mu(\mathcal{A}_S),$$

and we want to know when this lies in the image of the (injective) restriction map

$$V_\mu(\mathcal{A}_{\widetilde{S}}) \to V_\mu(\mathcal{A}_S). \tag{6.5.3}$$

Equivalently, when there is a (necessarily unique) dotted arrow

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{Z}(m,\mu) \\ \downarrow & \nearrow & \downarrow \\ \widetilde{S} & \longrightarrow & \mathcal{S}_K(G,\mathcal{D}) \end{array}$$

making the diagram commute.

Proposition 6.5.1. — *In the situation above, there is a canonical section*

$$\mathrm{obst}_x \in H^0\big(\widetilde{S}, \boldsymbol{\omega}|_{\widetilde{S}}^{-1}\big), \tag{6.5.4}$$

called the obstruction to deforming x, *such that x lies in the image of* (6.5.3) *if and only if* $\mathrm{obst}_x = 0$.

Proof. — Suppose first that $V_{\mathbb{Z}}$ is self-dual at p, so that we have an inclusion

$$\boldsymbol{V}_{\mathrm{dR}} \to \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})$$

as a local direct summand of vector bundles on $\mathcal{S}_K(G,\mathcal{D})$. The vector bundle $\boldsymbol{H}_{\mathrm{dR}}$ is identified with the first relative de Rham homology of the Kuga-Satake abelian scheme $\mathcal{S}$. As such, it is is endowed with its Gauss-Manin connection, which restricts to a flat connection

$$\nabla : \boldsymbol{V}_{\mathrm{dR}} \to \boldsymbol{V}_{\mathrm{dR}} \otimes \Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}.$$

Indeed, one can check this in the complex fiber, over which the connection becomes identified, using (3.1.3), with

$$\boldsymbol{V}_{\mathrm{Be}} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})} \xrightarrow{1\otimes d} \boldsymbol{V}_{\mathrm{Be}} \otimes_{\mathbb{Z}} \Omega^1_{\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})}.$$

The de Rham realization

$$x_{\mathrm{dR}} \in H^0(S, \boldsymbol{V}_{\mathrm{dR}}|_S) \tag{6.5.5}$$

is parallel, and therefore admits parallel transport (the algebraic theory of parallel transport can be extracted from [**3**, § 2], for example) to $\widetilde{S}$: there is a unique parallel extension of x_{dR} to

$$\widetilde{x}_{\mathrm{dR}} \in H^0(\widetilde{S}, \boldsymbol{V}_{\mathrm{dR}}|_{\widetilde{S}}). \tag{6.5.6}$$

We now define obst_x be the image of $\widetilde{x}_{\mathrm{dR}}$ under $\boldsymbol{V}_{\mathrm{dR}} \to \boldsymbol{V}_{\mathrm{dR}}/F^0\boldsymbol{V}_{\mathrm{dR}}$, and use the perfect bilinear pairing (4.2.1) to identify

$$\boldsymbol{V}_{\mathrm{dR}}/F^0\boldsymbol{V}_{\mathrm{dR}} \cong (F^1\boldsymbol{V}_{\mathrm{dR}})^{-1} = \boldsymbol{\omega}^{-1}.$$

The local sections of $F^0\boldsymbol{V}_{\mathrm{dR}}$ are precisely those local sections of $\boldsymbol{V}_{\mathrm{dR}} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})$ which preserve the Hodge filtration $F^0\boldsymbol{H}_{\mathrm{dR}} \subset \boldsymbol{H}_{\mathrm{dR}}$. The vanishing of obst_x is equivalent to

$$\widetilde{x}_{\mathrm{dR}} \in H^0(\widetilde{S}, F^0\boldsymbol{V}_{\mathrm{dR}}|_{\widetilde{S}}),$$

which is therefore equivalent to the endomorphism

$$\widetilde{x}_{\mathrm{dR}} \in \mathrm{End}(\boldsymbol{H}_{\mathrm{dR}}|_{\widetilde{S}})$$

respecting the Hodge filtration. Using the deformation theory of abelian schemes described in [**38**, Chapter 2], this is equivalent to

$$x \in V_\mu(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Z}_{(p)}$$

admitting an extension to

$$\widetilde{x} \in \mathrm{End}(\mathcal{A}_{\widetilde{S}}) \otimes \mathbb{Z}_{(p)}.$$

Using Proposition 6.4.2 it is easy to see that when such an extension exists it must lie in $V_\mu(\mathcal{A}_S)$. This proves the claim when $V_{\mathbb{Z}}$ is self-dual at p.

We now explain how to construct the section (6.5.4) in general. Fix an isometric embedding $V_{\mathbb{Z}} \subset V^\diamond_{\mathbb{Z}}$ as in § 6.2, and assume that $V^\diamond_{\mathbb{Z}}$ is self-dual at p, so that we have morphisms of integral models

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond).$$

In the notation of (6.4.11), the special quasi-endomorphism $x \in V_\mu(\mathcal{A}_S)$ determines another special quasi-endomorphism

$$x^\diamond = x + \bar{\mu}_p \in V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}},$$

and x extends to $V_\mu(\mathcal{A}_{\widetilde{S}})$ and only if $x^\diamond$ extends to $V(\mathcal{A}_{\widetilde{S}})_{\mathbb{Z}_{(p)}}$.

The self-dual-at-p case considered above determines an obstruction to deforming $x^\diamond$, denoted

$$\mathrm{obst}_{x^\diamond} \in H^0\big(\widetilde{S}, \boldsymbol{\omega}^\diamond|^{-1}_{\widetilde{S}}\big).$$

Recalling that $\boldsymbol{\omega}|_{\widetilde{S}} = \boldsymbol{\omega}^{\diamond}|_{\widetilde{S}}$ by definition, we now define (6.5.4) by

$$\mathrm{obst}_x = \mathrm{obst}_{x^\diamond}.$$

It is easy to check that this does not depend on the auxiliary choice of $V_{\mathbb{Z}}^{\diamond}$ used in its construction, and has the desired properties. □

Proposition 6.5.2. — *Every geometric point of $\mathcal{S}_K(G,\mathcal{D})$ admits an étale neighborhood $U \to \mathcal{S}_K(G,\mathcal{D})$ such that*

$$\mathcal{Z}(m,\mu)_{/U} \to U$$

restricts to a closed immersion on every connected component of its domain. Each such closed immersion is an effective Cartier divisor on U.

Proof. — The first claim is a formal consequence of Proposition 6.4.3, and holds for any finite, unramified, relatively representable morphism of Deligne-Mumford stacks. Indeed, if $\mathcal{O}_s$ denotes the étale local ring at a geometric point $s \to \mathcal{S}_K(G,\mathcal{D})$, then finiteness and relative representability imply that

$$\mathrm{Spec}(\mathcal{O}_s) \times_{\mathcal{S}_K(G,\mathcal{D})} \mathcal{Z}(m,\mu) \cong \bigsqcup_t \mathrm{Spec}(\mathcal{O}_t),$$

where t runs over the geometric points $t \to \mathcal{Z}(m,\mu)$ above s, and unramifiedness implies that each morphism $\mathcal{O}_s \to \mathcal{O}_t$ is surjective.

Fix one such t, set $J = \ker(\mathcal{O}_s \to \mathcal{O}_t)$, and consider the nilpotent thickening

$$\mathrm{Spec}(\mathcal{O}_t) = \mathrm{Spec}(\mathcal{O}_s/J) \hookrightarrow \mathrm{Spec}(\mathcal{O}_s/J^2).$$

In particular, we have a diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_t) & \longrightarrow & \mathcal{Z}(m,\mu) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_s/J^2) & \longrightarrow & \mathcal{S}_K(G,\mathcal{D}) \end{array}$$

exactly as in (6.5.2). The pullback of the Kuga-Satake abelian scheme to $\mathrm{Spec}(\mathcal{O}_t)$ acquires a tautological special quasi-endomorphism x. The obstruction to deforming x is, after choosing a trivialization of $\boldsymbol{\omega}|_{\mathrm{Spec}(\mathcal{O}_t)}$, an element

$$\mathrm{obst}_x \in \mathcal{O}_s/J^2$$

that generates J/J^2 as an $\mathcal{O}_s$-module. Nakayama's lemma now implies that $J \subset \mathcal{O}_s$ is a principal ideal, and so $\mathrm{Spec}(\mathcal{O}_t) \hookrightarrow \mathrm{Spec}(\mathcal{O}_s)$ is an effective Cartier divisor.

This proves the claim on the level of étale local rings, and the extension to étale neighborhoods is routine. □

Proposition 6.5.2 is what justifies referring to the morphisms (6.4.12) as divisors, even though they are not closed immersions. In the notation of that proposition, every connected component of the source of

$$\mathcal{Z}(m,\mu)_{/U} \to U$$

determines a Cartier divisor on U. Summing over all such components and then gluing as U varies over an étale cover defines an effective Cartier divisor on $\mathcal{S}_K(G,\mathcal{D})$ in the usual sense. When no confusion can arise (and perhaps even when it can), we denote this Cartier divisor again by $\mathcal{Z}(m,\mu)$.

We end this subsection by explaining the precise relation between the analytic obstruction (6.5.1) and the algebraic obstruction (6.5.4).

Fix a $g \in G(\mathbb{A}_f)$. If we pull back the diagram (6.5.2) via the morphism

$$\mathcal{D} \xrightarrow{z\mapsto(z,g)} \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}) \tag{6.5.7}$$

we obtain (at least if $\widetilde{S}$ is of finite type over $\mathbb{Q}$) a diagram

$$\begin{array}{ccc} \mathcal{S} = S^{\mathrm{an}} \times_{\mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{an}}} \mathcal{D} & \longrightarrow & \bigsqcup \mathcal{D}(x) \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{S}} = \widetilde{S}^{\mathrm{an}} \times_{\mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{an}}} \mathcal{D} & \longrightarrow & \mathcal{D}, \end{array}$$

of complex analytic spaces, in which the disjoint union is as in (6.4.13), and the vertical arrow on the left is defined by a coherent sheaf of ideals whose square is 0. In particular $\mathcal{S} \to \widetilde{\mathcal{S}}$ induces an isomorphism of underlying topological spaces.

For a fixed x, let $\mathcal{S}(x) \subset \mathcal{S}$ be the union of those connected components of whose image under the top horizontal arrow lies in the factor $\mathcal{D}(x)$. This determines a union of connected components $\widetilde{\mathcal{S}}(x) \subset \widetilde{\mathcal{S}}$, and gives us a diagram of complex analytic spaces

$$\begin{array}{ccccc} S^{\mathrm{an}} & \longleftarrow & \mathcal{S}(x) & \longrightarrow & \mathcal{D}(x) \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{S}^{\mathrm{an}} & \longleftarrow & \widetilde{\mathcal{S}}(x) & \longrightarrow & \mathcal{D}. \end{array}$$

Proposition 6.5.3. — *There is an equality of sections*

$$\mathrm{obst}_x^{\mathrm{an}}|_{\widetilde{\mathcal{S}}(x)} = \mathrm{obst}_x|_{\widetilde{\mathcal{S}}(x)},$$

where the left hand side is the pullback of (6.5.1) *via* $\widetilde{\mathcal{S}}(x) \to \mathcal{D}$ *and the right hand side is the pullback of* (6.5.4) *via* $\widetilde{\mathcal{S}}(x) \to \widetilde{S}^{\mathrm{an}}$.

Proof. — The pullback of $\boldsymbol{V}_{\mathrm{dR}}$ via (6.5.7) is canonically identified with the constant vector bundle

$$\boldsymbol{V}_{\mathrm{dR}}|_{\mathcal{D}} = V \otimes \mathcal{O}_{\mathcal{D}},$$

and under this identification the pullback of the connection ∇ is the induced by the usual $d : \mathcal{O}_{\mathcal{D}} \to \Omega^1_{\mathcal{D}/\mathbb{C}}$.

By the discussion leading to (6.4.13), the pullback of (6.5.5) via $\mathcal{S}(x) \to S^{\mathrm{an}}$ is identified with the constant section

$$x \otimes 1 \in H^0\big(\mathcal{S}(x), \boldsymbol{V}_{\mathrm{dR}}|_{\mathcal{S}(x)}\big),$$

and the pullback of (6.5.6) via $\tilde{\mathcal{S}}(x) \to \widetilde{S}^{\mathrm{an}}$ is its unique parallel extension

$$x \otimes 1 \in H^0\big(\widetilde{\mathcal{S}}(x), \boldsymbol{V}_{\mathrm{dR}}|_{\widetilde{\mathcal{S}}(x)}\big).$$

Thus $\mathrm{obst}_x|_{\widetilde{\mathcal{S}}(x)}$ is the image of $x \otimes 1$ under

$$V \otimes \mathcal{O}_{\widetilde{\mathcal{S}}(x)} \cong \boldsymbol{V}_{\mathrm{dR}}|_{\widetilde{\mathcal{S}}(x)} \to \big(\boldsymbol{V}_{\mathrm{dR}}/F^0\boldsymbol{V}_{\mathrm{dR}}\big)|_{\widetilde{\mathcal{S}}(x)} \cong \boldsymbol{\omega}^{-1}_{\widetilde{\mathcal{S}}(x)}.$$

On the other hand, the analytically defined obstruction (6.5.1) is, essentially by construction, the image of the constant section $x \otimes 1$ under

$$V \otimes \mathcal{O}_{\mathcal{D}} \cong \boldsymbol{V}_{\mathrm{dR}}|_{\mathcal{D}} \to \big(\boldsymbol{V}_{\mathrm{dR}}/F^0\boldsymbol{V}_{\mathrm{dR}}\big)|_{\mathcal{D}} \cong \boldsymbol{\omega}^{-1}_{\mathcal{D}}.$$

The stated equality of sections over $\widetilde{\mathcal{S}}(x)$ follows immediately. □

6.6. The pullback formula for special divisors. — Suppose we are in the general situation of § 6.2 (in particular, we impose no assumption of self-duality on $V^{\diamond}_{\mathbb{Z}}$), so that we have a morphism (6.2.1) of Shimura varieties

$$\mathrm{Sh}_K(G, \mathcal{D}) \to \mathrm{Sh}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}).$$

The larger Shimura variety $\mathrm{Sh}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ has its own integral model

$$\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}) \to \mathrm{Spec}(\mathbb{Z}_{(p)}),$$

obtained by repeating the construction of § 6.3 with $(G, \mathcal{D})$ replaced by $(G^{\diamond}, \mathcal{D}^{\diamond})$. That is, choose an isometric embedding $V^{\diamond} \subset V^{\diamond\diamond}$ into a larger quadratic space that admits an almost self-dual lattice at p, and define $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ as a normalization. Of course $\mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$ has its own line bundle $\boldsymbol{\omega}^{\diamond}$, its own Kuga-Satake abelian scheme, and its own collection of special divisors $\mathcal{Z}^{\diamond}(m, \mu)$.

Proposition 6.6.1. — *The above morphism of canonical models extends uniquely to a finite morphism*

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}) \tag{6.6.1}$$

of integral models. The line bundles of weight one modular forms on the source and target of (6.6.1) *are related by a canonical isomorphism*

$$\boldsymbol{\omega}^{\diamond}|_{\mathcal{S}_K(G,\mathcal{D})} \cong \boldsymbol{\omega}. \tag{6.6.2}$$

Proof. — The existence and uniqueness of (6.6.1) is proved in [**1**, Proposition 2.5.1].

If $V^{\diamond}_{\mathbb{Z}}$ is almost self-dual at p then (6.6.2) is just a restatement of the definition of $\boldsymbol{\omega}$. For the general case, one embeds $V^{\diamond}$ into a quadratic space $V^{\diamond\diamond}$ admitting a lattice that is almost self-dual at p. This allows one to identify both sides of (6.6.2) with the pullback of $\boldsymbol{\omega}^{\diamond\diamond}$ for some morphisms

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}) \to \mathcal{S}_{K^{\diamond\diamond}}(G^{\diamond\diamond}, \mathcal{D}^{\diamond\diamond})$$

into the larger Shimura variety determined by $V^{\diamond\diamond}$. □

Define a quadratic space

$$\Lambda = \{x \in L^\diamond_{\mathbb{Z}} : x \perp L\}$$

over $\mathbb{Z}$ of signature $(n^\diamond - n, 0)$. There are natural inclusions

$$V_{\mathbb{Z}} \oplus \Lambda \subset V^\diamond_{\mathbb{Z}} \subset (V^\diamond_{\mathbb{Z}})^\vee \subset V^\vee_{\mathbb{Z}} \oplus \Lambda^\vee$$

all of finite index, from which it follows that the orthogonal decomposition

$$V^\diamond = V \oplus \Lambda_{\mathbb{Q}}$$

identifies

$$\mu + V^\diamond_{\mathbb{Z}} = \bigsqcup_{\mu_1+\mu_2\in\mu} (\mu_1 + V_{\mathbb{Z}}) \times (\mu_2 + \Lambda).$$

Here the disjoint union over $\mu_1 + \mu_2 \in \mu$ is understood to mean the union over all pairs

$$(\mu_1, \mu_2) \in (V^\vee_{\mathbb{Z}}/V_{\mathbb{Z}}) \oplus (\Lambda^\vee/\Lambda)$$

satisfying $\mu_1 + \mu_2 \in (\mu + V^\diamond_{\mathbb{Z}})/(V_{\mathbb{Z}} \oplus \Lambda)$.

The following lemma gives a corresponding decomposition of special quasi-endomorphisms. For the proof see [**1**, Proposition 2.6.4].

Proposition 6.6.2. — *For any scheme S and any morphism $S \to \mathcal{S}_K(G, \mathcal{D})$ there is a canonical isometry*

$$V(\mathcal{A}^\diamond_S) \cong V(\mathcal{A}_S) \oplus \Lambda_{\mathbb{Q}},$$

which restricts to a bijection

$$V_\mu(\mathcal{A}^\diamond_S) \cong \bigsqcup_{\mu_1+\mu_2\in\mu} V_{\mu_1}(\mathcal{A}_S) \times (\mu_2 + \Lambda). \tag{6.6.3}$$

The relation between special divisors on the source and target of (6.6.1) is most easily expressed in terms of the line bundles associated to the divisors, rather than the divisors themselves. By abuse of notation, we now use $\mathcal{Z}(m, \mu)$ to denote also the line bundle on $\mathcal{S}_K(G, \mathcal{D})$ determined by the Cartier divisor of the same name, extend the definition to $m \le 0$ by

$$\mathcal{Z}(m, \mu) = \begin{cases} \boldsymbol{\omega}^{-1} & \text{if } (m, \mu) = (0, 0) \\ \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} & \text{otherwise,} \end{cases}$$

and use similar conventions for $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond)$.

Proposition 6.6.3. — *For any rational number $m \ge 0$ and any $\mu \in (V^\diamond_{\mathbb{Z}})^\vee/V^\diamond_{\mathbb{Z}}$, there is a canonical isomorphism of line bundles*

$$\mathcal{Z}^\diamond(m, \mu)|_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu}} \mathcal{Z}(m_1, \mu_1)^{\otimes r_\Lambda(m_2,\mu_2)}$$

on $\mathcal{S}_K(G, \mathcal{D})$. Here we have set

$$R_\Lambda(m, \mu) = \{\lambda \in \mu + \Lambda : Q(\lambda) = m\}$$

and $r_\Lambda(m, \mu) = \#R_\Lambda(m, \mu)$.

Proof. — If $m < 0$, or if $m = 0$ and $\mu \neq 0$, the tensor product on the right is empty, and both sides of the desired isomorphism are canonically trivial. If $(m, \mu) = (0, 0)$ the claim is just a restatement of Proposition 6.6.1. Thus we may assume that $m > 0$.

The decomposition (6.6.3) induces an isomorphism

$$\mathcal{Z}^{\diamond}(m,\mu)_{/\mathcal{S}_K(G,\mathcal{D})} \cong \bigsqcup_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0 \\ \lambda\in R_\Lambda(m_2,\mu_2)}} \mathcal{Z}(m_1,\mu_1) \sqcup \bigsqcup_{\substack{\mu_2\in\mu \\ \lambda\in R_\Lambda(m,\mu_2)}} \mathcal{S}_K(G,\mathcal{D}) \tag{6.6.4}$$

of $\mathcal{S}_K(G,\mathcal{D})$-stacks, where the condition $\mu_2 \in \mu$ means that

$$0 + \mu_2 \in (V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}) \oplus (\Lambda^{\vee}/\Lambda)$$

lies in the subset $(\mu + V_{\mathbb{Z}}^{\diamond})/(V_{\mathbb{Z}} \oplus \Lambda)$. Explicitly, given any connected scheme S and a morphism

$$S \to \mathcal{S}_K(G,\mathcal{D}),$$

a lift of the morphism to the first disjoint union on the right hand side of (6.6.4) determines a pair

$$(x,\lambda) \in V_{\mu_1}(\mathcal{A}_S) \times (\mu_2 + \Lambda)$$

satisfying $m_1 = Q(x)$ and $m_2 = Q(\lambda)$. Using (6.6.3) we obtain a special quasi-endomorphism

$$x^{\diamond} = x + \lambda \in V_{\mu}(\mathcal{A}_S^{\diamond}).$$

Similarly, a lift to the second disjoint union determines a vector $\lambda \in \mu_2 + \Lambda$ satisfying $m = Q(\lambda)$, which determines a special quasi-endomorphism

$$x^{\diamond} = 0 + \lambda \in V_{\mu}(\mathcal{A}_S^{\diamond}). \tag{6.6.5}$$

In either case $Q(x^{\diamond}) = m$, and so our lift determines an S-point of the left hand side of (6.6.4).

If $\Lambda^{\vee}$ does not represent m, then $R_\Lambda(m,\mu_2) = \emptyset$ for all choices of μ_2, and the desired isomorphism of line bundles

$$\begin{aligned}\mathcal{Z}^{\diamond}(m,\mu)|_{\mathcal{S}_K(G,\mathcal{D})} &\cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)} \\ &\cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)},\end{aligned}$$

on $\mathcal{S}_K(G,\mathcal{D})$ follows immediately from (6.6.4). In general, the decomposition (6.6.4) shows that the support of $\mathcal{Z}^{\diamond}(m,\mu)$ contains the image of (6.6.1) as soon as there is some $\mu_2 \in \mu$ for which $R_\Lambda(m,\mu_2)$ is nonempty. Thus we must compute an improper intersection.

Fix a geometric point $s \to \mathcal{S}_K(G,\mathcal{D})$ and, as in Proposition 6.5.2, an étale neighborhood

$$U^{\diamond} \to \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond})$$

of s small enough that the morphism

$$\mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}} \to U^{\diamond}$$

restricts to a closed immersion on every connected component of the domain. By shrinking $U^{\diamond}$ we may assume that these connected components are in bijection with the set of lifts

$$\begin{array}{ccc} & & \mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}} \\ & \nearrow & \downarrow \\ s & \longrightarrow & U^{\diamond}. \end{array}$$

Having so chosen $U^{\diamond}$, we then choose a connected étale neighborhood

$$U \to \mathcal{S}_K(G,\mathcal{D})$$

of s small enough that there exists a lift

$$\begin{array}{ccc} U & \dashrightarrow & U^{\diamond} \\ \downarrow & & \downarrow \\ \mathcal{S}_K(G,\mathcal{D}) & \longrightarrow & \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}), \end{array}$$

and so that in the cartesian diagram

$$\begin{array}{ccc} \mathcal{Z}^{\diamond}(m,\mu)_{/U} & \longrightarrow & \mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}} \\ \downarrow & & \downarrow \\ U & \longrightarrow & U^{\diamond} \end{array}$$

each of the vertical arrows restricts to a closed immersion on every connected component of its source, and the top horizontal arrow induces a bijection on connected components.

The decomposition (6.6.4) induces a decomposition of U-schemes

$$\mathcal{Z}^{\diamond}(m,\mu)_{/U} \cong \bigsqcup_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0 \\ \lambda\in R_\Lambda(m_2,\mu_2)}} \mathcal{Z}(m_1,\mu_1)_{/U} \sqcup \bigsqcup_{\substack{\mu_2\in\mu \\ \lambda\in R_\Lambda(m,\mu_2)}} U.$$

The first disjoint union defines a Cartier divisor on U. In the second disjoint union, the copy of U indexed by $\lambda \in R_\Lambda(m,\mu_2)$ is the image of the open and closed immersion

$$f_\lambda : U \to \mathcal{Z}^{\diamond}(m,\mu)_{/U}$$

obtained by endowing the Kuga-Satake abelian scheme $\mathcal{A}^{\diamond}_U$ with the special quasi-endomorphism $0+\lambda \in V_\mu(\mathcal{A}^{\diamond}_U)$ of (6.6.5).

There is a corresponding canonical decomposition of $U^{\diamond}$-schemes

$$\mathcal{Z}^{\diamond}(m,\mu)_{/U^{\diamond}} = \mathcal{Z}^{\diamond}_{\mathrm{prop}} \sqcup \bigsqcup_{\substack{\mu_2\in\mu \\ \lambda\in R_\Lambda(m,\mu_2)}} \mathcal{Z}^{\diamond}_{\lambda}, \tag{6.6.6}$$

in which

$$\mathcal{Z}^\diamond_\lambda \subset \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond} \tag{6.6.7}$$

is the connected component containing the image of

$$U \xrightarrow{f_\lambda} \mathcal{Z}^\diamond(m,\mu)_{/U} \to \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond}$$

and

$$\mathcal{Z}^\diamond_{\mathrm{prop}} \subset \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond}$$

is the union of all connected components not of this form.

It is clear from the definitions that the image of $U \to U^\diamond$ intersects the Cartier divisor $\mathcal{Z}^\diamond_{\mathrm{prop}} \to U^\diamond$ properly, and in fact

$$\mathcal{Z}^\diamond_{\mathrm{prop}/U} \cong \bigsqcup_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0 \\ \lambda\in R_\Lambda(m_2,\mu_2)}} \mathcal{Z}(m_1,\mu_1)_{/U}. \tag{6.6.8}$$

On the other hand, the image of $U \to U^\diamond$ is completely contained within the support of every $\mathcal{Z}^\diamond_\lambda \to U^\diamond$.

By mild abuse of notation, we denote again by $\mathcal{Z}^\diamond_{\mathrm{prop}}$ and $\mathcal{Z}^\diamond_\lambda$ the line bundles on $U^\diamond$ determined by the Cartier divisors of the same name.

Lemma 6.6.4. — *There is a canonical isomorphisms of line bundles*

$$\mathcal{Z}^\diamond_{\mathrm{prop}}|_U \cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)}|_U,$$

and a canonical isomorphism

$$\mathcal{Z}^\diamond_\lambda|_U \cong \boldsymbol{\omega}^{-1}|_U. \tag{6.6.9}$$

Proof. — The first isomorphism is clear from the isomorphism of U-schemes (6.6.8). The second isomorphism is more subtle, and is based on similar calculations in the context of unitary Shimura varieties; see especially [**11**, Theorem 7.10] and [**28**].

Our étale neighborhood $U^\diamond \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$ was chosen in such a way that $\mathcal{Z}^\diamond_\lambda \to U^\diamond$ is a closed immersion defined by a locally principal sheaf of ideals $J_\lambda \subset \mathcal{O}_{U^\diamond}$. The closed subscheme

$$\widetilde{\mathcal{Z}}^\diamond_\lambda \subset U^\diamond$$

defined by J_λ^2 is called the *first order tube* around $Z^\diamond_\lambda$. We now have morphisms

$$U \xrightarrow{f_\lambda} \mathcal{Z}^\diamond_\lambda \hookrightarrow \widetilde{\mathcal{Z}}^\diamond_\lambda \hookrightarrow U^\diamond \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond).$$

Tautologically, J_λ^{-1} is the line bundle on $U^\diamond$ determined by the Cartier divisor $\mathcal{Z}^\diamond_\lambda$. Denote by σ_λ the constant function 1, viewed as a section of $\mathcal{O}_{U^\diamond} \subset J_\lambda^{-1}$, so that $\mathrm{div}(\sigma_\lambda) = \mathcal{Z}^\diamond_\lambda$.

On the other hand, after restriction to the connected component (6.6.7) the Kuga-Satake abelian scheme $\mathcal{A}^\diamond$ acquires a tautological

$$x^\diamond \in V_\mu(\mathcal{A}^\diamond_{\mathcal{Z}^\diamond_\lambda}).$$

The discussion of § 6.5 then provides us with a canonical section

$$\mathrm{obst}_{x^\diamond} \in H^0\big(\widetilde{\mathcal{Z}}^\diamond_\lambda, \boldsymbol{\omega}^{-1}|_{\widetilde{\mathcal{Z}}^\diamond_\lambda}\big)$$

whose zero locus is the closed subscheme $\mathcal{Z}^\diamond_\lambda$.

The idea is roughly that the equality of divisors

$$\mathrm{div}(\sigma_\lambda) = \mathrm{div}(\mathrm{obst}_{x^\diamond})$$

should imply that there is a unique isomorphism of line bundles (6.6.9) over the first order tube sending $\sigma_\lambda \mapsto \mathrm{obst}_{x^\diamond}$, which we would then pull back via f_λ. This is a bit too strong. Instead, we argue that such an isomorphism exists Zariski locally on the first order tube, and that any two such local isomorphisms restrict to the same isomorphism over U.

Indeed, working Zariski locally, we can assume that

$$U = \mathrm{Spec}(R), \qquad U^\diamond = \mathrm{Spec}(R^\diamond)$$

for integral domains R and $R^\diamond$, and

$$\mathcal{Z}^\diamond_\lambda = \mathrm{Spec}(R^\diamond/J), \qquad \widetilde{\mathcal{Z}}^\diamond_\lambda = \mathrm{Spec}(R^\diamond/J^2).$$

The morphisms $U \to \mathcal{Z}^\diamond_\lambda \to U^\diamond$ then correspond to homomorphisms

$$R^\diamond \to R^\diamond/J \to R.$$

Let $\mathfrak{p} \subset R^\diamond$ be the kernel of this composition, so that $J \subset \mathfrak{p}$. Note that $p \notin \mathfrak{p}$, as the flatness of $\mathcal{S}_K(G,\mathcal{D})$ over $\mathbb{Z}_{(p)}$ implies that R has no p-torsion.

Assume that we have chosen trivializations of the line bundles $\boldsymbol{\omega}|_{U^\diamond}$ and $Z^\diamond_\lambda$ on $U^\diamond$, so that our sections obst_λ and σ_λ are identified with elements

$$a \in R^\diamond/J^2 \qquad \text{and} \qquad b \in R^\diamond,$$

respectively. Each of these elements generates the ideal $J/J^2 \subset R^\diamond/J^2$.

Lemma 6.6.5. — *There exists $u \in R^\diamond/J^2$ such that $ua = b$. The image of any such u in $R^\diamond/\mathfrak{p} \subset R$ is a unit. If also $u'a = b$, then $u = u'$ in $R^\diamond/\mathfrak{p} \subset R$.*

Proof. — Suppose we are given any $x \in R^\diamond/J^2$ with $bx = 0$. We claim that

$$x \in \mathfrak{p}/J^2.$$

If not, then any lift $x \in R^\diamond$ becomes a unit in the localization $R^\diamond_\mathfrak{p}$. As $bx \in J^2$, we obtain

$$b \in \mathfrak{p}^2 R^\diamond_\mathfrak{p}. \tag{6.6.10}$$

We have noted above that $p \notin \mathfrak{p}$, and so $R^\diamond_\mathfrak{p}$ is a $\mathbb{Q}$-algebra. The source and target of

$$\mathcal{Z}^\diamond(m,\mu) \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$$

have smooth generic fibers, and so $R^\diamond_\mathfrak{p} \to R^\diamond_\mathfrak{p}/bR^\diamond_\mathfrak{p}$ is a morphism of regular local rings. By (6.6.10), this morphism induces an isomorphism on tangent spaces, and so is itself an isomorphism. Thus $b=0$ in $R^\diamond_\mathfrak{p}$, and hence also in $R^\diamond$. This contradicts the fact that b generates the ideal J.

As a and b generate both generate J/J^2, there exist $u,v \in R^\diamond/J^2$ such that $ua=b$ and $vb=a$. Obviously $b\cdot(1-uv)=0$, and taking $x=1-uv$ the paragraph above implies $1-uv \in \mathfrak{p}/J^2$. Thus the image of u in $R^\diamond/\mathfrak{p}$ is a unit with inverse v. If also $u'a=b$, the same argument shows that the image of u' in $R^\diamond/\mathfrak{p}$ is a unit with inverse v, and hence $u=u'$ in $R^\diamond/\mathfrak{p}$. □

The discussion above provides us with a canonical isomorphism

$$\mathcal{Z}^\diamond_\lambda|_U \cong \boldsymbol{\omega}^{-1}|_U$$

Zariski locally on U, and gluing over an open cover completes the proof of Lemma 6.6.4. □

We now complete the proof of Proposition 6.6.3. If we interpret the isomorphism of $U^\diamond$-schemes (6.6.6) as an isomorphism

$$\mathcal{Z}^\diamond(m,\mu)|_{U^\diamond} \cong \mathcal{Z}^\diamond_{\mathrm{prop}} \otimes \bigotimes_{\substack{\mu_2\in\mu \\ \lambda\in R_\Lambda(m,\mu_2)}} \mathcal{Z}^\diamond_\lambda,$$

of line bundles on $U^\diamond$, pull back via $U \to U^\diamond$, and use Lemma 6.6.4, we obtain canonical isomorphisms

$$\begin{aligned}\mathcal{Z}^\diamond(m,\mu)|_U &\cong \Bigg(\bigotimes_{\substack{m_1+m_2=m\\ \mu_1+\mu_2\in\mu\\ m_1>0}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)}|_U\Bigg) \otimes \Bigg(\bigotimes_{\mu_2\in\mu} \boldsymbol{\omega}^{-r_\Lambda(m,\mu_2)}|_U\Bigg)\\ &\cong \bigotimes_{\substack{m_1+m_2=m\\ \mu_1+\mu_2=\mu}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)}|_U\end{aligned}$$

of line bundles over the étale neighborhood U of $s \to \mathcal{S}_K(G,\mathcal{D})$. Now let U vary over an étale cover and apply descent. □

7. Normality and flatness

Keep $V_\mathbb{Z} \subset V$ and $K \subset G(\mathbb{A}_f)$ as in § 6, and once again fix a prime p at which $V_\mathbb{Z}$ is maximal. After some technical preliminaries in § 7.1, we prove in § 7.2 that the special fiber of the integral model

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_{(p)})$$

is geometrically normal if $n\geq 6$, and that the special divisors are flat if $n \geq 4$. When $p\neq 2$ these results already appear [7] in [2]. Here we use similar ideas, but employ the

(7) With the sharper bounds $n\geq 5$ and $n\geq 3$, respectively.

methods of Ogus [44] to control the dimension of the supersingular locus, as these apply even when $p = 2$.

7.1. Local properties of special cycles. — Suppose in this subsection that $V_{\mathbb{Z}}$ is self-dual at p. As in the discussion of § 6.3, the smooth integral model $\mathcal{S}_K(G, \mathcal{D})$ comes with filtered vector bundles $0 \subset F^0\boldsymbol{H}_{\mathrm{dR}} \subset \boldsymbol{H}_{\mathrm{dR}}$ and

$$0 \subset F^1\boldsymbol{V}_{\mathrm{dR}} \subset F^0\boldsymbol{V}_{\mathrm{dR}} \subset \boldsymbol{V}_{\mathrm{dR}},$$

along with an injection $\boldsymbol{V}_{\mathrm{dR}} \to \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})$ onto a local direct summand. Composition in $\underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR}})$ endows $\boldsymbol{V}_{\mathrm{dR}}$ with a quadratic form

$$\boldsymbol{Q} : \boldsymbol{V}_{\mathrm{dR}} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})},$$

under which $F^1\boldsymbol{V}_{\mathrm{dR}}$ is an isotropic line with orthogonal subsheaf $F^0\boldsymbol{V}_{\mathrm{dR}}$.

Recall from (6.4.6) the $\mathbb{Z}$-module (6.4.6) of special quasi-endomoprhisms

$$V_0(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Z}_{(p)}$$

determined by the trivial coset $0 \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Any $x \in V_0(\mathcal{A}_S)$ has a *de Rham realization* $\boldsymbol{x}_{\mathrm{dR}}$, which is a global section of the subsheaf

$$\boldsymbol{V}_{\mathrm{dR},S} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{\mathrm{dR},S}).$$

In particular, de Rham realization defines a morphism of $\mathcal{O}_S$-modules

$$V_0(\mathcal{A}_S) \otimes \mathcal{O}_S \to \boldsymbol{V}_{\mathrm{dR},S}.$$

compatible with the quadratic forms on source and target. In fact, as is clear from the proof of Proposition 6.5.1, the image is contained in $F^0\boldsymbol{V}_{\mathrm{dR},S}$.

Fix a positive definite quadratic space Λ over $\mathbb{Z}$, and consider the stack

$$\mathcal{Z}(\Lambda) \to \mathcal{S}_K(G, \mathcal{D}) \tag{7.1.1}$$

with functor of points

$$\mathcal{Z}(\Lambda)(S) = \{\text{isometric embeddings } \iota : \Lambda \to V_0(\mathcal{A}_S)\}$$

for any morphism $S \to \mathcal{S}_K(G, \mathcal{D})$. As observed in [2, § 4.4] (see also Lemma 7.1.1 below), this is a Deligne-Mumford stack over $\mathbb{Z}_{(p)}$ whose generic fiber is smooth of dimension $n - \mathrm{rank}(\Lambda)$. Moreover, the morphism (7.1.1) is finite and unramified.

We now briefly recall the deformation theory of these stacks. As in the proof of Proposition 6.5.1, we have a canonical flat connection

$$\nabla : \boldsymbol{V}_{\mathrm{dR}} \to \boldsymbol{V}_{\mathrm{dR}} \otimes \Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}.$$

This connection satsfies Griffiths's transversality with respect to the Hodge filtration, and the Kodaira-Spencer map associated with it induces an isomorphism

$$F^1\boldsymbol{V}_{\mathrm{dR}} \otimes \big(\Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}\big)^{\vee} \cong F^0\boldsymbol{V}_{\mathrm{dR}}/F^1\boldsymbol{V}_{\mathrm{dR}}.$$

Dualizing, and using the bilinear pairing on $\boldsymbol{V}_{\mathrm{dR}}$, we obtain an isomorphism

$$F^0\boldsymbol{V}_{\mathrm{dR}}/F^1\boldsymbol{V}_{\mathrm{dR}} \cong (\boldsymbol{V}_{\mathrm{dR}}/F^0\boldsymbol{V}_{\mathrm{dR}}) \otimes \Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}.$$

This is [**39**, Proposition 4.16], whose proof applies also when $p = 2$; one only has to replace appeals to results from [**31**] with appeals to the analogous results from [**30**].

Now, suppose that we have a point s of $\mathcal{S}_K(G, \mathcal{D})$ valued in a field k. If $\tilde{s}$ is any lift of s to the ring of dual numbers $k[\epsilon]$, the connection ∇ induces a canonical isomorphism

$$\xi_{\tilde{s}} : \boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon] \cong \boldsymbol{V}_{\mathrm{dR},\tilde{s}},$$

and thus gives rise to an isotropic line

$$F^1_{\tilde{s}}(\boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon]) \stackrel{\mathrm{def}}{=} \xi_{\tilde{s}}^{-1}(F^1\boldsymbol{V}_{\mathrm{dR},\tilde{s}}) \subset \boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon].$$

By construction, this line lifts $F^1\boldsymbol{V}_{\mathrm{dR},s}$.

The properties of the Kodaira-Spencer map mentioned above can now be reinterpreted as saying that the association

$$\tilde{s} \mapsto F^1_{\tilde{s}}(\boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon])$$

is a bijection from the tangent space of $\mathcal{S}_K(G, \mathcal{D})$ at s to the space of isotropic lines in $\boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon]$ lifting $F^1\boldsymbol{V}_{\mathrm{dR},s}$. This latter space can be canonically identified with the k-vector space

$$\mathrm{Hom}_k(F^1\boldsymbol{V}_{\mathrm{dR},s}, F^0\boldsymbol{V}_{\mathrm{dR},s}/F^1\boldsymbol{V}_{\mathrm{dR},s})$$

as follows: Any lift $F^1_{\tilde{s}}(\boldsymbol{V}_{\mathrm{dR},t} \otimes_k k[\epsilon])$ will be contained in $F^0\boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon]$, and so we can consider the associated map

$$F^1_{\tilde{s}}(\boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon]) \to (F^0\boldsymbol{V}_{\mathrm{dR},s}/F^1\boldsymbol{V}_{\mathrm{dR},s}) \otimes_k k[\epsilon],$$

which will factor as

$$\begin{array}{ccc} F^1_{\tilde{s}}(\boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon]) & \longrightarrow & (F^0\boldsymbol{V}_{\mathrm{dR},s}/F^1\boldsymbol{V}_{\mathrm{dR},s}) \otimes_k k[\epsilon] \\ {\scriptstyle \epsilon\mapsto 0}\downarrow & & \uparrow{\scriptstyle 1\otimes\epsilon} \\ F^1\boldsymbol{V}_{\mathrm{dR},s} & \xrightarrow[\varphi_{\tilde{s}}]{} & F^0\boldsymbol{V}_{\mathrm{dR},s}/F^1\boldsymbol{V}_{\mathrm{dR},s}. \end{array}$$

The desired identification is now given by the assignment $F^1_{\tilde{s}} \mapsto \varphi_{\tilde{s}}$.

We can say more. Suppose that s lifts to a k-point of $\mathcal{Z}(\Lambda)$ corresponding to an embedding $\Lambda \hookrightarrow V(\mathcal{A}_s)$. We will continue to use s to denote this lift as well. The de Rham realization of the embedding gives a map

$$\Lambda \to F^0\boldsymbol{V}_{\mathrm{dR},s},$$

and we let

$$\boldsymbol{\Lambda}_{\mathrm{dR},s} \subset F^0\boldsymbol{V}_{\mathrm{dR},s}$$

be the k-subspace generated by its image. Now, the bijection from the previous paragraph identifies the tangent space of $\mathcal{Z}(\Lambda)$ at s with the space of isotropic lines in $\boldsymbol{V}_{\mathrm{dR},s} \otimes_k k[\epsilon]$ that lift $F^1\boldsymbol{V}_{\mathrm{dR},s}$ and are also orthogonal to $\boldsymbol{\Lambda}_{\mathrm{dR},s}$. This space in turn can be identified with the k-vector space

$$\mathrm{Hom}_k(F^1\boldsymbol{V}_{\mathrm{dR},s}, \overline{\boldsymbol{\Lambda}}^{\perp}_{\mathrm{dR},s}), \tag{7.1.2}$$

where

$$\overline{\mathbf{\Lambda}}_{\mathrm{dR},s} \subset F^0\boldsymbol{V}_{\mathrm{dR},s}/F^1\boldsymbol{V}_{\mathrm{dR},s}$$

is the the image of $\mathbf{\Lambda}_{\mathrm{dR},s}$ and $\overline{\mathbf{\Lambda}}^{\perp}_{\mathrm{dR},s}$ is its orthogonal complement.

For proofs of the above statements, which use the explicit description of the complete local rings of $\mathcal{S}_K(G,\mathcal{D})$, see [**39**, Prop. 5.16]. As observed there, they also apply more generally to arbitrary nilpotent divided power thickenings. We record some immediate consequences.

Lemma 7.1.1. — *Let the notation be as above, and set* $r = \mathrm{rank}(\Lambda)$.

1. *The completed étale local ring* $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$ *is a quotient of* $\widehat{\mathcal{O}}_{\mathcal{S}_K(G,\mathcal{D}),s}$ *by an ideal generated by* $\mathrm{rank}(\Lambda)$ *elements.*
2. $\mathcal{Z}(\Lambda)$ *is smooth at* s *if and only if* $\overline{\mathbf{\Lambda}}_{\mathrm{dR},s}$ *has* k*-dimension* $\mathrm{rank}(\Lambda)$. *In particular, the generic fiber of* $\mathcal{Z}(\Lambda)$ *is smooth.*
3. *Suppose that* k *has characteristic* p, *and that the Krull dimension of* $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)$ *is* $n-\mathrm{rank}(\Lambda)$. *Then* $\mathcal{Z}(\Lambda)$ *and* $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ *are local complete intersections at* s. *Moreover,* $\mathcal{Z}(\Lambda)$ *is flat over* $\mathbb{Z}_{(p)}$ *at* s.

Proof. — The first claim is a consequence of the deformation theory explained above (more precisely, of its generalization to arbitrary square-zero thickenings) and Nakayama's lemma. See [**39**, Corollary 5.17].

For the second claim, note that $\mathcal{Z}(\Lambda)$ will be smooth at s if and only if its tangent space at s has dimension $n-\mathrm{rank}(\Lambda)$. As we have identified the tangent space with (7.1.2), this is equivalent to $\overline{\mathbf{\Lambda}}_{\mathrm{dR},s}$ having dimension $\mathrm{rank}(\Lambda)$. For the assertion about the generic fiber, it suffices to check the criterion for smoothness at every $\mathbb{C}$-valued point. Now, note that the de Rham realization

$$V_0(\mathcal{A}_s)\otimes_{\mathbb{Z}}\mathbb{C} \to \boldsymbol{V}_{\mathrm{dR},s},$$

is injective, and also that the image of this realization is precisely the weight $(0,0)$ part of the Hodge structure on $\boldsymbol{V}_{\mathrm{dR},s}$, and hence is complementary to $F^1\boldsymbol{V}_{\mathrm{dR},s}$. This implies that $\overline{\mathbf{\Lambda}}_{\mathrm{dR},s}$ has dimension r over $\mathbb{C}$, and hence that $\mathcal{Z}(\Lambda)$ is smooth at s.

Now we come to the third claim. Note that $\widehat{\mathcal{O}}_{\mathcal{S}_K(G,\mathcal{D}),s}$ is formally smooth over $W(k)$ of Krull dimension $n+1$. Hence, $\widehat{\mathcal{O}}_{\mathcal{S}_K(G,\mathcal{D}),s}/(p)$ is also formally smooth over k of Krull dimension n, and $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)$, which is its quotient by an ideal generated by $\mathrm{rank}(\Lambda)$ element, is a complete intersection as soon as

$$\dim\big(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)\big) = n-\mathrm{rank}(\Lambda).$$

This is precisely our hypothesis.

Now, note that we have

$$n-\mathrm{rank}(\Lambda)+1 \le \dim(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}) \le \dim\big(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)\big)+1 = n-\mathrm{rank}(\Lambda)+1.$$

Here, the first two inequalities follow from Krull's Hauptidealsatz. This shows

$$\dim(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}) = n-\mathrm{rank}(\Lambda)+1,$$

and implies that $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$ is a complete intersection ring.

Finally, to see that $\mathcal{Z}(\Lambda)$ is flat over $\mathbb{Z}_{(p)}$ at s, note that p cannot be a zero divisor in $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$: Indeed, the equality

$$\dim\,(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)) = n - \mathrm{rank}(\Lambda) = \dim\ \widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s} - 1$$

implies that p is not contained in any minimal prime of $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$. Since $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$ is a complete intersection ring and hence Cohen-Macaulay, this implies that p is not a zero divisor. □

For any morphism $S \to \mathcal{Z}(\Lambda)$, de Rham realization defines a morphism

$$\Lambda \otimes \mathcal{O}_S \to \boldsymbol{V}_{\mathrm{dR},S}.$$

Let $\boldsymbol{\Lambda}_{\mathrm{dR},S} \subset \boldsymbol{V}_{\mathrm{dR},S}$ be the image of this morphism.

We will consider the canonical open substack

$$\mathcal{Z}^{\mathrm{pr}}(\Lambda) \hookrightarrow \mathcal{Z}(\Lambda) \tag{7.1.3}$$

characterized by the property that a morphism $S \to \mathcal{Z}(\Lambda)$ factors through $\mathcal{Z}^{\mathrm{pr}}(\Lambda)$ if and only if $\boldsymbol{\Lambda}_{\mathrm{dR},S} \subset \boldsymbol{V}_{\mathrm{dR},S}$ is a local direct summand of rank equal to $\mathrm{rank}(\Lambda)$.

Proposition 7.1.2. — *Consider the following assertions:*

1. *For any generic geometric point η of $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$, the Kuga-Satake abelian scheme $\mathcal{A}_\eta$ is ordinary, and the tautological map $\Lambda \to V_0(\mathcal{A}_\eta)$ is an isomorphism.*
2. *The special fiber $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is a generically smooth local complete intersection of dimension $n - \mathrm{rank}(\Lambda)$.*
3. *The special fiber $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is smooth outside of a codimension 2 subspace.*

Then (1) *and* (2) *hold whenever* $\mathrm{rank}(\Lambda) \le n/2$, *and* (3) *holds whenever* $\mathrm{rank}(\Lambda) \le (n-1)/2$.

Proof. — We will prove the proposition by induction on the rank of Λ. For any integer $r \ge 0$ and $i \in \{1,2,3\}$, let $P_i(r)$ be the statement that assertion (i) is valid whenever $\mathrm{rank}(\Lambda) = r$. We claim

(i) if $0 \le r \le (n-1)/2$ then $P_2(r)$ implies $P_1(r)$,
(ii) if $r \le (n-2)/2$ then $P_1(r)$ and $P_2(r)$ together imply $P_2(r+1)$,
(iii) if $r \le (n-3)/2$ then $P_1(r)$ and $P_2(r)$ together imply $P_3(r+1)$.

Once the claims are proved, the lemma will follow by induction. Indeed, the base case $P_2(0)$ is implied by the smoothness of $\mathcal{S}_K(G,\mathcal{D})$.

The claims themselves follow from an argument derived from [**44**], which was used in [**39**, Proposition 6.17], and exploits the following simple lemma.

Lemma 7.1.3. — *Let Z be an $\mathbb{F}_p$-scheme admitting an unramified map $Z \to \mathcal{S}_K(G,\mathcal{D})$. Suppose that we have a local direct summand $\boldsymbol{N} \subset \boldsymbol{V}_{\mathrm{dR}}|_Z$ that is horizontal for the integrable connection*

$$\boldsymbol{V}_{\mathrm{dR},Z} \to \boldsymbol{V}_{\mathrm{dR},Z} \otimes_{\mathcal{O}_Z} \Omega^1_{Z/\mathbb{F}_p}$$

induced from the one on $\boldsymbol{V}_{\mathrm{dR}}$*. Suppose also that*

$$F^1\boldsymbol{V}_{\mathrm{dR},Z} \subset \boldsymbol{N}.$$

Then $\dim(Z) \leq \mathrm{rank}(\boldsymbol{N}) - 1$*. If, in addition* $\boldsymbol{N} \cap F^0\boldsymbol{V}_{\mathrm{dR},Z}$ *is a local direct summand of* $\boldsymbol{V}_{\mathrm{dR},Z}$*, then we in fact have*

$$\dim(Z) \leq \mathrm{rank}(\boldsymbol{N} \cap F^0\boldsymbol{V}_{\mathrm{dR},Z}) - 1.$$

Proof. — For the first assertion, it is enough to show that, at any point $z \in Z(k)$ valued in a field k, the tangent space of Z at z has dimension at most $\mathrm{rank}(\boldsymbol{N}) - 1$. But our hypotheses imply that, if $\tilde{z} \in Z(k[\epsilon])$ is any lift of Z, then we must have

$$F^1_{\tilde{z}}(\boldsymbol{V}_{\mathrm{dR},Z} \otimes_k k[\epsilon]) \subset (\boldsymbol{N}_z \otimes_k k[\epsilon]) \cap (F^0\boldsymbol{V}_{\mathrm{dR},z} \otimes_k k[\epsilon]).$$

This, combined with the fact that Z is unramified over $\mathcal{S}_K(G,\mathcal{D})$, implies that the tangent space of Z at z can be identified with a subspace of

$$\mathrm{Hom}_k(F^1\boldsymbol{V}_{\mathrm{dR},z}, \overline{\boldsymbol{N}}_z) \subset \mathrm{Hom}_k(F^1\boldsymbol{V}_{\mathrm{dR},z}, F^0\boldsymbol{V}_{\mathrm{dR},z}/F^1\boldsymbol{V}_{\mathrm{dR},z}),$$

where $\overline{\boldsymbol{N}}_z$ is the image of $\boldsymbol{N}_z \cap F^0\boldsymbol{V}_{\mathrm{dR},z}$ in $F^0\boldsymbol{V}_{\mathrm{dR},z}/F^1\boldsymbol{V}_{\mathrm{dR},z}$. We are now done, since $\overline{\boldsymbol{N}}_z$ has dimension at most $\mathrm{rank}(\boldsymbol{N}) - 1$.

The second assertion is immediate from the proof of the first. □

We begin with claim (i). Assume $P_2(r)$, and suppose $\mathrm{rank}(\Lambda) = r$. Fix a geometric generic point η of $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$. Then $P_2(r)$ implies that there is a smooth $\mathbb{F}_p$-scheme U, equidimensional of dimension $n - r$, and an étale map $U \to \mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$, whose image contains η.

As explained in the proof of [**39**, Proposition 6.17], there is a canonical isotropic line

$$\boldsymbol{C} \subset \boldsymbol{V}_{\mathrm{dR},U},$$

called the *conjugate filtration*, which is horizontal for the connection on $\boldsymbol{V}_{\mathrm{dR},U}$, is contained in $\boldsymbol{\Lambda}^{\perp}_{\mathrm{dR},U}$, and is such that a point $t \in U(k)$ is non-ordinary if and only if $\boldsymbol{C}_t \subset F^0\boldsymbol{V}_{\mathrm{dR},t}$, or, equivalently, if and only if

$$F^1\boldsymbol{V}_{\mathrm{dR},t} \subset \boldsymbol{C}_t^{\perp} \cap \boldsymbol{\Lambda}^{\perp}_{\mathrm{dR},t}.$$

Now, we have

$$\boldsymbol{C}_t \subset \boldsymbol{\Lambda}_{\mathrm{dR},t}$$

only if $F^1\boldsymbol{V}_{\mathrm{dR},t} \subset \boldsymbol{\Lambda}_{\mathrm{dR},t}$. See for instance [**39**, Lemma 4.20]. Therefore, since we are assuming that U is smooth, the subsheaf

$$\boldsymbol{C}_U + \boldsymbol{\Lambda}_{\mathrm{dR},U} \subset \boldsymbol{V}_{\mathrm{dR},U}$$

is a horizontal local direct summand of rank $r + 1$.

By Lemma 7.1.3, if $Z \subset U$ is a closed subscheme with

$$F^1\boldsymbol{V}_{\mathrm{dR},Z} \subset \boldsymbol{C}_Z + \boldsymbol{\Lambda}_{\mathrm{dR},Z},$$

then $\dim Z \leq r$. Using $r \leq (n-1)/2$, we see that $r = \dim Z < \dim U = n - r$. Therefore, after shrinking U if necessary, we can assume that

$$F^1\boldsymbol{V}_{\mathrm{dR},U} + \boldsymbol{C}_U + \boldsymbol{\Lambda}_{\mathrm{dR},U} \subset \boldsymbol{V}_{\mathrm{dR},U}$$

is a direct summand of rank $r+2$, or, equivalently, that

$$F^0\boldsymbol{V}_{\mathrm{dR},U} \cap \boldsymbol{C}_U^{\perp} \cap \boldsymbol{\Lambda}_{\mathrm{dR},U}^{\perp} \subset \boldsymbol{V}_{\mathrm{dR},U}$$

is a direct summand of rank $n-r$. Therefore, once again by Lemma 7.1.3, the locus in U where $F^1\boldsymbol{V}_{\mathrm{dR},U}$ is contained in this direct summand has dimension at most $n-r-1$. But this is precisely the non-ordinary locus in U. As $\dim(U) = n-r$, this shows the first part of $P_1(r)$.

Suppose now that the map $\Lambda \to V_0(\mathcal{A}_\eta)$ is not a bijection, so that there exists $x \in V_0(\mathcal{A}_\eta)$ such that

$$\widetilde{\Lambda} = \Lambda + \langle x \rangle \subset V_0(\mathcal{A}_\eta)$$

is a direct summand of rank $r+1$, and its de Rham realization

$$\widetilde{\boldsymbol{\Lambda}}_{\mathrm{dR},\eta} = \boldsymbol{\Lambda}_{\mathrm{dR},\eta} + \langle \boldsymbol{x}_{\mathrm{dR},\eta} \rangle \subset \boldsymbol{V}_{\mathrm{dR},\eta}$$

is a $k(\eta)$-vector subspace of dimension $r+1$.

After shrinking U if necessary, we can assume that $x \in V_0(\mathcal{A}_U)$, and that de Rham realization gives us a local direct summand

$$\widetilde{\boldsymbol{\Lambda}}_{\mathrm{dR},U} = \boldsymbol{\Lambda}_{\mathrm{dR},U} + \langle \boldsymbol{x}_{\mathrm{dR},U} \rangle \subset \boldsymbol{V}_{\mathrm{dR},U}$$

of rank $r+1$ that is horizontal for the connection. However, the discussion of the deformation theory above Lemma 7.1.1 implies that, over U, the Kodaira-Spencer map factors through an isomorphism

$$(F^0\boldsymbol{V}_{\mathrm{dR},U}/F^1\boldsymbol{V}_{\mathrm{dR},U})/\overline{\boldsymbol{\Lambda}}_{\mathrm{dR},U} \cong (\boldsymbol{V}_{\mathrm{dR},U}/F^1\boldsymbol{V}_{\mathrm{dR},U}) \otimes_{\mathcal{O}_U} \Omega^1_{U/\mathbb{F}_p}.$$

However, the horizontality of $\widetilde{\boldsymbol{\Lambda}}_{\mathrm{dR},U}$ guarantees that its (non-trivial) image on the left-hand side is in the kernel of the Kodaira-Spencer map. This contradiction finishes the proof of claim (i).

We will prove claims (ii) and (iii). Suppose that $P_1(r)$ and $P_2(r)$ hold and that $\mathrm{rank}(\Lambda) = r+1$. Write $\Lambda = \Lambda_1 \oplus \Lambda_0$, where $\mathrm{rank}(\Lambda_0) = 1$. Then we have an obvious factorization

$$\mathcal{Z}^{\mathrm{pr}}(\Lambda) \to \mathcal{Z}^{\mathrm{pr}}(\Lambda_1) \to \mathcal{S}_K(G,\mathcal{D}).$$

The first arrow exhibits $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ as a divisor on $\mathcal{Z}^{\mathrm{pr}}(\Lambda_1)_{\mathbb{F}_p}$ (étale locally on the source, in the sense of Proposition 6.5.2). Indeed, the complete local rings of the former are cut out by one equation in those of the latter, and $P_1(r)$ shows that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ does not contain any generic points of $\mathcal{Z}^{\mathrm{pr}}(\Lambda_1)_{\mathbb{F}_p}$. Therefore, by Lemma 7.1.1 and $P_2(r)$, we find that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is a local complete intersection of dimension $n-(r+1)$.

Let $W \subset \mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ be the nonsmooth locus, with its reduced substack structure. We find from Lemma 7.1.1 that

$$F^1\boldsymbol{V}_{\mathrm{dR}}|_W \subset \boldsymbol{\Lambda}_{\mathrm{dR}}|_W.$$

By Lemma 7.1.3, this implies that $\dim(W) \le r$. This is bounded by $n-r-2$ under the hypothesis $r \le (n-2)/2$, and by $n-r-3$ if $r \le (n-3)/2$. This proves (ii) and (iii), and completes the proof of Proposition 7.1.2. □

It will be useful to recall some bounds on the dimension of the supersingular locus in the mod-p fiber of $\mathcal{S}_K(G,\mathcal{D})$ under the assumption that $V_{\mathbb{Z}_p}$ is almost self-dual.

Proposition 7.1.4. — *Suppose that $V_{\mathbb{Z}_p}$ is almost self-dual of rank $n+2$, and suppose that $Z \to \mathcal{S}_K(G,\mathcal{D})$ is an unramified morphism from an $\mathbb{F}_p$-scheme Z such that, for all points $z \in Z(k)$ valued in a field k, the abelian variety $\mathcal{A}_z$ is supersingular. Then* $\dim(Z) \le n/2$. *If $V_{\mathbb{Q}_p}$ is an orthogonal sum of hyperbolic planes, we have the sharper bound*

$$\dim(Z) \le \frac{n}{2} - 1.$$

Proof. — If $V_{\mathbb{Q}_p}$ is not an orthogonal sum of hyperbolic planes, then we can find an embedding

$$V_{\mathbb{Z}} \hookrightarrow V^{\diamond}_{\mathbb{Z}},$$

where $V^{\diamond}_{\mathbb{Q}_p}$ is of this form, and where the codimension of $V \subset V^{\diamond}$ is 1 if n is odd and 2 if n is even. Using such an embedding, the proposition can be reduced to proving the final assertion, and so we may assume that $V_{\mathbb{Q}_p}$ (and hence $V_{\mathbb{Z}_p}$) is an orthogonal sum of hyperbolic planes.

When $p > 2$, the proposition follows from the much finer results of [**29**], which give a complete description of the supersingular locus of $\mathcal{S}_K(G,\mathcal{D})_{\mathbb{F}_p}$. However, if one is only interested in upper bounds, one can appeal to the methods of [**45**], which apply even when $p = 2$ and $V_{\mathbb{Z}_p}$ is self-dual. See in particular Proposition 14 of [*loc. cit.*]

For the convenience of the reader, we sketch the basic idea here. First, we can replace Z with its underlying reduced scheme. Second, we can throw away its singular part, and assume that Z is smooth.

If $z \in Z(k)$ is a geometric point, then the *Artin invariant* of z is the k-codimension of the image of $V_0(\mathcal{A}_z) \otimes_{\mathbb{Z}} k \to \boldsymbol{V}_{\mathrm{dR},z}$. This is an integer between 1 and $n/2$. Ogus's argument shows that there is a canonical filtration of $F^0\boldsymbol{V}_{\mathrm{dR},Z}$ by coherent, isotropic, horizontal coherent subsheaves

$$\boldsymbol{E}_1 \subset \cdots \boldsymbol{E}_i \ \subset \cdots \subset \boldsymbol{E}_{n/2} \subset F^0\boldsymbol{V}_{\mathrm{dR},Z}$$

with the following properties:

— A geometric point $z \in Z(k)$ has Artin invariant $\le j$ if and only if

$$F^1\boldsymbol{V}_{\mathrm{dR},z} \subset \boldsymbol{E}_{j,z}.$$

— If $Z_{\ge j} \subset Z$ is the open subscheme where the Artin invariant is $\ge j$, then $\boldsymbol{E}_{j,Z_{\ge j}}$ is a rank j local direct summand of $\boldsymbol{V}_{\mathrm{dR},Z_{\ge j}}$.

Note that the first condition ensures that locus where the Artin invariant is bounded below by j is indeed an open subscheme of Z.

Given these two properties, it is immediate from Lemma 7.1.3 that the dimension of Z is bounded above by $r-1$, where r is the maximal Artin invariant attained by a geometric point of Z. This proves the proposition.

The construction of $\boldsymbol{E}_j$ is as follows. For $j=1$, $\boldsymbol{E}_1$ is just the conjugate filtration $\boldsymbol{C}\subset \boldsymbol{V}_{\mathrm{dR},Z}$ already encountered in the proof of Proposition 7.1.2. The crystalline Frobenius on the crystalline realization of $\mathcal{A}_Z$ induces an isometry

$$\gamma : \mathrm{Fr}_Z^*(F^0\boldsymbol{V}_{\mathrm{dR},Z}/F^1\boldsymbol{V}_{\mathrm{dR},Z})\cong \boldsymbol{C}^\perp/\boldsymbol{C},$$

where Fr_Z is the absolute Frobenius on Z. Now inductively define $\boldsymbol{E}_j\subset\boldsymbol{C}^\perp$ as the pre-image of the image of $\boldsymbol{E}_{j-1}$ under the composition

$$\mathrm{Fr}_Z^*\boldsymbol{E}_{j-1}\hookrightarrow \mathrm{Fr}_Z^*F^0\boldsymbol{V}_{\mathrm{dR},Z}\xrightarrow{\gamma}\boldsymbol{C}^\perp/\boldsymbol{C}.$$

It follows from the argument in [**45**, Lemma 5] that $\boldsymbol{E}_j$ is a subsheaf of $F^0\boldsymbol{V}_{\mathrm{dR},Z}$ for all j, so that the inductive procedure is well-defined. That it is isotropic, coherent and horizontal follows from the construction. That the filtration thus obtained has the desired properties follows from the arguments in Proposition 6 and Lemma 9 of [*loc. cit.*]. □

Lemma 7.1.5. — *Suppose that Λ is maximal at p. The complement of $\mathcal{Z}^{\mathrm{pr}}(\Lambda)$ in $\mathcal{Z}(\Lambda)$ lies above the supersingular locus of $\mathcal{S}_K(G,\mathcal{D})_{\mathbb{F}_p}$. If we let*

$$m=\begin{cases}\frac{n}{2} & \text{if } V_{\mathbb{Z}_p} \text{ is an orthogonal sum of hyperbolic planes}\\ \lfloor\frac{n}{2}\rfloor & \text{if } n \text{ is odd}\\ \frac{n}{2}-1 & \text{otherwise,}\end{cases}$$

then the following properties hold.

1. *If* $\mathrm{rank}(\Lambda)\le m$ *then* $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ *is dense in* $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$.
2. *If* $\mathrm{rank}(\Lambda)\le m-1$ *then the complement of* $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ *in* $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ *has codimension at least* 2.

Proof. — Once we know that the complement is supported above the supersingular locus of the mod-p fiber, the rest will follow from the bounds in Proposition 7.1.4.

To prove the assertion on the complement, we first note that the open immersion (7.1.3) induces an isomorphism of the generic fibers; see [**39**, Prop. 6.16]. Therefore, we only have to show that the mod-p fiber of the complement is supported on the supersingular locus. Equivalently, we must show that, for any non-supersingular point $s\in\mathcal{Z}(\Lambda)(k)$ valued in a field k of characteristic p, the subspace $\boldsymbol{\Lambda}_{\mathrm{dR},s}\subset\boldsymbol{V}_{\mathrm{dR},s}$ has k-dimension $\mathrm{rank}(\Lambda)$.

Arguing as in [**39**, § 6.27], we find that, for such a point s, the de Rham realization map

$$V_0(\mathcal{A}_s)\otimes k\to\boldsymbol{V}_{\mathrm{dR},s}$$

is injective. Moreover, by the maximality of Λ at p, the image of

$$\Lambda\otimes\mathbb{Z}_{(p)}\to V_0(\mathcal{A}_s)\otimes\mathbb{Z}_{(p)}$$

is a $\mathbb{Z}_{(p)}$-module direct summand of rank rank(Λ). Combining these two observations shows that the subspace $\mathbf{\Lambda}_{\mathrm{dR},s} \subset \boldsymbol{V}_{\mathrm{dR},s}$ has k-dimension rank(Λ), and completes the proof of the lemma. □

Proposition 7.1.6. — *Suppose that Λ is maximal at p, and let m be defined as in Lemma 7.1.5.*

1. *If* $\operatorname{rank}(\Lambda) \le m$ *then $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ is a generically smooth local complete intersection of dimension $n - \operatorname{rank}(\Lambda)$. Moreover, $\mathcal{Z}(\Lambda)$ is normal and flat over $\mathbb{Z}_{(p)}$.*
2. *If* $\operatorname{rank}(\Lambda) \le m - 1$ *then $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ is geometrically normal.*

Proof. — Note that we always have

$$m \le \frac{n}{2} \quad \text{and} \quad m - 1 \le \frac{n-1}{2}.$$

First suppose $\operatorname{rank}(\Lambda) \le m$. Combining Proposition 7.1.2 and Lemma 7.1.5 shows that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is a generically smooth local complete intersection of dimension $n - \operatorname{rank}(\Lambda)$, and is dense in $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$. Hence $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ is itself generically smooth of dimension $n - \operatorname{rank}(\Lambda)$.

It now follows from claim (3) of Lemma 7.1.1 that $\mathcal{Z}(\Lambda)$ is a local complete intersection, flat over $\mathbb{Z}_{(p)}$. In particular, it is Cohen-Macaualy and so satisfies Serre's property (S_k) for all $k \ge 1$. Recall from claim (2) of Lemma 7.1.1 that the generic fiber of $\mathcal{Z}(\Lambda)$ is smooth over $\mathbb{Q}$. As we have already proved that the special fiber is generically smooth, $\mathcal{Z}(\Lambda)$ is regular in codimension one, and hence satisfies Serre's property (R_1). Claim (2) now follows from Serre's criterion for normality.

Now suppose $\operatorname{rank}(\Lambda) \le m - 1$. We have already shown that the geometric fiber of $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ is a local complete intersection. So, just as above, to show that it is normal it is enough to show that it is regular in codimension one. This follows by combining Proposition 7.1.2 and Lemma 7.1.5, which shows that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is smooth outside of a codimension two subspace, and that its complement in $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ has codimension at least 2. □

7.2. Normality of the fibers, and flatness of divisors.

— We return to the general setting in which $V_{\mathbb{Z}} \subset V$ is any maximal lattice, and deduce two important consequences from the results of § 7.1.

Proposition 7.2.1. — *If $n \ge 6$, the special fiber of $\mathcal{S}_K(G, \mathcal{D})$ is geometrically normal.*

Proof. — When $p > 2$, this is part of [**2**, Theorem 4.4.5]. The same idea of proof works in general, bolstered now by Proposition 7.1.6

Using Lemma 6.2.1, we may choose an embedding $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^{\diamond}$ as in § 6.2 in such a way that $V_{\mathbb{Z}}^{\diamond}$ is self-dual at p, and

$$\Lambda = \{x \in V_{\mathbb{Z}}^{\diamond} : x \perp V_{\mathbb{Z}}\}$$

has rank at most r, where $r = 2$ if n is even and $r = 3$ otherwise. [8]

There is a commutative diagram

$$\begin{array}{ccc} & & \mathcal{Z}^{\diamond}(\Lambda) \\ & \nearrow & \downarrow \\ \mathcal{S}_K(G,\mathcal{D}) & \longrightarrow & \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}), \end{array}$$

in which the vertical morphism is defined as in (7.1.1), the horizontal morphism is (6.6.1), and the diagonal arrow is induced by the isometric embedding

$$\Lambda \to V_0(\mathcal{A}^{\diamond}_{\mathcal{S}_K(G,\mathcal{D})}),$$

determined by (6.6.3).

The self-duality of $V^{\diamond}_{\mathbb{Z}_p}$ gives us an isomorphism

$$V^{\vee}_{\mathbb{Z}_p}/V_{\mathbb{Z}_p} \cong \Lambda^{\vee}_{\mathbb{Z}_p}/\Lambda_{\mathbb{Z}_p}$$

of quadratic spaces over $\mathbb{Q}_p/\mathbb{Z}_p$, as in (6.4.10). The maximality of $V_{\mathbb{Z}}$ at p implies that the left hand side contains no nonzero isotropic vectors, and so neither does the right hand side. This implies the maximality of Λ at p. With this in hand, we may apply Proposition 7.1.6 and the inequality

$$r \leq \frac{n+r}{2} - 2,$$

which holds as $n \geq 6$, to see that $\mathcal{Z}^{\diamond}(\Lambda)$ has geometrically normal fibers.

Thus it suffices to show that the diagonal arrow is an open and closed immersion. This holds in the generic fiber by [**39**, Lemma 7.1], and hence also on the level of integral models as the source and target are both normal. □

Proposition 7.2.2. — *Assume that $n \geq 4$. For every positive $m \in \mathbb{Q}$ and $\mu \in V^{\vee}_{\mathbb{Z}}/V_{\mathbb{Z}}$, the special divisor $\mathcal{Z}(m,\mu)$ is flat over $\mathbb{Z}_{(p)}$.*

Proof. — When $p > 2$ this is [**2**, Proposition 4.5.8]. We explain how to extend the proof to the general case.

As in the proof of Proposition 7.2.1 fix an embedding $V_{\mathbb{Z}} \hookrightarrow V^{\diamond}_{\mathbb{Z}}$ with $V^{\diamond}_{\mathbb{Z}}$ self-dual at p, and so that

$$\Lambda = \{x \in V^{\diamond}_{\mathbb{Z}} : x \perp V_{\mathbb{Z}}\}$$

is maximal of rank at most r with $r = 2$ when n is even and $r = 3$ otherwise. [9]

Consider again the finite unramified morphism

$$\mathcal{Z}^{\diamond}(\Lambda) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond},\mathcal{D}^{\diamond}).$$

---

(8) If $p \neq 2$ we can choose $V^{\diamond}_{\mathbb{Z}}$ to be self-dual at p with $r = 2$. In this case, we can improve the bound to $n \geq 5$ as in [**2**, Theorem 4.4.5].

(9) Once again, if $p > 2$, then we can always take $r = 2$ and the result can be strengthened to only require $n \geq 3$.

By Proposition 7.1.6, this is normal and flat over $\mathbb{Z}_{(p)}$, as long as we have

$$2 \leq \frac{n+2}{2} - 1,$$

for n even and

$$3 \leq \frac{n+3}{2} - 1,$$

for n odd. These inequalities hold for $n \geq 4$.

Using the decomposition (6.6.4), we may choose a positive $m^\diamond \in \mathbb{Q}$ and a $\mu^\diamond \in (V_{\mathbb{Z}}^\diamond)^\vee / V_{\mathbb{Z}}^\diamond$ in such a way that

$$\mathcal{Z}(m,\mu) \subset \mathcal{Z}^\diamond(m^\diamond,\mu^\diamond) \times_{\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)} \mathcal{S}_K(G,\mathcal{D})$$

as an open and closed substack. Now use the open and closed immersion

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{Z}^\diamond(\Lambda)$$

from the proof of Proposition 7.2.1 to identify

$$\mathcal{Z}(m,\mu) \subset \mathcal{Z}^\diamond(m^\diamond,\mu^\diamond) \times_{\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)} \mathcal{Z}^\diamond(\Lambda) \tag{7.2.1}$$

as a union of connected components. In particular, by Proposition 6.5.2, the projection

$$\mathcal{Z}(m,\mu) \to \mathcal{Z}^\diamond(\Lambda) \tag{7.2.2}$$

is, étale locally on the target, a disjoint union of closed immersions each defined by a single equation.

Lemma 7.2.3. — *The image of* (7.2.2) *contains no irreducible component of* $\mathcal{Z}^\diamond(\Lambda)_{\mathbb{F}_p}$.

Proof. — An S-point of $\mathcal{Z}(m,\mu)$ determines a special quasi-endomorphism $x \in V(\mathcal{A}_S)$ with $Q(x) = m$. The image of such an S-point under the inclusion (7.2.1) determines an $x^\diamond \in V(\mathcal{A}_S^\diamond)$, as well as an isometric embedding $\iota : \Lambda \to V_0(\mathcal{A}_S)$. Unpacking the construction of the inclusion (7.2.1), we find that the orthogonal decomposition

$$V(\mathcal{A}_S^\diamond) = V(\mathcal{A}_S) \oplus \Lambda_{\mathbb{Q}},$$

of Proposition 6.6.2 identifies $x^\diamond = x + \iota(\lambda)$ for some $\lambda \in \Lambda_{\mathbb{Q}}$. In particular, x determines a nonzero element of $V(\mathcal{A}_S^\diamond)$ orthogonal to $\iota(\Lambda_{\mathbb{Q}})$, and

$$\iota : \Lambda_{\mathbb{Q}} \to V(\mathcal{A}_S^\diamond)$$

is not surjective.

In contrast, for every generic point η of $\mathcal{Z}^\diamond(\Lambda)_{\mathbb{F}_p}$ we have

$$\iota_\eta : \Lambda \cong V_0(\mathcal{A}_\eta^\diamond).$$

Indeed, this follows from the density $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p} \subset \mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ proved in Lemma 7.1.5, and assertion (1) of Proposition 7.1.2. It can be checked that the numerical hypotheses hold under our hypothesis $n \geq 4$.

Thus the image of (7.2.2) cannot contain the generic point of any irreducible component of $\mathcal{Z}^\diamond(\Lambda)_{\mathbb{F}_p}$, completing the proof of the lemma. □

To complete the proof of Proposition 7.2.2, we apply the following lemma to the complete local ring of the local complete intersection (and hence Cohen-Macaulay) stack $\mathcal{Z}^\diamond(\Lambda)$ at a point in the image of (7.2.2), and taking a to be the equation defining the complete local ring of $\mathcal{Z}(m,\mu)$ at a point in the pre-image.

Lemma 7.2.4. — *Let R be a complete local flat $\mathbb{Z}_{(p)}$-algebra that is Cohen-Macaulay. Suppose that $a \in R$ is such that $\mathrm{Spec}(R/aR) \subset \mathrm{Spec}\ R$ does not contain any irreducible component of $\mathrm{Spec}(R \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p)$. Then R/aR is also flat over $\mathbb{Z}_{(p)}$.*

Proof. — Since R is $\mathbb{Z}_{(p)}$-flat, R/pR is once again Cohen-Macaulay. Our hypotheses imply that the image $\overline{a} \in R/pR$ of a is not contained in any minimal prime of R/pR, which means that $\overline{a}$ is a non-zero divisor in R/pR. Since R is local, this is equivalent to saying that p is a non-zero divisor in R/aR, which shows that R/aR is $\mathbb{Z}_{(p)}$-flat. □

This completes the proof of Proposition 7.2.2 □

8. Integral theory of q-expansions

Keep the hypotheses and notation of § 6 and § 7. In particular, we fix a prime p at which $V_{\mathbb{Z}} \subset V$ is maximal. We now consider toroidal compactifications of the integral model

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_{(p)}).$$

If V is anisotropic then [**40**, Corollary 4.1.7] shows that the integral model is already proper. Therefore, in this subsection, we assume that V admits an isotropic vector.

8.1. Toroidal compactification. — Fix auxiliary data $V^\diamond_{\mathbb{Z}} \subset V^\diamond$ and $K^\diamond$ as in § 6.2, and choose this in such a way that $V^\diamond_{\mathbb{Z}}$ is almost self-dual at p. In particular, from (6.3.2) we have the finite morphism

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$$

of integral models, under which $\boldsymbol{\omega}^\diamond$ pulls back to $\boldsymbol{\omega}$.

We may choose the auxiliary $V^\diamond$ to have signature $(n^\diamond, 2)$ with $n^\diamond \geq 5$. By Lemma 6.1.3, this allows us to choose a symplectic form $\psi^\diamond$ on

$$H^\diamond = C(V^\diamond)$$

in such a way that the $\mathbb{Z}$-lattice $H^\diamond_{\mathbb{Z}} = C(V^\diamond_{\mathbb{Z}})$ is self-dual at p. As in § 4.3 we obtain an embedding

$$(G^\diamond,\mathcal{D}^\diamond) \to (G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$$

into the Siegel Shimura datum determined by $(H^\diamond,\psi^\diamond)$. Recalling the Shimura datum $(\mathbb{G}_m,\mathcal{H}_0)$ of § 3.5, this also fixes a morphism $(G^\diamond,\mathcal{D}^\diamond) \to (\mathbb{G}_m,\mathcal{H}_0)$.

Define reductive groups over $\mathbb{Z}_{(p)}$ by

$$\mathcal{G}^\diamond = \mathrm{GSpin}(V^\diamond_{\mathbb{Z}_{(p)}}), \quad \mathcal{G}^{\mathrm{Sg}} = \mathrm{GSp}(H^\diamond_{\mathbb{Z}_{(p)}}),$$

so that $G^\diamond \to G^{\mathrm{Sg}}$ extends to a closed immersion $\mathcal{G}^\diamond \to \mathcal{G}^{\mathrm{Sg}}$. Fix a compact open subgroup

$$K^{\mathrm{Sg}} = K_p^{\mathrm{Sg}} K^{\mathrm{Sg},p} \subset G^{\mathrm{Sg}}(\mathbb{A}_f)$$

containing $K^\diamond$ and satisfying $K_p^{\mathrm{Sg}} = \mathcal{G}^{\mathrm{Sg}}(\mathbb{Z}_p)$. After shrinking the prime-to-p parts of

$$K \subset K^\diamond \subset K^{\mathrm{Sg}},$$

we assume that all three are neat.

We can construct a toroidal compactification of $\mathcal{S}_K(G,\mathcal{D})$ as follows. Fix a finite, complete K^{Sg}-admissible cone decomposition Σ^{Sg} for $(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$. As explained in §2.5, it pulls back to a finite, complete, $K^\diamond$-admissible polyhedral cone decomposition $\Sigma^\diamond$ for $(G^\diamond,\mathcal{D}^\diamond)$, and a finite, complete, K-admissible polyhedral cone decomposition Σ for $(G,\mathcal{D})$. If Σ^{Sg} has the no self-intersection property, then so do the decompositions induced from it.

Assume that K^{Sg} and Σ^{Sg} are chosen so that Σ^{Sg} is smooth and satisfies the no self-intersection property. We obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_K(G,\mathcal{D},\Sigma) & \longleftarrow & \mathrm{Sh}_K(G,\mathcal{D},\Sigma) \\ \downarrow & & \downarrow \\ \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) & \longleftarrow & \mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) \\ \downarrow & & \downarrow \\ \mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}},\Sigma^{\mathrm{Sg}}) & \longleftarrow & \mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}},\Sigma^{\mathrm{Sg}}), \end{array} \tag{8.1.1}$$

where $\mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}},\Sigma^{\mathrm{Sg}})$ is the toroidal compactification of $\mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$ constructed by Faltings-Chai. Note that the neatness of K^{Sg} implies that it is an algebraic space, rather than a stack, but does not guarantee that it is a scheme. The two algebraic spaces above it are defined by normalization, exactly as in (6.3.2).

According to [**40**, Theorem 4.1.5], the algebraic space $\mathcal{S}_K(G,\mathcal{D},\Sigma)$ is proper over $\mathbb{Z}_{(p)}$ and admits a stratification

$$\mathcal{S}_K(G,\mathcal{D},\Sigma) = \bigsqcup_{(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)} \mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma) \tag{8.1.2}$$

by locally closed subspaces, extending (2.6.1), in which every stratum is flat over $\mathbb{Z}_{(p)}$. The unique open stratum is $\mathcal{S}_K(G,\mathcal{D})$, and its complement is a Cartier divisor.

Fix a toroidal stratum representative $(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)$ in such a way that the parabolic subgroup underlying Φ is the stabilizer of an isotropic line. As in § 2.3, the cusp label representative Φ determines a T_Φ-torsor

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \to \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0),$$

and the rational polyhedral cone σ determines a partial compactification

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \hookrightarrow \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma).$$

The base $\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0)$ of the T_Φ-torsor, being a zero dimensional étale scheme over $\mathbb{Q}$, has a canonical finite normal integral model defined as the normalization of $\mathrm{Spec}(\mathbb{Z}_{(p)})$. The picture is

$$\begin{array}{ccc} \mathcal{S}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0) & \longleftarrow & \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{Z}_{(p)}) & \longleftarrow & \mathrm{Spec}(\mathbb{Q}). \end{array}$$

Proposition 8.1.1. — *Define an integral model*

$$\mathcal{T}_\Phi = \mathrm{Spec}\Big(\mathbb{Z}_{(p)}[q_\alpha]_{\alpha \in \Gamma_\Phi^\vee(1)}\Big)$$

of the torus T_Φ of § 2.3.

1. *The $\mathbb{Q}$-scheme $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ admits a canonical integral model*

$$\mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \to \mathrm{Spec}(\mathbb{Z}_{(p)}),$$

endowed with the structure of a relative $\mathcal{T}_\Phi$-torsor

$$\mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \to \mathcal{S}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0)$$

compatible with the torsor structure (2.3.1) *in the generic fiber.*

2. *There is a canonical isomorphism*

$$\widehat{\mathcal{S}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)$$

of formal algebraic spaces extending (2.6.3).

Here $\mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \hookrightarrow \mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$ is the partial compactification determined by the rational polyhedral cone

$$\sigma \subset U_\Phi(\mathbb{R})(-1) = \mathrm{Hom}(\mathbb{G}_m, \mathcal{T}_\Phi)_\mathbb{R}$$

and the formal scheme on the left hand side is its completion along its unique closed stratum. On the right,

$$\widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) = \mathcal{S}_K(G, \mathcal{D}, \Sigma)^\wedge_{\mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)}$$

is the formal completion along the stratum indexed by (Φ, σ).

Proof. — This is a consequence of [**40**, Theorem 4.1.5]. □

By [**40**, Theorem 2] and [**15**], both $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$ and the Faltings-Chai compactification are proper. They admit stratifications

$$\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) = \bigsqcup_{(\Phi^\diamond, \sigma^\diamond) \in \mathrm{Strat}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)} \mathcal{Z}_{K^\diamond}^{(\Phi^\diamond, \sigma^\diamond)}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond),$$

and

$$\mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}, \Sigma^{\mathrm{Sg}}) = \bigsqcup_{(\Phi^{\mathrm{Sg}}, \sigma^{\mathrm{Sg}}) \in \mathrm{Strat}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})} \mathcal{Z}_{K^{\mathrm{Sg}}}^{(\Phi^{\mathrm{Sg}}, \sigma^{\mathrm{Sg}})}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}, \Sigma^{\mathrm{Sg}}),$$

analogous to (8.1.2). By [**40**, (4.1.13)], these stratifications satisfy a natural compatibility: if

$$(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)$$

has images $(\Phi^\diamond,\sigma^\diamond)$ and $(\Phi^{\mathrm{Sg}},\sigma^{\mathrm{Sg}})$, in the sense of § 2.5, then the maps in (8.1.1) induce maps on strata

$$\mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)\to \mathcal{Z}_{K^\diamond}^{(\Phi^\diamond,\sigma^\diamond)}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)\to \mathcal{Z}_{K^{\mathrm{Sg}}}^{(\Phi^{\mathrm{Sg}},\sigma^{\mathrm{Sg}})}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}},\Sigma^{\mathrm{Sg}}).$$

Applying the functor of Proposition 8.1.2 below to the $\mathcal{G}^\diamond$-representation $V^\diamond_{\mathbb{Z}_{(p)}}$ yields a line bundle $\boldsymbol{\omega}^\diamond = F^1\boldsymbol{V}^\diamond_{\mathrm{dR}}$ on $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$, which we pull back to a line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_K(G,\mathcal{D},\Sigma)$. This gives an extension of (6.3.3) to the toroidal compactification.

Proposition 8.1.2. — *There is a functor*

$$N\mapsto (\boldsymbol{N}_{\mathrm{dR}},F^\bullet\boldsymbol{N}_{\mathrm{dR}})$$

from representations $\mathcal{G}^\diamond\to\mathrm{GL}(N)$ on free $\mathbb{Z}_{(p)}$-modules of finite rank to filtered vectors bundles on $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$, extending the functor (6.3.1) *on the open stratum, and the functor of Theorem 3.4.1 in the generic fiber.*

Proof. — Consider the filtered vector bundle $(\boldsymbol{H}^\diamond_{\mathrm{dR}},F^\bullet\boldsymbol{H}^\diamond_{\mathrm{dR}})$ over $\mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$ obtained by applying the functor (3.3.2) to the representation

$$G^\diamond\to G^{\mathrm{Sg}}=\mathrm{GSp}(H^\diamond).$$

Now let $\nu^\diamond : G^\diamond\to\mathbb{G}_m$ be the spinor similitude, and let $\mathbb{Q}(\nu^\diamond)$ denote the corresponding one-dimensional representation of $G^\diamond$. It determines a line bundle on $\mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$, which is canonically a pullback via the morphism

$$\mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)\xrightarrow{\nu^\diamond}\mathrm{Sh}_{\nu^\diamond(K^\diamond)}(\mathbb{G}_m,\mathcal{H}_0).$$

Combining this with Remark 3.5.2, we see that the line bundle determined by $\mathbb{Q}(\nu^\diamond)$ is canonically identified with $\mathrm{Lie}(\mathbb{G}_m)$, and hence the $G^\diamond$-equivariant morphism $\psi^\diamond : H\otimes H\to\mathbb{Q}(\nu^\diamond)$ induces an alternating form

$$\psi^\diamond:\boldsymbol{H}^\diamond_{\mathrm{dR}}\otimes\boldsymbol{H}^\diamond_{\mathrm{dR}}\to\mathrm{Lie}(\mathbb{G}_m).$$

The nontrivial step $F^0\boldsymbol{H}^\diamond_{\mathrm{dR}}$ in the filtration is a Lagrangian subsheaf with respect to this pairing.

The vector bundle $\boldsymbol{H}^\diamond_{\mathrm{dR}}$ is canonically identified with the pullback via

$$\mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)\to\mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$$

of the first relative homology

$$\boldsymbol{H}^{\mathrm{Sg}}_{\mathrm{dR}}=\underline{\mathrm{Hom}}\big(R^1\pi_*\Omega^\bullet_{A^{\mathrm{Sg}}/\mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})},\mathcal{O}_{\mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})}\big)$$

of the universal polarized abelian scheme $\pi: A^{\mathrm{Sg}}\to\mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$. As the universal abelian scheme extends canonically to the integral model, so does $\boldsymbol{H}^{\mathrm{Sg}}_{\mathrm{dR}}$. Its pullback defines an extension of $\boldsymbol{H}^\diamond_{\mathrm{dR}}$, along with its filtration and alternating form, to the integral model $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$.

Now fix a family of tensors

$$\{s_\alpha\} \subset H^{\diamond,\otimes}_{\mathbb{Z}_{(p)}}$$

that cut out the reductive subgroup $\mathcal{G}^\diamond \subset \mathcal{G}^{\mathrm{Sg}}$. The functoriality of (3.3.2) implies that these tensors define global sections $\{\boldsymbol{s}_{\alpha,dR}\}$ of $\boldsymbol{H}^{\diamond,\otimes}_{\mathrm{dR}}$ over the generic fiber. By [**31**, Corollary 2.3.9], they extend (necessarily uniquely) to sections over the integral model $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond)$.

By [**40**, Proposition 4.3.7], the filtered vector bundle $(\boldsymbol{H}^\diamond_{\mathrm{dR}}, F^\bullet \boldsymbol{H}^\diamond_{\mathrm{dR}})$ admits a canonical extension to $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$. The alternating form $\psi^\diamond$ and the sections $\boldsymbol{s}_{\alpha,dR}$ also extend (necessarily uniquely).

This allows us to define a $\mathcal{G}^\diamond$-torsor

$$\mathcal{J}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) \xrightarrow{a} \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond),$$

whose functor of points assigns to a scheme $S \to \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$ the set of all pairs (f, f_0) of isomorphisms

$$f : \boldsymbol{H}^\diamond_{dR/S} \cong H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S, \quad f_0 : \mathrm{Lie}(\mathbb{G}_m)_{/S} \cong \mathcal{O}_S, \tag{8.1.3}$$

satisfying $f(\boldsymbol{s}_{\alpha,dR}) = s_\alpha \otimes 1$ for all α, and making the diagram

$$\begin{array}{ccc} \boldsymbol{H}^\diamond_{dR/S} \otimes \boldsymbol{H}^\diamond_{dR/S} & \xrightarrow{\psi^\diamond} & \mathrm{Lie}(\mathbb{G}_m) \\ \downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle f_0} \\ (H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S) \otimes (H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S) & \xrightarrow{\psi^\diamond} & \mathcal{O}_S \end{array}$$

commute.

Define smooth $\mathbb{Z}_{(p)}$-schemes $\check{\mathcal{M}}^\diamond$ and $\check{\mathcal{M}}^{\mathrm{Sg}}$ with functors of points

$$\check{\mathcal{M}}(G^\diamond, \mathcal{D}^\diamond)(S) = \{\text{isotropic lines } z \subset V^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S\}$$
$$\check{\mathcal{M}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})(S) = \{\text{Lagrangian subsheaves } F^0 \subset H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S\}.$$

These are integral models of the compact duals $\check{M}(G^\diamond, \mathcal{D}^\diamond)$ and $\check{M}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})$ of § 4.3, and are related, using (4.1.2), by a closed immersion

$$\check{\mathcal{M}}(G^\diamond, \mathcal{D}^\diamond) \to \check{\mathcal{M}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}), \tag{8.1.4}$$

sending the isotropic line $z \subset V^\diamond_{\mathbb{Z}_{(p)}}$ to the Lagrangian $zH^\diamond_{\mathbb{Z}_{(p)}} \subset H^\diamond_{\mathbb{Z}_{(p)}}$.

We now have a diagram

$$\begin{array}{ccc} \mathcal{J}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) & \xrightarrow{b} & \check{\mathcal{M}}(G^\diamond, \mathcal{D}^\diamond) \\ \downarrow{\scriptstyle a} & & \\ \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond), & & \end{array} \tag{8.1.5}$$

in which a is a $\mathcal{G}^\diamond$-torsor and b is $\mathcal{G}^\diamond$-equivariant, extending the diagram (3.4.1) already constructed in the generic fiber. To define the morphism b we first define a morphism

$$\mathcal{J}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) \to \check{\mathcal{M}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})$$

by sending an S-point (f, f_0) to the Lagrangian subsheaf

$$f(F^0 \boldsymbol{H}_{dR/S}) \subset H_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S.$$

This morphism factors through (8.1.4). Indeed, as (8.1.4) is a closed immersion, this is a formal consequence of the fact that we have such a factorization in the generic fiber, as can be checked using the analogous complex analytic construction.

With the diagram (8.1.5) in hand, the construction of the desired functor proceeds by simply imitating the construction (3.3.2) used in the generic fiber. □

8.2. Integral q-expansions. — Continue with the assumptions of § 8.1, and now fix a toroidal stratum representative

$$(\Phi, \sigma) \in \mathrm{Strat}_K(G, \mathcal{D}, \Sigma)$$

as in § 4.6. Thus $\Phi = (P, \mathcal{D}^\circ, h)$ with P the stabilizer of an isotropic line $I \subset V$, and $\sigma \in \Sigma_\Phi$ is a top dimensional rational polyhedral cone. Let

$$(\Phi^\diamond, \sigma^\diamond) \in \mathrm{Strat}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$$

be the image of (Φ, σ), in the sense of § 2.5.

The formal completions along the corresponding strata

$$\begin{aligned} \mathcal{Z}_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma) &\subset \mathcal{S}_K(G, \mathcal{D}, \Sigma) \\ \mathcal{Z}_{K^\diamond}^{(\Phi^\diamond,\sigma^\diamond)}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) &\subset \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) \end{aligned} \tag{8.2.1}$$

are denoted

$$\begin{aligned} \widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) &= \mathcal{S}_K(G, \mathcal{D}, \Sigma)^\wedge_{\mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)}, \\ \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) &= \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)^\wedge_{\mathcal{Z}_{K^\diamond}^{(\Phi^\diamond,\sigma^\diamond)}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}. \end{aligned}$$

These are formal algebraic spaces over $\mathbb{Z}_{(p)}$ related by a finite morphism

$$\widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) \to \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond). \tag{8.2.2}$$

Fix a $\mathbb{Z}_{(p)}$-module generator $\ell \in I \cap V_{\mathbb{Z}_{(p)}}$. Recall from the discussion leading to (4.6.10) that such an ℓ determines an isomorphism

$$[\boldsymbol{\ell}^{\otimes k}, -] : \boldsymbol{\omega}^{\otimes k} \to \mathcal{O}_{\widehat{\mathrm{Sh}}_K(G,\mathcal{D},\Sigma)}$$

of line bundles on $\widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)$.

Proposition 8.2.1. — *The above isomorphism extends uniquely to an isomorphism*

$$[\boldsymbol{\ell}^{\otimes k}, -] : \boldsymbol{\omega}^{\otimes k} \to \mathcal{O}_{\widehat{\mathcal{S}}_K(G,\mathcal{D},\Sigma)}$$

of line bundles on the integral model $\widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)$.

Proof. — The maximality of $V_{\mathbb{Z}_p}$ implies that $V_{\mathbb{Z}_p} \subset V^\diamond_{\mathbb{Z}_p}$ is a $\mathbb{Z}_p$-module direct summand. In particular,

$$I_{\mathbb{Z}_{(p)}} = I \cap V_{\mathbb{Z}_{(p)}} = I \cap V^\diamond_{\mathbb{Z}_{(p)}} \subset V^\diamond_{\mathbb{Z}_{(p)}}$$

is a $\mathbb{Z}_{(p)}$-module direct summand generated by ℓ. Because $\boldsymbol{\omega}$ is defined as the pullback of $\boldsymbol{\omega}^\diamond$, and because the uniqueness part of the claim is obvious, it suffices to construct an isomorphism

$$[\boldsymbol{\ell}, -] : \boldsymbol{\omega}^\diamond \to \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}, \tag{8.2.3}$$

extending the one in the generic fiber, and then pull back along (8.2.2).

We return to the notation of the proof of Proposition 8.1.2. Let $\mathcal{P}^\diamond \subset \mathcal{G}^\diamond$ be the stabilizer of the isotropic line $I_{\mathbb{Z}_{(p)}} \subset V^\diamond_{\mathbb{Z}_{(p)}}$, define a $\mathcal{P}^\diamond$-stable weight filtration

$$\mathrm{wt}_{-3} H^\diamond_{\mathbb{Z}_{(p)}} = 0, \quad \mathrm{wt}_{-2} H^\diamond_{\mathbb{Z}_{(p)}} = \mathrm{wt}_{-1} H^\diamond_{\mathbb{Z}_{(p)}} = I_{\mathbb{Z}_{(p)}} H^\diamond_{\mathbb{Z}_{(p)}}, \quad \mathrm{wt}_0 H^\diamond_{\mathbb{Z}_{(p)}} = H^\diamond_{\mathbb{Z}_{(p)}},$$

and set

$$\mathcal{Q}^\diamond_\Phi = \ker\big(\mathcal{P}^\diamond \to \mathrm{GL}(\mathrm{gr}_0(H^\diamond_{\mathbb{Z}_{(p)}}))\big).$$

Compare with the discussion of § 4.4.

The $\mathbb{Z}_{(p)}$-schemes of (8.1.4) sit in a commutative diagram

$$\begin{array}{ccc} \check{\mathcal{M}}^\diamond_\Phi & \longrightarrow & \check{\mathcal{M}}^{\mathrm{Sg}}_\Phi \\ \downarrow & & \downarrow \\ \check{\mathcal{M}}(G^\diamond, \mathcal{D}^\diamond) & \longrightarrow & \check{\mathcal{M}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}), \end{array}$$

in which the horizontal arrows are closed immersions, and the vertical arrows are open immersions. The $\mathbb{Z}_{(p)}$-schemes in the top row are defined by their functors of points, which are

$$\check{\mathcal{M}}^\diamond_\Phi(S) = \left\{ \begin{array}{c} \text{isotropic lines } z \subset V^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S \text{ such that} \\ V^\diamond_{\mathbb{Z}_{(p)}} \to V^\diamond_{\mathbb{Z}_{(p)}} / I^\perp_{\mathbb{Z}_{(p)}} \\ \text{identifies } z \cong (V^\diamond_{\mathbb{Z}_{(p)}} / I^\perp_{\mathbb{Z}_{(p)}}) \otimes \mathcal{O}_S \end{array} \right\}$$

and

$$\check{\mathcal{M}}^{\mathrm{Sg}}_\Phi(S) = \left\{ \begin{array}{c} \text{Lagrangian subsheaves } F^0 \subset H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S \text{ such that} \\ H^\diamond_{\mathbb{Z}_{(p)}} \to \mathrm{gr}_0(H^\diamond_{\mathbb{Z}_{(p)}}) \\ \text{identifies } F^0 \cong \mathrm{gr}_0(H^\diamond_{\mathbb{Z}_{(p)}}) \otimes \mathcal{O}_S \end{array} \right\}.$$

Passing to formal completions, the diagram (8.1.5) determines a diagram

$$\begin{array}{ccc} \widehat{\mathcal{J}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) & \xrightarrow{\ b\ } & \check{\mathcal{M}}(G^\diamond, \mathcal{D}^\diamond) \\ {\scriptstyle a}\downarrow & & \\ \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) & & \end{array} \tag{8.2.4}$$

of formal algebraic spaces over $\mathbb{Z}_{(p)}$, in which a is a $\mathcal{G}^\diamond$-torsor and b is $\mathcal{G}^\diamond$-equivariant, and $\widehat{\mathcal{J}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ is the formal completion of $\mathcal{J}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ along the fiber over the stratum (8.2.1).

Lemma 8.2.2. — *The $\mathcal{G}^\diamond$-torsor in* (8.2.4) *admits a canonical reduction of structure to a $\mathcal{Q}_\Phi^\diamond$-torsor $\mathcal{J}_\Phi^\diamond$, sitting in a diagram*

$$\begin{array}{ccc} \mathcal{J}_\Phi^\diamond & \xrightarrow{\ b\ } & \check{\mathcal{M}}_\Phi^\diamond \\ {\scriptstyle a}\downarrow & & \\ \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond). & & \end{array}$$

Proof. — The essential point is that the filtered vector bundle $(\boldsymbol{H}_{\mathrm{dR}}^\diamond, F^\bullet\boldsymbol{H}_{\mathrm{dR}}^\diamond)$ on $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ used in the construction of the $\mathcal{G}^\diamond$-torsor

$$\mathcal{J}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)\to\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$$

acquires extra structure after restriction to $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$. Namely, it acquires a weight filtration

$$\mathrm{wt}_{-3}\boldsymbol{H}_{\mathrm{dR}}^\diamond=0,\quad \mathrm{wt}_{-2}\boldsymbol{H}_{\mathrm{dR}}^\diamond=\mathrm{wt}_{-1}\boldsymbol{H}_{\mathrm{dR}}^\diamond,\quad \mathrm{wt}_{0}\boldsymbol{H}_{\mathrm{dR}}^\diamond=\boldsymbol{H}_{\mathrm{dR}}^\diamond,$$

along with distinguished isomorphisms

$$\begin{aligned}\mathrm{gr}_{-2}(\boldsymbol{H}_{\mathrm{dR}}^\diamond)&\cong\mathrm{gr}_{-2}(H_{\mathbb{Z}_{(p)}}^\diamond)\otimes\mathrm{Lie}(\mathbb{G}_m)\\ \mathrm{gr}_{0}(\boldsymbol{H}_{\mathrm{dR}}^\diamond)&\cong\mathrm{gr}_{0}(H_{\mathbb{Z}_{(p)}}^\diamond)\otimes\mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}.\end{aligned}$$

This follows from the discussion of [**40**, (4.3.1)]. The essential point is that over the formal completion $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ there is a canonical degenerating abelian scheme, and the desired extension of $\boldsymbol{H}_{\mathrm{dR}}^\diamond$ is its de Rham realization. The extension of the weight and Hodge filtrations is also a consequence of this observation; see [**40**, § 1], and in particular [**40**, Proposition 1.3.5].

The desired reduction of structure $\mathcal{J}_\Phi^\diamond\subset\widehat{\mathcal{J}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ is now defined as the closed formal algebraic subspace parametrizing pairs of isomorphisms (f,f_0) as in (8.1.3) that respect this additional structure.

Moreover, after restricting $\boldsymbol{H}_{\mathrm{dR}}^\diamond$ to $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$, the surjection $\boldsymbol{H}_{\mathrm{dR}}^\diamond\to\mathrm{gr}_0\boldsymbol{H}_{\mathrm{dR}}^\diamond$ identifies $F^0\boldsymbol{H}_{\mathrm{dR}}^\diamond\cong\mathrm{gr}_0\boldsymbol{H}_{\mathrm{dR}}^\diamond$. Indeed, in the language of [**40**, § 1], this just amounts to the observation that the de Rham realization of a 1-motive with trivial abelian part has trivial weight and Hodge filtrations.

As the composition

$$\mathcal{J}_\Phi^\diamond\subset\widehat{\mathcal{J}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)\xrightarrow{b}\check{\mathcal{M}}(G^\diamond,\mathcal{D}^\diamond)\subset\check{\mathcal{M}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$$

sends $(f,f_0)\mapsto f(F^0\boldsymbol{H}_{\mathrm{dR}}^\diamond)$, it takes values in the open subscheme

$$\check{\mathcal{M}}_\Phi^{\mathrm{Sg}}\subset\check{\mathcal{M}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}}).$$

It therefore take values in the closed subscheme $\check{\mathcal{M}}^\diamond_\Phi \subset \check{\mathcal{M}}^{\mathrm{Sg}}_\Phi$, as this can be checked in the generic fiber, where it follows from the analogous complex analytic constructions. □

Returning to the main proof, let $\check{I} \subset \check{V}^\diamond$ be the constant $\mathcal{Q}^\diamond_\Phi$-equivariant line bundles on $\check{\mathcal{M}}^\diamond_\Phi$ determined by the representations $I_{\mathbb{Z}_{(p)}} \subset V^\diamond_{\mathbb{Z}_{(p)}}$, and let $\check{\omega}^\diamond \subset \check{V}^\diamond$ be the tautological line bundle. The self-duality of $V^\diamond_{\mathbb{Z}_{(p)}}$ guarantees that the bilinear pairing on $\check{V}^\diamond$ restricts to an isomorphism

$$[-,-] : \check{I} \otimes \check{\omega}^\diamond \to \mathcal{O}_{\check{\mathcal{M}}^\diamond_\Phi}.$$

Pulling back these line bundles to $\mathcal{J}^\diamond_\Phi$ and taking the quotient by $\mathcal{Q}^\diamond_\Phi$, we obtain an isomorphism

$$[-,-] : \boldsymbol{I}_{\mathrm{dR}} \otimes \boldsymbol{\omega}^\diamond \to \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}$$

of line bundles on $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$.

On the other hand, the action of $\mathcal{Q}^\diamond_\Phi$ on $I_{\mathbb{Z}_{(p)}}$ is through the character $\nu^\diamond_\Phi$, which agrees with the restriction of $\nu^\diamond : \mathcal{G}^\diamond \to \mathbb{G}_m$ to $\mathcal{Q}^\diamond_\Phi$. The canonical morphism

$$\mathcal{J}^\diamond_\Phi \to \widehat{\mathcal{J}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) \xrightarrow{(f,f_0)\mapsto f_0} \underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m), \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}\big)$$

of formal algebraic spaces over $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$ identifies $\ker(\nu^\diamond_\Phi)\backslash\mathcal{J}^\diamond_\Phi$ with the trivial $\mathbb{G}_m$-torsor

$$\underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m), \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}\big) \cong \underline{\mathrm{Aut}}\big(\mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}\big)$$

over $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$. As the action of $\mathcal{G}^\diamond$ on $I_{\mathbb{Z}_{(p)}}$ is via $\nu^\diamond_\Phi$, this trivialization fixes an isomorphism

$$\begin{aligned} \boldsymbol{I}_{\mathrm{dR}} &= \mathcal{Q}^\diamond_\Phi\backslash\big(I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\mathcal{J}^\diamond_\Phi}\big) \\ &= \mathbb{G}_m\backslash\big(I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\ker(\nu^\diamond_\Phi)\backslash\mathcal{J}^\diamond_\Phi}\big) \\ &\cong I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}. \end{aligned}$$

The generator $\ell \in I_{\mathbb{Z}_{(p)}}$ now determines a trivializing section $\boldsymbol{\ell} = \ell \otimes 1$ of $\boldsymbol{I}_{\mathrm{dR}}$, defining the desired isomorphism (8.2.3). This completes the proof of Proposition 8.2.1. □

Let $I_* \subset V$ and

$$(Q_\Phi, \mathcal{D}_\Phi) \underset{\nu_\Phi}{\overset{s}{\rightleftarrows}} (\mathbb{G}_m, \mathcal{H}_0).$$

be as in the discussion preceding Proposition 4.6.2. Choose a compact open subgroup $K_0 \subset \mathbb{A}_f^\times$ small enough that $s(K_0) \subset K_\Phi$, and assume that K_0 factors as

$$K_0 = \mathbb{Z}_p^\times \cdot K_0^p.$$

Let $F/\mathbb{Q}$ be the abelian extension of $\mathbb{Q}$ determined by

$$\mathrm{rec} : \mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/K_0 \cong \mathrm{Gal}(F/\mathbb{Q}).$$

Fix a prime $\mathfrak{p} \subset \mathcal{O}_F$ above p, and let $R \subset F$ be the localization of $\mathcal{O}_F$ at $\mathfrak{p}$. Note that the above assumption on K implies that p is unramified in F.

Proposition 8.2.3. — *If we set*

$$\widehat{\mathcal{T}}_\Phi(\sigma) = \mathrm{Spf}\Big(\mathbb{Z}_{(p)}[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geq 0}}\Big),$$

there is a unique morphism

$$\bigsqcup_{a\in\mathbb{Q}_{>0}^\times\backslash\mathbb{A}_f^\times/K_0} \widehat{\mathcal{T}}_\Phi(\sigma)_{/R} \to \widehat{\mathcal{S}}_K(G,\mathcal{D},\Sigma)_{/R}$$

of formal algebraic spaces over R whose base change to $\mathbb{C}$ agrees with the morphism of Proposition 4.6.2. Moreover, if t is any point of the source and s is its image in $\widehat{\mathcal{S}}_K(G,\mathcal{D},\Sigma)_{/R}$, the induced map on étale local rings $\mathcal{O}_s^{et}\to\mathcal{O}_t^{et}$ is faithfully flat.

Proof. — The uniqueness of such a morphism is clear. We have to show existence. The proof of this proceeds just as that of Proposition 4.6.2, except that it uses Proposition 8.1.1 as input. The only additional observation required is that we have an isomorphism

$$\bigsqcup_{a\in\mathbb{Q}_{>0}^\times\backslash\mathbb{A}_f^\times/K_0} \mathrm{Spec}(R) \cong \mathcal{S}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)_{/R} \tag{8.2.5}$$

of R-schemes, which realizes (4.6.8) on $\mathbb{C}$-points. Here $\mathcal{S}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)$ is defined as the normalization of $\mathrm{Spec}(\mathbb{Z}_{(p)})$ in $\mathrm{Sh}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)$.

To see this, note that the defining property of canonical models provides an isomorphism

$$\mathrm{Spec}(F)\cong\mathrm{Sh}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)$$

of $\mathbb{Q}$-schemes, and hence an isomorphism F-schemes

$$\bigsqcup_{a\in\mathbb{Q}_{>0}^\times\backslash\mathbb{A}_f^\times/K_0} \mathrm{Spec}(F) \cong \mathrm{Sh}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)_{/F}.$$

Using the fact that p is unramified in F, one can see that this isomorphism extends to (8.2.5). □

Suppose ψ is a section of the line bundle $\boldsymbol{\omega}^{\otimes k}$ on $\mathrm{Sh}_K(G,\mathcal{D})_{/F}$. It follows from Proposition 4.6.3 that the q-expansion (4.6.10) of ψ has coefficients in F for every $a\in\mathbb{A}_f^\times$. If we view ψ as a rational section on $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$, the following result gives a criterion for testing flatness of its divisor.

Corollary 8.2.4. — *Assume that the special fiber of $\mathcal{S}_K(G,\mathcal{D})_{/R}$ is geometrically normal, and for every $a\in\mathbb{A}_f^\times$ the q-expansion* (4.6.10) *satisfies*

$$\mathrm{FJ}^{(a)}(\psi)\in R[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geq 0}}.$$

If this q-expansion is nonzero modulo $\mathfrak{p}$ for all a, then $\mathrm{div}(\psi)$ *is R-flat.*

Proof. — As $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$ is flat over R, to show that $\mathrm{div}(\psi)$ is R-flat it is enough to show that its support does not contain any irreducible components of the special fiber of $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$.

Every connected component

$$C \subset \mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$$

has irreducible special fiber. Indeed, we have assumed that the special fiber of $\mathcal{S}_K(G,\mathcal{D})_{/R}$ is geometrically normal. It therefore follows from [**40**, Theorem 1] that the special fiber of $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$ is also geometrically normal. On the other hand, [**40**, Corollary 4.1.11] shows that C has geometrically connected special fiber. Therefore the special fiber of C is both connected and normal, and hence is irreducible.

As in the proof of Proposition 4.6.3, the closed stratum

$$\mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)_{/R} \subset \mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$$

meets every connected component. Pick a closed point s of this stratum lying on the connected component C. By the definition of $\mathrm{FJ}^{(a)}(\psi)$, and from Proposition 8.2.3, our hypothesis on the q-expansion implies that the restriction of ψ to the completed local ring $\mathcal{O}_s$ of s defines a rational section of $\boldsymbol{\omega}^{\otimes k}$ whose divisor is an R-flat Cartier divisor on $\mathrm{Spf}(\mathcal{O}_s)$.

It follows that $\mathrm{div}(\psi)$ does not contain the special fiber of C, and varying C shows that $\mathrm{div}(\psi)$ contains no irreducible components of the special fiber of $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$. □

Remark 8.2.5. — If $V_{\mathbb{Z}_p}$ is almost self-dual, then $\mathcal{S}_K(G,\mathcal{D})$ is smooth over $\mathbb{Z}_{(p)}$, and hence has geometrically normal special fiber. Without the assumption of almost self-duality, Proposition 7.2.1 tells us that the special fiber is geometrically normal whenever $n \geq 6$.

9. Borcherds products on integral models

Keep $V_{\mathbb{Z}} \subset V$ of signature $(n,2)$ with $n \geq 1$, and let $(G,\mathcal{D})$ be the associated GSpin Shimura datum. As in the introduction, let Ω be a finite set of prime numbers containing all primes at which $V_{\mathbb{Z}}$ is not maximal, and choose (1.1.2) to be factorizable $K = \prod_p K_p$ with

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times$$

for all $p \notin \Omega$. Set $\mathbb{Z}_\Omega = \mathbb{Z}[1/p : p \in \Omega]$.

9.1. Statement of the main result. — In § 6.3 and § 6.4 we constructed, for every prime $p \notin \Omega$, an integral model over $\mathbb{Z}_{(p)}$ of the Shimura variety $\mathrm{Sh}_K(G,\mathcal{D})$, along with a family of special divisors and a line bundle of weight one modular forms. As explained in [**1**, § 2.4] and [**2**, § 4.5], as p varies these models arise as the localizations of a flat and normal integral model

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_\Omega),$$

endowed with a family of special divisors $\mathcal{Z}(m,\mu)$ indexed by positive $m \in \mathbb{Q}$ and $\mu \in L^\vee/L$, and a line bundle of weight one modular forms $\boldsymbol{\omega}$.

Theorem 9.1.1. — *Suppose*

$$f(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m \gg -\infty}} c(m)\cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_\mathbb{Z}})$$

is a weakly holomorphic form as in (5.1.1), *and assume f is integral in the sense of Definition 5.1.2. After multiplying f by any sufficiently divisible positive integer, there is a rational section $\psi(f)$ of $\boldsymbol{\omega}^{\otimes c(0,0)}$ over $\mathcal{S}_K(G,\mathcal{D})$ whose norm under the metric* (4.2.3) *is related to the regularized theta lift of § 5.2 by*

$$-2\log\|\psi(f)\| = \Theta^{\mathrm{reg}}(f), \tag{9.1.1}$$

and whose divisor is

$$\mathrm{div}(\psi(f)) = \sum_{\substack{m>0 \\ \mu\in V_\mathbb{Z}^\vee/V_\mathbb{Z}}} c(-m,\mu)\cdot \mathcal{Z}(m,\mu). \tag{9.1.2}$$

The remainder of this subsection is devoted to proving Theorem 9.1.1 under some restrictive hypotheses on the pair $V_\mathbb{Z} \subset V$. These will allow us to deduce algebraicity of the Borcherds product from Proposition 5.2.3, prove its descent to $\mathbb{Q}$ using the q-expansion principle of Proposition 4.6.3, and deduce the equality of divisors (9.1.2) from the flatness of both sides over $\mathbb{Z}_\Omega$.

Proposition 9.1.2. — *If $n \geq 6$, and if there exists an $h \in G(\mathbb{A}_f)$ and isotropic vectors $\ell, \ell_* \in hV_\mathbb{Z}$ such that $[\ell,\ell_*] = 1$, then Theorem 9.1.1 holds.*

Proof. — It suffices to treat the case where

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times,$$

for then we can pull back $\psi(f)$ to any smaller level structure.

The vectors $\ell, \ell_* \in V$ satisfy the relation (5.3.1) with $k = \ell_*$. Let I and I_* be the isotropic lines in V spanned by ℓ and ℓ_*, respectively. Let P be the stabilizer of I, and let $\mathcal{D}^\circ \subset \mathcal{D}$ be a connected component. This determines a cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h).$$

Although we will not use this fact explicitly, the following lemma implies that the 0-dimensional stratum of the Baily-Borel compactification $\mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{BB}}$ indexed by Φ is geometrically connected. In other words, Baily-Borel compactification has a cusp defined over $\mathbb{Q}$.

Lemma 9.1.3. — *The complex orbifold* $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ *is connected, and the section* (4.6.6) *determined by* I_* *satisfies*

$$s(\widehat{\mathbb{Z}}^\times) \subset K_\Phi. \tag{9.1.3}$$

Proof. — We first prove (9.1.3). Consider the hyperbolic place

$$W = \mathbb{Q}\ell + \mathbb{Q}\ell_* \subset V.$$

Its corresponding spinor similitude group $\mathrm{GSpin}(W)$ is just the unit group of the even Clifford algebra $C^+(W)$. The natural inclusion $\mathrm{GSpin}(W) \to G$ takes values in the subgroup Q_Φ, and the cocharacter (4.6.6) factors as

$$\mathbb{G}_m \xrightarrow{s} \mathrm{GSpin}(W) \to Q_\Phi,$$

where the first arrow sends $a \in \mathbb{Q}^\times$ to

$$s(a) = a^{-1}\ell_*\ell + \ell\ell_* \in C^+(W)^\times.$$

From this explicit formula and the inclusion

$$H_{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Z}}\ell \oplus \widehat{\mathbb{Z}}\ell_* \subset hV_{\widehat{\mathbb{Z}}},$$

it is clear that (4.6.6) satisfies

$$s(\widehat{\mathbb{Z}}^\times) \subset C^+(W_{\widehat{\mathbb{Z}}})^\times \subset Q_\Phi(\mathbb{A}_f) \cap C(hV_{\widehat{\mathbb{Z}}})^\times = K_\Phi.$$

Now we prove the connectedness claim. From (9.1.3) it follows that

$$\widehat{\mathbb{Z}}^\times = \nu_\Phi(s(\widehat{\mathbb{Z}}^\times)) \subset \nu_\Phi(K_\Phi) \subset \nu(K),$$

and hence the 0-dimensional Shimura variety

$$\mathrm{Sh}_{\nu(K)}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \backslash \mathcal{H}_0 \times \mathbb{A}_f^\times / \nu(K)$$

consists of a single point. The proof of Proposition 4.6.3 shows that the fibers of

$$\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}) \to \mathrm{Sh}_{\nu(K)}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$$

are connected, completing the proof. □

Applying Theorem 5.2.2 and Proposition 5.2.3 to the form $2f$ gives us a rational section

$$\psi(f) = (2\pi i)^{c(0,0)}\Psi(2f) \tag{9.1.4}$$

of $\boldsymbol{\omega}^{\otimes c(0,0)}$ over $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$. We first prove that $\psi(f)$ can be rescaled by a constant of absolute value 1 to make it defined over $\mathbb{Q}$.

Fix a neat compact open subgroup $\tilde{K} \subset K$ small enough that there is a $\tilde{K}$-admissible complete cone decomposition Σ for $(G,\mathcal{D})$ satisfying the conclusion of Lemma 5.4.1. In particular, we have a top-dimensional rational polyhedral cone $\sigma \in \Sigma_\Phi$ whose interior is contained in a fixed Weyl chamber

$$\mathscr{W} \subset \mathrm{LightCone}^\circ(V_{0\mathbb{R}}) \cong C_\Phi.$$

Let $\tilde{\psi}(f)$ denote the pullback of $\psi(f)$ to $\mathrm{Sh}_{\tilde{K}}(G,\mathcal{D},\Sigma)_{/\mathbb{C}}$. Recalling the construction of q-expansions of (4.6.10), the toroidal stratum representative

$$(\Phi,\sigma)\in \mathrm{Strat}_{\tilde{K}}(G,\mathcal{D},\Sigma)$$

determines a collection of formal q-expansions

$$\mathrm{FJ}^{(a)}(\tilde{\psi}(f))\in \mathbb{C}[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geq 0}} \tag{9.1.5}$$

indexed by $a\in\mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/\tilde{K}_0$, where $\tilde{K}_0\subset\mathbb{G}_m(\mathbb{A}_f)$ is chosen small enough that its image under (4.6.6) is contained in $\tilde{K}_\Phi$.

We can read off these q-expansions from Proposition 5.4.2, which implies

$$\mathrm{FJ}^{(a)}(\tilde{\psi}(f))=\big(\kappa^{(a)}\cdot q_{\alpha(\varrho)}\cdot \mathrm{BP}(f)\big)^2, \tag{9.1.6}$$

for an explicit

$$\mathrm{BP}(f)\in\mathbb{Z}[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geq 0}}, \tag{9.1.7}$$

and some constants $\kappa^{(a)}\in\mathbb{C}$ of absolute value 1. Indeed, the hypotheses on $\ell,\ell_*\in V_\mathbb{Z}$ imply that the constants N and A appearing in (5.3.6) are equal to 1, and our choice of $k=\ell_*\in hV_\mathbb{Z}$ implies that $\zeta_\mu=1$ for all $\mu\in hV_\mathbb{Z}^\vee/hV_\mathbb{Z}$.

Moreover, it is clear from the presentation of $\mathrm{BP}(f)$ as a product that its constant term is equal to 1.

The q-expansion (9.1.5) is actually independent of a. Indeed, using the notation of (5.4.5), with K replaced by $\tilde{K}$ throughout, these q-expansions can be computed in terms of the pullback of $\psi(f)$ to the upper left corner in

$$\begin{array}{ccc}
\displaystyle\bigsqcup_{a\in\mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/\tilde{K}_0}\tilde{\Gamma}_\Phi^{(a)}\backslash\mathcal{D}^\circ & \xrightarrow{z\mapsto(z,s(a)h)} & \mathrm{Sh}_{\tilde{K}}(G,\mathcal{D})(\mathbb{C})\\
\downarrow & & \downarrow\\
(K_\Phi\cap U_\Phi(\mathbb{Q}))\backslash\mathcal{D}^\circ & \xrightarrow{z\mapsto(z,h)} & \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}).
\end{array}$$

Here we have chosen our coset representatives $a\in\widehat{\mathbb{Z}}^\times$. This implies, by Lemma 9.1.3, that $s(a)\in K_\Phi\subset hKh^{-1}$, and so

$$\tilde{\Gamma}_\Phi^{(a)}=s(a)\tilde{K}_\Phi s(a)^{-1}\cap U_\Phi(\mathbb{Q})\subset K_\Phi\cap U_\Phi(\mathbb{Q})$$

and $s(a)hK=hK$. It follows that the pullback of $\psi(f)$ to the upper left corner is the same on every copy of $\mathcal{D}^\circ$.

Having proved that all of the $\kappa^{(a)}$ are equal, we may rescale $\psi(f)$ by a constant of absolute value 1 to make all of them equal to 1. The q-expansion principle of Proposition 4.6.3 now implies that $\tilde{\psi}(f)$ is defined over $\mathbb{Q}$, and the same is therefore true of $\psi(f)$. The equality (9.1.1) follows from the equality (5.2.2).

It only remains to prove the equality of divisors (9.1.2). In the generic fiber, this follows from (9.1.1) and the analysis of the singularities of $\Theta^{\mathrm{reg}}(f)$ found in **[5]** or **[8]**. To prove equality on the integral model, it therefore suffices to prove that both sides of the desired equality are flat over $\mathbb{Z}_\Omega$. Flatness of the special divisors $\mathcal{Z}(m,\mu)$ is Proposition 7.2.2.

To prove the flatness of $\mathrm{div}(\psi(f))$ it suffices to show, for every prime $p \notin \Omega$, that $\mathrm{div}(\psi(f))$ has no irreducible components supported in characteristic p. This follows from Corollary 8.2.4 and the observation made above that (9.1.7) has nonzero reduction at p.

The only technical point is that to apply Corollary 8.2.4 to the integral model of $\mathrm{Sh}_{\tilde{K}}(G,\mathcal{D},\Sigma)$ over $\mathbb{Z}_{(p)}$, we must choose $\tilde{K}$ to have p-component

$$\tilde{K}_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times,$$

and similarly choose $\tilde{K}_0$ to have p-component $\mathbb{Z}_p^\times$. As p varies, this forces us to vary $\tilde{K}$. As we need $\tilde{K}$ to satisfy the conclusion of Lemma 5.4.1, this may require us to also vary both Σ and the rational polyhedral cone $\sigma \in \Sigma_\Phi$. Thus, having rescaled the Borcherds product to eliminate the constants $\kappa^{(a)}$ at one boundary stratum, we may be forced to apply Corollary 8.2.4 at a different boundary stratum of a different toroidal compactification at different level structure, at which we must deal with new constants $\kappa^{(a)}$.

This is not really a problem. For a given p, one can check using Remark 2.4.9 that it is possible to choose $\tilde{K}$ (and hence Σ and $\sigma \in \Sigma_\Phi$) as in Lemma 5.4.1 by shrinking only the prime-to-p part of K. Using Lemma 9.1.3, we may then choose $\tilde{K}_0$ to have p-component $\mathbb{Z}_p^\times$. Now pull back $\psi(f)$ via the resulting étale cover

$$\mathcal{S}_{\tilde{K}}(G,\mathcal{D})_{/\mathbb{Z}_{(p)}} \to \mathcal{S}_K(G,\mathcal{D})_{/\mathbb{Z}_{(p)}}$$

over integral models over $\mathbb{Z}_{(p)}$ to obtain a section $\tilde{\psi}(f)$ whose q-expansion again has the form (9.1.6) for some constants $\kappa^{(a)}$ of absolute value 1.

The point is simply that our $\psi(f)$, hence also $\tilde{\psi}(f)$, has been rescaled so that it is defined over $\mathbb{Q}$. This allows us to use the q-expansion principle of Proposition 4.6.3 to deduce that each $\kappa^{(a)}$ is rational, hence is ± 1. Thus the power series (9.1.6) has integer coefficients and nonzero reduction at p. Corollary 8.2.4 implies that the divisor of $\tilde{\psi}(f)$ has no irreducible components in characteristic p, so the same holds for $\psi(f)$. □

9.2. Proof of Theorem 9.1.1. — In this subsection we complete the proof of Theorem 9.1.1 by developing a purely algebraic analogue of the embedding trick of Borcherds. This allows us to deduce the general case from the special case proved in Proposition 9.1.2.

According to [**5**, Lemma 8.1] there exist self-dual $\mathbb{Z}$-quadratic spaces $\Lambda^{[1]}$ and $\Lambda^{[2]}$ of signature $(24,0)$ whose corresponding theta series

$$\vartheta^{[i]}(\tau) = \sum_{x \in \Lambda^{[i]}} q^{Q(x)} \in M_{12}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{C})$$

are related by

$$\vartheta^{[2]} - \vartheta^{[1]} = 24\Delta. \tag{9.2.1}$$

Here Δ is Ramanujan's modular discriminant, and Q is the quadratic form on $\Lambda^{[i]}$. Denote by

$$r^{[i]}(m) = \#\{x \in \Lambda^{[i]} : Q(x) = m\}$$

the m-th Fourier coefficient of $\vartheta^{[i]}$. Set

$$V_{\mathbb{Z}}^{[i]} = V_{\mathbb{Z}} \oplus \Lambda^{[i]} \qquad \text{and} \qquad V^{[i]} = V \oplus \Lambda_{\mathbb{Q}}^{[i]}. \tag{9.2.2}$$

In the notation of § 5.1, the inclusion $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^{[i]}$ identifies

$$V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \cong (V_{\mathbb{Z}}^{[i]})^{\vee}/V_{\mathbb{Z}}^{[i]},$$

and the induced isomorphism

$$S_{V_{\mathbb{Z}}} \cong S_{V_{\mathbb{Z}}^{[i]}}$$

is compatible with the Weil representations on source and target. The fixed weakly holomorphic form f of (5.1.1) therefore determines a form

$$f^{[i]}(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c^{[i]}(m) \cdot q^m \in M^!_{-11-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}^{[i]}})$$

by setting $f^{[i]} = f/(24\Delta)$. The relation

$$f = \vartheta^{[2]} f^{[2]} - \vartheta^{[1]} f^{[1]}$$

implies the equality of Fourier coefficients

$$c(m,\mu) = \sum_{k \geq 0} r^{[2]}(k) \cdot c^{[2]}(m-k,\mu) - \sum_{k \geq 0} r^{[1]}(k) \cdot c^{[1]}(m-k,\mu). \tag{9.2.3}$$

Each $V^{[i]}$ determines a GSpin Shimura datum $(G^{[i]}, \mathcal{D}^{[i]})$. By choosing

$$K^{[i]} = G^{[i]}(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}}^{[i]})^{\times}$$

for our compact open subgroups, we put ourselves in the situation of § 6.6. Note that in § 6.6 the integral models were over $\mathbb{Z}_{(p)}$, but everything extends verbatim to $\mathbb{Z}_{\Omega}$. In particular, we have finite morphisms of integral models

$$\begin{array}{ccccc} & & \mathcal{S}_K(G,\mathcal{D}) & & \\ & \swarrow^{j^{[1]}} & & \searrow^{j^{[2]}} & \\ \mathcal{S}^{[1]} & & & & \mathcal{S}^{[2]} \end{array}$$

over $\mathbb{Z}_{\Omega}$, where we abbreviate

$$\mathcal{S}^{[i]} = \mathcal{S}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]}).$$

Each $\mathcal{S}^{[i]}$ has its own line bundle of weight one modular forms $\boldsymbol{\omega}^{[i]}$ and its own family $\mathcal{Z}^{[i]}(m,\mu)$ of special divisors.

The following lemma shows that each $V_{\mathbb{Z}}^{[i]} \subset V^{[i]}$ satisfies the hypotheses of Proposition 9.1.2. Thus, after replacing f (and hence both $f^{[1]}$ and $f^{[2]}$) by a positive integer multiple, we obtain a Borcherds product $\psi(f^{[i]})$ on $\mathcal{S}^{[i]}$ with divisor

$$\operatorname{div}(\psi(f^{[i]}) = \sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c^{[i]}(-m,\mu) \cdot \mathcal{Z}^{[i]}(m,\mu). \tag{9.2.4}$$

Lemma 9.2.1. — *There exist isotropic vectors $\ell, \ell_* \in V_{\mathbb{Z}}^{[i]}$ with $[\ell, \ell_*] = 1$.*

Proof. — Let $\mathbb{H} = \mathbb{Z}\ell \oplus \mathbb{Z}\ell_*$ be the integral hyperbolic plane, so that ℓ and ℓ_* are isotropic with $[\ell, \ell_*] = 1$. To prove the existence of an isometric embedding $\mathbb{H} \to V_{\mathbb{Z}}^{[i]}$, we first prove the existence everywhere locally.

At the archimedean place this is clear from the signature, so fix a prime p. The $\mathbb{Q}_p$-quadratic space $\Lambda^{[i]} \otimes_{\mathbb{Z}} \mathbb{Q}_p$ has dimension ≥ 5, so admits an isometric embedding

$$\mathbb{H} \otimes \mathbb{Q}_p \to \Lambda^{[i]} \otimes \mathbb{Q}_p.$$

Enlarging the image of $\mathbb{H} \otimes \mathbb{Z}_p$ to a maximal lattice, and invoking Eichler's theorem that all maximal lattices in a $\mathbb{Q}_p$-quadratic space are isometric [**16**, Theorem 8.8], we find that $\mathbb{H} \otimes \mathbb{Z}_p$ embeds into the (self-dual, hence maximal) lattice $\Lambda^{[i]} \otimes \mathbb{Z}_p$. A fortiori, it embeds into $V_{\mathbb{Z}}^{[i]} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

The existence of the desired embeddings everywhere locally implies that there exist isometric embeddings

$$a : \mathbb{H} \otimes \mathbb{Q} \to V_{\mathbb{Z}}^{[i]} \otimes \mathbb{Q}, \tag{9.2.5}$$

and

$$\alpha : \mathbb{H} \otimes \widehat{\mathbb{Z}} \to V_{\mathbb{Z}}^{[i]} \otimes \widehat{\mathbb{Z}}.$$

We may choose these in such a way that a and α induce the same embedding of $\mathbb{Q}_p$-quadratic spaces at all but finitely many primes p. All embeddings

$$\mathbb{H} \otimes \mathbb{Q}_p \to V_{\mathbb{Z}}^{[i]} \otimes \mathbb{Q}_p$$

lie in a single $\mathrm{SO}(V^{[i]})(\mathbb{Q}_p)$-orbit, and so there exists a

$$g \in \mathrm{SO}(V^{[i]})(\mathbb{A}_f)$$

such that

$$ga(\mathbb{H} \otimes \widehat{\mathbb{Z}}) = \alpha(\mathbb{H} \otimes \widehat{\mathbb{Z}}). \tag{9.2.6}$$

Fix a subspace $W \subset V_{\mathbb{Z}}^{[i]} \otimes \mathbb{Q}$ of signature $(2,1)$ perpendicular to the image of (9.2.5). There exists an isomorphism $\mathrm{SO}(W) \cong \mathrm{PGL}_2$ identifying the spinor norm

$$\mathrm{SO}(W)(\mathbb{A}_f) \to \mathbb{A}_f^{\times}/(\mathbb{A}_f^{\times})^2$$

with the determinant, and hence the spinor norm is surjective. This allows us to modify g by an element of $\mathrm{SO}(W)(\mathbb{A}_f)$, which does not change the relation (9.2.6), in order to arrange that g has trivial spinor norm. Now choose any lift

$$g \in \mathrm{Spin}(V^{[i]})(\mathbb{A}_f),$$

and note that (9.2.6) implies

$$ga(\mathbb{H}\otimes\widehat{\mathbb{Z}})\subset V_{\mathbb{Z}}^{[i]}\otimes\widehat{\mathbb{Z}}.$$

As the spin group is simply connected, it satisfies strong approximation. By choosing $\gamma\in\mathrm{Spin}(V^{[i]})(\mathbb{Q})$ sufficiently close to g, we find an isometric embedding $\gamma a:\mathbb{H}\to V_{\mathbb{Z}}^{[i]}$. □

At least formally, we wish to define

$$\psi(f)=\frac{(j^{[2]})^{*}\psi(f^{[2]})}{(j^{[1]})^{*}\psi(f^{[1]})}.$$

As noted in § 1.4, the image of $j^{[i]}$ will typically be contained in the support of the divisor of $\psi(f^{[i]})$, and so the quotient on the right will typically be either $0/0$ or ∞/∞.

The key to making sense of this quotient is to combine the following lemma, which is really just a restatement of (9.2.4), with the pullback formula of Proposition 6.6.3. As in the pullback formula, we use $\mathcal{Z}^{[i]}(m,\mu)$ to denote both the special divisor and its corresponding line bundle, and extend the definition to $m\leq 0$ by

$$\mathcal{Z}^{[i]}(m,\mu)=\begin{cases}(\boldsymbol{\omega}^{[i]})^{-1} & \text{if } (m,\mu)=(0,0)\\ \mathcal{O}_{\mathcal{S}^{[i]}} & \text{otherwise.}\end{cases}$$

Lemma 9.2.2. — *The Borcherds product $\psi(f^{[i]})$ determines an isomorphism of line bundles*

$$\mathcal{O}_{\mathcal{S}^{[i]}}\cong\bigotimes_{\substack{m\geq 0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}}\mathcal{Z}^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)}.$$

Proof. — If $m>0$ there is a canonical section

$$s^{[i]}(m,\mu)\in H^{0}\big(\mathcal{S}^{[i]},\mathcal{Z}^{[i]}(m,\mu)\big)$$

with divisor the Cartier divisor $\mathcal{Z}^{[i]}(m,\mu)$ of the same name. This is just the constant function 1, viewed as a section of

$$\mathcal{O}_{\mathcal{S}^{[i]}}\subset\mathcal{Z}^{[i]}(m,\mu).$$

The equality of divisors (9.2.4) implies that there is a unique isomorphism

$$(\boldsymbol{\omega}^{[i]})^{\otimes c^{[i]}(0,0)}\cong\bigotimes_{\substack{m>0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}}\mathcal{Z}^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)}$$

sending

$$\psi(f^{[i]})\mapsto\bigotimes_{\substack{m>0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}}s^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)},$$

and so the claim is immediate from the definition of $\mathcal{Z}^{[i]}(0,\mu)$. □

Proof of Theorem 9.1.1. — If we pull back the isomorphism of Lemma 9.2.2 via $j^{[i]}$ and use the pullback formula of Proposition 6.6.3, we obtain isomorphisms of line bundles

$$\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m_1,m_2\geq 0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m_1,\mu)^{\otimes r^{[i]}(m_2)\cdot c^{[i]}(-m_1-m_2,\mu)}$$

for $i \in \{1,2\}$. These two isomorphisms, along with (9.2.3), determine an isomorphism

$$\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m\geq 0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu)^{\otimes c(-m,\mu)}.$$

Now simply reverse the reasoning in the proof of Lemma 9.2.2. Rewrite the isomorphism above as

$$\boldsymbol{\omega}^{c(0,0)} \cong \bigotimes_{\substack{m>0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu)^{\otimes c(-m,\mu)}.$$

Each line bundle on the right admits a canonical section $s(m,\mu)$ whose divisor is the Cartier divisor $\mathcal{Z}(m,\mu)$ of the same name, and so the rational section of $\boldsymbol{\omega}^{c(0,0)}$ defined by

$$\psi(f) = \bigotimes_{\substack{m>0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} s(m,\mu)^{\otimes c(-m,\mu)} \tag{9.2.7}$$

has divisor

$$\operatorname{div}(\psi(f)) = \sum_{\substack{m>0\\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu)\cdot\mathcal{Z}(m,\mu).$$

To complete the proof of Theorem 9.1.1, we need only prove that the section defined by (9.2.7) satisfies the norm relation (9.1.1).

Fix a $g \in G(\mathbb{A}_f)$, and consider the complex uniformizations

$$\begin{array}{ccccc}
 & \mathcal{D}^{[1]} & \xrightarrow{\hspace{6em}} & & \mathcal{S}^{[1]}(\mathbb{C}) \\
 {\scriptstyle j^{[1]}}\nearrow & & & \nearrow{\scriptstyle j^{[1]}} & \\
\mathcal{D} & \xrightarrow{\hspace{3em}} & \mathcal{S}_K(G,\mathcal{D})(\mathbb{C}) & & \\
 {\scriptstyle j^{[2]}}\searrow & & & \searrow{\scriptstyle j^{[2]}} & \\
 & \mathcal{D}^{[2]} & \xrightarrow{\hspace{6em}} & & \mathcal{S}^{[2]}(\mathbb{C}),
\end{array}$$

in which all horizontal arrows send $z \mapsto (z,g)$.

Denote by $\psi_g(f)$ the pullback of $\psi(f)$ to $\mathcal{D}$. The similarly defined meromorphic sections $\psi_g(f^{[i]})$ on $\mathcal{D}^{[i]}$ are already assumed to satisfy the norm relation

$$-2\log\|\psi_g(f^{[i]})\| = \Theta_g^{\mathrm{reg}}(f^{[i]})$$

on $\mathcal{D}^{[i]}$, where

$$\Theta_g^{\mathrm{reg}}(f^{[i]}) = \Theta^{\mathrm{reg}}(f^{[i]},g)$$

is the regularized theta lift of § 5.2.

Recall from § 6.5 that every $x \in V^{[i]}$ with $Q(x) > 0$ determines a global section

$$\mathrm{obst}_x^{\mathrm{an}} \in H^0\big(\mathcal{D}^{[i]}, \boldsymbol{\omega}_{\mathcal{D}^{[i]}}^{-1}\big),$$

with zero locus the analytic divisor

$$\mathcal{D}^{[i]}(x) = \{z \in \mathcal{D}^{[i]} : [z, x] = 0\}.$$

The pullback of $\mathcal{Z}^{[i]}(m, \mu)(\mathbb{C})$ to $\mathcal{D}^{[i]}$ is given by the locally finite sum of analytic divisors

$$\sum_{\substack{x \in g\mu + gV_{\mathbb{Z}}^{[i]} \\ Q(x) = m}} \mathcal{D}^{[i]}(x).$$

Define the *renormalized* Borcherds product

$$\tilde{\psi}_g(f^{[i]}) = \psi_g(f^{[i]}) \otimes \bigotimes_{m>0} \bigotimes_{\substack{\lambda \in \Lambda^{[i]} \\ Q(\lambda) = m}} (\mathrm{obst}_\lambda^{\mathrm{an}})^{\otimes - c^{[i]}(-m,0)}.$$

This is a meromorphic section of $\bigotimes_{m \geq 0} \big(\boldsymbol{\omega}_{\mathcal{D}^{[i]}}\big)^{\otimes r^{[i]}(m) c^{[i]}(-m,0)}$ with divisor

$$\mathrm{div}\big(\tilde{\psi}_g(f^{[i]})\big) = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c^{[i]}(-m, \mu) \sum_{\substack{x \in g\mu + gV_{\mathbb{Z}}^{[i]} \\ Q(x) = m \\ x \notin \Lambda^{[i]}}} \mathcal{D}^{[i]}(x).$$

Note that each divisor $\mathcal{D}^{[i]}(x)$ appearing on the right hand side intersects the subspace $\mathcal{D} \subset \mathcal{D}^{[i]}$ properly. Indeed, If we decompose $x = y + \lambda$ with $y \in g\mu + gV_{\mathbb{Z}}$ and $\lambda \in \Lambda$, then

$$\mathcal{D}^{[i]}(x) \cap \mathcal{D} = \begin{cases} \mathcal{D}(y) & \text{if } Q(y) > 0 \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\mathcal{D}(y) = \{z \in \mathcal{D} : [z, y] = 0\}$.

This is the point: by renormalizing the Borcherds product we have removed precisely the part of its divisor that intersects $\mathcal{D}$ improperly, and so the renormalized Borcherds product has a well-defined pullback to $\mathcal{D}$. Indeed, using the relation (9.2.3), we see that

$$\psi_g(f) = \frac{(j^{[2]})^* \tilde{\psi}_g(f^{[2]})}{(j^{[1]})^* \tilde{\psi}_g(f^{[1]})} \tag{9.2.8}$$

is a section of the line bundle $\boldsymbol{\omega}_{\mathcal{D}}^{\otimes c(0,0)}$. By directly comparing the algebraic and analytic constructions, which ultimately boils down to the comparison of algebraic and analytic obstructions found in Proposition 6.5.3, one can check that it agrees with the $\psi_g(f)$ defined at the beginning of the proof.

Define the *renormalized* regularized theta lift

$$\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[i]}) = \Theta_g^{\mathrm{reg}}(f^{[i]}) + 2\sum_{m>0} c^{[i]}(-m,0) \sum_{\substack{\lambda\in\Lambda^{[i]} \\ Q(\lambda)=m}} \log \|\mathrm{obst}_\lambda^{\mathrm{an}}\|$$

so that

$$-2\log\|\tilde{\psi}_g(f^{[i]})\| = \widetilde{\Theta}_g^{\mathrm{reg}}(f^{[i]}). \tag{9.2.9}$$

Combining this with (9.2.8) yields

$$-2\log\|\psi_g(f)\| = (j^{[2]})^*\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[2]}) - (j^{[1]})^*\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[1]}).$$

As was noted in Remark 5.2.1, the regularized theta lift $\Theta_g^{\mathrm{reg}}(f^{[i]})$ is *over-regularized*, in the sense that its definition makes sense at every point of $\mathcal{D}^{[i]}$, even at points of the divisor along which $\Theta_g^{\mathrm{reg}}(f^{[i]})$ has its logarithmic singularities. As in [**1**, Proposition 5.5.1], its values along the discontinuity agree with the values of $\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[i]})$, and in fact we have

$$(j^{[i]})^*\Theta_g^{\mathrm{reg}}(f^{[i]}) = (j^{[i]})^*\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[i]})$$

as functions on $\mathcal{D}$.

On the other hand, for each $i\in\{1,2\}$, the regularized theta lift has the form

$$\Theta_g^{\mathrm{reg}}(f^{[i]})(z) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f^{[i]}(\tau)\vartheta^{[i]}(\tau,z,g)\ \frac{du\,dv}{v^2}$$

as in (5.2.1). As in [**12**, (4.16)], when we restrict the variable z to $\mathcal{D}\subset\mathcal{D}^{[i]}$ the theta kernel factors as

$$\vartheta^{[i]}(\tau,z,g) = \vartheta(\tau,z,g)\cdot\vartheta^{[i]}(\tau),$$

where $\vartheta(\tau,z,g)$ is the theta kernel defining $\Theta_g^{\mathrm{reg}}(f)$. Thus

$$(j^{[i]})^*\Theta_g^{\mathrm{reg}}(f^{[i]}) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\vartheta(\tau,z,g)\cdot\frac{\vartheta^{[i]}(\tau)}{24\Delta}\ \frac{du\,dv}{v^2}.$$

Combining this last equality with (9.2.1) proves the first equality in

$$\begin{aligned}\Theta_g^{\mathrm{reg}}(f) &= (j^{[2]})^*\Theta_g^{\mathrm{reg}}(f^{[2]}) - (j^{[1]})^*\Theta_g^{\mathrm{reg}}(f^{[1]})\\ &= (j^{[2]})^*\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[2]}) - (j^{[1]})^*\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[1]}),\end{aligned}$$

which is just a more explicit statement of [**5**, Lemma 8.1]. Combining this with (9.2.8) and (9.2.9) shows that $\psi(f)$ satisfies the norm relation (9.1.1), and completes the proof of Theorem 9.1.1. □

9.3. A remark on sufficient divisibility. — In order to obtain a Borcherds product $\psi(f)$ on the integral model $\mathcal{S}_K(G,\mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_\Omega)$, Theorem 9.1.1 requires that we first multiply the integral form

$$f(\tau) = \sum_{\substack{m\in\mathbb{Q} \\ m\gg-\infty}} c(m)\cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_\mathbb{Z}})$$

by some unspecified positive integer N. In fact, examination of the proof shows that $N = N(V_\mathbb{Z})$ depends only on the quadratic lattice $V_\mathbb{Z}$, and not on the choice of f, the finite set of primes Ω, or the level subgroup K.

Indeed, one first checks this in the situation of Proposition 9.1.2. Thus we assume that $n \geq 6$, and that there exists an $h \in G(\mathbb{A}_f)$ and isotropic vectors $\ell, \ell_* \in hV_\mathbb{Z}$ with $[\ell,\ell_*] = 1$. As in the proof of that proposition, one can reduce to the case

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times.$$

The only point in the proof of Proposition 9.1.2 where one must replace f by Nf is when Theorem 5.2.2 and Proposition 5.2.3 are invoked to obtain the Borcherds product (9.1.4) over the complex fiber $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$. Thus we only need to require that N be chosen divisible enough that the multipliers

$$\xi_g(f) : G(\mathbb{Q})^\circ \cap gKg^{-1} \to \mathbb{C}^\times$$

of (5.2.5) satisfy $\xi_g(f)^N = 1$, as f varies over all integral forms as above and $g \in G(\mathbb{A}_f)$ runs over the finite set of indices in

$$\bigsqcup_g (G(\mathbb{Q})^\circ \cap gKg^{-1})\backslash\mathcal{D}^\circ \cong \mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}.$$

This is possible, as the natural map

$$G(\mathbb{Q})^\circ \cap gKg^{-1} \to \mathrm{SO}(gV_\mathbb{Z})$$

has kernel $\{\pm 1\}$, and its image has finite abelianization; see [7].

The general case follows by examining the constructions of § 9.2. Applying the special case above to the lattices in (9.2.2) yields positive integers $N(V^{[1]}_\mathbb{Z})$ and $N(V^{[2]}_\mathbb{Z})$. Any multiple of

$$N(V_\mathbb{Z}) = N(V^{[1]}_\mathbb{Z}) \cdot N(V^{[2]}_\mathbb{Z})$$

is then "sufficiently divisible" for the purposes of Theorem 9.1.1.

9.4. Modularity of the generating series. — For any positive $m \in \mathbb{Q}$ and any $\mu \in V^\vee_\mathbb{Z}/V_\mathbb{Z}$, we denote again by

$$\mathcal{Z}(m,\mu) \in \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$

the line bundle defined by the Cartier divisor of the same name. Extend the definition to $m = 0$ by

$$\mathcal{Z}(0,\mu) = \begin{cases} \boldsymbol{\omega}^{-1} & \text{if } \mu = 0 \\ \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} & \text{otherwise.} \end{cases}$$

Recall from § 5.1 the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}})$$

on $S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}]$.

Theorem 9.4.1. — *Let $\phi_\mu \in S_{V_{\mathbb{Z}}}$ be the characteristic function of the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. For any $\mathbb{Z}$-linear map $\alpha : \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$ we have*

$$\sum_{\substack{m\geq 0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \alpha(\mathcal{Z}(m,\mu)) \cdot \phi_\mu \cdot q^m \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}}).$$

Proof. — According to the modularity criterion of [**6**, Theorem 3.1], a formal q-expansion

$$\sum_{\substack{m\geq 0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} a(m,\mu) \cdot \phi_\mu \cdot q^m$$

with coefficients in $S_{V_{\mathbb{Z}}}$ defines an element of $M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$ if and only if

$$0 = \sum_{\substack{m\geq 0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot a(m,\mu) \tag{9.4.1}$$

for every

$$f(\tau) = \sum_{\substack{m\in \mathbb{Q} \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(m,\mu) \cdot q^m \in M^{!}_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}).$$

By the main result of [**41**], it suffices to verify (9.4.1) only for $f(\tau)$ that are integral, in the sense of Definition 5.1.2.

For any integral $f(\tau)$, Theorem 9.1.1 implies that

$$\boldsymbol{\omega}^{c(0,0)} = \sum_{\substack{m> 0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu)$$

up to a torsion element in $\mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$, and hence

$$\sum_{\substack{m\geq 0 \\ \mu\in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu) \in \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$

is killed by any $\mathbb{Z}$-linear map $\alpha : \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$. Thus the claimed modularity follows from the result of Borcherds cited above. □

9.5. Modularity of the arithmetic generating series. — Bruinier [8] has defined a Green function $\Theta^{\mathrm{reg}}(F_{m,\mu})$ for the divisor $\mathcal{Z}(m,\mu)$. This Green function is constructed, as in (5.2.1), as the regularized theta lift of a harmonic Hejhal-Poincare series

$$F_{m,\mu} \in H_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}),$$

whose holomorphic part, in the sense of [10, § 3], has the form

$$F^{+}_{m,\mu}(\tau) = \left(\frac{\phi_{\mu} + \phi_{-\mu}}{2}\right) \cdot q^{-m} + O(1),$$

where $\phi_{\mu} \in S_{V_{\mathbb{Z}}}$ is the characteristic function of the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. See [1, § 3.2] and the references therein.

This Green function determines a metric on the corresponding line bundle, and so determines a class

$$\widehat{\mathcal{Z}}(m,\mu) = (\mathcal{Z}(m,\mu), \Theta^{\mathrm{reg}}(F_{m,\mu})) \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G,\mathcal{D}))$$

for every positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Recall that that we have defined a metric (4.2.3) on the line bundle $\boldsymbol{\omega}$, and so obtain a class

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G,\mathcal{D})).$$

We define

$$\widehat{\mathcal{Z}}(0,\mu) = \begin{cases} \widehat{\boldsymbol{\omega}}^{-1} & \text{if } \mu = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 9.5.1. — *Suppose $n \geq 3$. For any $\mathbb{Z}$-linear functional*

$$\alpha : \widehat{\mathrm{Pic}}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$$

we have

$$\sum_{\substack{m \geq 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \alpha\big(\widehat{\mathcal{Z}}(m,\mu)\big) \cdot \phi_{\mu} \cdot q^{m} \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}}).$$

Proof. — The assumption that $n \geq 3$ implies that any form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(m,\mu) \cdot q^{m} \in M^{!}_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

has negative weight. As in [10, Remark 3.10], this implies that any such f is a linear combination of the Hejhal-Poincare series $F_{m,\mu}$, and in fact

$$f = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot F_{m,\mu}.$$

This last equality follows, as in the proof of [11, Lemma 3.10], from the fact that the difference between the two sides is a harmonic weak Maass form whose holomorphic part is $O(1)$. In particular, we have the equality of regularized theta lifts

$$\Theta^{\mathrm{reg}}(f) = \sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \Theta^{\mathrm{reg}}(F_{m,\mu}).$$

Now assume that f is integral. After replacing f by a positive integer multiple, Theorem 9.1.1 provides us with a Borcherds product $\psi(f)$ with arithmetic divisor

$$\widehat{\operatorname{div}}(\psi(f)) = (\operatorname{div}(\psi(f)), -\log \|\psi(f)\|^2) = \sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \widehat{\mathscr{Z}}(m,\mu).$$

On the other hand, in the group of metrized line bundles we have

$$\widehat{\operatorname{div}}(\psi(f)) = \widehat{\boldsymbol{\omega}}^{\otimes c(0,0)} = -c(0,0) \cdot \widehat{\mathscr{Z}}(0,0).$$

The above relations show that

$$\sum_{\substack{m\geq 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \widehat{\mathscr{Z}}(m,\mu) \in \widehat{\operatorname{Pic}}(\mathcal{S}_K(G,\mathcal{D}))$$

is a torsion element for any integral f. Exactly as in the proof of Theorem 9.4.1, the claim follows from the modularity criterion of Borcherds. □

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