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2023

PARAMETRIX FOR WAVE EQUATIONS ON A ROUGH BACKGROUND

I
REGULARITY OF THE PHASE AT INITIAL TIME
II
CONSTRUCTION AND CONTROL AT INITIAL TIME

Jérémie SZEFTEL

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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PARAMETRIX FOR WAVE EQUATIONS ON A ROUGH BACKGROUND

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by Jérémie SZEFTEL

Abstract. — This book is dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\square_{\mathbf{g}}\phi = 0$, where $\mathbf{g}$ is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor $\mathbf{R}$ of $\mathbf{g}$ is a major step of the proof of the bounded L^2 curvature conjecture proposed in [**10**], and solved jointly with S. Klainerman and I. Rodnianski in [**17**]. On a more general level, this book deals with the control of the eikonal equation on a rough background, and with the derivation of L^2 bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

Résumé. **(Parametrix pour l'équation des ondes sur un espace-temps peu régulier : I. Régularité de la phase à l'instant initial. II. Construction et contrôle à l'instant initial)** — Cet ouvrage est dédié à la construction et au contrôle d'une paramétrix pour l'équation des ondes homogène $\square_{\mathbf{g}}\phi = 0$, où $\mathbf{g}$ est une métrique peu régulière satisfaisant les équations d'Einstein dans le vide. Le contrôle d'une telle paramétrix ainsi que du terme d'erreur associé lorsque l'on suppose seulement des bornes L^2 sur le tenseur de courbure $\mathbf{R}$ de $\mathbf{g}$ est une étape cruciale de la preuve de la conjecture de courbure L^2 proposée dans [**10**], et résolue conjointement avec S. Klainerman et I. Rodnianski dans [**17**]. Plus généralement, cet ouvrage concerne le contrôle de l'équation eikonale sur un espace-temps peu régulier et la dérivation de bornes L^2 pour des opérateurs intégraux de Fourier sur des variétés avec une phase et un symbole peu réguliers, et possède de ce point de vue un intérêt propre.

CONTENTS

PART I

REGULARITY OF THE PHASE AT INITIAL TIME

CHAPTER 1

INTRODUCTION

We consider the Einstein vacuum equations,

$$\mathbf{R}_{\alpha\beta} = 0, \tag{1.1}$$

where $\mathbf{R}_{\alpha\beta}$ denotes the Ricci curvature tensor of a four dimensional Lorentzian space time $(\mathscr{M}, \mathbf{g})$. The Cauchy problem consists in finding a metric $\mathbf{g}$ satisfying (1.1) such that the metric induced by $\mathbf{g}$ on a given space-like hypersurface Σ_0 and the second fundamental form of Σ_0 are prescribed. The initial data then consists of a Riemannian three dimensional metric g_{ij} and a symmetric tensor k_{ij} on the space-like hypersurface $\Sigma_0 = \{t = 0\}$. Now, (1.1) is an overdetermined system and the initial data set (Σ_0, g, k) must satisfy the constraint equations

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \mathrm{Tr} k = 0, \\ R - |k|^2 + (\mathrm{Tr} k)^2 = 0, \end{cases} \tag{1.2}$$

where the covariant derivative ∇ is defined with respect to the metric g, R is the scalar curvature of g, and $\mathrm{Tr}k$ is the trace of k with respect to the metric g.

The fundamental problem in general relativity is to study the long term regularity and asymptotic properties of the Cauchy developments of general, asymptotically flat, initial data sets (Σ_0, g, k). As far as local regularity is concerned it is natural to ask what are the minimal regularity properties of the initial data which guarantee the existence and uniqueness of local developments. In **[17]**, we obtain the following result which solves bounded L^2 curvature conjecture proposed in **[10]**:

***Theorem 1.1* (Theorem 1.10 in [17]).** — *Let $(\mathscr{M}, \mathbf{g})$ an asymptotically flat solution to the Einstein vacuum Equations* (1.1) *together with a maximal foliation by space-like hypersurfaces Σ_t defined as level hypersurfaces of a time function t. Let $r_{\mathrm{vol}}(\Sigma_t, 1)$ the volume radius on scales ≤ 1 of Σ_t* [1]. *Assume that the initial slice (Σ_0, g, k) is such that:*

$$\|R\|_{L^2(\Sigma_0)} \leq \varepsilon,\ \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \varepsilon \text{ and } r_{\mathrm{vol}}(\Sigma_0, 1) \geq \frac{1}{2}.$$

1. See Remark 1.5 below for a definition.

Then, there exists a small universal constant $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$*, then the following control holds on* $0 \le t \le 1$*:*

$$\|\mathbf{R}\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \varepsilon,\ \|k\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \varepsilon \text{ and } \inf_{0\le t\le 1} r_{\mathrm{vol}}(\Sigma_t, 1) \ge \frac{1}{4}.$$

Remark 1.2. — While the first nontrivial improvements for well posedness for quasilinear hyperbolic systems (in spacetime dimensions greater than $1+1$), based on Strichartz estimates, were obtained in **[2]**, **[1]**, **[27]**, **[28]**, **[11]**, **[14]**, **[20]**, Theorem 1.1, is the first result in which the full nonlinear structure of the quasilinear system, not just its principal part, plays a crucial role. We note that though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to its causal geometry, i.e., L^2 bounds on the curvature is the minimum requirement necessary to obtain lower bounds on the radius of injectivity of null hypersurfaces. We refer the reader to Section 1 in **[17]** for more motivations and historical perspectives concerning Theorem 1.1.

Remark 1.3. — The regularity assumptions on Σ_0 in Theorem 1.1—i.e., R and ∇k bounded in $L^2(\Sigma_0)$—correspond to an initial data set $(g,\, k) \in H^2_{\mathrm{loc}}(\Sigma_0) \times H^1_{\mathrm{loc}}(\Sigma_0)$.

Remark 1.4. — In **[17]**, our main result is stated for corresponding large data. We then reduce the proof to the small data statement of Theorem 1.1 relying on a truncation and rescaling procedure, the control of the harmonic radius of Σ_0 based on Cheeger-Gromov convergence of Riemannian manifolds together with the assumption on the lower bound of the volume radius of Σ_0, and the gluing procedure in **[7]**, **[6]**. We refer the reader to Section 2.3 in **[17]** for the details.

Remark 1.5. — We recall for the convenience of the reader the definition of the volume radius of the Riemannian manifold Σ_t. Let $B_r(p)$ denote the geodesic ball of center p and radius r. The volume radius $r_{\mathrm{vol}}(p,r)$ at a point $p \in \Sigma_t$ and scales $\le r$ is defined by

$$r_{\mathrm{vol}}(p,r) = \inf_{r'\le r} \frac{|B_{r'}(p)|}{r^3},$$

with $|B_r|$ the volume of B_r relative to the metric g_t on Σ_t. The volume radius $r_{\mathrm{vol}}(\Sigma_t, r)$ of Σ_t on scales $\le r$ is the infimum of $r_{\mathrm{vol}}(p,r)$ over all points $p \in \Sigma_t$.

The proof of Theorem 1.1, obtained in the sequence of papers **[17]**, **[23]**, **[24]**, **[25]**, **[26]**, **[22]**, relies on the following ingredients (2):

- **A** *Provide a system of coordinates relative to which* (1.1) *exhibits a null structure.*
- **B** *Prove appropriate bilinear estimates for solutions to* $\square_{\mathbf{g}}\phi = 0$*, on a fixed Einstein vacuum background* (3)*.*

2. We also need trilinear estimates and an $L^4(\mathcal{M})$ Strichartz estimate (see the introduction in **[17]**).
3. Note that the first bilinear estimate of this type was obtained in **[12]**.

C *Construct a parametrix for solutions to the homogeneous wave equations $\square_{\mathbf{g}}\phi = 0$ on a fixed Einstein vacuum background, and obtain control of the parametrix and of its error term only using the fact that the curvature tensor is bounded in L^2.*

Steps **A** and **B** are carried out in **[17]**. In particular, the proof of the bilinear estimates rests on a representation formula for the solutions of the wave equation using the following plane wave parametrix [(4)]:

$$Sf(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega,\ (t,x)\in\mathscr{M}, \tag{1.3}$$

where $u(.,.,\omega)$ is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathscr{M}$ such that $u(0,x,\omega) \sim x.\omega$ when $|x| \to +\infty$ on Σ_0 [(5)]. Therefore, in order to complete the proof of the bounded L^2 curvature conjecture, we need to carry out step **C** with the parametrix defined in (1.3).

Remark 1.6. — Note that the parametrix (1.3) is invariantly defined [(6)], i.e., without reference to any coordinate system. This is crucial since coordinate systems consistent with L^2 bounds on the curvature would not be regular enough to control a parametrix.

Remark 1.7. — In addition to their relevance to the resolution of the bounded L^2 curvature conjecture, the methods and results of step **C** are also of independent interest. Indeed, they deal on the one hand with the control of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ at a critical level [(7)], and on the other hand with the derivation of L^2 bounds for Fourier integral operators with significantly lower differentiability assumptions both for the corresponding phase and symbol compared to classical methods (see for example **[21]** and references therein).

In view of the energy estimates for the wave equation, it suffices to control the parametrix at $t=0$ (i.e., restricted to Σ_0)

$$Sf(0,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(0,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega,\ x\in\Sigma_0 \tag{1.4}$$

and the error term

$$Ef(t,x) = \square_{\mathbf{g}}Sf(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)}\square_{\mathbf{g}}u(t,x,\omega) f(\lambda\omega)\lambda^3 d\lambda d\omega,\ (t,x)\in\mathscr{M}. \tag{1.5}$$

4. (1.3) actually corresponds to a half-wave parametrix. The full parametrix corresponds to the sum of two half-parametrix. See **[24]** for the construction of the full parametrix.

5. The asymptotic behavior for $u(0,x,\omega)$ when $|x|\to+\infty$ will be used in **[24]** to generate with the parametrix any initial data set for the wave equation.

6. Our choice is reminiscent of the one used in **[20]** in the context of $H^{2+\epsilon}$ solutions of quasilinear wave equations. Note however that the construction in that paper is coordinate dependent.

7. We need at least L^2 bounds on the curvature to obtain a lower bound on the radius of injectivity of the null level hypersurfaces of the solution u of the eikonal equation, which in turn is necessary to control the local regularity of u (see **[25]**).

This requires the following ingredients, the two first being related to the control of the parametrix restricted to Σ_0 (1.4), and the two others being related to the control of the error term (1.5):

C1 *Make an appropriate choice for the equation satisfied by* $u(0,x,\omega)$ *on* Σ_0*, and control the geometry of the foliation generated by the level surfaces of* $u(0,x,\omega)$ *on* Σ_0*.*

C2 *Prove that the parametrix at* $t=0$ *given by* (1.4) *is bounded in* $\mathscr{L}(L^2(\mathbb{R}^3), L^2(\Sigma))$ *using the estimates for* $u(0,x,\omega)$ *obtained in* **C1**.

C3 *Control the geometry of the foliation generated by the level hypersurfaces of* u *on* $\mathcal{M}$*.*

C4 *Prove that the error term* (1.5) *satisfies the estimate* $\|Ef\|_{L^2(\mathcal{M})} \leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}$ *using the estimates for* u *and* $\square_{\mathbf{g}} u$ *proved in* **C3**.

Step **C3** was initiated in the sequence of papers **[13]**, **[15]**, **[16]** where the authors prove the estimate $\square_{\mathbf{g}} u \in L^\infty(\mathcal{M})$, which is crucial for step **C3** and **C4**. In the present paper, we focus on step **C1**. Remember that u is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathcal{M}$. To define u in a unique manner, we still have to prescribe u on Σ_0. Having in mind steps **C2** and **C3**, we look for $u(0,x,\omega)$ satisfying the three following conditions:

C1a $u(0,x,\omega) \sim x.\omega$ *when* $|x| \to +\infty$ *on* Σ_0*.*

C1b $\square_{\mathbf{g}} u(0,x,\omega)$ *is in* $L^\infty(\Sigma)$*. In fact, the estimate* $\square_{\mathbf{g}} u \in L^\infty(\mathcal{M})$ *is obtained in* **[13]** *using a transport equation (the Raychadhouri equation) so that one needs the corresponding estimate on* Σ_0 *(i.e., at* $t=0$*).*

C1c $u(0,x,\omega)$ *has enough regularity in* x *and* ω *to achieve step* **C2**, *i.e., to control the parametrix at* $t=0$ *given by* (1.4).

Such a choice turns out to be a difficult task. This is due to the fact that the initial data set (Σ_0, g, k) has very little regularity. In fact, to be consistent with the bounded L^2 curvature conjecture, one should only assume that the curvature tensor R of g and ∇k are in $L^2(\Sigma)$. Together with **C1b**, this drastically limits the regularity in x of $u(0,x,\omega)$. Although (Σ_0, g, k) is independent of ω (which only intervenes in **C1a** to prescribe the asymptotic behavior of $u(0,x,\omega)$), the function $u(0,x,\omega)$ has also very limited regularity in ω. We will thus have to make a very careful choice of $u(0,x,\omega)$ to be able to satisfy the three conditions **C1a C1b C1c** at the same time.

Let us note that the typical choice $u(0,x,\omega) = x\cdot\omega$ in a given coordinate system would not work for us, since we don't have enough control on the regularity of a given coordinate system within our framework. Instead, we need to find a geometric definition of $u(0,x,\omega)$. A natural choice would be

$$\square_{\mathbf{g}} u = 0 \text{ on } \Sigma_0,$$

which by a simple computation turns out to be the following simple variant of the minimal surface equation [8]

$$\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$

Unfortunately, this choice does not allow us to have enough control of the derivatives of u in the normal direction to the level surfaces of u. This forces us to look for an alternate equation for u:

$$\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 1 - \frac{1}{|\nabla u|} + k\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right) \text{ on } \Sigma_0.$$

In the time symmetric case, i.e., $k = 0$, this choice simply means that the mean curvature of the level surfaces of u is equal to 1 minus the lapse of u. In this context, this construction has not appeared in the literature. It is closest in spirit to the mean curvature flow equation, as it can be recast in an alternative form

$$\frac{dx}{du} = (1 + H + k_{NN})N,$$

where N is the mean curvature of the level surface of u. Its main advantage is that it turns out to be parabolic in the normal direction to the level surfaces of u. Consequently, this construction retains the regularity of the leaves of the foliation of the minimal surface choice, but also additionally gives stronger control in the normal direction to the leaves.

The rest of the paper is as follows. In Chapter 2, we motivate our choice for $u(0, x, \omega)$ and we state the main results. In Chapter 3, we assume the existence of $u(0, x, \omega)$ and prove calculus inequalities with respect to the foliation generated by $u(0, x, \omega)$ on Σ_0, which will be needed in the sequel. In Chapter 4, we investigate the regularity of $u(0, x, \omega)$ with respect to x. In Chapter 5, we recall the properties of the geometric Littlewood-Paley decompositions established in [**15**], and we derive useful commutator estimates, product estimates, as well as parabolic estimates. In Chapter 6, we derive additional regularity for $u(0, x, \omega)$ with respect to x. In Chapter 7, we investigate the regularity of $u(0, x, \omega)$ with respect to ω. In Chapter 7, we construct a global coordinate system on the leaves of the foliation generated by $u(0, x, \omega)$ on Σ_0. Finally, we derive additional estimates for $u(0, x, \omega)$ in Chapter 8.

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8. In the time symmetric case $k = 0$, this is exactly the minimal surface equation.

CHAPTER 2

MAIN RESULTS

From now on, there will be no further reference to Σ_t for $t > 0$. Since there is no confusion, we will denote Σ_0 simply by Σ in the rest of the paper.

2.1. Modification of R and k near the asymptotic end

Recall from Theorem 1.1 that our assumptions on the initial data set (Σ, g, k) are the following

$$\|R\|_{L^2(\Sigma)} + \|k\|_{L^2(\Sigma)} + \|\nabla k\|_{L^2(\Sigma)} \leq \varepsilon, \tag{2.1}$$

where $\varepsilon > 0$ is small enough. Now, as a byproduct of the reduction to these small initial data outlined in Remark 1.4 and performed in Section 2.3 of **[17]**, we may also assume the existence of a global coordinate system on (Σ, g, k) relative to which we have

$$\frac{1}{2}|\xi|^2 \leq g_{ij}\xi^i\xi^j \leq 2|\xi|^2 \tag{2.2}$$

and (Σ, g, k) is smooth in $|x| \geq 1$.

In order to construct $u(0, x, \omega)$ satisfying the asymptotic behavior **C1a**, we need to modify (Σ, g, k) outside of $|x| \leq 1$. We can glue it to $(\mathbb{R}^3, \delta, 0)$ so that the new initial data set is still smooth outside of $|x| \leq 1$, satisfies (2.1), and coincides with $(\mathbb{R}^3, \delta, 0)$ outside of a slightly larger neighborhood. We still denote this initial data set (Σ, g, k). Of course, (Σ, g, k) does not satisfies the constraint equations in the annulus where the gluing takes place. However, for the construction of $u(0, x, \omega)$, we only require (Σ, g, k) to satisfy the constraint equations in $|x| \leq 1$. Outside of $|x| \leq 1$, (Σ, g, k) is smooth, so things are much easier.

Finally, in order to be consistent with the statement of Theorem 1.1, we consider a maximal foliation, i.e.,

$$\mathrm{Tr} k = 0.$$

2.2. Geometry of the foliations generated by u on $\mathcal{M}$ and by $u|_{\Sigma}$ on Σ

Let u a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u = 0$ on $\mathcal{M}$. Let $L = -\mathbf{g}^{\alpha\beta}\partial_\alpha u\,\partial_\beta$ be the corresponding null generator vector field and s its affine parameter, i.e., $L(s) = 1$. Let us introduce the level hypersurfaces of u

$$\mathcal{H}_{u_0} = \{(t,x) \text{ in } \mathcal{M} \text{ such that } u = u_0\},$$

which generate a foliation on $\mathcal{M}$. The level surfaces $P_{s,u}$ of s generate the geodesic foliation on $\mathcal{H}_u$.

The geometry of $\mathcal{H}_u$ depends in particular of the null second fundamental form

$$\chi(X,Y) = \mathbf{g}(\mathbf{D}_X L, Y), \tag{2.3}$$

with X, Y arbitrary vector fields tangent to the s-foliation $P_{s,u}$ and where $\mathbf{D}$ is the covariant differentiation with respect to $\mathbf{g}$. We denote by $\operatorname{tr}\chi$ the trace of χ, i.e., $\operatorname{tr}\chi = \delta^{AB}\chi_{AB}$ where χ_{AB} are the components of χ relative to an orthonormal frame $(e_A)_{A=1,2}$ on the leaves of the s-foliation. An easy computation yields:

$$\square_{\mathbf{g}} u = \operatorname{tr}\chi, \tag{2.4}$$

so that ones needs to prove enough regularity for $\operatorname{tr}\chi$ to control the error term (1.5) of the parametrix (1.3). $\operatorname{tr}\chi$ satisfies the well known Raychadhouri equation

$$\frac{d}{ds}\operatorname{tr}\chi + \frac{1}{2}(\operatorname{tr}\chi)^2 = -|\widehat{\chi}|^2, \tag{2.5}$$

with $\widehat{\chi}_{AB} = \chi_{AB} - 1/2\operatorname{tr}\chi\delta_{AB}$ the traceless part of χ. This transport equation is used in **[13]** to prove the crucial estimate $\operatorname{tr}\chi \in L^\infty(\mathcal{M})$ provided that $\operatorname{tr}\chi$ is in $L^\infty(\Sigma)$ at $t = 0$.

Let us now recall the link between $u|_{\Sigma}$ and $\operatorname{tr}\chi|_{\Sigma}$. We define the lapse $a = |\nabla u|^{-1}$, and the unit vector N such that $\nabla u = a^{-1}N$. We also define the level surfaces

$$P_{u_0} = \{x \text{ in } \Sigma \text{ such that } u = u_0\},$$

so that N is the normal to P_u in Σ. The second fundamental form θ of P_u is defined by

$$\theta(X,Y) = g(\nabla_X N, Y), \tag{2.6}$$

with X, Y arbitrary vector fields tangent to the u-foliation P_u on Σ and where ∇ denotes the covariant differentiation with respect to g. We extend θ as a tensor on Σ by setting

$$\theta(N,.) = \theta(.,N) = 0. \tag{2.7}$$

We denote by $\operatorname{tr}\theta$ the trace of θ, i.e., $\operatorname{tr}\theta = \delta^{AB}\theta_{AB}$ where θ_{AB} are the components of θ relative to an orthonormal frame $(e_A)_{A=1,2}$ on P_u. We then have the following equality on Σ:

$$\operatorname{tr}\chi = \operatorname{tr}\theta + \operatorname{tr}k. \tag{2.8}$$

Now, $\mathrm{Tr}k = \mathrm{tr}\, k + k_{NN}$. Recall from Section 2.1 that we impose $\mathrm{Tr}k = 0$ which corresponds to a maximal foliation. Thus, we obtain the following relation between u and $\mathrm{tr}\,\chi$ on Σ:

$$\mathrm{tr}\,\chi = \mathrm{tr}\,\theta - k_{NN} \quad \text{on} \quad \Sigma. \tag{2.9}$$

Finally, using (2.4) and (2.9), we may reformulate **C1b** as:

$$\mathrm{tr}\,\theta - k_{NN} \in L^\infty(\Sigma). \tag{2.10}$$

2.3. Structure equations of the foliation generated by a function u on Σ

We recall the structure equations of the foliation generated by a scalar function u on Σ (see for example [**4**]).

Proposition 2.1. — *The orthonormal frame frame $N, e_A, A = 1, 2$ of Σ satisfies the following system:*

$$\begin{cases} \nabla_N e_A = \nabla\!\!\!/_N e_A + a^{-1}(\nabla\!\!\!/_A a)N, \\ \nabla_A N = \theta_{AB} e_B, \\ \nabla_B e_A = \nabla\!\!\!/_B e_A - \theta_{AB} N, \\ \nabla_N N = -a^{-1}\nabla\!\!\!/ a. \end{cases} \tag{2.11}$$

Also, the lapse a and the second fundamental form θ satisfy the following system:

$$\begin{cases} a^{-1}\Delta\!\!\!/(a) = -\nabla_N \mathrm{tr}\,\theta - |\theta|^2 + R_{NN}, \\ \nabla\!\!\!/^B \widehat{\theta}_{AB} = \frac{1}{2}\nabla\!\!\!/_A \mathrm{tr}\,\theta + R_{NA}, \\ a^{-1}\nabla\!\!\!/_A\nabla\!\!\!/_B a + \nabla_N \theta_{AB} + 2\theta_A^C\theta_{CB} - \mathrm{tr}\,\theta\theta_{AB} + K\gamma_{AB} = R_{AB}, \end{cases} \tag{2.12}$$

where $\widehat{\theta}_{AB} = \theta_{AB} - 1/2\mathrm{tr}\,\theta\delta_{AB}$ is the traceless part of θ, K is the Gauss curvature of P_u, γ is the metric on P_u induced by g, and $\nabla\!\!\!/$ is the intrinsic covariant derivative on P_u. Finally, we have:

$$2K - \mathrm{tr}\,\theta^2 + |\theta|^2 = R - 2R_{NN}. \tag{2.13}$$

Proof. — We start with (2.11). Note that the second equality in (2.11) follows from the definition of the second fundamental form θ. Also, the first and the third equality follow from the second and the fourth equality and the fact that the frame is orthonormal. Thus, it remains to prove the fourth equality in (2.11).

Since $\nabla u = a^{-1}N$, we have $N(u) = a^{-1}$. Thus, using $e_A(u) = 0$ using the fact that the frame is orthonormal, we obtain:

$$\begin{aligned}
\nabla_A(a^{-1}) &= \nabla_A(N(u)) \\
&= [e_A, N](u) \\
&= \nabla_A N(u) - \nabla_N e_A(u) \\
&= a^{-1} g(N, \nabla_A N - \nabla_N e_A) \\
&= a^{-1} g(\nabla_N N, e_A),
\end{aligned}$$

which concludes the proof of (2.11).

We now turn to the proof of (2.12) starting with the first equation. Using the definition of the curvature tensor R, we have:

$$\begin{aligned}
g([\nabla_A, \nabla_N]N, e_B) &= g(\slashed{\nabla}_A \nabla_N N, e_B) - g(\nabla_N \slashed{\nabla}_A N, e_B) + g(\nabla_{\nabla_N e_A} N, e_B) \\
&= g(\nabla_A \nabla_N N, e_B) - g(\nabla_N \nabla_A N, e_B) + g(\nabla_{\nabla_N e_A} N, e_B) \\
&= -R_{ANBN} + g(\nabla_{\nabla_A N - \nabla_N e_A} N, e_B) + g(\nabla_{\nabla_N e_A} N, e_B) \\
&= -R_{ANBN} + \theta_{AC}\theta_{CB},
\end{aligned}$$

where we used (2.11) in the last inequality. Taking the trace yields:

$$[\mathrm{div}, \nabla_N]N = -R_{NN} + |\theta|^2,$$

which together with (2.11) implies:

$$\mathrm{div}(\nabla_N N) = \nabla_N(\mathrm{div}(N)) + [\mathrm{div}, \nabla_N]N = \nabla_N \mathrm{tr}\,\theta - R_{NN} + |\theta|^2. \tag{2.14}$$

Using (2.11), we have:

$$\mathrm{div}(\nabla_N N) = -\mathrm{div}(a^{-1}\slashed{\nabla} a) = -\slashed{\mathrm{div}}(a^{-1}\slashed{\nabla} a) - |a^{-1}\slashed{\nabla} a|^2 = -a^{-1}\slashed{\Delta}(a),$$

which together with (2.14) proves the first equality of (2.12).

Next, we turn to the second equality of (2.12). Using the definition of the curvature tensor R, we have:

$$\begin{aligned}
\slashed{\nabla}_A \theta_{BC} - \slashed{\nabla}_B \theta_{AC} &= e_A(g(\nabla_B N, e_C)) - \theta(\slashed{\nabla}_A e_B, e_C) - \theta(e_B, \slashed{\nabla}_A e_C) \\
&\quad - e_B(g(\nabla_A N, e_C)) + \theta(\slashed{\nabla}_B e_A, e_C) + \theta(e_A, \slashed{\nabla}_B e_C) \\
&= g((\nabla_A \nabla_B - \nabla_B \nabla_A)N, e_C) + g(\nabla_B N, \nabla_A e_C - \slashed{\nabla}_A e_C) \\
&\quad - \theta(\slashed{\nabla}_A e_B, e_C) - g(\nabla_A N, \nabla_B e_C - \slashed{\nabla}_B e_C) + \theta(\slashed{\nabla}_B e_A, e_C) \\
&= R_{ABNC} + g(\nabla_{\nabla_A e_B - \slashed{\nabla}_A e_B - \nabla_B e_A + \slashed{\nabla}_B e_A} N, e_C) \\
&= R_{ABNC},
\end{aligned}$$

where we used (2.11), the fact that θ is symmetric, and the fact that the frame is orthonormal. Taking the trace yields:

$$\slashed{\mathrm{div}}(\theta)_A = \slashed{\nabla}_A \mathrm{tr}\,\theta + R_{ABNB} = \slashed{\nabla}_A \mathrm{tr}\,\theta + R_{AN},$$

which together with the definition of $\widehat{\theta}$ proves the second equality of (2.12).

We now turn to the last equality of (2.12). Using the definition of the curvature tensor R and the property (2.7) of θ, we have:

$$\begin{aligned}\nabla_N\theta_{AB} &= \nabla_N(g(\nabla_A N, e_B)) - \theta(\nabla_N e_A, e_B) - \theta(e_A, \nabla_N e_B)\\ &= g(\nabla_N\nabla_A N, e_B) - \theta(\nabla\!\!\!/_N e_A, e_B)\\ &= g(\nabla_A\nabla_N N, e_B) + R_{ANBN} + g(\nabla_{\nabla_N e_A - \nabla_A N} N, e_B) - \theta(\nabla\!\!\!/_N e_A, e_B)\\ &= g(\nabla_A\nabla_N N, e_B) + R_{ANBN} + g(\nabla_{\nabla_N e_A - \nabla\!\!\!/_N e_A - \nabla_A N} N, e_B),\end{aligned}$$

which together with (2.11) yields:

$$\nabla_N\theta_{AB} = -a^{-1}\nabla\!\!\!/_A\nabla\!\!\!/_B a - \theta_{AC}\theta_{CB} + R_{ANBN}. \tag{2.15}$$

Now, the Gauss equation of the foliation generated by u on Σ reads:

$$R_{AB} = R_{ANBN} + K\gamma_{AB} + \theta_{AC}\theta_{CB} - \operatorname{tr}\theta\theta_{AB}, \tag{2.16}$$

which together with (2.15) proves the last equation of (2.12).

Finally, we turn to (2.13). This follows from taking the trace of the Gauss Equation (2.16). Note that it also follows form taking the trace of the last equality of (2.12) and using the first equality. □

2.4. Commutation formulas

Let Π the projection operator from the tangent space of Σ to the tangent space P_u, which is defined in an arbitrary orthonormal frame on Σ by

$$\Pi^i_j = \delta^i_j - N^i N_j.$$

Then, for any P_u-tangent tensor F, we define $\nabla\!\!\!/_N F$ as the projection of $\nabla_N F$ on P_u:

$$\nabla\!\!\!/_N U_{i_1\cdots i_n} = \Pi^{i_1}_{j_1}\cdots\Pi^{i_n}_{j_n}\nabla_N U_{j_1\cdots j_n}.$$

We have the following useful commutation formulas between $\nabla\!\!\!/$ and $\nabla\!\!\!/_N$ (see **[4]** page 64).

Lemma 2.2. — *For any P_u-tangent tensor F on Σ, we have schematically:*

$$[\nabla\!\!\!/_N, \nabla\!\!\!/]F = a^{-1}\nabla\!\!\!/ a\cdot\nabla_N F - \theta\cdot\nabla\!\!\!/ F + R_{N.}\cdot F + \theta\cdot a^{-1}\nabla\!\!\!/ a\cdot F. \tag{2.17}$$

In particular, we obtain for any scalar f on Σ:

$$[\nabla\!\!\!/_N, \nabla\!\!\!/]f = a^{-1}\nabla\!\!\!/ a\,\nabla_N f - \theta\cdot\nabla\!\!\!/ f \tag{2.18}$$

and:

(2.19)

$$\begin{aligned}[\nabla_N, \Delta\!\!\!/]f = &-\operatorname{tr}\theta\Delta\!\!\!/ f - 2\widehat{\theta}\cdot\nabla\!\!\!/^2 f + 2a^{-1}\nabla\!\!\!/ a\cdot\nabla\!\!\!/\nabla_N f + a^{-1}\Delta\!\!\!/ a\nabla_N f - 2R_{N.}\cdot\nabla\!\!\!/ f\\ &- \nabla\!\!\!/\operatorname{tr}\theta\cdot\nabla\!\!\!/ f - 2\widehat{\theta}\cdot a^{-1}\nabla\!\!\!/ a\cdot\nabla\!\!\!/ f.\end{aligned}$$

We will use some variants of the commutator Formulas (2.17), (2.18) and (2.19). In particular, for any scalar function f on Σ, (2.19) yields:

$$a[\nabla_N, a^{-1}\not\Delta]f = -(\operatorname{tr}\theta + a^{-1}\nabla_N a)\not\Delta f - 2\widehat{\theta}\cdot\not\nabla^2 f + 2a^{-1}\not\nabla a\cdot\not\nabla\nabla_N f + a^{-1}\not\Delta(a)\nabla_N f$$

$$- 2R_{N.}\cdot\not\nabla f - \not\nabla\operatorname{tr}\theta\cdot\not\nabla f - 2\widehat{\theta}\cdot a^{-1}\not\nabla a\cdot\not\nabla f. \tag{2.20}$$

Also, for some applications we have in mind, we would like to get rid of the terms containing $\not\nabla_N$ in the right-hand side of (2.17), (2.18) and (2.19). This is achieved by considering the commutators with $\not\nabla_{aN}$ instead of $\not\nabla_N$. (2.17) implies for any P_u-tangent tensor F on Σ, schematically:

$$[\not\nabla_{aN}, \not\nabla]F = -a\theta\cdot\not\nabla F + aR_{N.}\cdot F + \theta\cdot\not\nabla(a)\cdot F. \tag{2.21}$$

Using twice the commutator Formula (2.21), we obtain, schematically:

$$[\not\nabla_{aN}, \not\Delta]F = \not\nabla\cdot(-\theta\cdot\not\nabla F + R_{N.}\cdot F + \theta\cdot\not\nabla(a)F) - \not\nabla\theta\cdot\not\nabla F + R_{N.}\cdot\not\nabla F + \theta\cdot\not\nabla(a)\cdot\not\nabla F. \tag{2.22}$$

In view of (2.21), we also have for any scalar function f on Σ:

$$[\nabla_{aN}, \not\Delta]f = -a\operatorname{tr}\theta\not\Delta f - 2a\widehat{\theta}\cdot\not\nabla^2 f + (-2aR_{N.} - a\not\nabla\operatorname{tr}\theta + 2\widehat{\theta}\cdot\not\nabla a)\cdot\not\nabla f. \tag{2.23}$$

Finally, we conclude this section with the following commutator formula on P_u. For any scalar function f on P_u, we have:

$$[\not\nabla, \not\Delta]f = K\not\nabla f. \tag{2.24}$$

2.5. The choice of $u(0, x, \omega)$

In view of (2.10), we may reformulate **C1a C1b C1c**. We look for $u(0, x, \omega)$ satisfying the three following conditions:

C1a $u(0, x, \omega) \sim x.\omega$ *when* $|x| \to +\infty$ *on* Σ

C1b $\operatorname{tr}\theta - k_{NN} \in L^\infty(\Sigma)$

C1c $u(0, x, \omega)$ *has as enough regularity in* x *and* ω *to achieve step* **C2**, *i.e., to control the parametrix at* $t = 0$ *given by* (1.4),

where the initial data set (Σ, g, k) satisfies:

$$\begin{cases} \nabla^j k_{ij} = 0, \\ R = |k|^2, \\ \operatorname{Tr}k = 0 \end{cases} \tag{2.25}$$

and where R and ∇k are in $L^2(\Sigma)$ and satisfy the smallness assumption (2.1).

In order to motivate our choice of $u(0, x, \omega)$, we investigate the regularity of the lapse a, which by (2.12) satisfies the following equation:

$$a^{-1}\not\Delta(a) = -\nabla_N\operatorname{tr}\theta - |\theta|^2 - R_{NN}. \tag{2.26}$$

Since R is in $L^2(\Sigma)$, (2.26) implies that a has at most two derivatives in $L^2(\Sigma)$. Thus, $u(0, x, \omega)$ has at most three derivatives with respect to x in $L^2(\Sigma)$. This is not enough to satisfy **C1c** (i.e., to obtain the boundedness of the parametrix at $t = 0$ in L^2). In fact, the classical T^*T argument (see for example [**21**]) relies on integrations by parts in x and would require at least one more derivative since Σ has dimension 3.

Alternatively, we could try to use the TT^* argument which relies on integrations by parts in ω. Indeed, R being independent of ω, one would expect the regularity of $u(0,x,\omega)$ with respect to ω to be better. Differentiating (2.26) with respect to ω, we obtain:

$$a^{-1}\not{\Delta}(\partial_\omega a) = 2\not{\nabla}\nabla_N a + \cdots , \tag{2.27}$$

where the term on the right-hand side comes from the commutator $[\partial_\omega, \not{\Delta}]$ (see Chapter 7). Thus, obtaining an estimate for $\partial_\omega a$ from (2.27) requires to control $\nabla_N a$. Unfortunately, (2.26) seems to give control of tangential derivatives of a only. This is where the specific choice of $u(0,x,\omega)$ comes into play.

Having in mind the equation of minimal surfaces (i.e., $\operatorname{tr}\theta = 0$), condition **C1b** suggest the choice $\operatorname{tr}\theta - k_{NN} = 0$. Unfortunately, this equation together with (2.26) does not provide any control of $\nabla_N a$. We might propose as a second guess natural guess to take instead $\operatorname{tr}\theta - k_{NN} = \nabla_N a$. Plugging in (2.26) yields an elliptic equation for a: $\nabla_N^2 a + a^{-1}\not{\Delta}(a) = -|\theta|^2 - \nabla_N(k_{NN}) - R_{NN}$. This allows us to control $\nabla_N^2 a$ in $L^2(\Sigma)$. However, $\nabla_N a$ is at most in $H^1(\Sigma)$ which does not embed in $L^\infty(\Sigma)$—since Σ has dimension 3—so that condition **C1b** is not satisfied. To sum up, the first guess $\operatorname{tr}\theta - k_{NN} = 0$ satisfies **C1b**, but not **C1c**, whereas the second guess $\operatorname{tr}\theta - k_{NN} = \nabla_N a$ might satisfy **C1c**, but does not satisfy **C1b**.

The correct choice is the intermediate one:

$$\operatorname{tr}\theta - k_{NN} = 1 - a. \tag{2.28}$$

We will see in Chapter 4 that $a - 1$ belongs to $L^\infty(\Sigma)$ so that **C1b** is satisfied. Also, plugging (2.28) in (2.26) yields:

$$\nabla_N a - a^{-1}\not{\Delta}(a) = |\theta|^2 + \nabla_N(k_{NN}) + R_{NN}. \tag{2.29}$$

This parabolic equation will allow us to control normal derivatives of a. In turn, we will control derivatives of a with respect to ω using (2.27). Ultimately, we will prove enough regularity with respect to both x and ω for **C1c** to be satisfied.

2.6. Main results

From now on, we will not make any further reference to the space-time $\mathscr{M}$. Instead, we will work only with the initial data set (Σ, g, k). Thus, since there can be no more confusion, we will denote $u(0,x,\omega)$ simply by $u(x,\omega)$. To u, we associate P_u, a, N, θ and K as in Section 2.2. For $1 \leq p, q \leq +\infty$, we define the spaces $L^p_u L^q(P_u)$ for tensors F on Σ using the norm:

$$\|F\|_{L^p_u L^q(P_u)} = \left(\int_u \|F\|^p_{L^q(P_u)} du\right)^{1/p}.$$

Remark 2.3. — In the rest of the paper, all inequalities hold for any $\omega \in \mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.

We first state a result of existence and regularity with respect to x for u.

Theorem 2.4. — *Let (Σ, g, k) chosen as in Section 2.1. There exists a scalar function u on $\Sigma \times \mathbb{S}^2$ satisfying assumption* **C1a** *and such that:*

$$\begin{aligned} \|a-1\|_{L^\infty_u L^2(P_u)} + \|\nabla a\|_{L^\infty_u L^2(P_u)} + \|a-1\|_{L^\infty(\Sigma)} + \|\nabla\!\!\!/ \nabla a\|_{L^2(\Sigma)} &\lesssim \varepsilon, \\ \|\mathrm{tr}\,\theta - k_{NN}\|_{L^\infty(\Sigma)} + \|\nabla \theta\|_{L^2(\Sigma)} + \|K\|_{L^2(\Sigma)} &\lesssim \varepsilon, \end{aligned} \tag{2.30}$$

where P_u, a, N, θ and K are associated to u as in Section 2.2.

Notice that condition **C1b** is implied by (2.30). In order to state our second result, we introduce fractional Sobolev spaces $H^b(P_u)$ on the surfaces P_u for any $b \in \mathbb{R}$ (see Section 5.6 for their definition). We have the following estimate for $\nabla_N^2 a$, and improved estimate for $\nabla_N a$.

Theorem 2.5. — *Let (Σ, g, k) chosen as in Section 2.1. Let u the scalar function on $\Sigma \times \mathbb{S}^2$ constructed in Theorem 2.4, and let P_u, a, N, θ and K be associated to u as in Section 2.2. We have:*

$$\|\nabla_N a\|_{L^\infty_u L^4(P_u)} + \|\nabla_N^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)} \lesssim \varepsilon. \tag{2.31}$$

The third theorem investigates the regularity of u with respect to ω:

Theorem 2.6. — *Let (Σ, g, k) chosen as in Section 2.1. Let u the scalar function on $\Sigma \times \mathbb{S}^2$ constructed in Theorem 2.4, and let P_u, a, N, θ and K be associated to u as in Section 2.2. We have:*

$$\begin{aligned} \|\partial_\omega a\|_{L^\infty(\Sigma)} + \|\nabla\!\!\!/ \partial_\omega a\|_{L^\infty_u L^2(P_u)} + \|\nabla\!\!\!/^2 \partial_\omega a\|_{L^2(\Sigma)} + \|\nabla_N \partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)} & \\ + \|\nabla_N \partial_\omega a\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)} + \|\nabla_N^2 \partial_\omega a\|_{L^2_u H^{-\frac{3}{2}}(P_u)} + \|\nabla \partial_\omega \theta\|_{L^2(\Sigma)} &\lesssim \varepsilon, \\ \|\partial_\omega N\|_{L^\infty(\Sigma)} &\lesssim 1, \end{aligned} \tag{2.32}$$

$$\begin{aligned} \|\partial_\omega^2 a\|_{L^2_u H^{\frac{3}{2}}(P_u)} + \|\partial_\omega^2 a\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N \partial_\omega^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)} + \|\nabla \partial_\omega^2 \theta\|_{L^2(\Sigma)} &\lesssim \varepsilon, \\ \|\partial_\omega^2 N\|_{L^\infty(\Sigma)} &\lesssim 1 \end{aligned} \tag{2.33}$$

and

$$\|\partial_\omega^3 u\|_{L^\infty_{\mathrm{loc}}(\Sigma)} \lesssim 1. \tag{2.34}$$

Remark 2.7. — In order to prove Theorem 2.4, Theorem 2.5 and Theorem 2.6, we will rely in a fundamental way on the choice (2.28) for u, and on the structure of the constraint equations in the maximal foliation (2.25).

2.7. Coordinate systems on P_u and Σ

In order to prove Theorem 2.4, Theorem 2.5 and Theorem 2.6, we will use embeddings on the level surfaces P_u of u. These embeddings are discussed in Chapter 3, and their proof will require in particular, the existence of a suitable coordinate system. The following proposition establishes the existence of a global coordinate system on P_u.

Proposition 2.8. — *Let $\omega \in \mathbb{S}^2$. Let $\Phi_u : P_u \to T_\omega \mathbb{S}^2$ defined by:*

$$\Phi_u(x) := \partial_\omega u(x, \omega), \tag{2.35}$$

where $T_\omega \mathbb{S}^2$ is the tangent space to $\mathbb{S}^2$ at ω. Then Φ_u is a global C^1 diffeomorphism from P_u to $T_\omega \mathbb{S}^2$.

The following proposition establishes the existence of a global coordinate system on Σ and provides the control of the determinant of the corresponding Jacobian. This will turn out to be useful to control the parametrix at $t = 0$ given by (1.4), which corresponds to step **C2** (see [**24**]).

Proposition 2.9. — *Let $\omega \in \mathbb{S}^2$. Let $\Phi : \Sigma \to \mathbb{R}^3$ defined by:*

$$\Phi(x) := u(x, \omega)\omega + \partial_\omega u(x, \omega) = u(x, \omega)\omega + \Phi_u(x), \tag{2.36}$$

where Φ_u has been defined in (2.35). Then Φ is a bijection, and the determinant of its Jacobian satisfies the following estimate:

$$\| |\det(\mathrm{Jac}\, \Phi)| - 1 \|_{L^\infty(\Sigma)} \lesssim \varepsilon. \tag{2.37}$$

2.8. Additional estimates

Below, we provide several additional estimates. These are consequences of Theorem 2.4, Theorem 2.5 and Theorem 2.6 that will be needed in steps **C2** and **C3** (see respectively [**24**] and [**25**]). We start with a first proposition.

Proposition 2.10. — *Let (Σ, g, k) chosen as in Section 2.1. Let u the scalar function on $\Sigma \times \mathbb{S}^2$ constructed in Theorem 2.4, and let N be associated to u as in Section 2.2. For all $x \in \Sigma$ and $\omega \in \mathbb{S}^2$, we have:*

$$|N(x, \omega) + N(x, -\omega)| \lesssim \varepsilon. \tag{2.38}$$

Also, we have:

$$||N(x, \omega) - N(x, \omega')| - |\omega - \omega'|| \lesssim |\omega - \omega'|(\varepsilon + |\omega - \omega'|), \forall x \in \Sigma, \omega, \omega' \in \mathbb{S}^2. \tag{2.39}$$

Finally, let $\nu \in \mathbb{S}^2$ and Φ_ν the map defined in (2.36). Then, we have:

$$\begin{aligned} u(x, \omega) - \Phi_\nu(x) \cdot \omega &= O(\varepsilon|\omega - \nu|^2), \\ \partial_\omega u(x, \omega) - \partial_\omega(\Phi_\nu(x) \cdot \omega) &= O(\varepsilon|\omega - \nu|), \\ \partial^2_\omega u(x, \omega) - \partial^2_\omega(\Phi_\nu(x) \cdot \omega) &= O(\varepsilon). \end{aligned} \tag{2.40}$$

We introduce the family of intrinsic Littlewood-Paley projections P_j which have been constructed in **[15]** using the heat flow on the surfaces P_u (see Section 5.1 for their main properties). This allows us to define the following Besov space $\mathcal{B}$ for tensors F on Σ:

$$\|F\|_{\mathcal{B}} = \sum_{j\geq 0} 2^j \|P_j F\|_{L^\infty_u L^2(P_u)} + \|P_{<0}F\|_{L^\infty_u L^2(P_u)}, \tag{2.41}$$

where $P_{<0} = \sum_{j<0} P_j$. In particular, one can show that a scalar function belonging to $\mathcal{B}$ also belongs to $L^\infty(\Sigma)$ (see **[15]**). Now, as recalled in the introduction, the reason for requiring condition **C1b** for u is that a crucial space-time quantity has been proved to be in L^∞ in **[13]** relying on a transport equation (the Raychadhouri equation) so that the corresponding quantity at $t = 0$ should be in $L^\infty(\Sigma)$. However, pseudodifferential operators of order 0 do not map L^∞ to L^∞ which forces the authors in **[13]** to actually prove a stronger estimate. In fact, they work with a Besov space which both embeds in L^∞ and is stable relative to operators of order 0. In turn, this forces us to obtain a stronger version of condition **C1b**. This is the aim of the following proposition:

Proposition 2.11. — *Let (Σ, g, k) chosen as in Section 2.1. Let u the scalar function on $\Sigma \times \mathbb{S}^2$ constructed in Theorem 2.4, and let P_u, N and θ be associated to u as in Section 2.2. We have:*

$$\|\operatorname{tr}\theta - k_{NN}\|_{\mathcal{B}} \lesssim \varepsilon. \tag{2.42}$$

Using the geometric Littlewood Paley projections P_j together with the estimates for $\nabla_N a$ in (2.30), and the estimate for $\nabla^2_N a$ in (2.31), we obtain the following proposition:

Proposition 2.12. — *Let (Σ, g, k) chosen as in Section 2.1. Let u the scalar function on $\Sigma \times \mathbb{S}^2$ constructed in Theorem 2.4, and let a and N be associated to u as in Section 2.2. For all $j \geq 0$, there are scalar functions a^j_1 and a^j_2 such that:*

$$\nabla_N a = a^j_1 + a^j_2 \text{ where } \|a^j_1\|_{L^2(\Sigma)} \lesssim 2^{-\frac{j}{2}}\varepsilon \text{ and } \|\nabla_N a^j_2\|_{L^2(\Sigma)} \lesssim 2^{\frac{j}{4}}\varepsilon. \tag{2.43}$$

Remark 2.13. — Recall from Section 2.5 that we do not have enough regularity in x to apply the T^*T method. Alternatively, we could try the TT^* method which relies on integration by parts in ω. But $\partial^3_\omega u \in L^\infty_{\text{loc}}(\Sigma)$ is also not enough and we would need at least one more derivative in ω (see also Remark 7.6). Nevertheless, we will prove in **[24]** that the regularity of u both with respect to x and ω obtained in this paper is enough to show that condition **C1c** is satisfied.

The rest of the paper is as follows. In Chapter 3, we prove various embeddings and estimates on P_u and Σ which are compatible with the regularity for u obtained in Theorem 2.4. In Chapter 4, we prove Theorem 2.4. In Chapter 5, we recall the properties of the geometric Littlewood-Paley projections P_j introduced in **[15]**. We then prove several commutator and product estimates, as well as estimates for some parabolic equations on Σ. In Chapter 6, we prove Theorem 2.5. In Chapter 7, we prove

Theorem 2.6. In Chapter 8, we prove Proposition 2.8 and Proposition 2.9. Finally, Proposition 2.10, Proposition 2.11 and Proposition 2.12 are proved in Chapter 9.

CHAPTER 3

CALCULUS INEQUALITIES

3.1. The Sobolev embedding on Σ

Recall from Section 2.1 that there is a global coordinate system on (Σ, g, k) relative to which we have

$$\frac{1}{2}|\xi|^2 \leq g_{ij}\xi^i\xi^j \leq 2|\xi|^2. \tag{3.1}$$

Lemma 3.1. — *Let f a real scalar function on Σ. Then:*

$$\|f\|_{L^{\frac{3}{2}}(\Sigma)} \lesssim \|\nabla f\|_{L^1(\Sigma)}. \tag{3.2}$$

Proof. — We may assume that f has compact support in Σ. In the global coordinate system $x = (x_1, x_2, x_3)$ on Σ satisfying (3.1), we have:

$$\begin{aligned}|f(x_1,x_2,x_3)|^{\frac{3}{2}} &= \left|\int_{-\infty}^{x_1}\partial_1 f(y,x_2,x_3)dy\int_{-\infty}^{x_2}\partial_2 f(x_1,y,x_3)dy\int_{-\infty}^{x_3}\partial_3 f(x_1,x_2,y)dy\right|^{\frac{1}{2}}\\ &\lesssim \left(\int_{\mathbb{R}}|\partial_1 f(y,x_2,x_3)|dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|\partial_2 f(x_1,y,x_3)|dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|\partial_3 f(x_1,x_2,y)|dy\right)^{\frac{1}{2}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^3}|f(x_1,x_2,x_3)|^{\frac{3}{2}}dx_1dx_2dx_3\\ &\quad\lesssim \left(\int_{\mathbb{R}^3}|\partial_1 f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}|\partial_2 f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{1}{2}}\\ &\quad\quad\left(\int_{\mathbb{R}^3}|\partial_3 f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{1}{2}} \lesssim \left(\int_{\mathbb{R}^3}|\nabla f(x_1,x_2,x_3)|dx_1dx_2dx_3\right)^{\frac{3}{2}}.\end{aligned}$$

Now in view of the coordinates system property (3.1), we deduce from the previous estimate:

$$\left(\int_{\mathbb{R}^3}|f(x)|^{\frac{3}{2}}\sqrt{|g_t|}dx_1dx_2dx_3\right)^{\frac{2}{3}} \lesssim \int_{\mathbb{R}^3}|\nabla f(x)|\sqrt{|g_t|}dx_1dx_2dx_3$$

as desired. □

As a corollary of the estimate (3.2), we may derive the following Sobolev embeddings.

Corollary 3.2. — *Given an arbitrary tensorfield F on Σ, we have*

$$\|F\|_{L^6(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)}. \tag{3.3}$$

Proof. — We use (3.2) with $f = |F|^4$:

$$\|F\|^4_{L^6(\Sigma)} = \||F|^4\|_{L^{\frac{3}{2}}(\Sigma)} \lesssim \||F|^2 F\nabla F\|_{L^1(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)}\|F\|^3_{L^6(\Sigma)},$$

which yields (3.3). □

3.2. Embeddings compatible with the foliation generated by u on Σ

We assume the existence of a real function u on Σ. We define the lapse $a = |\nabla u|^{-1}$, and the unit vector N such that $\nabla u = a^{-1}N$. We also define the level surfaces $P_u = \{x \,/\, u(x) = u\}$ so that N is the normal to P_u. In this section we establish some basic calculus inequalities with respect to the foliation generated by u on Σ in the strip S defined by:

$$S = \{x \text{ such that } -2 < u(x) < 2\}.$$

These calculus inequalities will be used in all subsequent sections of the present paper. We will use the following assumptions, which are consistent with our assumption on R and our choice of bootstrap assumptions (see (4.9), (4.10), (4.11), (4.12)):

(3.4)
$$\begin{aligned}\|R\|_{L^2(S)} + \|a-1\|_{L^\infty(S)} + \|\nabla\!\!\!/\nabla a\|_{L^2(S)} + \|\nabla a\|_{L^2(S)} + \|\operatorname{tr}\theta\|_{L^6(S)}\\ + \|\nabla\theta\|_{L^2(S)} + \|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|K\|_{L^2(S)} \leq \delta\end{aligned}$$

for some small enough constant $\delta > 0$.

Let μ_u denote the area element of P_u. Then, for all integrable function f on S, the coarea formula implies:

$$\int_S f d\Sigma = \int_{-2}^{2}\int_{P_u} f a d\mu_u du. \tag{3.5}$$

It is also well-known that for a scalar function f:

$$\frac{d}{du}\left(\int_{P_u} f d\mu_u\right) = \int_{P_u}\left(\frac{df}{du} + \operatorname{tr}\theta f\right) d\mu_u. \tag{3.6}$$

For $1 \leq p, q \leq +\infty$, we define the spaces $L^p_{[-2,2]}L^q(P_u)$ using the norm

$$\|F\|_{L^p_{[-2,2]}L^q(P_u)} = \left(\int_{-2}^{2} \|F\|^p_{L^q(P_u)} du\right)^{1/p}.$$

In particular, in view of the assumptions (3.4) for a, $L^p_{[-2,2]}L^p(P_u)$ coincides with $L^p(S)$ for all $1 \leq p \leq +\infty$. We denote by γ the metric induced by g on P_u, and by $\nabla\!\!\!/$

the induced covariant derivative. We define the space $H^1(S)$ for tensors F on S using the norm

$$\|F\|_{H^1(S)} = (\|F\|^2_{L^2(S)} + \|\nabla F\|^2_{L^2(S)})^{1/2}.$$

A coordinate chart $U \subset P_u$ with coordinates x^1, x^2 is admissible if, relative to these coordinates, there exists a constant $c > 0$ such that,

$$c^{-1}|\xi|^2 \leq \gamma_{AB}(p)\xi^A\xi^B \leq c|\xi|^2, \qquad \text{uniformly for all } \ p \in U. \tag{3.7}$$

We assume that P_u can be covered by a finite number of admissible coordinate charts, i.e., charts satisfying the conditions (3.7). Furthermore, we assume that the constant c in (3.7) and the number of charts is independent of u.

Remark 3.3. — The existence of a covering of P_u by coordinate charts satisfying (3.7) with a constant $c > 0$ and the number of charts independent of u follows from Proposition 2.8.

Under these assumptions, the following calculus inequality has been proved in **[15]**:

Proposition 3.4. — *Let f be a real scalar function. Then,*

$$\|f\|_{L^2(P_u)} \lesssim \|\nabla\!\!\!/ f\|_{L^1(P_u)} + \|f\|_{L^1(P_u)}. \tag{3.8}$$

As a corollary of the estimate (3.8), the following Gagliardo-Nirenberg inequality is derived in **[15]**:

Corollary 3.5. — *Given an arbitrary tensorfield F on P_u and any $2 \leq p < \infty$, we have:*

$$\|F\|_{L^p(P_u)} \lesssim \|\nabla\!\!\!/ F\|^{1-\frac{2}{p}}_{L^2(P_u)}\|F\|^{\frac{2}{p}}_{L^2(P_u)} + \|F\|_{L^2(P_u)}. \tag{3.9}$$

As a corollary to (3.8) it is also classical to derive the following inequality (for a proof, see for example **[8]** page 157):

Corollary 3.6. — *For any tensorfield F on P_u and any $p > 2$,*

$$\|F\|_{L^\infty(P_u)} \lesssim \|\nabla\!\!\!/ F\|_{L^p(P_u)} + \|F\|_{L^p(P_u)}. \tag{3.10}$$

Below, we state and prove several embeddings with respect to the foliation generated by u on Σ. The difficulty is to obtain these estimates while using only assumptions that are compatible with the regularity for u obtained in Theorem 2.4.

Proposition 3.7. — *Let F be a tensorfield on S such that $F \in H^1(S)$. Assume also (3.4). Then F belongs to $L^\infty_{[-2,2]}L^4(P_u)$.*

Proof. — We have

(3.11)

$$\begin{aligned}\|F(u,.)\|^4_{L^4(P_u)} &= \|F(-2,.)\|^4_{L^4(P_{-2})}\\ &\quad + 4\int_{-2}^{u}\int_{P_{u'}} \nabla_N F(u',x')\cdot F(u',x')|F(u',x')|^2 du' d\mu_{u'}\\ &\quad + \int_{-2}^{u}\int_{P_{u'}} \operatorname{tr}\theta|F(u',x')|^4 du' d\mu_{u'}\\ &\lesssim \|F(-2,.)\|^4_{L^4(P_{-2})} + \|\nabla_N F\|_{L^2(S)}\|F\|^3_{L^6(S)}\\ &\quad + \|\operatorname{tr}\theta\|_{L^6(S)}\|F\|^4_{L^{24/5}(S)}\\ &\lesssim \|F(-2,.)\|^4_{L^4(P_{-2})} + \|\nabla_N F\|_{L^2(S)}\|F\|^3_{L^6(S)} + \|F\|^4_{L^{12}_{[-2,2]}L^{24/5}(P_u)},\end{aligned}$$

where we used the assumption (3.4) for $\operatorname{tr}\theta$ in the last inequality. Replacing F with $\varphi(u)F$ where φ is a smooth function such that $\varphi(-2)=1$ and $\varphi(2)=0$, and proceeding as in (3.11), we obtain:

$$\|F(-2,.)\|^4_{L^4(P_{-2})} \lesssim \|\nabla_N F\|_{L^2(S)}\|F\|^3_{L^6(S)} + \|F\|^4_{L^{12}_{[-2,2]}L^{24/5}(P_u)} + \|F\|^4_{L^4(S)}, \tag{3.12}$$

which together with (3.11) yields:

(3.13)

$$\|F(u,.)\|^4_{L^4(P_u)} \lesssim \|\nabla_N F\|_{L^2(S)}\|F\|^3_{L^6(S)} + \|F\|^4_{L^{12}_{[-2,2]}L^{24/5}(P_u)} + \|F\|_{L^2(S)}\|F\|^3_{L^6(S)}.$$

This concludes the proof by taking the supremum in u on the left-hand side, and by using the Sobolev embedding (3.3) and the following estimate:

$$\|F\|_{L^{12}_{[-2,2]}L^{24/5}(P_u)} \lesssim \|F\|^{\frac{1}{2}}_{L^6(S)}\|F\|^{\frac{1}{2}}_{L^\infty_{[-2,2]}L^4(P_u)}.$$

□

In Proposition 3.7, we can get rid of the assumption that $F\in L^2(S)$. This is done in the following corollary.

Corollary 3.8. — *Let F be a tensorfield on S such that $\nabla F \in L^2(S)$ and $F(-2,.)\in L^4(P_{-2})$. Assume also (3.4). Then F belongs to $L^\infty_{[-2,2]}L^4(P_u)$ and $L^6(S)$. Moreover, if $F(-2,.)\in L^2(P_{-2})$, then F also belongs to $L^\infty_{[-2,2]}L^2(P_u)$ and $H^1(S)$.*

Proof. — The proof of Proposition 3.7 yields:

$$\begin{aligned}\|F\|_{L^\infty_{[-2,2]}L^4(P_u)} &\lesssim \|F(-2,.)\|_{L^4(P_{-2})} + \|\nabla_N F\|^{\frac{1}{4}}_{L^2(S)}\|F\|^{\frac{3}{4}}_{L^6(S)}\\ &\quad + \|\operatorname{tr}\theta\|_{L^6(S)}\big(\|F\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|F\|_{L^6(S)}\big),\end{aligned}$$

which together with the Sobolev embedding (3.3), and the assumption (3.4) for $\operatorname{tr}\theta$, yields for δ small enough:

$$\|F\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \|F(-2,.)\|_{L^4(P_{-2})} + \|\nabla F\|_{L^2(S)}. \tag{3.14}$$

This proves the first statement of the corollary.

Now, we also assume that $F(-2,.) \in L^2(P_{-2})$.

$$
\begin{aligned}
\|F(u,.)\|^2_{L^2(P_u)} &= \|F(-2,.)\|^2_{L^2(P_{-2})} \\
&\quad + 2\int_{-2}^{u}\int_{P_{u'}} \nabla_N F(u',x')\cdot F(u',x')du'd\mu_{u'} \\
&\quad + \int_{-2}^{u}\int_{P_{u'}} \operatorname{tr}\theta |F(u',x')|^2 du'd\mu_{u'} \\
&\lesssim \|F(-2,.)\|^2_{L^2(P_{-2})} + \|\nabla_N F\|_{L^2(S)}\|F\|_{L^2(S)} \\
&\quad + \|\operatorname{tr}\theta\|_{L^6(S)}\|F\|^2_{L^{12/5}(S)} \\
&\lesssim \|F(-2,.)\|^2_{L^2(P_{-2})} + \|\nabla_N F\|_{L^2(S)}\|F\|_{L^\infty_{[-2,2]}L^2(P_u)} \\
&\quad + \|\operatorname{tr}\theta\|_{L^6(S)}\|F\|^{3/2}_{L^\infty_{[-2,2]}L^2(P_u)}\|F\|^{1/2}_{L^6(S)},
\end{aligned}
\tag{3.15}
$$

which proves that $F \in L^\infty_{[-2,2]}L^2(P_u)$ by taking the supremum in u on the left-hand side and using the Sobolev embedding (3.3) and the assumption (3.4) for $\operatorname{tr}\theta$. This concludes the proof of the corollary. □

Proposition 3.9. — *Let F be a tensorfield on S such that $F \in L^\infty_{[-2,2]}L^2(P_u)$ and $\nabla\!\!\!/ F \in L^2(S)$. Then F belongs to $L^4(S)$.*

Proof. — We have

$$
\begin{aligned}
\|F\|^4_{L^4(S)} = \int_{-2}^{2}\|F\|^4_{L^4(P_u)}a du &\lesssim \int_{-2}^{2}\|F\|^4_{L^4(P_u)}du \\
&\lesssim \int_{-2}^{2}(\|F\|^2_{L^2(P_u)}\|\nabla\!\!\!/ F\|^2_{L^2(P_u)} + \|F\|^4_{L^2(P_u)})du \\
&\lesssim \|F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla\!\!\!/ F\|^2_{L^2(S)} + \|F\|^4_{L^4_{[-2,2]}L^2(P_u)} \\
&\lesssim \|F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}(\|\nabla\!\!\!/ F\|^2_{L^2(S)} + \|F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}),
\end{aligned}
\tag{3.16}
$$

where we have used (3.9) with $p = 4$. □

Proposition 3.10. — *Let F be a tensorfield on S such that $F \in H^1(S)$ and $\nabla\!\!\!/ \nabla F \in L^2(S)$. Assume also (3.4). Then F belongs to $L^\infty(S)$ and $\nabla_N \nabla\!\!\!/ F$ belongs to $L^2(S)$. Moreover, the conclusion still holds if instead of $F \in H^1(S)$ we assume $\nabla F \in L^2(S)$ and $F(-2,.) \in L^4(P_{-2})$.*

Proof. — Using (3.10) with $p = 4$ and Proposition 3.7, we obtain:

$$
\|F\|_{L^\infty(S)} \lesssim \|\nabla\!\!\!/ F\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|F\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \|F\|_{H^1(S)} + \|\nabla\nabla\!\!\!/ F\|_{L^2(S)}.
\tag{3.17}
$$

Thus, we just need to prove that $\nabla_N\nabla\!\!\!/ F$ belongs to $L^2(S)$ to conclude the proof. Since $\nabla\!\!\!/ \nabla_N F$ belongs to $L^2(S)$, it remains to prove that $[\nabla\!\!\!/, \nabla_N]F$ is in $L^2(S)$. The commutation Formula (2.17) yields:

$$\|[\nabla_N, \nabla\!\!\!/]F\|_{L^2(S)} \leq (\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)})\|\nabla F\|_{L^2_{[-2,2]}L^4(P_u)} + (\|R\|_{L^2(S)} + \|\nabla\!\!\!/ a\|_{L^2_{[-2,2]}L^4(P_u)}\|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)})\|F\|_{L^\infty(S)}. \tag{3.18}$$

Using the Gagliardo-Nirenberg inequality (3.9) and Proposition 3.7 to bound the norm in $L^2_{[-2,2]}L^4(P_u)$ and $L^\infty_{[-2,2]}L^4(P_u)$ of ∇F and $\nabla\!\!\!/ a$, together with the estimate (3.17) and the estimate (3.4), we finally obtain:

$$\|\nabla_N \nabla\!\!\!/ F\|_{L^2(S)} \lesssim (1 + \|\nabla\!\!\!/ a\|_{H^1(S)} + \|\nabla\!\!\!/ a\|_{L^2(S)} + \|\nabla\!\!\!/^2 a\|_{L^2(S)}) \times (\|F\|_{H^1(S)} + \|\nabla\nabla\!\!\!/ F\|_{L^2(S)}). \tag{3.19}$$

Next, we evaluate $\nabla_N \nabla\!\!\!/ a$. The commutation formula for scalars (2.18) yields:

$$\|[\nabla_N, \nabla\!\!\!/]a\|_{L^2(S)} \leq (\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)})\|\nabla a\|_{L^2_{[-2,2]}L^4(P_u)},$$

which together with the Gagliardo-Nirenberg inequality (3.9) and Proposition 3.7 to bound the norm in $L^2_{[-2,2]}L^4(P_u)$ of $\nabla\!\!\!/ a$, and the estimate (3.4), implies

$$\|\nabla_N \nabla\!\!\!/ a\|_{L^2(S)} \lesssim \|\nabla\!\!\!/ \nabla_N a\|_{L^2(S)} + (\|\nabla\!\!\!/ a\|_{H^1(S)} + \delta)(\|\nabla\!\!\!/ \nabla a\|_{L^2(S)} + \|\nabla a\|_{L^2(S)}). \tag{3.20}$$

Using again (3.4), we deduce for $\delta > 0$ small enough:

$$\|\nabla_N \nabla\!\!\!/ a\|_{L^2(S)} \lesssim \delta. \tag{3.21}$$

Finally, we conclude the proof in the case where $F \in H^1(S)$ using (3.19) together with the smallness assumption (3.4) and (3.21). In the case where $\nabla F \in L^2(S)$ and $F(-2,.) \in L^4(P_{-2})$, we proceed in the same way except that we use Corollary 3.8 to bound F in $L^\infty_{[-2,2]}L^4(P_u)$. □

Proposition 3.11. — *Let F be a tensorfield on S such that $\nabla\!\!\!/^2 F \in L^2(S)$, $\nabla_N F \in L^2(S)$ and $\nabla\!\!\!/ F(-2,.) \in L^2(P_{-2})$. Assume also (3.4). Then $\nabla\!\!\!/ F$ belongs to $L^\infty_{[-2,2]}L^2(P_u)$ and to $L^4(S)$.*

Proof. — We start with the estimate of $\nabla\!\!\!/ F$ in $L^\infty_{[-2,2]}L^2(P_u)$. We have:

$$\begin{aligned}
\|\nabla\!\!\!/ F(u,.)\|^2_{L^2(P_u)} &= \|\nabla\!\!\!/ F(-2,.)\|^2_{L^2(P_{-2})} + 2\int_{-2}^u \int_{P_{u'}} \nabla_N \nabla\!\!\!/ F(u',x') \cdot \nabla\!\!\!/ F(u',x') du' d\mu_{u'} \\
&\quad + \int_{-2}^u \int_{P_{u'}} \operatorname{tr}\theta |\nabla\!\!\!/ F(u',x')|^2 du' d\mu_{u'} \\
&\lesssim \|\nabla\!\!\!/ F(-2,.)\|^2_{L^2(P_{-2})} + \left| \int_{-2}^u \int_{P_{u'}} \nabla\!\!\!/ \nabla_N F(u',x') \cdot \nabla\!\!\!/ F(u',x') du' d\mu_{u'} \right| \\
&\quad + \left| \int_{-2}^u \int_{P_{u'}} [\nabla_N, \nabla\!\!\!/]F(u',x') \cdot \nabla\!\!\!/ F(u',x') du' d\mu_{u'} \right| \\
&\quad + \|\operatorname{tr}\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} \|\nabla\!\!\!/ F\|^2_{L^2_{[-2,2]}L^{\frac{8}{3}}(P_u)}
\end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&\lesssim \|\nabla\!\!\!/ F(-2,.)\|^2_{L^2(P_{-2})} + \|\Delta\!\!\!/ F\|_{L^2(S)}\|\nabla_N F\|_{L^2(S)}\\
&\quad + \|[\nabla\!\!\!/, \nabla_N]F\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)}\|\nabla\!\!\!/ F\|_{L^2_{[-2,2]}L^4(P_u)}\\
&\quad + \|\mathrm{tr}\,\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/^2 F\|^2_{L^2(S)},
\end{aligned}
$$

where we used in the last inequality an integration by parts and the Gagliardo- Nirenberg inequality (3.9). Now, using the commutator Formula (2.17), we have:

$$
\begin{aligned}
\|[\nabla\!\!\!/, \nabla_N]F\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} &\lesssim \|a^{-1}\nabla\!\!\!/ a\,\nabla_N F - \theta\nabla\!\!\!/ F + R_{N.}F + \theta a^{-1}\nabla\!\!\!/ a\,F\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)}\\
&\lesssim \|a^{-1}\nabla\!\!\!/ a\,\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N F\|_{L^2(S)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/ F\|_{L^2(S)}\\
&\quad + (\|R\|_{L^2(S)} + \|\theta a^{-1}\nabla\!\!\!/ a\,\|_{L^2(S)})\|F\|_{L^\infty_{[-2,2]}L^4(P_u)}\\
&\lesssim (\|R\|_{L^2(S)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\\
&\quad + \|a^{-1}\nabla\!\!\!/ a\,\|_{L^\infty_{[-2,2]}L^4(P_u)})(\|\nabla_N F\|_{L^2(S)} + \|\nabla\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)}),
\end{aligned}
\tag{3.23}
$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (3.9). (3.22) and (3.23) yields:

$$
\begin{aligned}
\|\nabla\!\!\!/ F(u,.)\|^2_{L^2(P_u)} &\lesssim \|\nabla\!\!\!/ F(-2,.)\|^2_{L^2(P_{-2})}\\
&\quad + (1 + \|R\|_{L^2(S)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|a^{-1}\nabla\!\!\!/ a\,\|_{L^\infty_{[-2,2]}L^4(P_u)})\\
&\quad \times (\|\nabla_N F\|^2_{L^2(S)} + \|\nabla\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla\!\!\!/^2 F\|_{L^2(S)} + \|\nabla\!\!\!/^2 F\|^2_{L^2(S)}).
\end{aligned}
$$

Finally, taking the supremum in u and using the assumption (3.4) implies:

$$
\|\nabla\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|\nabla\!\!\!/ F(-2,.)\|_{L^2(P_{-2})} + \|\nabla\!\!\!/^2 F\|_{L^2(S)} + \|\nabla_N F\|_{L^2(S)}. \tag{3.24}
$$

Next, we estimate of $\nabla\!\!\!/ F$ in $L^4(S)$. In view of Proposition 3.9, we have:

$$
\|\nabla\!\!\!/ F\|_{L^4(S)} \lesssim \|\nabla\!\!\!/^2 F\|_{L^2(S)} + \|\nabla\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)}.
$$

Together with (3.24), this concludes the proof of the proposition. □

Proposition 3.12. — *Let F be a vector field on S such that $F(-2,.) \in L^2(P_{-2})$, $\nabla\!\!\!/ F \in L^2(S)$, and $\nabla_N F = \nabla\!\!\!/ f_1 + F_2$ where f_1 is a scalar function on S such that $f_1 \in L^2(S)$ and F_2 is a vector field on S such that $F_2 \in L^{4/3}(S)$. Assume also (3.4). Then F belongs to $L^\infty_{[-2,2]}L^2(P_u)$.*

Proof. — We have

(3.25)

$$
\begin{aligned}
\|F(u,.)\|^2_{L^2(P_u)} &\lesssim \|F(-2,.)\|^2_{L^2(P_{-2})} + \int_{-2}^{u}\int_{P_{u'}} \nabla_N F\cdot F d\mu_{u'}du'\\
&\quad + \int_{-2}^{u}\int_{P_{u'}} \mathrm{tr}\,\theta|F|^2 d\mu_{u'}du'\\
&\lesssim \|F(-2,.)\|^2_{L^2(P_{-2})} + \int_{-2}^{u}\int_{P_{u'}} (\nabla\!\!\!/ f_1 + F_2)\cdot F d\mu_{u'}du'
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\mathrm{tr}\,\theta\|_{L^6(S)}\|F\|^2_{L^{\frac{12}{5}}(S)}\\
&\lesssim \|F(-2,.)\|^2_{L^2(P_{-2})} + \int_{-2}^{u}\int_{P_{u'}} f_1 \mathrm{d}\!\!/\mathrm{iv} F d\mu_{u'}du'\\
&\quad + \|F_2\|_{L^{4/3}(S)}\|F\|_{L^4(S)} + \|\mathrm{tr}\,\theta\|_{L^6(S)}\|F\|^2_{L^{\frac{12}{5}}(S)}\\
&\lesssim \|F(-2,.)\|^2_{L^2(P_{-2})} + (\|f_1\|_{L^2(S)} + \|F_2\|_{L^{4/3}(S)})\\
&\quad \times (\|\nabla\!\!\!/ F\|_{L^2(S)} + \|F\|_{L^\infty_{[-2,2]}L^2(P_u)})\\
&\quad + \|\mathrm{tr}\,\theta\|_{L^6(S)}(\|\nabla\!\!\!/ F\|^{\frac{1}{3}}_{L^2(S)}\|F\|^{\frac{5}{3}}_{L^\infty_{[-2,2]}L^2(P_u)} + \|F\|_{L^\infty_{[-2,2]}L^2(P_u)}),
\end{aligned}
$$

where we have used Proposition 3.9 to bound $\|F\|_{L^4(S)}$, and (3.9) with $p = 12/5$ to bound $\|F\|_{L^{12/5}(S)}$. This concludes the proof by taking the supremum in u on the left-hand side and using the assumption (3.4). □

3.3. The Bochner identity and consequences

We recall the Bochner identity on P_u (which has dimension 2). This allows us to control the L^2 norm of the second derivatives of a tensorfield in terms of the L^2 norm of the laplacian and geometric quantities associated with P_u (see for example **[15]** for a proof).

Proposition 3.13. — *Let K denote the Gauss curvature of P_u. Then*
(i) *For a scalar function f:*

$$\int_{P_u} |\nabla\!\!\!/^2 f|^2\mu_u = \int_{P_u}|\Delta\!\!\!/ f|^2\mu_u - \int_{P_u} K|\nabla\!\!\!/ f|^2\mu_u. \tag{3.26}$$

(ii) *For a vector field F_a:*
(3.27)

$$\int_{P_u} |\nabla\!\!\!/^2 F|^2\mu_u = \int_{P_u}|\Delta\!\!\!/ F|^2\mu_u - \int_{P_u} K(2\,|\nabla\!\!\!/ F|^2 - |\mathrm{d}\!\!/\mathrm{iv} F|^2 - |\mathrm{cu}\!\!/\mathrm{rl} F|^2)\mu_u + \int_{P_u} K^2|F|^2\mu_u,$$

where $\mathrm{d}\!\!/\mathrm{iv} F = \gamma^{ab}\nabla\!\!\!/_b F_a$, $\mathrm{cu}\!\!/\mathrm{rl} F = \mathrm{d}\!\!/\mathrm{iv}({}^*F) = \in_{ab}\nabla\!\!\!/_a F_b$.

Remark 3.14. — As a consequence of (3.27) together with a $L^\infty(P_u)$ estimate for tensors, we have the following Bochner inequality for tensors F on P_u (see **[15]** for a proof):

$$\|\nabla\!\!\!/^2 F\|_{L^2(P_u)} \lesssim \|\Delta\!\!\!/ F\|_{L^2(P_u)} + \|K\|_{L^2(P_u)}\|\nabla\!\!\!/ F\|_{L^2(P_u)} + \|K\|^2_{L^2(P_u)}\|F\|_{L^2(P_u)}. \tag{3.28}$$

Using Proposition 3.13, we obtain the following proposition:

Proposition 3.15. — *Let f be a scalar function on S such that $\nabla\!\!\!/ f(-2,.) \in L^2(P_{-2})$, $\nabla_N f \in L^2(S)$ and $\Delta\!\!\!/ f \in L^2(S)$. Assume also (3.4). Then $\nabla\!\!\!/^2 f$ belongs to $L^2(S)$ and $\nabla\!\!\!/ f$ belongs to $L^\infty_{[-2,2]}L^2(P_u)$.*

Proof. — The Bochner identity (3.26) implies:

$$\begin{aligned}\|\nabla\!\!\!/^2 f\|_{L^2(S)} &\lesssim \|\Delta\!\!\!/ f\|_{L^2(S)} + \|K|\nabla\!\!\!/ f|^2\|^{1/2}_{L^1(S)}\\ &\lesssim \|\Delta\!\!\!/ f\|_{L^2(S)} + \|K\|^{1/2}_{L^2(S)}\|\nabla\!\!\!/ f\|_{L^4(S)}\\ &\lesssim \|\Delta\!\!\!/ f\|_{L^2(S)} + \|K\|^{1/2}_{L^2(S)}(\|\nabla\!\!\!/ f\|^{1/2}_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla\!\!\!/^2 f\|^{1/2}_{L^2(S)}\\ &\quad + \|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)}),\end{aligned} \tag{3.29}$$

where we have used Proposition 3.9. Thus, it just remains to prove that $\nabla\!\!\!/ f$ belongs to $L^\infty_{[-2,2]}L^2(P_u)$. In order to use Proposition 3.12 , we have first to estimate $[\nabla_N, \nabla\!\!\!/]f$, which is given by the commutator Formula (2.18). We estimate $[\nabla_N, \nabla\!\!\!/]f$ in $L^2_{[-2,2]}L^{4/3}(P_u)$:

$$\begin{aligned}\|[\nabla_N, \nabla\!\!\!/]f\|_{L^2_{[-2,2]}L^{4/3}(P_u)} &\lesssim \|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N f\|_{L^2(S)}\\ &\quad + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)}.\end{aligned}$$

Thus, $\nabla_N\nabla\!\!\!/ f = \nabla\!\!\!/ f_1 + F_2$ where $f_1 = \nabla_N f$ belongs to $L^2(S)$ and $F_2 = [\nabla_N, \nabla\!\!\!/]f$ belongs to $L^2_{[-2,2]}L^{4/3}(P_u)$. According to Proposition 3.12, and using assumption (3.4), this implies:

$$\|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|\nabla f\|_{L^2(S)} + \|\nabla\!\!\!/^2 f\|_{L^2(S)} + \delta\|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)}.$$

For $\delta > 0$ small enough, this yields:

$$\|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|\nabla f\|_{L^2(S)} + \|\nabla\!\!\!/^2 f\|_{L^2(S)}.$$

Together with (3.29), this implies:

$$\|\nabla\!\!\!/^2 f\|_{L^2(S)} \lesssim \|\nabla f\|_{L^2(S)} + \|K\|^{1/2}_{L^2(S)}\|\nabla\!\!\!/^2 f\|_{L^2(S)},$$

which concludes the proof since $\|K\|_{L^2(S)} \leq \delta$ for a small $\delta > 0$ in view of assumption (3.4). □

3.4. Parabolic and elliptic estimates

In the proof of Theorem 2.4 and Theorem 2.6, we will often encounter parabolic equations of the following type:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)f = h \quad \text{on} \quad -2 < u < 2$$

(see for Example (2.29)). In Proposition 3.16 and Proposition 3.17 below, we obtain estimates for such equations.

Proposition 3.16. — *Let f be a scalar function on S such that:*

$$(\nabla_N - a^{-1}\Delta\!\!\!/)f = h \quad \textit{on} \quad -2 < u < 2, \tag{3.30}$$

where h is in $L^2(S)$. Assume also that $f(-2,.)$ and $\nabla\!\!\!\!/ f(-2,.)$ both belong to $L^2(P_{-2})$. Finally, assume (3.4)*. Then, we have:*

$$\|f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N f\|_{L^2(S)} + \|\nabla\!\!\!\!/^2 f\|_{L^2(S)} \lesssim \|h\|_{L^2(S)} + \|f(-2,.)\|_{L^2(P_{-2})} + \|\nabla\!\!\!\!/ f(-2,.)\|_{L^2(P_{-2})}. \tag{3.31}$$

Proof. — We multiply (3.30) by f and integrate on $-2 < u' < u$ where $u \leq 2$. Using integration by parts together with (3.5) and (3.6), we obtain:

$$\begin{aligned}
&\frac{1}{2}\|f(u,.)\|^2_{L^2(P_u)} + \|a^{-1/2}\nabla\!\!\!\!/ f\|^2_{L^2(S)}\\
&= \frac{1}{2}\|f(-2,.)\|^2_{L^2(P_{-2})} + \frac{1}{2}\int_{-2}^u\int_{P_{u'}} a^{-1}\mathrm{tr}\,\theta f^2 d\mu_{u'}du' + \int_{-2}^u\int_{P_{u'}} hf d\mu_{u'}du'\\
&\lesssim \|f(-2,.)\|^2_{L^2(P_{-2})} + \|\mathrm{tr}\,\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|f\|^2_{L^2_{[-2,2]}L^{\frac{8}{3}}(P_u)}\\
&\quad + \|h\|_{L^2(S)}\|f\|_{L^\infty_{[-2,2]}L^2(P_u)}.
\end{aligned}\tag{3.32}$$

Together with (3.9) and (3.4), we get:

$$\|f\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!\!/ f\|^2_{L^2(S)} \lesssim \|h\|^2_{L^2(S)} + \|f(-2,.)\|^2_{L^2(P_{-2})}. \tag{3.33}$$

We multiply (3.30) by $\Delta\!\!\!\!/ f$ and integrate on $-2 < u' < u$ where $u \leq 2$:

$$\begin{aligned}
&\int_{-2}^u\int_{P_{u'}} \nabla\!\!\!\!/(a\nabla_N f)\nabla\!\!\!\!/ f d\mu_{u'}du' + \|a^{-1/2}\Delta\!\!\!\!/ f\|^2_{L^2(S)}\\
&\qquad = \int_{-2}^u\int_{P_{u'}} h\Delta\!\!\!\!/ f a d\mu_{u'}du' \lesssim \|h\|_{L^2(S)}\|\Delta\!\!\!\!/ f\|_{L^2(S)}.
\end{aligned}\tag{3.34}$$

Using integration by parts together with (3.5) and (3.6), we obtain:

$$\begin{aligned}
&\frac{1}{2}\|\nabla\!\!\!\!/ f(u,.)\|^2_{L^2(P_u)} + \|a^{-1/2}\Delta\!\!\!\!/ f\|^2_{L^2(S)}\\
&\leq \frac{1}{2}\|\nabla\!\!\!\!/ f(-2,.)\|^2_{L^2(P_{-2})} + \int_{-2}^u\int_{P_{u'}} [\nabla_N, \nabla\!\!\!\!/]f\nabla\!\!\!\!/ f d\mu_{u'}du'\\
&\quad + \int_{-2}^u\int_{P_{u'}} |\nabla\!\!\!\!/ f|^2\mathrm{tr}\,\theta d\mu_{u'}du' - \int_{-2}^u\int_{P_{u'}} \nabla\!\!\!\!/ a\nabla\!\!\!\!/ f\nabla_N f d\mu_{u'}du' + \|h\|_{L^2(S)}\|\Delta\!\!\!\!/ f\|_{L^2(S)}.
\end{aligned}\tag{3.35}$$

Using the commutator Formula (2.18), we get:

$$\begin{aligned}
&\|\nabla\!\!\!\!/ f\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + \|a^{-1/2}\Delta\!\!\!\!/ f\|^2_{L^2(S)}\\
&\lesssim \|\nabla\!\!\!\!/ f(-2,.)\|^2_{L^2(P_{-2})} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!\!/ f\|^2_{L^2_{[-2,2]}L^{8/3}(P_u)} + \|h\|_{L^2(S)}\|\Delta\!\!\!\!/ f\|_{L^2(S)},
\end{aligned}\tag{3.36}$$

which together with (3.9) and (3.4) yields:

$$\|\nabla\!\!\!/ f\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + \|\Delta\!\!\!/ f\|^2_{L^2(S)} \lesssim \delta(\|\nabla\!\!\!/ f\|^2_{L^2(S)} + \|\nabla\!\!\!/^2 f\|^2_{L^2(S)}) + \|\nabla\!\!\!/ f(-2,.)\|^2_{L^2(P_{-2})} + \|h\|^2_{L^2(S)}. \tag{3.37}$$

Since $\nabla_N f = a^{-1}\Delta\!\!\!/ f + h$, (3.37) yields:

$$\|\nabla_N f\|^2_{L^2(S)} \lesssim \delta(\|\nabla\!\!\!/ f\|^2_{L^2(S)} + \|\nabla\!\!\!/^2 f\|^2_{L^2(S)}) + \|\nabla\!\!\!/ f(-2,.)\|^2_{L^2(P_{-2})} + \|h\|^2_{L^2(S)}, \tag{3.38}$$

which together with Proposition 3.15, (3.4) and (3.37) implies:

$$\|\nabla\!\!\!/^2 f\|^2_{L^2(S)} \lesssim \delta\|\nabla\!\!\!/ f\|^2_{L^2(S)} + \|\nabla\!\!\!/ f(-2,.)\|^2_{L^2(P_{-2})} + \|h\|^2_{L^2(S)}. \tag{3.39}$$

Finally, (3.33), (3.37), (3.38) and (3.39) yield (3.31) for $\delta > 0$ small enough. □

Proposition 3.17. — *Let f be a scalar function on S such that:*

$$(\nabla_N - a^{-1}\Delta\!\!\!/)f = h \quad on \quad -2 < u < 2. \tag{3.40}$$

Assume that there exists a vector field H on S tangent to P_u and a scalar function h_1 on S such that:

$$h = \mathrm{di\!\!\!/v}(H) + h_1 \ \ with\ H \in L^2(S)\ and\ h_1 \in L^{\frac{4}{3}}(S). \tag{3.41}$$

Assume also that $f(-2,.)$ belongs to $L^2(P_{-2})$. Finally, assume (3.4). Then, we have:

$$\|f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/ f\|_{L^2(S)} \lesssim \|H\|_{L^2(S)} + \|h_1\|_{L^{\frac{4}{3}}(S)} + \|f(-2,.)\|_{L^2(P_{-2})}. \tag{3.42}$$

Proof. — We multiply (3.40) by f and integrate on $-2 < u' < u$ where $u \leq 2$. Using integration by parts together with (3.5) and (3.6), we obtain:

$$\frac{1}{2}\|f(u,.)\|^2_{L^2(P_u)} + \|a^{-1/2}\nabla\!\!\!/ f\|^2_{L^2(S)} \tag{3.43}$$

$$= \frac{1}{2}\|f(-2,.)\|^2_{L^2(P_{-2})} + \frac{1}{2}\int_{-2}^{u}\int_{P_{u'}} a^{-1}\mathrm{tr}\,\theta f^2 d\mu_{u'}du'$$

$$+ \int_{-2}^{u}\int_{P_{u'}} hf d\mu_{u'}du' \tag{3.44}$$

$$\lesssim \|f(-2,.)\|^2_{L^2(P_{-2})} + \|\mathrm{tr}\,\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|f\|^2_{L^2_{[-2,2]}L^{\frac{8}{3}}(P_u)}$$

$$+ \int_{-2}^{u}\int_{P_{u'}} hf d\mu_{u'}du'.$$

Taking (3.41) into account, we have:

(3.45)

$$\int_{-2}^{u}\int_{P_{u'}} hf d\mu_{u'}du' = \int_{-2}^{u}\int_{P_{u'}} \mathrm{di\!\!\!/v}(H) fa d\mu_{u'}du' + \int_{-2}^{u}\int_{P_{u'}} fh_1 d\mu_{u'}du'$$

$$= -\int_{-2}^{u}\int_{P_{u'}} H\nabla\!\!\!/ f d\mu_{u'}du' - \int_{-2}^{u}\int_{P_{u'}} a^{-1}\nabla\!\!\!/ a\, Hf d\mu_{u'}du'$$

$$+ \int_{-2}^{u} \int_{P_{u'}} f h_1 d\mu_{u'} du'$$
$$\lesssim \|\nabla\!\!\!/ f\|_{L^2(S)} \|H\|_{L^2(S)} + \|f\|_{L^4(S)} (\|a^{-1} \nabla\!\!\!/ a\|_{L^4(S)} \|H\|_{L^2(S)}$$
$$+ \|h_1\|_{L^{\frac{4}{3}}(S)}),$$

which together with Proposition 3.9 and (3.4) yields:

$$\int_{-2}^{u} \int_{P_{u'}} h f d\mu_{u'} du' \lesssim (\|f\|_{L^\infty_{[-2,2]} L^2(P_u)} + \|\nabla\!\!\!/ f\|_{L^2(S)})(\|H\|_{L^2(S)} + \|h_1\|_{L^{\frac{4}{3}}(S)}). \tag{3.46}$$

Finally, (3.9), (3.4), (3.43) and (3.46) imply (3.42). □

In Chapter 4, we will have among other things to control $\widehat{\theta}$ (the traceless part of θ). Now, according to the second equation of (2.12), $\widehat{\theta}$ satisfies an equation of the type $\text{di}\!\!\!/\text{v}(F) = h$. Thus, we conclude this section with an estimate that will allow us to control the solution to such equations.

Proposition 3.18. — *Let F a symmetric 2-tensor such that $\text{tr}F = 0$. Then:*

$$\|\nabla\!\!\!/ F\|_{L^2(S)} \lesssim \|\text{di}\!\!\!/\text{v} F\|_{L^2(S)} + \|K\|^{\frac{1}{2}}_{L^2(S)} \|F\|_{L^4(S)}. \tag{3.47}$$

Proof. — This follows immediately from the following identity for Hodge systems (see for example [**13**]):

$$\int_{P_u} (|\nabla\!\!\!/ F|^2 + 2K|F|^2) = 2 \int_{P_u} |\text{di}\!\!\!/\text{v} F|^2. \tag{3.48}$$

□

CHAPTER 4

CONSTRUCTION OF THE FOLIATION AND REGULARITY WITH RESPECT TO x

This chapter deals with the proof of Theorem 2.4. By Section 2.1, we may assume that (Σ, g, k) coincides with $(\mathbb{R}^3, \delta, 0)$ outside of a compact, say $|x| \geq 1$. Notice that in $|x| \geq 1$ and for all $\omega \in \mathbb{S}^2$, the scalar function $x.\omega$ satisfies the Equation (2.28) and the estimate (2.30), since $a \equiv 1, \theta \equiv 0$ and $N \equiv \omega$ in this region. Thus, we would like to construct a function u solution of (2.28) satisfying (2.30) in a region containing $|x| \leq 2$ and to glue it to $x.\omega$ in $1 \leq |x| \leq 2$. Now, (2.28) is of parabolic type—see (2.29)—where u plays the role of time. Therefore, for each $\omega \in \mathbb{S}^2$, we will construct $u(., \omega)$ on a strip of type $S = \{x \in \Sigma \text{ such that } -2 < u(x, \omega) < 2\}$ solution of:

$$\begin{cases} \operatorname{tr}\theta - k_{NN} = 1 - a, \quad \text{on} \quad -2 < u < 2, \\ \qquad u(., \omega) = -2 \quad \text{on} \quad x.\omega = -2. \end{cases} \tag{4.1}$$

The rest of the chapter is as follows. We first prove a priori estimates consistent with the estimate (2.30) and valid on $-2 < u < 2$ for the solution u of (4.1). We also prove on $-2 < u < 2$ a priori estimates for higher derivatives of the solution u of (4.1). We then extend a solution u of (2.28) on $u \leq \alpha$ to a solution on the strip $\alpha < u < \alpha + T$:

$$\operatorname{tr}\theta - k_{NN} = 1 - a, \quad \text{on} \quad \alpha < u < \alpha + T, \tag{4.2}$$

where $-2 \leq \alpha \leq 2$, and $T > 0$ is small enough. Together with the a priori estimates, this allows us to control the solution of (4.2) on $-2 + kT < u < -2 + (k+1)T$ uniformly with respect to $k = 0, \ldots, [4/T]$ in order to obtain a solution u of (4.1) on $-2 < u < 2$. Finally, we conclude the proof of Theorem 2.4 by showing how to glue the solution u of (4.1) to $x.\omega$ in $1 \leq |x| \leq 2$ in order to obtain a solution on Σ satisfying (2.30).

Remark 4.1. — In order to obtain higher order derivatives estimates for (4.1), and in order to construct the solution of (4.2), we need to assume that (Σ, g, k) is more regular. We would like to insist on the fact that this additional regularity is only assumed to obtain the existence of u solution of (4.1). On the other hand, we only rely on the control of $\|R\|_{L^2(\Sigma)}$ and $\|\nabla k\|_{L^2(\Sigma)}$ given by (2.1) to prove the estimate (2.30).

4.1. A priori estimates for lower order derivatives

Let (Σ, g, k) chosen as in Section 2.1. In particular, we assume:

$$\|\nabla k\|_{L^2(\Sigma)} + \|R\|_{L^2(\Sigma)} \leq \varepsilon. \tag{4.3}$$

Let u a scalar function on $\Sigma \times \mathbb{S}^2$, and let P_u, a, N, θ and K be associated to u as in Section 2.2. Assume that u satisfies the additional Equation (2.28). The Equations (2.11) (2.12) (2.13) may be rewritten:

$$\begin{cases} \nabla_A N = \theta_{AB} e_B, \\ \nabla_N N = -\nabla\!\!\!\!/ a, \end{cases} \tag{4.4}$$

$$\begin{cases} \operatorname{tr}\theta - k_{NN} = 1 - a, \\ \nabla_N a - a^{-1}\Delta\!\!\!\!/(a) = |\theta|^2 + \nabla_N(k_{NN}) + R_{NN}, \\ \nabla\!\!\!\!/^B \widehat{\theta}_{AB} = \frac{1}{2}\nabla\!\!\!\!/_A \operatorname{tr}\theta + R_{NA}, \\ a^{-1}\nabla\!\!\!\!/_A \nabla\!\!\!\!/_B a + \nabla_N \theta_{AB} + \theta_A^C \theta_{CB} + K\gamma_{AB} = R_{AB} \end{cases} \tag{4.5}$$

and

$$2K - \operatorname{tr}\theta^2 + |\theta|^2 = R - 2R_{NN}. \tag{4.6}$$

In this section, we establish a priori estimates for a, N, θ and K corresponding to (2.30) in the region S of Σ between P_{-2} and P_2 (i.e., $S = \{x /\ -2 < u(x,\omega) < 2\}$) where u is initialized on $x.\omega = -2$ by:

$$u(x,\omega) = -2 \ \text{ on } \ x.\omega = -2. \tag{4.7}$$

Note that the first equation of (4.5), (4.7) and the fact that (g, k, Σ) coincides with $(\delta, 0, \mathbb{R}^3)$ for $|x| \geq 2$ yields:

$$\nabla^p(a-1) = 0,\ \nabla^p\theta = 0,\ \nabla^p(N-\omega) = 0 \text{ for all } p \in \mathbb{N} \ \text{ on } \ u = -2, \tag{4.8}$$

so that the subsequent integrations by parts will not create boundary terms at $u = -2$. We will assume:

$$\|a-1\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|a-1\|_{L^\infty(S)} + \|\nabla\!\!\!\!/ \nabla a\|_{L^2(S)} + \|K\|_{L^2(S)} \leq D\varepsilon \tag{4.9}$$

and

$$\|\nabla\theta\|_{L^2(S)} \leq D^2\varepsilon, \tag{4.10}$$

where D is a large enough constant. We will then try to improve on these estimates. Let us note that (4.8), (4.9) and (4.10) together with Corollary 3.8, Proposition 3.9 and Proposition 3.10 yield:

$$\|\nabla_N a\|_{L^4(S)} + \|\nabla\!\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!\!/ a\|_{L^6(S)} \leq D^2\varepsilon \tag{4.11}$$

and

$$\|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\theta\|_{L^6(S)} \leq D^3\varepsilon. \tag{4.12}$$

Also, using Corollary 3.8, (4.3), and the fact that $k \equiv 0$ on $x.\omega = -2$ by Section 2.1 yields:

$$\|k\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|k\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|k\|_{L^6(S)} \leq D\varepsilon. \tag{4.13}$$

4.1.1. Improvement of the bootstrap assumptions (4.10). — We start by estimating θ. Since $\mathrm{tr}\,\theta - k_{NN} = 1 - a$, we have from (4.9):

$$\|\mathrm{tr}\,\theta - k_{NN}\|_{L^\infty(S)} \leq D\varepsilon. \tag{4.14}$$

Also, the first equation of (4.5) together with (4.4) yields, schematically:

$$\nabla_N \mathrm{tr}\,\theta = \nabla_N k_{NN} - 2k_{\nabla\!\!\!/ aN} - \nabla_N a,\ \nabla\!\!\!/ \mathrm{tr}\,\theta = \nabla\!\!\!/ k_{NN} + 2k_{N\cdot} \cdot \theta - \nabla\!\!\!/ a,$$

so that:

$$\begin{aligned} \|\nabla \mathrm{tr}\,\theta\|_{L^2(S)} &\lesssim \|\nabla a\|_{L^2(S)} + \|\nabla k\|_{L^2(S)} \\ &\quad + \|k\|_{L^\infty_{[-2,2]}L^4(P_u)}(\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}) \\ &\lesssim (D + D^4\varepsilon)\varepsilon, \end{aligned} \tag{4.15}$$

where we have used the bootstrap assumption (4.9), (4.11), (4.12) and (4.13) to obtain the last inequality. We continue with the estimates for $\widehat{\theta}$. The third equation in (4.5) and Proposition 3.18 yield:

$$\|\nabla\!\!\!/ \widehat{\theta}\|_{L^2(S)} \lesssim \|\nabla\!\!\!/ \mathrm{tr}\,\theta\|_{L^2(S)} + \|R_{N\cdot}\|_{L^2(S)} + \|K\|^{\frac{1}{2}}_{L^2(S)}\|\widehat{\theta}\|_{L^4(S)}, \tag{4.16}$$

which together with (4.3), (4.9), (4.12) and (4.15) yields:

$$\|\nabla\!\!\!/ \widehat{\theta}\|_{L^2(S)} \lesssim (D + D^4\varepsilon^{\frac{1}{2}})\varepsilon. \tag{4.17}$$

Also, using the last equation of (4.5), we have:

$$\|\nabla_N \theta\|_{L^2(S)} \lesssim \|\nabla\!\!\!/^2 a\|_{L^2(S)} + \|K\|_{L^2(S)} + \|R\|_{L^2(S)} + \|\theta\|^2_{L^4(S)}, \tag{4.18}$$

which together with (4.3), (4.9) and (4.12) yields:

$$\|\nabla_N \theta\|_{L^2(S)} \lesssim (D + D^6\varepsilon)\varepsilon. \tag{4.19}$$

Finally, (4.15), (4.17) and (4.19) yield:

$$\|\nabla \theta\|_{L^2(S)} \lesssim (D + D^6\varepsilon^{\frac{1}{2}})\varepsilon, \tag{4.20}$$

which is an improvement of (4.10).

4.1.2. Improvement of the bootstrap assumptions (4.9). — We now try to improve (4.9). Note first that (4.6) yields:

$$\|K\|_{L^2(S)} \lesssim \|\mathrm{tr}\,\theta\|^2_{L^4(S)} + \|\theta\|^2_{L^4(S)} + \|R\|_{L^2(S)}. \tag{4.21}$$

Together with (4.3) and (4.12), this yields:

$$\|K\|_{L^2(S)} \lesssim (1 + D^6\varepsilon)\varepsilon. \tag{4.22}$$

We rewrite the second equation of (4.5) as:

$$(\nabla_N - a^{-1}\not\Delta)(a-1) = h, \tag{4.23}$$

where h is given by:

$$h = |\theta|^2 + \nabla_N(k_{NN}) + R_{NN}. \tag{4.24}$$

Using the second equation of (4.4) implies:

$$\nabla_N(k_{NN}) = \nabla_N k_{NN} + k(\nabla_N N, N) = \nabla_N k_{NN} - k(\not\nabla a, N), \tag{4.25}$$

which together with (4.24) yields:

$$h = |\theta|^2 + \nabla_N k_{NN} - k(\not\nabla a, N) + R_{NN}. \tag{4.26}$$

Using (4.3), (4.11), (4.12), (4.13) and (4.26), we obtain:

$$\|h\|_{L^2(S)} \lesssim (1 + D^6\varepsilon)\varepsilon. \tag{4.27}$$

Using Proposition 3.16, (4.8), (4.9), (4.11), (4.12), (4.23) and (4.27) we obtain:

$$\|a-1\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\not\nabla a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N a\|_{L^2(S)} + \|\not\nabla^2 a\|_{L^2(S)} \lesssim (1 + D^6\varepsilon)\varepsilon. \tag{4.28}$$

In order to obtain estimates for $\not\nabla\nabla_N a$ and $\nabla_N^2 a$, we differentiate the second equation of (4.5) by ∇_N:

$$(\nabla_N - a^{-1}\not\Delta)\nabla_N a = [\nabla_N, a^{-1}\not\Delta]a + 2\theta\nabla_N\theta + \nabla_N^2 k_{NN} + \nabla_N R_{NN}. \tag{4.29}$$

Using (4.25), we have:

$$\nabla_N^2(k_{NN}) = \nabla_N(\nabla_N k_{NN} - k(\not\nabla a, N)). \tag{4.30}$$

The Commutator Formula (2.18) and the second equation of (4.4) yield:

$$\begin{aligned} \nabla_N(k(\not\nabla a, N)) &= -\nabla_N k(\not\nabla a, N) - k(\nabla_N\not\nabla a, N) \\ &\quad + k(\not\nabla a, \not\nabla a) \\ &= -\nabla_N k(\not\nabla a, N) - k(\not\nabla\nabla_N a, N) \\ &\quad - \nabla_N a k(\not\nabla a, N) + \theta(\not\nabla a, e_A)k_{AN} + k(\not\nabla a, \not\nabla a). \end{aligned} \tag{4.31}$$

Using the constraint Equations (1.2) and the fact that we have a maximal foliation yields:

$$\begin{aligned} \nabla_N k(NN) &= -\nabla_A k_{AN} \\ &= -\not{\mathrm{div}}(k_{N.}) - \mathrm{tr}\,\theta k_{NN} + \theta_{AB}k_{AB}, \end{aligned} \tag{4.32}$$

which together with the commutator Formula (2.17), the second equation of (4.4), (4.30) and (4.31) implies, schematically:

$$\begin{aligned} \nabla_N^2(k_{NN}) = &-\not{\mathrm{div}}(\nabla_N k_{N.}) + a^{-1}\not\nabla a\,\nabla_N k_{N.} + \theta\not\nabla k_{N.} + R_{N.}k + \theta a^{-1}\not\nabla a\,k_{N.} \\ &+ \nabla_N\theta k + \theta\nabla_N k + \theta k\not\nabla a + \nabla_N k(\not\nabla a, N) \\ &+ k(\not\nabla\nabla_N a, N) + \nabla_N a k(\not\nabla a, N) \\ &+ \theta k + k(\not\nabla a, \not\nabla a). \end{aligned} \tag{4.33}$$

We use the twice-contracted Bianchi identity on Σ

$$\nabla^j R_{ij} = \frac{1}{2}\nabla_i R, \tag{4.34}$$

together with the constraint Equations (1.2) to express $\nabla_N R_{NN}$:

$$\begin{aligned} \nabla_N R_{NN} &= -\nabla_A R_{AN} + k\cdot\nabla_N k \\ &= -\mathrm{d}\!\!/\mathrm{iv}(R_{.N}) + \mathrm{tr}\,\theta R_{NN} - \theta_{AB}R_{AB} + k\cdot\nabla_N k. \end{aligned} \tag{4.35}$$

Finally, we use the commutator Formula (2.20) for a scalar f:

$$\begin{aligned} a[\nabla_N, a^{-1}\Delta\!\!\!/]a = {} & -\mathrm{tr}\,\theta\Delta\!\!\!/ a - 2\widehat{\theta}\cdot\nabla\!\!\!/^2 a + 2a^{-1}\nabla\!\!\!/ a\cdot\nabla\!\!\!/\nabla_N a - 2R_{N.}\cdot\nabla\!\!\!/ a \\ & - \nabla\!\!\!/\mathrm{tr}\,\theta\cdot\nabla\!\!\!/ a - 2\widehat{\theta}\cdot a^{-1}\nabla\!\!\!/ a\cdot\nabla\!\!\!/ a. \end{aligned} \tag{4.36}$$

(4.29), (4.33), (4.35) and (4.36) yield:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N a = \mathrm{d}\!\!/\mathrm{iv}(H) + h_1, \tag{4.37}$$

where the tensor H is given by

$$H = -\nabla_N k_{.N} - R_{.N} \tag{4.38}$$

and where the scalar h_1 is given schematically by

$$\begin{aligned} h_1 = {} & -a^{-1}\mathrm{tr}\,\theta\Delta\!\!\!/ a - 2a^{-1}\widehat{\theta}\nabla\!\!\!/^2 a + 2a^{-2}\nabla\!\!\!/ a\nabla\!\!\!/\nabla_N a - 2R_{N.}a^{-1}\nabla\!\!\!/ a - \nabla\!\!\!/\mathrm{tr}\,\theta a^{-1}\nabla\!\!\!/ a \\ & + 2\widehat{\theta}|a^{-1}\nabla\!\!\!/ a|^2 + 2\theta\nabla_N\theta + a^{-1}\nabla\!\!\!/ a\,\nabla_N k_{N.} + \nabla\theta k + \theta\nabla k + R_{N.}k \\ & + \theta a^{-1}\nabla\!\!\!/ a\, k_{N.} + 2k\nabla_N k + \nabla_N k(\nabla\!\!\!/ a, N) + k(\nabla\!\!\!/\nabla_N a, N) \\ & + \nabla_N a k(\nabla\!\!\!/ a, N) + \theta k + k(\nabla\!\!\!/ a, \nabla\!\!\!/ a) + \theta R. \end{aligned} \tag{4.39}$$

We estimate H in $L^2(S)$ using (4.3):

$$\|H\|_{L^2(S)} \leq \|\nabla k\|_{L^2(S)} + \|R\|_{L^2(S)} \leq 2\varepsilon. \tag{4.40}$$

We estimate h in $L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)$:

$$\begin{aligned} \|h\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} \lesssim {} & (\|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|k\|_{L^\infty_{[-2,2]}L^4(P_u)}) \\ & \times (\|\nabla\!\!\!/^2 a\|_{L^2(S)} + \|\nabla\!\!\!/\nabla_N a\|_{L^2(S)} + \|R\|_{L^2(S)} + \|\nabla\theta\|_{L^2(S)} \\ & + \|\nabla k\|_{L^2(S)} + \|\nabla\!\!\!/ a\|^2_{L^4(S)} + \|\theta\|^2_{L^4(S)} + \|\nabla_N a\|^2_{L^4(S)} + \|k\|^2_{L^4(S)}), \end{aligned} \tag{4.41}$$

which together with (4.3), (4.9), (4.10), (4.11), (4.12) and (4.13) yields:

$$\|h\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} \lesssim D^9\varepsilon^2. \tag{4.42}$$

Using Proposition 3.17, (4.8), (4.11), (4.12), (4.37), (4.40) and (4.42) we obtain:

$$\|\nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/\nabla_N a\|_{L^2(S)} \lesssim (1 + D^9\varepsilon)\varepsilon. \tag{4.43}$$

Now, Proposition 3.10 together with (4.28) and (4.43) yields:

$$\|a - 1\|_{L^\infty(S)} \lesssim (1 + D^9\varepsilon)\varepsilon. \tag{4.44}$$

Finally, (4.22), (4.28), (4.43) and (4.44) imply:

$$\begin{aligned}(4.45)\qquad \|a-1\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|\nabla a\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|a-1\|_{L^\infty(S)}\\ +\|\nabla\!\!\!/\nabla a\|_{L^2(S)}+\|K\|_{L^2(S)} &\lesssim (1+D^9\varepsilon)\varepsilon,\end{aligned}$$

which is an improvement of (4.9).

Thus, there is a universal constant D such that (4.9) and (4.10) hold. Together with (4.4) and (4.14), this yields (2.30).

4.2. A priori estimates for higher order derivatives

In addition to (4.3), we assume the following control on R and k:

$$(4.46)\qquad \|\nabla^j R\|_{L^2(S)}+\|\nabla^{1+j}k\|_{L^2(S)}\leq M,\ \text{for all } 1\leq j\leq 5,$$

where M is a large constant. The goal of this section is to prove the following proposition:

Proposition 4.2. — *Let* (Σ, g, k) *chosen as in Section 2.1, and satisfying* (4.46). *Let* u *a scalar function defined on* $S=\{x/-2<u(x,\omega)<2\}$, *and let* P_u, a, N, θ *and* K *be associated to* u *as in Section 2.2. Assume that* u *satisfies the additional Equation* (2.28) *and is initialized on* $x.\omega=-2$ *by* (4.7). *Then,* a *and* θ *satisfy the following estimates:*

$$(4.47)\qquad \|\nabla\!\!\!/^2\nabla^{j-1}a\|_{L^2(S)}+\|\nabla^j a\|_{L^2(S)}+\|\nabla^j\theta\|_{L^2(S)}\leq C(M),\ \textit{for all } 1\leq j\leq 5.$$

Remark 4.3. — In connection with Remark 4.1, let us insist again on the fact that the assumption (4.46) is only used to obtain the existence of u solution to (4.1).

The proof of Proposition 4.2 is postponed to Appendix A.2.

4.3. Construction of the foliation on a small strip

Let $-2\leq\alpha\leq 2$. In the following theorem, we assume that the u-foliation satisfying (2.28) exists on $-2<u<\alpha$, and we show how to extend u to a solution on the strip $\alpha\leq u<\alpha+T$ provided $T>0$ is chosen small enough.

Theorem 4.4. — *Assume that we have the following control on* R *and* k:

$$(4.48)\qquad \|\nabla^j R\|_{L^2(\Sigma)}+\|\nabla^{1+j}k\|_{L^2(\Sigma)}<+\infty,\ \textit{for all } j=1\leq j\leq 5.$$

Also, assume that u *is a solution to* (2.28) *for* $-2\leq u\leq\alpha$ *such that*

$$(4.49)\qquad |a-1|\leq 1/4,\ \textit{for } u\leq\alpha$$

and

$$(4.50)\qquad \|\nabla^j\theta\|_{L^2(S\cap\{u\leq\alpha\})}<+\infty,\ \textit{for all } j=1\leq j\leq 5.$$

Then, there exists a constant $T > 0$ such that we may extend the solution u of (2.28) *to $\alpha < u < \alpha + T$. Furthermore, T only depends on the size of the norms appearing in* (4.48) *and* (4.50).

Remark 4.5. — We do not claim any sharpness in the Sobolev exponents appearing in the statement of Theorem 4.4. Our goal is to obtain an existence result with $T > 0$ depending only on a fixed number of derivatives of (R, k), and of θ in $u \leq \alpha$, no matter how large this fixed number is.

The proof of Theorem 4.4 is postponed to Appendix A.5.

4.4. Proof of Theorem 2.4

We apply here the strategy explained in the introduction of Chapter 4.

Let $0 < \alpha \leq 4$. We look for a solution $u(.,\omega)$ to:

$$\begin{cases} \operatorname{tr}\theta - k_{NN} = 1 - a, \quad \text{on} \quad -2 < u < -2 + \alpha, \\ \qquad u(.,\omega) = -2 \quad \text{on} \quad x.\omega = -2. \end{cases} \tag{4.51}$$

Theorem 4.4 ensures that $u(.,\omega)$ solution of (4.51) exists as long as $|a - 1| \leq 1/4$ and $\|\nabla^j \theta(.,\omega)\|_{L^2(S \cap \{u \leq -2+\alpha\})}$, $1 \leq j \leq 5$, stay under control. Now, the a priori estimates (4.9) and (4.47) yield $|a - 1| \leq 1/4$ and the control of the norm of $\|\nabla^j \theta(.,\omega)\|_{L^2(S)}$, $1 \leq j \leq 5$. Thus, we deduce the existence of $u(.,\omega)$ solution of:

$$\begin{cases} \operatorname{tr}\theta - k_{NN} = 1 - a, \quad \text{on} \quad -2 < u < 2, \\ \qquad u(.,\omega) = -2 \quad \text{on} \quad x.\omega = -2, \end{cases} \tag{4.52}$$

satisfying (4.9), (4.10) and (4.47) on $-2 < u < 2$.

Now, we would like to glue the solution $u(.,\omega)$ of (4.52) to $x.\omega$ in the region $1 \leq |x| \leq 2$ where (Σ, g, k) coincides with $(\mathbb{R}^3, \delta, 0)$ by Section 2.1. We will use the following lemma.

Lemma 4.6. — *Let $u(.,\omega)$ the solution of* (4.52) *satisfying* (4.9), (4.10) *and* (4.47) *on $-2 < u < 2$. Then, we have:*

$$(1 + |x|)^{-1}|u - x.\omega| + |\nabla u - \omega| \lesssim \varepsilon, \ \text{in } \{|x| \geq 1\} \cap \{-2 < u < 2\}. \tag{4.53}$$

The proof of Lemma 4.6 is postponed to the end of the section. We now conclude the proof of Theorem 2.4 by showing how to glue u and $x.\omega$ together in $\{1 \leq |x| \leq 2\}$. Let φ a smooth function with compact support which is equal to 1 on $|x| \leq 1$ and to 0 on $|x| \geq 2$. Let $\tilde{u}$ be defined on Σ by:

$$\tilde{u} = \varphi u + (1 - \varphi) x.\omega. \tag{4.54}$$

Then, $\tilde{u}$ satisfies **C1a**. Also, since u satisfies (4.9) and (4.10) in $\{-2 < u < 2\}$, since $x.\omega$ satisfies the same estimates in $|x| \geq 1$, and since we have (4.53) on $1 \leq |x| \leq 2$, $\tilde{u}$ satisfies (2.30) on Σ. This concludes the proof of Theorem 2.4.

Proof of Lemma 4.6. — We first show that $u(.,\omega)$ satisfies better estimates in this region due to the hypoellipticity of the parabolic-elliptic system (4.5). In particular, we obtain the following improvement of (4.47) for $j=2$:

$$\|\nabla\!\!\!/^2\nabla a\|_{L^2(S)} + \|\nabla^2 a\|_{L^2(S)} + \|\nabla^2\theta\|_{L^2(S)} \lesssim \varepsilon, \text{ in } \{|x|\geq 1\}\cap\{-2<u<2\}. \tag{4.55}$$

In fact, $C(M)$ in (4.47) comes from the assumption (4.46) on the norms of R and k. However, since R and k vanish in $|x|\geq 1$, we may take $M=0$ in this region. Let us prove for example the estimate for $\|\nabla\!\!\!/^2\nabla_N a\|_{L^2(S)}$ in (4.55), the others being similar. Let φ a smooth function with compact support which is equal to 1 on $|x|\leq 1$. Using (A.1), we obtain an equation for $(1-\varphi)\nabla_N a$:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)[(1-\varphi)\nabla_N a] = (1-\varphi)h + \tilde{h}, \tag{4.56}$$

where h is given by (A.2) and $\tilde{h}$ is given by:

$$\tilde{h} = -\nabla_N\varphi\nabla_N a + a^{-1}\Delta\!\!\!/\varphi\nabla_N a + 2a^{-1}\nabla\!\!\!/\varphi\nabla\!\!\!/\nabla_N a. \tag{4.57}$$

(A.5) and the fact that R and k vanish on the support of $1-\varphi$ yield:

$$\|(1-\varphi)h\|_{L^2(S)} \lesssim \varepsilon\|(1-\varphi)\nabla^2\theta\|_{L^2(S)} + \varepsilon. \tag{4.58}$$

(4.9) and the fact that φ is smooth yields:

$$\|\tilde{h}\|_{L^2(S)} \lesssim \varepsilon. \tag{4.59}$$

Proposition 3.16, (4.56), (4.58) and (4.59) yield:

$$\|(1-\varphi)\nabla_N^2 a\|_{L^2(S)} + \|(1-\varphi)\nabla\!\!\!/^2\nabla_N a\|_{L^2(S)} \lesssim \sqrt{\varepsilon}\|(1-\varphi)\nabla^2\theta\|_{L^2(S)} + \varepsilon. \tag{4.60}$$

In the same fashion, we adapt the analysis of (A.7)-(A.24) and we use the fact that R and k vanish on the support of $1-\varphi$ to obtain estimates for $\|(1-\varphi)\nabla\!\!\!/^3 a\|_{L^2(S)}$ and $\|(1-\varphi)\nabla^2\theta\|_{L^2(S)}$ which yield (4.55).

We now use (4.55) and the fact that $u=-2$ on $x.\omega=-2$ to show that u and $x.\omega$ are close to each other in the region $\{|x|\geq 1\}\cap\{-2<u<2\}$. Proposition 3.10, (4.4) and (4.55) yield:

$$|\nabla N| \lesssim \varepsilon, \text{ in } \{|x|\geq 1\}\cap\{-2<u<2\}. \tag{4.61}$$

Since $N=\omega$ on $x.\omega=-2$, (4.61) yields:

$$|N-\omega| \lesssim \varepsilon, \text{ in } \{|x|\geq 1\}\cap\{-2<u<2\}. \tag{4.62}$$

We have $u=x.\omega$ on $x.\omega=-2$, so since $\nabla u = a^{-1}N$, (4.9) and (4.62) yield the desired estimate (4.53). This concludes the proof of Lemma 4.6. ☐

CHAPTER 5

LITTLEWOOD-PALEY THEORY ON P_u AND CONSEQUENCES

In this chapter, we introduce several tools which will be needed to prove Theorem 2.5 and Theorem 2.6. We introduce and recall the main properties of the family of intrinsic Littlewood-Paley projections P_j which has been constructed in [**15**] using the heat flow on the surfaces P_u. We then prove a crucial bound for K. This allows us to derive suitable commutator estimates, product estimates and estimates for parabolic equations.

Remark 5.1. — Recall that (Σ, g, k) coincides with $(\mathbb{R}^3, \delta, 0)$ in $|x| \geq 2$. Also, $u(x, \omega)$ coincides with $x.\omega$ in $|x| \geq 2$, and so $a \equiv 1$, $N \equiv \omega$, $\theta \equiv 0$ and $K \equiv 0$ in this region. Therefore, u clearly satisfies the estimates of Theorem 2.5, Theorem 2.6 and of the propositions thereafter in the region $|x| \geq 2$. Thus, in the rest of the paper, we will restrict the proof all our estimates in the strip $S = \{x/ -2 < u < 2\}$ where $u(x, \omega)$ is solution to:

$$\begin{cases} \operatorname{tr}\theta - k_{NN} = 1 - a, \text{ on } -2 < u < 2, \\ \qquad u(., \omega) = -2 \text{ on } x.\omega = -2. \end{cases}$$

5.1. Properties of the geometric Littlewood-Paley projections P_j

In this section, we introduce and recall the main properties of the family of intrinsic Littlewood-Paley projections P_j which has been constructed in [**15**] using the heat flow on the surfaces P_u. We recall the properties of the heat equation for arbitrary tensorfields F on P_u.

$$\partial_\tau U(\tau)F - \Delta\!\!\!/ U(\tau)F = 0, \; U(0)F = F.$$

The following L^2 estimates for the operator $U(\tau)$ are proved in [**15**].

Proposition 5.2. — *We have the following estimates for the operator $U(\tau)$:*

$$\|U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_u)} d\tau' \lesssim \|F\|^2_{L^2(S)}, \tag{5.1}$$

$$\|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau \|\Delta\!\!\!/ U(\tau')F\|^2_{L^2(P_u)} d\tau' \lesssim \|\nabla\!\!\!/ F\|^2_{L^2(S)}, \tag{5.2}$$

$$\tau\|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau \tau'\|\Delta\!\!\!/ U(\tau')F\|^2_{L^2(P_u)}d\tau' \lesssim \|F\|^2_{L^2(S)}. \tag{5.3}$$

We also introduce the nonhomogeneous heat equation:

$$\partial_\tau V(\tau) - \Delta\!\!\!/ V(\tau) = F(\tau),\ V(0) = 0,$$

for which we easily derive the following estimates:

Proposition 5.3. — *Let $\beta > 0$. We have the following estimates for the operator $V(\tau)$:*

$$\|\nabla\!\!\!/ V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\Delta\!\!\!/ V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau \|F(\tau')\|^2_{L^2(P_u)}d\tau', \tag{5.4}$$

$$\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau\int_{P_u} V(\tau')F(\tau')d\mu_u d\tau', \tag{5.5}$$

$$\tau\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \tau'\|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau\int_{P_u} \tau' V(\tau')F(\tau')d\mu_u d\tau', \tag{5.6}$$

$$\begin{aligned}\tau^{2\beta}\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau {\tau'}^{2\beta}\|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_u)}d\tau' &\lesssim \int_0^\tau\int_{P_u} {\tau'}^{2\beta} V(\tau')F(\tau')d\mu_u d\tau',\\ &+ \int_0^\tau {\tau'}^{2\beta-1}\|V(\tau')\|^2_{L^2(P_u)}d\tau'.\end{aligned} \tag{5.7}$$

We now recall the definition of the geometric Littlewood-Paley projections P_j constructed in [**15**]:

Definition 5.4. — *Consider a smooth function m on $[0,\infty)$, vanishing sufficiently fast at ∞, verifying the vanishing moments property:*

$$\int_0^\infty \tau^{k_1}\partial_\tau^{k_2} m(\tau)d\tau = 0, \quad |k_1| + |k_2| \le N. \tag{5.8}$$

We set, $m_j(\tau) = 2^{2j}m(2^{2j}\tau)$ and define the geometric Littlewood-Paley (LP) projections P_j, for arbitrary tensorfields F on S to be

$$P_jF = \int_0^\infty m_j(\tau)U(\tau)Fd\tau. \tag{5.9}$$

Given an interval $I \subset \mathbb{Z}$ we define

$$P_I = \sum_{j\in I} P_jF.$$

In particular we shall use the notation $P_{<k}, P_{\le k}, P_{>k}, P_{\ge k}$.

Observe that P_j are selfadjoint, i.e., $P_j = P_j^*$, in the sense,

$$<P_jF, G> = <F, P_jG>,$$

where, for any given m-tensors F, G

$$<F, G> = \int_{P_u} \gamma^{i_1 j_1} \dots \gamma^{i_m j_m} F_{i_1 \dots i_m} G_{j_1 \dots j_m} d\mu_u$$

denotes the usual L^2 scalar product. Recall also from **[15]** that there exists a function m satisfying (5.8) such that the LP-projections associated to m verify:

$$\sum_j P_j = I. \tag{5.10}$$

The following properties of the LP-projections P_j have been proved in **[15]**:

Theorem 5.5. — *The LP-projections P_j verify the following properties:*

i) L^p-boundedness *For any $1 \le p \le \infty$, and any interval $I \subset \mathbb{Z}$,*

$$\|P_I F\|_{L^p(P_u)} \lesssim \|F\|_{L^p(P_u)}. \tag{5.11}$$

ii) Bessel inequality

$$\sum_j \|P_j F\|^2_{L^2(P_u)} \lesssim \|F\|^2_{L^2(P_u)}.$$

iii) Finite band property *For any $1 \le p \le \infty$.*

$$\begin{aligned} \|\Delta\!\!\!/ P_j F\|_{L^p(P_u)} &\lesssim 2^{2j} \|F\|_{L^p(P_u)} \\ \|P_j F\|_{L^p(P_u)} &\lesssim 2^{-2j} \|\Delta\!\!\!/ F\|_{L^p(P_u)}. \end{aligned} \tag{5.12}$$

In addition, the L^2 estimates

$$\begin{aligned} \|\nabla\!\!\!\!/ P_j F\|_{L^2(P_u)} &\lesssim 2^{j} \|F\|_{L^2(P_u)} \\ \|P_j F\|_{L^2(P_u)} &\lesssim 2^{-j} \|\nabla\!\!\!\!/ F\|_{L^2(P_u)}, \end{aligned} \tag{5.13}$$

hold together with the dual estimate

$$\|P_j \nabla\!\!\!\!/ F\|_{L^2(P_u)} \lesssim 2^{j} \|F\|_{L^2(P_u)}.$$

iv) Weak Bernstein inequality *For any $2 \le p < \infty$*

$$\begin{aligned} &\|P_j F\|_{L^p(P_u)} \lesssim (2^{(1-\frac{2}{p})j} + 1) \|F\|_{L^2(P_u)}, \\ &\|P_{<0} F\|_{L^p(P_u)} \lesssim \|F\|_{L^2(P_u)}, \end{aligned}$$

together with the dual estimates

$$\begin{aligned} &\|P_j F\|_{L^2(P_u)} \lesssim (2^{(1-\frac{2}{p})j} + 1) \|F\|_{L^{p'}(P_u)}, \\ &\|P_{<0} F\|_{L^2(P_u)} \lesssim \|F\|_{L^{p'}(P_u)}. \end{aligned}$$

We use the Littlewood-Paley projections P_j to define Sobolev spaces $H^b(P_u)$.

Definition 5.6. — *Let $b \in \mathbb{R}$. Then, we define the Sobolev space $H^b(P_u)$ as follows:*

$$\|F\|^2_{H^b(P_u)} = \sum_{j \ge 0} 2^{2jb} \|P_j F\|^2_{L^2(P_u)} + \|P_{<0} F\|^2_{L^2(P_u)}.$$

Let us state a lemma about the action of $\nabla\!\!\!\!/$ on $H^b(P_u)$.

Lemma 5.7. — *Let $0 < b < 1$. Let F a tensor on P_u such that $F \in H^b(P_u)$. Then, $\nabla\!\!\!/ F \in H^{b-1}(P_u)$.*

Proof. — We have:

$$\|P_j \nabla\!\!\!/ F\|_{L^2(P_u)} \lesssim \sum_{l\geq 0} \|P_j \nabla\!\!\!/ P_l F\|_{L^2(P_u)}. \tag{5.14}$$

If $l \leq j$, we use the boundedness of P_j on $L^2(P_u)$ and the finite band property for P_l to obtain:

$$\begin{aligned} 2^{j(b-1)}\|P_j \nabla\!\!\!/ P_l F\|_{L^2(P_u)} &\lesssim 2^{j(b-1)}\|\nabla\!\!\!/ P_l F\|_{L^2(P_u)} \\ &\lesssim 2^{j(b-1)}2^l\|P_l F\|_{L^2(P_u)} \\ &\lesssim 2^{-|j-l|(1-b)}2^{bl}\|P_l F\|_{L^2(P_u)}, \end{aligned} \tag{5.15}$$

where we used in the last inequality the fact that $l \leq j$ and $b < 1$.

If $l > j$, we use the finite band property for P_j to obtain:

$$\begin{aligned} 2^{j(b-1)}\|P_j \nabla\!\!\!/ P_l F\|_{L^2(P_u)} &\lesssim 2^{j(b-1)}2^j\|P_l F\|_{L^2(P_u)} \\ &\lesssim 2^{-|j-l|b}2^{bl}\|P_l F\|_{L^2(P_u)}, \end{aligned} \tag{5.16}$$

where we used in the last inequality the fact that $l > j$ and $b > 0$. Finally, (5.14), (5.15) and (5.16) imply:

$$\begin{aligned} \sum_{j\geq 0} 2^{2(b-1)j}\|P_j \nabla\!\!\!/ F\|^2_{L^2(P_u)} &\lesssim \sum_{j\geq 0}\left(\sum_{l\geq 0} 2^{-\min(b,1-b)|j-l|}2^{lb}\|P_l F\|_{L^2(P_u)}\right)^2 \\ &\lesssim \sum_{l\geq 0} 2^{2lb}\|P_l F\|^2_{L^2(P_u)} \\ &\lesssim \|F\|^2_{H^b(P_u)}, \end{aligned}$$

where we used the fact that $\min(b, 1-b) > 0$. This concludes the proof of the lemma. □

We also recall the definition of the negative fractional powers of $\Lambda^2 = I - \Delta\!\!\!/$ on any smooth tensorfield F on P_u used in [**15**].

$$\Lambda^\alpha F = \frac{1}{\Gamma(-\alpha/2)}\int_0^\infty \tau^{-\frac{\alpha}{2}-1}e^{-\tau}U(\tau)F d\tau, \tag{5.17}$$

where α is an arbitrary complex number with $\Re(\alpha) < 0$ and Γ denotes the Gamma function. We extend the definition of fractional powers of Λ to the range of α with $\Re(\alpha) > 0$, on smooth tensorfields F, by defining first

$$\Lambda^\alpha F = \Lambda^{\alpha-2}\cdot(I - \Delta\!\!\!/)F$$

for $0 < \Re(\alpha) \leq 2$ and then, in general, for $0 < \Re(\alpha) \leq 2n$, with an arbitrary positive integer n, according to the formula

$$\Lambda^\alpha F = \Lambda^{\alpha-2n}\cdot(I - \Delta\!\!\!/)^n F.$$

With this definition, Λ^α is symmetric and verifies the group property $\Lambda^\alpha\Lambda^\beta = \Lambda^{\alpha+\beta}$. We also have by standard complex interpolation the following inequality:

$$\|\Lambda^{\mu\alpha+(1-\mu)\beta}F\|_{L^2(P_u)} \lesssim \|\Lambda^\alpha F\|^\mu_{L^2(P_u)}\|\Lambda^\beta F\|^{1-\mu}_{L^2(P_u)}. \tag{5.18}$$

Using the operators Λ^α, we complete (5.1)-(5.3) with:

$$\|\Lambda^{-1}U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau \tau'\|\nabla\!\!\!/\Lambda^{-1}U(\tau')F\|^2_{L^2(P_u)}d\tau' \lesssim \|\Lambda^{-1}F\|^2_{L^2(S)}, \tag{5.19}$$

$$\tau\|U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau \tau'\|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_u)}d\tau' \lesssim \|\Lambda^{-1}F\|^2_{L^2(S)}, \tag{5.20}$$

for $\alpha \in \mathbb{R}$,

$$\|\Lambda^\alpha V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^\alpha V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau\int_{P_u}\Lambda^{2\alpha}V(\tau')F(\tau')d\mu_u d\tau' \tag{5.21}$$

and for $0 < \eta < \delta < 1$:

$$\tau^{1+\delta}\|\nabla\!\!\!/ U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau {\tau'}^{1+\delta}\|\Delta\!\!\!/ U(\tau')F\|^2_{L^2(P_u)}d\tau' \lesssim \|\Lambda^{-\eta}F\|^2_{L^2(S)}, \tag{5.22}$$

$$\tau^{1+\delta}\|U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau {\tau'}^{1+\delta}\|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_u)}d\tau' \lesssim \|\Lambda^{-1-\eta}F\|^2_{L^2(S)}, \tag{5.23}$$

$$\tau^{\delta}\|U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau {\tau'}^{\delta}\|\nabla\!\!\!/ U(\tau')F\|^2_{L^2(P_u)}d\tau' \lesssim \|\Lambda^{-\eta}F\|^2_{L^2(S)}, \tag{5.24}$$

$$\tau^{\delta}\|\Lambda^{-1}U(\tau)F\|^2_{L^2(P_u)} + \int_0^\tau {\tau'}^{\delta}\|U(\tau')F\|^2_{L^2(P_u)}d\tau' \lesssim \|\Lambda^{-1-\eta}F\|^2_{L^2(S)}. \tag{5.25}$$

We now investigate the boundedness of $\Lambda^{-\alpha}$ on $L^p(P_u)$ spaces for $0 \leq \alpha \leq 1$. For any tensor F on P_u and any $\alpha \in \mathbb{R}$, integrating by parts and using the definition of Λ, we get:

(5.26)
$$\begin{aligned}\|\Lambda^\alpha F\|^2_{L^2(P_u)} + \|\nabla\!\!\!/\Lambda^\alpha F\|^2_{L^2(P_u)} &= \int_{P_u}\Lambda^\alpha F\Lambda^\alpha F d\mu_u + \int_{P_u}\nabla\!\!\!/\Lambda^\alpha F\nabla\!\!\!/\Lambda^\alpha F d\mu_u\\ &= \int_{P_u}(1-\Delta\!\!\!/)\Lambda^\alpha F\Lambda^\alpha F d\mu_u = \int_{P_u}\Lambda^2\Lambda^\alpha F\Lambda^\alpha F d\mu_u\\ &= \|\Lambda^{\alpha+1}F\|^2_{L^2(P_u)}.\end{aligned}$$

Taking $\alpha = -1$ in (5.26), we obtain:

$$\|\nabla\!\!\!/\Lambda^{-1}F\|_{L^2(P_u)} \lesssim \|F\|_{L^2(P_u)}. \tag{5.27}$$

Below, we deduce several estimates from (5.27). Taking the adjoint of (5.27), we obtain for any vector field F:

$$\|\Lambda^{-1}\mathrm{d}\!\!\!/\mathrm{iv} F\|_{L^2(P_u)} \lesssim \|F\|_{L^2(P_u)}. \tag{5.28}$$

Also, (3.9) and (5.27) imply for any tensor F on P_u:

$$\|\Lambda^{-1}F\|_{L^p(P_u)} \lesssim \|F\|_{L^2(P_u)} \text{ for all } 2 \leq p < +\infty. \tag{5.29}$$

Taking the adjoint of (5.29) yields:

$$\|\Lambda^{-1}F\|_{L^2(P_u)} \lesssim \|F\|_{L^p(P_u)} \text{ for all } 1 < p \leq 2. \tag{5.30}$$

Interpolating between the identity and Λ^{-1}, we deduce form (5.30):

$$\|\Lambda^{-\alpha}F\|_{L^2(P_u)} \lesssim \|F\|_{L^p(P_u)} \text{ for all } 0 < \alpha < 1,\ \frac{2}{1+\alpha} < p \leq 2. \tag{5.31}$$

Finally, we conclude this section by recalling the sharp Bernstein inequality for scalars obtained in [15]. It is derived under the additional assumption that the Christoffel symbols Γ^A_{BC} of the coordinate system (3.7) on P_u verify:

$$\sum_{A,B,C}\int_U |\Gamma^A_{BC}|^2 dx^1 dx^2 \leq c^{-1}, \tag{5.32}$$

with a constant $c > 0$ independent of u and where U is a coordinate chart.

Remark 5.8. — The existence of a covering of P_u by coordinate charts satisfying (3.7) and (5.32) with a constant $c > 0$ and the number of charts independent of u will be established in Proposition 8.1.

Let $0 \leq \gamma < 1$, and let K_γ be defined by:

$$K_\gamma := \|\Lambda^{-\gamma}K\|_{L^2(P_u)}. \tag{5.33}$$

Then, we have the following sharp Bernstein inequality for any scalar function f on P_u, $0 \leq \gamma < 1$, any $j \geq 0$, and an arbitrary $2 \leq p < \infty$ (see [15]):

$$\|P_j f\|_{L^\infty(P_u)} \lesssim 2^j\Big(1 + 2^{-\frac{j}{p}}\big(K_\gamma^{\frac{1}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big) + 1\Big)\|f\|_{L^2(P_u)}, \tag{5.34}$$

$$\|P_{<0} f\|_{L^\infty(P_u)} \lesssim \big(1 + K_\gamma^{\frac{2}{p(1-\gamma)}} + K_\gamma^{\frac{1}{2p}}\big)\|f\|_{L^2(P_u)}. \tag{5.35}$$

Also, the Bochner identity (3.26) together with the properties of Λ implies the following inequality (see [15]):

$$\int_{P_u} |\nabla\!\!\!/^2 f|^2 \lesssim \int_{P_u} |\Delta\!\!\!/ f|^2 + \big(K_\gamma^{\frac{2}{1-\gamma}} + K_\gamma\big)\int_{P_u} |\nabla\!\!\!/ f|^2. \tag{5.36}$$

Thus, we need to bound K_γ in order to be able to use (5.34), (5.35), and (5.36). For $\Re(\alpha) < 0$, we will use the fact that for any tensor F on P_u:

$$\|\Lambda^{-\alpha}F\|^2_{L^2(P_u)} \lesssim \|P_{<0}F\|^2_{L^2(P_u)} + \sum_{j=0}^{+\infty} 2^{-2\alpha j}\|P_j F\|^2_{L^2(P_u)}. \tag{5.37}$$

which follows from the methods in [15]. Therefore, we would like to control K in $L^\infty_u H^{-\alpha}(P_u)$ for some $\alpha < 1$. This is the goal of the next section.

5.2. Control of K in $L^\infty_u H^{-\frac{1}{2}}(P_u)$

The goal of this section is to prove the following estimate.

Proposition 5.9. — *Let (Σ, g, k) chosen as in Section 2.1. Let u the scalar function on $\Sigma\times\mathbb{S}^2$ constructed in Theorem 2.4, and let P_u, N, θ and K be associated to u as in Section 2.2. We have:*

$$\sum_{j\geq 0} 2^{-j}\|P_jK\|^2_{L^\infty_u L^2(P_u)} + \|P_{<0}K\|^2_{L^\infty_u L^2(P_u)} \lesssim \varepsilon^2. \tag{5.38}$$

Proof. — Recall from (4.9) that:

$$\|K\|_{L^2(S)} \lesssim \varepsilon. \tag{5.39}$$

Also, (2.25) and (4.6) yield:

$$\nabla_N K = \operatorname{tr}\theta\nabla_N \operatorname{tr}\theta - \theta\nabla_N\theta + k\nabla_N k - \nabla_N R_{NN},$$

which together with (4.35) implies:

$$\nabla_N K = \operatorname{d\!\!/iv}(B) + b,$$

where

$$B = R_{N.},\ b = \operatorname{tr}\theta\nabla_N\operatorname{tr}\theta - \theta\nabla_N\theta + R(\nabla\!\!\!/ a, N).$$

Multiplying by a, this implies:

$$\nabla_{aN}K = \operatorname{d\!\!/iv}(B_1) + b_1, \tag{5.40}$$

where

$$B_1 = aR_{N.},\ b_1 = -\nabla\!\!\!/ a\cdot B + a\operatorname{tr}\theta\nabla_N\operatorname{tr}\theta - a\theta\nabla_N\theta + aR(\nabla\!\!\!/ a, N). \tag{5.41}$$

Using (4.3), (4.10), (4.11) and (4.12), we obtain:

$$\|B_1\|_{L^2(S)} \leq \varepsilon \tag{5.42}$$

and

$$\begin{aligned}\|b_1\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} &\lesssim \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N\theta\|_{L^2(S)}\\ &\quad + (\|B_1\|_{L^2(S)}\|R\|_{L^2(S)})\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\\ &\lesssim \varepsilon.\end{aligned} \tag{5.43}$$

In particular, (5.28), (5.30), (5.40), (5.42) and (5.43) yield:

$$\|\Lambda^{-1}\nabla_N K\|_{L^2(S)} \lesssim \varepsilon. \tag{5.44}$$

We may assume the existence of $\widetilde{P_j}$ with the same properties than P_j such that $P_j = \widetilde{P_j}^2$ (see [**15**]), and for simplicity we write $P_j = P_j^2$. Also, using the fact that $\Lambda\Lambda^{-1} = I$ and that Λ commutes with P_j, we obtain:

$$P_j\nabla_{aN}K = \Lambda P_j(P_j\Lambda^{-1}\nabla_{aN}K),$$

which together with property (iii) of Theorem 5.5 yields:

$$\|P_j\nabla_{aN}K\|_{L^2(S)} \lesssim \|\Lambda P_j(P_j\Lambda^{-1}\nabla_{aN}K)\|_{L^2(S)} \lesssim 2^j\|P_j\Lambda^{-1}\nabla_{aN}K\|_{L^2(S)}.$$

Using property (ii) of Theorem 5.5, we get:

$$\sum_{j\geq 0} 2^{-2j}\|P_j\nabla_{aN}K\|^2_{L^2(S)} \lesssim \sum_{j\geq 0}\|P_j\Lambda^{-1}\nabla_{aN}K\|^2_{L^2(S)} \lesssim \|\Lambda^{-1}\nabla_{aN}K\|^2_{L^2(S)}.$$

Together with (5.44), we finally obtain:

$$\sum_{j\geq 0} 2^{-2j}\|P_j\nabla_{aN}K\|^2_{L^2(S)} \lesssim \varepsilon^2. \tag{5.45}$$

To prove Proposition 5.9, we assume:

$$\sum_{j\geq 0} 2^{-j}\|P_jK\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + \|P_{<0}K\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim D^2\varepsilon^2, \tag{5.46}$$

where D is a large enough constant. We will then try to improve (5.46). Note that (5.36), (5.37) and (5.46) yield for any scalar function f on P_u:

$$\|\nabla\!\!\!/^2 f\|^2_{L^2(P_u)} \lesssim \|\Delta\!\!\!/ f\|^2_{L^2(P_u)} + (D\varepsilon + D^4\varepsilon^4)\|\nabla\!\!\!/ f\|^2_{L^2(P_u)}. \tag{5.47}$$

The term $\|P_{<0}K\|_{L^\infty_{[-2,2]}L^2(P_u)}$ is easier to bound, so we concentrate on estimating the sum $\sum_{j\geq 0} 2^{-j}\|P_jK\|_{L^2(S)}$. We will use the following variant of (3.15) where we do not yet use Cauchy-Schwarz in u for the integral containing $\nabla_N F$:

$$\begin{aligned}\|F\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|F(-2,.)\|^2_{L^2(P_{-2})} + \int_{-2}^{2}\|\nabla_N F\|_{L^2(P_u)}\|F\|_{L^2(P_u)}du \\ + \|\nabla\!\!\!/ F\|_{L^2(S)}\|F\|_{L^2(S)}.\end{aligned} \tag{5.48}$$

Using (5.48), the fact that $P_jK \equiv 0$ on $u=-2$, and properties (ii) and (iii) of Theorem 5.5, we have:

(5.49)

$$\begin{aligned}&\sum_{j\geq 0} 2^{-j}\|P_jK\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\\
&\lesssim \sum_{j\geq 0} 2^{-j}\left(\int_{-2}^{2}\|P_jK\|_{L^2(P_u)}\|\nabla_N P_jK\|_{L^2(P_u)}du + \|P_jK\|_{L^2(S)}\|\nabla\!\!\!/ P_jK\|_{L^2(S)}\right)\\
&\lesssim \sum_{j\geq 0} 2^{-j}\left(\int_{-2}^{-2}\|P_jK\|_{L^2(P_u)}\|\nabla_N P_jK\|_{L^2(P_u)}du\right) + \sum_{j\geq 0}\|P_jK\|^2_{L^2(S)}\\
&\lesssim \sum_{j\geq 0} 2^{-j}\left(\int_{-2}^{-2}\|P_jK\|_{L^2(P_u)}\|\nabla_{aN} P_jK\|_{L^2(P_u)}du\right) + \varepsilon^2,\end{aligned}$$

where we used in the last inequality the estimate (4.9) for a and the estimate (5.39) for K. We inject the estimate:

$$\|\nabla_{aN}P_jK\|_{L^2(P_u)} \lesssim \|P_j\nabla_{aN}K\|_{L^2(P_u)} + \|[\nabla_{aN}, P_j]K\|_{L^2(P_u)}$$

in (5.49). We obtain:

$$\sum_{j\geq 0} 2^{-j}\|P_jK\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \sum_{j\geq 0}(\|P_jK\|^2_{L^2(S)} + 2^{-2j}\|P_j\nabla_{aN}K\|^2_{L^2(S)})$$

$$+\sum_{j\geq 0}2^{-j}\|P_jK\|_{L^\infty_{[-2,2]}L^2(P_u)}\|[\nabla_{aN},P_j]K\|_{L^1_{[-2,2]}L^2(P_u)}+\varepsilon^2,$$

which together with the estimates (5.39) and (5.45) for K implies:

$$\sum_{j\geq 0}2^{-j}\|P_jK\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim\sum_{j\geq 0}2^{-j}\|[\nabla_{aN},P_j]K\|^2_{L^1_{[-2,2]}L^2(P_u)}+\varepsilon^2. \tag{5.50}$$

Now, we will prove:

$$\|[\nabla_{aN},P_j]K\|_{L^1_{[-2,2]}L^2(P_u)}\lesssim 2^{\frac{j}{3}}(\varepsilon+D\varepsilon^2). \tag{5.51}$$

Together with (5.50), this yields:

$$\sum_{j\geq 0}2^{-j}\|P_jK\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim\varepsilon^2+\left(\sum_{j\geq 0}2^{-\frac{j}{3}}\right)(\varepsilon^2+D^2\varepsilon^4)\lesssim\varepsilon^2+D^2\varepsilon^4,$$

which is an improvement of (5.46). Thus we have:

$$\sum_{j\geq 0}2^{-j}\|P_jK\|^2_{L^\infty_{[-2,2]}L^2(P_u)}+\|P_{<0}K\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim\varepsilon^2,$$

which concludes the proof of Proposition 5.9 provided (5.51) holds.

In the rest of the proof, we focus on obtaining (5.51). We have:

$$[\nabla_{aN},P_j]K=\int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{5.52}$$

where $V(\tau)$ is satisfies:

$$(\partial_\tau-\not\Delta)V(\tau)=[\nabla_{aN},\not\Delta]U(\tau)K,\ V(0)=0. \tag{5.53}$$

In view of (5.52), we have:

$$\|[\nabla_{aN},P_j]K\|_{L^1_{[-2,2]}L^2(P_u)}\lesssim\int_0^\infty m_j(\tau)\|V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)}d\tau. \tag{5.54}$$

Now, using (5.18) and (5.26), we have:

$$\begin{aligned}
&\int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(P_u)}d\tau\lesssim\int_0^\infty m_j(\tau)\|\Lambda^{-\frac13}V(\tau)\|^{\frac23}_{L^2(P_u)}\|\not\nabla\Lambda^{-\frac13}V(\tau)\|^{\frac13}_{L^2(P_u)}d\tau\\
&\lesssim\left(\int_0^\infty m_j(\tau)\|\Lambda^{-\frac13}V(\tau)\|_{L^2(P_u)}d\tau\right)^{\frac23}\left(\int_0^\infty\|\not\nabla\Lambda^{-\frac13}V(\tau)\|^2_{L^2(P_u)}d\tau\right)^{\frac16}\\
&\quad\times\left(\int_0^\infty m_j(\tau)^2d\tau\right)^{\frac16}\\
&\lesssim 2^{\frac{j}{3}}\left(\sup_\tau\|\Lambda^{-\frac13}V(\tau)\|_{L^2(P_u)}+\left(\int_0^\infty\|\not\nabla\Lambda^{-\frac13}V(\tau)\|^2_{L^2(P_u)}d\tau\right)^{\frac12}\right).
\end{aligned} \tag{5.55}$$

Integrating in u and using (5.54), we obtain:

$$\|[\nabla_{aN}, P_j]K\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^{\frac{j}{3}}\left(\sup_\tau\|\Lambda^{-\frac{1}{3}}V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)} + \int_{-2}^{2}\left(\int_0^\infty\|\nabla\!\!\!/\Lambda^{-\frac{1}{3}}V(\tau)\|^2_{L^2(P_u)}d\tau\right)^{\frac{1}{2}}du\right). \tag{5.56}$$

Now, we will prove:

$$\|\Lambda^{-\frac{1}{3}}V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)} + \int_{-2}^{2}\left(\int_0^\tau\|\nabla\!\!\!/\Lambda^{-\frac{1}{3}}V(\tau')\|^2_{L^2(P_u)}d\tau'\right)^{\frac{1}{2}}du \lesssim \varepsilon + \varepsilon^2 D. \tag{5.57}$$

Together with (5.56), this yields the wanted estimate (5.51).

In the rest of the proof, we focus on obtaining (5.57). In view of (5.53) and the heat flow estimate (5.21), we have:

$$\|\Lambda^{-\frac{1}{3}}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau\|\nabla\!\!\!/\Lambda^{-\frac{1}{3}}V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau\int_{P_u}\Lambda^{-\frac{2}{3}}V(\tau')[\nabla_{aN}, \Delta\!\!\!/]U(\tau')d\mu_u d\tau'.$$

Injecting the commutator Formula (2.23), integrating by parts, we obtain the following estimate:

$$\begin{aligned}\|\Lambda^{-\frac{1}{3}}V(\tau)\|^2_{L^2(P_u)} &+ \int_0^\tau\|\nabla\!\!\!/\Lambda^{-\frac{1}{3}}V(\tau')\|^2_{L^2(P_u)}d\tau' \\ &\lesssim (\|a\nabla\!\!\!/(\theta)\|_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|_{L^2(P_u)} + \|aR\|_{L^2(P_u)}) \\ &\quad\times\int_0^\tau\|\nabla\!\!\!/U(\tau')\|_{L^p(P_u)}\|\nabla\!\!\!/\Lambda^{-\frac{2}{3}}V(\tau')\|_{L^2(P_u)}d\tau',\end{aligned} \tag{5.58}$$

where

$$2 < p < 3.$$

Now, we have in view of (5.26) and (5.18):

$$\|\nabla\!\!\!/\Lambda^{-\frac{2}{3}}V(\tau')\|_{L^2(P_u)} \lesssim \|\Lambda^{-\frac{1}{3}}V(\tau')\|^{\frac{1}{3}}_{L^2(P_u)}\|\nabla\!\!\!/\Lambda^{-\frac{2}{3}}V(\tau')\|^{\frac{2}{3}}_{L^2(P_u)},$$

which together with (5.58) implies:

$$\begin{aligned}&\|\Lambda^{-\frac{1}{3}}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau\|\nabla\!\!\!/\Lambda^{-\frac{1}{3}}V(\tau')\|^2_{L^2(P_u)}d\tau' \\ &\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)})\int_0^\tau\tau'^{(\frac{1}{3})-}\|\nabla\!\!\!/U(\tau')\|^2_{L^p(P_u)}d\tau'.\end{aligned} \tag{5.59}$$

The Gagliardo-Nirenberg inequality (3.9) implies:

$$\begin{aligned}&\int_0^\tau\tau'^{(\frac{1}{3})-}\|\nabla\!\!\!/U(\tau')\|^2_{L^p(P_u)}d\tau' \\ &\lesssim \int_0^\tau\tau'^{(\frac{1}{3})-}\|\nabla\!\!\!/U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}\|\nabla\!\!\!/^2U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}d\tau'\end{aligned}$$

$$\lesssim \left(\int_0^\tau \|\nabla U(\tau')\|^2_{L^2(P_u)} d\tau'\right)^{\frac{2}{p}} \left(\int_0^\tau \tau' \|\nabla^2 U(\tau')\|^2_{L^2(P_u)} d\tau'\right)^{1-\frac{2}{p}},$$

where we used in the last inequality the fact that:

$$\left(\frac{1}{3}\right)_- - 1 + \frac{2}{p} > 0$$

in view of the restriction $p < 3$. Together with the Bochner inequality (5.47), we obtain:

$$\begin{aligned}&\int_0^\tau \tau'^{(\frac{1}{3})-} \|\nabla U(\tau')\|^2_{L^p(P_u)} d\tau' \\ &\quad \lesssim (1 + D\varepsilon + D^4\varepsilon^4)^{1-\frac{2}{p}} \left(\int_0^\tau \|\nabla U(\tau')\|^2_{L^2(P_u)} d\tau' + \int_0^\tau \tau' \|\Delta U(\tau')\|^2_{L^2(P_u)} d\tau'\right).\end{aligned}$$

Thus, we obtain in view of the heat flow estimates (5.1) and (5.3):

$$\int_0^\tau \tau'^{(\frac{1}{3})-} \|\nabla U(\tau')\|^2_{L^p(P_u)} d\tau' \lesssim \|K\|^2_{L^2(P_u)}.$$

Together with (5.59), this yields:

$$\begin{aligned}&\|\Lambda^{-\frac{1}{3}} V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla \Lambda^{-\frac{1}{3}} V(\tau')\|^2_{L^2(P_u)} d\tau' \\ &\lesssim (1 + D\varepsilon + D^4\varepsilon^4)^{1-\frac{2}{p}} (\|a\nabla(\theta)\|^2_{L^2(P_u)} + \|\nabla(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)}) \|K\|^2_{L^2(P_u)}.\end{aligned}$$

Integrating in u, this yields:

$$\begin{aligned}\|\Lambda^{-\frac{1}{3}} V(\tau)\|_{L^1_{[-2,2]} L^2(P_u)} + \int_{-2}^2 \left(\int_0^\tau \|\nabla \Lambda^{-\frac{1}{3}} V(\tau')\|^2_{L^2(P_u)} d\tau'\right)^{\frac{1}{2}} du \\ \lesssim (1 + D\varepsilon + D^4\varepsilon^4)^{\frac{1}{2}-\frac{1}{p}} (\|a\nabla(\theta)\|_{L^2(S)} + \|\nabla(a)\theta\|_{L^2(S)} \\ + \|aR\|_{L^2(S)}) \|K\|_{L^2(S)} \\ \lesssim (1 + D\varepsilon + D^4\varepsilon^4)^{\frac{1}{2}-\frac{1}{p}} \varepsilon^2,\end{aligned}$$

where we used in the last inequality the estimate (2.30) for a and θ, the smallness assumption (2.1) for R, and the estimate (5.39) for K. Now, since $2 < p < 3$, we obtain:

$$\|\Lambda^{-\frac{1}{3}} V(\tau)\|_{L^1_{[-2,2]} L^2(P_u)} + \int_{-2}^2 \left(\int_0^\tau \|\nabla \Lambda^{-\frac{1}{3}} V(\tau')\|^2_{L^2(P_u)} d\tau'\right)^{\frac{1}{2}} du \lesssim \varepsilon^2 + D^{\frac{2}{3}} \varepsilon^{\frac{8}{3}},$$

which implies (5.57). This concludes the proof of the proposition. □

Remark 5.10. — The following consequence of Proposition 5.9 will be useful in the next two sections. Proposition 5.9 and (5.37) with the choice $\alpha = 1/2$ imply:

(5.60) $$\|K_{\frac{1}{2}}\|_{L^\infty(-2,2)} = \|\Lambda^{-\frac{1}{2}} K\|_{L^\infty_{[-2,2]} L^2(P_u)} \lesssim \varepsilon,$$

where $K_{1/2}$ has been defined in (5.33). Together with (5.34) and (5.35) with the choice $\gamma = 1/2$, we obtain for any scalar function f on P_u and any $j \geq 0$:

$$\|P_j f\|_{L^\infty(P_u)} \lesssim 2^j \|f\|_{L^2(P_u)}, \tag{5.61}$$

$$\|P_{<0} f\|_{L^\infty(P_u)} \lesssim \|f\|_{L^2(P_u)}. \tag{5.62}$$

Also, (5.60) and (5.36) with the choice $\gamma = 1/2$ imply:

$$\int_{P_u} |\nabla\!\!\!/^2 f|^2 \lesssim \int_{P_u} |\Delta\!\!\!/ f|^2 + \varepsilon \int_{P_u} |\nabla\!\!\!/ f|^2. \tag{5.63}$$

Using the Bochner inequality (5.63), we may prove the following lemma.

Lemma 5.11. — *For any 1-form F on P_u, for any $1 < p \leq 2$ and for all $j \geq 0$, we have:*

$$\|P_j \mathrm{d}\!\!\!/\mathrm{iv}(F)\|_{L^2(P_u)} \lesssim 2^{\frac{2}{p}j} \|F\|_{L^p(P_u)}. \tag{5.64}$$

Proof. — By duality, it suffices to prove for any scalar function f on P_u, for any $2 \leq p < +\infty$ and for all $j \geq 0$ the following inequality:

$$\|\nabla\!\!\!/ P_j f\|_{L^p(P_u)} \lesssim 2^{2(1-\frac{1}{p})j} \|f\|_{L^2(P_u)}. \tag{5.65}$$

Now, using the Gagliardo-Nirenberg inequality (3.9), the Bochner inequality for scalar functions (5.63), and the property iii) of Theorem 5.5 for Littlewood-Paley projections, we have:

$$\begin{aligned}
\|\nabla\!\!\!/ P_j f\|_{L^p(P_u)} &\lesssim \|\nabla\!\!\!/^2 P_j f\|_{L^2(P_u)}^{1-\frac{2}{p}} \|\nabla\!\!\!/ P_j f\|_{L^2(P_u)}^{\frac{2}{p}} \\
&\lesssim \left(\|\Delta\!\!\!/ P_j f\|_{L^2(P_u)} + \|\nabla\!\!\!/ P_j f\|_{L^2(P_u)}\right)^{1-\frac{2}{p}} \|\nabla\!\!\!/ P_j f\|_{L^2(P_u)}^{\frac{2}{p}} \\
&\lesssim 2^{2j(1-\frac{1}{p})} \|f\|_{L^2(P_u)},
\end{aligned}$$

which is (5.65). This concludes the proof of Lemma 5.11. □

Let us state another consequence of the Bochner inequality (5.63).

Lemma 5.12. — *Let $0 < b < 2$. Let f a scalar on P_u such that $f \in H^b(P_u)$. Then, $\nabla\!\!\!/ f \in H^{b-1}(P_u)$.*

Proof. — We have:

$$\|P_j \nabla\!\!\!/ f\|_{L^2(P_u)} \lesssim \sum_{l \geq 0} \|P_j \nabla\!\!\!/ P_l f\|_{L^2(P_u)}. \tag{5.66}$$

If $l \leq j$, we use the finite band property of P_j and P_l and the Bochner inequality (5.63) for scalars to obtain:

$$\begin{aligned}
2^{j(b-1)} \|P_j \nabla\!\!\!/ P_l f\|_{L^2(P_u)} &\lesssim 2^{j(b-2)} \|\nabla\!\!\!/^2 P_l f\|_{L^2(P_u)} \\
&\lesssim 2^{j(b-2)} 2^{2l} \|P_l f\|_{L^2(P_u)} \\
&\lesssim 2^{-|j-l|(2-b)} 2^{bl} \|P_l f\|_{L^2(P_u)},
\end{aligned} \tag{5.67}$$

where we used in the last inequality the fact that $l \leq j$ and $b < 2$.

If $l > j$, we use the finite band property for P_j to obtain:

$$2^{j(b-1)}\|P_j \nabla\!\!\!/ P_l f\|_{L^2(P_u)} \lesssim 2^{jb}\|P_l f\|_{L^2(P_u)} \tag{5.68}$$
$$\lesssim 2^{-|j-l|b}2^{bl}\|P_l f\|_{L^2(P_u)},$$

where we used in the last inequality the fact that $l > j$ and $b > 0$. Finally, (5.66), (5.67) and (5.68) imply:

$$\sum_{j\geq 0} 2^{2(b-1)j}\|P_j \nabla\!\!\!/ f\|^2_{L^2(P_u)} \lesssim \sum_{j\geq 0}\left(\sum_{l\geq 0} 2^{-\min(b,2-b)|j-l|}2^{lb}\|P_l f\|_{L^2(P_u)}\right)^2$$
$$\lesssim \sum_{l\geq 0} 2^{2lb}\|P_l f\|^2_{L^2(P_u)}$$
$$\lesssim \|f\|^2_{H^b(P_u)},$$

where we used the fact that $\min(b, 2-b) > 0$. This concludes the proof of the lemma. □

Finally, the bound (5.60) allows us to prove the following Hodge inequality.

Lemma 5.13. — *Let F a symmetric 2-tensor such that $trF = 0$. Then:*

$$\|\nabla\!\!\!/ F\|_{L^2(P_u)} \lesssim \|\mathrm{d}\!\!/\mathrm{iv} F\|_{L^2(P_u)} + \varepsilon\|F\|_{L^2(P_u)}. \tag{5.69}$$

Proof. — Recall the identity (3.48) for Hodge systems:

$$\int_{P_u} (|\nabla\!\!\!/ F|^2 + 2K|F|^2) = 2\int_{P_u} |\mathrm{d}\!\!/\mathrm{iv} F|^2. \tag{5.70}$$

We have:

$$\left|\int_{P_u} K|F|^2\right| \lesssim \|K\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)}\||F|^2\|_{H^{\frac{1}{2}}(P_u)}$$
$$\lesssim \varepsilon\||F|^2\|_{H^{\frac{1}{2}}(P_u)},$$

where we used the bound (5.60) in the last inequality. Together with (5.70), this implies:

$$\|\nabla\!\!\!/ F\|_{L^2(P_u)} \lesssim \|\mathrm{d}\!\!/\mathrm{iv} F\|_{L^2(P_u)} + \varepsilon^{\frac{1}{2}}\||F|^2\|^{\frac{1}{2}}_{H^{\frac{1}{2}}(P_u)}. \tag{5.71}$$

Next, we estimate the last term in the right-hand side of (5.71). We have:

$$P_j(|F|^2) = 2^{-2j}P_j \Delta\!\!\!/(|F|^2) = 2^{-2j}P_j \mathrm{d}\!\!/\mathrm{iv}(\nabla\!\!\!/(|F|^2)).$$

Together with (5.64), we obtain:

$$2^{\frac{j}{2}}\|P_j(|F|^2)\|_{L^2(P_u)} \lesssim 2^{\frac{j}{2}}2^{-2j}2^{\frac{4j}{3}}\|F\cdot \nabla\!\!\!/ F\|_{L^{\frac{3}{2}}(P_u)}$$
$$\lesssim 2^{-\frac{j}{6}}\|F\|_{L^6(P_u)}\|\nabla\!\!\!/ F\|_{L^2(P_u)}$$
$$\lesssim 2^{-\frac{j}{6}}\|F\|^{\frac{1}{3}}_{L^2(P_u)}\|\nabla\!\!\!/ F\|^{\frac{5}{3}}_{L^2(P_u)},$$

where we used in the last inequality the Gagliardo-Nirenberg estimate (3.9). This yields:

$$\||F|^2\|_{H^{\frac{1}{2}}(P_u)} \lesssim \|F\|_{L^2(P_u)}^{\frac{1}{3}}\|\nabla\!\!\!/ F\|_{L^2(P_u)}^{\frac{5}{3}}.$$

Together with (5.71), we obtain (5.69). This concludes the proof of the lemma. □

5.3. Estimates for the commutator $[\nabla_{aN}, P_j]$

In this section, we state several estimates for the commutator $[\nabla_{aN}, P_j]$. To simplify the exposition, the proof are postponed to Appendix B. The reason we prefer to consider $[\nabla_{aN}, P_j]$ instead of $[\nabla_{aN}, P_j]$ is because the former does not contain any N derivative in view of the commutator estimates (2.22) and (2.23). We start with a first commutator estimate.

Proposition 5.14. — *Let f a scalar function on S. Then, for any $j \geq 0$ and for any $\delta > 0$, we have the following commutator estimate:*

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim \varepsilon\|\Lambda^{\frac{1}{2}+\delta}f\|_{L^2(S)} + \varepsilon\|\Lambda^{\delta}f\|_{L^\infty_{[-2,2]}L^2(P_u)}. \tag{5.72}$$

We state a second commutator estimate.

Proposition 5.15. — *Let F a tensor on S. Then, for any $j \geq 0$ and for any $\delta > 0$, we have the following commutator estimate:*

$$\|[\nabla_{aN}, P_j]F\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^{-j(1-\delta)}\varepsilon\left(\|\nabla\!\!\!/ F\|_{L^2(S)} + \|F\|_{L^\infty_{[-2,2]}L^2(P_u)}\right). \tag{5.73}$$

Proposition 5.15 yields the following corollary.

Corollary 5.16. — *For any P_u-tangent tensor F on S such that $F \equiv 0$ on $u = -2$, and for all $j \geq 0$, we have:*

$$\|P_jF\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} \lesssim \|F\|_{H^1(S)}. \tag{5.74}$$

We state a third commutator estimate.

Proposition 5.17. — *Let f a scalar function on S. Then, for any $j \geq 0$ and for any $0 < \delta < \alpha < 1$, we have the following commutator estimate:*

$$\|[\nabla_{aN}, P_j]f\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^{j\alpha}\varepsilon\|\Lambda^{-\delta}f\|_{L^2(S)}. \tag{5.75}$$

We state a fourth commutator estimate.

Proposition 5.18. — *Let f a scalar function on S. Then, for any $j \geq 0$ and for any $\delta > 0$, we have the following commutator estimate:*

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim 2^{-(1-\delta)j}\varepsilon(\|\Delta\!\!\!/ f\|_{L^2(S)} + \|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)}). \tag{5.76}$$

Proposition 5.18 yields the following corollary.

Corollary 5.19. — *Let a tensor F on S such that $F \equiv 0$ on $u = -2$, $\nabla\!\!\!/^2 F \in L^2(S)$ and $\nabla_N F \in L^2_u H^b(P_u)$ for $b > 0$. Then, $F \in L^\infty(S)$.*

We state a fifth commutator estimate.

Proposition 5.20. — *Let f a scalar function on S. Then, for any $j \geq 0$ and for any $\delta > 0$, we have the following commutator estimate:*

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim 2^j \varepsilon \|\Lambda^{-(1-\delta)} f\|_{L^\infty_{[-2,2]} L^2(P_u)}. \tag{5.77}$$

5.4. Product estimates

In this section, we derive several product estimates. To simplify the exposition, the proof are postponed to Appendix C. Note that all product estimates in this section are sharp except the first one.

Proposition 5.21. — *Let $0 < b < \frac{1}{2}$. For any tensors F, G and H on P_u such that $F \cdot G \cdot H$ is a scalar, we have:*

$$\|F \cdot G \cdot H\|_{H^b(P_u)} \lesssim \|F\|_{H^1(P_u)} \|G\|_{H^1(P_u)} \|H\|_{H^{\frac{1}{2}}(P_u)}. \tag{5.78}$$

Proposition 5.22. — *For any P_u-tangent tensor G and H on S such that $G \cdot H$ is a scalar, we have:*

$$\|G \cdot H\|_{L^2(P_u)} \lesssim \|G\|_{H^{\frac{1}{2}}(P_u)} \|H\|_{H^{\frac{1}{2}}(P_u)}. \tag{5.79}$$

Proposition 5.23. — *For any scalars f and h on P_u, we have:*

$$\|fh\|_{H^{-\frac{1}{2}}(P_u)} \lesssim (\|f\|_{L^\infty(P_u)} + \|\nabla\!\!\!/ f\|_{L^2(P_u)}) \|h\|_{H^{-\frac{1}{2}}(P_u)}. \tag{5.80}$$

Proposition 5.24. — *For any P_u-tangent tensor G and H on S such that $G \cdot H$ is a scalar, and for all $j \geq 0$, we have:*

$$\sum_{j \geq 0} 2^{-j} \|P_j(G \cdot H)\|^2_{L^2(S)} \lesssim \|G\|^2_{H^1(S)} \|H\|^2_{L^2(S)}. \tag{5.81}$$

Lemma 5.25. — *Let F and G two tensors on P_u such that the contraction $F \cdot G$ is a scalar. Then, we have:*

$$\sup_{j \geq 0} 2^{-j} \|P_j(F \cdot G)\|_{L^2(P_u)} \lesssim \|F\|_{H^{\frac{1}{2}}(P_u)} \|G\|_{H^{-\frac{1}{2}}(P_u)}. \tag{5.82}$$

Lemma 5.26. — *Let $-1 < b < 1$. Let f a scalar function on P_u, and G a 1-form on P_u. Then, we have:*

$$\|\mathrm{div}\!\!\!/(fG)\|_{H^{b-1}(P_u)} \lesssim \|f\|_{H^b(P_u)} (\|G\|_{L^\infty(P_u)} + \|\nabla\!\!\!/ G\|_{L^2(P_u)}). \tag{5.83}$$

Lemma 5.27. — *Let $1 \leq b < 2$. Let f a scalar function on S, and G a 1-form on S. Then, we have:*

(5.84)
$$\|\mathrm{div}\!\!\!/(fG)\|_{L^2_u H^{b-1}(P_u)} \lesssim (\|f\|_{L^2_u H^b(P_u)} + \|f\|_{L^\infty_u H^{b-1}(P_u)})(\|G\|_{L^\infty(S)} + \|\nabla\!\!\!/ G\|_{L^\infty_{[-2,2]} L^2(P_u)}).$$

Lemma 5.28. — *Let* $0 < b < 1$. *Let* F *a tensor on* P_u *and* h *a scalar function on* P_u. *Then, we have:*

$$\|Fh\|_{H^b(P_u)} \lesssim (\|F\|_{L^\infty(P_u)} + \|\nabla\!\!\!/ F\|_{L^2(P_u)})\|h\|_{H^b(P_u)}. \tag{5.85}$$

5.5. Estimates for parabolic equations on S

Consider the following parabolic equation:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)f = h \text{ on } S, \tag{5.86}$$

where f and h are scalar functions on S. In Proposition 3.16 and Proposition 3.17, we obtained estimates for such equations. In this section, we derive additional estimates involving the Littlewood Paley projections of Section 5.1. We start with the following commutation lemma.

Lemma 5.29. — *Let* f *satisfying Equation* (5.86). *Then,* $P_j f$ *satisfies the following parabolic equation:*

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(P_j f) = a^{-1}P_j(ah) + a^{-1}[\nabla_{aN}, P_j]f \text{ on } S. \tag{5.87}$$

Proof. — We multiply Equation (5.86) with a. We obtain:

$$(\nabla_{aN} - \Delta\!\!\!/)f = ah.$$

Next, we commute with P_j, using the fact that P_j commutes with $\Delta\!\!\!/$. We obtain:

$$(\nabla_{aN} - \Delta\!\!\!/)(P_j f) = P_j(ah) + [\nabla_{aN}, P_j]f.$$

Finally, multiplying with a^{-1}, we get (5.87). This concludes the proof of Lemma 5.29. □

Proposition 5.30. — *Let* f *be a scalar function on* S *satisfying* (5.86) *and such that* $f \equiv 0$ *on* $u = -2$. *Assume that there exists two tensors* G *and* H *on* S *on* S *on* S *tangent to* P_u *such that:*

$$h = G \cdot H \text{ with } \|H\|_{L^2(S)} \lesssim \varepsilon \text{ and } \|G\|_{H^1(S)} \lesssim \varepsilon. \tag{5.88}$$

Then, we have:

$$\|f\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \varepsilon \tag{5.89}$$

and:

$$\sum_{j\geq 0}\left(2^{3j}\|P_j f\|^2_{L^2(S)} + 2^j\|P_j f\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{-j}\|P_j(\nabla_N f)\|^2_{L^2(S)}\right) \lesssim \varepsilon^2. \tag{5.90}$$

Proof. — We multiply (5.86) by f^3 and integrate on $-2 < u' < u$ where $u \leq 2$. Using integration by parts together with (3.5) and (3.6), we obtain:

(5.91)
$$\begin{aligned}
&\frac{1}{4}\|f(u,.)\|^4_{L^4(P_u)} + \|a^{-1/2} f \nabla\!\!\!/ f\|^2_{L^2(S)}\\
&= \frac{1}{4}\|f(-2,.)\|^4_{L^4(P_{-2})} + \frac{1}{2}\int_{-2}^{u}\int_{P_{u'}} a^{-1}\mathrm{tr}\,\theta f^4 d\mu_{u'}du' + \int_{-2}^{u}\int_{P_{u'}} hf^3 d\mu_{u'}du'\\
&\lesssim \|\mathrm{tr}\,\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|f\|^4_{L^4_{[-2,2]}L^{\frac{16}{3}}(P_u)}\\
&\quad + \|h\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)}\|f\|^3_{L^6_{[-2,2]}L^{12}(P_u)},
\end{aligned}$$

where we used in the last inequality the fact that $f \equiv 0$ on $u = -2$. In view of the assumptions (5.88) on h, we have $h = G \cdot H$, and thus:

$$\begin{aligned}
\|h\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} &\lesssim \|G\|_{L^\infty_{[-2,2]}L^4(P_u)}\|H\|_{L^2(S)}\\
&\lesssim \|G\|_{H^1(S)}\|H\|_{L^2(S)}\\
&\lesssim \varepsilon,
\end{aligned}$$

where we used Proposition 3.7, and the estimates for G and H provided by (5.88). Together with (5.91), and the estimate (2.30) for $\mathrm{tr}\,\theta$, we obtain:

$$\|f(u,.)\|_{L^4(P_u)} \lesssim \varepsilon + \varepsilon(\|f\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|f\|_{L^6_{[-2,2]}L^{12}(P_u)}).$$

Taking the supremum in u on the left-hand side, we get:

$$\|f\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \varepsilon + \varepsilon\|f\|_{L^6_{[-2,2]}L^{12}(P_u)}. \tag{5.92}$$

Next, we derive an estimate for $P_j f$. In view of Lemma 5.29 and since f satisfies (5.86), $P_j f$ satisfies the following parabolic equation:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(P_j f) = a^{-1}P_j(ah) + a^{-1}[\nabla_{aN}, P_j]f \text{ on } S. \tag{5.93}$$

Together with the estimate (3.31), we obtain:

$$\begin{aligned}
\|P_j f\|_{L^\infty_{[-2,2]}L^2(P_u)} &+ \|\nabla\!\!\!/(P_j f)\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N(P_j f)\|_{L^2(S)} + \|\nabla\!\!\!/^2(P_j f)\|_{L^2(S)}\\
&\lesssim \|a^{-1}P_j(ah)\|_{L^2(S)} + \|a^{-1}[\nabla_{aN}, P_j]f\|_{L^2(S)} + \|P_j f(-2,.)\|_{L^2(P_{-2})}\\
&\quad + \|\nabla\!\!\!/(P_j f)(-2,.)\|_{L^2(P_{-2})}\\
&\lesssim \|a^{-1}P_j(ah)\|_{L^2(S)} + \|a^{-1}[\nabla_{aN}, P_j]f\|_{L^2(S)},
\end{aligned}$$

where we used in the last inequality the fact that $P_f \equiv 0$ in $u = -2$. Using the finite band property for P_j and the estimate (2.30) for a, we obtain:

$$\begin{aligned}
2^j\|P_j f\|_{L^\infty_{[-2,2]}L^2(P_u)} &+ \|\nabla_N(P_j f)\|_{L^2(S)} + 2^{2j}\|P_j f\|_{L^2(S)}\\
&\lesssim \|P_j(ah)\|_{L^2(S)} + \|[\nabla_{aN}, P_j]f\|_{L^2(S)}.
\end{aligned} \tag{5.94}$$

Next, we estimate the two terms in the right-hand side in (5.94) starting with the first one. Since $ah = G \cdot (aH)$, and in view of Proposition 5.24, we have:

$$\sum_{j\geq 0} 2^{-j}\|P_j(ah)\|^2_{L^2(S)} \lesssim \|G\|^2_{H^1(S)}\|aH\|^2_{L^2(S)} \lesssim \varepsilon^2, \tag{5.95}$$

where we used in the last inequality the assumption (5.88) for G and H, and the estimate (2.30) for a. For the second term in the right-hand side in (5.94), we used the commutator estimate (5.72), which yields for any $\delta > 0$:

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim \varepsilon\|\Lambda^{\frac{1}{2}+\delta}f\|_{L^2(S)} + \varepsilon\|\Lambda^\delta f\|_{L^\infty_{[-2,2]}L^2(P_u)}.$$

Together with (5.94) and (5.95), we obtain:

$$\begin{aligned}\sum_{j\geq 0}\Big(2^{3j}\|P_jf\|^2_{L^2(S)} + 2^j\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{-j}\|P_j(\nabla_N f)\|^2_{L^2(S)}\Big)\\ \lesssim \varepsilon^2(1 + \|\Lambda^{\frac{1}{2}+\delta}f\|^2_{L^2(S)} + \|\Lambda^\delta f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}).\end{aligned}\tag{5.96}$$

Now, since $\delta > 0$, we have:

$$\begin{aligned}&\|\Lambda^{\frac{1}{2}+\delta}f\|^2_{L^2(S)} + \|\Lambda^\delta f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\\ &\lesssim \left(\sum_{j\geq 0}(2^{j(\frac{1}{2}+\delta)}\|P_jf\|_{L^2(S)} + 2^{j\delta}\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)})\right)^2\\ &\lesssim \sum_{j\geq 0}2^{j(1+3\delta)}\|P_jf\|^2_{L^2(S)} + \sum_{j\geq 0}2^{j3\delta}\|P_jf\|^2_{L^{infty}_{[-2,2]}L^2(P_u)}\\ &\lesssim \sum_{j\geq 0}\Big(2^{3j}\|P_jf\|^2_{L^2(S)} + 2^j\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\Big),\end{aligned}\tag{5.97}$$

where we chose in the last estimate $0 < \delta \leq \frac{1}{3}$. Finally, (5.96) and (5.97) imply:

$$\sum_{j\geq 0}\Big(2^{3j}\|P_jf\|^2_{L^2(S)} + 2^j\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{-j}\|P_j(\nabla_N f)\|^2_{L^2(S)}\Big) \lesssim \varepsilon^2. \tag{5.98}$$

Next, we estimate the $L^4_{[-2,2]}H^1(p)$ norm of f. Using the Bessel inequality for P_j, we have:

$$\begin{aligned}\|f\|^4_{L^4_{[-2,2]}H^1(P_u)} &= \Big\|\|f\|^2_{H^1(P_u)}\Big\|^2_{L^2_{[-2,2]}}\\ &\lesssim \left\|\sum_{j\geq 0}2^{2j}\|P_jf\|^2_{L^2(S)}\right\|^2_{L^2_{[-2,2]}} \mathbb{S}^2\\ &\lesssim \left(\sum_{j\geq 0}2^{2j}\|P_jf\|^2_{L^4_{[-2,2]}L^2(P_u)}\right)^2.\end{aligned}$$

Thus, in view of (5.98), we have:

$$\|f\|^4_{L^4_{[-2,2]}H^1(P_u)} \lesssim \left(\sum_{j\geq 0} 2^{3j}\|P_jf\|^2_{L^2(S)}\right)\left(\sum_{j\geq 0} 2^{j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\right) \lesssim \varepsilon^4. \tag{5.99}$$

In view of (5.92), we need to estimate $\|f\|_{L^6_{[-2,2]}L^{12}(P_u)}$. Now, note that applying the Sobolev embedding (3.8) with f^q for some integer $q \geq 2$ yields:

$$\begin{aligned}\|f\|^q_{L^{2q}(P_u)} = \|f^q\|_{L^2(P_u)} &\lesssim \|\nabla\!\!\!/(f^{q-1})\|_{L^1(P_u)} + \|f^q\|_{L^1(P_u)}\\ &\lesssim \|f\|^{q-1}_{L^{2(q-1)}(P_u)}\|\nabla\!\!\!/ f\|_{L^2(P_u)} + \|f\|^q_{L^q(P_u)}.\end{aligned}$$

Using the previous inequality successively with $q = 3, 4, 5, 6$ implies the following variant of the Gagliardo-Nirenberg inequality (3.9):

$$\|f\|_{L^{12}(P_u)} \lesssim \|f\|^{\frac{1}{3}}_{L^4(P_u)}\|\nabla\!\!\!/ f\|^{\frac{2}{3}}_{L^2(P_u)} + \|f\|_{L^2(P_u)}.$$

In particular, we obtain:

$$\|f\|_{L^6_{[-2,2]}L^{12}(P_u)} \lesssim \|f\|^{\frac{1}{3}}_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/ f\|^{\frac{2}{3}}_{L^4_{[-2,2]}L^2(P_u)} \lesssim \varepsilon^{\frac{2}{3}}\|f\|^{\frac{1}{3}}_{L^\infty_{[-2,2]}L^4(P_u)}, \tag{5.100}$$

where we used (5.99) in the last inequality. Finally, (5.100) and (5.92) yield:

$$\|f\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \varepsilon,$$

which together with (5.98) implies (5.89) and (5.90). This concludes the proof of the proposition. □

We have the following extension of Proposition 3.16.

Proposition 5.31. — *Let f be a scalar function on S such that P_jf satisfies:*

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(P_jf) = h \quad \text{on} \quad S. \tag{5.101}$$

Assume also that $f \equiv 0$ on $u = -2$, and that we have a decomposition for h:

$$h = h_1 + h_2.$$

Then, we have:

$$2^j\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{2j}\|P_jf\|_{L^2(S)} \lesssim \|h_1\|_{L^2(S)} + 2^j\|h_2\|_{L^1_{[-2,2]}L^2(P_u)}. \tag{5.102}$$

Proof. — We multiply (5.101) by $\Delta\!\!\!/ P_jf$ and integrate on $-2 < u' < u$ where $u \leq 2$. We proceed as in (3.34) (3.35) (3.36) (3.37), except that we estimate the integral in of (3.32) involving h as:

$$\begin{aligned}\left|\int_{-2}^u\int_{P_{u'}} h\Delta\!\!\!/ P_jf a d\mu_{u'}du'\right| &\lesssim \left|\int_{-2}^u\int_{P_{u'}} h_1\Delta\!\!\!/ P_jf a d\mu_{u'}du'\right| + \left|\int_{-2}^u\int_{P_{u'}} h_2\Delta\!\!\!/ P_jf a d\mu_{u'}du'\right|\\ &\lesssim \|h_1\|_{L^2(S)}\|\Delta\!\!\!/ P_jf\|_{L^2(S)} + \|h_2\|_{L^1_{[-2,2]}L^2(P_u)}\|\Delta\!\!\!/(P_jf)\|_{L^\infty_{[-2,2]}L^2(P_u)}\\ &\lesssim \|h_1\|_{L^2(S)}\|\Delta\!\!\!/ P_jf\|_{L^2(S)} + 2^j\|h_2\|_{L^1_{[-2,2]}L^2(P_u)}\|\nabla\!\!\!/(P_jf)\|_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned}$$

where we used the estimate (2.30) for a and the finite band property for P_j. We obtain the analog of (3.37):

$$\begin{aligned}&\|\nabla\!\!\!/ P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}+\|\Delta\!\!\!/ P_jf\|^2_{L^2(S)}\\ &\lesssim\varepsilon(\|\nabla\!\!\!/ P_jf\|^2_{L^2(S)}+\|\nabla\!\!\!/^2(P_jf)\|^2_{L^2(S)})+\|h_1\|^2_{L^2(S)}+2^{2j}\|h_2\|^2_{L^1_{[-2,2]}L^2(P_u)}.\end{aligned}\tag{5.103}$$

Finally, (5.103) together with the Bochner inequality (5.63) and the finite band property for P_j yields (5.102). This concludes the proof of the proposition. □

Proposition 5.32. — *Let f be a scalar function on S satisfying* (5.86) *and such that $f\equiv 0$ on $u=-2$. Assume that h satisfies:*

$$h=h_1+h_2\ \textit{with}\ \sup_{j\geq 0}\|P_j(ah_1)\|_{L^2(S)}\lesssim 2^{2j}\varepsilon\ \textit{and}\ \sup_{j\geq 0}\|P_j(ah_2)\|_{L^1_{[-2,2]}L^2(P_u)}\lesssim\varepsilon 2^j.\tag{5.104}$$

Then, we have:

$$\sup_{j\geq 0}\|P_jf\|_{L^2(S)}+\sup_{j\geq 0}2^{-j}\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim\varepsilon.\tag{5.105}$$

Proof. — Recall from (5.93) that P_jf satisfies the following parabolic equation:

$$(\nabla_N-a^{-1}\Delta\!\!\!/)(P_jf)=a^{-1}P_j(ah)+a^{-1}[\nabla_{aN},P_j]f\ \text{ on }\ S.$$

Together with the estimate (5.102), we obtain:

$$\begin{aligned}&2^j\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)}+2^{2j}\|P_jf\|_{L^2(S)}\\ &\lesssim\|a^{-1}P_j(ah_1)\|_{L^2(S)}+2^j\|a^{-1}P_j(ah_2)\|_{L^1_{[-2,2]}L^2(P_u)}+2^j\|a^{-1}[\nabla_{aN},P_j]f\|_{L^1_{[-2,2]}L^2(P_u)}\\ &\lesssim\varepsilon 2^{2j}+2^j\|[\nabla_{aN},P_j]f\|_{L^1_{[-2,2]}L^2(P_u)},\end{aligned}$$

where we used the estimate (2.30) for a and the assumption (5.104) on h. This yields:

$$2^{-j}\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|P_jf\|_{L^2(S)}\lesssim\varepsilon+2^{-j}\|[\nabla_{aN},P_j]f\|_{L^1_{[-2,2]}L^2(P_u)}.\tag{5.106}$$

Next, we use the commutator estimate (5.75). We have:

$$\|[\nabla_{aN},P_j]f\|_{L^1_{[-2,2]}L^2(P_u)}\lesssim 2^{j\alpha}\varepsilon\|\Lambda^{-\delta}F\|_{L^2(S)},\tag{5.107}$$

for any $0<\delta<\alpha<1$. Now, for any $\delta>0$, we have:

$$\begin{aligned}\|\Lambda^{-\delta}F\|_{L^2(S)}&\lesssim\sum_{j\geq 0}\|\Lambda^{-\delta}P_jF\|_{L^2(S)}\\ &\lesssim\sum_{j\geq 0}2^{-j\delta}\|P_jF\|_{L^2(S)}\\ &\lesssim\sup_{j\geq 0}\|P_jf\|_{L^2(S)}.\end{aligned}\tag{5.108}$$

Finally, (5.106), (5.107) and (5.108) imply for any $j\geq 0$:

$$2^{-j}\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|P_jf\|_{L^2(S)}\lesssim\varepsilon+\varepsilon\sup_{j\geq 0}\|P_jf\|_{L^2(S)},$$

which yields (5.105). This concludes the proof of the proposition. □

Proposition 5.33. — *Let f be a scalar function on S satisfying* (5.86) *and such that $f\equiv 0$ on $u=-2$. Assume that h satisfies:*

$$\sum_{j\geq 0}2^{-3j}\|P_j(ah)\|^2_{L^2(S)}\lesssim\varepsilon^2. \tag{5.109}$$

Then, we have:

$$\sum_{j\geq 0}\Big(2^j\|P_jf\|^2_{L^2(S)}+2^{-j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}+2^{-3j}\|P_j(\nabla_Nf)\|^2_{L^2(S)}\Big)\lesssim\varepsilon^2. \tag{5.110}$$

Proof. — Recall the estimate (5.94):

$$\begin{aligned}2^j\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)}&+\|\nabla_N(P_jf)\|_{L^2(S)}+2^{2j}\|P_jf\|_{L^2(S)}\\&\lesssim\|P_j(ah)\|_{L^2(S)}+\|[\nabla_{aN},P_j]f\|_{L^2(S)}.\end{aligned}$$

This yields:

$$\begin{aligned}&\sum_{j\geq 0}\Big(2^j\|P_jf\|^2_{L^2(S)}+2^{-j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}+2^{-3j}\|P_j(\nabla_Nf)\|^2_{L^2(S)}\Big)\\&\lesssim\sum_{j\geq 0}2^{-3j}\|P_j(ah)\|^2_{L^2(S)}+\sum_{j\geq 0}2^{-3j}\|[\nabla_{aN},P_j]f\|^2_{L^2(S)}\\&\lesssim\varepsilon^2+\sum_{j\geq 0}2^{-3j}\|[\nabla_{aN},P_j]f\|^2_{L^2(S)},\end{aligned} \tag{5.111}$$

where we used the assumption (5.109) in the last inequality.

Next, we use the commutator estimate (5.77), which yields for any $\delta>0$:

$$\|[\nabla_{aN},P_j]f\|_{L^2(S)}\lesssim 2^j\varepsilon\|\Lambda^{-(1-\delta)}f\|_{L^\infty_{[-2,2]}L^2(P_u)}.$$

Together with (5.111), we obtain:

$$\begin{aligned}&\sum_{j\geq 0}\Big(2^j\|P_jf\|^2_{L^2(S)}+2^{-j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}+2^{-3j}\|P_j(\nabla_Nf)\|^2_{L^2(S)}\Big)\\&\lesssim\varepsilon^2+\left(\sum_{j\geq 0}2^{-j}\right)\varepsilon^2\|\Lambda^{-(1-\delta)}f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\\&\lesssim\varepsilon^2(1+\|\Lambda^{-(1-\delta)}f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}).\end{aligned} \tag{5.112}$$

Now, since $\delta>0$, we have:

$$\begin{aligned}\|\Lambda^{-(1-\delta)}f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}&\lesssim\left(\sum_{j\geq 0}2^{-j(1-\delta)}\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)})\right)^2\\&\lesssim\sum_{j\geq 0}2^{-(2-3\delta)j}\|P_jf\|^2_{L^{infty}_{[-2,2]}L^2(P_u)}\\&\lesssim\sum_{j\geq 0}2^{-j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned} \tag{5.113}$$

where we chose in the last estimate $0 < \delta \leq \frac{1}{3}$. Finally, (5.112) and (5.113) imply (5.110). This concludes the proof of the proposition. □

Proposition 5.34. — *Let f be a scalar function on S satisfying* (5.86) *and such that $f \equiv 0$ on $u = -2$. Assume that h satisfies:*

$$\sum_{j\geq 0} 2^{-j}\|P_j(ah)\|^2_{L^2(S)} \lesssim \varepsilon^2. \tag{5.114}$$

Then, we have:

$$\sum_{j\geq 0}\Big(2^{3j}\|P_jf\|^2_{L^2(S)} + 2^j\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{-j}\|P_j(\nabla_N f)\|^2_{L^2(S)}\Big) \lesssim \varepsilon^2. \tag{5.115}$$

Proof. — The proof follows from (5.93), (5.94), (5.95), (5.96), (5.97), (5.98). □

Proposition 5.35. — *Let $0 < b < 1$. Let f be a scalar function on S satisfying* (5.86) *and such that $f \equiv 0$ on $u = -2$. Assume that h satisfies:*

$$\sum_{j\geq 0} 2^{2jb}\|P_j(ah)\|^2_{L^2(S)} \lesssim \varepsilon^2. \tag{5.116}$$

Then, we have:

(5.117)

$$\sum_{j\geq 0}\Big(2^{(4+2b)j}\|P_jf\|^2_{L^2(S)} + 2^{(2+2b)j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{2bj}\|P_j(\nabla_N f)\|^2_{L^2(S)}\Big) \lesssim \varepsilon^2.$$

Proof. — Recall the estimate (5.94):

$$\begin{aligned}2^j\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N(P_jf)\|_{L^2(S)} + 2^{2j}\|P_jf\|_{L^2(S)}&\\ \lesssim \|P_j(ah)\|_{L^2(S)} + \|[\nabla_{aN}, P_j]f\|_{L^2(S)}&.\end{aligned}$$

This yields:

$$\begin{aligned}&\sum_{j\geq 0}\Big(2^{(4+2b)j}\|P_jf\|^2_{L^2(S)} + 2^{(2+2b)j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{2bj}\|P_j(\nabla_N f)\|^2_{L^2(S)}\Big)\\ &\lesssim \sum_{j\geq 0} 2^{2bj}\|P_j(ah)\|^2_{L^2(S)} + \sum_{j\geq 0} 2^{2bj}\|[\nabla_{aN}, P_j]f\|^2_{L^2(S)}\\ &\lesssim \varepsilon^2 + \sum_{j\geq 0} 2^{2bj}\|[\nabla_{aN}, P_j]f\|^2_{L^2(S)}, \qquad (5.118)\end{aligned}$$

where we used the assumption (5.116) in the last inequality.

Next, we use the commutator estimate (5.76), which yields for any $\delta > 0$:

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim 2^{-(1-\delta)j}\varepsilon(\|\not{\Delta} f\|_{L^2(S)} + \|\not{\nabla} f\|_{L^\infty_{[-2,2]}L^2(P_u)}).$$

Together with (5.118), we obtain:

$$\sum_{j\geq 0}\Big(2^{(4+2b)j}\|P_jf\|^2_{L^2(S)}+2^{(2+2b)j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}+2^{2bj}\|P_j(\nabla_N f)\|^2_{L^2(S)}\Big)$$

$$\lesssim \varepsilon^2+\left(\sum_{j\geq 0}2^{-2j(1-b-\delta)}\right)\varepsilon^2(\|\not{\Delta} f\|^2_{L^2(S)}+\|\not{\nabla} f\|^2_{L^\infty_{[-2,2]}L^2(P_u)})$$

$$\lesssim \varepsilon^2(1+\|\not{\Delta} f\|^2_{L^2(S)}+\|\not{\nabla} f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}), \tag{5.119}$$

where we chose in the last inequality $0<\delta<1-b$ which is possible since $b<1$. Now, the finite band property for P_j yields:

$$\|\not{\Delta} f\|^2_{L^2(S)}+\|\not{\nabla} f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim\left(\sum_{j\geq 0}(2^{2j}\|P_jf\|_{L^2(S)}+2^j\|P_jf\|_{L^\infty_{[-2,2]}L^2(P_u)})\right)^2$$

$$\lesssim\sum_{j\geq 0}\Big(2^{(4+2b)j}\|P_jf\|^2_{L^2(S)}+2^{(2+2b)j}\|P_jf\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\Big),$$

where we used in the last inequality the fact that $b>0$. Together with (5.119), this implies (5.117). This concludes the proof of the proposition. □

CHAPTER 6

ESTIMATES FOR $\nabla_N a$ AND $\nabla_N^2 a$ (PROOF OF THEOREM 2.5)

This chapter is dedicated to the proof of Theorem 2.5. We recall the decomposition (4.37) (4.38) (4.39):

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N a = \mathrm{di}\!\!/\mathrm{v}(H) + h, \tag{6.1}$$

where the tensor H is given by

$$H = -\nabla_N k_{.N} - R_{.N} \tag{6.2}$$

and where the scalar h is given schematically by

$$\begin{aligned} h = & -a^{-1}\mathrm{tr}\,\theta\Delta\!\!\!/ a - 2a^{-1}\widehat{\theta}\nabla\!\!\!\!/^2 a + 2a^{-2}\nabla\!\!\!\!/ a\nabla\!\!\!\!/\nabla_N a - 2R_{N.}a^{-1}\nabla\!\!\!\!/ a \ - \nabla\!\!\!\!/\mathrm{tr}\,\theta a^{-1}\nabla\!\!\!\!/ a \\ & + 2\widehat{\theta}|a^{-1}\nabla\!\!\!\!/ a\,|^2 + 2\theta\nabla_N\theta + a^{-1}\nabla\!\!\!\!/ a\,\nabla_N k_{N.} + \nabla\theta k + \theta\nabla k + R_{N.}k \\ & + \theta a^{-1}\nabla\!\!\!\!/ a\, k_{N.} + 2k\nabla_N k + \nabla_N k(\nabla\!\!\!\!/ a, N) + k(\nabla\!\!\!\!/\nabla_N a, N) \\ & + \nabla_N a k(\nabla\!\!\!\!/ a, N) + \theta k + k(\nabla\!\!\!\!/ a, \nabla\!\!\!\!/ a) + \theta R. \end{aligned} \tag{6.3}$$

We introduce the scalar functions on S a_1 and a_2 solutions of:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)a_1 = h \mathrm{onS},\ a_1(-2,.) = 0 \tag{6.4}$$

and:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)a_2 = \mathrm{di}\!\!/\mathrm{v}(H)\mathrm{onS},\ a_2(-2,.) = 0, \tag{6.5}$$

which yields, in view of (6.1), the fact that $\nabla_N a(-2,.) = 0$, the decomposition:

$$\nabla_N a = a_1 + a_2. \tag{6.6}$$

Remark 6.1. — In the right-hand side of (6.1), the regularity of h is better than the regularity of $\mathrm{di}\!\!/\mathrm{v}(H)$ (see (6.12)). On the other hand, we can not make sense of $\nabla_N h$, while the contracted Bianchi identities on Σ allow us to make sense of $\nabla_N \mathrm{di}\!\!/\mathrm{v}(H)$ (see in particular (6.33)). Thus, the idea behind the decomposition (6.6) is to take advantage of the regularity of h for a_1, and to use the structure of $\mathrm{di}\!\!/\mathrm{v}(H)$ to obtain a useful equation for $\nabla_N a_2$ (see (6.39)). We carry out this strategy in the rest of the chapter.

The following two propositions state the regularity of a_1 and a_2.

Proposition 6.2. — *Let a_1 be the solution of* (6.4)*, where h is defined in* (6.3)*. Then, we have:*

$$\|a_1\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \varepsilon \tag{6.7}$$

and:

$$\sum_{j\geq 0}\Big(2^{3j}\|P_j a_1\|^2_{L^2(S)} + 2^j\|P_j a_1\|^2_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{-j}\|P_j(\nabla_N a_1)\|^2_{L^2(S)}\Big) \lesssim \varepsilon^2. \tag{6.8}$$

Proposition 6.3. — *Let a_2 be the solution of* (6.5)*, where H is defined in* (6.2)*. Then, we have:*

$$\|a_2\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \varepsilon, \tag{6.9}$$

$$\sum_{j\geq 0} 2^{2j}\|P_j a_2\|^4_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \varepsilon^4 \tag{6.10}$$

and

$$\sup_{j\geq 0}\|P_j(\nabla_N a_2)\|_{L^2(S)} \lesssim \varepsilon. \tag{6.11}$$

The proof of Proposition 6.2 is postponed to Section 6.1, while the proof of Proposition 6.3 is postponed to Section 6.2. In view of the decomposition (6.6) for $\nabla_N a$, the estimates (6.7) (6.8) for a_1, and the estimates (6.9) (6.10) (6.11) for a_2, we immediately obtain the estimate (2.31) for $\nabla_N a$ and $\nabla_N^2 a$. This concludes the proof of Theorem 2.5.

6.1. Proof of Proposition 6.2

In view of the Definition (6.3), the scalar function h may be written as a linear combination of terms of the form $h = F\cdot G$, where F is schematically given by:

$$F = \nabla\!\!\!/^2 a + \nabla\!\!\!/\nabla_N a + R + \nabla\theta + |a^{-1}\nabla\!\!\!/ a|^2 + \nabla k + a^{-1}\nabla\!\!\!/ a\, k + \nabla(a)k$$

and G is schematically given by:

$$G = \theta + a^{-1}\nabla\!\!\!/ a + k.$$

In view of the estimate (2.30) for a and θ, and in view of the assumption (2.1) for R and k, we deduce:

$$h = F\cdot G \text{ with } \|F\|_{L^2(S)} \lesssim \varepsilon \text{ and } \|G\|_{H^1(S)} \lesssim \varepsilon. \tag{6.12}$$

Now, in view of the Equation (6.4) satisfied by a_1, and the decomposition (6.12) for h, the estimates (6.7) and (6.8) are a consequence of the estimates (5.89) and (5.90) of Proposition 5.30. This concludes the proof of Proposition 6.2.

6.2. Proof of Proposition 6.3

In view of the decomposition (6.6) of $\nabla_N a$, we have:

$$a_2 = \nabla_N a - a_1,$$

which together with the estimate (2.30) for a and the estimates (6.7) and (6.8) for a_1 implies:

$$\|\nabla\!\!\!/ a_2\|_{L^2(S)} + \|a_2\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \varepsilon. \tag{6.13}$$

Next, we derive en equation for $\nabla_{aN} a_2$. We use the following commutation lemma.

Lemma 6.4. — *Let f satisfying the following parabolic equation:*

$$(\nabla_{aN} - \Delta\!\!\!/)f = ah.$$

Then, $\nabla_{aN} f$ satisfies the following parabolic equation:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(\nabla_{aN} f) = \nabla_N(ah) + a^{-1}[\nabla_{aN}, \Delta\!\!\!/]f \quad \text{on } S. \tag{6.14}$$

Proof. — We multiply equation the equation satisfied by f with a. We obtain:

$$(\nabla_{aN} - \Delta\!\!\!/)f = ah.$$

Next, we commute with ∇_{aN}. We obtain:

$$(\nabla_{aN} - \Delta\!\!\!/)(\nabla_{aN} f) = \nabla_{aN}(ah) + [\nabla_{aN}, \Delta\!\!\!/]f.$$

Finally, multiplying with a^{-1}, we get (6.14). This concludes the proof of Lemma 5.29. □

In view of the Equation (6.5) satisfied by a_2, and in view of the commutation Lemma 6.4, $\nabla_{aN} a_2$ satisfies:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(\nabla_{aN} a_2) = \nabla_N(a\,\mathrm{d}\!\!\!/\mathrm{iv}(H)) + a^{-1}[\nabla_{aN}, \Delta\!\!\!/]a_2. \tag{6.15}$$

Next, we evaluate both terms in the right-hand side of (6.15) starting with the second one. Using the commutation Formula (2.23), we have:

$$a^{-1}[\nabla_{aN}, \Delta\!\!\!/]a_2 = h_1 + h_2, \tag{6.16}$$

where h_1 and h_2 are given schematically by:

$$h_1 = \mathrm{d}\!\!\!/\mathrm{iv}(\theta \cdot \nabla\!\!\!/ a_2) \tag{6.17}$$

and:

$$h_2 = (R + \nabla\!\!\!/\theta + \widehat{\theta}\nabla\!\!\!/ a)\nabla\!\!\!/ a_2. \tag{6.18}$$

We first estimate h_1. We have:

$$ah_1 = \mathrm{d}\!\!\!/\mathrm{iv}(a\theta \cdot \nabla\!\!\!/ a_2) - \nabla\!\!\!/(a) \cdot \theta\nabla\!\!\!/ a_2.$$

In view of the estimate (5.64) and the sharp Bernstein inequality (5.61), we obtain:

$$\begin{aligned}\|P_j(ah_1)\|_{L^2(S)} &\lesssim 2^{\frac{3j}{2}}\|a\theta\cdot\nabla\!\!\!/ a_2\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} + 2^j\|\nabla\!\!\!/(a)\cdot\theta\nabla\!\!\!/ a_2\|_{L^2(S)}\\ &\lesssim 2^{\frac{3j}{2}}\|a\|_{L^\infty(S)}\|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/ a_2\|_{L^2(S)}\\ &\quad + 2^j\|\nabla\!\!\!/(a)\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/ a_2\|_{L^2(S)}\\ &\lesssim 2^{\frac{3j}{2}}\varepsilon,\end{aligned}\tag{6.19}$$

where we used in the last inequality the estimate (2.30) for θ and a, and the estimate (6.13) for $\nabla\!\!\!/ a_2$. Next, we estimate h_2. In view of the estimate (2.30) for θ and a, the assumption (2.1) for R, and the estimate (6.13) for $\nabla\!\!\!/ a_2$, we have:

$$\|h_2\|_{L^1(S)} \lesssim (\|R\|_{L^2(S)} + \|\nabla\!\!\!/\theta\|_{L^2(S)} + \|\widehat{\theta}\|_{L^4(S)}\|\nabla\!\!\!/ a\|_{L^4(S)})\|\nabla\!\!\!/ a_2\|_{L^2(S)} \lesssim \varepsilon^2,$$

which together with the dual of the sharp Bernstein inequality (5.61) yields:

$$\|P_j(ah_2)\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^j\|a\|_{L^\infty(S)}\|h_1\|_{L^1(S)} \lesssim 2^j\varepsilon,\tag{6.20}$$

where we used in the last inequality the estimate (2.30) for a.

Next, we estimate the first term in the right-hand side of (6.15). We have:

$$\begin{aligned}a\nabla_N(a\,\mathrm{div}\!\!\!/(H)) &= \nabla_{aN}(a\,\mathrm{div}\!\!\!/(H))\\ &= \nabla_{aN}(a)\mathrm{div}\!\!\!/(H) + a[\nabla_{aN},\mathrm{div}\!\!\!/](H) + a\,\mathrm{div}\!\!\!/(\nabla_{aN}H),\end{aligned}$$

which together with the commutator Formula (2.21) yields:

$$a\nabla_N(a\,\mathrm{div}\!\!\!/(H)) = a\,\mathrm{div}\!\!\!/(\nabla_{aN}H) + h_3 + h_4,\tag{6.21}$$

where h_3 and h_4 are given by:

$$h_3 = \mathrm{div}\!\!\!/(\nabla_N(a)H - a\theta H)\tag{6.22}$$

and

$$h_4 = -(\nabla\!\!\!/(\nabla_{aN}(a)) + \nabla\!\!\!/(a\theta) + aR_{N\cdot} + a\theta\cdot\nabla\!\!\!/(a))\cdot H.\tag{6.23}$$

Now, in view of the definition of H (6.2), we have:

$$\|H\|_{L^2(S)} \lesssim \|\nabla k\|_{L^2(S)} + \|R\|_{L^2(S)} \lesssim \varepsilon,\tag{6.24}$$

where we used in the last inequality the assumption (2.1) on R and k. We first estimate h_3. In view of the estimate (5.64), we obtain:

(6.25)
$$\begin{aligned}\|P_j(h_3)\|_{L^2(S)} &\lesssim 2^{\frac{3j}{2}}(\|\nabla_N(a)H\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} + \|a\theta H\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)})\\ &\lesssim 2^{\frac{3j}{2}}\|a\|_{L^\infty(S)}(\|\nabla_N(a)\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)})\|H\|_{L^2(S)}\\ &\lesssim 2^{\frac{3j}{2}}\varepsilon(1+\|a_2\|_{L^\infty_{[-2,2]}L^4(P_u)}),\end{aligned}$$

where we used in the last inequality the estimate (2.30) for θ and a, the estimate (6.24) for H, the decomposition $\nabla_N a = a_1 + a_2$, and the estimate (6.7) for a_1. Next, we estimate h_4. In view of the estimate (2.30) for θ and a, the assumption (2.1) for R, and the estimate (6.24) for H, we have:

$$\begin{aligned}\|h_4\|_{L^1(S)} &\lesssim (\|\nabla\!\!\!/\nabla_N a\|_{L^2(S)} + \|\nabla\!\!\!/(a\theta)\|_{L^2(S)} + \|aR\|_{L^2(S)} + \|\widehat{\theta}\|_{L^4(S)}\|\nabla\!\!\!/ a\|_{L^4(S)})\\ &\quad \times \|H\|_{L^2(S)}\\ &\lesssim \varepsilon^2,\end{aligned}$$

which together with the dual of the sharp Bernstein inequality (5.61) yields:

$$\|P_j(h_4)\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^j\|h_4\|_{L^1(S)} \lesssim 2^j\varepsilon. \tag{6.26}$$

Next, we estimate the first term in the right-hand side of (6.21), i.e., $a\mathrm{d}\!\!/\mathrm{iv}(\nabla_{aN}H)$. Recall the definition of H (6.2):

$$H = -\nabla_N k_{.N} - R_{.N}.$$

We take the ∇_{aN} derivative of each of the two terms in the definition of H starting with the first one. Using the constraint Equations (1.2) and the fact that we have a maximal foliation yields:

$$\nabla_N k_{NA} = -\nabla^B k_{BA}. \tag{6.27}$$

Now, we have:

$$\nabla^B k_{BA} = \mathrm{d}\!\!/\mathrm{iv} k_A + \mathrm{tr}\,\theta k_{NA} + \theta_{AB}k_{NB}.$$

Together with (6.27), we obtain, schematically:

$$\nabla_N k_{N.} = -\mathrm{d}\!\!/\mathrm{iv} k - \theta\cdot k_{N.}. \tag{6.28}$$

Taking the ∇_{aN} derivative, we obtain:

$$\nabla_{aN}(\nabla_N k_{N.}) = -\mathrm{d}\!\!/\mathrm{iv}(\nabla_{aN}k) - [\nabla_{aN}, \mathrm{d}\!\!/\mathrm{iv}]k - \nabla_{aN}(\theta)\cdot k_{N.} - \theta\cdot\nabla_{aN}k_{N.} - \theta\cdot k_{.\nabla_{aN}N}.$$

Using the commutation Formula (2.21) and the structure Equation (4.4), we obtain, schematically:

$$\nabla_{aN}(\nabla_N k_{N.}) = -\mathrm{d}\!\!/\mathrm{iv}(\nabla_{aN}k) + a\theta\nabla k + a(R + \nabla\theta + \theta\nabla\!\!\!/(a))k. \tag{6.29}$$

Next, we take the ∇_{aN} derivative of the second term in the definition of H. The twice-contracted Bianchi identity on Σ yields:

$$\nabla_N R_{NA} = -\nabla^B R_{AB} + \frac{1}{2}\nabla_A R,$$

which together with the constraint Equations (1.2) implies:

$$\nabla_N R_{NA} = -\nabla^B R_{AB} + k\cdot\nabla_A k. \tag{6.30}$$

Now, we have:

$$\nabla^B R_{AB} = \nabla\!\!\!/^B R_{AB} + \mathrm{tr}\,\theta R_{NA} + \theta_{AB}R_{NB}.$$

Together with (6.30), we obtain schematically:

$$\nabla_N R_{N.} = -\mathrm{d}\!\!/\mathrm{iv} R - \theta\cdot R + k\cdot\nabla k. \tag{6.31}$$

This yields:

$$\nabla_{aN}(R_{N.}) = -\mathrm{d}\!\!/\mathrm{iv}(aR) + \nabla\!\!\!/(a)\cdot R - a\theta\cdot R + ak\cdot\nabla k. \tag{6.32}$$

Finally, in view of the Definition (6.2) of H, (6.29) and (6.32), we obtain schematically:

$$\nabla_{aN}H = \text{d}\!\!/\text{iv}(\nabla_{aN}k + aR) + a(\theta+k)\nabla k + a(R+\nabla\theta+\theta\nabla\!\!\!/(a))k + (\nabla\!\!\!/(a)+a\theta)\cdot R.$$

This yields:

$$\nabla_{aN}H = \text{d}\!\!/\text{iv}(H_1) + H_2, \tag{6.33}$$

where H_1 is a 2-tensor given by:

$$H_1 = \nabla_{aN}k + aR$$

and H_2 is a 1-form given by:

$$H_2 = a(\theta+k)\nabla k + a(R+\nabla\theta+\theta\nabla\!\!\!/(a))k + (\nabla\!\!\!/(a)+a\theta)\cdot R.$$

In particular, we have:

$$\|H_1\|_{L^2(S)} \lesssim \|a\|_{L^\infty(S)}(\|\nabla k\|_{L^2(S)} + \|R\|_{L^2(S)}) \lesssim \varepsilon, \tag{6.34}$$

where we used in the last inequality the assumption (2.1) on k and R. Also, we have:

$$\begin{aligned}\|H_2\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} &\lesssim \|a\|_{L^\infty(S)}(\|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|k\|_{L^\infty_{[-2,2]}L^4(P_u)}\\ &\quad + \|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)})(\|\nabla k\|_{L^2(S)} + \|R\|_{L^2(S)} + \|\nabla\theta\|_{L^2(S)})\\ &\lesssim \varepsilon,\end{aligned} \tag{6.35}$$

where we used in the last inequality the assumption (2.1) on k and R and the estimate (2.30) for a and θ. In view of (6.33), we obtain schematically:

$$\begin{aligned}a\text{d}\!\!/\text{iv}(\nabla_{aN}H) &= \text{d}\!\!/\text{iv}\text{d}\!\!/\text{iv}(aH_1) + \text{d}\!\!/\text{iv}(\nabla\!\!\!/(a)\cdot H_1) + \nabla\!\!\!/^2(a)\cdot H_1 + \text{d}\!\!/\text{iv}(aH_2) + \nabla(a)\cdot H_2\\ &= h_5 + h_6,\end{aligned} \tag{6.36}$$

where the scalar functions on S h_5 and h_6 are given by:

$$h_5 = \text{d}\!\!/\text{iv}\text{d}\!\!/\text{iv}(aH_1) + \text{d}\!\!/\text{iv}(\nabla\!\!\!/(a)\cdot H_1) + \text{d}\!\!/\text{iv}(aH_2) + \nabla(a)\cdot H_2$$

and:

$$h_6 = \nabla\!\!\!/^2(a)\cdot H_1.$$

Using the finite band property for P_j, the sharp Bernstein inequality (5.61) and the estimate (5.64), we obtain:

$$\begin{aligned}\|P_j(h_5)\|_{L^2(S)} &\lesssim 2^{2j}\|\Delta\!\!\!/^{-1}\text{d}\!\!/\text{iv}\text{d}\!\!/\text{iv}\|_{\mathcal{L}(L^2(P_u))}\|aH_1\|_{L^2(S)} + 2^{\frac{3j}{2}}\|\nabla\!\!\!/(a)\cdot H_1\|_{L^2(S)}\\ &\quad + 2^{\frac{3j}{2}}\|aH_2\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} + 2^j\|\nabla(a)\cdot H_2\|_{L^2_{[-2,2]}L^1(P_u)}\\ &\lesssim 2^{2j}\|\nabla\!\!\!/^2\Delta\!\!\!/^{-1}\|_{\mathcal{L}(L^2(P_u))}\|a\|_{L^\infty(S)}\|H_1\|_{L^2(S)} + 2^{\frac{3j}{2}}\|\nabla\!\!\!/(a)\|_{L^\infty_{[-2,2]}L^4(P_u)}\|H_1\|_{L^2(S)}\\ &\quad + 2^{\frac{3j}{2}}\|a\|_{L^\infty(S)}\|H_2\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} + 2^j\|\nabla(a)\|_{L^\infty_{[-2,2]}L^4(P_u)}\|H_2\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)},\end{aligned}$$

which together with the Bochner inequality for scalars (5.63), the estimate (2.30) for a, and the estimates (6.34) and (6.35) for H_1 and H_2 implies:

$$\|P_j(h_5)\|_{L^2(S)} \lesssim 2^{2j}\varepsilon. \tag{6.37}$$

Next, we estimate h_6. In view of the estimate (2.30) for a and the estimate (6.34) for H_1, we have:

$$\|h_6\|_{L^1(S)} \lesssim \|\nabla\!\!\!/^2 a\|_{L^2(S)}\|H_1\|_{L^2(S)} \lesssim \varepsilon.$$

Together with the sharp Bernstein inequality (5.61), we obtain:

$$\|P_j(h_6)\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^j\varepsilon. \tag{6.38}$$

Finally, in view of (6.15), (6.16), (6.21) and (6.36), we have:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)(\nabla_{aN}a_2) = h_7 + h_8, \tag{6.39}$$

where h_7 and h_8 are defined by:

$$h_7 = h_1 + a^{-1}h_3 + a^{-1}h_5$$

and:

$$h_8 = h_2 + a^{-1}h_4 + a^{-1}h_6.$$

In view of (6.19), (6.25) and (6.37), we have:

$$\|P_j(ah_7)\|_{L^2(S)} \lesssim 2^{2j}\varepsilon(1 + \|a_2\|_{L^\infty_{[-2,2]}L^4(P_u)}). \tag{6.40}$$

Also, in view of (6.20), (6.26) and (6.38), we have:

$$\|P_j(ah_8)\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^j\varepsilon. \tag{6.41}$$

Now, in view of (6.39), (6.40) and (6.41), (5.105) implies:

$$\sup_{j\geq 0}\|P_j(\nabla_{aN}a_2)\|_{L^2(S)} \lesssim \varepsilon(1 + \|a_2\|_{L^\infty_{[-2,2]}L^4(P_u)}). \tag{6.42}$$

Next, we state three lemma.

Lemma 6.5. — *For any scalar function f on S, and for any $0 \leq b < 1$, we have:*

$$\sup_{j\geq 0} 2^{jb}\|P_j(a^{\pm 1}f)\|_{L^2(S)} \lesssim \sup_{j\geq 0} 2^{jb}\|P_j f\|_{L^2(S)}. \tag{6.43}$$

Lemma 6.6. — *For any scalar function f on S such that $f \equiv 0$ on $u = -2$, we have:*

$$\left(\sum_{j\geq 0} 2^{2j}\|P_j f\|^4_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac{1}{4}} \lesssim \|\nabla\!\!\!/ f\|_{L^2(S)} + \|f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \sup_{j\geq 0}\|P_j(\nabla_N f)\|_{L^2(S)}. \tag{6.44}$$

Lemma 6.7. — *For any scalar function f on S such that $f \equiv 0$ on $u = -2$, we have:*

$$\|f\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \|\nabla\!\!\!/ f\|_{L^2(S)} + \sup_{j\geq 0}\|P_j(\nabla_N f)\|_{L^2(S)}. \tag{6.45}$$

The proof of Lemma 6.5 is postponed to Section 6.3, the proof of Lemma 6.6 is postponed to Section 6.4 and the proof of Lemma 6.5 is postponed to Section 6.5. Let us now conclude the proof of Proposition 6.3. In view of (6.13) and Lemma 6.7, we have:

$$\|a_2\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \varepsilon + \sup_{j\geq 0}\|P_j(\nabla_N a_2)\|_{L^2(S)}. \tag{6.46}$$

Together with (6.42), we obtain:

$$\sup_{j\geq 0}\|P_j(\nabla_{aN}a_2)\|_{L^2(S)} \lesssim \varepsilon(1+\sup_{j\geq 0}\|P_j(\nabla_N a_2)\|_{L^2(S)}).$$

Together with Lemma 6.5 and (6.46), we obtain:

$$\sup_{j\geq 0}\|P_j(\nabla_N a_2)\|_{L^2(S)} + \|a_2\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \varepsilon. \tag{6.47}$$

Finally, (6.13), (6.47) and Lemma 6.6 imply:

$$\left(\sum_{j\geq 0} 2^{2j}\|P_j f\|^4_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac{1}{4}} \lesssim \varepsilon.$$

This concludes the proof of Proposition 6.3.

6.3. Proof of Lemma 6.5

We have:

$$\|P_j(a^{\pm 1}f)\|_{L^2(S)} \lesssim \sum_{l\geq 0}\|P_j(a^{\pm 1}P_l f)\|_{L^2(S)}. \tag{6.48}$$

If $l\leq j$, we use the finite band property for P_j, and we obtain:

(6.49)

$$\begin{aligned}
2^{jb}\|P_j(a^{\pm 1}P_l f)\|_{L^2(S)} &\lesssim 2^{-j(1-b)}\|\nabla\!\!\!/(a^{\pm 1}P_l f)\|_{L^2(S)}\\
&\lesssim 2^{-j(1-b)}\|a^{\pm 1}\nabla\!\!\!/ P_l f\|_{L^2(S)} + 2^{-j(1-b)}\|\nabla\!\!\!/(a^{\pm 1})P_l f\|_{L^2(S)}\\
&\lesssim 2^{-j(1-b)}\|a^{\pm 1}\|_{L^\infty(S)}\|\nabla\!\!\!/ P_l f\|_{L^2(S)}\\
&\quad + 2^{-j(1-b)}\|\nabla\!\!\!/(a^{\pm 1})\|_{L^\infty_{[-2,2]}L^4(P_u)}\|P_l f\|_{L^2_{[-2,2]}L^4(P_u)}\\
&\lesssim 2^{-(j-l)(1-b)}\sup_{q\geq 0}2^{qb}\|P_q f\|_{L^2(S)},
\end{aligned}$$

where we used in the last inequality the finite band property and Bernstein for P_l, and the estimate (2.30) for a.

If $l>j$, we use the fact that:

$$\begin{aligned}
P_j(a^{\pm 1}P_l f) &= 2^{-2l}P_j(a^{\pm 1}\Delta\!\!\!/ P_l f)\\
&= 2^{-2l}P_j(\mathrm{d}\!\!\!/\mathrm{iv}(a^{\pm 1}\nabla\!\!\!/ P_l f)) - 2^{-2l}P_j(\nabla\!\!\!/(a^{\pm 1})\nabla\!\!\!/ P_l f),
\end{aligned}$$

which together with the finite band property and Bernstein for P_j yields:

$$\begin{aligned}
2^{jb}\|P_j(a^{\pm 1}P_l f)\|_{L^2(S)} &\lesssim 2^{-2l+j(1+b)}\|a^{\pm 1}\nabla\!\!\!/ P_l f\|_{L^2(S)} + 2^{-2l+\frac{j}{2}}\|\nabla\!\!\!/(a^{\pm 1})\nabla\!\!\!/ P_l f\|_{L^2_{[-2,2]}L^4(P_u)}\\
&\lesssim 2^{-2l+j(1+b)}(\|a^{\pm 1}\|_{L^\infty(S)} + \|\nabla\!\!\!/(a^{\pm 1})\|_{L^\infty_{[-2,2]}L^4(P_u)})\|\nabla\!\!\!/ P_l f\|_{L^2(S)}\\
&\lesssim 2^{-(l-j)(1+b)}\sup_{q\geq 0}2^{qb}\|P_q f\|_{L^2(S)}, \qquad (6.50)
\end{aligned}$$

where we used in the last inequality the finite band property for P_l, and the estimate (2.30) for a.

Finally, (6.48), (6.49) and (6.50) yield for all $j \geq 0$:

$$2^{jb}\|P_j(a^{\pm 1}f)\|_{L^2(S)} \lesssim \left(\sum_{l\geq 0}(2^{-|j-l|(1-b)} + 2^{-|j-l|(1+b)})\right)\sup_{q\geq 0}\|P_q f\|_{L^2(S)}$$
$$\lesssim \sup_{q\geq 0}\|P_q f\|_{L^2(S)},$$

where we used in the last inequality the fact that $0 \leq b < 1$. Taking the supremum in j yields (6.43). This concludes the proof of Lemma 6.5.

6.4. Proof of Lemma 6.6

We follow the proof of Corollary 5.16. Proceeding as in (B.12), we obtain for all $j \geq 0$:

(6.51)

$$2^j\|P_j f\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim 2^j\left(\int_{-2}^{-2}\|P_j f\|_{L^2(P_u)}\|\nabla_{aN}P_j f\|_{L^2(P_u)}du\right) + 2^{2j}\|P_j f\|^2_{L^2(S)}.$$

Then, proceeding as in (B.13), we obtain in view of (6.51):

$$2^j\|P_j f\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim 2^j\|P_j f\|_{L^2(S)}\|P_j(\nabla_{aN}f)\|_{L^2(S)}$$
$$+2^j\|P_j f\|_{L^\infty_{[-2,2]}L^2(P_u)}\|[\nabla_{aN}, P_j]F\|_{L^1_{[-2,2]}L^2(P_u)} + 2^{2j}\|P_j f\|^2_{L^2(S)}.$$

This implies:

$$2^j\|P_j f\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim 2^j\|P_j f\|_{L^2(S)}\|P_j(\nabla_{aN}f)\|_{L^2(S)}$$
$$+ 2^j\|[\nabla_{aN}, P_j]F\|^2_{L^1_{[-2,2]}L^2(P_u)} + 2^{2j}\|P_j f\|^2_{L^2(S)}.$$

Taking the square on both side, summing in $j \geq 0$, and using the Bessel inequality yields:

$$\sum_{j\geq 0}2^{2j}\|P_j f\|^4_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \left(\sum_{j\geq 0}2^{2j}\|P_j f\|^2_{L^2(S)}\right)\left(\sup_{j\geq 0}\|P_j(\nabla_{aN}f)\|^2_{L^2(S)}\right)$$
$$+ \sum_{æ\geq 0}2^{2j}\|[\nabla_{aN}, P_j]F\|^4_{L^1_{[-2,2]}L^2(P_u)} + \sum_{j\geq 0}2^{4j}\|P_j f\|^4_{L^2(S)}.$$

Using the Bessel inequality for P_j and Lemma 6.5, we finally obtain:

$$\sum_{j\geq 0}2^{2j}\|P_j f\|^4_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|\nabla\!\!\!/ f\|^4_{L^2(S)} + \left(\sup_{j\geq 0}\|P_j(\nabla_N f)\|_{L^2(S)}\right)^4 \tag{6.52}$$
$$+ \sum_{æ\geq 0}2^{2j}\|[\nabla_{aN}, P_j]F\|^4_{L^1_{[-2,2]}L^2(P_u)}.$$

Now, we have in view of the commutator estimate (5.73):

$$\|[P_j, \nabla_{aN}]f\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^{-j(1-\delta)}\varepsilon(\|\nabla\!\!\!/ f\|_{L^2(S)} + \|f\|_{L^\infty_{[-2,2]}L^2(P_u)}),$$

for any $\delta > 0$. Proceeding as in (B.15), we obtain:

$$\sum_{j\geq 0} 2^{2j}\|P_j f\|^4_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|\nabla\!\!\!/ f\|^4_{L^2(S)} + \|f\|^4_{L^\infty_{[-2,2]}L^2(P_u)}. \tag{6.53}$$

Finally, (6.52) and (6.53) yield (6.44). This concludes the proof of Lemma 6.7.

6.5. Proof of Lemma 6.7

Lemma 6.7 is an improvement of Proposition 3.7 where one has a slightly weaker assumption on $\nabla_N f$. Proceeding as in (3.11), we have:

$$\begin{aligned}
\|f(u,.)\|^4_{L^4(P_u)} &= \|f(-2,.)\|^4_{L^4(P_{-2})} + 4\int_{-2}^{u}\int_{P_{u'}} \nabla_N f(u',x')f^3 du' d\mu_{u'} \\
&\quad + \int_{-2}^{u}\int_{P_{u'}} \mathrm{tr}\,\theta f(u',x')^4 du' d\mu_{u'} \\
&\lesssim \sum_{j\geq 0}\left|\int_{-2}^{u}\int_{P_{u'}} P_j(\nabla_N f)(u',x')P_j(f^3) du' d\mu_{u'}\right| \\
&\quad + \|\mathrm{tr}\,\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|f\|^4_{L^4_{[-2,2]}L^{\frac{16}{3}}(P_u)} \\
&\lesssim \left(\sup_{j\geq 0}\|P_j(\nabla_N f)\|_{L^2(S)}\right)\left(\sum_{j\geq 0}\|P_j(f^3)\|_{L^2(S)}\right) \\
&\quad + \varepsilon\|f\|^2_{L^\infty_{[-2,2]}L^4(P_u)}\|f\|^2_{L^2_{[-2,2]}L^8(P_u)},
\end{aligned} \tag{6.54}$$

where we used the fact that $f(-2,.) = 0$ and the estimate (2.30) for $\mathrm{tr}\,\theta$. (6.54) yields:

$$\|f(u,.)\|^4_{L^4(P_u)} \lesssim \left(\sup_{j\geq 0}\|P_j(\nabla_N f)\|_{L^2(S)}\right)^4 + \left(\sum_{j\geq 0}\|P_j(f^3)\|_{L^2(S)}\right)^{\frac{4}{3}} + \|\nabla\!\!\!/ f\|^4_{L^2(S)},$$

which after taking the supremum in u on the left-hand side implies:

$$\|f\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim \sup_{j\geq 0}\|P_j(\nabla_N f)\|_{L^2(S)} + \left(\sum_{j\geq 0}\|P_j(f^3)\|_{L^2(S)}\right)^{\frac{1}{3}} + \|\nabla\!\!\!/ f\|_{L^2(S)}. \tag{6.55}$$

Next, we estimate the second term in the right-hand side of (6.55). We have:

$$\|P_j(f^3)\|_{L^2(S)} \lesssim \sum_{l,m,q}\|P_j(P_l f P_m f P_q f)\|_{L^2(S)}. \tag{6.56}$$

We may assume:

$$l \geq m \geq q$$

and we consider the following three cases:

$$q > j,\ q \leq j < l \text{ and } l \leq j.$$

We start with the case $q > j$. Then, using the strong Bernstein inequality for scalars (5.34) for P_j, we have:

(6.57)

$$\begin{aligned}\|P_j(P_l f P_m f P_q f)\|_{L^2(S)} &\lesssim 2^j \|P_l f P_m f P_q f\|_{L^2_{[-2,2]}L^1(P_u)}\\ &\lesssim 2^j \|P_l f\|_{L^2(S)} \|P_m f\|_{L^\infty_{[-2,2]}L^4(P_u)} \|P_q f\|_{L^\infty_{[-2,2]}L^4(P_u)}\\ &\lesssim 2^{j+\frac{m}{2}+\frac{q}{2}} \|P_l f\|_{L^2(S)} \|P_m f\|_{L^\infty_{[-2,2]}L^2(P_u)} \|P_q f\|_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned}$$

where we used Bernstein for P_m and P_q in the last inequality.

Next, we consider the case $q \leq j < l$. Then, the boundedness of P_j on $L^2(P_u)$ yields:

(6.58)

$$\begin{aligned}\|P_j(P_l f P_m f P_q f)\|_{L^2(S)} &\lesssim \|P_l f P_m f P_q f\|_{L^2(S)}\\ &\lesssim \|P_l f\|_{L^2(S)} \|P_m f\|_{L^\infty(S)} \|P_q f\|_{L^\infty(S)}\\ &\lesssim 2^{m+q} \|P_l f\|_{L^2(S)} \|P_m f\|_{L^\infty_{[-2,2]}L^2(P_u)} \|P_q f\|_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned}$$

where we used in the last inequality the strong Bernstein inequality for scalars (5.34) for P_m and P_q.

Finally, we consider the case $l \leq j$. Using the finite band property for P_j, we have:

$$\begin{aligned}(6.59)\quad P_j(P_l f P_m f P_q f) &= 2^{-2j} P_j(\not\Delta(P_l f P_m f P_q f))\\ &= 2^{-2j} P_j(\not\Delta(P_l f) P_m f P_q f) + 2^{-2j} P_j(\not\nabla(P_l f)\not\nabla(P_m f) P_q f)\\ &\quad + \text{ permutations of } (l, m, p).\end{aligned}$$

Using the boundedness of P_j on $L^2(P_u)$, we have:

(6.60)

$$\begin{aligned}\|P_j(\not\Delta(P_l f) P_m f P_q f)\|_{L^2(S)} &\lesssim \|\not\Delta(P_l f) P_m f P_q f\|_{L^2(S)}\\ &\lesssim \|\not\Delta P_l f\|_{L^2(S)} \|P_m f\|_{L^\infty(S)} \|P_q f\|_{L^\infty(S)}\\ &\lesssim 2^{2l+m+q} \|P_l f\|_{L^2(S)} \|P_m f\|_{L^\infty_{[-2,2]}L^2(P_u)} \|P_q f\|_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned}$$

where we used in the last inequality the finite band property for P_l, and the strong Bernstein inequality for scalars (5.34) for P_m and P_q. Using again the boundedness of P_j on $L^2(P_u)$, we have:

(6.61)

$$\begin{aligned}\|P_j(\not\nabla(P_l) f \not\nabla(P_m f) P_q f)\|_{L^2(S)} &\lesssim \|\not\nabla(P_l f)\not\nabla(P_m f) P_q f\|_{L^2(S)}\\ &\lesssim \|\not\nabla P_l f\|_{L^2_{[-2,2]}L^4(P_u)} \|\not\nabla P_m f\|_{L^\infty_{[-2,2]}L^4(P_u)} \|P_q f\|_{L^\infty(S)}\\ &\lesssim 2^{\frac{3l}{2}+\frac{3m}{2}+q} \|P_l f\|_{L^2(S)} \|P_m f\|_{L^\infty_{[-2,2]}L^2(P_u)} \|P_q f\|_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (3.9), the Bochner inequality for scalars (5.63), the finite band property for P_l and P_m, and the strong Bernstein inequality for scalars (5.34) for P_q. Now, since we assumed that $l \geq m$, (6.59), (6.60) and (6.61) imply:

$$\|P_j(P_lfP_mfP_qf)\|_{L^2(S)} \lesssim 2^{-2j+2l+m+q}\|P_lf\|_{L^2(S)}\|P_mf\|_{L^\infty_{[-2,2]}L^2(P_u)}\|P_qf\|_{L^\infty_{[-2,2]}L^2(P_u)}. \tag{6.62}$$

Finally, in view of (6.56), (6.57), (6.58) and (6.62), and since we assumed that $l \geq m \geq q$, we obtain:

$$\begin{aligned}\|P_j(f^3)\|_{L^2(S)} \lesssim & \sum_{l,m,q} 2^{-\frac{|j-l|}{6}-\frac{|j-m|}{6}-\frac{|j-q|}{6}}(2^l\|P_lf\|_{L^2(S)}) \\ & \times (2^{\frac{m}{2}}\|P_mf\|_{L^\infty_{[-2,2]}L^2(P_u)})(2^{\frac{q}{2}}\|P_qf\|_{L^\infty_{[-2,2]}L^2(P_u)}).\end{aligned}$$

This yields:

$$\begin{aligned}\sum_{j\geq 0}\|P_j(f^3)\|_{L^2(S)} \lesssim & \sum_{j\geq 0}\left(\sum_{l\geq 0}2^{-\frac{|j-l|}{6}}2^{2l}\|P_lf\|^2_{L^2(S)}\right)^{\frac{1}{2}} \\ & \times \left(\sum_{m\geq 0}2^{-\frac{|j-m|}{6}}2^{2m}\|P_mf\|^4_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac{1}{4}} \\ & \times \left(\sum_{q\geq 0}2^{-\frac{|j-q|}{6}}2^{2q}\|P_qf\|^4_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac{1}{4}} \\ \lesssim & \left(\sum_{j\geq 0}2^{2j}\|P_jf\|^2_{L^2(S)}\right)^{\frac{3}{2}} + \left(\sum_{j\geq 0}2^{2j}\|P_jf\|^4_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac{3}{4}} \\ \lesssim & \|\nabla f\|_{L^2(S)}\left(\sum_{j\geq 0}2^{2j}\|P_jf\|^4_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac{1}{2}},\end{aligned} \tag{6.63}$$

where we used in the last inequality the Bessel inequality. Now, (6.55) and (6.63) imply:

$$\begin{aligned}\|f\|_{L^\infty_{[-2,2]}L^4(P_u)} \lesssim & \sup_{j\geq 0}\|P_j(\nabla_N f)\|_{L^2(S)} \\ & + \|\nabla f\|^{\frac{1}{3}}_{L^2(S)}\left(\sum_{j\geq 0}2^{2j}\|P_jf\|^4_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac{1}{6}} + \|\nabla f\|_{L^2(S)},\end{aligned}$$

which together with Lemma 6.6 yields (6.45). This concludes the proof of Lemma 6.7.

CHAPTER 7

REGULARITY OF THE FOLIATION WITH RESPECT TO ω

Let $u(x,\omega)$ the function constructed in Chapter 4. In this chapter, we prove Theorem 2.6 which deals with the control of the derivatives with respect to ω of the foliation generated by $u(x,\omega)$ on Σ. Recall that (Σ, g, k) coincides with $(\mathbb{R}^3, \delta, 0)$ in $|x| \geq 2$. Also, $u(x,\omega)$ coincides with $x.\omega$ in $|x| \geq 2$, and so $a \equiv 1$, $N \equiv \omega$ and $\theta \equiv 0$ in this region. Thus, u clearly satisfies the estimates of Theorem 2.6 in $|x| \geq 2$ and it is enough to control the derivatives with respect to ω of the function $u(x,\omega)$ solution to:

$$\begin{cases} \operatorname{tr}\theta - k_{NN} = 1 - a, \quad \text{on} \quad -2 < u < 2, \\ \qquad u(.,\omega) = -2 \quad \text{on} \quad x.\omega = -2, \end{cases} \tag{7.1}$$

in the strip $S = \{x / -2 < u < 2\}$.

To $u(x,\omega)$, we associate the quantities N, a, θ and K as in Section 2.2. We will have to differentiate these quantities several times with respect to ω. Since a and K (resp. N) are scalars (resp. is a vector field) defined on $-2 < u < 2$, the meaning of $\partial_\omega N$, $\partial_\omega a$ and $\partial_\omega K$ is clear. On the other hand, θ is a 2-tensor on P_u, and we need to extend it to a 2-tensor on $-2 < u < 2$ for $\partial_\omega \theta$ to be properly defined. We choose the trivial extension:

$$\theta(N, .) = \theta(., N) \equiv 0, \tag{7.2}$$

so that θ is a symmetric 2-tensor on $-2 < u < 2$. For consistency, we extend its traceless part $\widehat{\theta}$ in the same way:

$$\widehat{\theta}(N, .) = \widehat{\theta}(., N) \equiv 0, \tag{7.3}$$

so that $\widehat{\theta}$ is a symmetric 2-tensor on $-2 < u < 2$ satisfying:

$$\widehat{\theta}(X, Y) = \theta(X, Y) - \frac{1}{2}\operatorname{tr}\theta(X.Y - (X.N)(Y.N)), \tag{7.4}$$

where X, Y are two vector fields on Σ.

7.1. First order derivatives with respect to ω

The goal of this section is to prove (2.32). We first give an outline of the proof. Differentiating the second equation of (4.5) with respect to ω, we obtain:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega a = 2\nabla\!\!\!/\nabla_N a + 2R_{N\partial_\omega N} + \cdots, \tag{7.5}$$

where the first term on the right-hand side comes from the commutator $[\partial_\omega, \Delta\!\!\!/]$ (see (7.13)). Since $\nabla\!\!\!/\nabla_N a$ and R are in $L^2(S)$ respectively by (4.9) and (4.3), this suggests in view of Proposition 3.16 that:

$$\|\nabla_N\partial_\omega a\|_{L^2(S)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} \lesssim \varepsilon. \tag{7.6}$$

Next, we differentiate (7.5) with respect to ∇_N. We obtain:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N\partial_\omega a = 2\nabla\!\!\!/\nabla_N^2 a + 2\nabla_N R_{N\partial_\omega N} + \cdots. \tag{7.7}$$

The term $\nabla_N R_{N\partial_\omega N}$ may be treated using the contracted Bianchi identity for R-–as we did for $\nabla_N R_{NN}$ in Section 2.4—and turns out to be in $L^2_u H^{-1-\delta}(P_u)$. On the other hand, in view of the estimate (2.31) for $\nabla_N^2 a$, $\nabla\!\!\!/\nabla_N^2 a$ belongs to $L^2_u H^{-\frac{3}{2}}(P_u)$. This suggest in view of Proposition 5.33 that:

$$\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N^2\partial_\omega a\|_{L^2_u H^{-\frac{3}{2}}(P_u)} \lesssim \varepsilon. \tag{7.8}$$

By interpolation between (7.6) and (7.8), we should obtain $\partial_\omega a$ in $L^\infty_u H^{\frac{5}{4}}(P_u)$ which embeds in $L^\infty(S)$ since P_u has dimension 2.

We now turn to the estimates for $\partial_\omega\theta$. Since $\operatorname{tr}\theta = a - 1 + k_{NN}$, we differentiate in ω, and we easily obtain from the assumption on k (4.3) and the estimate (7.6) that $\nabla\partial_\omega\operatorname{tr}\theta \in L^2(S)$. To obtain estimates for $\partial_\omega\theta$, we differentiate the last two equations of (4.5) with respect to ω:

$$\begin{cases} \nabla\!\!\!/^B\partial_\omega\widehat{\theta}_{AB} = \frac{1}{2}\nabla\!\!\!/_A\partial_\omega\operatorname{tr}\theta + \cdots, \\ \nabla_N\partial_\omega\theta_{AB} = -\nabla\!\!\!/\nabla_N a - \nabla\!\!\!/_A\nabla\!\!\!/_B\partial_\omega a + \cdots, \end{cases} \tag{7.9}$$

where the first term on the right-hand side of the second equation comes from the commutator $[\partial_\omega, \nabla\!\!\!/^2]$ (see (7.12)). Using the fact that $\nabla\partial_\omega\operatorname{tr}\theta \in L^2(S)$, $\nabla\!\!\!/\nabla_N a \in L^2(S)$ and $\nabla\!\!\!/^2\partial_\omega a \in L^2(S)$, we then obtain $\nabla\partial_\omega\theta \in L^2(S)$.

Finally, we turn to the estimates for $\partial_\omega N$. Differentiating (4.4) with respect to ω, we obtain:

$$\begin{cases} \nabla\!\!\!/\partial_\omega N = \partial_\omega\theta + \cdots, \\ \nabla_N\partial_\omega N = -\nabla\!\!\!/\partial_\omega a + \cdots. \end{cases} \tag{7.10}$$

Together with the fact that $\nabla\!\!\!/\partial_\omega\theta$ and $\nabla\!\!\!/^2\partial_\omega a$ belong to $L^2(S)$, this implies that $\nabla\!\!\!/^2\partial_\omega N$ and $\nabla\!\!\!/\nabla_N\partial_\omega N$ belong to $L^2(S)$. Using Proposition 3.10, we obtain that $\partial_\omega N$ belongs to $L^\infty(S)$.

The rest of this section is as follows. We start by deriving commutator formulas for $[\partial_\omega, \nabla\!\!\!/]$, $[\partial_\omega, \Delta\!\!\!/]$ and $[\partial_\omega, \nabla\!\!\!/^2]$. Then, we prove the estimates for $\partial_\omega a$. We continue with the estimates for $\partial_\omega\theta$. And we conclude with the estimates for $\partial_\omega N$.

7.1.1. Commutator formulas. — We have the following commutator formulas:

Lemma 7.1. — *Let f a scalar on Σ. We have:*

$$[\partial_\omega, \nabla\!\!\!/]f = -\nabla\!\!\!/_{\partial_\omega N} f N - \nabla_N f \partial_\omega N, \tag{7.11}$$

$$\begin{aligned}[\partial_\omega, \nabla\!\!\!/^2]f(e_A, e_B) = &-(\partial_\omega N)_A \nabla^2 f(N, e_B) - (\partial_\omega N)_B \nabla^2 f(N, e_A)\\ &- \partial_\omega \theta_{AB} \nabla_N f - \theta_{AB} \nabla\!\!\!/_{\partial_\omega N} f\end{aligned} \tag{7.12}$$

and

$$[\partial_\omega, \Delta\!\!\!/]f = -2\nabla^2 f(N, \partial_\omega N) - \partial_\omega \mathrm{tr}\,\theta \nabla_N f - \mathrm{tr}\,\theta \nabla\!\!\!/_{\partial_\omega N} f. \tag{7.13}$$

Proof. — Differentiating with respect to ω the equality

$$\nabla\!\!\!/ f = \nabla f - \nabla_N f N \tag{7.14}$$

and using the fact that ∂_ω commutes with ∇ since g is independent of ω, we obtain:

$$[\partial_\omega, \nabla\!\!\!/]f = -\nabla_{\partial_\omega N} f N - \nabla_N f \partial_\omega N. \tag{7.15}$$

Now, we have:

$$g(\partial_\omega N, N) = 0, \tag{7.16}$$

which follows from the differentiation of $g(N, N) = 1$ with respect to ω. Thus, $\partial_\omega N$ is tangent to P_u which implies that $\nabla_{\partial_\omega N} f = \nabla\!\!\!/_{\partial_\omega N} f$. Together with (7.15), this yields (7.11).

We now turn to the proof of (7.12). Differentiating (7.14) by $\nabla\!\!\!/$, we obtain:

$$\nabla\!\!\!/^2 f(e_A, e_B) = \nabla^2 f(e_A, e_B) - \nabla_N f \theta_{AB}. \tag{7.17}$$

Let Π denote the projection of vector fields of Σ on vector fields tangent to P_u:

$$\Pi X = X - (X.N)N. \tag{7.18}$$

The commutator $[\partial_\omega, \Pi]$ satisfies:

$$[\partial_\omega, \Pi]X = -(X.\partial_\omega N)N - (X.N)\partial_\omega N. \tag{7.19}$$

For X, Y two vector fields on Σ independent of ω, we differentiate $\nabla\!\!\!/^2 f(\Pi X, \Pi Y)$ with respect to ω using (7.17), (7.19) and the fact that ∂_ω commutes with ∇:

$$\begin{aligned}\partial_\omega(\nabla\!\!\!/^2 f(\Pi X, \Pi Y)) = &\nabla^2 \partial_\omega f(\Pi X, \Pi Y) - (X.\partial_\omega N)\nabla^2 f(N, \Pi Y)\\ &- (X.N)\nabla^2 f(\partial_\omega N, \Pi Y) - (Y.\partial_\omega N)\nabla^2 f(N, \Pi X)\\ &- (Y.N)\nabla^2 f(\partial_\omega N, \Pi X) - \nabla_N \partial_\omega f \theta_{XY} - \nabla\!\!\!/_{\partial_\omega N} f \theta_{XY} - \nabla_N f \partial_\omega \theta_{XY},\end{aligned} \tag{7.20}$$

where we have used the fact that $\theta_{\Pi X \Pi Y} = \theta_{XY}$ from (7.2). evaluating (7.20) at $X = e_A$, $Y = e_B$ yields (7.12). Finally, taking the trace of (7.12) yields (7.13). ☐

Lemma 7.2. — *Let ρ a symmetric 2-tensor on Σ such that $\rho(N, .) \equiv 0$. Then, we have:*

$$\begin{aligned}([\partial_\omega, \mathrm{di}\!\!/\mathrm{v}]\rho)_A = &-\mathrm{tr}\,\theta \rho_{\partial_\omega N A} - \theta_{AB}\rho_{B \partial_\omega N} - \nabla_N \rho_{\partial_\omega N B} + \theta_{\partial_\omega N C}\rho_{CA}\\ &+ (\partial_\omega N)_A \theta_{BC} \rho_{CB}.\end{aligned} \tag{7.21}$$

Proof. — For any symmetric 2-tensor ν on Σ, we have:

(7.22)
$$\begin{aligned}\nabla\!\!\!/_C\nu_{AB} &= e_C(\nu_{AB}) - \nu(\nabla\!\!\!/_C e_A, e_B) - \nu(e_A, \nabla\!\!\!/_C e_B)\\ &= e_C(\nu_{AB}) - \nu(\nabla_C e_A - g(\nabla_C e_A, N)N, e_B) - \nu(e_A, \nabla e_B - g(\nabla_C e_B, N)N)\\ &= e_C(\nu_{AB}) - \nu(\nabla_C e_A, e_B) - \nu(e_A, \nabla_C e_B) - \theta_{AC}\nu_{NB} - \theta_{BC}\nu_{NA}\\ &= \nabla_C\nu_{AB} - \theta_{AC}\nu_{NB} - \theta_{BC}\nu_{NA}.\end{aligned}$$

Applying (7.22) to ρ and using the fact that $\rho(N,.) = \rho(.,N) \equiv 0$, we obtain:

$$\nabla\!\!\!/_C\rho_{AB} = \nabla_C\rho_{AB}. \tag{7.23}$$

Let X a vector field on Σ independent of ω. Using (7.19) and (7.23), we have:

(7.24)
$$\begin{aligned}\partial_\omega((\mathrm{div}\!\!\!/\,\rho)_{\Pi X}) &= \partial_\omega(\nabla\!\!\!/_A\rho_{A\Pi X}) = \partial_\omega(\nabla_A\rho_{A\Pi X})\\ &= \nabla_{\partial_\omega e_A}\rho_{A\Pi X} + \nabla_A\rho_{\partial_\omega e_A\Pi X} + \nabla_A\partial_\omega\rho_{A\Pi X} - (X.N)\nabla_A\rho_{A\partial_\omega N}\\ &\quad -(X.\partial_\omega N)\nabla_A\rho_{AN}.\end{aligned}$$

Now, differentiating $g(e_A, e_B) = \delta_{AB}$ and $g(e_A, N) = 0$ with respect to ω, we obtain:

$$\left\{\begin{aligned}\partial_\omega e_1 &= g(\partial_\omega e_1, e_2)e_2 - g(\partial_\omega N, e_1)N,\\ \partial_\omega e_2 &= -g(\partial_\omega e_1, e_2)e_1 - g(\partial_\omega N, e_2)N.\end{aligned}\right. \tag{7.25}$$

This yields

$$\nabla_{\partial_\omega e_A}\rho_{A\Pi X} + \nabla_A\rho_{\partial_\omega e_A\Pi X} = -\nabla_N\rho_{\partial_\omega N\Pi X} - \nabla_{\partial_\omega N}\rho_{N\Pi X}. \tag{7.26}$$

Since $\rho(N,.) = \rho(.,N) \equiv 0$, we have:

$$\nabla_A\rho_{NB} = e_A(\rho_{NB}) - \rho_{\nabla_A NB} - \rho_{N\nabla_A e_B} = -\theta_{AC}\rho_{CB}. \tag{7.27}$$

Using again $\rho(N,.) = \rho(.,N) \equiv 0$, we have:

$$\partial_\omega\rho(N,.) = -\rho(\partial_\omega N,.), \tag{7.28}$$

which together with (7.22) applied to $\partial_\omega\rho$ yields:

$$\begin{aligned}\nabla_A\partial_\omega\rho_{AB} &= \nabla\!\!\!/_A\partial_\omega\rho_{AB} + \theta_{AA}\partial_\omega\rho_{NB} + \theta_{AB}\partial_\omega\rho_{AN}\\ &= (\mathrm{div}\!\!\!/\,\partial_\omega\rho)_B - \mathrm{tr}\,\theta\rho_{\partial_\omega NB} - \theta_{AB}\rho_{\partial_\omega NA}.\end{aligned} \tag{7.29}$$

Finally, (7.24), (7.26), (7.27) and (7.29) yield:

$$\begin{aligned}\partial_\omega((\mathrm{div}\!\!\!/\,\rho)_{\Pi X}) &= (\mathrm{div}\!\!\!/\,\partial_\omega\rho)_{\Pi X} - \mathrm{tr}\,\theta\rho_{\partial_\omega N\Pi X} - \theta_{B\Pi X}\rho_{B\partial_\omega N} - \nabla_N\rho_{\partial_\omega N\Pi X}\\ &\quad +\theta_{\partial_\omega NB}\rho_{B\Pi X} - (X.N)\nabla_B\rho_{B\partial_\omega N} + (X.\partial_\omega N)\theta_{BC}\rho_{CB}.\end{aligned} \tag{7.30}$$

Taking $X = e_A$ in (7.30) yields (7.21). □

We conclude this section by recalling the link between $\nabla\!\!\!/_A\nabla_N f$ and $\nabla^2 f(e_A, N)$ for a scalar function f:

$$\nabla\!\!\!/_A\nabla_N f = \nabla^2 f(e_A, N) + \theta(e_A, \nabla\!\!\!/ f). \tag{7.31}$$

7.1.2. Estimates for $\partial_\omega a$. — Note that the first equation of (4.5), (4.7) and the fact that (g,k,Σ) coincides with $(\delta,0,\mathbb{R}^3)$ for $|x|\geq 2$ yields:

$$\nabla^p(\partial_\omega a)=0,\ \nabla^p\partial_\omega\theta=0,\ \nabla^p(\partial_\omega N-\partial_\omega\omega)=0 \text{ for all } p\in\mathbb{N} \text{ on } u=-2, \tag{7.32}$$

so that integrations by parts will not create boundary terms at $u=-2$.

Differentiating the second equation of (4.5) with respect to ω, and using the commutator Formula (7.11), (7.13), the fact that $\partial_\omega N$ is tangent to P_u by (7.16), and (7.31), we obtain:

$$\nabla_N\partial_\omega a-a^{-1}\not{\Delta}\partial_\omega a=h, \tag{7.33}$$

where h is given by:

$$\begin{aligned}h=&-\not{\nabla}_{\partial_\omega N}a-a^{-2}\partial_\omega a\not{\Delta}a-2\not{\nabla}_{\partial_\omega N}\nabla_N a+2\theta(\partial_\omega N,\not{\nabla}a)\\&-\partial_\omega\operatorname{tr}\theta\nabla_N a-\operatorname{tr}\theta\not{\nabla}_{\partial_\omega N}a+2\theta\partial_\omega\theta+\partial_\omega(\nabla_N(k_{NN}))+2R_{N\partial_\omega N}.\end{aligned} \tag{7.34}$$

Using (4.4), we have:

$$\nabla_N(k_{NN})=\nabla_N k_{NN}-2k(\not{\nabla}a,N), \tag{7.35}$$

which together with (7.11) yields:

$$\begin{aligned}\partial_\omega(\nabla_N(k_{NN}))=&\nabla_{\partial_\omega N}k_{NN}+2\nabla_N k_{N\partial_\omega N}-2k(\not{\nabla}a,\partial_\omega N)\\&-2k(\not{\nabla}\partial_\omega a,N)+2\not{\nabla}_{\partial_\omega N}ak_{NN}+2\nabla_N ak_{N\partial_\omega N}.\end{aligned} \tag{7.36}$$

Using (7.34) and (7.36), we estimate the norm of h in $L^2(S)$:

$$\begin{aligned}\|h\|_{L^2(S)}\lesssim&\|\not{\nabla}a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}+\|a^{-2}\|_{L^\infty}\|\not{\Delta}a\|_{L^2(S)}\|\partial_\omega a\|_{L^\infty}\\&+\|\not{\nabla}\nabla_N a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}+\|\theta\|_{L^4(S)}\|\not{\nabla}a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\\&+\|\partial_\omega\operatorname{tr}\theta\|_{L^4(S)}\|\nabla_N a\|_{L^4(S)}+\|\operatorname{tr}\theta\|_{L^4(S)}\|\not{\nabla}a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\\&\times\|\partial_\omega N\|_{L^\infty(S)}+\|\theta\|_{L^4(S)}\|\partial_\omega\theta\|_{L^4(S)}+\|\nabla k\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}\\&+\|k\|_{L^4(S)}\|\nabla a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}+\|k\|_{L^4(S)}\|\not{\nabla}\partial_\omega a\|_{L^4(S)}\\&+\|R\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}.\end{aligned} \tag{7.37}$$

Together with (4.3), (4.9), (4.11), (4.12) and (4.13), this yields:

$$\|h\|_{L^2(S)}\lesssim\varepsilon\|\partial_\omega N\|_{L^\infty(S)}+\varepsilon\|\partial_\omega a\|_{L^\infty(S)}+\varepsilon\|\not{\nabla}\partial_\omega a\|_{L^4(S)}+\varepsilon\|\partial_\omega\theta\|_{L^4(S)}. \tag{7.38}$$

Proposition 3.16, (7.32), (7.33) and (7.38) yield:

$$\begin{aligned}\|\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|\nabla\partial_\omega a\|_{L^2(S)}+\|\not{\nabla}\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|\not{\nabla}^2\partial_\omega a\|_{L^2(S)}\\ \lesssim\varepsilon(\|\partial_\omega a\|_{L^\infty}+\|\partial_\omega N\|_{L^\infty(S)}+\|\partial_\omega\theta\|_{L^4(S)}).\end{aligned} \tag{7.39}$$

Next, we differentiate Equation (7.33) by ∇_N. We obtain:

$$\nabla_N(\nabla_N\partial_\omega a)-a^{-1}\not{\Delta}(\nabla_N\partial_\omega a)=[\nabla_N,a^{-1}\not{\Delta}](\partial_\omega a)+\nabla_N(h), \tag{7.40}$$

where h is given by (7.34). Next, we estimate each term in the right-hand side of (7.40) starting with the first one. In view of the commutator Formula (2.20), we have:

$$a[\nabla_N,a^{-1}\not{\Delta}]\partial_\omega a=h_1+2a^{-1}\not{\nabla}a\not{\nabla}\nabla_N\partial_\omega a+a^{-1}\not{\Delta}a\nabla_N\partial_\omega a, \tag{7.41}$$

where the scalar function h_1 is given by:

$$h_1 = -(\mathrm{tr}\,\theta + a^{-1}\nabla_N a)\mbox{$\Delta \mkern-13mu /$\,}\partial_\omega a - 2\widehat{\theta}\cdot\nabla\mkern-13mu /\,^2\partial_\omega a - 2R_{N.}\cdot\nabla\mkern-13mu /\,\partial_\omega a - \nabla\mkern-13mu /\,\mathrm{tr}\,\theta\cdot\nabla\mkern-13mu /\,\partial_\omega a + 2\widehat{\theta}\cdot a^{-1}\nabla\mkern-13mu /\, a\cdot\nabla\mkern-13mu /\,\partial_\omega a.$$

h_1 satisfies the following estimate:

$$\begin{aligned}\|h_1\|_{L^2_{[-2,2]}L^1(P_u)} \lesssim{}& (\|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|a^{-1}\nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)})\|\nabla\mkern-13mu /\,^2\partial_\omega a\|_{L^2(S)}\\ &+ (\|R\|_{L^2(S)} + \|\nabla\mkern-13mu /\,\mathrm{tr}\,\theta\|_{L^2(S)} + \|\widehat{\theta}a^{-1}\nabla\mkern-13mu /\, a\|_{L^2(S)})\|\nabla\mkern-13mu /\,\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)}.\end{aligned}$$

Together with the estimates (7.39) for $\partial_\omega a$, (4.3) for R, (4.11) for a, and (4.12) for θ, we obtain:

$$\|h_1\|_{L^2_{[-2,2]}L^1(P_u)} \lesssim \varepsilon(\|\partial_\omega a\|_{L^\infty(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega\theta\|_{L^4(S)}). \tag{7.42}$$

Also, the second term in (7.41) satisfies in view of the product estimate (5.82):

$$\sup_{j\geq 0} 2^{-j}\|P_j(a^{-1}\nabla\mkern-13mu /\, a\nabla\mkern-13mu /\,\nabla_N\partial_\omega a)\|_{L^2(S)} \lesssim \|\nabla\mkern-13mu /\,\nabla_N\partial_\omega a\|_{L^2_u H^{-\frac{1}{2}}(P_u)}\|a^{-1}\nabla\mkern-13mu /\, a\|_{L^\infty_u H^{\frac{1}{2}}(P_u)},$$

which together with the Lemma 5.7, the embedding (5.74), and the estimate (4.9) for a yield:

$$\sup_{j\geq 0} 2^{-j}\|P_j(a^{-1}\nabla\mkern-13mu /\, a\nabla\mkern-13mu /\,\nabla_N\partial_\omega a)\|_{L^2(S)} \lesssim \varepsilon\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}. \tag{7.43}$$

The third term in the right-hand side of (7.41) satisfies in view of the product estimate (5.82):

$$\begin{aligned}\sup_{j\geq 0} 2^{-j}\|P_j(a^{-1}\mbox{$\Delta \mkern-13mu /$\,} a\nabla_N\partial_\omega a)\|_{L^2(S)} &\lesssim \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}\|a^{-1}\mbox{$\Delta \mkern-13mu /$\,} a\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)}\\ &\lesssim \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}(\|\mathrm{d}\mkern-10mu /\,\mathrm{iv}(a^{-1}\nabla\mkern-13mu /\, a)\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)}\\ &\quad + \|a^{-2}|\nabla\mkern-13mu /\, a|^2\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)}),\end{aligned}$$

which together with the Lemma 5.7, the embedding (5.74), and the estimate (4.9) for a yield:

$$\sup_{j\geq 0} 2^{-j}\|P_j(a^{-1}\mbox{$\Delta \mkern-13mu /$\,} a\nabla_N\partial_\omega a)\|_{L^2(S)} \lesssim \varepsilon\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}. \tag{7.44}$$

Next, we estimate the second term in (7.40), i.e., $\nabla_N(h)$. In view of (7.34), we have:

$$\nabla_N(h) = h_2 + h_3 - 2a^{-1}\mathrm{d}\mkern-10mu /\,\mathrm{iv}(a\partial_\omega N\nabla_N^2 a) + \nabla_N(\partial_\omega(\nabla_N(k_{NN}))) + 2\nabla_N R_{N\partial_\omega N}, \tag{7.45}$$

where h_2 and h_3 are given respectively by:

$$\begin{aligned}h_2 ={}& a^{-2}\nabla\mkern-13mu /\,\partial_\omega a\cdot\nabla\mkern-13mu /\,\nabla_N a + a^{-3}\partial_\omega a\nabla\mkern-13mu /\, a\cdot\nabla\mkern-13mu /\,\nabla_N a + a^{-2}\partial_\omega a[\nabla_N, \mbox{$\Delta \mkern-13mu /$\,}]a - 2a^{-3}\nabla_N a\partial_\omega a\mbox{$\Delta \mkern-13mu /$\,} a\\ &+ 2a^{-3}|\nabla\mkern-13mu /\, a|^2\nabla_N\partial_\omega a + 2\nabla_N\theta(\partial_\omega N, \nabla\mkern-13mu /\, a) + 2\theta(\nabla_N\partial_\omega N, \nabla\mkern-13mu /\, a) + \theta(\partial_\omega N, \nabla_N\nabla\mkern-13mu /\, a)\\ &- \nabla_N(\partial_\omega\mathrm{tr}\,\theta)\nabla_N a - \nabla_N\mathrm{tr}\,\theta\nabla\mkern-13mu /\,_{\partial_\omega N}(a) - \mathrm{tr}\,\theta\nabla_N\nabla\mkern-13mu /\,_{\partial_\omega N}a + 2\nabla_N\nabla\mkern-13mu /\, a\cdot\nabla\mkern-13mu /\,\partial_\omega a\\ &- 2\nabla_N\nabla\mkern-13mu /\,_{\partial_\omega N}a\nabla_N a + 2\nabla_N\theta\cdot\partial_\omega\theta + 2\theta\nabla_N\partial_\omega\theta - 2R_{a^{-1}\nabla\mkern-13mu /\, a\,\partial_\omega N} + 2R_{N\nabla_N\partial_\omega N}\end{aligned}$$

and:

$$h_3 = a^{-1}\mathrm{di\!/\!v}(a^{-1}\partial_\omega a\nabla\!\!\!/\nabla_N a) + a^{-2}\nabla\!\!\!/ a\cdot\nabla\!\!\!/\nabla_N\partial_\omega a + a^{-2}\nabla_N(\partial_\omega a)\Delta\!\!\!/ a + 2\mathrm{di\!/\!v}(\partial_\omega N)\nabla_N^2 a - 6a^{-1}\nabla\!\!\!/_{\partial_\omega N}a\nabla_N^2 a - \partial_\omega \mathrm{tr}\,\theta\nabla_N^2 a + a\nabla\!\!\!/ a\cdot\nabla_N\nabla\!\!\!/(\partial_\omega a).$$

Let us first estimate h_2 and h_3. In view of the definition of h_2, we have:

$$\begin{aligned}
&\|h_2\|_{L^2_{[-2,2]}L^1(P_u)}\\
&\lesssim \|a^{-2}\|_{L^\infty}(\|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\partial_\omega a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)})\|\nabla\!\!\!/\nabla a\|_{L^2(S)}\\
&\quad + \|a^{-3}\partial_\omega a\|_{L^\infty(S)}(\|[\nabla_N,\Delta\!\!\!/]a\|_{L^2_{[-2,2]}L^1(P_u)} + \|\nabla_N a\|_{L^\infty(S)2}\|\Delta\!\!\!/ a\|_{L^2(S)})\\
&\quad + \|a^{-3}\|_{L^\infty}\|\nabla\!\!\!/ a\|^2_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N\partial_\omega a\|_{L^2(S)}\\
&\quad + \|\nabla_N\theta\|_{L^2(S)}\|\partial_\omega N\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla_N\partial_\omega N\|_{L^4(S)}\|\nabla\!\!\!/ a)\|_{L^4(S)}\\
&\quad + \|\theta\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla_N\nabla\!\!\!/ a\|_{L^2(S)} + \|\nabla_N(\partial_\omega \mathrm{tr}\,\theta)\|_{L^2(S)}\|\nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)}\\
&\quad + \|\nabla_N \mathrm{tr}\,\theta\|_{L^2(S)}\|\nabla\!\!\!/_{\partial_\omega N}(a)\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\mathrm{tr}\,\theta\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla_N\nabla\!\!\!/_{\partial_\omega N}a\|_{L^2(S)}\\
&\quad + \|\nabla_N\nabla\!\!\!/ a\|_{L^2(S)}\|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N\nabla\!\!\!/_{\partial_\omega N}a\|_{L^2(S)}\|\nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)}\\
&\quad + \|\nabla_N\theta\|_{L^2(S)}\|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla_N\partial_\omega\theta\|_{L^2(S)}\\
&\quad + \|R\|_{L^2(S)}\|a^{-1}\nabla\!\!\!/ a\,\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\partial_\omega N\|_{L^\infty(S)} + \|R\|_{L^2(S)}\|\nabla_N\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)},
\end{aligned}$$

which together with the commutator Formula (2.19) for $[\nabla_N,\Delta\!\!\!/]$, the estimates (4.9) (4.11) for a, (4.10) (4.12) for θ, the estimate (4.3) for R, and the estimate (7.39) for $\partial_\omega a$ yields:

$$\begin{aligned}
\|h_2\|_{L^2_{[-2,2]}L^1(P_u)} \lesssim\ & \varepsilon(\|\partial_\omega a\|_{L^\infty(S)} + \|\nabla_N\partial_\omega N\|_{L^4(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega\theta\|_{L^4(S)}\\
& + \|\nabla_N\partial_\omega\theta\|_{L^2(S)} + \|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^2(P_u)}).
\end{aligned}\tag{7.46}$$

Also, in view of the definition of h_3 and the product estimate (5.82), the finite band property for P_j and the estimates (7.43) and (7.44), we have:

$$\begin{aligned}
\sup_{j\geq 0}2^{-j}\|P_j(ah_3)\|_{L^2(P_u)} \lesssim\ & \|a^{-1}\partial_\omega a\|_{L^\infty(S)}\|\nabla\!\!\!/\nabla_N a\|_{L^2(S)} + \varepsilon\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}\\
& + (\|\mathrm{di\!/\!v}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\nabla\!\!\!/_{\partial_\omega N}a\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\\
& + \|a\partial_\omega \mathrm{tr}\,\theta\|_{L^\infty_u H^{\frac{1}{2}}(P_u)})\|\nabla_N^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)},
\end{aligned}$$

which together with the estimate (2.31) for $\nabla_N a$ and $\nabla_N^2 a$, the estimates (4.9) (4.11) for a, the estimate (7.39) for $\partial_\omega a$, the embedding (5.74) and the product estimate (5.85) yields:

$$\begin{aligned}
\sup_{j\geq 0}2^{-j}\|P_j(ah_3)\|_{L^2(P_u)} \lesssim\ & \varepsilon(\|\partial_\omega a\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\partial_\omega N\|_{L^\infty(S)}\\
& + \|\mathrm{di\!/\!v}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\partial_\omega\theta\|_{L^4(S)} + \|\nabla\partial_\omega \mathrm{tr}\,\theta\|_{L^2(S)}).
\end{aligned}\tag{7.47}$$

Next, we estimate the third term in (7.45). In view of the product estimate (5.83) with $b=\frac{1}{2}$, $f=\nabla_N^2 a$ and $G=a\partial_\omega N$, we have:

$$\begin{aligned}&\|\mathrm{d}\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N^2 a)\|_{L^2_u H^{-\frac{3}{2}}(P_u)}\\ &\qquad\lesssim \|\nabla_N^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)}(\|a\partial_\omega N\|_{L^\infty(S)}+\|\nabla\!\!\!/(a\partial_\omega N)\|_{L^\infty_{[-2,2]}L^2(P_u)}).\end{aligned}$$

Together with the estimate (2.31) for $\nabla_N^2 a$, the estimate (2.31) for $\nabla_N a$, and the estimates (4.9) (4.11) for a, this yields:

$$\|\mathrm{d}\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N^2 a)\|_{L^2_u H^{-\frac{3}{2}}(P_u)}\lesssim\varepsilon(\|\partial_\omega N\|_{L^\infty(S)}+\|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}).\tag{7.48}$$

Next, we estimate the third term in (7.45). We have:

$$\begin{aligned}&\nabla_N(\partial_\omega(\nabla_N(k_{NN})))=\nabla_{\partial_\omega N}(\nabla_N k_{NN})+[\nabla_N,\nabla_{\partial_\omega N}](k_{NN})+2\nabla_N(\nabla_N(k_{N\partial_\omega N}))\\
&=\mathrm{d}\!\!/\mathrm{iv}(\partial_\omega N\nabla_N(k_{NN}))-\mathrm{d}\!\!/\mathrm{iv}(\partial_\omega N)\nabla_N(k_{NN})+\nabla_{\nabla_N\partial_\omega N}(k_{NN})+a^{-1}\nabla\!\!\!/_{\partial_\omega N}a\nabla_N(k_{NN})\\
&\quad-\theta(\partial_\omega N,A)\nabla\!\!\!/_A(k_{NN})+2\nabla_N(\nabla_N k_{N\partial_\omega N}-k(a^{-1}\nabla\!\!\!/ a,\partial_\omega N)+k(N,\nabla_N\partial_\omega N))\\
&=\mathrm{d}\!\!/\mathrm{iv}(\partial_\omega N(\nabla_N k_{NN}-2k(a^{-1}\nabla\!\!\!/ a,N)))-\mathrm{d}\!\!/\mathrm{iv}(\partial_\omega N)(\nabla_N k_{NN}-2k(a^{-1}\nabla\!\!\!/ a,N))\\
&\quad+\nabla_{\nabla_N\partial_\omega N}(k_{NN})+a^{-1}\nabla\!\!\!/_{\partial_\omega N}a(\nabla_N k_{NN}-2k(a^{-1}\nabla\!\!\!/ a,N))\\
&\quad-\theta(\partial_\omega N,A)(\nabla_A k_{NN}+2\theta_{AB}k_{BN})+2\nabla_N(\nabla_N k_{N\partial_\omega N})-2\nabla_N k(a^{-1}\nabla\!\!\!/ a,\partial_\omega N)\\
&\quad-2k(\nabla_N(a^{-1}\nabla\!\!\!/ a),\partial_\omega N)-4k(a^{-1}\nabla\!\!\!/ a,\nabla_N\partial_\omega N)+2\nabla_N k(N,\nabla_N\partial_\omega N)\\
&\quad+2k(N,\nabla_N\nabla_N\partial_\omega N),\end{aligned}$$

where we used the structure Equations (4.4) for N and the commutator Formula (2.18). This yields:

$$\begin{aligned}&\nabla_N(\partial_\omega(\nabla_N(k_{NN})))\\ &\qquad=a^{-1}\mathrm{d}\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N k_{NN})+2\nabla_N(\nabla_N k_{N\partial_\omega N})+2k(N,\nabla_N\nabla_N\partial_\omega N)+h_4,\end{aligned}\tag{7.49}$$

with h_4 satisfying:

$$\begin{aligned}\|h_4\|_{L^2_{[-2,2]}L^1(P_u)}\lesssim&\Big(\|\nabla k\|_{L^2(S)}(\|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)})\\
&\quad+\|k\|_{L^\infty_{[-2,2]}L^2(P_u)}(\|\nabla\!\!\!/\nabla a\|_{L^2(S)}+\|a^{-1}\nabla\!\!\!/ a\|^2_{L^4(S)}+\|\theta\|^2_{L^4(S)})\Big)\\
&\quad\times\|\partial_\omega N\|_{L^\infty(S)}\\
&\quad+\Big(\|\nabla k\|_{L^2(S)}+\|k\|_{L^\infty_{[-2,2]}L^4(P_u)}\|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\Big)\\
&\quad\times\|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)})\\
\lesssim&\,\varepsilon(\|\partial_\omega N\|_{L^\infty(S)}+\|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}),\end{aligned}\tag{7.50}$$

where we used in the last inequality the estimate (4.3) for k, the estimate (4.9) for a and the estimate (4.12) for θ. Now, in view of the constraint Equations (2.25), we have

$$\nabla_N k_{NA}=-\nabla_B k_{BA}.$$

This yields:

$$\begin{aligned}\nabla_N k_{N\partial_\omega N} &= -\nabla_B k_{B\partial_\omega N}\\ &= -\mathrm{d}i\!\!/v(k_{\partial_\omega N.}) + k_{AB}(\nabla_A\partial_\omega N)_B - \mathrm{tr}\,\theta k_{N\partial_\omega N}.\end{aligned}$$

Differentiating with respect to ∇_N, we obtain:

$$\begin{aligned}&\nabla_N(\nabla_N k_{N\partial_\omega N})\\ &= -\mathrm{d}i\!\!/v(\nabla_N(k_{\partial_\omega N.})) - [\nabla_N, \mathrm{d}i\!\!/v](k_{\partial_\omega N.}) + \nabla_N k_{AB}(\nabla_A\partial_\omega N)_B + k_{AB}(\nabla_N\nabla_A\partial_\omega N)_B\\ &\quad - \nabla_N \mathrm{tr}\,\theta k_{N\partial_\omega N} - \mathrm{tr}\,\theta\nabla_N k_{N\partial_\omega N} + \mathrm{tr}\,\theta k(a^{-1}\nabla\!\!\!/ a\,, \partial_\omega N) - \mathrm{tr}\,\theta k_{N\nabla_N\partial_\omega N}\\ &= -\mathrm{d}i\!\!/v(\nabla_N k_{\partial_\omega N.}) - (a^{-1}\nabla\!\!\!/ a\,\nabla_N + \theta\nabla\!\!\!/ + R + a^{-1}\nabla\!\!\!/ a\,\cdot\theta)\cdot k_{\partial_\omega N.}\\ &\quad + \nabla_N k_{AB}(\nabla_A\partial_\omega N)_B + k_{AB}([\nabla_N, \nabla\!\!\!/]_A\partial_\omega N)_B - \nabla_N \mathrm{tr}\,\theta k_{N\partial_\omega N} - \mathrm{tr}\,\theta\nabla_N k_{N\partial_\omega N}\\ &\quad + \mathrm{tr}\,\theta k(a^{-1}\nabla\!\!\!/ a\,, \partial_\omega N) - \mathrm{tr}\,\theta k_{N\nabla_N\partial_\omega N}\\ &= -\mathrm{d}i\!\!/v(\nabla_N k_{\partial_\omega N.}) - (a^{-1}\nabla\!\!\!/ a\,\nabla_N + \theta\nabla\!\!\!/ + R + a^{-1}\nabla\!\!\!/ a\,\cdot\theta)\cdot k_{\partial_\omega N.}\\ &\quad + \nabla_N k_{AB}(\nabla_A\partial_\omega N)_B + k\cdot(a^{-1}\nabla\!\!\!/ a\,\nabla_N + \theta\nabla\!\!\!/ + R + a^{-1}\nabla\!\!\!/ a\,\cdot\theta)\cdot\partial_\omega N\\ &\quad - \nabla_N \mathrm{tr}\,\theta k_{N\partial_\omega N} - \mathrm{tr}\,\theta\nabla_N k_{N\partial_\omega N} + \mathrm{tr}\,\theta k(a^{-1}\nabla\!\!\!/ a\,, \partial_\omega N) - \mathrm{tr}\,\theta k_{N\nabla_N\partial_\omega N},\end{aligned}$$

where we used the commutator estimate (2.17). This yields:

$$\nabla_N(\nabla_N k_{N\partial_\omega N}) = -a^{-1}\mathrm{d}i\!\!/v(a\nabla_N k_{\partial_\omega N.}) + h_5, \tag{7.51}$$

with h_5 satisfying:

(7.52)

$$\begin{aligned}\|h_5\|_{L^2_{[-2,2]}L^1(P_u)} \lesssim{}& \|\nabla k\|_{L^2(S)}((\|a^{-1}\nabla\!\!\!/ a\,\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)})\|\partial_\omega N\|_{L^\infty(S)}\\ &+ \|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)})\\ &+ \|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}\|k\|_{L^4(S)}(\|\theta\|_{L^4(S)} + \|a^{-1}\nabla\!\!\!/ a\,\|_{L^4(S)})\\ &+ \|\nabla_N\theta\|_{L^2(S)}\|k\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\partial_\omega N\|_{L^\infty(S)}\\ \lesssim{}& \varepsilon(\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}),\end{aligned}$$

where we used in the last inequality the estimate (4.3) for k, the estimates (4.9) (4.11) for a, and the estimates (4.10) (4.12) for θ.

Next, we estimate the fourth term in (7.45). Using the twice contracted Bianchi identities (4.34), we have

$$\nabla_N R_{AN} = -\nabla_B R_{AB} + k\cdot\nabla_A k.$$

This yields:

$$\nabla_N R_{N\partial_\omega N} = -\mathrm{d}i\!\!/v(R_{\partial_\omega N.}) + \mathrm{tr}\,\theta R_{N\partial_\omega N} - R_{AB}(\nabla_A\partial_\omega N)_B + k\cdot\nabla_{\partial_\omega N}k.$$

We obtain:

$$\nabla_N R_{N\partial_\omega N} = -a^{-1}\mathrm{d}i\!\!/v(aR_{\partial_\omega N.}) + h_6, \tag{7.53}$$

with h_6 satisfying:

$$(7.54)\quad \begin{aligned}\|h_6\|_{L^2_{[-2,2]}L^1(P_u)} &\lesssim \|R\|_{L^2(S)}(\|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)} + (\|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\quad + \|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)})\|\partial_\omega N\|_{L^\infty(S)}) \\ &\quad + \|\nabla k\|_{L^2(S)}\|k\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\partial_\omega N\|_{L^\infty(S)} \\ &\lesssim \varepsilon(\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}),\end{aligned}$$

where we used in the last inequality the estimate (4.3) for R and k, the estimate (4.13) for k, and the estimate (4.12) for θ.

Finally, (7.45), (7.49), (7.51) and (7.53) imply:

$$\begin{aligned}\nabla_N(h) &= h_2 + h_3 + h_4 + 2h_5 + 2h_6 - 2a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N^2 a) + 2k(N, \nabla_N\nabla_N\partial_\omega N) \\ &\quad + a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N k_{NN}) - 2a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(a\nabla_N k_{\partial_\omega N.}) - 2a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(aR_{\partial_\omega N.}).\end{aligned}$$

Together with (7.40) and (7.41), this implies:

$$(7.55)\qquad \nabla_N(\nabla_N\partial_\omega a) - a^{-1}\Delta\!\!\!/(\nabla_N\partial_\omega a) = h_7,$$

where h_7 is given by:

$$\begin{aligned}h_7 &= a^{-1}(h_1 + 2a^{-1}\nabla\!\!\!/ a\nabla\!\!\!/\nabla_N\partial_\omega a + a^{-1}\Delta\!\!\!/ a\nabla_N\partial_\omega a) + h_2 + h_3 + h_4 + 2h_5 + 2h_6 \\ &\quad - 2a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N^2 a) + 2k(N, \nabla_N\nabla_N\partial_\omega N) + a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N k_{NN}) \\ &\quad - 2a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(a\nabla_N k_{\partial_\omega N.}) - 2a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(aR_{\partial_\omega N.}).\end{aligned}$$

Together with the strong Bernstein inequality for scalars (5.61), the finite band property for P_j, and the estimate (4.9) for a, this yields:

$$\begin{aligned}&\|P_j(ah_7)\|_{L^2(S)} \\ &\lesssim 2^j\Big(\|h_1\|_{L^2_{[-2,2]}L^1(P_u)} + \|h_2\|_{L^2_{[-2,2]}L^1(P_u)} + \|h_4\|_{L^2_{[-2,2]}L^1(P_u)} + \|h_5\|_{L^2_{[-2,2]}L^1(P_u)} \\ &\quad + \|h_6\|_{L^2_{[-2,2]}L^1(P_u)}\Big) + \|P_j(ah_3)\|_{L^2(S)} + \|P_j(a^{-1}\nabla\!\!\!/ a\cdot\nabla\!\!\!/\nabla_N\partial_\omega a)\|_{L^2(S)} \\ &\quad + \|P_j(a^{-1}\Delta\!\!\!/ a\nabla_N\partial_\omega a)\|_{L^2(S)} + \|P_j(\mathrm{d}\!\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N^2 a))\|_{L^2(S)} \\ &\quad + \|P_j(\mathrm{d}\!\!\!/\mathrm{iv}(ak(N, \nabla_N\nabla_N\partial_\omega N)))\|_{L^2(S)} + 2^j(\|\nabla_N k\|_{L^2(S)} \\ &\quad + \|R\|_{L^2(S)})\|a\|_{L^\infty(S)}\|\partial_\omega N\|_{L^\infty(S)}.\end{aligned}$$

Together with the estimates (7.38), (7.42), (7.43), (7.44), (7.46), (7.50), (7.52) and (7.54), the estimate (4.9) for a, the estimate (7.39) for $\partial_\omega a$, and the estimate (4.3) for R and k, we obtain:

$$\begin{aligned}\|P_j(ah_7)\|_{L^2(S)} &\lesssim \varepsilon 2^j\Big(\|\partial_\omega a\|_{L^\infty(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\nabla_N\partial_\omega N\|_{L^4(S)} + \|\nabla(\partial_\omega N)\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\quad + \|\mathrm{d}\!\!\!/\mathrm{iv}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\partial_\omega\theta\|_{L^4(S)} + \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\quad + \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}\Big) + \|P_j(\mathrm{d}\!\!\!/\mathrm{iv}(a\partial_\omega N\nabla_N^2 a))\|_{L^2(S)} \\ &\quad + \|P_j(ak(N, \nabla_N\nabla_N\partial_\omega N))\|_{L^2(S)}.\end{aligned}$$

Together with the estimate (7.48), we finally obtain:

$$\begin{aligned}(7.56)\quad & \|ah_7\|_{L^2_u H^{-\frac{3}{2}}(P_u)}\\ &\lesssim \varepsilon\Big(\|\partial_\omega a\|_{L^\infty(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\nabla_N\partial_\omega N\|_{L^4(S)} + \|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}\\ &\quad + \|\mathrm{div\!\!/}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}\Big)\\ &\quad + \|ak(N, \nabla_N\nabla_N\partial_\omega N)\|_{L^2_u H^{-\frac{3}{2}}(P_u)}.\end{aligned}$$

Now, in view of (7.55) and Proposition 5.33, we have:

$$\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N\partial_\omega a\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)} + \|\nabla_N^2\partial_\omega a\|_{L^2_u H^{-\frac{3}{2}}(P_u)} \lesssim \|ah_7\|_{L^2_u H^{-\frac{3}{2}}(P_u)}.$$

Together with (7.56), this yields:

$$\begin{aligned}(7.57)\quad & \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N\partial_\omega a\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)} + \|\nabla_N^2\partial_\omega a\|_{L^2_u H^{-\frac{3}{2}}(P_u)}\\ &\lesssim \varepsilon\Big(\|\partial_\omega a\|_{L^\infty(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\nabla_N\partial_\omega N\|_{L^4(S)} + \|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}\\ &\quad + \|\mathrm{div\!\!/}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}\Big)\\ &\quad + \|ak(N, \nabla_N\nabla_N\partial_\omega N)\|_{L^2_u H^{-\frac{3}{2}}(P_u)}.\end{aligned}$$

In view of Corollary 5.19, we have

$$(7.58)\qquad \|\partial_\omega a\|_{L^\infty(S)} \lesssim \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}.$$

Thus, we finally obtain, in view of (7.57) and (7.58):

(7.59)

$$\begin{aligned}&\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N\partial_\omega a\|_{L^\infty_u H^{-\frac{1}{2}}(P_u)} + \|\nabla_N^2\partial_\omega a\|_{L^2_u H^{-\frac{3}{2}}(P_u)}\\ &\lesssim \varepsilon\Big(\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla_N\partial_\omega N\|_{L^4(S)} + \|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}\\ &\quad + \|\mathrm{div\!\!/}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\nabla\partial_\omega\theta\|_{L^2(S)}\Big) + \|ak(N, \nabla_N\nabla_N\partial_\omega N)\|_{L^2_u H^{-\frac{3}{2}}(P_u)}.\end{aligned}$$

7.1.3. Estimates for $\partial_\omega\theta$. — Let us start by showing that $\partial_\omega\widehat{\theta}$ is traceless when seen as a tensor on P_u. Differentiating (7.4) with respect to ω, we obtain:

$$\begin{aligned}(7.60)\quad \partial_\omega\widehat{\theta}(X,Y) &= \partial_\omega\theta(X,Y) - \frac{1}{2}\partial_\omega \mathrm{tr}\,\theta(X.Y - (X.N)(Y.N))\\ &\quad + \frac{1}{2}\mathrm{tr}\,\theta((X.\partial_\omega N)(Y.N) + (X.N)(Y.\partial_\omega N)),\end{aligned}$$

which yields:

$$(7.61)\qquad \partial_\omega\widehat{\theta}_{AB} = \partial_\omega\theta_{AB} - \frac{1}{2}\partial_\omega \mathrm{tr}\,\theta\delta_{AB},$$

so that:

$$(7.62)\qquad \mathrm{tr}\,(\partial_\omega\widehat{\theta}) = \mathrm{tr}\,(\partial_\omega\theta) - \partial_\omega \mathrm{tr}\,\theta.$$

We compute $\partial_\omega \mathrm{tr}\,\theta$:

$$\partial_\omega \mathrm{tr}\,\theta = \partial_\omega(\theta_{AA}) = \mathrm{tr}\,(\partial_\omega\theta) + 2\theta(e_A, \partial_\omega e_A). \tag{7.63}$$

Together with (7.2), (7.63) and (7.25), this yields

$$\partial_\omega \mathrm{tr}\,\theta = \mathrm{tr}\,(\partial_\omega\theta). \tag{7.64}$$

Finally, (7.62) and (7.64) imply that $\partial_\omega\widehat{\theta}$ is traceless:

$$\mathrm{tr}\,(\partial_\omega\widehat{\theta}) = 0. \tag{7.65}$$

We now turn to the estimates for $\partial_\omega \mathrm{tr}\,\theta$. Differentiating the first equation of (4.5) with respect to ω, we obtain:

$$\partial_\omega \mathrm{tr}\,\theta = -\partial_\omega a + 2k_{N\partial_\omega N}, \tag{7.66}$$

so that:

$$\nabla\partial_\omega \mathrm{tr}\,\theta = -\nabla\partial_\omega a + 2\nabla k(N, \partial_\omega N) + 2k(\nabla N, \partial_\omega N) + 2k(N, \nabla\partial_\omega N), \tag{7.67}$$

which in turn yields:

$$\begin{aligned}\|\nabla\partial_\omega \mathrm{tr}\,\theta\|_{L^2(S)} \lesssim{}& \|\nabla\partial_\omega a\|_{L^2(S)} + \|\nabla k\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}\\ &+ \|k\|_{L^4(S)}\|\nabla N\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|k\|_{L^4(S)}\|\nabla\partial_\omega N\|_{L^4(S)}.\end{aligned} \tag{7.68}$$

Together with Proposition 3.9, (4.3), (4.11) and (4.13), we obtain:

$$\|\nabla\partial_\omega \mathrm{tr}\,\theta\|_{L^2(S)} \lesssim \|\nabla\partial_\omega a\|_{L^2(S)} + \varepsilon\|\partial_\omega N\|_{L^\infty(S)} + \varepsilon\|\nabla\partial_\omega N\|_{L^4(S)}. \tag{7.69}$$

We now turn to the estimates for $\nabla\!\!\!/\partial_\omega\widehat{\theta}$. We differentiate the third equation of (4.5) with respect to ω. Using (7.11), and (7.21), we obtain:

$$(\mathrm{d}\!\!/\mathrm{iv}\,\partial_\omega\widehat{\theta})_A = h, \tag{7.70}$$

where h is given by:

$$\begin{aligned}h ={}& \mathrm{tr}\,\theta\widehat{\theta}_{\partial_\omega NA} + \theta_{AB}\widehat{\theta}_{B\partial_\omega N} + \nabla_N\widehat{\theta}_{\partial_\omega NA} - \theta_{\partial_\omega NC}\widehat{\theta}_{CA}\\ &- (\partial_\omega N)_A\theta_{BC}\widehat{\theta}_{CB} + \frac{1}{2}\nabla\!\!\!/_A\partial_\omega \mathrm{tr}\,\theta - \frac{1}{2}\nabla_N \mathrm{tr}\,\theta(\partial_\omega N)_A + R_{A\partial_\omega N} - (\partial_\omega N)_A R_{NN}.\end{aligned} \tag{7.71}$$

Differentiating (7.4) with respect to ∇_N, we obtain:

$$\nabla_N\widehat{\theta}_{AB} = \nabla_N\theta_{AB} - \frac{1}{2}\nabla_N \mathrm{tr}\,\theta\delta_{AB}. \tag{7.72}$$

Also, the definition of $\mathrm{tr}\,\theta$ and $\widehat{\theta}$ implies:

$$\theta_{AC}\widehat{\theta}_{CB} - \widehat{\theta}_{AC}\theta_{CB} = 0, \tag{7.73}$$

which together with (7.71) and (7.72) yields:

$$\begin{aligned}h ={}& \tfrac{1}{2}\nabla\!\!\!/_A\partial_\omega \mathrm{tr}\,\theta + \nabla_N\theta_{\partial_\omega NB} - \tfrac{1}{2}\nabla_N \mathrm{tr}\,\theta(\partial_\omega N)_A + \mathrm{tr}\,\theta\widehat{\theta}_{\partial_\omega NA} - (\partial_\omega N)_A\theta_{BC}\widehat{\theta}_{CB}\\ &+ R_{A\partial_\omega N} - (\partial_\omega N)_A R_{NN}.\end{aligned} \tag{7.74}$$

We estimate the norm of h in $L^2(S)$:

(7.75)
$$\|h\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega \mathrm{tr}\,\theta\|_{L^2(S)} + \|\nabla_N\theta\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\theta\|^2_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|R\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}.$$

Together with (4.3), (4.10) and (4.12), this yields:

$$\|h\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega \mathrm{tr}\,\theta\|_{L^2(S)} + \varepsilon\|\partial_\omega N\|_{L^\infty(S)}. \tag{7.76}$$

Proposition 3.18, (7.65), (7.70) and (7.76) imply:

$$\|\nabla\!\!\!/\partial_\omega\widehat{\theta}\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega \mathrm{tr}\,\theta\|_{L^2(S)} + \varepsilon\|\partial_\omega N\|_{L^\infty(S)} + \|K\|^{\frac{1}{2}}_{L^2(S)}\|\partial_\omega\widehat{\theta}\|_{L^4(S)}. \tag{7.77}$$

Finally, Corollary 3.8, (4.9), (7.32), (7.69) and (7.77) yield:

$$\begin{aligned}\|\nabla\!\!\!/\partial_\omega\widehat{\theta}\|_{L^2(S)} \lesssim{}& \varepsilon\|\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\partial_\omega a\|_{L^2(S)}\\ &+ \varepsilon\|\partial_\omega N\|_{L^\infty(S)} + \varepsilon\|\nabla\partial_\omega N\|_{L^4(S)} + \varepsilon^{\frac{1}{2}}\|\nabla\partial_\omega\theta\|_{L^2(S)}.\end{aligned} \tag{7.78}$$

We now turn to the estimates for $\nabla_N\partial_\omega\theta$. Let X, Y two vector fields on Σ independent of ω. (7.2) and the last equation of (4.5) imply:

$$a^{-1}\nabla\!\!\!/_{\Pi X}\nabla\!\!\!/_{\Pi Y}a + \nabla_N\theta_{\Pi X\Pi Y} + \theta_X^j\theta_{jY} + Kg(\Pi X, \Pi Y) = R(\Pi X, \Pi Y). \tag{7.79}$$

We differentiate (7.79) with respect to ω. Using (7.12), (7.19) and , and evaluating at $X = e_A$, $Y = e_B$, we obtain:

$$\nabla_N\partial_\omega\theta_{AB} = h, \tag{7.80}$$

where h is given by:

(7.81)
$$\begin{aligned}h ={}& -a^{-1}\nabla\!\!\!/_A\nabla\!\!\!/_B\partial_\omega a + (\partial_\omega N)_A a^{-1}\nabla^2 a(e_B, N) + (\partial_\omega N)_B a^{-1}\nabla^2 a(e_A, N)\\ &+ \partial_\omega\theta_{AB}a^{-1}\nabla_N a + \theta_{AB}a^{-1}\nabla\!\!\!/_{\partial_\omega N}a + a^{-2}\partial_\omega a\nabla\!\!\!/_A\nabla\!\!\!/_B\partial_\omega a - \nabla_{\partial_\omega N}\theta_{AB}\\ &- (\partial_\omega N)_A\nabla_N\theta_{NB} - (\partial_\omega N)_B\nabla_N\theta_{NA} - \partial_\omega\theta_A^C\theta_{CB} - \theta_A^C\partial_\omega\theta_{CB} - \partial_\omega K\gamma_{AB}\\ &- (\partial_\omega N)_A R_{NB} - (\partial_\omega N)_B R_{NA}.\end{aligned}$$

Using (4.4) and (7.2), we have:

$$\nabla_N\theta_{NA} = \theta(\nabla\!\!\!/ a, e_A). \tag{7.82}$$

Using (7.22), (7.31) and (7.82), we rewrite (7.81) as:

$$\begin{aligned}h ={}& -a^{-1}\nabla\!\!\!/_A\nabla\!\!\!/_B\partial_\omega a + (\partial_\omega N)_A a^{-1}\nabla\!\!\!/_B\nabla_N a + (\partial_\omega N)_B a^{-1}\nabla\!\!\!/_A\nabla_N a\\ &+ \partial_\omega\theta_{AB}a^{-1}\nabla_N a + \theta_{AB}a^{-1}\nabla\!\!\!/_{\partial_\omega N}a + a^{-2}\partial_\omega a\nabla\!\!\!/_A\nabla\!\!\!/_B\partial_\omega a - \nabla\!\!\!/_{\partial_\omega N}\theta_{AB}\\ &- 2(\partial_\omega N)_A\theta(\nabla\!\!\!/ a, e_B) - 2(\partial_\omega N)_B\theta(\nabla\!\!\!/ a, e_A) - \partial_\omega\theta_A^C\theta_{CB} - \theta_A^C\partial_\omega\theta_{CB}\\ &- \partial_\omega K\gamma_{AB} - (\partial_\omega N)_A R_{NB} - (\partial_\omega N)_B R_{NA}.\end{aligned} \tag{7.83}$$

Thus, we estimate the norm of h in $L^2(S)$ by:
(7.84)

$$\begin{aligned}\|h\|_{L^2(S)} \lesssim{}& \|a^{-1}\|_{L^\infty(S)}\Big(\|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \|\nabla\!\!\!/\nabla_N a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}\\ &+ \|\partial_\omega\theta\|_{L^4(S)}\|\nabla_N a\|_{L^4(S)} + \|\theta\|_{L^4(S)}\|\nabla\!\!\!/ a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\Big)\\ &+ \|a^{-2}\|_{L^\infty(S)}\|\partial_\omega a\|_{L^\infty(S)}\|\nabla\!\!\!/ a\|_{L^2(S)} + \|\nabla\!\!\!/\theta\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}\\ &+ \|\theta\|_{L^4(S)}\|\nabla\!\!\!/ a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega\theta\|_{L^4(S)}\|\theta\|_{L^4(S)} + \|\partial_\omega K\|_{L^2(S)}\\ &+ \|R\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}.\end{aligned}$$

Differentiating (4.6) with respect to ω and using Corollary 3.8, (4.3), (4.12) and (7.32), we obtain:

$$(7.85)\qquad\begin{aligned}\|\partial_\omega K\|_{L^2(S)} &\lesssim \|R\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\theta\|_{L^4(S)}\|\partial_\omega\theta\|_{L^4(S)}\\ &\lesssim \varepsilon\|\partial_\omega N\|_{L^\infty(S)} + \varepsilon\|\partial_\omega\theta\|_{L^4(S)}\\ &\lesssim \varepsilon\|\partial_\omega N\|_{L^\infty(S)} + \varepsilon\|\nabla\partial_\omega\theta\|_{L^2(S)}.\end{aligned}$$

(4.3), (4.9), (4.11), (4.12), (7.84) and (7.85) yield:
(7.86)

$$\|h\|_{L^2(S)} \lesssim \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)}+\varepsilon\Big(\|\partial_\omega a\|_{L^\infty(S)}+\|\partial_\omega N\|_{L^\infty(S)}+\|\partial_\omega\theta\|_{L^4(S)}+\|\nabla\partial_\omega\theta\|_{L^2(S)}\Big).$$

Corollary 3.8, Proposition 3.9, (7.32), (7.80) and (7.86) yield:
(7.87)

$$\|\nabla_N\partial_\omega\theta\|_{L^2(S)} \lesssim \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \varepsilon\Big(\|\partial_\omega a\|_{L^\infty(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\partial_\omega\theta\|_{L^2(S)}\Big).$$

Finally, (7.69), (7.78) and (7.87) yield:
(7.88)

$$\|\nabla\partial_\omega\theta\|_{L^2(S)} \lesssim \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \varepsilon\Big(\|\partial_\omega a\|_{L^\infty(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\partial_\omega N\|_{L^4(S)}\Big).$$

7.1.4. Estimates for $\partial_\omega N$. — We start by estimating the norm of $\nabla\partial_\omega N$ in $L^4(S)$. Let X, Y two vector fields on Σ independent of ω. We rewrite the first equation of (4.4) as:

$$(7.89)\qquad g(\nabla_{\Pi X}N, \Pi Y) = \theta(\Pi X, \Pi Y).$$

We differentiate (7.89) with respect to ω. Using (7.19) and evaluating at $X = e_A$, $Y = e_B$, we obtain:

$$(7.90)\qquad g(\nabla_A\partial_\omega N, e_B) = \partial_\omega\theta_{AB} - (\partial_\omega N)_A a^{-1}\nabla\!\!\!/_B a.$$

Also, using (7.16), we have:

$$(7.91)\qquad g(\nabla_A\partial_\omega N, N) = -g(\partial_\omega N, \nabla_A N) = -\theta(\partial_\omega N, e_A).$$

Differentiating the second equation of (4.4) and using (7.11), we obtain:
(7.92)

$$\nabla_N\partial_\omega N = -\theta(\partial_\omega N, e_A)e_A - a^{-1}\nabla\!\!\!/\partial_\omega a + a^{-1}\nabla_N a\partial_\omega N + a^{-1}\nabla\!\!\!/_{\partial_\omega N}aN + a^{-2}\partial_\omega a\nabla\!\!\!/ a.$$

(7.90), (7.91) and (7.92) yield:

$$\begin{aligned}\|\nabla\partial_\omega N\|_{L^4(S)} \lesssim{}& \|\partial_\omega\theta\|_{L^4(S)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)} + (\|\nabla\!\!\!/ a\|_{L^4(S)} + \|\theta\|_{L^4(S)} \\ &+ \|\nabla_N a\|_{L^4(S)})(\|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|_{L^\infty(S)}).\end{aligned} \tag{7.93}$$

Together with (4.11) and (4.12), this yields:

$$\|\nabla\partial_\omega N\|_{L^4(S)} \lesssim \|\partial_\omega\theta\|_{L^4(S)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)} + \varepsilon\|\partial_\omega N\|_{L^\infty(S)} + \varepsilon\|\partial_\omega a\|_{L^\infty(S)}. \tag{7.94}$$

Finally, using Corollary 3.8 and Proposition 3.9, we obtain:

$$\begin{aligned}\|\nabla\partial_\omega N\|_{L^4(S)} \lesssim{}& \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} \\ &+ \varepsilon\|\partial_\omega N\|_{L^\infty(S)} + \varepsilon\|\partial_\omega a\|_{L^\infty(S)}.\end{aligned} \tag{7.95}$$

Next, we estimate the norm of $\nabla\partial_\omega N$ in $L^\infty_{[-2,2]}L^2(P_u)$. In view of (7.90), (7.91) and (7.92), we have:

$$\begin{aligned}&\|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\lesssim \|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|a^{-1}\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + (\|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\quad + \|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|a^{-1}\nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)})(\|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|_{L^\infty(S)}).\end{aligned}$$

Together with (4.11) for a and (4.12) for θ, this yields:

$$\begin{aligned}&\|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\quad\lesssim \|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \varepsilon(\|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|_{L^\infty(S)}).\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}&\|\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\quad\lesssim \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \varepsilon(\|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|_{L^\infty(S)}).\end{aligned} \tag{7.96}$$

Next, we estimate the norm of $\mathrm{div}\!\!\!/(\partial_\omega N)$ in $L^\infty_u H^{\frac{1}{2}}(P_u)$. In view of (7.90), we have:

$$\mathrm{div}\!\!\!/(\partial_\omega N) = \mathrm{tr}\,(\partial_\omega\theta) - a^{-1}\nabla\!\!\!/_{\partial_\omega N}a.$$

This yields:

$$\|\mathrm{div}\!\!\!/(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} \lesssim \|\partial_\omega\theta\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|a^{-1}\nabla\!\!\!/_{\partial_\omega N}a\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}.$$

In view of Corollary 5.16, we finally obtain:

$$\begin{aligned}&\|\mathrm{div}\!\!\!/(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} \\ &\quad\lesssim \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla(a^{-1}\nabla\!\!\!/_{\partial_\omega N}a)\|_{L^2(S)} \\ &\quad\lesssim \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla(a^{-1}\nabla\!\!\!/ a)\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|a^{-1}\nabla\!\!\!/ a\|_{L^4(S)}\|\nabla\partial_\omega N\|_{L^4(S)} \\ &\quad\lesssim \|\nabla\partial_\omega\theta\|_{L^2(S)} + \varepsilon(\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\partial_\omega N\|_{L^4(S)}),\end{aligned} \tag{7.97}$$

where we used in the last inequality the estimates (4.9) and (4.11) for a.

In view of the right-hand side of (7.59), we need to control

$$\|ak(N, \nabla_N\nabla_N\partial_\omega N)\|_{L^2_u H^{-\frac{3}{2}}(P_u)}.$$

In view of (7.92), we have:

$$\begin{aligned}
&\nabla_N\nabla_N\partial_\omega N\\
&\quad = -\nabla_N\theta(\partial_\omega N, e_A)e_A - \theta(\nabla_N\partial_\omega N, e_A)e_A - a^{-1}\nabla\!\!\!/\nabla_N\partial_\omega a - a^{-1}[\nabla_N, \nabla\!\!\!/]\partial_\omega a\\
&\qquad + a^{-2}\nabla_N a\nabla\!\!\!/\partial_\omega a + a^{-1}\nabla_N^2 a\partial_\omega N + a^{-1}\nabla_N a\nabla_N\partial_\omega N - a^{-2}(\nabla_N a)^2\partial_\omega N\\
&\qquad + \nabla_N(a^{-1}\nabla\!\!\!/ a)_{\partial_\omega N}N + a^{-1}\nabla\!\!\!/_{\nabla_N\partial_\omega N}aN - a^{-1}\nabla\!\!\!/_{\partial_\omega N}aa^{-1}\nabla\!\!\!/ a + \nabla_N(a^{-2}\nabla\!\!\!/ a)\partial_\omega a\\
&\qquad + a^{-2}\nabla_N\partial_\omega a\nabla\!\!\!/ a.
\end{aligned}$$

This yields

$$\nabla_N\nabla_N\partial_\omega N = -a^{-1}\nabla\!\!\!/\nabla_N\partial_\omega a - a^{-2}\nabla\!\!\!/ a\nabla_N\partial_\omega a + a^{-1}\nabla_N^2 a\partial_\omega N + H, \tag{7.98}$$

where, in view of the commutator Formula (2.18), the vector field H is given by

$$\begin{aligned}
H = &-\nabla_N\theta(\partial_\omega N, e_A)e_A - \theta(\nabla_N\partial_\omega N, e_A)e_A + a^{-1}\theta\cdot\nabla\!\!\!/\partial_\omega a + a^{-2}\nabla_N a\nabla\!\!\!/\partial_\omega a\\
&+ a^{-1}\nabla_N a\nabla_N\partial_\omega N - a^{-2}(\nabla_N a)^2\partial_\omega N + \nabla_N(a^{-1}\nabla\!\!\!/ a)_{\partial_\omega N}N + a^{-1}\nabla\!\!\!/_{\nabla_N\partial_\omega N}aN\\
&- a^{-1}\nabla\!\!\!/_{\partial_\omega N}aa^{-1}\nabla\!\!\!/ a + \nabla_N(a^{-2}\nabla\!\!\!/ a)\partial_\omega a.
\end{aligned}$$

We have

$$\begin{aligned}
\|H\|_{L^2(S)} \lesssim &\Big(\|\nabla\theta\|_{L^2(S)} + \|\nabla_N(a^{-2}\nabla\!\!\!/ a)\|_{L^2(S)} + (\|\theta\|_{L^4(S)} + \|\nabla\partial_\omega N\|_{L^4(S)}\\
&\quad + \|a^{-1}\nabla\!\!\!/\partial_\omega a\|_{L^4(S)} + \|a^{-1}\nabla a\|_{L^4(S)})^2\Big)(1 + \|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|_{L^\infty(S)}),
\end{aligned}$$

which together with (4.9), (4.10), (4.11) and (4.12) yields:

$$\|H\|_{L^2(S)} \lesssim \varepsilon(\|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)} + \|\nabla\partial_\omega N\|_{L^4(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|_{L^\infty(S)}). \tag{7.99}$$

Using the finite band property for P_j, we obtain

$$\begin{aligned}
&\|P_j(ak(N, H))\|_{L^2(S)}\\
&\quad \lesssim 2^j\|ak(N, H)\|_{L^2_{[-2,2]}L^1(P_u)}\\
&\quad \lesssim \|a\|_{L^\infty(S)}\|k\|_{L^\infty_{[-2,2]}L^2(P_u)}\|H\|_{L^2(S)}\\
&\quad \lesssim \varepsilon(\|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)} + \|\nabla\partial_\omega N\|_{L^4(S)} + \|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|_{L^\infty(S)}),
\end{aligned} \tag{7.100}$$

where we used in the last inequality the estimate (7.99) for H, the estimate (4.9) for a and the estimate (4.13) for k. Next, we estimate the other terms generated by the right-hand side of (7.98). In view of the product estimate (5.82), the embeddings (5.74) and Lemma 5.7, we have

$$\begin{aligned}
\|P_j(ak(N, a^{-1}\nabla\!\!\!/\nabla_N\partial_\omega a))\|_{L^2(S)} &\lesssim 2^j\|kN\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\|\nabla\!\!\!/\nabla_N\partial_\omega a\|_{L^2_u H^{-\frac{1}{2}}(P_u)}\\
&\lesssim 2^j\|kN\|_{H^1(S)}\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)}\\
&\lesssim 2^j\varepsilon\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac{1}{2}}(P_u)},
\end{aligned} \tag{7.101}$$

where we used in the last inequality the estimates (4.3) and (4.13) for k, (4.11) and (4.4). Using the finite band property for P_j, we have

(7.102)
$$\begin{aligned}\|P_j(ak(N,a^{-2}\nabla\!\!\!/ a\nabla_N\partial_\omega a))\|_{L^2(S)} &\lesssim 2^j\|ak(N,a^{-2}\nabla\!\!\!/ a\nabla_N\partial_\omega a)\|_{L^2_{[-2,2]}L^1(P_u)}\\ &\lesssim 2^j\|ka^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla_N\partial_\omega a\|_{L^2(S)}\\ &\lesssim 2^j\|k\|_{L^\infty_{[-2,2]}L^4(P_u)}\|a^{-1}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N\partial_\omega a\|_{L^2(S)}\\ &\lesssim 2^j\varepsilon\|\nabla_N\partial_\omega a\|_{L^2(S)},\end{aligned}$$

where we used in the last inequality the estimate (4.13) for k and the estimate (4.11) for a. In view of the product estimate (5.82) and the embeddings (5.74), we have

$$\begin{aligned}(7.103)\quad \|P_j(ak(N,a^{-1}\nabla^2_N a\partial_\omega N))\|_{L^2(S)} &\lesssim 2^j\|kN\partial_\omega N\|_{L^\infty_u H^{\frac12}(P_u)}\|\nabla^2_N a\|_{L^2_u H^{-\frac12}(P_u)}\\ &\lesssim 2^j\|kN\partial_\omega N\|_{H^1(S)}\|\nabla^2_N a\|_{L^2_u H^{-\frac12}(P_u)}\\ &\lesssim 2^j\varepsilon(\|\partial_\omega N\|_{L^\infty(S)}+\|\nabla\partial_\omega N\|_{L^4(S)}),\end{aligned}$$

where we used in the last inequality the estimates (4.3) and (4.13) for k, (4.11), (4.4), and the estimate (2.31) for $\nabla^2_N a$. Finally, (7.98)–(7.103) imply

(7.104)
$$\begin{aligned}&\|ak(N,\nabla_N\nabla_N\partial_\omega N)\|_{L^2_u H^{-\frac32}(P_u)}\\ &\lesssim \varepsilon\Big(\|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)}+\|\nabla\partial_\omega N\|_{L^4(S)}+\|\partial_\omega N\|_{L^\infty(S)}+\|\partial_\omega a\|_{L^\infty(S)}+\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac12}(P_u)}\Big).\end{aligned}$$

We now estimate the norm of $\partial_\omega N$ in $L^\infty(S)$. Using (7.32) and the fact that (Σ,g) coincides with $(\mathbb{R}^3,\delta)$ for $|x|\geq 1$ by Section 2.1, we have:

$$(7.105)\qquad g(\partial_\omega N,\partial_\omega N)=I\ \text{ on }\ x.\omega=-2,$$

where I is the 2×2 identity matrix.

We will estimate the $L^\infty(S)$ norm of $g(\partial_\omega N,\partial_\omega N)-I$ using Proposition 3.10. To this end, we need to estimate the norm of:

$$(7.106)\qquad \nabla\!\!\!/^2(g(\partial_\omega N,\partial_\omega N)-I)=2\nabla\!\!\!/(g(\nabla\!\!\!/\partial_\omega N,\partial_\omega N)),$$
$$(7.107)\qquad \text{and}\quad \nabla\!\!\!/\nabla_N(g(\partial_\omega N,\partial_\omega N)-I)=2\nabla\!\!\!/(g(\nabla_N\partial_\omega N,\partial_\omega N)),$$

in $L^2(S)$. First we estimate the norm of (7.106) in $L^2(S)$. Using (7.90), we have:

$$(7.108)\qquad g(\nabla\!\!\!/_A\partial_\omega N,\partial_\omega N)=\partial_\omega\theta(\partial_\omega N,e_A)-(\partial_\omega N)_A a^{-1}\nabla\!\!\!/_{\partial_\omega N}a,$$

which together with (7.106) yields:

(7.109)
$$\begin{aligned}\nabla\!\!\!/^2_{AB}(g(\partial_\omega N,\partial_\omega N)-I)=&\,2\nabla\!\!\!/_A(\partial_\omega\theta)(e_B,\partial_\omega N)+2\partial_\omega\theta(e_B,\nabla\!\!\!/_A\partial_\omega N)\\ &-2g(\nabla\!\!\!/_A\partial_\omega N,e_B)\nabla\!\!\!/_{\partial_\omega N}a-2(\partial_\omega N)_B\nabla\!\!\!/(a^{-1}\nabla\!\!\!/ a)(e_A,\partial_\omega N)\\ &-2(\partial_\omega N)_B g(\nabla\!\!\!/_A\partial_\omega N,a^{-1}\nabla\!\!\!/ a).\end{aligned}$$

We estimate the norm of $\nabla\!\!\!/^2(g(\partial_\omega N,\partial_\omega N)-I)$:

$$
\begin{aligned}
\|\nabla\!\!\!/^2(g(\partial_\omega N,\partial_\omega N)-I)\|_{L^2(S)} \lesssim{}& \|\nabla\!\!\!/\partial_\omega\theta\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}+(\|\partial_\omega\theta\|_{L^4(S)}\\
&+\|\partial_\omega N\|_{L^\infty(S)}\|a^{-1}\nabla\!\!\!/ a\|_{L^4(S)})\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}\\
&+\|\partial_\omega N\|^2_{L^\infty(S)}\|\nabla\!\!\!/(a^{-1}\nabla\!\!\!/ a)\|_{L^2(S)},
\end{aligned} \tag{7.110}
$$

which together with (4.9) and (4.11) yields:

$$
\begin{aligned}
\|\nabla\!\!\!/^2(g(\partial_\omega N,\partial_\omega N)-I)\|_{L^2(S)} \lesssim{}& \|\nabla\!\!\!/\partial_\omega\theta\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}+(\|\partial_\omega\theta\|_{L^4(S)}\\
&+\varepsilon\|\partial_\omega N\|_{L^\infty(S)})\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}+\varepsilon\|\partial_\omega N\|^2_{L^\infty(S)}.
\end{aligned} \tag{7.111}
$$

We turn to the estimate of the norm of (7.107) in $L^2(S)$. Using (7.92), we have:

$$
g(\nabla_N\partial_\omega N,\partial_\omega N)=-\theta(\partial_\omega N,\partial_\omega N)-a^{-1}\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a)+a^{-1}\nabla_N a|\partial_\omega N|^2+a^{-2}\partial_\omega a\nabla\!\!\!/_{\partial_\omega N}a, \tag{7.112}
$$

which together with (7.107) yields:

$$
\begin{aligned}
&\nabla\!\!\!/_A\nabla_N(g(\partial_\omega N,\partial_\omega N)-I)\\
&=-2\nabla\!\!\!/_A\theta(\partial_\omega N,\partial_\omega N)-4\theta(\nabla\!\!\!/_A\partial_\omega N,\partial_\omega N)-2\nabla\!\!\!/(a^{-1}\nabla\!\!\!/(\partial_\omega a))(\partial_\omega N,e_A)\\
&\quad-2g(\nabla\!\!\!/_A\partial_\omega N,a^{-1}\nabla\!\!\!/\partial_\omega a)+2\nabla\!\!\!/(a^{-1}\nabla_N a)|\partial_\omega N|^2+4a^{-1}\nabla_N a g(\nabla\!\!\!/\partial_\omega N,\partial_\omega N)\\
&\quad+\nabla\!\!\!/(a^{-2}\nabla\!\!\!/ a)\partial_\omega a\cdot\partial_\omega N+a^{-2}\partial_\omega a\nabla\!\!\!/_{\nabla\!\!\!/\partial_\omega N}a+a^{-2}\nabla\!\!\!/(\partial_\omega a)\nabla\!\!\!/_{\partial_\omega N}a.
\end{aligned} \tag{7.113}
$$

We estimate the norm of $\nabla\!\!\!/\nabla_N(g(\partial_\omega N,\partial_\omega N)-I)$:

$$
\begin{aligned}
&\|\nabla\!\!\!/\nabla_N(g(\partial_\omega N,\partial_\omega N)-I)\|_{L^2(S)}\\
&\lesssim\|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}+(\|\nabla\!\!\!/\theta\|_{L^2(S)}+\|\nabla\!\!\!/\nabla_N a\|_{L^2(S)})\|\partial_\omega N\|^2_{L^\infty(S)}\\
&\quad+(\|\theta\|_{L^4(S)}+\|\nabla_N a\|_{L^4(S)})\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}+\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}\|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)},
\end{aligned} \tag{7.114}
$$

which together with (4.9), (4.10), (4.11) and (4.12) yields:

$$
\begin{aligned}
&\|\nabla\!\!\!/\nabla_N(g(\partial_\omega N,\partial_\omega N)-I)\|_{L^2(S)}\\
&\lesssim\|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}+\varepsilon\|\partial_\omega N\|^2_{L^\infty(S)}+\varepsilon\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\\
&\quad+\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}\|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)}.
\end{aligned} \tag{7.115}
$$

Proposition 3.10, (7.111) and (7.115) yield:

$$
\begin{aligned}
&\|g(\partial_\omega N,\partial_\omega N)-I\|_{L^\infty(S)}\\
&\lesssim\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}(\|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)}+\|\partial_\omega\theta\|_{L^4(S)})+(\|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)}+\varepsilon\|\partial_\omega N\|_{L^\infty(S)}\\
&\quad+\varepsilon\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}+\|\nabla\!\!\!/\partial_\omega\theta\|_{L^2(S)})\|\partial_\omega N\|_{L^\infty(S)}.
\end{aligned} \tag{7.116}
$$

(7.116) implies:

$$\|\partial_\omega N\|_{L^\infty(S)} \lesssim 1 + \|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^4(S)} + \|\partial_\omega\theta\|_{L^4(S)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \|\nabla\!\!\!/\partial_\omega\theta\|_{L^2(S)}. \tag{7.117}$$

Together with Corollary 3.8, Proposition 3.9 and (7.95), we obtain:

$$\|\partial_\omega N\|_{L^\infty(S)} \lesssim 1 + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \|\nabla\partial_\omega\theta\|_{L^2(S)}. \tag{7.118}$$

Finally, (7.39), (7.59), (7.88), (7.58), (7.95), (7.96), (7.97), (7.104) and (7.118) yield:

$$\begin{aligned}&\|\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \|\nabla_N\partial_\omega a\|_{L^2_uH^{\frac12}(P_u)}\\ &\quad + \|\nabla_N\partial_\omega a\|_{L^\infty_uH^{-\frac12}(P_u)} + \|\nabla_N^2\partial_\omega a\|_{L^2_uH^{-\frac32}(P_u)} + \|\partial_\omega a\|_{L^\infty(S)} + \|\nabla\partial_\omega\theta\|_{L^2(S)}\\ &\quad + \|\nabla\partial_\omega N\|_{L^4(S)}\\ &\lesssim \varepsilon\end{aligned} \tag{7.119}$$

and

$$\|\partial_\omega N\|_{L^\infty(S)} \lesssim 1, \tag{7.120}$$

which concludes the proof of (2.32).

7.2. Second order derivatives with respect to ω

The goal of this section is to prove (2.33). We first give an outline of the proof. Differentiating the Equation (7.33) for $\partial_\omega a$ with respect to ω, we obtain:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega^2 a = 2\nabla_N^2 a + \nabla\!\!\!/\nabla_N a + 2R_{\partial_\omega N\partial_\omega N} + \cdots, \tag{7.121}$$

where the first two terms on the right-hand side come respectively from the commutators $[\partial_\omega, \nabla\!\!\!/]$ and $[\partial_\omega, \Delta\!\!\!/]$ (see (7.11) and (7.13)). Since R is in $L^2(S)$ by (4.3), $\nabla_N^2 a$ is in $L^2_uH^{-\frac12}(P_u)$ by (2.31), and $\nabla_N\partial_\omega a$ is in $L^2_uH^{\frac12}(P_u)$ by (2.32), this suggests in view of Proposition 5.34 that:

$$\|\partial_\omega^2 a\|_{L^2_uH^{\frac32}(P_u)} + \|\partial_\omega^2 a\|_{L^\infty_uH^{\frac12}(P_u)} + \|\nabla_N\partial_\omega^2 a\|_{L^2_uH^{-\frac12}(P_u)} \lesssim \varepsilon. \tag{7.122}$$

Remark 7.3. — Note that we may not differentiate the Equation (7.121) for $\partial_\omega^2 a$ with respect to ∇_N. Indeed, the term $\nabla_N R_{\partial_\omega N\partial_\omega N}$ has no structure: unlike R_{NN} and $R_{N\partial_\omega N}$ which were involved in the equation for a and $\partial_\omega a$, $R_{\partial_\omega N\partial_\omega N}$ does not contain any contraction with N since $\partial_\omega N$ is tangent to P_u. Thus, unlike $\nabla_N R_{NN}$ and $\nabla_N R_{N\partial_\omega N}$, we can not write $\nabla_N R_{\partial_\omega N\partial_\omega N}$ as a tangential derivative using the contracted Bianchi identities for R. In turn, we can not obtain any estimate for $\nabla_N^2\partial_\omega^2 a$.

Next, we turn to the estimates for $\partial_\omega^2\theta$. Differentiating the Equation (7.66) for $\partial_\omega \mathrm{tr}\,\theta$ and the Equation (7.70) for $\partial_\omega\widehat{\theta}$ with respect to ω, we obtain:

$$\begin{cases} \nabla\!\!\!/ \partial_\omega^2 \mathrm{tr}\,\theta = \nabla k_{N\partial_\omega^2 N} + \cdots, \\ \nabla\!\!\!/^B \partial_\omega^2 \widehat{\theta}_{AB} = R_{N\partial_\omega^2 N} + \cdots, \end{cases} \tag{7.123}$$

which together with the estimate (4.3) for R and k yields $\nabla\!\!\!/\partial_\omega^2\theta \in L^2(S)$ provided $\partial_\omega^2 N$ belongs to $L^\infty(S)$.

Finally, we turn to the estimates for $\partial_\omega^2 N$. Differentiating the Equations (7.90), (7.91) and (7.92) for $\partial_\omega N$ with respect to ω, we obtain:

$$\begin{cases} \nabla\!\!\!/ \partial_\omega^2 N = \partial_\omega^2\theta + \cdots, \\ \nabla_N \partial_\omega^2 N = -a^{-1}\nabla\!\!\!/\partial_\omega^2 a + \cdots. \end{cases} \tag{7.124}$$

Together with the fact that $\nabla\!\!\!/\partial_\omega\theta$ belong to $L^2(S)$ and $\partial_\omega a$ belongs to $L^1_u H^{\frac{3}{2}}(P_u)$, this suggests using interpolation that $\partial_\omega^2 N$ belongs to $L^\infty_u H^{\frac{5}{4}}(P_u)$. Since $\frac{5}{4} > 1$, and since P_u is 2 dimensional, we obtain that $\partial_\omega^2 N$ belongs indeed to $L^\infty(S)$.

The rest of this chapter is as follows. We first prove the estimates for $\partial_\omega^2 a$. Then, we prove the estimates for $\partial_\omega^2\theta$. Finally, we conclude with the estimates for $\partial_\omega^2 N$.

7.2.1. Estimates for $\partial_\omega^2 a$. — Recall (7.33) and (7.34). $\partial_\omega a$ satisfies:

$$\nabla_N \partial_\omega a - a^{-1}\Delta\!\!\!/\partial_\omega a = a^{-1}h, \tag{7.125}$$

where h is given by:

$$\begin{aligned} h = & -\nabla\!\!\!/_{\partial_\omega N} a - a^{-2}\partial_\omega a \Delta\!\!\!/ a - 2\nabla\!\!\!/_{\partial_\omega N}\nabla_N a + 2\theta(\partial_\omega N, \nabla\!\!\!/ a) \\ & - \partial_\omega \mathrm{tr}\,\theta \nabla_N a - \mathrm{tr}\,\theta \nabla\!\!\!/_{\partial_\omega N} a + 2\theta\partial_\omega\theta + \partial_\omega(\nabla_N(k_{NN})) + 2R_{N\partial_\omega N}. \end{aligned} \tag{7.126}$$

Now, differentiating (7.125) with respect to ω and using the commutator Formula (7.13), we obtain:

$$\nabla_N \partial_\omega^2 a - a^{-1}\Delta\!\!\!/\partial_\omega^2 a = -2a^{-1}\mathrm{d}i\!\!\!/v(\nabla_N(\partial_\omega a)\partial_\omega N) + \partial_\omega h + a^{-1}h_1, \tag{7.127}$$

where h_1 is given by:

$$\begin{aligned} h_1 = & -a\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a) - a^{-1}\partial_\omega a \Delta\!\!\!/(\partial_\omega a) + 2\mathrm{d}i\!\!\!/v(\partial_\omega N)\nabla_N(\partial_\omega a) \\ & + 2\theta(\partial_\omega N, \nabla\!\!\!/(\partial_\omega a)) - \partial_\omega \mathrm{tr}\,\theta \nabla_N(\partial_\omega a) - \mathrm{tr}\,\theta\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a). \end{aligned}$$

Together with the product estimate (5.79), this yields:

$$\begin{aligned} & \|h_1\|_{L^2(S)} \\ & \lesssim \|a\|_{L^\infty(S)}\|\partial_\omega N\|_{L^\infty(S)}\|\nabla\!\!\!/(\partial_\omega a)\|_{L^2(S)} + \|\partial_\omega a\|_{L^\infty(S)}\|\Delta\!\!\!/(\partial_\omega a)\|_{L^2(S)} \\ & \quad + \|\mathrm{d}i\!\!\!/v(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\|\nabla_N(\partial_\omega a)\|_{L^2_u H^{\frac{1}{2}}(P_u)} \\ & \quad + \|\theta\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\|\nabla\!\!\!/(\partial_\omega a)\|_{L^4(S)} + \|\partial_\omega \mathrm{tr}\,\theta\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\|\nabla_N(\partial_\omega a)\|_{L^2_u H^{\frac{1}{2}}(P_u)} \\ & \quad + \|\mathrm{tr}\,\theta\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\|\nabla\!\!\!/(\partial_\omega a)\|_{L^4(S)} + \|\partial_\omega a\|_{L^\infty(S)}\|h\|_{L^2(S)} \end{aligned} \tag{7.128}$$

$$\lesssim \varepsilon(1 + \|\mathrm{d}\!\!/\mathrm{iv}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\partial_\omega \mathrm{tr}\,\theta\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}),$$

where we used in the last inequality the estimate (4.9) for a, the estimate (4.12) for θ, and the estimate (2.32) for $\partial_\omega a$ and $\partial_\omega N$. In view of Corollary 5.16 and the estimate (7.97), we have:

$$\begin{aligned}
&\|\mathrm{d}\!\!/\mathrm{iv}(\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\partial_\omega \mathrm{tr}\,\theta\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\\
&\quad\lesssim \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla(\nabla\!\!\!/_{\partial_\omega N} a)\|_{L^2(S)}\\
&\quad\lesssim \|\nabla\partial_\omega\theta\|_{L^2(S)} + \|\nabla\nabla\!\!\!/ a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\!\!\!/ a\|_{L^4(S)}\|\nabla\partial_\omega N\|_{L^4(S)}\\
&\quad\lesssim \varepsilon,
\end{aligned}$$

where we used in the last inequality the estimates (4.9) and (4.11) for a, and the estimate (2.32) for $\partial_\omega\theta$ and $\partial_\omega N$. Together with (7.128), this finally yields:

$$\|h_1\|_{L^2(S)} \lesssim \varepsilon. \tag{7.129}$$

Next, we estimate the first term in the right-hand side of (7.127). In view of the product estimate (5.83), we have:

$$\begin{aligned}
&\|\mathrm{d}\!\!/\mathrm{iv}(\nabla_N(\partial_\omega a)\partial_\omega N)\|_{L^2_u H^{-\frac{1}{2}}(P_u)}\\
&\quad\lesssim \|\nabla_N(\partial_\omega a)\|_{L^2_u H^{\frac{1}{2}}(P_u)}(\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)})\\
&\quad\lesssim \varepsilon,
\end{aligned} \tag{7.130}$$

where we used in the last inequality the estimate (2.32) for $\partial_\omega a$ and $\partial_\omega N$.

Finally, we estimate the second term in the right-hand side of (7.127). We first provide a decomposition of $\partial_\omega^2 N$. Differentiating (7.16) with respect to ω, we obtain:

$$g(\partial_\omega^2 N, N) = -g(\partial_\omega N, \partial_\omega N), \tag{7.131}$$

which yields:

$$\partial_\omega^2 N = \Pi(\partial_\omega^2 N) - |\partial_\omega N|^2 N. \tag{7.132}$$

Next, we compute $\partial_\omega h$. Differentiating (7.126) with respect to ω and using (7.132) and the commutator Formula (7.11), we obtain:

$$\partial_\omega h = 2|\partial_\omega N|^2\nabla_N^2 a - 2a^{-1}\mathrm{d}\!\!/\mathrm{iv}(\nabla_N(\partial_\omega a)\partial_\omega N) + h_2, \tag{7.133}$$

where h_2 is given by:

$$\begin{aligned}
h_2 = &-\nabla\!\!\!/_{\Pi(\partial_\omega^2 N)}a - \nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a) - a^{-2}\partial_\omega a\Delta\!\!\!/(\partial_\omega a) + 2a^{-3}(\partial_\omega a)^2\Delta\!\!\!/ a - a^{-2}\partial_\omega a[\partial_\omega, \Delta\!\!\!/]a\\
&- 2\nabla\!\!\!/_{\Pi(\partial_\omega^2 N)}\nabla_N a - 2\nabla\!\!\!/_{\partial_\omega N}\nabla\!\!\!/_{\partial_\omega N}a + 2\theta(\partial_\omega^2 N, \nabla\!\!\!/ a) + 2\partial_\omega\theta(\partial_\omega N, \nabla\!\!\!/ a) + 2\theta(\partial_\omega N, \nabla\!\!\!/\partial_\omega a\\
&- \nabla_N(a)\partial_\omega N) - \partial_\omega^2\mathrm{tr}\,\theta\nabla_N a - \partial_\omega\mathrm{tr}\,\theta\nabla_N(\partial_\omega a) - 2\partial_\omega\mathrm{tr}\,\theta\nabla\!\!\!/_{\partial_\omega N}a - \mathrm{tr}\,\theta\nabla_{\partial_\omega^2 N}a\\
&- \mathrm{tr}\,\theta\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a) + 2\nabla\!\!\!/ a\nabla\!\!\!/\partial_\omega^2 a + 2|\nabla\!\!\!/\partial_\omega a|^2 - 2\nabla_N a\nabla\!\!\!/_{\partial_\omega N}\partial_\omega a - 2\nabla\!\!\!/_{\partial_\omega N}(a)\nabla_N\partial_\omega a\\
&- 2\nabla\!\!\!/_{\partial_\omega^2 N}a\nabla_N a - 2|\nabla\!\!\!/_{\partial_\omega N}a|^2 - 2\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a)\nabla_N a - 2\nabla\!\!\!/_{\partial_\omega N}a\nabla_N(\partial_\omega a) + 2\theta\partial_\omega^2\theta\\
&+ 2|\partial_\omega\theta|^2 + \nabla_{\partial_\omega^2 N}k_{NN} + 2\nabla_N k_{N\partial_\omega^2 N} + 4\nabla_{\partial_\omega N}k_{N\partial_\omega N} + 2\nabla_N k_{\partial_\omega N\partial_\omega N} + 2R_{N\partial_\omega^2 N}\\
&+ 2R_{\partial_\omega N\partial_\omega N}.
\end{aligned}$$

Together with the product estimate (5.79), this yields:

$$
\begin{aligned}
&\|h_2\|_{L^2(S)}\\
&\lesssim \|a^{-2}\partial_\omega a\|_{L^\infty(S)}\|\not\Delta\partial_\omega a\|_{L^2(S)} + \|a^{-3}(\partial_\omega a)^2\|_{L^\infty(S)}\|\not\Delta a\|_{L^2(S)}\\
&\quad + \|a^{-2}\partial_\omega a\|_{L^\infty(S)}\|[\partial_\omega, \not\Delta]a\|_{L^2(S)} + \|\not\nabla a\|_{L^2(S)}\|\partial_\omega^2 N\|_{L^\infty(S)}\\
&\quad + \|\not\nabla\partial_\omega a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\not\nabla\nabla_N a\|_{L^2(S)}\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\not\nabla^2 a\|_{L^2(S)}\|\partial_\omega N\|^2_{L^\infty(S)}\\
&\quad + \|\not\nabla a\|_{L^4(S)}\|\not\nabla\partial_\omega N\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\theta\|_{L^4(S)}\|\partial_\omega^2 N\|_{L^\infty(S)}\|\not\nabla a\|_{L^4(S)}\\
&\quad + \|\partial_\omega\theta\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\|\not\nabla a\|_{L^4(S)} + \|\theta\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}(\|\not\nabla\partial_\omega a\|_{L^4(S)}\\
&\quad + \|\nabla_N a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}) + \|\partial_\omega^2 \mathrm{tr}\,\theta\|_{L^2_{[-2,2]}L^4(P_u)}\|\nabla_N a\|_{L^\infty_{[-2,2]}L^4(P_u)}\\
&\quad + \|\partial_\omega \mathrm{tr}\,\theta\|_{L^\infty_u H^{\frac12}(P_u)}\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac12}(P_u)} + \|\partial_\omega \mathrm{tr}\,\theta\|_{L^4(S)}\|\not\nabla a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\\
&\quad + \|\mathrm{tr}\,\theta\|_{L^4(S)}\|\nabla a\|_{L^4(S)}\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\mathrm{tr}\,\theta\|_{L^4(S)}\|\not\nabla\partial_\omega a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\\
&\quad + \|\not\nabla a\|_{L^\infty_u H^{\frac12}(P_u)}\|\not\nabla\partial_\omega^2 a\|_{L^2_u H^{\frac12}(P_u)} + \|\not\nabla\partial_\omega a\|^2_{L^4(S)}\\
&\quad + \|\nabla_N a\|_{L^4(S)}\|\not\nabla\partial_\omega a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\not\nabla_{\partial_\omega N}(a)\|_{L^\infty_u H^{\frac12}(P_u)}\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac12}(P_u)}\\
&\quad + \|\not\nabla a\|_{L^4(S)}\|\nabla_N a\|_{L^4(S)}\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\not\nabla a\|^2_{L^4(S)}\|\partial_\omega N\|^2_{L^\infty(S)}\\
&\quad + \|\not\nabla\partial_\omega a\|_{L^4(S)}\|\nabla_N a\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\not\nabla_{\partial_\omega N}(a)\|_{L^\infty_u H^{\frac12}(P_u)}\|\nabla_N\partial_\omega a\|_{L^2_u H^{\frac12}(P_u)}\\
&\quad + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\partial_\omega^2\theta\|_{L^2_{[-2,2]}L^4(P_u)} + \|\partial_\omega\theta\|^2_{L^4(S)}\\
&\quad + (\|\nabla k\|_{L^2(S)} + \|R\|_{L^2(S)})(\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\partial_\omega N\|^2_{L^\infty(S)})\\
&\lesssim \varepsilon(1 + \|\partial_\omega^2 a\|_{L^2_{[-2,2]}L^4(P_u)} + \|\not\nabla\partial_\omega^2 a\|_{L^2_u H^{\frac12}(P_u)} + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\partial_\omega^2\theta\|_{L^2_{[-2,2]}L^4(P_u)}),
\end{aligned}
$$

where we used in the last inequality the commutator Formula (7.13) and the identity (7.31), the estimates (4.9) (4.11) for a, the estimate (2.31) for $\nabla_N a$, the estimate (4.12) for $\partial_\omega\theta$, the estimate (4.3) for k and R, the estimate (2.32) for $\partial_\omega a$, $\partial_\omega\theta$ and $\partial_\omega N$, and Corollary 5.16. Together with the Gagliardo-Nirenberg inequality (3.9) and the Lemma 5.12, this yields:

$$
\|h_2\|_{L^2(S)} \lesssim \varepsilon(1 + \|\partial_\omega^2 a\|_{L^2_u H^{\frac32}(P_u)} + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\not\nabla\partial_\omega^2\theta\|_{L^2(S)}). \tag{7.134}
$$

Next, we evaluate the first term in the right-hand side of (7.133). In view of Proposition 5.23, we have:

$$
\begin{aligned}
\|a|\partial_\omega N|^2\nabla_N^2 a\|_{L^2_u H^{-\frac12}(P_u)} &\lesssim (\|a|\partial_\omega N|^2\|_{L^\infty(S)} + \|\not\nabla(a|\partial_\omega N|^2)\|_{L^\infty_{[-2,2]}L^2(P_u)})\|\nabla_N^2 a\|_{H^{-\frac12}(P_u)}\\
&\lesssim \varepsilon(\|a\|_{L^\infty(S)}\|\partial_\omega N\|^2_{L^\infty(S)} + \|\not\nabla a\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\partial_\omega N\|^2_{L^\infty(S)}\\
&\qquad + \|\not\nabla\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}\|a\|_{L^\infty(S)}\|\partial_\omega N\|_{L^\infty(S)})\\
&\lesssim \varepsilon, \qquad (7.135)
\end{aligned}
$$

where we used the estimate (2.31) for $\nabla_N^2 a$, the estimates (4.9) (4.11) for a, and the estimate (2.32) for $\partial_\omega N$.

Finally, in view of (7.127) and (7.133), we have:

$$\nabla_N\partial_\omega^2 a - a^{-1}\not{\Delta}\partial_\omega^2 a = h_3, \tag{7.136}$$

where h_3 is given by:

$$h_3 = -4a^{-1}\not{\mathrm{div}}(\nabla_N(\partial_\omega a)\partial_\omega N) + 2|\partial_\omega N|^2\nabla_N^2 a + a^{-1}h_1 + h_2.$$

Together with the estimates (7.129), (7.130), (7.134), (7.135) and the estimate (4.9) for a, this yields:

(7.137)
$$\begin{aligned}\|ah_3\|_{L^2_uH^{-\frac12}(P_u)} &\lesssim \|\not{\mathrm{div}}(\nabla_N(\partial_\omega a)\partial_\omega N)\|_{L^2_uH^{-\frac12}(P_u)} + \|a|\partial_\omega N|^2\nabla_N^2 a\|_{L^2_uH^{-\frac12}(P_u)}\\ &\quad + \|h_1\|_{L^2(S)} + \|ah_2\|_{L^2(S)}\\ &\lesssim \varepsilon(1+\|\partial_\omega^2 a\|_{L^2_uH^{\frac32}(P_u)} + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\not{\nabla}\partial_\omega^2\theta\|_{L^2(S)}).\end{aligned}$$

Now, in view of (7.136) and Proposition 5.34, we have:

$$\|\partial_\omega^2 a\|_{L^2_uH^{\frac32}(P_u)} + \|\partial_\omega^2 a\|_{L^\infty_uH^{\frac12}(P_u)} + \|\nabla_N\partial_\omega^2 a\|_{L^2_uH^{-\frac12}(P_u)} \lesssim \|h_3\|_{L^2_uH^{-\frac12}(P_u)}.$$

Together with (7.137), this yields:

$$\begin{aligned}&\|\partial_\omega^2 a\|_{L^2_uH^{\frac32}(P_u)} + \|\partial_\omega^2 a\|_{L^\infty_uH^{\frac12}(P_u)} + \|\nabla_N\partial_\omega^2 a\|_{L^2_uH^{-\frac12}(P_u)}\\ &\qquad\lesssim \varepsilon(1+\|\partial_\omega^2 a\|_{L^2_uH^{\frac32}(P_u)} + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\not{\nabla}\partial_\omega^2\theta\|_{L^2(S)}).\end{aligned}$$

Thus, we finally obtain:

$$\begin{aligned}&\|\partial_\omega^2 a\|_{L^2_uH^{\frac32}(P_u)} + \|\partial_\omega^2 a\|_{L^\infty_uH^{\frac12}(P_u)} + \|\nabla_N\partial_\omega^2 a\|_{L^2_uH^{-\frac12}(P_u)}\\ &\qquad\lesssim \varepsilon(1+\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\not{\nabla}\partial_\omega^2\theta\|_{L^2(S)}).\end{aligned} \tag{7.138}$$

7.2.2. Estimates for $\partial_\omega^2\theta$. — Let us start by computing the trace of $\partial_\omega\widehat{\theta}$ when seen as a tensor on P_u. Differentiating (7.60) with respect to ω, we obtain:

$$\begin{aligned}\partial_\omega^2\widehat{\theta}(X,Y) &= \partial_\omega^2\theta(X,Y) - \frac12\partial_\omega^2\mathrm{tr}\,\theta(X.Y-(X.N)(Y.N))\\ &\quad + \partial_\omega\mathrm{tr}\,\theta((X.\partial_\omega N)(Y.N)+(X.N)(Y.\partial_\omega N))\\ &\quad + \frac12\mathrm{tr}\,\theta((X.\partial_\omega^2 N)(Y.N)+(X.N)(Y.\partial_\omega^2 N)+2(\partial_\omega N\cdot X)(\partial_\omega N\cdot Y)),\end{aligned}$$

which yields:

$$\partial_\omega^2\widehat{\theta}_{AB} = \partial_\omega^2\theta_{AB} - \frac12\partial_\omega^2\mathrm{tr}\,\theta\delta_{AB} + \mathrm{tr}\,\theta(\partial_\omega N)_A(\partial_\omega N)_B, \tag{7.139}$$

so that:

$$\mathrm{tr}\,(\partial_\omega^2\widehat{\theta}) = \mathrm{tr}\,(\partial_\omega^2\theta) - \partial_\omega^2\mathrm{tr}\,\theta + \mathrm{tr}\,\theta|\partial_\omega N|^2. \tag{7.140}$$

We compute $\partial_\omega^2\mathrm{tr}\,\theta$. In view of (7.64), we have:

$$\partial_\omega^2\mathrm{tr}\,\theta = \partial_\omega(\partial_\omega\theta_{AA}) = \mathrm{tr}\,(\partial_\omega^2\theta) + 2\partial_\omega\theta(e_A,\partial_\omega e_A). \tag{7.141}$$

Now, in view of (7.2), we have:

$$\partial_\omega\theta(N,.) = -\theta(\partial_\omega N,.). \tag{7.142}$$

(7.141), (7.142) and (7.25) yield

$$\partial_\omega^2 \mathrm{tr}\,\theta = \mathrm{tr}\,(\partial_\omega^2\theta) + 2\theta(\partial_\omega N, \partial_\omega N). \tag{7.143}$$

Finally, (7.140) and (7.143) imply:

$$\mathrm{tr}\,(\partial_\omega^2\widehat{\theta}) = \mathrm{tr}\,\theta|\partial_\omega N|^2 - 2\theta(\partial_\omega N, \partial_\omega N). \tag{7.144}$$

We now turn to the estimates for $\partial_\omega^2 \mathrm{tr}\,\theta$. Differentiating (7.66) with respect to ω, we obtain:

$$\partial_\omega^2 \mathrm{tr}\,\theta = -\partial_\omega^2 a + 2k_{N\partial_\omega^2 N} + 2k_{\partial_\omega N\partial_\omega N},$$

so that:

$$\nabla\!\!\!\!/\,\partial_\omega^2 \mathrm{tr}\,\theta = -\nabla\!\!\!\!/\,\partial_\omega^2 a + 2k_{N\nabla\!\!\!\!/\,\partial_\omega^2 N} + 2k_{\nabla\!\!\!\!/\, N\partial_\omega^2 N} + 2\nabla\!\!\!\!/\, k_{N\partial_\omega^2 N} + 4k_{\partial_\omega N\nabla\!\!\!\!/\,\partial_\omega N} + 2\nabla\!\!\!\!/\, k_{\partial_\omega N\partial_\omega N},$$

which in turn yields:

$$\begin{aligned}\|\nabla\!\!\!\!/\,\partial_\omega^2 \mathrm{tr}\,\theta\|_{L^2(S)} \lesssim{}& \|\nabla\!\!\!\!/\,\partial_\omega^2 a\|_{L^2(S)} + \|k\|_{L^\infty_{[-2,2]}L^4(P_u)}(\|\nabla\!\!\!\!/\,\partial_\omega^2 N\|_{L^2_{[-2,2]}L^4(P_u)}\\ &+ \|\nabla\!\!\!\!/\, N\|_{L^2_{[-2,2]}L^4(P_u)}\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!\!/\,\partial_\omega N\|_{L^2_{[-2,2]}L^4(P_u)}\|\partial_\omega N\|_{L^\infty(S)})\\ &+ \|\nabla\!\!\!\!/\, k\|_{L^2(S)}(\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\partial_\omega N\|^2_{L^\infty(S)}).\end{aligned}$$

Together with the Gagliardo-Nirenberg inequality (3.9), Lemma 5.12, the estimate (2.32) for $\partial_\omega a$ and $\partial_\omega N$, and the estimate (4.3) for k, we obtain:

$$\|\nabla\!\!\!\!/\,\partial_\omega^2 \mathrm{tr}\,\theta\|_{L^2(S)} \lesssim \|\nabla\!\!\!\!/\,\partial_\omega^2 a\|_{L^2(S)} + \varepsilon(1 + \|\nabla\!\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)} + \|\partial_\omega^2 N\|_{L^\infty(S)}). \tag{7.145}$$

We now turn to the estimates for $\nabla\!\!\!\!/\,\partial_\omega^2\widehat{\theta}$. We differentiate the third equation of (4.5) with respect to ω. We introduce the symmetric tensor σ on S defined by:

$$\begin{aligned}\sigma(X,Y) = \partial_\omega\widehat{\theta}(X,Y) &+ \left(\theta(\partial_\omega N, Y) - \frac{1}{2}\mathrm{tr}\,\theta\partial_\omega N\cdot Y\right) N\cdot X\\ &+ \left(\theta(\partial_\omega N, X) - \frac{1}{2}\mathrm{tr}\,\theta\partial_\omega N\cdot X\right) N\cdot Y,\end{aligned} \tag{7.146}$$

which in view of (7.60) and (7.142) satisfies:

$$\sigma(N,.) = \sigma(.,N) = 0.$$

We may thus apply the commutator Formula (7.21) to σ. We obtain:

$$\begin{aligned}([\partial_\omega, \mathrm{div}\!\!\!\!/\,]\sigma)_A = -\mathrm{tr}\,\theta\sigma_{\partial_\omega NA} - \theta_{AB}\sigma_{B\partial_\omega N} - \nabla_N\sigma_{\partial_\omega NB} + \theta_{\partial_\omega NC}\sigma_{CA}\\ + (\partial_\omega N)_A\theta_{BC}\sigma_{CB}.\end{aligned} \tag{7.147}$$

Now, in view of the Definition (7.146) of σ, and the structure equation for N (4.4), we have:

$$\begin{aligned}\sigma_{AB} &= \partial_\omega\widehat{\theta}_{AB},\\ \nabla_N\sigma_{AB} &= \nabla_N\partial_\omega\widehat{\theta}_{AB} - \widehat{\theta}_{\partial_\omega NB}\nabla\!\!\!\!/_A a - \widehat{\theta}_{\partial_\omega NA}\nabla\!\!\!\!/_B a\end{aligned} \tag{7.148}$$

$$\nabla\!\!\!/_C\sigma_{AB} = \nabla\!\!\!/_C\partial_\omega\widehat{\theta}_{AB} + \widehat{\theta}_{\partial_\omega NB}\theta_{AC} + \widehat{\theta}_{\partial_\omega NA}\theta_{BC},$$

which together with (7.147) implies:

$$\begin{aligned}([\partial_\omega, \mathrm{di}\!\!/\mathrm{v}]\sigma)_A =& -\mathrm{tr}\,\theta\partial_\omega\widehat{\theta}_{\partial_\omega NA} - \theta_{AB}\partial_\omega\widehat{\theta}_{B\partial_\omega N} - \nabla_N\partial_\omega\widehat{\theta}_{\partial_\omega NB} + \widehat{\theta}_{\partial_\omega NB}\nabla\!\!\!/_{\partial_\omega N}a\\ &+ \widehat{\theta}_{\partial_\omega N\partial_\omega N}\nabla\!\!\!/_A a + \theta_{\partial_\omega NC}\partial_\omega\widehat{\theta}_{CA} + (\partial_\omega N)_A\theta_{BC}\partial_\omega\widehat{\theta}_{CB}.\end{aligned} \tag{7.149}$$

Now, we have in view of (7.148), (7.70) and (7.74):

$$(\mathrm{di}\!\!/\mathrm{v}\sigma)_A = h, \tag{7.150}$$

where h is given by:

$$\begin{aligned}h =& \frac{1}{2}\nabla\!\!\!/_A\partial_\omega\mathrm{tr}\,\theta + \nabla_N\theta_{\partial_\omega NB} - \frac{1}{2}\nabla_N\mathrm{tr}\,\theta(\partial_\omega N)_A + 2\mathrm{tr}\,\theta\widehat{\theta}_{\partial_\omega NA} + \theta_{AB}\widehat{\theta}_{B\partial_\omega N}\\ &- (\partial_\omega N)_A\theta_{BC}\widehat{\theta}_{CB} + R_{A\partial_\omega N} - (\partial_\omega N)_A R_{NN}.\end{aligned} \tag{7.151}$$

Differentiating (7.150) and (7.151) with respect to ω, and using (7.149) and the commutator Formula (7.11), we obtain:

$$(\mathrm{di}\!\!/\mathrm{v}\partial_\omega\sigma)_A = h_1, \tag{7.152}$$

where h_1 is given by:

$$\begin{aligned}h_1 =& \frac{1}{2}\nabla\!\!\!/_A\partial^2_\omega\mathrm{tr}\,\theta - \frac{1}{2}\nabla_N\partial_\omega\mathrm{tr}\,\theta(\partial_\omega N)_A + \nabla_N\theta_{\partial^2_\omega NA} + 2\nabla_N\partial_\omega\theta_{\partial_\omega NA} + \nabla\!\!\!/_{\partial_\omega N}\theta_{\partial_\omega NA}\\ &- \frac{1}{2}\nabla_N\mathrm{tr}\,\theta(\partial^2_\omega N)_A - \frac{1}{2}\nabla_N\partial_\omega\mathrm{tr}\,\theta(\partial_\omega N)_A - \frac{1}{2}\nabla\!\!\!/_{\partial_\omega N}\mathrm{tr}\,\theta(\partial_\omega N)_A + 2\mathrm{tr}\,\theta\widehat{\theta}_{\partial^2_\omega NA}\\ &+ 3\mathrm{tr}\,\theta\partial_\omega\widehat{\theta}_{\partial_\omega NA} + 2\partial_\omega\mathrm{tr}\,\theta\widehat{\theta}_{\partial_\omega NA} + \theta_{AB}\widehat{\theta}_{B\partial^2_\omega N} + \partial_\omega\theta_{AB}\widehat{\theta}_{B\partial_\omega N} + 2\theta_{AB}\partial_\omega\widehat{\theta}_{B\partial_\omega N}\\ &- (\partial^2_\omega N)_A\theta_{BC}\widehat{\theta}_{CB} - (\partial_\omega N)_A\partial_\omega\theta_{BC}\widehat{\theta}_{CB} - 2(\partial_\omega N)_A\theta_{BC}\partial_\omega\widehat{\theta}_{CB} - \theta_{\partial_\omega NC}\partial_\omega\widehat{\theta}_{CA}\\ &- \widehat{\theta}_{\partial_\omega NB}\nabla\!\!\!/_{\partial_\omega N}a - \widehat{\theta}_{\partial_\omega N\partial_\omega N}\nabla\!\!\!/_A a + R_{A\partial^2_\omega N} - (\partial^2_\omega N)_A R_{NN} - 2(\partial_\omega N)_A R_{N\partial_\omega N}.\end{aligned}$$

h_1 satisfies:

$$\begin{aligned}\|h_1\|_{L^2(S)} \lesssim& \|\nabla\!\!\!/\partial^2_\omega\mathrm{tr}\,\theta\|_{L^2(S)} + \|\nabla\theta\|_{L^2(S)}(\|\partial^2_\omega N\|_{L^\infty(S)} + \|\partial_\omega N\|^2_{L^\infty(S)})\\ &+ \|\nabla\partial_\omega\theta\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}\\ &+ \|\theta\|_{L^4(S)}\|\widehat{\theta}\|_{L^4(S)}(\|\partial^2_\omega N\|_{L^\infty(S)}\\ &+ \|\partial_\omega N\|^2_{L^\infty(S)}) + \|\partial_\omega\widehat{\theta}\|_{L^4(S)}\|\theta\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}\\ &+ \|\partial_\omega\theta\|_{L^4(S)}\|\widehat{\theta}\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)} + \|\widehat{\theta}\|_{L^4(S)}\|\nabla\!\!\!/a\|_{L^4(S)}\|\partial_\omega N\|^2_{L^\infty(S)}\\ &+ \|R\|_{L^2(S)}(\|\partial^2_\omega N\|_{L^\infty(S)} + \|\partial_\omega N\|^2_{L^\infty(S)}).\end{aligned}$$

Together with the estimate (7.145) for $\partial^2_\omega\mathrm{tr}\,\theta$, the estimates (4.10) (4.12) for θ, the estimate (2.32) for $\partial_\omega\theta$ and $\partial_\omega N$, the estimate (4.11) for a, and the estimate (4.3) for R, we obtain:

$$\|h_1\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial^2_\omega a\|_{L^2(S)} + \varepsilon(1 + \|\nabla\!\!\!/^2\partial^2_\omega N\|_{L^2(S)} + \|\partial^2_\omega N\|_{L^\infty(S)}). \tag{7.153}$$

Next, we compare $\mbox{div}\!\!\!/\,\partial_\omega\sigma$ to $\mbox{div}\!\!\!/\,\partial_\omega^2\theta$. Differentiating the Definition (7.146) of σ with respect to ω first, and then $\mbox{div}\!\!\!/$, we obtain:

$$\mbox{div}\!\!\!/\,\partial_\omega\sigma = \mbox{div}\!\!\!/\,\partial_\omega^2\widehat{\theta} + h_2, \tag{7.154}$$

where h_2 is given schematically by:

$$h_2 = \nabla\!\!\!/\theta(\partial_\omega^2 N + (\partial_\omega N)^2) + \nabla\!\!\!/\partial_\omega\theta\partial_\omega N + \partial_\omega\theta\nabla\!\!\!/\partial_\omega N + \theta(\nabla\!\!\!/\partial_\omega^2 N + \partial_\omega N\nabla\!\!\!/\partial_\omega N).$$

h_2 satisfies:

$$\begin{aligned}\|h_2\|_{L^2(S)} \lesssim\ & \|\nabla\!\!\!/\theta\|_{L^2(S)}(\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\partial_\omega N\|^2_{L^\infty(S)}) + \|\nabla\!\!\!/\partial_\omega\theta\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}\\ &+ \|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/\partial_\omega N\|_{L^2_{[-2,2]}L^4(P_u)}\\ &+ \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}(\|\nabla\!\!\!/\partial_\omega^2 N\|_{L^2_{[-2,2]}L^4(P_u)} + \|\partial_\omega N\|_{L^\infty(S)}\|\nabla\!\!\!/\partial_\omega N\|_{L^2_{[-2,2]}L^4(P_u)}).\end{aligned}$$

Together with the Gagliardo-Nirenberg inequality (3.9), the estimates (4.10) (4.12) for θ and the estimate (2.32) for $\partial_\omega\theta$ and $\partial_\omega N$, we obtain

$$\|h_2\|_{L^2(S)} \lesssim \varepsilon(1 + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}). \tag{7.155}$$

Finally, in view of (7.152), (7.153), (7.154) and (7.155), we have:

$$\|\mbox{div}\!\!\!/(\partial_\omega^2\widehat{\theta})\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2 a\|_{L^2(S)} + \varepsilon(1 + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}). \tag{7.156}$$

Next, we estimate $\nabla\!\!\!/\mathrm{tr}\,(\partial_\omega^2\widehat{\theta})$. In view of (7.144), we have:

$$\nabla\!\!\!/\mathrm{tr}\,(\partial_\omega^2\widehat{\theta}) = |\partial_\omega N|^2\nabla\!\!\!/\mathrm{tr}\,\theta + 2\mathrm{tr}\,\theta\partial_\omega N\nabla\!\!\!/\partial_\omega N - 2\nabla\!\!\!/\theta(\partial_\omega N, \partial_\omega N) - 4\theta(\partial_\omega N, \nabla\!\!\!/\partial_\omega N).$$

This yields:

$$\|\nabla\!\!\!/\mathrm{tr}\,(\partial_\omega^2\widehat{\theta})\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\theta\|_{L^2(S)}\|\partial_\omega N\|^2_{L^\infty(S)} + \|\theta\|_{L^4(S)}\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)},$$

which together with the estimates (4.10) (4.12) for θ, and the estimate (2.32) for $\partial_\omega N$ and $\partial_\omega\theta$ implies:

$$\|\nabla\!\!\!/\mathrm{tr}\,(\partial_\omega^2\widehat{\theta})\|_{L^2(S)} \lesssim \varepsilon. \tag{7.157}$$

Together with (7.156), we obtain:

$$\|\mbox{div}\!\!\!/(\partial_\omega^2\widehat{\theta} - \mathrm{tr}\,(\partial_\omega^2\widehat{\theta}))\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2 a\|_{L^2(S)} + \varepsilon(1 + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}).$$

In view of the Hodge estimate (5.69), this implies:

$$\|\nabla\!\!\!/(\partial_\omega^2\widehat{\theta} - \mathrm{tr}\,(\partial_\omega^2\widehat{\theta}))\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2 a\|_{L^2(S)} + \varepsilon(1 + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}),$$

which together with (7.157) yields:

$$\|\nabla\!\!\!/\partial_\omega^2\widehat{\theta}\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2 a\|_{L^2(S)} + \varepsilon(1 + \|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}). \tag{7.158}$$

Now, in view of (7.139), we have:

$$\begin{aligned}\|\nabla\!\!\!/\partial_\omega^2\theta\|_{L^2(S)} \lesssim\ & \|\nabla\!\!\!/\partial_\omega^2\widehat{\theta}\|_{L^2(S)} + \|\nabla\!\!\!/\partial_\omega^2\mathrm{tr}\,\theta\|_{L^2(S)} + \|\nabla\!\!\!/\mathrm{tr}\,\theta\|_{L^2(S)}\|\partial_\omega N\|^2_{L^\infty(S)}\\ &+ \|\mathrm{tr}\,\theta\|_{L^4(S)}\|\nabla\!\!\!/\partial_\omega N\|_{L^4(S)}\|\partial_\omega N\|_{L^\infty(S)}.\end{aligned}$$

Together with the estimate (7.145) for $\partial_\omega^2 \mathrm{tr}\,\theta$, the estimate (7.158) for $\partial_\omega^2\widehat{\theta}$, the estimates (4.10) (4.12) for $\mathrm{tr}\,\theta$ and the estimate (2.32) for $\partial_\omega N$, we finally obtain:

$$\|\nabla\!\!\!/\partial_\omega^2\theta\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2 a\|_{L^2(S)} + \varepsilon(1+\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}). \tag{7.159}$$

7.2.3. Estimates for $\partial_\omega^2 N$. — Let X, Y two vector fields on Σ independent of ω. We rewrite (7.90):

$$g(\nabla_{\Pi X}\partial_\omega N, \Pi Y) = \partial_\omega\theta_{\Pi X\Pi Y} - (\partial_\omega N)_{\Pi X}\nabla_{\Pi Y} a. \tag{7.160}$$

We differentiate (7.160) with respect to ω. Using (7.19) and evaluating at $X=e_A$, $Y=e_B$, we obtain:

$$\begin{aligned} g(\nabla_A\partial_\omega^2 N, e_B) &- g(\nabla_N\partial_\omega N, e_B)(\partial_\omega N)_A - g(\nabla_A\partial_\omega N, N)(\partial_\omega N)_B\\ &= \partial_\omega^2\theta_{AB} - \partial_\omega\theta_{NB}(\partial_\omega N)_A - \partial_\omega\theta_{AN}(\partial_\omega N)_B - (\partial_\omega^2 N)_A\nabla\!\!\!/_B a\\ &\quad - (\partial_\omega N)_A\nabla\!\!\!/_B(\partial_\omega a) + (\partial_\omega N)_A(\partial_\omega N)_B\nabla_N a. \end{aligned}$$

Together with the identities (7.91), (7.92), (7.142), we obtain:

$$\begin{aligned} &g(\nabla_A\partial_\omega^2 N, e_B)\\ &\quad= \partial_\omega^2\theta_{AB} - (\partial_\omega^2 N)_A\nabla\!\!\!/_B a - 2(\partial_\omega N)_A\nabla\!\!\!/_B(\partial_\omega a) + 2(\partial_\omega N)_A(\partial_\omega N)_B\nabla_N a. \end{aligned} \tag{7.161}$$

Next, we differentiate the identity (7.131). We obtain:

$$g(\nabla_A\partial_\omega^2 N, N) + g(\partial_\omega^2 N, \nabla_A N) = -2g(\nabla_A\partial_\omega N, \partial_\omega N).$$

Together with (4.4) and (7.90), we obtain:

$$g(\nabla_A\partial_\omega^2 N, N) = -\theta_{A\partial_\omega^2 N} - 2\partial_\omega\theta_{A\partial_\omega N} + (\partial_\omega N)_A\nabla\!\!\!/_{\partial_\omega N} a. \tag{7.162}$$

Finally, differentiating (7.92), and using the commutator Formula (7.11), and the identities (7.2) and (7.25), we obtain:

$$\begin{aligned} \nabla_N\partial_\omega^2 N = &-\theta(\partial_\omega^2 N, e_A)e_A - \nabla\!\!\!/(\partial_\omega^2 a) + \nabla_{\partial_\omega^2 N} a + \nabla_N a\,\partial_\omega^2 N - \nabla_{\partial_\omega N}\partial_\omega N\\ &- \partial_\omega\theta(\partial_\omega N, e_A)e_A + 2\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a)N + 2\nabla_N(\partial_\omega a)\partial_\omega N + 2\nabla\!\!\!/_{\partial_\omega N}(a)\partial_\omega N. \end{aligned} \tag{7.163}$$

Next, we estimate $\nabla\!\!\!/^2\partial_\omega^2 N$. Differentiating (7.161) and (7.162), we obtain:

$$\begin{aligned} &\|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}\\ &\lesssim \|\nabla\!\!\!/\partial_\omega^2\theta\|_{L^2(S)} + (\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega^2 N\|_{L^2_{[-2,2]}L^4(P_u)})(\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\\ &\quad + \|\nabla\!\!\!/^2 a\|_{L^2(S)} + \|\nabla\!\!\!/\theta\|_{L^2(S)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}) + (\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^4(P_u)})\\ &\times(\|\nabla\!\!\!/\partial_\omega a\|_{L^2_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)} + \|\nabla\!\!\!/\partial_\omega\theta\|_{L^2(S)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)})\\ &\quad + (\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^4(P_u)})^2(\|\nabla\!\!\!/\nabla a\|_{L^2(S)} + \|\nabla a\|_{L^2_{[-2,2]}L^4(P_u)}). \end{aligned}$$

Together with the Gagliardo-Nirenberg inequality (3.9), the estimates (4.9) (4.11) for a, the estimates (4.10) (4.12) for θ, and the estimate (2.32) for $\partial_\omega a$, $\partial_\omega\theta$ and $\partial_\omega N$, we obtain:

$$\|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2\theta\|_{L^2(S)} + \varepsilon(1+\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)})$$

and thus:

$$\|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2\theta\|_{L^2(S)} + \varepsilon(1+\|\partial_\omega^2 N\|_{L^\infty(S)}). \tag{7.164}$$

Next, we estimate $\nabla_N\partial_\omega^2 N$. In view of (7.163), we have:

$$\nabla_N\partial_\omega^2 N = -\nabla\!\!\!/(\partial_\omega^2 a) + 2\nabla_N(\partial_\omega a)\partial_\omega N + H, \tag{7.165}$$

where H is given by:

$$\begin{aligned} H = & -\theta(\partial_\omega^2 N, e_A)e_A + \nabla_{\partial_\omega^2 N} a + \nabla_N a\partial_\omega^2 N - \nabla_{\partial_\omega N}\partial_\omega N \\ & - \partial_\omega\theta(\partial_\omega N, e_A)e_A + 2\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega a)N + 2\nabla\!\!\!/_{\partial_\omega N}(a)\partial_\omega N. \end{aligned}$$

We have:

$$\begin{aligned} \|\nabla\!\!\!/ H\|_{L^2(S)} \lesssim & (\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega^2 N\|_{L^2_{[-2,2]}L^4(P_u)})(\|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} \\ & + \|\nabla\!\!\!/\theta\|_{L^2(S)} + \|\nabla a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/\nabla a\|_{L^2(S)}) + \|\nabla\!\!\!/\nabla_{\partial_\omega N}\partial_\omega N\|_{L^2(S)} \\ & + (\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^4(P_u)})(\|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/\partial_\omega\theta\|_{L^2(S)} \\ & + \|\nabla\!\!\!/\partial_\omega a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/^2\partial_\omega a\|_{L^2(S)}) + (\|\partial_\omega N\|_{L^\infty(S)} \\ & + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^4(P_u)})^2(\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/^2 a\|_{L^2(S)}). \end{aligned}$$

Together with the Gagliardo-Nirenberg inequality (3.9), the estimates (4.9) (4.11) for a, the estimates (4.10) (4.12) for θ, and the estimate (2.32) for $\partial_\omega a$, $\partial_\omega\theta$ and $\partial_\omega N$, we obtain:

$$\|\nabla\!\!\!/ H\|_{L^2(S)} \lesssim \varepsilon(1+\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}). \tag{7.166}$$

Also, Lemma 5.12 yields:

$$\|\nabla\!\!\!/(\partial_\omega^2 a)\|_{L^2_u H^{\frac{1}{2}}(P_u)} \lesssim \|\partial_\omega^2 a\|_{L^2_u H^{\frac{3}{2}}(P_u)}. \tag{7.167}$$

The product estimate (5.85) implies:

$$\|\nabla_N(\partial_\omega a)\partial_\omega N\|_{L^2_u H^{\frac{1}{2}}(P_u)} \lesssim \|\nabla_N(\partial_\omega a)\|_{L^2_u H^{\frac{1}{2}}(P_u)}(\|\partial_\omega N\|_{L^\infty(S)} + \|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^2(P_u)}),$$

which together with the estimate (2.32) for $\partial_\omega a$ and $\partial_\omega N$ yields:

$$\|\nabla_N(\partial_\omega a)\partial_\omega N\|_{L^2_u H^{\frac{1}{2}}(P_u)} \lesssim \varepsilon. \tag{7.168}$$

Finally, (7.165), (7.166), (7.167) and (7.168) imply:

$$\|\nabla_N\partial_\omega^2 N\|_{L^2_u H^{\frac{1}{2}}(P_u)} \lesssim \|\partial_\omega^2 a\|_{L^2_u H^{\frac{3}{2}}(P_u)} + \varepsilon(1+\|\partial_\omega^2 N\|_{L^\infty(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)}). \tag{7.169}$$

Next, we estimate the $L^\infty(S)$ norm of $\partial_\omega^2 N$. In view of Corollary 5.19, we have:

$$\|\partial_\omega^2 N\|_{L^\infty(S)} \lesssim \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)} + \|\nabla_N\partial_\omega^2 N\|_{L^2_u H^{\frac{1}{2}}(P_u)}.$$

Together with (7.164) and (7.169), we finally obtain:

$$\|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)} + \|\nabla_N\partial_\omega^2 N\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|\partial_\omega^2 N\|_{L^\infty(S)} \lesssim \|\nabla\!\!\!/\partial_\omega^2\theta\|_{L^2(S)} + \|\partial_\omega^2 a\|_{L^2_u H^{\frac{3}{2}}(P_u)} + \varepsilon. \tag{7.170}$$

Finally, (7.138), (7.159) and (7.170) yield:

$$\|\partial_\omega^2 a\|_{L^2_u H^{\frac{3}{2}}(P_u)} + \|\partial_\omega^2 a\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|\nabla_N\partial_\omega^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)} + \|\nabla\!\!\!/\partial_\omega^2\theta\|_{L^2(S)} + \|\nabla\!\!\!/^2\partial_\omega^2 N\|_{L^2(S)} + \|\nabla_N\partial_\omega^2 N\|_{L^2_u H^{\frac{1}{2}}(P_u)} \lesssim \varepsilon \tag{7.171}$$

and:

$$\|\partial_\omega^2 N\|_{L^\infty(S)} \lesssim 1, \tag{7.172}$$

which concludes the proof of (2.33).

7.3. Third order derivatives with respect to ω

The goal of this section is to prove (2.34). We first give an outline of the proof. We start with the derivation of an equation for $\partial_\omega^3 u$. Recall that $\mathrm{div}(N) = \mathrm{tr}\,\theta$, $N = \nabla u/|\nabla u|$, $a = 1/|\nabla u|$ and $\mathrm{tr}\,\theta = 1 - a + k_{NN}$, so that:

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 1 - \frac{1}{|\nabla u|} + k_{NN}. \tag{7.173}$$

Differentiating (7.173) three times with respect to ω yields:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega^3 u = \nabla\!\!\!/\partial_\omega^2 a + \cdots. \tag{7.174}$$

In view of the estimate (2.33) for $\partial_\omega^2 a$ and the parabolic estimate (5.117), this suggests that $\partial_\omega^3 u$ satisfies the following estimate:

$$\|\partial_\omega^3 u\|_{L^2_u H^{\frac{5}{2}}(P_u)} + \|\partial_\omega^3 u\|_{L^\infty_u H^{\frac{3}{2}}(P_u)} + \|\nabla_N\partial_\omega^3 u\|_{L^2_u H^{\frac{1}{2}}(P_u)} \lesssim 1. \tag{7.175}$$

Now, since $\partial_\omega^3 u \in L^\infty_u H^{\frac{3}{2}}(P_u)$ and P_u is 2-dimensional, we obtain that $\partial_\omega^3 u$ belongs to $L^\infty(S)$.

The rest of this section is as follows. We start by deriving the equations for $\partial_\omega^3 u$ and $\partial_\omega^3 N$. Then, we prove the estimates for $\partial_\omega^3 u$.

Remark 7.4. — Note that $\partial_\omega^3 u = \partial_\omega^3(x.\omega)$ on $x.\omega = -2$, which yields:

$$|\partial_\omega^3 u| \sim |x| \text{ when } |x| \to +\infty \text{ on } x.\omega = -2. \tag{7.176}$$

This lack of decay is a problem when one tries to solve (7.174). However, recall from Chapter 4 that the final solution will be equal to $x.\omega$ in the region $|x| \geq 2$ so that the estimate (2.34) is clearly satisfied there. Thus, we may estimate $\varphi\partial_\omega^3 u$ instead of $\partial_\omega^3 u$, where φ is a smooth function on Σ equal to 1 on $|x| \leq 2$, $\varphi > 0$ on Σ, and $\varphi \sim |x|^{-3}$ when $|x|$ goes to infinity. Then, $\varphi\partial_\omega^3 u$ is L^2 on $x.\omega = -2$. Also, the lower order terms generated by commuting (7.122) with the multiplication by φ are all under control since they are localized in a compact region of $|x| \geq 2$ where u is explicitly given by $u = x \cdot \omega$. In the rest of the section, we omit this detail and we assume that the decay of $\partial_\omega^3 u$ is sufficient at $x.\omega = -2$.

Remark 7.5. — One may ask whether it is possible to obtain estimates for higher order derivatives of u and a with respect to ω. Consider first $\partial_\omega^4 a$. Differentiating the Equation (7.127) for $\partial_\omega^2 a$ twice would yield:

$$(\nabla_N - a^{-1}\not\Delta)\partial_\omega^4 a = \nabla_N^2 \partial_\omega^2 a + \cdots .$$

Now, we notice in Remark 7.3 that one can not obtain an estimate for $\nabla_N^2 a$, so that the above equation for $\partial_\omega^4 a$ is useless. On the other hand, differentiating the Equation (7.174) with respect to ω, we obtain:

$$(\nabla_N - a^{-1}\not\Delta)\partial_\omega^4 u = \not\nabla \partial_\omega^3 a + \cdots . \tag{7.177}$$

Now, differentiating $\nabla u = a^{-1}N$ three times with respect to ω, we obtain:

$$\partial_\omega^3 a = -a\nabla_N \partial_\omega^3 u + \cdots ,$$

which together with (7.175) suggests that $\partial_\omega^3 a$ belongs to $L_u^2 H^{\frac{1}{2}}(P_u)$. Thus, in view of (7.177) and the parabolic estimate (5.115), we see that $\partial_\omega^4 u$ is at best in $L_u^\infty H^{\frac{1}{2}}(P_u)$ which does not embed in $L^\infty(S)$. Interpolating with (7.175), we see that the best estimate we might hope for is:

$$\partial_\omega^{3+\delta} u \in L^\infty(S) \text{ for all } \delta < \frac{1}{2}. \tag{7.178}$$

Remark 7.6. — Note in conjunction with Remark 2.13 that the estimate (7.178) would still be at least half a derivative away from allowing to apply the TT^* method in step **C2**.

7.3.1. Derivation of the equation for $\partial_\omega^3 N$ and $\partial_\omega^3 u$. — We first establish the link between $\partial_\omega^2 \log(a)$ and $\partial_\omega^2 u$:

Lemma 7.7. — *$\partial_\omega^2 \log(a)$ and $\partial_\omega^2 u$ are linked by the following equality:*

$$\partial_\omega^2 \log(a) = -a\nabla_N(\partial_\omega^2 u) - |\partial_\omega N|^2 + (\partial_\omega \log(a))^2. \tag{7.179}$$

Proof. — We start with the equality $\nabla u = a^{-1}N$. Differentiating it with respect to ω, we obtain:

$$\nabla \partial_\omega u = a^{-1}\partial_\omega N - a^{-1}\partial_\omega \log(a) N, \tag{7.180}$$

which together with (7.16) yields:

$$\begin{cases} \not\nabla \partial_\omega u = a^{-1}\partial_\omega N, \\ \nabla_N \partial_\omega u = -a^{-1}\partial_\omega \log(a). \end{cases} \tag{7.181}$$

Differentiating the second equation of (7.181) with respect to ω yields:

$$\nabla_N \partial_\omega^2 u + \not\nabla_{\partial_\omega N}\partial_\omega u = -a^{-1}\partial_\omega^2 \log(a) + a^{-1}(\partial_\omega \log(a))^2. \tag{7.182}$$

Together with (7.181), this yields (7.179). ☐

Next, we establish the link between $\partial_\omega^3 N$ and $\partial_\omega^3 u$:

Lemma 7.8. — *$\partial_\omega^3 N$ and $\partial_\omega^3 u$ are linked by the following equality:*

$$\begin{aligned}\partial_\omega^3 N = {}& a\nabla\!\!\!/(\partial_\omega^3 u) + (3\partial_\omega^2\log(a) - 3(\partial_\omega\log(a))^2)\partial_\omega N + 3\partial_\omega\log(a)\partial_\omega^2 N\\ &+ (-3g(\partial_\omega N, \partial_\omega^2 N) + 3\partial_\omega\log(a)|\partial_\omega N|^2)N.\end{aligned}\tag{7.183}$$

Proof. — Differentiating the first equation of (7.181) with respect to ω and using (7.11) yields:

$$\nabla\!\!\!/\partial_\omega^2 u - \nabla\!\!\!/_{\partial_\omega N}(\partial_\omega u)N - \nabla_N(\partial_\omega u)\partial_\omega N = a^{-1}\partial_\omega^2 N - a^{-1}\partial_\omega\log(a)\partial_\omega N.\tag{7.184}$$

Together with (7.181), this yields:

$$\partial_\omega^2 N = a\nabla\!\!\!/(\partial_\omega^2 u) + 2\partial_\omega\log(a)\partial_\omega N - |\partial_\omega N|^2 N.\tag{7.185}$$

Differentiating (7.185) with respect to ω, we obtain:

$$\begin{aligned}\partial_\omega^3 N = {}& a\partial_\omega(\nabla\!\!\!/(\partial_\omega^2 u)) + a\partial_\omega\log(a)\nabla\!\!\!/(\partial_\omega^2 u) + 2\partial_\omega^2\log(a)\partial_\omega N + 2\partial_\omega\log(a)\partial_\omega^2 N\\ &- 2g(\partial_\omega^2 N, \partial_\omega N)N - |\partial_\omega N|^2\partial_\omega N.\end{aligned}\tag{7.186}$$

(7.11), (7.185), (7.179) and (7.186) yield:

$$\begin{aligned}\partial_\omega^3 N = {}& a(\nabla\!\!\!/(\partial_\omega^3 u) - \nabla\!\!\!/_{\partial_\omega N}(\partial_\omega^2 u)N - \nabla_N(\partial_\omega^2 u)\partial_\omega N) + a\partial_\omega\log(a)\nabla\!\!\!/(\partial_\omega^2 u)\\ &+ 2\partial_\omega^2\log(a)\partial_\omega N + 2\partial_\omega\log(a)\partial_\omega^2 N - 2g(\partial_\omega^2 N, \partial_\omega N)N - |\partial_\omega N|^2\partial_\omega N\\ = {}& a\nabla\!\!\!/(\partial_\omega^3 u) + (3\partial_\omega^2\log(a) - 3(\partial_\omega\log(a))^2)\partial_\omega N + 3\partial_\omega\log(a)\partial_\omega^2 N\\ &+ (-3g(\partial_\omega N, \partial_\omega^2 N) + 3\partial_\omega\log(a)|\partial_\omega N|^2)N,\end{aligned}\tag{7.187}$$

which implies (7.183). □

We finally derive an equation for $\partial_\omega^3 u$:

Lemma 7.9. — *$\partial_\omega^3 u$ satisfies the following equation:*

$$\begin{aligned}&(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega^3 u\\ &= 2a^{-2}\nabla_{\partial_\omega^3 N}(\log(a)) - 2a^{-2}k(N, \partial_\omega^3 N) - a^{-1}\partial_\omega\log(a)\Delta\!\!\!/\partial_\omega^2 u + 2a^{-1}\nabla\!\!\!/_{\partial_\omega N}\nabla_N\partial_\omega^2 u\\ &\quad + \partial_\omega^2\log(a)(3a^{-2}\partial_\omega\mathrm{tr}\,\theta - 2a^{-1}\partial_\omega\log(a) - 8a^{-2}\nabla\!\!\!/_{\partial_\omega N}\log(a) + 4a^{-2}k(N, \partial_\omega N))\\ &\quad + 4a^{-2}\nabla\!\!\!/_{\partial_\omega N}\partial_\omega^2\log(a) + 6a^{-2}\nabla_{\partial_\omega^2 N}\partial_\omega\log(a) - 12a^{-2}\partial_\omega\log(a)\nabla_{\partial_\omega^2 N}\log(a)\\ &\quad + 12a^{-2}\partial_\omega\log(a)k(N, \partial_\omega^2 N) + 6a^{-2}\theta(\partial_\omega N, \partial_\omega^2 N) - 3a^{-2}\mathrm{tr}\,\theta g(\partial_\omega N, \partial_\omega^2 N)\\ &\quad - 6a^{-2}k(\partial_\omega N, \partial_\omega^2 N) - 3a^{-1}g(\partial_\omega N, \partial_\omega^2 N) - 5a^{-2}(\partial_\omega\log(a))^2\partial_\omega\mathrm{tr}\,\theta\\ &\quad + 2a^{-2}\partial_\omega\theta(\partial_\omega N, \partial_\omega N) - 16a^{-2}\partial_\omega\log(a)\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega\log(a)) - 2a^{-1}(\partial_\omega\log(a))^3\\ &\quad + a^{-2}(\partial_\omega\log(a))^2(16\nabla\!\!\!/_{\partial_\omega N}(\log(a)) - 8k(N, \partial_\omega N)) + \partial_\omega\log(a)(4a^{-2}\mathrm{tr}\,\theta|\partial_\omega N|^2\\ &\quad - 8a^{-2}\theta(\partial_\omega N, \partial_\omega N) + 12a^{-2}k(\partial_\omega N, \partial_\omega N) + 3a^{-1}|\partial_\omega N|^2).\end{aligned}\tag{7.188}$$

Proof. — We start by obtaining an equation for $\partial_\omega u$. We differentiate the first equation of (4.5) by ω:

$$\partial_\omega \mathrm{tr}\,\theta - 2k(N, \partial_\omega N) = -a\partial_\omega \log(a). \tag{7.189}$$

By (4.4), we have $\mathrm{tr}\,\theta = \mathrm{div}(N)$, and differentiating with respect to ω, we obtain:

$$\partial_\omega \mathrm{tr}\,\theta = \mathrm{div}(\partial_\omega N). \tag{7.190}$$

Now, for any vector field X tangent to P_u, we have:

$$\mathrm{div}(X) = \mathrm{d}\not{\mathrm{i}}\mathrm{v}(X) + \nabla\!\!\!/_X \log(a), \tag{7.191}$$

which together with (7.16) and (7.190) yields:

$$\partial_\omega \mathrm{tr}\,\theta = \mathrm{d}\not{\mathrm{i}}\mathrm{v}(\partial_\omega N) + \nabla\!\!\!/_{\partial_\omega N} \log(a). \tag{7.192}$$

(7.181), (7.189) and (7.192) imply:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega u = a^{-1}\nabla\!\!\!/ \log(a)\nabla\!\!\!/\partial_\omega u + a^{-2}\nabla\!\!\!/_{\partial_\omega N} \log(a) - 2a^{-2}k(N, \partial_\omega N), \tag{7.193}$$

which together with the first equation of (7.181) yields:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega u = 2a^{-2}\nabla\!\!\!/_{\partial_\omega N} \log(a) - 2a^{-2}k(N, \partial_\omega N). \tag{7.194}$$

We differentiate (7.194) with respect to ω to obtain an equation for ∂_ω^2:

$$\begin{aligned}
&\nabla_N(\partial_\omega^2 u) + \nabla\!\!\!/_{\partial_\omega N}(\partial_\omega u) - a^{-1}\Delta\!\!\!/(\partial_\omega^2 u) - a^{-1}[\partial_\omega, \Delta\!\!\!/](\partial_\omega u) + a^{-1}\partial_\omega(\log(a))\Delta\!\!\!/(\partial_\omega u)\\
&\quad = 2a^{-2}\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega \log(a)) + 2a^{-2}\nabla_{\partial_\omega^2 N}(\log(a)) - 4a^{-2}\partial_\omega \log(a)\nabla\!\!\!/_{\partial_\omega N} \log(a)\\
&\qquad - 2a^{-2}k(N, \partial_\omega^2 N) - 2a^{-2}k(\partial_\omega N, \partial_\omega N) + 4a^{-2}\partial_\omega \log(a)k(N, \partial_\omega N).
\end{aligned} \tag{7.195}$$

The first equation of (7.181) and (7.192) yield:

$$\Delta\!\!\!/(\partial_\omega u) = a^{-1}\partial_\omega \mathrm{tr}\,\theta - 2a^{-1}\nabla\!\!\!/_{\partial_\omega N} \log(a). \tag{7.196}$$

(7.13), (7.181), (7.195) and (7.196) imply:

$$\begin{aligned}
(\nabla_N - a^{-1}\Delta\!\!\!/)\partial_\omega^2 u = {} & -2a^{-1}\nabla^2(\partial_\omega u)(N, \partial_\omega N) + 2a^{-2}\nabla_{\partial_\omega^2 N}(\log(a))\\
& - 2a^{-2}k(N, \partial_\omega^2 N) + 2a^{-2}\partial_\omega(\log(a))\partial_\omega \mathrm{tr}\,\theta + 2a^{-2}\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega \log(a))\\
& - 6a^{-2}\partial_\omega \log(a)\nabla\!\!\!/_{\partial_\omega N} \log(a) + 4a^{-2}\partial_\omega \log(a)k(N, \partial_\omega N)\\
& - a^{-2}\mathrm{tr}\,\theta|\partial_\omega N|^2 - 2a^{-2}k(\partial_\omega N, \partial_\omega N) - a^{-1}|\partial_\omega N|^2.
\end{aligned} \tag{7.197}$$

Using (7.181), we rewrite $\nabla^2(\partial_\omega u)(N, \partial_\omega N)$ as:

$$\begin{aligned}
\nabla^2(\partial_\omega u)(N, \partial_\omega N) &= \nabla_{\partial_\omega N}(\nabla_N(\partial_\omega u)) - \nabla_{\nabla_{\partial_\omega N} N}(\partial_\omega u)\\
&= \nabla\!\!\!/_{\partial_\omega N}(-a^{-1}\partial_\omega \log(a)) - \theta(\partial_\omega N, e_A)\nabla\!\!\!/_A(\partial_\omega u)\\
&= -a^{-1}\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega \log(a)) + a^{-1}\partial_\omega \log(a)\nabla\!\!\!/_{\partial_\omega N}(\log(a))\\
&\quad - a^{-1}\theta(\partial_\omega N, \partial_\omega N).
\end{aligned} \tag{7.198}$$

(7.197) and (7.198) yield:

(7.199)
$$\begin{aligned}&(\nabla_N - a^{-1}\not\Delta)\partial_\omega^2 u\\ &= 2a^{-2}\nabla_{\partial_\omega^2 N}(\log(a)) - 2a^{-2}k(N,\partial_\omega^2 N) + 2a^{-2}\partial_\omega(\log(a))\partial_\omega \mathrm{tr}\,\theta\\ &\quad + 4a^{-2}\not\nabla_{\partial_\omega N}(\partial_\omega \log(a)) - 8a^{-2}\partial_\omega \log(a)\not\nabla_{\partial_\omega N}\log(a) + 4a^{-2}\partial_\omega\log(a)k(N,\partial_\omega N)\\ &\quad - a^{-2}\mathrm{tr}\,\theta|\partial_\omega N|^2 + 2a^{-2}\theta(\partial_\omega N,\partial_\omega N) - 2a^{-2}k(\partial_\omega N,\partial_\omega N) - a^{-1}|\partial_\omega N|^2.\end{aligned}$$

Differentiating (7.199) with respect to ω, we obtain:

(7.200)
$$\begin{aligned}&(\nabla_N - a^{-1}\not\Delta)\partial_\omega^3 u + \not\nabla_{\partial_\omega N}\partial_\omega^2 u + a^{-1}\partial_\omega\log(a)\not\Delta\partial_\omega^2 u - a^{-1}[\partial_\omega,\not\Delta]\partial_\omega^2 u\\ &\quad = 2a^{-2}\nabla_{\partial_\omega^3 N}(\log(a)) - 2a^{-2}k(N,\partial_\omega^3 N) + 2a^{-2}\partial_\omega\log(a)\partial_\omega^2\mathrm{tr}\,\theta\\ &\quad\quad + \partial_\omega^2\log(a)(2a^{-2}\partial_\omega\mathrm{tr}\,\theta - 8a^{-2}\not\nabla_{\partial_\omega N}\log(a) + 4a^{-2}k(N,\partial_\omega N))\\ &\quad\quad + 4a^{-2}\not\nabla_{\partial_\omega N}\partial_\omega^2\log(a) + 6a^{-2}\nabla_{\partial_\omega^2 N}\partial_\omega\log(a) - 12a^{-2}\partial_\omega\log(a)\nabla_{\partial_\omega^2 N}\log(a)\\ &\quad\quad + 8a^{-2}\partial_\omega\log(a)k(N,\partial_\omega^2 N) + 4a^{-2}\theta(\partial_\omega N,\partial_\omega^2 N) - 2a^{-2}\mathrm{tr}\,\theta g(\partial_\omega N,\partial_\omega^2 N)\\ &\quad\quad - 6a^{-2}k(\partial_\omega N,\partial_\omega^2 N) - 2a^{-1}g(\partial_\omega N,\partial_\omega^2 N) - 4a^{-2}(\partial_\omega\log(a))^2\partial_\omega\mathrm{tr}\,\theta\\ &\quad\quad - a^{-2}\partial_\omega\mathrm{tr}\,\theta|\partial_\omega N|^2 + 2a^{-2}\partial_\omega\theta(\partial_\omega N,\partial_\omega N) - 16a^{-2}\partial_\omega\log(a)\not\nabla_{\partial_\omega N}(\partial_\omega\log(a))\\ &\quad\quad + a^{-2}(\partial_\omega\log(a))^2(16\not\nabla_{\partial_\omega N}(\log(a)) - 8k(N,\partial_\omega N)) + \partial_\omega\log(a)(2a^{-2}\mathrm{tr}\,\theta|\partial_\omega N|^2\\ &\quad\quad - 4a^{-2}\theta(\partial_\omega N,\partial_\omega N) + 8a^{-2}k(\partial_\omega N,\partial_\omega N) + a^{-1}|\partial_\omega N|^2).\end{aligned}$$

Using (7.185), we have:

$$\not\nabla_{\partial_\omega N}\partial_\omega^2 u = a^{-1}g(\partial_\omega N,\partial_\omega^2 N) - 2a^{-1}\partial_\omega\log(a)|\partial_\omega N|^2. \tag{7.201}$$

(7.13) and (7.31) yield:

(7.202)
$$\begin{aligned}[\partial_\omega,\not\Delta]\partial_\omega^2 u &= -2\nabla^2\partial_\omega^2 u(N,\partial_\omega N) - \partial_\omega\mathrm{tr}\,\theta\nabla_N\partial_\omega^2 u - \mathrm{tr}\,\theta\not\nabla_{\partial_\omega N}\partial_\omega^2 u\\ &= -2\not\nabla_{\partial_\omega N}\nabla_N\partial_\omega^2 u + 2\theta(\partial_\omega N,\not\nabla\partial_\omega^2 u) - \partial_\omega\mathrm{tr}\,\theta\nabla_N\partial_\omega^2 u - \mathrm{tr}\,\theta\not\nabla_{\partial_\omega N}\partial_\omega^2 u,\end{aligned}$$

which together with (7.185) and (7.179) implies:

(7.203)
$$\begin{aligned}[\partial_\omega,\not\Delta]\partial_\omega^2 u &= -2\not\nabla_{\partial_\omega N}\nabla_N\partial_\omega^2 u + a^{-1}\partial_\omega\mathrm{tr}\,\theta\partial_\omega^2\log(a) + 2a^{-1}\theta(\partial_\omega N,\partial_\omega^2 N)\\ &\quad - a^{-1}\mathrm{tr}\,\theta g(\partial_\omega N,\partial_\omega^2 N) - a^{-1}\partial_\omega\mathrm{tr}\,\theta(\partial_\omega\log(a))^2 + a^{-1}\partial_\omega\mathrm{tr}\,\theta|\partial_\omega N|^2\\ &\quad - 4a^{-1}\partial_\omega\log(a)\theta(\partial_\omega N,\partial_\omega N) + 2\mathrm{tr}\,\theta\partial_\omega\log(a)|\partial_\omega N|^2.\end{aligned}$$

Differentiating the first equation of (4.5) twice with respect to ω, we obtain:

$$\partial_\omega^2\mathrm{tr}\,\theta = -a\partial_\omega^2\log(a) - a(\partial_\omega\log(a))^2 + 2k(N,\partial_\omega^2 N) + 2k(\partial_\omega N,\partial_\omega N). \tag{7.204}$$

Finally, (7.200), (7.201), (7.203) and (7.204) imply (7.188). □

7.3.2. Estimates for $\partial_\omega^3 u$. — The Equation (7.188) takes the form:

$$(\nabla_N - a^{-1}\not\Delta)\partial_\omega^3 u = h, \tag{7.205}$$

where h is given by:

$$\begin{aligned}
h = {} & 2a^{-2}\nabla_{\partial_\omega^3 N}(\log(a)) - 2a^{-2}k(N,\partial_\omega^3 N)\\
& - a^{-1}\partial_\omega \log(a)\not\Delta\partial_\omega^2 u + 2a^{-1}\not\nabla_{\partial_\omega N}\nabla_N\partial_\omega^2 u + \partial_\omega^2\log(a)(3a^{-2}\partial_\omega \mathrm{tr}\,\theta - 2a^{-1}\partial_\omega\log(a)\\
& - 8a^{-2}\not\nabla_{\partial_\omega N}\log(a) + 4a^{-2}k(N,\partial_\omega N)) + 4a^{-2}\not\nabla_{\partial_\omega N}\partial_\omega^2\log(a)\\
& + 6a^{-2}\nabla_{\partial_\omega^2 N}\partial_\omega\log(a) - 12a^{-2}\partial_\omega\log(a)\nabla_{\partial_\omega^2 N}\log(a)\\
& + 12a^{-2}\partial_\omega\log(a)k(N,\partial_\omega^2 N) + 6a^{-2}\theta(\partial_\omega N,\partial_\omega^2 N) - 3a^{-2}\mathrm{tr}\,\theta g(\partial_\omega N,\partial_\omega^2 N)\\
& - 6a^{-2}k(\partial_\omega N,\partial_\omega^2 N) - 3a^{-1}g(\partial_\omega N,\partial_\omega^2 N) - 5a^{-2}(\partial_\omega\log(a))^2\partial_\omega\mathrm{tr}\,\theta\\
& + 2a^{-2}\partial_\omega\theta(\partial_\omega N,\partial_\omega N) - 16a^{-2}\partial_\omega\log(a)\not\nabla_{\partial_\omega N}(\partial_\omega\log(a)) - 2a^{-1}(\partial_\omega\log(a))^3\\
& + a^{-2}(\partial_\omega\log(a))^2(16\not\nabla_{\partial_\omega N}(\log(a)) - 8k(N,\partial_\omega N)) + \partial_\omega\log(a)(4a^{-2}\mathrm{tr}\,\theta|\partial_\omega N|^2\\
& - 8a^{-2}\theta(\partial_\omega N,\partial_\omega N) + 12a^{-2}k(\partial_\omega N,\partial_\omega N) + 3a^{-1}|\partial_\omega N|^2).
\end{aligned}$$

Let $0 < b < \frac{1}{2}$. We estimate the norm of h in $L^2_u H^b(P_u)$. Using the product estimate (5.78), we have:

$$\begin{aligned}
& \|h\|_{L^2_u H^b(P_u)}\\
& \lesssim \|a^{-2}\nabla_{\partial_\omega^3 N}(\log(a))\|_{L^2_u H^b(P_u)} + \|\partial_\omega^3 N\|_{L^2_u H^1(P_u)}\|a^{-2}k\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\\
& \quad + \|\partial_\omega a\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\|\not\Delta\partial_\omega^2 u\|_{L^2_u H^1(P_u)} + \|a^{-1}\not\nabla_{\partial_\omega N}\nabla_N\partial_\omega^2 u\|_{L^2_u H^b(P_u)}\\
& \quad + \|\partial_\omega^2\log(a)\|_{L^2_u H^1(P_u)}(\|a^{-2}\partial_\omega\mathrm{tr}\,\theta\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|a^{-1}\partial_\omega\log(a)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\\
& \quad + \|a^{-2}\not\nabla_{\partial_\omega N}\log(a)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|a^{-2}k(N,\partial_\omega N)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)})\\
& \quad + \|a^{-2}\not\nabla_{\partial_\omega N}\partial_\omega^2\log(a)\|_{L^2_u H^b(P_u)} + \|a^{-2}\nabla_{\partial_\omega^2 N}\partial_\omega\log(a)\|_{L^2_u H^b(P_u)}\\
& \quad + \|a^{-2}\partial_\omega\log(a)\|_{L^\infty_u H^1(P_u)}\|\partial_\omega^2 N\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\|\nabla\log(a)\|_{L^2_u H^1(P_u)}\\
& \quad + \|\partial_\omega^2 N\|_{L^2_u H^1(P_u)}(\|a^{-2}\partial_\omega\log(a)\|_{L^\infty_u H^1(P_u)}\|k\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|a^{-2}\theta\partial_\omega N\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\\
& \quad + \|a^{-2}\mathrm{tr}\,\theta\partial_\omega N\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|a^{-2}k\partial_\omega N\|_{L^\infty_u H^{\frac{1}{2}}(P_u)})\\
& \quad + \|a^{-1}\partial_\omega\log(a)\|^2_{L^\infty_u H^1(P_u)}\|\partial_\omega\mathrm{tr}\,\theta\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|\partial_\omega\theta\|_{L^2_u H^{\frac{1}{2}}(P_u)}\|a^{-1}\partial_\omega N\|^2_{L^\infty_u H^1(P_u)}\\
& \quad + \|\partial_\omega\log(a)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\|a^{-2}\partial_\omega N\|_{L^\infty_{[-2,2]}L^1(P_u)}\|\not\nabla(\partial_\omega\log(a))\|_{L^2_u H^1(P_u)}\\
& \quad + \|a^{-1}\partial_\omega\log(a)\|_{L^2_u H^{\frac{1}{2}}(P_u)}\|\partial_\omega\log(a)\|^2_{L^\infty_u H^1(P_u)} + \|a^{-1}\partial_\omega\log(a)\|^2_{L^\infty_u H^1(P_u)}\\
& \quad\quad \times(\|\not\nabla_{\partial_\omega N}(\log(a)\|_{L^2_u H^{\frac{1}{2}}(P_u)} + \|k\partial_\omega N\|_{L^2_u H^{\frac{1}{2}}(P_u)})\\
& \quad + \|\partial_\omega\log(a)\|_{L^\infty_u H^1(P_u)}(\|\theta\|_{L^\infty_u H^{\frac{1}{2}}(P_u)} + \|k\|_{L^\infty_u H^{\frac{1}{2}}(P_u)})\|a^{-2}(\partial_\omega N)^2\|_{L^2_u H^1(P_u)}.
\end{aligned}$$

Together with the embedding (5.74), the estimates (4.9) and (4.11) for a, the estimate (4.10) and (4.12) for θ, the estimates (4.3) (4.13) for k, the estimate (2.32) for $\partial_\omega a$,

$\partial_\omega N$ and $\partial_\omega\theta$, and the estimate (2.33) for $\partial_\omega^2\log(a)$ and $\partial_\omega^2 N$, we obtain:

(7.206)
$$\begin{aligned}&\|h\|_{L^2_u H^b(P_u)}\\ &\lesssim \|a^{-2}\nabla_{\partial^3_\omega N}(\log(a))\|_{L^2_u H^b(P_u)} + \varepsilon\|\Delta\!\!\!/\partial^2_\omega u\|_{L^2_u H^1(P_u)} + \|a^{-1}\nabla\!\!\!/_{\partial_\omega N}\nabla_N\partial^2_\omega u\|_{L^2_u H^b(P_u)}\\ &\quad + \|a^{-2}\nabla\!\!\!/_{\partial_\omega N}\partial^2_\omega\log(a)\|_{L^2_u H^b(P_u)} + \|a^{-2}\nabla_{\partial^2_\omega N}\partial_\omega\log(a)\|_{L^2_u H^b(P_u)}\\ &\quad + \varepsilon + \varepsilon\|\partial^3_\omega N\|_{L^2_u H^1(P_u)}.\end{aligned}$$

Next, we estimate the various terms in the right-hand side of (7.206) starting with the fifth one. Using the decomposition (7.132) of $\partial^2_\omega N$, we have:

$$a^{-2}\nabla_{\partial^2_\omega N}\partial_\omega\log(a) = -a^{-2}|\partial_\omega N|^2\nabla_N\partial_\omega\log(a) + a^{-2}\nabla\!\!\!/_{\Pi(\partial^2_\omega N)}\partial_\omega\log(a).$$

Together with the product estimate (5.78), this yields:

(7.207)
$$\begin{aligned}\|a^{-2}\nabla_{\partial^2_\omega N}\partial_\omega\log(a)\|_{L^2_u H^b(P_u)} &\lesssim \|a^{-1}\partial_\omega N\|^2_{L^\infty_u H^1(P_u)}\|\nabla_N\partial_\omega\log(a)\|_{L^2_u H^{\frac12}(P_u)}\\ &\quad + \|a^{-2}\partial^2_\omega N\|_{L^\infty_u H^{\frac12}(P_u)}\|\nabla\!\!\!/\partial_\omega\log(a)\|_{L^2_u H^1(P_u)}\\ &\lesssim \varepsilon,\end{aligned}$$

where we used in the last inequality the embedding (5.74), the estimate (4.9) for a, the estimate (2.32) for $\partial_\omega a$, $\partial_\omega N$, and the estimate (2.33) for $\partial^2_\omega N$.

Next, we estimate the first term in the right-hand side of (7.206). We first provide a decomposition of $\partial^3_\omega N$. Differentiating (7.131) with respect to ω, we obtain:

$$g(\partial^3_\omega N, N) = -3g(\partial^2_\omega N, \partial_\omega N),$$

which yields:

$$\partial^3_\omega N = \Pi(\partial^3_\omega N) - 3g(\partial^2_\omega N,\partial_\omega N)N. \tag{7.208}$$

We obtain:

$$a^{-2}\nabla_{\partial^3_\omega N}(\log(a)) = -3a^{-2}g(\partial^2_\omega N,\partial_\omega N)\nabla_N(\log(a)) + a^{-2}\nabla\!\!\!/_{\Pi(\partial^3_\omega N)}(\log(a)).$$

Together with the product estimate (5.78), this yields:

$$\begin{aligned}\|a^{-2}\nabla_{\partial^3_\omega N}(\log(a))\|_{L^2_u H^b(P_u)} &\lesssim \|\partial^2_\omega N\|_{L^\infty_u H^{\frac12}(P_u)}\|a^{-2}\partial_\omega N\|_{L^\infty_u H^1(P_u)}\|\nabla_N(\log(a))\|_{L^2_u H^1(P_u)}\\ &\quad + \|\partial^3_\omega N\|_{L^2_u H^1(P_u)}\|a^{-2}\nabla\!\!\!/\log(a)\|_{L^\infty_u H^{\frac12}(P_u)}\\ &\lesssim \varepsilon(1+\|\partial^3_\omega N\|_{L^2_u H^1(P_u)}),\end{aligned} \tag{7.209}$$

where we used in the last inequality the embedding (5.74), the estimates (4.9) (4.11) for a, the estimate (2.32) for $\partial_\omega N$, and the estimate (2.33) for $\partial^2_\omega N$.

Next, we estimate the fourth term in the right-hand side of (7.206). We have:

$$a^{-2}\nabla\!\!\!/_{\partial_\omega N}\partial^2_\omega\log(a) = \mathrm{div}\!\!\!/(a^{-2}\partial^2_\omega\log(a)\partial_\omega N) - \mathrm{div}\!\!\!/(a^{-2}\partial_\omega N)\partial^2_\omega\log(a).$$

Together with the product estimates (5.84) and (5.78), this yields:

(7.210) $\quad \|a^{-2}\nabla\!\!\!/_{\partial_\omega N}\partial^2_\omega\log(a)\|_{L^2_u H^b(P_u)}$

$$
\begin{aligned}
&\lesssim \|\mathrm{d}\!\!\!/\mathrm{iv}(a^{-2}\partial_\omega^2\log(a)\partial_\omega N)\|_{L^2_u H^b(P_u)} + \|\mathrm{d}\!\!\!/\mathrm{iv}(a^{-2}\partial_\omega N)\partial_\omega^2\log(a)\|_{L^2_u H^b(P_u)}\\
&\lesssim (\|\partial_\omega^2\log(a)\|_{L^2_u H^{\frac{3}{2}}(P_u)} + \|\partial_\omega^2\log(a)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)})(\|a^{-2}\partial_\omega N\|_{L^\infty(S)}\\
&\quad + \|\nabla\!\!\!/(a^{-2}\partial_\omega N)\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\mathrm{d}\!\!\!/\mathrm{iv}(a^{-2}\partial_\omega N)\|_{L^2_u H^1(P_u)})\\
&\lesssim \varepsilon,
\end{aligned}
$$

where we used in the last inequality the estimate (4.9) for a, the estimate (2.32) for $\partial_\omega N$, and the estimate (2.33) for $\partial_\omega^2\log(a)$.

Next, we estimate the second term in the right-hand side of (7.206). In view of (7.185), we have:

$$\Pi(\partial_\omega^2 N) = a\nabla\!\!\!/(\partial_\omega^2 u) + 2\partial_\omega\log(a)\partial_\omega N.$$

Differentiating, we obtain:

$$\mathrm{d}\!\!\!/\mathrm{iv}(\Pi(\partial_\omega^2 N)) = a\Delta\!\!\!\!/(\partial_\omega^2 u) + \nabla\!\!\!/(a)\cdot\nabla\!\!\!/(\partial_\omega^2 u) + 2\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega\log(a)) + 2\partial_\omega\log(a)\mathrm{d}\!\!\!/\mathrm{iv}(\partial_\omega N),$$

which together with (7.185) implies:

$$
\begin{aligned}
\Delta\!\!\!\!/(\partial_\omega^2 u) &= a^{-1}\mathrm{d}\!\!\!/\mathrm{iv}(\Pi(\partial_\omega^2 N)) - a^{-2}\nabla\!\!\!/_{\Pi(\partial_\omega^2 N)}a + 2a^{-1}\nabla\!\!\!/_{\partial_\omega N}(a)\partial_\omega\log(a)\\
&\quad - 2a^{-1}\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega\log(a)) - 2a^{-1}\partial_\omega\log(a)\mathrm{d}\!\!\!/\mathrm{iv}(\partial_\omega N).
\end{aligned}
$$

This yields:

(7.211)

$$
\begin{aligned}
&\|\Delta\!\!\!\!/(\partial_\omega^2 u)\|_{L^2_u H^1(P_u)}\\
&\lesssim (\|a^{-1}\|_{L^\infty(S)} + \|\nabla\!\!\!/(a^{-1})\|_{L^\infty_{[-2,2]}L^4(P_u)})(\|\mathrm{d}\!\!\!/\mathrm{iv}(\Pi(\partial_\omega^2 N))\|_{L^2_u H^1(P_u)}\\
&\quad + \|\nabla\!\!\!/^2\partial_\omega\log(a)\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}) + \|\nabla\!\!\!/^2 a\|_{L^2(S)}(\|\partial_\omega^2 N\|_{L^\infty(S)}\\
&\quad + \|\partial_\omega\log(a)\|_{L^\infty(S)}\|\partial_\omega N\|_{L^\infty(S)}) + \|\nabla\!\!\!/^2\partial_\omega\log(a)\|_{L^2(S)}\|a^{-1}\partial_\omega N\|_{L^\infty(S)}\\
&\quad + \|\partial_\omega\log(a)\|_{L^\infty(S)}\|\nabla\!\!\!/\partial_\omega N\|_{L^2_u H^1(P_u)} + \|\nabla\!\!\!/^2\partial_\omega\log(a)\|_{L^2(S)}\|\nabla\!\!\!/\partial_\omega N\|_{L^\infty_{[-2,2]}L^4(P_u)}\\
&\lesssim \varepsilon,
\end{aligned}
$$

where we used in the last inequality the estimates (4.9) (4.12) for a, the estimate (2.32) for $\partial_\omega a$ and $\partial_\omega N$, and the estimate (2.33) for $\partial_\omega^2 N$.

Next, we estimate the third term in the right-hand side of (7.206). Differentiating the identity (7.179), we have:

$$
\begin{aligned}
&a^{-1}\nabla\!\!\!/_{\partial_\omega N}\nabla_N\partial_\omega^2 u\\
&= a^{-1}\nabla\!\!\!/_{\partial_\omega N}\left(-a^{-1}\partial_\omega^2\log(a) - a^{-1}|\partial_\omega N|^2 + a^{-1}(\partial_\omega\log(a))^2\right)\\
&= -a^{-2}\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega^2\log(a)) + a^{-3}\nabla\!\!\!/_{\partial_\omega N}\log(a)(\partial_\omega^2\log(a) + |\partial_\omega N|^2 - (\partial_\omega\log(a))^2)\\
&\quad - 2a^{-2}\partial_\omega N\cdot\nabla\!\!\!/_{\partial_\omega N}\partial_\omega N + 2a^{-2}\partial_\omega\log(a)\nabla\!\!\!/_{\partial_\omega N}(\partial_\omega\log(a)).
\end{aligned}
$$

Together with the product estimate (5.78), we obtain:

(7.212)

$$\|a^{-1}\nabla\!\!\!/_{\partial_\omega N}\nabla_N\partial_\omega^2 u\|_{L^2_u H^b(P_u)}$$

$$
\begin{aligned}
&\lesssim \|a^{-2}\nabla\!\!\!\!/_{\partial_\omega N}(\partial_\omega^2\log(a))\|_{L^2_u H^b(P_u)} + \|a^{-3}\nabla\!\!\!\!/_{\partial_\omega N}\log(a)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}(\|\partial_\omega^2\log(a)\|_{L^2_u H^1(P_u)}\\
&\quad + \|\partial_\omega N\|_{L^\infty_u H^1(P_u)}\|\partial_\omega N\|_{L^2_u H^1(P_u)} + \|\partial_\omega\log(a)\|_{L^\infty_u H^1(P_u)}\|\partial_\omega\log(a)\|_{L^2_u H^1(P_u)})\\
&\quad + \|\partial_\omega N\|_{L^\infty_u H^1(P_u)}\|\nabla\!\!\!\!/\partial_\omega N\|_{L^2_u H^1(P_u)}\\
&\quad + \|\partial_\omega\log(a)\|_{L^\infty_u H^{\frac{1}{2}}(P_u)}\|\nabla\!\!\!\!/\partial_\omega\log(a)\|_{L^2_u H^1(P_u)}\|\partial_\omega N\|_{L^\infty_u H^1(P_u)}\\
&\lesssim \varepsilon,
\end{aligned}
$$

where we used in the last inequality the estimate (7.210), the embedding (5.74), the estimates (4.9) (4.11) for a, the estimate (2.32) for $\partial_\omega a$ and $\partial_\omega N$, and the estimate (2.33) for $\partial_\omega^2\log(a)$.

Finally, in view of (7.206), (7.207), (7.209), (7.210), (7.211) and (7.212), we obtain:

$$
\|h\|_{L^2_u H^b(P_u)} \lesssim \varepsilon + \varepsilon\|\partial_\omega^3 N\|_{L^2_u H^1(P_u)}. \tag{7.213}
$$

In view of the Equation (7.205) for $\partial_\omega^3 u$, the estimate (7.213), and the estimate (5.117) for parabolic equations, we obtain:

(7.214)

$$
\|\partial_\omega^3 u\|_{L^2_u H^{2+b}(P_u)} + \|\partial_\omega^3 u\|_{L^\infty_u H^{1+b}(P_u)} + \|\nabla_N\partial_\omega^3 u\|_{L^2_u H^b(P_u)} \lesssim \varepsilon + \varepsilon\|\partial_\omega^3 N\|_{L^2_u H^1(P_u)},
$$

for any $0 < b < \frac{1}{2}$.

Next, we estimate $\partial_\omega^3 N$. Recall (7.183):

$$
\begin{aligned}
\partial_\omega^3 N = {}& a\nabla\!\!\!\!/(\partial_\omega^3 u) + (3\partial_\omega^2\log(a) - 3(\partial_\omega\log(a))^2)\partial_\omega N + 3\partial_\omega\log(a)\partial_\omega^2 N\\
&+ (-3g(\partial_\omega N, \partial_\omega^2 N) + 3\partial_\omega\log(a)|\partial_\omega N|^2)N.
\end{aligned}
$$

Together with the Gagliargdo-Nirenberg inequality (3.9), this yields:

(7.215)

$$
\begin{aligned}
&\|\partial_\omega^3 N\|_{L^2_u H^1(P_u)}\\
&\lesssim (\|a\|_{L^\infty(S)} + \|\nabla\!\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)})\|\partial_\omega^3 u\|_{L^2_u H^2(P_u)}\\
&\quad + (\|\partial_\omega^2\log(a)\|_{L^2_u H^1(P_u)} + (\|\partial_\omega\log(a)\|_{L^\infty_u H^1(P_u)} + \|\partial_\omega\log(a)\|_{L^\infty(S)})^2)\\
&\times(\|\partial_\omega N\|_{L^\infty_u H^1(P_u)} + \|\partial_\omega N\|_{L^\infty(S)}) + (\|\partial_\omega\log(a)\|_{L^\infty_u H^1(P_u)} + \|\partial_\omega\log(a)\|_{L^\infty(S)}\\
&\quad + \|\partial_\omega N\|_{L^\infty_u H^1(P_u)} + \|\partial_\omega N\|_{L^\infty(S)})(\|\partial_\omega^2 N\|_{L^2_u H^1(P_u)} + \|\partial_\omega^2 N\|_{L^\infty(S)})\\
&\quad + (\|\partial_\omega\log(a)\|_{L^\infty_u H^1(P_u)} + \|\partial_\omega\log(a)\|_{L^\infty(S)})(\|\partial_\omega N\|_{L^\infty_u H^1(P_u)} + \|\partial_\omega N\|_{L^\infty(S)})^2\\
&\lesssim \varepsilon(1 + \|\partial_\omega^3 u\|_{L^2_u H^2(P_u)}),
\end{aligned}
$$

where we used in the last inequality the estimates (4.9) (4.12) for a, the estimate (2.32) for $\partial_\omega a$ and $\partial_\omega N$, and the estimate (2.33) for $\partial_\omega^2\log(a)$ and $\partial_\omega^2 N$. (7.214) and (7.215) imply:

$$
\|\partial_\omega^3 u\|_{L^2_u H^{2+b}(P_u)} + \|\partial_\omega^3 u\|_{L^\infty_u H^{1+b}(P_u)} + \|\nabla_N\partial_\omega^3 u\|_{L^2_u H^b(P_u)} \lesssim \varepsilon(1 + \|\partial_\omega^3 u\|_{L^2_u H^2(P_u)}),
$$

for any $0 < b < \frac{1}{2}$. This yields:

$$
\|\partial_\omega^3 u\|_{L^2_u H^{2+b}(P_u)} + \|\partial_\omega^3 u\|_{L^\infty_u H^{1+b}(P_u)} + \|\nabla_N\partial_\omega^3 u\|_{L^2_u H^b(P_u)} \lesssim \varepsilon, \tag{7.216}
$$

for any $0 < b < \frac{1}{2}$. Now, the strong Bernstein inequality for scalars (5.61) yields:

$$\begin{aligned}
\|\partial_\omega^3 u\|_{L^\infty(S)} &\lesssim \sum_{j\geq 0} \|P_j \partial_\omega^3 u\|_{L^\infty(S)} \\
&\lesssim \sum_{j\geq 0} 2^j \|P_j \partial_\omega^3 u\|_{L^\infty_{[-2,2]} L^2(P_u)} \\
&\lesssim \left(\sum_{j\geq 0} 2^{-jb}\right) \|\partial_\omega^3 u\|_{L^\infty_u H^{1+b}(P_u)} \\
&\lesssim \|\partial_\omega^3 u\|_{L^\infty_u H^{1+b}(P_u)},
\end{aligned}$$

where the last inequality hods for any $b > 0$. Together with (7.216), we finally obtain:

$$\|\partial_\omega^3 u\|_{L^\infty(S)} \lesssim 1.$$

This concludes the proof of (2.34).

CHAPTER 8

A GLOBAL COORDINATE SYSTEM ON P_u AND Σ

The inequalities in Chapters 3 and 9 have been derived under the assumption that P_u can be covered by a finite number of charts satisfying the conditions (3.7) and (5.32) such that the constant $c > 0$ in (3.7) and (5.32) and the number of charts is independent of u. In this chapter, we prove that a covering of P_u by such coordinate systems exists. We first prove the existence of a global coordinate system on P_u, which corresponds to the proof of Proposition 2.8. We then show that (3.7) and (5.32) hold for this global coordinate system on P_u with a constant $c > 0$ independent of u. Finally, we also introduce a global coordinate system on Σ for which we control the determinant of the corresponding Jacobian, which corresponds to the proof of Proposition 2.9.

8.1. Proof of Proposition 2.8

Recall the Definition (2.35) of $\Phi_u : P_u \to T_\omega \mathbb{S}^2$:

$$\Phi_u(x) := \partial_\omega u(x, \omega),$$

where $T_\omega \mathbb{S}^2$ is the tangent space to $\mathbb{S}^2$ at ω.

Step 1. Φ_u is a local C^1 diffeomorphism. — We first prove that Φ_u is a local C^1 diffeomorphism. Using (7.181) we obtain a formula for $d\Phi_u$:

$$d\Phi_u = \nabla\!\!\!/ \partial_\omega u = a^{-1} \partial_\omega N. \tag{8.1}$$

In particular, if e_1, e_2 is an orthonormal frame on TP_u and (φ, ψ) are the usual spherical coordinates on $\mathbb{S}^2$, we have:

$$\mathrm{Jac}\Phi_u = a^{-1} \begin{pmatrix} g(\partial_\varphi N, e_1) g(\partial_\psi N, e_1) \\ g(\partial_\varphi N, e_2) g(\partial_\psi N, e_2) \end{pmatrix}. \tag{8.2}$$

Our estimates for a and $\partial_\omega N$ together with (8.2) imply that we control Φ_u in C^1. We deduce a formula for $(\mathrm{Jac}\Phi_u)^* \mathrm{Jac}\Phi_u$ from (8.2):

$$(\mathrm{Jac}\Phi_u)^* \mathrm{Jac}\Phi_u = a^{-2} \begin{pmatrix} g(\partial_\varphi N, \partial_\varphi N) g(\partial_\psi N, \partial_\varphi N) \\ g(\partial_\psi N, \partial_\varphi N) g(\partial_\psi N, \partial_\psi N) \end{pmatrix}, \tag{8.3}$$

which we denote for simplicity by:

$$(\mathrm{Jac}\Phi_u)^*\mathrm{Jac}\Phi_u = a^{-2}g(\partial_\omega N, \partial_\omega N). \tag{8.4}$$

Recall that u coincides with $x.\omega$ in $|x| \geq 2$, so that $(\mathrm{Jac}\Phi_u)^*\mathrm{Jac}\Phi_u$ is equal to the 2×2 identity matrix I in this region. According to (2.30) and (7.116), we have:

$$\|(\mathrm{Jac}\Phi_u)^*\mathrm{Jac}\Phi_u - I\|_{L^\infty(\Sigma)} \lesssim \varepsilon, \tag{8.5}$$

so that $|\det((\mathrm{Jac}\Phi_u)^*\mathrm{Jac}\Phi_u) - 1| \lesssim \varepsilon$. In turn, this yields:

$$\||\det(\mathrm{Jac}\Phi_u)| - 1\|_{L^\infty(\Sigma)} \lesssim \varepsilon. \tag{8.6}$$

From the fact that Φ_u is C^1 and (8.6), we deduce that Φ_u is a C^1 local diffeomorphism.

Step 2. Φ_u is onto. — We continue by showing that Φ_u is onto. The image of Φ_u is a nonempty subset of $T_\omega\mathbb{S}^2$ which is open since it is a local diffeomorphism at each point in P_u. Let us show that the image of Φ_u is also closed in $T_\omega\mathbb{S}^2$. Indeed, consider a subsequence $\Phi_u(x_n) = y_n$ that converges to some $\underline{y}$ in $T_\omega\mathbb{S}^2$. In particular, y_n is a bounded sequence. Since u coincides with $x.\omega$ in the region $|x| \geq 2$, it is easy to check that

$$\lim_{x\in P_u,\, |x|\to+\infty} |\Phi_u(x)| = +\infty,$$

so that x_n must be a bounded sequence too. Thus, we may extract a subsequence from x_n that converges towards some $\underline{x} \in P_u$. Finally, we have $\Phi_u(\underline{x}) = \underline{y}$ by the continuity of Φ_u, so that the image of Φ_u is closed. Thus, the image of Φ_u is a nonempty open and closed subset of $T_\omega\mathbb{S}^2$. Since $T_\omega\mathbb{S}^2$ is connex, the image of Φ_u coincides with $T_\omega\mathbb{S}^2$, and Φ_u is onto.

Step 3. Φ_u is one-to-one. — We conclude the proof of Proposition 2.8 by showing that Φ_u is one-to-one. Let us assume the contrary. Then, there exists x_1 and x_2 in P_u such that $x_1 \neq x_2$ and $\Phi_u(x_1) = \Phi_u(x_2)$. In particular, using the Definition (2.35) of Φ_u and the usual spherical coordinates (φ, ψ) on $\mathbb{S}^2$, we have:

$$\partial_\varphi u(x_1,\omega) = \partial_\varphi u(x_2,\omega) \text{ and } \partial_\psi u(x_1,\omega) = \partial_\psi u(x_2,\omega). \tag{8.7}$$

We define $\alpha := \partial_\varphi u(x_1,\omega)$ and $\beta := \partial_\psi u(x_1,\omega)$. (8.7) implies that:

(8.8) $\{\partial_\varphi u(.,\omega) = \alpha\}$ and $\{\partial_\psi u(.,\omega) = \beta\}$ intersect at two distinct points in P_u.

Our goal from now on is to prove that the situation described in (8.8) can not happen. Let us first show that the level curve $\{\partial_\varphi u(.,\omega) = \alpha\}$ is connex in P_u. Note that $\{\partial_\varphi u(.,\omega) = \alpha\}$ coincides with the union of two half straight lines in the region $|x| \geq 2$ since u coincides with $x.\omega$ there. Let us call C_- and C_+ the connex component containing each of these half straight lines. Let x_0 a point on $\{\partial_\varphi u(.,\omega) = \alpha\}$. We consider the following curve:

$$\frac{d\mu}{d\tau} = \partial_\varphi N(\mu(\tau)),\ \mu(0) = x_0. \tag{8.9}$$

Since $\partial_\varphi N$ is tangent to $\{\partial_\varphi u(.,\omega) = \alpha\}$, we see that the curve μ is contained inside $\{\partial_\varphi u(.,\omega) = \alpha\}$. Note also that according to (8.3) and (8.5), we have $|\partial_\varphi N| \simeq 1$

everywhere, so that μ exists for all $\tau \in \mathbb{R}$ and does not have a limit in P_u when $\tau \to \pm\infty$. Let us prove that:

$$\lim_{\tau \to \pm\infty} |\mu(\tau)| = +\infty. \tag{8.10}$$

Indeed, if (8.10) does not hold, then we can construct a sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \to \pm\infty$ and $\mu(\tau_n) \to \underline{x}$ for some $\underline{x}$ in $\{\partial_\varphi u(.,\omega) = \alpha\}$. Now, since $\nabla\!\!\!/\,\partial_\varphi u = a^{-1}\partial_\varphi N(\underline{x}) \neq 0$, the implicit function theorem implies the existence of a neighborhood V of $\underline{x}$ in P_u such that $\{\partial_\varphi u(.,\omega) = \alpha\}$ coincides with a single arc of curve in V. Let $n_0 \in \mathbb{N}$ large enough such that $\mu(\tau_n) \in V$ for all $n \geq n_0$. Then, for each $n \geq n_0$ and for τ sufficiently close to τ_n, $\mu(\tau)$ lies inside V and is therefore on this arc of curve. Since μ does not have a limit in P_u when $\tau \to \pm\infty$, this implies that $\mu(\tau)$ covers the whole arc of curve inside V for each $n \geq n_0$ and for τ sufficiently close to τ_n. Thus, $\mu(\tau)$ must be periodic.

Let us now consider the connex components of $P_u \setminus \{\mu(\tau),\, \tau \in \mathbb{R}\}$. If there is only one such component, then there is a neighborhood W in P_u of $\{\mu(\tau),\, \tau \in \mathbb{R}\}$ where $\partial_\varphi u \neq \alpha$ on $W \setminus \{\mu(\tau),\, \tau \in \mathbb{R}\}$ and $W \setminus \{\mu(\tau),\, \tau \in \mathbb{R}\}$ is connex. Thus, either $\partial_\varphi u > \alpha$ everywhere on $W \setminus \{\mu(\tau),\, \tau \in \mathbb{R}\}$, or $\partial_\varphi u < \alpha$ everywhere on $W \setminus \{\mu(\tau),\, \tau \in \mathbb{R}\}$. In both cases, $\partial_\varphi u$ reaches a local extrema on $\{\mu(\tau),\, \tau \in \mathbb{R}\}$, and its gradient vanishes. This is impossible since $\nabla\!\!\!/\,\partial_\varphi u = a^{-1}\partial_\varphi N(\underline{x}) \neq 0$ everywhere.

Assume now that $P_u \setminus \{\mu(\tau),\, \tau \in \mathbb{R}\}$ has at least two connex components. Since $\{\mu(\tau),\, \tau \in \mathbb{R}\}$ is periodic, it is compact, and at least one connex component must be precompact. The boundary of this connex component is $\{\mu(\tau),\, \tau \in \mathbb{R}\}$ where $\partial_\varphi u = \alpha$. So $\partial_\varphi u$ reaches a local extrema inside this precompact connex component, and its gradient vanishes there. This is impossible since $a^{-1}\partial_\varphi N(\underline{x}) \neq 0$ everywhere. This concludes the proof of (8.10).

Since (8.10) holds, this means that any point x_0 in $\{\partial_\varphi u(.,\omega) = \alpha\}$ belongs either to C_- or to C_+. We now prove that $C_- = C+$. Assume the contrary. Consider x_0 for example on C_+. Then since C_+ coincides with a half straight line in the region $|x| \geq 2$, (8.10) implies that $C_+ \cap \{|x| \geq 2\}$ is covered at least twice by $\mu(\tau)$ (when $\tau \to -\infty$ and when $\tau \to +\infty$). Thus, $\mu(\tau)$ takes at least one value twice and must be periodic, which is in contradiction with (8.10). Thus $C_- = C_+$ and the level curve $\{\partial_\varphi u(.,\omega) = \alpha\}$ is connex in P_u.

We now prove that the situation described in (8.8) can not happen. Let x_1 and x_2 the two distinct points of (8.8) where $\{\partial_\varphi u(.,\omega) = \alpha\}$ and $\{\partial_\psi u(.,\omega) = \beta\}$ intersect. Since the level curve $\{\partial_\varphi u(.,\omega) = \alpha\}$ is connex in P_u, $\{\partial_\varphi u(.,\omega) = \alpha\} \setminus (\{x_1\} \cup \{x_2\})$ has three connex components in P_u. Also, since $\{\partial_\varphi u(.,\omega) = \alpha\}$ coincides with the union of two half straight lines in the region $|x| \geq 2$, one of these three connex components is precompact. Let us call $\underline{C}$ this precompact connex component of $\{\partial_\varphi u(.,\omega) = \alpha\} \setminus (\{x_1\} \cup \{x_2\})$. Note that its boundary $\partial\underline{C}$ consists of $\{x_1\} \cup \{x_2\}$. Then, since $\partial_\psi u(x_1) = \partial_\psi u(x_2) = \beta$ by (8.7), $\partial_\psi u$ reaches a local extrema at a point $\underline{x}$ inside $\underline{C}$. Thus, the tangent vector to $\{\partial_\varphi u(.,\omega) = \alpha\}$ and $\{\partial_\psi u(.,\omega) = \beta\}$ at $\underline{x}$ must be collinear. This implies that $\partial_\varphi N(\underline{x})$ and $\partial_\psi N(\underline{x})$ must be collinear. It is impossible since (8.3) and (8.5) yield $|\partial_\varphi N| \simeq 1$, $|\partial_\psi N| \simeq 1$ and $|g(\partial_\varphi N, \partial_\psi N)| \lesssim \varepsilon$.

Finally, we have proved that the situation in (8.8) can not happen so that Φ_u is one-to-one. This concludes the proof of Proposition 2.8.

8.2. The control of the Christoffel symbols

We now show that the global coordinate system induced by Φ_u on P_u satisfies (3.7) and (5.32) such that the constant $c > 0$ in (3.7) and (5.32) is independent of u.

Proposition 8.1. — *Let $\omega \in \mathbb{S}^2$. Let $\Phi_u : P_u \to T_\omega \mathbb{S}^2$ defined by (2.35). Then, it induces a global coordinate system on P_u which satisfies:*

$$|\gamma_{AB}(p)\xi^A\xi^B - |\xi|^2| \lesssim \varepsilon|\xi|^2, \qquad \textit{uniformly for all } p \in \mathbb{R}^2. \tag{8.11}$$

Moreover, the Christoffel symbols Γ^A_{BC} verify,

$$\sum_{A,B,C} \int_{\mathbb{R}^2} |\Gamma^A_{BC}|^2 dx^1 dx^2 \lesssim \varepsilon^2. \tag{8.12}$$

Proof. — The coordinates functions on P_u induced by the global C^1 diffeomorphism Φ_u defined in (2.35) are given by:

$$x_1 = \partial_\varphi u(.,\omega),\ x_2 = \partial_\psi u(.,\omega), \tag{8.13}$$

which using (8.1) implies:

$$\frac{\partial}{\partial x_1} = a^{-1}\partial_\varphi N,\ \frac{\partial}{\partial x_2} = a^{-1}\partial_\psi N. \tag{8.14}$$

Since $\gamma_{AB} = g(\frac{\partial}{\partial x_A}, \frac{\partial}{\partial x_B})$, (8.3), (8.5) and (8.14) imply (8.11).

We now turn to the proof of (8.12). By definition of the Christoffel symbols Γ^A_{BC}, we have:

$$\Gamma^A_{BC} = g\left(\nabla_B \frac{\partial}{\partial x_C}, \frac{\partial}{\partial x_A}\right). \tag{8.15}$$

In view of (8.14) and (8.15), the Christoffel symbols are of the form:

$$\Gamma = a^{-3} g(\nabla_{\partial_\omega N}\partial_\omega N, \partial_\omega N) - a^{-3}\nabla\!\!\!\!/_{\partial_\omega N} a g(\partial_\omega N, \partial_\omega N), \tag{8.16}$$

which together with (7.108) implies:

$$\Gamma = a^{-3}\partial_\omega\theta(\partial_\omega N, \partial_\omega N) - 2a^{-3}\nabla\!\!\!\!/_{\partial_\omega N} a|\partial_\omega N|^2. \tag{8.17}$$

(2.30), (2.32) and (8.17) imply:

$$\|\Gamma\|_{L^\infty_u L^2(P_u)} \lesssim \|\partial_\omega\theta\|_{L^\infty_u L^2(P_u)}\|\partial_\omega N\|^2_{L^\infty(\Sigma)} + \|\nabla\!\!\!\!/ a\|_{L^\infty_u L^2(P_u)}\|\partial_\omega N\|^3_{L^\infty(\Sigma)} \lesssim \varepsilon, \tag{8.18}$$

which is (8.12). This concludes the proof of Proposition 8.1. ☐

8.3. Proof of Proposition 2.9

Let $\omega \in \mathbb{S}^2$. Recall the Definition (2.36) of $\Phi : \Sigma \to \mathbb{R}^3$:

$$\Phi(x) := u(x,\omega)\omega + \partial_\omega u(x,\omega) = u(x,\omega)\omega + \Phi_u(x),$$

where Φ_u has been defined in Proposition 2.8.

We start by showing that Φ is one-to-one. Assume that $\Phi(x_1) = \Phi(x_2)$ for x_1 and x_2 in Σ. Then, since the image of Φ_u is contained in $T_\omega\mathbb{S}^2$, $\omega.\Phi_u(x) = 0$ for all $x \in \Sigma$, so we have from (2.36):

$$u(x_1,\omega) = u(x_2,\omega) \text{ and } \Phi_{u(x_1,\omega)}(x_1) = \Phi_{u(x_2,\omega)}(x_2). \tag{8.19}$$

Since Φ_u is one-to-one by Proposition 2.8, (8.19) implies $x_1 = x_2$. Thus, Φ is one-to-one.

We now prove that Φ is onto. Let $y \in \mathbb{R}^3$. Then $y = (y.\omega)\omega + y'$ where y' belongs to $T_\omega\mathbb{S}^2$. Let $u = y.\omega$. Since Φ_u is onto by Proposition 2.8, there exists $x \in P_u$ such that $\Phi_u(x) = y'$. Thus, $u(x,\omega) = y.\omega$ and $\Phi_u(x) = y'$ so that $\Phi(x) = y$ by (2.36). Therefore, Φ is onto.

We now turn to the proof of (2.37). Using the fact that $\nabla u = a^{-1}N$ together with (7.181) we obtain a formula for $d\Phi$:

$$d\Phi = (\nabla u)\omega + \nabla\partial_\omega u = a^{-1}N\omega + a^{-1}\partial_\omega N - a^{-1}\partial_\omega a N. \tag{8.20}$$

In particular, if e_1, e_2 is an orthonormal frame on TP_u and (φ,ψ) are the usual spherical coordinates on $\mathbb{S}^2$, we have:

$$\mathrm{Jac}\Phi = a^{-1}\begin{pmatrix} 1 & -\partial_\varphi a & -\partial_\psi a \\ 0 & g(\partial_\varphi N, e_1) & g(\partial_\psi N, e_1) \\ 0 & g(\partial_\varphi N, e_2) & g(\partial_\psi N, e_2)\end{pmatrix}. \tag{8.21}$$

We deduce from (8.21) a formula for $(\mathrm{Jac}\Phi_u)^*\mathrm{Jac}\Phi_u$:

$$(\mathrm{Jac}\Phi_u)^*\mathrm{Jac}\Phi_u = a^{-2}\times\begin{pmatrix} 1 & -\partial_\varphi a & -\partial_\psi a \\ -\partial_\varphi a & (\partial_\varphi a)^2 + g(\partial_\varphi N, \partial_\varphi N) & \partial_\varphi a\partial_\psi a g(\partial_\psi N, \partial_\varphi N) \\ -\partial_\psi a & \partial_\varphi a\partial_\psi a + g(\partial_\psi N, \partial_\varphi N) & (\partial_\psi a)^2 + g(\partial_\psi N, \partial_\psi N)\end{pmatrix}.$$

Taking the determinant yields:

$$\det((\mathrm{Jac}\Phi)^*\mathrm{Jac}\Phi) = a^{-2}\det((\mathrm{Jac}\Phi_u)^*\mathrm{Jac}\Phi_u), \tag{8.22}$$

which together with (8.5) implies:

$$\|\det((\mathrm{Jac}\Phi)^*\mathrm{Jac}\Phi) - 1\|_{L^\infty(\Sigma)} \lesssim \varepsilon. \tag{8.23}$$

(8.23) yields (2.37). This concludes the proof of Proposition 2.9.

CHAPTER 9

ADDITIONAL ESTIMATES

This chapter is dedicated to the proof of Proposition 2.10, Proposition 2.11 and Proposition 2.12.

9.1. Proof of Proposition 2.10

We start with the proof of the estimate (2.38). We first derive an estimate for ∇N and $\nabla^2 N$. In view of the structure Equation (4.4), we have:

$$\begin{aligned}\|\nabla N\|_{L^2(S)} + \|\nabla^2 N\|_{L^2(S)} &\lesssim \|\theta\|_{L^2(S)} + \|a^{-1}\nabla\!\!\!/ a\,\|_{L^2(S)} \\ &\quad + \|\nabla\theta\|_{L^2(S)} + \|\nabla(a^{-1}\nabla\!\!\!/ a\,)\|_{L^2(S)} \\ &\lesssim \varepsilon + \|\nabla(a^{-1}\nabla\!\!\!/ a\,)\|_{L^2(S)},\end{aligned} \tag{9.1}$$

where we used in the last inequality the estimate (4.9) for a and the estimate (4.10) for θ. Now, we have:

$$\begin{aligned}\|\nabla(a^{-1}\nabla\!\!\!/ a\,)\|_{L^2(S)} &\lesssim \|\nabla\!\!\!/(a^{-1}\nabla\!\!\!/ a\,\|_{L^2(S)} + \|\nabla_N a^{-1}\nabla\!\!\!/ a\,\|_{L^2(S)} \\ &\lesssim \|a^{-1}\|_{L^\infty(S)}(\|\nabla\!\!\!/\nabla a\|_{L^2(S)} + \|[\nabla_N,\nabla\!\!\!/]a\|_{L^2(S)}) \\ &\quad + \|a^{-2}\|_{L^\infty(S)}\|\nabla a\|^2_{L^4(S)} \\ &\lesssim \varepsilon + \|[\nabla_N,\nabla\!\!\!/]a\|_{L^2(S)},\end{aligned}$$

where we used in the last inequality the estimates (4.9) (4.11) for a. Together with the commutator estimate (2.18), we deduce:

$$\begin{aligned}\|\nabla(a^{-1}\nabla\!\!\!/ a\,)\|_{L^2(S)} &\lesssim \varepsilon + \|[\nabla_N,\nabla\!\!\!/]a\|_{L^2(S)} \\ &\lesssim \varepsilon + (\|\theta\|_{L^4(S)} + \|a^{-1}\nabla\!\!\!/ a\,\|_{L^4(S)})\|\nabla a\|_{L^4(S)} \\ &\lesssim \varepsilon,\end{aligned}$$

where we used in the last inequality the estimates (4.9) (4.11) for a and the estimate (4.10) for θ. In view of (9.1), we finally obtain:

$$\|\nabla N\|_{L^2(S)} + \|\nabla^2 N\|_{L^2(S)} \lesssim \varepsilon. \tag{9.2}$$

Next, recall from Proposition 3.10 the following bound on the $L^\infty(S)$ norm of a tensor F on S. We have:

$$\|F\|_{L^\infty(S)} \lesssim \|F(-2,.)\|_{L^4(P_{-2})} + \|\nabla F\|_{L^2(S)} + \|\nabla\!\!\!/ \nabla F\|_{L^2(S)}. \tag{9.3}$$

Now, recall that $u = x \cdot \omega$ in $|x| \geq 2$, and thus $P_{u=-2} = \{x \cdot \omega = -2\}$. Therefore, P_{-2} is included in the region $|x| \geq 2$. In particular, if $F \equiv 0$ in $|x| \geq 2$, we may use (9.3) and obtain:

$$\|F\|_{L^\infty(\Sigma)} \lesssim \|\nabla F\|_{L^2(\Sigma)} + \|\nabla\!\!\!/ \nabla F\|_{L^2(\Sigma)} \text{ for all } F \text{ such that } F = 0 \text{ in } |x| \geq 2. \tag{9.4}$$

Also, working in the global coordinate system on P_u given by Proposition 2.8, we easily derive

$$\|f\|_{L^2(P_u)} \lesssim \|\nabla\!\!\!/ f\|_{L^2(P_u)} \text{ for any scalar } f \text{ such that } f = 0 \text{ in } P_u \cap \{|x| \geq 2\}.$$

Integrating in u, and in view of coarea formula (3.5), we deduce

$$\|f\|_{L^2(\Sigma)} \lesssim \|\nabla\!\!\!/ f\|_{L^2(\Sigma)} \text{ for any scalar } f \text{ such that } f = 0 \text{ in } |x| \geq 2.$$

With the choice $f = |F|$, this yields

$$\|F\|_{L^2(\Sigma)} \lesssim \|\nabla\!\!\!/ F\|_{L^2(\Sigma)} \text{ for any tensor } F \text{ such that } F = 0 \text{ in } |x| \geq 2.$$

Together with (9.4), we finally obtain

$$\|F\|_{L^\infty(\Sigma)} \lesssim \|\nabla\!\!\!/ \nabla F\|_{L^2(\Sigma)} \text{ for all } F \text{ such that } F = 0 \text{ in } |x| \geq 2. \tag{9.5}$$

Since $u = x \cdot \omega$ in $|x| \geq 2$, we have in particular $N = \omega$ in $|x| \geq 2$. This yields:

$$N(x, \omega) + N(x, -\omega) = \omega - \omega = 0 \text{ in } |x| \geq 2.$$

Thus, using the estimate (9.5) with $F = N(., \omega) + N(., -\omega)$ implies:

$$\begin{aligned}\|N(., \omega) + N(., -\omega)\|_{L^\infty(\Sigma)} &\lesssim \|\nabla^2 N(., \omega)\|_{L^2(\Sigma)} + \|\nabla^2 N(., -\omega)\|_{L^2(\Sigma)} \\ &\lesssim \varepsilon,\end{aligned}$$

where we used the fact that $\nabla N \equiv 0$ in $|x| \geq 2$ and the estimate (9.2). This concludes the proof of the estimate (2.38).

Next, we prove the estimate (2.39). We have:

$$N(x, \omega') = N(x, \omega) + \partial_\omega N(x, \omega)(\omega - \omega') + \int_{[\omega, \omega']} \partial_\omega^2 N(., \omega'') d\omega'' (\omega - \omega')^2.$$

This yields:

$$\begin{aligned}||N(x, \omega) - N(x, \omega')| - |\partial_\omega N(x, \omega)(\omega - \omega')|| &\lesssim \|\partial_\omega^2 N\|_{L^\infty(S)} |\omega - \omega'|^2 \\ &\lesssim |\omega - \omega'|^2,\end{aligned} \tag{9.6}$$

where we used in the last inequality the estimate (2.33) for $\partial_\omega^2 N$. Now, the estimate (7.116) implies:

$$\|g(\partial_\omega N, \partial_\omega N) - I\|_{L^\infty(S)} \lesssim \varepsilon.$$

This yields:

$$||\partial_\omega N(x, \omega)(\omega - \omega')| - |\omega - \omega'|| \lesssim \varepsilon |\omega - \omega'|.$$

Together with (9.6), we obtain:

$$||N(x,\omega) - N(x,\omega')| - |\omega - \omega'|| \lesssim |\omega - \omega'|(\varepsilon + |\omega - \omega'|).$$

This concludes the proof of the estimate (2.39).

Finally, we prove the estimate (2.40). We first estimate u, $\partial_\omega u$ and $\partial_\omega^2 u$. Differentiating the equality $\nabla u = a^{-1}N$, and using the structure Equation (4.4), we obtain:

$$\|\nabla^2 u\|_{L^2(S)} \lesssim \|a^{-2}\nabla a\|_{L^2(S)} + \|a^{-1}\theta\|_{L^2(S)} \lesssim \varepsilon, \tag{9.7}$$

where we used in the last inequality the estimate (4.9) for a and the estimate (4.10) for θ. Also, differentiating the identity (7.180) for $\nabla\partial_\omega u$, and using the structure Equation (4.4), we obtain:

$$\begin{aligned}\|\nabla^2\partial_\omega u\|_{L^2(S)} &\lesssim \|a^{-1}\nabla\partial_\omega N\|_{L^2(S)} + \|a^{-1}\nabla\partial_\omega a\|_{L^2(S)}\\ &\quad + (\|a^{-2}\nabla a\|_{L^4(S)} + \|a^{-1}\theta\|_{L^4(S)})(\|\partial_\omega a\|_{L^\infty(S)} + \|\partial_\omega N\|_{L^\infty(S)})\\ &\lesssim \varepsilon,\end{aligned} \tag{9.8}$$

where we used in the last inequality the estimate (4.9) for a, the estimate (4.10) for θ and the estimate (2.32) for $\partial_\omega N$ and $\partial_\omega a$. Finally, differentiating (7.180) with respect to ω, we obtain:

$$\nabla(\partial_\omega^2 u) = a^{-1}\partial_\omega^2 N - a^{-1}\partial_\omega^2 a N - 2a^{-1}\partial_\omega N\partial_\omega a + a^{-2}(\partial_\omega a)^2 N.$$

Differentiating with respect to $\nabla\!\!\!/$, we obtain:

$$\begin{aligned}\|\nabla\!\!\!/\nabla\partial_\omega^2 u\|_{L^2(S)} &\lesssim \|a^{-1}\|_{L^\infty(S)}(\|\nabla\partial_\omega^2 N\|_{L^2(S)} + \|\nabla\!\!\!/\partial_\omega^2 a\|_{L^2(S)}) + (\|a^{-2}\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\\ &\quad + \|a^{-1}\theta\|_{L^\infty_{[-2,2]}L^4(P_u)})(\|\partial_\omega^2 N\|_{L^2_{[-2,2]}L^4(P_u)} + \|\partial_\omega^2 a\|_{L^2_{[-2,2]}L^4(P_u)}\\ &\quad + \|\partial_\omega a\|_{L^2_{[-2,2]}L^4(P_u)}\|\partial_\omega N\|_{L^\infty(S)} + \|\partial_\omega a\|^2_{L^4_{[-2,2]}L^8(P_u)})\\ &\quad + \|\nabla\!\!\!/\partial_\omega a\|_{L^2(S)}\|\partial_\omega N\|_{L^\infty(S)}\\ &\quad + \|\partial_\omega a\|_{L^\infty(S)}(\|a^{-2}\nabla\!\!\!/ a\|_{L^2(S)} + \|a^{-1}\nabla\!\!\!/\partial_\omega N\|_{L^2(S)})\\ &\lesssim \varepsilon,\end{aligned} \tag{9.9}$$

where we used in the last inequality the estimates (4.9) (4.11) for a, the estimates (4.10) (4.12) for θ, the estimate (2.32) for $\partial_\omega N$ and $\partial_\omega a$, and the estimate (2.33) for $\partial_\omega^2 N$ and $\partial_\omega^2 a$.

Recall that for $\omega \in \mathbb{S}^2$, the map $\Phi_\omega : \Sigma \to \mathbb{R}^3$ is defined by:

$$\Phi_\omega(x) := u(x,\omega)\omega + \partial_\omega u(x,\omega).$$

Since $u = x\cdot\omega$ in $|x| \geq 2$, we have:

$$\Phi_\omega(x) = x \text{ for } |x| \geq 2,$$

which yields:

$$u(x,\omega) - \Phi_\nu(x)\cdot\omega = 0,\ \partial_\omega u(x,\omega) - \partial_\omega(\Phi_\nu(x)\cdot\omega) = 0, \tag{9.10}$$

$$\text{and} \quad \partial_\omega^2 u(x,\omega) - \partial_\omega^2(\Phi_\nu(x)\cdot\omega) = 0 \text{ in } |x|\geq 2.$$

Now, let $\nu\in\mathbb{S}^2$. We first estimate $\partial_\omega^2 u(x,\omega)-\partial_\omega^2(\Phi_\nu(x)\cdot\omega)$. In view of (9.10) and (9.5), we have:

(9.11)

$$\begin{aligned}\|\partial_\omega^2 u(.,\omega)-\partial_\omega^2(\Phi_\nu(.)\cdot\omega)\|_{L^\infty(\Sigma)} &\lesssim \|\nabla\!\!\!/\nabla\partial_\omega^2 u(.,\omega)\|_{L^2(\Sigma)}+\|\nabla^2\partial_\omega u(.,\nu)\|_{L^2(\Sigma)}\\ &\quad+\|\nabla^2 u(.,\nu)\|_{L^2(\Sigma)}\\ &\lesssim \varepsilon,\end{aligned}$$

where we used in the last inequality the estimate (9.7) for u, the estimate (9.8) for $\partial_\omega u$, and the estimate (9.9) for $\partial_\omega^2 u$.

Next, we estimate $\partial_\omega u(x,\omega)-\partial_\omega(\Phi_\nu(x)\cdot\omega)$. We have:

$$\begin{aligned}\Phi_\nu(x)\cdot\omega &= u(x,\nu)\nu\cdot\omega+\partial_\omega u(x,\nu)\omega\\ &= u(x,\nu)+\partial_\omega u(x,\nu)(\omega-\nu)-\frac{|\omega-\nu|^2}{2}u(x,\nu),\end{aligned}\tag{9.12}$$

where we used in the last equality the fact that $\partial_\omega u(x,\nu)\nu=0$. Thus, we obtain:

$$\begin{aligned}\partial_\omega u(x,\omega)-\partial_\omega(\Phi_\nu(x)\cdot\omega) &= \partial_\omega u(x,\omega)-\partial_\omega u(x,\nu)-(\omega-\nu)u(x,\nu)\\ &= \int\partial_\omega^2 u(x,\omega')(\omega-\nu)-(\omega-\nu)u(x,\nu),\end{aligned}$$

where ω' is on the arc $[\omega,\nu]$ of $\mathbb{S}^2$. Together with (9.10) and (9.5), this implies:

(9.13)

$$\begin{aligned}\|\partial_\omega u(.,\omega)-\partial_\omega(\Phi_\nu(.)\cdot\omega)\|_{L^\infty(\Sigma)} &\lesssim |\omega-\nu|(\|\nabla\!\!\!/'\nabla\partial_\omega^2 u(.,\omega')\|_{L^2(\Sigma)}+\|\nabla^2 u(.,\nu)\|_{L^2(\Sigma)})\\ &\lesssim \varepsilon|\omega-\nu|,\end{aligned}$$

where we used in the last inequality the estimate (9.7) for u and the estimate (9.9) for $\partial_\omega^2 u$.

Finally, we estimate $u(x,\omega)-\Phi_\nu(x)\cdot\omega$. In view of (9.12), we have:

$$\begin{aligned}u(x,\omega)-\Phi_\nu(x)\cdot\omega &= u(x,\omega)-u(x,\nu)-\partial_\omega u(x,\nu)(\omega-\nu)+\frac{|\omega-\nu|^2}{2}u(x,\nu)\\ &= \int\partial_\omega^2 u(x,\omega')(\omega-\nu)^2+\frac{|\omega-\nu|^2}{2}u(x,\nu),\end{aligned}$$

where ω' is on the arc $[\omega,\nu]$ of $\mathbb{S}^2$. Together with (9.10) and (9.5), this implies:

(9.14)

$$\begin{aligned}\|u(.,\omega)-\Phi_\nu(.)\cdot\omega\|_{L^\infty(\Sigma)} &\lesssim |\omega-\nu|^2(\|\nabla\!\!\!/'\nabla\partial_\omega^2 u(.,\omega')\|_{L^2(\Sigma)}+\|\nabla^2 u(.,\nu)\|_{L^2(\Sigma)})\\ &\lesssim \varepsilon|\omega-\nu|^2,\end{aligned}$$

where we used in the last inequality the estimate (9.7) for u and the estimate (9.9) for $\partial_\omega^2 u$.

Finally, (9.11), (9.13) and (9.14) imply (2.40). This concludes the proof of Proposition 2.10.

9.2. Proof of Proposition 2.11

Recall from the first equation of (4.5) that $\operatorname{tr}\theta - k_{NN} = 1 - a$. Now, since a satisfies (2.30), $\operatorname{tr}\theta - k_{NN}$ satisfies:

$$\|\nabla_N(\operatorname{tr}\theta - k_{NN})\|_{L^2(S)} + \|\nabla\!\!\!/(\operatorname{tr}\theta - k_{NN})\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/\nabla(\operatorname{tr}\theta - k_{NN})\|_{L^2(S)} \lesssim \varepsilon.$$

Thus, Proposition 2.11 is a direct consequence of the following proposition:

Proposition 9.1. — *Let a scalar function f on Σ such that $f \equiv 0$ on $u = -2$ and:*

$$\|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N f\|_{L^2(S)} + \|\nabla\!\!\!/\nabla f\|_{L^2(S)} \lesssim \varepsilon. \tag{9.15}$$

Then, we have:

$$\|f\|_{\mathcal{B}} \lesssim \varepsilon. \tag{9.16}$$

The rest of this section is dedicated to the proof of Proposition 9.1.

Proof. — Using the Definition (2.41), (3.15), property (iii) of Theorem 5.5 and (9.15), we have:

$$\begin{aligned}
\|f\|_{\mathcal{B}} &= \sum_{j\geq 0} 2^j \|P_j f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|P_{<0} f\|_{L^\infty_{[-2,2]}L^2(P_u)} \\
&\lesssim \sum_{j\geq 0} 2^j \|P_j f\|^{\frac12}_{L^2(S)} (\|\nabla_N P_j f\|_{L^2(S)} + \|\nabla\!\!\!/ P_j f\|_{L^2(S)})^{\frac12} \\
&\quad + \|P_{<0} f\|^{\frac12}_{L^2(S)} (\|\nabla_N P_{<0} f\|_{L^2(S)} + \|\nabla\!\!\!/ P_{<0} f\|_{L^2(S)})^{\frac12} \\
&\lesssim \sum_{j\geq 0} \|\Delta\!\!\!/ f\|^{\frac12}_{L^2(S)} (\|\nabla_N P_j f\|_{L^2(S)} + 2^{-j}\|\Delta\!\!\!/ f\|_{L^2(S)})^{\frac12} \\
&\quad + \|f\|^{\frac12}_{L^2(S)} (\|\nabla_N P_{<0} f\|_{L^2(S)} + \|f\|_{L^2(S)})^{\frac12} \\
&\lesssim \varepsilon^{\frac12} \left(\sum_{j\geq 0} \|\nabla_{aN} P_j f\|^{\frac12}_{L^2(S)} + \|\nabla_{aN} P_{<0} f\|^{\frac12}_{L^2(S)} \right) + \varepsilon,
\end{aligned} \tag{9.17}$$

where we used the estimate (2.32) for a in the last estimate. The term $\|\nabla_{aN} P_{<0} f\|_{L^2(S)}$ is easier to bound, so we concentrate on estimating the sum $\sum_{j\geq 0}\|\nabla_{aN} P_j f\|^{\frac12}_{L^2(S)}$.

Let $0 < \delta < 1$. In view of the finite band property for P_j, and the commutator estimate (5.76), we have:

$$\begin{aligned}
\|\nabla_{aN} P_j f\|_{L^2(S)} &\lesssim \|P_j(\nabla_{aN} f)\|_{L^2(S)} + \|[\nabla_{aN}, P_j] f\|_{L^2(S)} \\
&\lesssim 2^{-j}\|\nabla\!\!\!/(\nabla_{aN} f)\|_{L^2(S)} + 2^{-(1-\delta)j}\varepsilon(\|\Delta\!\!\!/ f\|_{L^2(S)} + \|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)}) \\
&\lesssim 2^{-(1-\delta)j}\varepsilon(\|a\|_{L^\infty(S)}\|\nabla\!\!\!/\nabla_N f\|_{L^2(S)} + \|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N f\|_{L^2_{[-2,2]}L^4(P_u)} \\
&\quad + \|\Delta\!\!\!/ f\|_{L^2(S)} + \|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)}) \\
&\lesssim 2^{-(1-\delta)j}\varepsilon,
\end{aligned} \tag{9.18}$$

where we used in the last inequality the Gagliardo-Nirenberg inequality (3.9), the estimate (2.30) for a and the estimate (9.15) for f. Since $\delta < 1$, (9.17) and (9.18) imply (9.16). This concludes the proof of the proposition. □

9.3. Proof of Proposition 2.12

We decompose $\nabla_N a$ in the following way:

$$\nabla_N a = a_1^j + a_2^j, \text{ where } a_1^j = P_{>j/2}(\nabla_N a) \text{ and } a_2^j = P_{\leq j/2}(\nabla_N a). \tag{9.19}$$

Using the estimate (4.9) for a and the finite band property for P_j, we obtain:

$$\|a_1^j\|_{L^2(S)} \leq \sum_{l>j/2} \|P_l \nabla_N a\|_{L^2(S)} \lesssim \sum_{l>j/2} 2^{-l}\|\nabla\!\!\!/ \nabla_N a\|_{L^2(S)} \lesssim 2^{-j/2}\varepsilon. \tag{9.20}$$

We also have:

$$\|\nabla_N a_2^j\|_{L^2(S)} \leq \sum_{l\leq j/2} \|\nabla_N P_l \nabla_N a\|_{L^2(S)} \lesssim \sum_{l\leq j/2} \|\nabla_{aN} P_l \nabla_N a\|_{L^2(S)}, \tag{9.21}$$

where we used in the last inequality the estimate (4.9) for a.

Next, we estimate $\nabla_{aN} P_l \nabla_N a$. Let $\delta > 0$. In view of the finite band property for P_l and the commutator estimate (5.72), we have:

$$\begin{aligned}\|\nabla_{aN} P_l \nabla_N a\|_{L^2(S)} &\lesssim \|P_l(a\nabla_N^2 a)\|_{L^2(S)} + \|[\nabla_{aN}, P_l]\nabla_N a\|_{L^2(S)}\\ &\lesssim 2^{\frac{l}{2}}\|a\nabla_N^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)} + \varepsilon\|\Lambda^{\frac{1}{2}+\delta}\nabla_N a\|_{L^2(S)}\\ &\quad + \varepsilon\|\Lambda^\delta \nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)}.\end{aligned}$$

Together with the product estimate (5.80), we obtain:

$$\begin{aligned}\|\nabla_{aN} P_l \nabla_N a\|_{L^2(S)} &\lesssim 2^{\frac{l}{2}}(\|a\|_{L^\infty(S)} + \|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^2(P_u)})\|\nabla_N^2 a\|_{L^2_u H^{-\frac{1}{2}}(P_u)}\\ &\quad + \varepsilon\|\nabla\!\!\!/ \nabla_N a\|_{L^2(S)} + \varepsilon\|\nabla_N a\|_{L^\infty_u H^\delta(P_u)}\\ &\lesssim 2^{\frac{l}{2}}\varepsilon + \varepsilon\|\nabla_N a\|_{L^\infty_u H^\delta(P_u)},\end{aligned} \tag{9.22}$$

where we used in the last inequality the estimate (2.30) for a and the estimate (2.31) for $\nabla_N^2 a$. Now, in view of the decomposition (6.6) of $\nabla_N a$, and the estimates (6.8) and (6.10), we have for all $j \geq 0$:

$$\|P_j \nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim 2^{-\frac{j}{2}}\varepsilon,$$

which yields:

$$\|\nabla_N a\|_{L^\infty_u H^{(\frac{1}{2})-}(P_u)} \lesssim \varepsilon. \tag{9.23}$$

Choosing $0 < \delta < \frac{1}{2}$ in (9.22) and using (9.23) finally yields:

$$\|\nabla_{aN} P_l \nabla_N a\|_{L^2(S)} \lesssim 2^{\frac{l}{2}}\varepsilon.$$

Together with (9.21), we obtain:

$$\|\nabla_N a_2^j\|_{L^2(S)} \lesssim \sum_{l \le j/2} 2^{\frac{l}{2}} \varepsilon \lesssim 2^{\frac{j}{4}} \varepsilon. \tag{9.24}$$

Finally, in view of (9.19), (9.20) and (9.24), we obtain the conclusion of the proposition.

APPENDIX A

PROOF OF PROPOSITION 4.2 AND THEOREM 4.4

A.1. Proof of (4.47) for $j = 2$

Remark first that (4.47) for $j = 1$ has already been obtained in Section 4.1. We prove (4.47) by iteration on j. Let us first start in this section with the case $j = 2$.

We start by estimating $\|\nabla_N^2 a\|_{L^2(S)}$ and $\|\nabla\!\!\!/^2 \nabla_N a\|_{L^2(S)}$. By (4.29) and (2.20), we have:

$$(\nabla_N - a^{-1}\Delta\!\!\!/)\nabla_N a = h, \tag{A.1}$$

where h is defined by:

$$\begin{aligned} h = & -a^{-1}\mathrm{tr}\,\theta\Delta\!\!\!/ a - 2a^{-1}\widehat{\theta}\nabla\!\!\!/^2 a + 2a^{-2}\nabla\!\!\!/ a\nabla\!\!\!/\nabla_N a - 2R_{N.}a^{-1}\nabla\!\!\!/ a \,-\, \nabla\!\!\!/\mathrm{tr}\,\theta a^{-1}\nabla\!\!\!/ a \\ & + 2\widehat{\theta}|a^{-1}\nabla\!\!\!/ a\,|^2 + 2\theta\nabla_N\theta + \nabla_N^2 k_{NN} + \nabla_N R_{NN}. \end{aligned} \tag{A.2}$$

We estimate $\|h\|_{L^2(S)}$:

$$\begin{aligned} \|h\|_{L^2(S)} \lesssim\, & \|\theta\|_{L^\infty(S)}\|\nabla\!\!\!/^2 a\|_{L^2(S)} + \|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/\nabla_N a\|_{L^2_{[-2,2]}L^4(P_u)} \\ & + \|R\|_{L^\infty(S)}\|\nabla\!\!\!/ a\|_{L^2(S)} + \|\nabla\!\!\!/\mathrm{tr}\,\theta\|_{L^2_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)} \\ & + \|\theta\|_{L^6(S)}\|\nabla\!\!\!/ a\|^2_{L^6(S)} + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N\theta\|_{L^2_{[-2,2]}L^4(P_u)} \\ & + \|\nabla_N^2 k_{NN}\|_{L^2(S)} + \|\nabla_N R_{NN}\|_{L^2(S)}, \end{aligned} \tag{A.3}$$

which together with (4.9), (4.11), (4.12), (4.46) and (A.3) yields:

$$\begin{aligned} \|h\|_{L^2(S)} \lesssim\, & \varepsilon(\|\theta\|_{L^\infty(S)} + \|\nabla\!\!\!/\nabla_N a\|_{L^2_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/\mathrm{tr}\,\theta\|_{L^2_{[-2,2]}L^4(P_u)} \\ & + \|\nabla_N\theta\|_{L^2_{[-2,2]}L^4(P_u)} + \varepsilon^2) + M. \end{aligned} \tag{A.4}$$

Together with (3.9), Proposition 3.10, (4.9) and (4.10), this implies:

$$\|h\|_{L^2(S)} \lesssim \varepsilon(\|\nabla^2\theta\|_{L^2(S)} + \|\nabla\!\!\!/^2\nabla_N a\|_{L^2(S)} + \varepsilon) + M. \tag{A.5}$$

Proposition 3.16, (4.8), (4.9), (4.11), (4.12), (A.1) and (A.5) yield:

$$\|\nabla\!\!\!/\nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N^2 a\|_{L^2(S)} + \|\nabla\!\!\!/^2\nabla_N a\|_{L^2(S)} \lesssim \varepsilon\|\nabla^2\theta\|_{L^2(S)} + M. \tag{A.6}$$

Let us now estimate $\|\nabla\!\!\!/^3 a\|_{L^2(S)}$. We differentiate the second equation of (4.5) with respect to $\nabla\!\!\!/$ and we obtain, in view of the commutator Formula (2.24):

$$a^{-1}\Delta\!\!\!/\nabla\!\!\!/ a = h + \nabla\!\!\!/\nabla_N a, \tag{A.7}$$

where h is defined by:

$$h = -a^{-2}\nabla\!\!\!/ a\Delta\!\!\!/ a + Ka^{-1}\nabla\!\!\!/ a + 2\theta\nabla\!\!\!/\theta + \nabla\!\!\!/\nabla_N k_{NN} + \nabla\!\!\!/ R_{NN}. \tag{A.8}$$

(A.7) yields:

$$\|a^{-1}\Delta\!\!\!/\nabla\!\!\!/ a\|_{L^2(S)} \leq \|h\|_{L^2(S)} + \|\nabla\!\!\!/\nabla_N a\|_{L^2(S)}. \tag{A.9}$$

We estimate $\|h\|_{L^2(S)}$:

$$\begin{aligned}\|h\|_{L^2(S)} \lesssim\ & \|\nabla\!\!\!/ a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/^2 a\|_{L^2_{[-2,2]}L^4(P_u)} + \|K\|_{L^3(S)}\|\nabla\!\!\!/ a\|_{L^6(S)}\\ & + \|\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla\!\!\!/\theta\|_{L^2_{[-2,2]}L^4(P_u)} + \|\nabla\!\!\!/\nabla_N k_{NN}\|_{L^2(S)}\\ & + \|\nabla\!\!\!/ R_{NN}\|_{L^2(S)}.\end{aligned} \tag{A.10}$$

Together with (4.6), (4.11), (4.12) and (4.46), this yields:

$$\|h\|_{L^2(S)} \lesssim \varepsilon(\|\nabla\!\!\!/^2 a\|_{L^2_{[-2,2]}L^4(P_u)} + \varepsilon^2 + M) + \varepsilon\|\nabla\!\!\!/\theta\|_{L^2_{[-2,2]}L^4(P_u)} + M. \tag{A.11}$$

Together with (3.9), (4.9) and (4.10), this implies:

$$\|h\|_{L^2(S)} \lesssim \varepsilon(\|\nabla\!\!\!/^3 a\|_{L^2(S)} + \|\nabla\!\!\!/^2\theta\|_{L^2(S)} + M) + M. \tag{A.12}$$

Now, (4.6), (4.12) and (4.46) imply:

$$\|K\|_{L^3(S)} + \|K\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|\theta\|^2_{L^6(S)} + \|\theta\|^2_{L^\infty_{[-2,2]}L^4(P_u)} + M \lesssim M + \varepsilon^2. \tag{A.13}$$

(3.27) and (A.13) yield:

(A.14)

$$\begin{aligned}\|\nabla\!\!\!/^3 a\|_{L^2(S)} &\lesssim \|\Delta\!\!\!/\nabla\!\!\!/ a\|_{L^2(S)} + \|K\|^{1/2}_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla\!\!\!/^2 a\|_{L^2_{[-2,2]}L^4(P_u)} + \|K\|_{L^3(S)}\|\nabla\!\!\!/ a\|_{L^6(S)}\\ &\lesssim \|\Delta\!\!\!/\nabla\!\!\!/ a\|_{L^2(S)} + (M+\varepsilon^2)\|\nabla\!\!\!/^2 a\|_{L^2_{[-2,2]}L^4(P_u)} + (M+\varepsilon^2)\varepsilon.\end{aligned}$$

Together with (3.9) and (4.9), this implies:

$$\|\nabla\!\!\!/^3 a\|_{L^2(S)} \lesssim \|\Delta\!\!\!/\nabla\!\!\!/ a\|_{L^2(S)} + (M^2+\varepsilon^2)\varepsilon. \tag{A.15}$$

(4.9), (A.9), (A.12) and (A.15) yield:

$$\|\nabla\!\!\!/^3 a\|_{L^2(S)} \lesssim \varepsilon\|\nabla\!\!\!/^2\theta\|_{L^2(S)} + M. \tag{A.16}$$

Let us now estimate $\|\nabla\!\!\!/^2\theta\|_{L^2(S)}$. We differentiate the first equation of (4.1), which yields together with (4.9) and (4.46):

$$\|\nabla\!\!\!/^2 \mathrm{tr}\,\theta\|_{L^2(S)} \leq \|\nabla\!\!\!/^2 a\|_{L^2(S)} + \|\nabla\!\!\!/^2 k_{NN}\| \lesssim \varepsilon + M. \tag{A.17}$$

Let us now estimate $\|\nabla\!\!\!/^2\widehat{\theta}\|_{L^2(S)}$. We consider the Hodge operator $\mathcal{D}_2$ which takes any symmetric traceless 2-tensor F on P_u into the 1-form $\mathrm{div}\!\!\!/ F$. Let ${}^*\mathcal{D}_2$ its adjoint which takes 1-forms on P_u into the 2-covariant symmetric traceless tensor

$({}^*\mathscr{D}_2 F)_{AB} = \nabla\!\!\!/_B F_A + \nabla\!\!\!/_A F_B - (\mathrm{di}\!\!/\!\mathrm{v} F)\gamma_{AB}$. We have the following identity:

$$ {}^*\mathscr{D}_2\mathscr{D}_2 = -\frac{1}{2}\Delta\!\!\!/ + K. \tag{A.18}$$

Thus, applying ${}^*\mathscr{D}_2$ to the third equation of (4.5), we obtain:

$$\Delta\!\!\!/\widehat{\theta} = 2K\widehat{\theta} -^* \mathscr{D}_2(\nabla\!\!\!/\mathrm{tr}\,\theta) - 2^*\mathscr{D}_2(R_{N.}). \tag{A.19}$$

(A.19) together with (4.46) and (A.17) yields:

$$\|\Delta\!\!\!/\widehat{\theta}\|_{L^2(S)} \lesssim \|K\widehat{\theta}\|_{L^2(S)} + M + \varepsilon. \tag{A.20}$$

The analog of (3.27) for 2-tensors, (4.12), (A.13) and (A.20) yield:

(A.21)
$$\begin{aligned}\|\nabla\!\!\!/^2\widehat{\theta}\|_{L^2(S)} &\lesssim \|K\|^{1/2}_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla\!\!\!/\widehat{\theta}\|_{L^2_{[-2,2]}L^4(P_u)} + \|K\|_{L^3(S)}\|\widehat{\theta}\|_{L^6(S)} + M + \varepsilon\\ &\lesssim (M+\varepsilon^2)\|\nabla\!\!\!/\widehat{\theta}\|_{L^2_{[-2,2]}L^4(P_u)} + (M+\varepsilon^2)\varepsilon + M + \varepsilon.\end{aligned}$$

Together with (3.9) and (4.10), this implies:

$$\|\nabla\!\!\!/^2\widehat{\theta}\|_{L^2(S)} \lesssim M. \tag{A.22}$$

Finally, (A.17) and (A.22) yield:

$$\|\nabla\!\!\!/^2\theta\|_{L^2(S)} \lesssim M. \tag{A.23}$$

Let us now estimate $\|\nabla\nabla_N\theta\|_{L^2(S)}$. Differentiating the last equation of (4.5) by ∇, taking the norm in $L^2(S)$, using (2.17), and estimating the various quantities in the same fashion as previously, we obtain:

$$\|\nabla\nabla_N\theta\|_{L^2(S)} \lesssim \|\nabla\!\!\!/^2\nabla_N a\|_{L^2(S)} + \varepsilon\|\nabla^2\theta\|_{L^2(S)} + M. \tag{A.24}$$

Finally, (A.6), (A.16), (A.23) and (A.24) yield the proof of (4.47) for $j = 2$.

A.2. End of the proof of Proposition 4.2

In this appendix, we end the proof of Proposition 4.2 by arguing by iteration, i.e., by proving (4.47) for $j+1$ assuming (4.47) for j with $2 \leq j \leq 4$.

We state two lemmas which will be used in the course of the proof.

Lemma A.1. — *Let F a tensor on S and $l \in \mathbb{N}$. Assume that* (4.47) *holds with $j = 2$. Assume also that $\|\nabla\!\!\!/\nabla_N^2 a\|_{L^2(S)} \leq C(M)$. Then, we have the following inequality:*

$$\|\nabla^l F\|_{L^2(S)} \leq C(M)\left(\|\nabla_N^l F\|_{L^2(S)} + \|\nabla\!\!\!/^l F\|_{L^2(S)} + \sum_{m=0}^{l-1}\|\nabla^m F\|_{L^2(S)}\right). \tag{A.25}$$

Lemma A.2. — *Let f a scalar function on S. We have the following commutator formula:*

(A.26)
$$[\nabla_N^j, a^{-1}\not\Delta]f = j(2a^{-1}\not\nabla a\not\nabla + a^{-1}\not\Delta a + (j-1)|a^{-1}\not\nabla a|^2)\nabla_N^j f + \left(\prod_{l=1}^{p}\not\nabla^{t_l^1}\nabla_N^{t_l^2}a\right)\left(\prod_{m=1}^{q}\not\nabla^{v_m^1}\nabla_N^{v_m^2}\theta\right)\left(\prod_{n=1}^{r}\not\nabla^{w_n^1}\nabla_N^{w_n^2}R\right)\not\nabla^{s_1}\nabla_N^{s_2}f,$$

where $t_l^1, t_l^2, v_m^1, v_m^2, w_n^1$ and w_n^2 satisfy:

$$\begin{aligned} &t_1^1+\cdots+t_p^1+v_1^1+\cdots+v_q^1+w_1^1+\cdots+w_r^1+s_1=2,\\ &t_1^2+\cdots+t_p^2+v_1^2+\cdots+v_q^2+w_1^2+\cdots+w_r^2+s_2=j-q-r,\\ &t_l^2\le j-1, 0\le l\le p,\ s_2\le j-1.\end{aligned} \tag{A.27}$$

We postpone the proof of Lemma A.1 to Section A.3, and the proof of Lemma A.2 to Section A.4. We now continue the proof of Proposition 4.2. We differentiate the second equation of (4.5) by ∇_N^j:

$$(\nabla_N - a^{-1}\not\Delta)\nabla_N^j a = h, \tag{A.28}$$

where h is defined by:

$$h = [\nabla_N^j, a^{-1}\not\Delta]a + \nabla_N^j(|\theta|^2) + \nabla_N^{j+1}k_{NN} + \nabla_N^j R_{NN}. \tag{A.29}$$

We estimate $\|h\|_{L^2(S)}$. Using (4.46) and (A.29), we obtain:

$$\|h\|_{L^2(S)} \lesssim \|[\nabla_N^j, a^{-1}\not\Delta]a\|_{L^2(S)} + \|\nabla_N^j(|\theta|^2)\|_{L^2(S)} + M. \tag{A.30}$$

If $j=2$, we have:

$$\|\nabla_N^2(|\theta|^2)\|_{L^2(S)} \lesssim \|\theta\|_{L^\infty(S)}\|\nabla^2\theta\|_{L^2(S)} + \|\nabla\theta\|^2_{L^4(S)} \lesssim M^2, \tag{A.31}$$

where we have used Proposition 3.10 to bound the $L^\infty(S)$ norm. If $j\ge 3$, using (4.47) for j and Leibnitz formula yields:

$$\|\nabla_N^j(|\theta|^2)\|_{L^2(S)} \lesssim \sum_{0\le p\le j/2}\|\nabla^p\theta\|_{L^\infty(S)}\|\nabla^{j-p}\theta\|_{L^2(S)} \le C(M). \tag{A.32}$$

We now estimate $\|[\nabla_N^j, a^{-1}\not\Delta]a\|_{L^2(S)}$ with the help of (A.26). We have:

$$\begin{aligned}\|a^{-1}\not\nabla a\not\nabla\nabla_N^j a\|_{L^2(S)} &\lesssim \|a^{-1}\not\nabla a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|a^{-1}\not\nabla a\not\nabla\nabla_N^j a\|_{L^2_{[-2,2]}L^4(P_u)}\\ &\lesssim \varepsilon\|\not\nabla^2\nabla_N^j a\|^{1/2}_{L^2(S)}\|\not\nabla\nabla_N^j a\|^{1/2}_{L^2(S)},\end{aligned} \tag{A.33}$$

where we have used (3.9) and (4.11). Using the estimate (4.9) for a, and the Gagliardo-Nirenberg inequality (3.9), we have:

$$\begin{aligned}\|a^{-1}\not\Delta a\nabla_N^j a\|_{L^2(S)} &\lesssim \|a^{-1}\|_{L^\infty(S)}\|\not\Delta a\|_{L^\infty_{[-2,2]}L^4(P_u)}\|\nabla_N^j a\|_{L^2_{[-2,2]}L^4(P_u)}\\ &\lesssim \|\not\Delta a\|^{\frac12}_{L^2(S)}\|\not\nabla\not\Delta a\|^{\frac12}_{L^2(S)}\|\nabla_N^j a\|^{\frac12}_{L^2(S)}\|\not\nabla\nabla_N^j a\|^{\frac12}_{L^2(S)}\\ &\lesssim C(M)\|\not\nabla\nabla_N^j a\|^{\frac12}_{L^2(S)},\end{aligned} \tag{A.34}$$

where we used in the last inequality (4.47) for j and for 2. Using (4.47) for j and for 2 yields:

$$\| |a^{-1}\nabla\!\!\!/ a|^2\nabla_N^j a\|_{L^2(S)} \lesssim \|a^{-1}\nabla\!\!\!/ a\|^2_{L^\infty(S)}\|\nabla_N^j a\|_{L^2(S)} \leq C(M), \tag{A.35}$$

where we have used Proposition 3.10 to bound $\|a^{-1}\nabla\!\!\!/ a\|_{L^\infty(S)}$. Using (4.47) for j and for 2 together with (A.27) yields:

(A.36)

$$\left\|\left(\prod_{l=1}^p \nabla\!\!\!/^{t_l^1}\nabla_N^{t_l^2}a\right)\left(\prod_{m=1}^q \nabla\!\!\!/^{v_l^1}\nabla_N^{v_l^2}\theta\right)\left(\prod_{n=1}^r \nabla\!\!\!/^{w_l^1}\nabla_N^{w_l^2}R\right)\nabla\!\!\!/^{s_1}\nabla_N^{s_2}a\right\|_{L^2(S)} \leq C(M).$$

(A.26), (A.33), (A.34), (A.35) and (A.36) yield:

(A.37)

$$\|[\nabla_N^j, a^{-1}\Delta\!\!\!/]a\|_{L^2(S)} \lesssim C(M)(1+\|\nabla\!\!\!/\nabla_N^j a\|^{\frac{1}{2}}_{L^2(S)}) + \varepsilon(\|\nabla\!\!\!/^2\nabla_N^j a\|_{L^2(S)} + \|\nabla\!\!\!/\nabla_N^j a\|_{L^2(S)}).$$

Finally, (A.30), (A.31), (A.32) and (A.37) yield:

(A.38)

$$\|h\|_{L^2(S)} \lesssim C(M)(1+C(M)\|\nabla\!\!\!/\nabla_N^j a\|^{\frac{1}{2}}_{L^2(S)}) + \varepsilon(\|\nabla\!\!\!/^2\nabla_N^j a\|_{L^2(S)} + \|\nabla\!\!\!/\nabla_N^j a\|_{L^2(S)}).$$

Proposition 3.16, (A.28) and (A.38) yield:

$$\|\nabla\!\!\!/\nabla_N^j a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/^2\nabla_N^j a\|_{L^2(S)} + \|\nabla_N^{j+1}a\|_{L^2(S)} \leq C(M). \tag{A.39}$$

Now, (4.47) for j, (A.25) and (A.39) yield:

$$\|\nabla^{j+1}a\|_{L^2(S)} \leq C(M). \tag{A.40}$$

Let us now estimate $\|\nabla\!\!\!/^{j+1}\theta\|_{L^2(S)}$. We differentiate the first equation of (4.5) by $\nabla\!\!\!/^{j+1}$, which yields together with (4.46) and (4.47) for j:

$$\|\nabla\!\!\!/^{j+1}\mathrm{tr}\,\theta\|_{L^2(S)} \leq \|\nabla\!\!\!/^{j+1}a\|_{L^2(S)} + \|\nabla\!\!\!/^{j+1}k_{NN}\|_{L^2(S)} \leq C(M). \tag{A.41}$$

Differentiating (A.19) by $\nabla\!\!\!/^{j-1}$, we obtain:

$$\nabla\!\!\!/^{j-1}\Delta\!\!\!/\widehat{\theta} = 2\nabla\!\!\!/^{j-1}(K\widehat{\theta}) - \nabla\!\!\!/^{j-1}({}^*\mathscr{D}_2(\nabla\!\!\!/\mathrm{tr}\,\theta)) - 2\nabla\!\!\!/^{j-1}({}^*\mathscr{D}_2(R_{N\cdot})). \tag{A.42}$$

(4.46) and (A.41) yield:

$$\|\nabla\!\!\!/^{j-1}({}^*\mathscr{D}_2(\nabla\!\!\!/\mathrm{tr}\,\theta))\|_{L^2(S)} + \|\nabla\!\!\!/^{j-1}({}^*\mathscr{D}_2(R_{N\cdot}))\|_{L^2(S)} \leq C(M). \tag{A.43}$$

Using Leibnitz formula together with (4.6), (4.46) and (4.47) for j, we obtain:

$$\|\nabla\!\!\!/^{j-1}(K\widehat{\theta})\|_{L^2(S)} \leq C(M). \tag{A.44}$$

(A.42), (A.43) and (A.44) yield:

$$\|\nabla\!\!\!/^{j-1}\Delta\!\!\!/\widehat{\theta}\|_{L^2(S)} \leq C(M). \tag{A.45}$$

Now, (2.24) yields:

$$\Delta\!\!\!/\nabla\!\!\!/^{j-1}\widehat{\theta} = \nabla\!\!\!/^{j-1}\Delta\!\!\!/\widehat{\theta} + \sum_{p=1}^{j-1}\nabla\!\!\!/^{j-1-p}K\nabla\!\!\!/^p\widehat{\theta}, \tag{A.46}$$

which together with (4.6), (4.46), (4.47) and (A.45) implies:

$$\|\not\Delta\not\nabla^{j-1}\widehat{\theta}\|_{L^2(S)} \leq C(M). \tag{A.47}$$

The analog of (3.27) for 2-tensors, (A.13) and (A.47) yield:

$$\begin{aligned}
&\|\not\nabla^{j+1}\widehat{\theta}\|_{L^2(S)} \\
&\lesssim \|\not\Delta\not\nabla^{j-1}\widehat{\theta}\|_{L^2(S)} + \|K\|^{1/2}_{L^\infty_{[-2,2]}L^2(P_u)}\|\not\nabla^j\widehat{\theta}\|_{L^2_{[-2,2]}L^4(P_u)} + \|K\|_{L^3(S)}\|\not\nabla^{j-1}\widehat{\theta}\|_{L^6(S)} \\
&\lesssim (M+\varepsilon^2)(\|\not\nabla^j\widehat{\theta}\|_{L^2_{[-2,2]}L^4(P_u)} + \|\not\nabla^{j-1}\widehat{\theta}\|_{L^6(S)}) + C(M).
\end{aligned} \tag{A.48}$$

Together with (3.9) and (4.47) for j, this implies:

$$\|\not\nabla^{j+1}\widehat{\theta}\|_{L^2(S)} \leq C(M). \tag{A.49}$$

Finally, (A.41) and (A.49) yield:

$$\|\not\nabla^{j+1}\theta\|_{L^2(S)} \leq C(M). \tag{A.50}$$

Let us now estimate $\|\nabla_N^{j+1}\theta\|_{L^2(S)}$. Differentiating the last equation of (4.5) by ∇_N^j, taking the norm in $L^2(S)$, using the computation (A.64) of $[\nabla_N^j, \not\nabla^2]$ proved in the appendix, (4.47) for j, (A.39), and estimating the various quantity in the same fashion as previously, we obtain:

$$\|\nabla_N^{j+1}\theta\|_{L^2(S)} \leq C(M). \tag{A.51}$$

Now, (4.47) for j, (A.25), (A.50) and (A.51) yield:

$$\|\nabla^{j+1}\theta\|_{L^2(S)} \leq C(M). \tag{A.52}$$

Also, differentiating the last equation of (4.5) by ∇^j, taking the norm in $L^2(S)$, (4.47) for j, (A.52), and estimating the various quantity in the same fashion as previously, we obtain:

$$\|\not\nabla^2\nabla^{j-1}a\|_{L^2(S)} \leq C(M). \tag{A.53}$$

Finally, (A.40), (A.52) and (A.53) yield (4.47) for $j+1$ so that (4.47) is true for all $1 \leq j \leq 5$. This concludes the proof of Proposition 4.2.

A.3. Proof of Lemma A.1

Let us first recall the following result (see for instance [**5**]). If the symbol $a(x,\xi)$ satisfies:

$$\sup_\xi \|a(.,\xi)\|_{H^{3/2+\delta}(\mathbb{R}^3)} < +\infty \tag{A.54}$$

for some $\delta > 0$, then the pseudodifferential operator $a(x,D)$ acting on $\mathbb{R}^3$ is bounded on $L^2(\mathbb{R}^3)$. Now, assume that the symbol $a(x,\xi)$ satisfies:

$$\sup_\xi \|a(.,\xi)\|_{H^{5/2+\delta}(\mathbb{R}^3)} < +\infty \tag{A.55}$$

for some $\delta > 0$ and:

$$a(x, \xi) \geq 1 \text{ for all } (x, \xi). \tag{A.56}$$

Then, using the previous result and the symbolic calculus for the adjoint and the composition of pseudodifferential operators, one can show that:

$$a(x, D) - \sqrt{a}(x, D)^* \sqrt{a}(x, D) \text{ is bounded from } H^{-1}(\mathbb{R}^3) \text{ to } L^2(\mathbb{R}^3). \tag{A.57}$$

Thus, under the assumptions (A.55) (A.56), the Garding inequality holds:

$$(a(x, D)v, v) \geq -C\|v\|_{H^{-1}(\mathbb{R}^3)}, \tag{A.58}$$

where v is in $L^2(\mathbb{R}^3)$ and $C > 0$ is a constant depending in the quantity in (A.55).

Now, consider

$$a(x, \xi) = 2^{l+1}\left(\left(N.\frac{\xi}{|\xi|}\right)^{2l} + \left(e.\frac{\xi}{|\xi|}\right)^{2l}\right) - 1. \tag{A.59}$$

Then, we clearly have (A.56). We also have (A.55):

$$\sup_{\xi}\|a(., \xi)\|_{H^{5/2+\delta}} \leq C(\|N\|_{H^{5/2+\delta}}) \leq C(\|\nabla^3 N\|_{L^2(S)}) \leq C(M), \tag{A.60}$$

where we have used (4.4), (4.47) with $j = 2$ and $\|\nabla\!\!\!/\nabla_N^2 a\|_{L^2(S)} \leq C(M)$. Thus, a defined by (A.59) satisfies (A.58), which together with the choice $v = |D|^l F$ concludes the proof of Lemma A.1.

A.4. Proof of Lemma A.2

We start by deriving a formula for the commutator $[\nabla_N^j, \nabla\!\!\!/]$. Let F a tensor on S. Using (2.17), one proves the following commutator formula by iteration:

$$\begin{aligned}[\nabla_N^j, \nabla\!\!\!/]F = {} & j\nabla\!\!\!/ a\nabla_N^j F \\ & + \nabla\!\!\!/\nabla_N^{t_1} a\left(\prod_{l=2}^{p}\nabla_N^{t_l} a\right)\left(\prod_{m=1}^{q}\nabla_N^{v_m}\theta\right)\left(\prod_{n=1}^{r}\nabla_N^{w_n} R\right)\nabla_N^s F \\ & + \left(\prod_{l=1}^{p}\nabla_N^{t_l} a\right)\left(\prod_{m=1}^{q}\nabla_N^{v_m}\theta\right)\left(\prod_{n=1}^{r}\nabla_N^{w_n} R\right)\nabla\!\!\!/\nabla_N^s F,\end{aligned} \tag{A.61}$$

where t_l, v_m and w_n satisfy:

$$\begin{aligned}t_1 + \cdots + t_p + v_1 + \cdots + v_q + w_1 + \cdots + w_r + s = j - q - r, \\ t_l \leq j - 1, 1 \leq l \leq p, s \leq j - 1.\end{aligned} \tag{A.62}$$

Then, using the fact that:

$$[\nabla_N^j, \nabla\!\!\!/^2] = [\nabla_N^j, \nabla\!\!\!/]\nabla\!\!\!/ + \nabla\!\!\!/[\nabla_N^j, \nabla\!\!\!/], \tag{A.63}$$

we deduce from (A.61) and (A.62) the following commutator formula:

$$[\nabla_N^j, \nabla\!\!\!/^2]F = \left(\prod_{l=1}^{p}\nabla\!\!\!/^{t_l^1}\nabla_N^{t_l^2} a\right)\left(\prod_{m=1}^{q}\nabla\!\!\!/^{v_m^1}\nabla_N^{v_m^2}\theta\right)\left(\prod_{n=1}^{r}\nabla\!\!\!/^{w_n^1}\nabla_N^{w_n^2} R\right)\nabla\!\!\!/^{s_1}\nabla_N^{s_2} F, \tag{A.64}$$

where $t_l^1, t_l^2, v_m^1, v_m^2, w_n^1$ and w_n^2 satisfy:

$$
\begin{aligned}
&t_1^1+\cdots+t_p^1+v_1^1+\cdots+v_q^1+w_1^1+\cdots+w_r^1+s_1=2,\\
&t_1^2+\cdots+t_p^2+v_1^2+\cdots+v_q^2+w_1^2+\cdots+w_r^2+s_2=j-q-r,\\
&\qquad t_l^2\leq j-1, 0\leq l\leq p,\ s_1+s_2\leq j+1.
\end{aligned}
\tag{A.65}
$$

Now, using (2.20), we have for any scalar f on S:

$$
\begin{aligned}
[\nabla_N^j, a^{-1}\slashed{\Delta}]f &= \sum_{l=1}^{j}\nabla_N^{l-1}[\nabla_N, a^{-1}\slashed{\Delta}]\nabla_N^{j-l}f\\
&= \sum_{l=1}^{j}\nabla_N^{l-1}(-(\mathrm{tr}\,\theta+a^{-1}\nabla_N a)\slashed{\Delta}-2\widehat{\theta}\cdot\slashed{\nabla}^2+2a^{-1}\slashed{\nabla}a\cdot\slashed{\nabla}\nabla_N\\
&\quad +a^{-1}\slashed{\Delta}\nabla_N-2R_{N.}\cdot\slashed{\nabla}-\slashed{\nabla}\mathrm{tr}\,\theta\cdot\slashed{\nabla}+2\widehat{\theta}\cdot a^{-1}\slashed{\nabla}a\cdot\slashed{\nabla})\nabla_N^{j-l}f\\
&= 2\sum_{l=1}^{j}\nabla_N^{l-1}(a^{-1}\slashed{\nabla}a\slashed{\nabla}\nabla_N^{j+1-l}f+a^{-1}\slashed{\Delta}a\nabla_N^{j+1-l}f)\\
&\quad +\sum_{l=1}^{j}\nabla_N^{l-1}(-(\mathrm{tr}\,\theta+a^{-1}\nabla_N a)\slashed{\Delta}-2\widehat{\theta}\cdot\slashed{\nabla}^2-2R_{N.}\cdot\slashed{\nabla}\\
&\quad -\slashed{\nabla}\mathrm{tr}\,\theta\cdot\slashed{\nabla}+2\widehat{\theta}\cdot a^{-1}\slashed{\nabla}a\cdot\slashed{\nabla})\nabla_N^{j-l}f.
\end{aligned}
\tag{A.66}
$$

We rewrite the first term in the right-hand side of (A.66):

$$
\begin{aligned}
&\sum_{l=1}^{j}\nabla_N^{l-1}(a^{-1}\slashed{\nabla}a\slashed{\nabla}\nabla_N^{j+1-l}f+a^{-1}\slashed{\Delta}a\nabla_N^{j+1-l}f)\\
&\quad = ja^{-1}\slashed{\nabla}a\slashed{\nabla}\nabla_N^j f+ja^{-1}\slashed{\Delta}a\nabla_N^j f\\
&\qquad +\sum_{l=1}^{j}a^{-1}\slashed{\nabla}a[\nabla_N^{l-1},\slashed{\nabla}]\nabla_N^{j+1-l}f+\sum_{l=1}^{j}\sum_{m=1}^{l-1}\nabla_N^m(a^{-1}\slashed{\nabla}a)\nabla_N^{l-1-m}\slashed{\nabla}\nabla_N^{j+1-l}f\\
&\qquad +\sum_{l=1}^{j}\sum_{m=1}^{l-1}\nabla_N^m(a^{-1}\slashed{\Delta}a)\nabla_N^{j-m}f\\
&\quad = ja^{-1}\slashed{\nabla}a\slashed{\nabla}\nabla_N^j f+ja^{-1}\slashed{\Delta}a\nabla_N^j f+\frac{j(j-1)}{2}|\slashed{\nabla}a|^2\nabla_N^j f\\
&\qquad +\sum_{l=1}^{j}a^{-1}\slashed{\nabla}a([\nabla_N^{l-1},\slashed{\nabla}]-(l-1)\slashed{\nabla}a\nabla_N^{l-1})\nabla_N^{j+1-l}f\\
&\qquad +\sum_{l=1}^{j}\sum_{m=1}^{l-1}\nabla_N^m(a^{-1}\slashed{\nabla}a)\nabla_N^{l-1-m}\slashed{\nabla}\nabla_N^{j+1-l}f+\sum_{l=1}^{j}\sum_{m=1}^{l-1}\nabla_N^m(a^{-1}\slashed{\Delta}a)\nabla_N^{j-m}f.
\end{aligned}
\tag{A.67}
$$

Finally, (A.61), (A.62), (A.64), (A.65), (A.66) and (A.67) yield (A.26) and (A.27).

A.5. Proof of Theorem 4.4

Recall that for $-2 \leq \alpha \leq 2$, we assume that the u-foliation satisfying (2.28) exists on $-2 < u < \alpha$. Recall also that proving Theorem 4.4 consists in extending u to a solution on the strip $\alpha \leq u < \alpha + T$ provided $T > 0$ is chosen small enough.

We start by constructing an auxiliary foliation initialized on $P_{u=\alpha}$.

Lemma A.3. — *Assume that R and k satisfy* (4.48). *Also, assume that u is a solution to* (2.28) *for $u \leq \alpha$ satisfying* (4.49) *and* (4.50). *Let the scalar function $\underline{u}$ satisfying the Eikonal equation and initialized on $P_{u=\alpha}$ by the u-foliation, i.e.,*

$$g(\nabla \underline{u}, \nabla \underline{u}) = 1, \qquad \underline{u}|_{P_{u=\alpha}} = \alpha. \tag{A.68}$$

Then, the foliation by $\underline{u}$ is defined on the region $\underline{S}_{\underline{\delta}}$ of Σ given by

$$\underline{S}_{\underline{\delta}} = \{\alpha \leq \underline{u} < \alpha + \underline{\delta}\}$$

for some $\underline{\delta} > 0$ small enough only depending on the norms appearing in (4.48) (4.50). *Furthermore, denoting by $\underline{P}_{\underline{u}}$, $\underline{a}$ and $\underline{\theta}$ respectively the corresponding leaves, lapse and second fundamental form of the $\underline{u}$-foliation, we have*

$$\underline{a} = 1, \tag{A.69}$$

$$|\mathrm{tr}\, \underline{\theta} - k(\underline{N}, \underline{N})| \leq \frac{3}{8} \tag{A.70}$$

and

$$\|\underline{\theta}\|_{L^\infty_{\underline{u}} L^4(\underline{P}_{\underline{u}})} + \max_{1 \leq j \leq 4} \|\nabla^j \underline{\theta}\|_{L^\infty_{\underline{u}} L^2(\underline{P}_{\underline{u}})} < +\infty. \tag{A.71}$$

We postpone the proof of Lemma A.3 to Section A.6.

Definition A.4. — *The map $\Phi_{\underline{u}}(p)$ from $P_{u=\alpha}$ to $\underline{S}_{\underline{\delta}}$ is given by*

$$\Phi_\alpha(p) = p, \qquad \underline{N}(\Phi_{\underline{u}}(p)) = 0.$$

Similarly to the proof of local existence for the mean curvature flow in **[9]**, we will write the solutions to (2.28) as graphs over $P_{u=\alpha}$. More precisely, consider a scalar function f on a neighborhood of $u = \alpha$ in $[\alpha, +\infty) \times P_{u=\alpha}$. Then, where we look u as

$$P_u = \{\Phi_{\underline{u}}(p) \in \underline{S}_{\underline{\delta}} \,/\, \underline{u} = f(u, p)\}, \qquad f(\alpha, p) = \alpha, \tag{A.72}$$

where we note that any point $q \in \underline{S}_{\underline{\delta}}$ is of the form $\Phi_{\underline{u}}(p)$ for a unique $\underline{u} \in [\alpha, \alpha + \underline{\delta})$ and a unique $p \in P_{u=\alpha}$.

We will rely on the following three lemmas.

Lemma A.5. — *Let a coordinates system (x^1, x^2) on a chart of P_α, and extend (x^1, x^2) to $\underline{S}_{\underline{\delta}}$ by $\underline{N}(x^A) = 0$ for $A = 1, 2$ so that $(\underline{u}, x^1, x^2)$ forms a coordinates system on a chart of $\underline{S}_{\underline{\delta}}$. Also, introduce a coordinates system (u, y^1, y^2) on $\bigcup_u P_u$ with*

$$y^1 = x^1, \quad y^2 = x^2.$$

Then, we have

$$\begin{aligned}\gamma_{AB} &= \underline{\gamma}_{AB} + \partial_{x^A} f \partial_{x^B} f, \quad A, B = 1, 2,\\ g_{Au} &= \partial_u f \partial_{x^A} f, \quad A = 1, 2,\\ g_{uu} &= (\partial_u f)^2,\end{aligned}$$

where γ *and* $\underline{\gamma}$ *denote respectively the induced metrics of* P_u *and* $\underline{P}_{\underline{u}}$*, and where*

$$\gamma_{AB} := \gamma(\partial_{y^A}, \partial_{y^B}), \quad \underline{\gamma}_{AB} := g(\partial_{x^A}, \partial_{x^B}), \quad g_{Au} := g(\partial_{y^A}, \partial_u), \quad g_{uu} := g(\partial_u, \partial_u).$$

Also, the unit normal N *to* P_u *is given by* $N^u\partial_u + N^A\partial_{y^A}$ *with*

$$\begin{aligned}N^A &= -\frac{\gamma^{AB}\partial_{x^B} f}{\sqrt{1-\gamma^{AB}\partial_{x^A} f \partial_{x^B} f}}, \quad A = 1, 2,\\ N^u &= \frac{1}{\partial_u f\sqrt{1-\gamma^{AB}\partial_{x^A} f \partial_{x^B} f}},\end{aligned}$$

provided

$$\partial_u f > 0, \qquad \gamma^{AB}\partial_{x^A} f \partial_{x^B} f < 1.$$

Next, we compute the mean curvature $\operatorname{tr}\theta$ and the lapse a of the u-foliation.

Lemma A.6. — *We have*

$$\begin{aligned}\operatorname{tr}\theta = {}& \frac{1}{2}\gamma^{AB}\left\{\frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A} f \partial_{x^B} f\right)}{\left(1-\gamma^{AB}\partial_{x^A} f \partial_{x^B} f\right)^{\frac{3}{2}}}\partial_{x^B} f + \frac{\partial_{\underline{u}}(\underline{\gamma}_{AB})}{\sqrt{1-\gamma^{AB}\partial_{x^A} f \partial_{x^B} f}}\right\}\\ & + \gamma^{AB}\gamma_{CB}\partial_{y^A}(N^C) + \gamma^{AB}N^C g(D_{\partial_{y^A}}\partial_{y^C}, \partial_{y^B})\end{aligned}$$

and

$$a = \sqrt{1 - \frac{\gamma_{11}(\partial_{x^2} f)^2 + \gamma_{22}(\partial_{x^1} f)^2 - 2\gamma_{21}\partial_{x^1} f \partial_{x^2} f}{\det(\gamma)}}\,\partial_u f,$$

provided

$$\partial_u f > 0, \qquad \gamma^{AB}\partial_{x^A} f \partial_{x^B} f < 1$$

and

$$\frac{\gamma_{11}(\partial_{x^2} f)^2 + \gamma_{22}(\partial_{x^1} f)^2 - 2\gamma_{21}\partial_{x^1} f \partial_{x^2} f}{\det(\gamma)} < 1.$$

Next, we rewrite Equation (2.28).

Lemma A.7. — *Decompose* f *as*

$$f(u, x^1, x^2) = u + \widetilde{f}(x^1, x^2, u).$$

Then, the Equation (2.28) *takes the following form*

$$\partial_u \widetilde{f} - \Delta\!\!\!\!/_{\underline{\gamma}(u)}\widetilde{f} - F_1^{AB}\left(\nabla\!\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\partial_{x^A}\partial_{x^B}\widetilde{f}$$

$$= k(\underline{N}, N) - \operatorname{tr}\underline{\theta} + F_2^{ij}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)k_{ij} + F_3^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\underline{\theta}_{AB} + F_4\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right),$$

where F_1^{AB}, F_2^{ij}, F_3^{AB} and F_4 are smooth in a neighborhood of the origin such that

$$F_1^{AB}(0) = 0, \qquad F_2^{ij}(0) = 0, \qquad F_3^{AB}(0) = 0, \qquad F_4(0,0) = 0$$

and where the metric $\underline{\gamma}(u)$ on $P_{u=\alpha}$ is given by

$$[\underline{\gamma}(u)]_{AB}(x^1, x^2) = [g(\partial_{x^A}, \partial_{x^B})](\underline{u} = f(u, x^1, x^2), x^1, x^2).$$

We postpone the proof of Lemmas A.5, A.6 and A.7 respectively to Sections A.7, A.8 and A.9.

We are now ready to prove Theorem 4.4:

Proof. — Recall from Lemma A.7 that (2.28) takes the following form

$$\begin{aligned}&\partial_u\widetilde{f} - \Delta\!\!\!/_{\underline{\gamma}(u)}\widetilde{f} - F_1^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\partial_{x^A}\partial_{x^B}\widetilde{f}\\ &= k(\underline{N}, N) - \operatorname{tr}\underline{\theta} + F_2^{ij}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)k_{ij} + F_3^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\underline{\theta}_{AB} + F_4\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right),\end{aligned}$$

where F_1^{AB}, F_2^{ij} and F_3 are smooth functions in a neighborhood of the origin such that

$$F_1^{AB}(0) = 0, \qquad F_2^{ij}(0) = 0, \qquad F_3^{AB}(0) = 0, \qquad F_4(0,0) = 0$$

and where

$$f(u, x^1, x^2) = u + \widetilde{f}(x^1, x^2, u).$$

We consider the above quasilinear scalar parabolic equation for $\widetilde{f}$ on $\alpha \leq u \leq \alpha+T$ with $T \leq \underline{\delta}$ so that we may rely on the control of the $\underline{u}$ foliation of Lemma A.3. Note that

$$\widetilde{f}(\alpha, \cdot) = f(\alpha, \cdot) - \alpha = 0,$$

i.e., $\widetilde{f}$ has trivial initial data at $u = \alpha$. We look for $\widetilde{f}$ in the following set

$$X_{\alpha,\widetilde{\delta},T} = \left\{\widetilde{f}\,\middle|\, \max_{0\leq j\leq 4}\sup_{\alpha<u<\alpha+T}\|\nabla\!\!\!/^j_{\underline{\gamma}(u)}\widetilde{f}\|_{L^2(P_\alpha)} \leq \widetilde{\delta}\right\},$$

where $\widetilde{\delta}$ will be chosen small enough below. Local existence then follows for

$$T = T\left(\widetilde{\delta}, \max_{1\leq j\leq 4}\|\nabla^j k\|_{L^2(\Sigma)}, \max_{1\leq j\leq 4}\|\nabla^j\underline{\theta}\|_{L^2(\Sigma)}\right) > 0$$

small enough by Banach fixed point and standard parabolic estimates, where we used in particular the control of k provided by (4.48) and the one for $\underline{\theta}$ provided by Lemma A.3.

Next, we need to check that, with such $\widetilde{f}$, f satisfies the conditions

$$\partial_u f > 0, \qquad \gamma^{AB}\partial_{x^A}f\partial_{x^B}f < 1$$

and

$$\frac{\gamma_{11}(\partial_{x^2}f)^2+\gamma_{22}(\partial_{x^1}f)^2-2\gamma_{21}\partial_{x^1}f\partial_{x^2}f}{\det(\gamma)}<1$$

used in the derivation of the parabolic equation for $\widetilde{f}$. First, note from the above Banach fixed point procedure that we have

$$|\partial_{x^A}f|=|\partial_{x^A}\widetilde{f}|\lesssim\max_{0\leq j\leq 3}\sup_{\alpha<u<\alpha+T}\|\nabla\!\!\!/^j_{\underline{\gamma}(u)}\widetilde{f}\|_{L^2(P_\alpha)}\lesssim\widetilde{\delta},\quad A=1,2,$$

which implies the last two conditions for f provided $\widetilde{\delta}$ is chosen small enough. Concerning the condition $\partial_u f>0$, notice from the parabolic equation for $\widetilde{f}$ that

$$\begin{aligned}|\partial_u\widetilde{f}|&\leq|k(\underline{N},\underline{N})-\operatorname{tr}\underline{\theta}|+\left|\Delta\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}+F_1^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\partial_{x^A}\partial_{x^B}\widetilde{f}\right|\\&\quad+\left|F_2^{ij}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)k_{ij}+F_3^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\underline{\theta}_{AB}+F_4\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\right|\\&\leq|k(\underline{N},\underline{N})-\operatorname{tr}\underline{\theta}|+O(\widetilde{\delta})\left(\max_{1\leq j\leq 2}\|\nabla^j k\|_{L^2(\Sigma)}+\max_{1\leq j\leq 2}\|\nabla^j\underline{\theta}\|_{L^2(\Sigma)}\right).\end{aligned}$$

In particular, choosing $\widetilde{\delta}$ small enough, we have

$$|\partial_u\widetilde{f}|\leq|k(\underline{N},\underline{N})-\operatorname{tr}\underline{\theta}|+\frac{1}{8}.$$

In view of the control for $k(\underline{N},\underline{N})-\operatorname{tr}\underline{\theta}$ provided by Lemma A.3, we deduce

$$|\partial_u\widetilde{f}|\leq\frac{1}{2}$$

and hence

$$|\partial_u f|=|1+\partial_u\widetilde{f}|\geq 1-|\partial_u\widetilde{f}|\geq\frac{1}{2},$$

so that we indeed have $\partial_u f>0$.

Finally, having established the existence of f, and checked that the necessary conditions for the derivation of the parabolic equation for $\widetilde{f}$ are satisfied by f, it remains to recover u from the equation

$$f(x^1,x^2,u(x^1,x^2,\underline{u}))=\underline{u}.$$

Since $\partial_u f>0$, the existence of u follows immediately from applying the implicit function theorem to the above equation in the neighborhood of $(x^1,x^2,u=\alpha)$ which satisfies $f(x^1,x^2,\alpha)=\alpha$ thanks to the initialization of f. This concludes the proof of Theorem 4.4. □

A.6. Proof of Lemma A.3

Since

$$g(\nabla\underline{u},\nabla\underline{u})=1,$$

we immediately have the identity $\underline{a}=1$.

Next, we focus of the estimates for the second fundamental form $\underline{\theta}$ of the $\underline{u}$-foliation. Since local existence for the Eikonal equation is classical, we only sketch the proof. In view of (2.12) and (2.13), and using the fact that $\underline{a} = 1$, we have

$$\nabla_{\underline{N}}\underline{\theta}_{AB} + 2\underline{\theta}_A^C\underline{\theta}_{CB} - \operatorname{tr}\underline{\theta}\,\underline{\theta}_{AB} + \frac{1}{2}\big((\operatorname{tr}\underline{\theta})^2 - |\underline{\theta}|^2 + R - 2R_{\underline{N}\,\underline{N}}\big)\underline{\gamma}_{AB} = R_{AB}.$$

One easily derives from this system of transport equations the following estimates

$$\begin{aligned}
&\|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 4}\|\underline{\nabla}^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})}\\
&\lesssim \|\theta\|_{L^4(P_{u=\alpha})} + \max_{1\leq j\leq 4}\|\underline{\nabla}^j\theta\|_{L^2(P_{u=\alpha})} + \sqrt{\underline{\delta}}\left(1 + \|\underline{\theta}\|_{L^\infty} + \max_{1\leq j\leq 2}\|\underline{\nabla}^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})}\right)\\
&\quad\times\left(\|\nabla^{\leq 2}R\|_{L^2(\Sigma)} + \|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 4}\|\underline{\nabla}^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})}\right),
\end{aligned}$$

where we used the fact that $\underline{\theta} = \theta$ on $\underline{P}_{\underline{u}=\alpha} = P_{u=\alpha}$, as well as the the commutation Formula (2.21) and the fact that $\underline{a} = 1$ to differentiate the equation for $\underline{\theta}_{AB}$ with $\underline{\nabla}$. We infer

$$\begin{aligned}
&\|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 4}\|\underline{\nabla}^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})}\\
&\lesssim \max_{1\leq j\leq 5}\|\nabla^j\theta\|_{L^2(S\cap\{u\leq\alpha\})} + \sqrt{\underline{\delta}}\left(1 + \|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 2}\|\underline{\nabla}^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})}\right)\\
&\quad\times\left(\|\nabla^{\leq 2}R\|_{L^2(\Sigma)} + \|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 4}\|\underline{\nabla}^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})}\right)
\end{aligned}$$

and hence

$$\|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 4}\|\underline{\nabla}^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})} < +\infty$$

for some $\underline{\delta} > 0$ small enough only depending on the norms appearing in (4.48) (4.50). Using the above system of transport equations for $\underline{\theta}$ to recover $\nabla_{\underline{N}}$ derivates, we deduce

$$\|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 4}\|\nabla^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})} < +\infty.$$

Finally, we consider the estimate for $\operatorname{tr}\underline{\theta} - k(\underline{N},\underline{N})$. We have

$$\begin{aligned}
&\|\operatorname{tr}\underline{\theta} - k(\underline{N},\underline{N})\|_{L^\infty(\underline{S}_{\underline{\delta}})}\\
&\quad\leq \|\operatorname{tr}\underline{\theta} - k(\underline{N},\underline{N})\|_{L^\infty(\underline{P}_{\underline{u}=\alpha})} + \|\nabla_{\underline{N}}\underline{\theta}\|_{L^1_{\underline{u}}L^\infty(\underline{P}_{\underline{u}})} + \|\nabla k\|_{L^1_{\underline{u}}L^\infty(\underline{P}_{\underline{u}})}\\
&\quad\leq \|\operatorname{tr}\underline{\theta} - k(\underline{N},\underline{N})\|_{L^\infty(\underline{P}_{\underline{u}=\alpha})} + \sqrt{\underline{\delta}}\Big(\|\nabla_{\underline{N}}\underline{\theta}\|_{L^2_{\underline{u}}L^\infty(\underline{P}_{\underline{u}})} + \|\nabla k\|_{L^2_{\underline{u}}L^\infty(\underline{P}_{\underline{u}})}\Big)\\
&\quad\leq \|\operatorname{tr}\underline{\theta} - k(\underline{N},\underline{N})\|_{L^\infty(\underline{P}_{\underline{u}=\alpha})}\\
&\qquad + C\sqrt{\underline{\delta}}\left(\|\underline{\theta}\|_{L^\infty_{\underline{u}}L^4(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 3}\|\nabla^j\underline{\theta}\|_{L^\infty_{\underline{u}}L^2(\underline{P}_{\underline{u}})} + \max_{1\leq j\leq 3}\|\nabla^j k\|_{L^2(\Sigma)}\right).
\end{aligned}$$

In view of the above, we infer, for $\underline{\delta} > 0$ small enough,

$$\|\operatorname{tr}\underline{\theta} - k(\underline{N},\underline{N})\|_{L^\infty(\underline{S}_{\underline{\delta}})} \leq \|\operatorname{tr}\underline{\theta} - k(\underline{N},\underline{N})\|_{L^\infty(\underline{P}_{\underline{u}=\alpha})} + \frac{1}{8}.$$

Since we have $\underline{N} = N$, $\operatorname{tr}\underline{\theta} = \operatorname{tr}\theta$ and $\underline{P}_{\underline{u}=\alpha} = P_{u=\alpha}$ on $\underline{u} = \alpha$, we infer

$$\|\operatorname{tr}\underline{\theta} - k(\underline{N}, \underline{N})\|_{L^\infty(\underline{S}_\delta)} \leq \|\operatorname{tr}\theta - k_{NN}\|_{L^\infty(P_{u=\alpha})} + \frac{1}{8}.$$

Also, since u is a solution to (2.28) for $u \leq \alpha$, this yields

$$\|\operatorname{tr}\underline{\theta} - k(\underline{N}, \underline{N})\|_{L^\infty(\underline{S}_\delta)} \leq \|a - 1\|_{L^\infty(P_{u=\alpha})} + \frac{1}{8},$$

which together with (4.49) implies

$$\|\operatorname{tr}\underline{\theta} - k(\underline{N}, \underline{N})\|_{L^\infty(\underline{S}_\delta)} \leq \frac{3}{8}$$

as stated. This concludes the proof of Lemma A.3.

A.7. Proof of Lemma A.5

To perform computations, we consider a coordinates system (x^1, x^2) on a chart of P_α. Then, we extend (x^1, x^2) to $\underline{S}_\delta$ by

$$\underline{N}(x^A) = 0, \qquad A = 1, 2,$$

so that $(\underline{u}, x^1, x^2)$ forms a coordinates system on a chart of $\underline{S}_\delta$, and in view of the definition of the map $\Phi_{\underline{u}}(p)$, the coordinates of $\Phi_{\underline{u}}(p)$ are given by $(\underline{u}, x^1, x^2)$ if (x^1, x^2) are the coordinates of p on P_α. In particular, we have in this coordinates system

$$\partial_{\underline{u}} = \underline{N}, \qquad g(\partial_{\underline{u}}, \partial_{x^A}) = 0, \qquad g(\partial_{\underline{u}}, \partial_{\underline{u}}) = 1.$$

Also, note that P_u is given in these local coordinates by

$$(\underline{u}, x^1, x^2), \qquad \underline{u} = f(u, x^1, x^2).$$

Next, we introduce a coordinates system (u, y^1, y^2) on $\bigcup_u P_u$ with

$$y^1 = x^1, \quad y^2 = x^2$$

and notice that

$$\partial_{y^A} = \partial_{x^A} + \partial_{x^A} f \partial_{\underline{u}}, \qquad \partial_u = \partial_u f \partial_{\underline{u}}.$$

In this coordinates system, we compute

$$\begin{aligned}
g(\partial_{y^A}, \partial_{y^B}) &= g(\partial_{x^A} + \partial_{x^A} f \partial_{\underline{u}}, \partial_{x^B} + \partial_{x^B} f \partial_{\underline{u}}) \\
&= g(\partial_{x^A}, \partial_{x^B}) + \partial_{x^A} f \partial_{x^B} f \\
g(\partial_{y^A}, \partial_u) &= g(\partial_{x^A} + \partial_{x^A} f \partial_{\underline{u}}, \partial_u f \partial_{\underline{u}}) \\
&= \partial_{x^A} f \partial_u f, \\
g(\partial_u, \partial_u) &= g(\partial_u f \partial_{\underline{u}}, \partial_u f \partial_{\underline{u}}) = (\partial_u f)^2
\end{aligned}$$

and hence

$$\begin{aligned}
&\gamma_{AB} = \underline{\gamma}_{AB} + \partial_{x^A} f \partial_{x^B} f, \quad A, B = 1, 2, \\
&g_{Au} = \partial_u f \partial_{x^A} f, \quad A = 1, 2
\end{aligned}$$

$$g_{uu} = (\partial_u f)^2,$$

as stated, where γ and $\underline{\gamma}$ denote respectively the induced metrics of P_u and $\underline{P}_{\underline{u}}$, and where

$$\gamma_{AB} := \gamma(\partial_{y^A}, \partial_{y^B}), \quad \underline{\gamma}_{AB} := g(\partial_{x^A}, \partial_{x^B}), \quad g_{Au} := g(\partial_{y^A}, \partial_u), \quad g_{uu} := g(\partial_u, \partial_u).$$

Next, we introduce the following vector field

$$\widetilde{N} = \partial_u + \widetilde{N}^A \partial_{y^A}$$

and choose $(\widetilde{N}^1, \widetilde{N}^2)$, such that $\widetilde{N}$ is normal to P_u, i.e., for $A = 1, 2$,

$$\begin{aligned} 0 = g(\widetilde{N}, \partial_{y^A}) &= g(\partial_u + \widetilde{N}^B \partial_{y^B}, \partial_{y^A}) \\ &= g_{uA} + \widetilde{N}^B \gamma_{BA} \end{aligned}$$

and hence

$$\widetilde{N}^A = -\gamma^{AB} g_{uB}.$$

Also, we have

$$\begin{aligned} g(\widetilde{N}, \widetilde{N}) &= g(\partial_u + \widetilde{N}^A \partial_{y^A}, \partial_u + \widetilde{N}^B \partial_{y^B}) \\ &= g_{uu} + 2\widetilde{N}^A g_{uA} + \widetilde{N}^A \widetilde{N}^B \gamma_{AB} \\ &= g_{uu} - 2\gamma^{AB} g_{uB} g_{uA} + \gamma^{AC} g_{uC} \gamma^{BD} g_{uD} \gamma_{AB} \\ &= g_{uu} - \gamma^{AB} g_{uB} g_{uA}. \end{aligned}$$

We deduce that the unit norma N to P_u is given by $N^u \partial_u + N^A \partial_{y^A}$ with

$$N^A = -\frac{\gamma^{AB} g_{uB}}{\sqrt{g_{uu} - \gamma^{CD} g_{uC} g_{uD}}}, \quad A = 1, 2, \qquad N^u = \frac{1}{\sqrt{g_{uu} - \gamma^{CD} g_{uC} g_{uD}}},$$

which we rewrite as

$$\begin{aligned} N^A &= -\frac{\gamma^{AB} \partial_{x^B} f}{\sqrt{1 - \gamma^{AB} \partial_{x^A} f \partial_{x^B} f}}, \quad A = 1, 2, \\ N^u &= \frac{1}{\partial_u f \sqrt{1 - \gamma^{AB} \partial_{x^A} f \partial_{x^B} f}}, \end{aligned}$$

provided

$$\partial_u f > 0, \qquad \gamma^{AB} \partial_{x^A} f \partial_{x^B} f < 1.$$

This concludes the proof of Lemma A.5.

A.8. Proof of Lemma A.6

The mean curvature $\operatorname{tr}\theta$ of the u-foliation is given by

$$\begin{aligned} \operatorname{tr}\theta &= \gamma^{AB} g(D_{\partial_{y^A}} N, \partial_{y^B}) \\ &= \gamma^{AB} g(D_{\partial_{y^A}}(N^u \partial_u), \partial_{y^B}) + \gamma^{AB} g(D_{\partial_{y^A}}(N^C \partial_{y^C}), \partial_{y^B}). \end{aligned}$$

Since

$$\begin{aligned}\gamma^{AB} g(D_{\partial_{y^A}}(N^u\partial_u), \partial_{y^B}) &= \gamma^{AB}\partial_{y^A}(N^u) g_{uB} + N^u\gamma^{AB} g(D_{\partial_{y^A}}\partial_u, \partial_{y^B}) \\ &= \gamma^{AB}\partial_{y^A}(N^u) g_{uB} + N^u\gamma^{AB} g(D_{\partial_u}\partial_{y^A}, \partial_{y^B}) \\ &= \gamma^{AB}\partial_{y^A}(N^u) g_{uB} + \frac{1}{2}N^u\gamma^{AB}\partial_u(\gamma_{AB}),\end{aligned}$$

we infer

$$\begin{aligned}\operatorname{tr}\theta = {}& \gamma^{AB}\partial_{y^A}(N^u) g_{uB} + \frac{1}{2}N^u\gamma^{AB}\partial_u(\gamma_{AB}) \\ & + \gamma^{AB}\gamma_{CB}\partial_{y^A}(N^C) + \gamma^{AB}N^C g(D_{\partial_{y^A}}\partial_{y^C}, \partial_{y^B}).\end{aligned}$$

Next, we compute

$$\begin{aligned}\partial_{y^A}(N^u) &= -\frac{\partial_{x^A}\partial_u f}{(\partial_u f)^2\sqrt{1-\gamma^{AB}\partial_{x^A} f\partial_{x^B} f}} + \frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}{2\partial_u f\left(1-\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)^{\frac{3}{2}}} \\ &= -\frac{\partial_{x^A}\partial_u f}{\partial_u f}N^u + \frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}{2\left(1-\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}N^u\end{aligned}$$

and

$$\begin{aligned}\partial_u(\gamma_{AB}) &= \partial_u\left(\underline{\gamma}_{AB} + \partial_{x^A} f\partial_{x^B} f\right) \\ &= \partial_u(\underline{\gamma}_{AB}) + \partial_{x^A} f\partial_u\partial_{x^B} f + \partial_{x^B} f\partial_u\partial_{x^A} f,\end{aligned}$$

which yields

$$\begin{aligned}& \gamma^{AB}\partial_{y^A}(N^u) g_{uB} + \frac{1}{2}N^u\gamma^{AB}\partial_u(\gamma_{AB}) \\ &= \frac{1}{2}\gamma^{AB}N^u\Bigg\{-2\left(\frac{\partial_{x^A}\partial_u f}{\partial_u f} - \frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}{2\left(1-\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}\right) g_{uB} \\ &\qquad + \partial_u(\underline{\gamma}_{AB}) + \partial_{x^A} f\partial_u\partial_{x^B} f + \partial_{x^B} f\partial_u\partial_{x^A} f\Bigg\} \\ &= \frac{1}{2}\gamma^{AB}N^u\Bigg\{-2\partial_{x^B} f\partial_{x^A}\partial_u f + \partial_{x^A} f\partial_u\partial_{x^B} f + \partial_{x^B} f\partial_u\partial_{x^A} f \\ &\qquad + \frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}{\left(1-\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}\partial_{x^B} f\partial_u f + \partial_u(\underline{\gamma}_{AB})\Bigg\} \\ &= \frac{1}{2}\gamma^{AB}N^u\left\{\frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}{\left(1-\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}\partial_{x^B} f\partial_u f + \partial_u(\underline{\gamma}_{AB})\right\}\end{aligned}$$

and hence

$$\begin{aligned}\operatorname{tr}\theta = {}& \frac{1}{2}\gamma^{AB}N^u\left\{\frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}{\left(1-\gamma^{AB}\partial_{x^A} f\partial_{x^B} f\right)}\partial_{x^B} f\partial_u f + \partial_u(\underline{\gamma}_{AB})\right\} \\ & + \gamma^{AB}\gamma_{CB}\partial_{y^A}(N^C) + \gamma^{AB}N^C g(D_{\partial_{y^A}}\partial_{y^C}, \partial_{y^B}).\end{aligned}$$

Recalling that $\partial_u = \partial_u f \partial_{\underline{u}}$ we infer

$$\begin{aligned}\operatorname{tr}\theta = {} & \frac{1}{2}\gamma^{AB}N^u\partial_u f\left\{\frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A}f\partial_{x^B}f\right)}{\left(1-\gamma^{AB}\partial_{x^A}f\partial_{x^B}f\right)}\partial_{x^B}f + \partial_{\underline{u}}(\underline{\gamma}_{AB})\right\}\\ & + \gamma^{AB}\gamma_{CB}\partial_{y^A}(N^C) + \gamma^{AB}N^C g(D_{\partial_{y^A}}\partial_{y^C}, \partial_{y^B}).\end{aligned}$$

Since

$$N^u = \frac{1}{\partial_u f\sqrt{1-\gamma^{AB}\partial_{x^A}f\partial_{x^B}f}},$$

we deduce

$$\begin{aligned}\operatorname{tr}\theta = {} & \frac{1}{2}\gamma^{AB}\left\{\frac{\partial_{y^A}\left(\gamma^{AB}\partial_{x^A}f\partial_{x^B}f\right)}{\left(1-\gamma^{AB}\partial_{x^A}f\partial_{x^B}f\right)^{\frac{3}{2}}}\partial_{x^B}f + \frac{\partial_{\underline{u}}(\underline{\gamma}_{AB})}{\sqrt{1-\gamma^{AB}\partial_{x^A}f\partial_{x^B}f}}\right\}\\ & + \gamma^{AB}\gamma_{CB}\partial_{y^A}(N^C) + \gamma^{AB}N^C g(D_{\partial_{y^A}}\partial_{y^C}, \partial_{y^B}),\end{aligned}$$

as stated.

Next, we compute the lapse a of the u-foliation. We have

$$\begin{aligned}\det(g) &= g_{uu}\det(\gamma) - g_{u2}(\gamma_{11}g_{2u} - \gamma_{21}g_{1u}) + g_{u1}(\gamma_{12}g_{2u} - \gamma_{22}g_{1u})\\ &= \left(g_{uu} + \frac{-g_{u2}(\gamma_{11}g_{2u} - \gamma_{21}g_{1u}) + g_{u1}(\gamma_{12}g_{2u} - \gamma_{22}g_{1u})}{\det(\gamma)}\right)\det(\gamma).\end{aligned}$$

On the other hand, we have by the coarea formula, written in the (y^1, y^2, u) coordinates system,

$$\sqrt{\det(g)} = a\sqrt{\det(\gamma)},$$

where a is the lapse, and hence

$$a = \sqrt{g_{uu} + \frac{-g_{u2}(\gamma_{11}g_{2u} - \gamma_{21}g_{1u}) + g_{u1}(\gamma_{12}g_{2u} - \gamma_{22}g_{1u})}{\det(\gamma)}}.$$

Since

$$g_{Au} = \partial_u f\partial_{x^A}f, \quad A = 1, 2, \qquad g_{uu} = (\partial_u f)^2,$$

we may rewrite a as

$$a = \sqrt{1 - \frac{\gamma_{11}(\partial_{x^2}f)^2 + \gamma_{22}(\partial_{x^1}f)^2 - 2\gamma_{21}\partial_{x^1}f\partial_{x^2}f}{\det(\gamma)}}\partial_u f,$$

provided $\partial_u f > 0$ and

$$\frac{\gamma_{11}(\partial_{x^2}f)^2 + \gamma_{22}(\partial_{x^1}f)^2 - 2\gamma_{21}\partial_{x^1}f\partial_{x^2}f}{\det(\gamma)} < 1.$$

This concludes proof of Lemma A.6.

A.9. Proof of Lemma A.7

We look for f under the form

$$f(u,x^1,x^2) = u + \widetilde{f}(x^1,x^2,u),$$

so that

$$\partial_u f = 1 + \partial_u \widetilde{f}, \qquad \partial_{x^A} f = \partial_{x^A}\widetilde{f}, \quad A = 1,2.$$

Plugging in the formula for γ_{AB} and N^A of Lemma A.5, we infer

$$\gamma_{AB} = \underline{\gamma}_{AB} + \partial_{x^A}\widetilde{f}\partial_{x^B}\widetilde{f}$$

and

$$N^A = -\frac{(\underline{\gamma}_{AB} + \partial_{x^A}\widetilde{f}\partial_{x^B}\widetilde{f})\partial_{x^B}\widetilde{f}}{\sqrt{1-(\underline{\gamma}_{AB} + \partial_{x^A}\widetilde{f}\partial_{x^B}\widetilde{f})\partial_{x^A}\widetilde{f}\partial_{x^B}\widetilde{f}}}, \quad A = 1,2.$$

Then, plugging in the formula for $\mathrm{tr}\,\theta$ of Lemma A.6, and using the fact that

$$\partial_{\underline{u}}(\underline{\gamma}_{AB}) = 2\underline{\theta}_{AB}, \qquad \gamma^{AB}\partial_{\underline{u}}(\underline{\gamma}_{AB}) = 2\mathrm{tr}\,\underline{\theta},$$

we infer

$$\mathrm{tr}\,\theta = -\Delta\!\!\!\!/_{\underline{\gamma}(u)}\widetilde{f} - F_5^{AB}\left(\nabla\!\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\partial_{x^A}\partial_{x^B}\widetilde{f} + \mathrm{tr}\,\underline{\theta} + F_6^{AB}\left(\nabla\!\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\underline{\theta}_{AB},$$

where F_5^{AB} and F_6^{AB} are smooth in a neighborhood of the origin such that

$$F_5^{AB}(0) = 0, \qquad F_6^{AB}(0) = 0.$$

Also, plugging in the formula for a of Lemma A.6, we infer

$$a = \left(1 + F_7\left(\nabla\!\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\right)(1+\partial_u\widetilde{f}),$$

where F_7 is a smooth function in a neighborhood of the origin such that

$$F_7(0) = 0.$$

Next, we use

$$\begin{aligned} N &= N^u\partial_u + N^A\partial_{y^A} \\ &= N^u\partial_u f\partial_{\underline{u}} + N^A(\partial_{x^A} + \partial_{x^A}f\partial_{\underline{u}}) \\ &= (N^u\partial_u f + N^A\partial_{x^A}f)\partial_{\underline{u}} + N^A\partial_{x^A} \end{aligned}$$

and plug the formula for N of Lemma A.5 to obtain

$$N = \frac{1-\gamma^{AB}\partial_{x^B}f\partial_{x^A}f}{\sqrt{1-\gamma^{AB}\partial_{x^A}f\partial_{x^B}f}}\partial_{\underline{u}} - \frac{\gamma^{AB}\partial_{x^B}f}{\sqrt{1-\gamma^{AB}\partial_{x^B}f\partial_{x^A}f}}\partial_{x^A}.$$

We infer

$$k_{NN} = k(\underline{N},\underline{N}) + F_8^{ij}\left(\nabla\!\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)k_{ij},$$

where F_8^{ij} is smooth in a neighborhood of the origin such that

$$F_8^{ij}(0) = 0.$$

In view of the above identities for $\operatorname{tr}\theta$, a and k_{NN}, we may thus rewrite (2.28), given by

$$\operatorname{tr}\theta - k_{NN} = 1 - a,$$

as

$$\begin{aligned}&\left(1 + F_7\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\right)(1+\partial_u\widetilde{f}) - 1 - \Delta\!\!\!/_{\underline{\gamma}(u)}\widetilde{f} - F_5^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\partial_{x^A}\partial_{x^B}\widetilde{f}\\&= k(\underline{N},\underline{N}) + F_8^{ij}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)k_{ij} - \operatorname{tr}\underline{\theta} - F_6^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\underline{\theta}_{AB},\end{aligned}$$

or

$$\begin{aligned}&\partial_u\widetilde{f} - \Delta\!\!\!/_{\underline{\gamma}(u)}\widetilde{f} - F_1^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\partial_{x^A}\partial_{x^B}\widetilde{f}\\&= k(\underline{N},\underline{N}) - \operatorname{tr}\underline{\theta} + F_2^{ij}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)k_{ij} + F_3^{AB}\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right)\underline{\theta}_{AB} + F_4\left(\nabla\!\!\!/_{\underline{\gamma}(u)}\widetilde{f}\right),\end{aligned}$$

where F_1^{AB}, F_2^{ij}, F_3^{AB} and F_4 are smooth in a neighborhood of the origin such that

$$F_1^{AB}(0) = 0, \qquad F_2^{ij}(0) = 0, \qquad F_3^{AB}(0) = 0, \qquad F_4(0,0) = 0.$$

This concludes the proof of Lemma A.7.

APPENDIX B

PROOF OF THE ESTIMATES FOR THE COMMUTATOR $[\nabla_{aN}, P_j]$

In this appendix, we prove the commutator estimates stated in Section 5.3.

B.1. Proof of Proposition 5.14

Proceeding as in (5.50), (5.51), (5.52), we have:

$$[\nabla_{aN}, P_j]f = \int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{B.1}$$

where V is given by:

$$(\partial_\tau - \not{\Delta})V(\tau) = [\nabla_{aN}, \not{\Delta}]U(\tau)f,\ V(0) = 0. \tag{B.2}$$

In view of (B.1), we have:

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim \int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(S)}d\tau.$$

Thus, to obtain (5.72), it suffices to show:

$$\sup_\tau \|V(\tau)\|_{L^2(S)} \lesssim \varepsilon\|\Lambda^{\frac{1}{2}+\delta}f\|_{L^2(S)} + \varepsilon\|\Lambda^\delta f\|_{L^\infty_{[-2,2]}L^2(P_u)}. \tag{B.3}$$

From now on, we focus on proving (B.3).

In view of (B.2) and the heat flow estimate (5.5), we have:

$$\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau \int_{P_u} V(\tau')[\nabla_{aN}, \not{\Delta}]U(\tau')d\mu_u d\tau'.$$

Using the commutator Formula (2.23), and integrating the second order derivative by parts, we obtain the following estimate:

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau' \\
&\lesssim (\|a\not{\nabla}(\theta)\|_{L^2(P_u)} + \|\not{\nabla}(a)\theta\|_{L^2(P_u)} + \|aR\|_{L^2(P_u)}) \int_0^\tau \|\not{\nabla}U(\tau')\|_{L^4(P_u)}\|V(\tau')\|_{L^4(P_u)}d\tau' \\
&\|a\theta\|_{L^\infty_u L^4(P_u)} \int_0^\tau \|\not{\nabla}U(\tau')\|_{L^4(P_u)}\|\not{\nabla}V(\tau')\|_{L^2(P_u)}d\tau.
\end{aligned}$$

Together with the Gagliardo-Nirenberg inequality (3.9), Proposition 3.7, and the estimate (2.30) for a and θ, we obtain:

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_u)} d\tau' \\
&\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)}) \\
&\quad \times \int_0^\tau {\tau'}^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau \\
&\quad + \varepsilon^2 \int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau + \frac{1}{2}\int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_u)} d\tau' \\
&\quad + \frac{1}{2}\int_0^\tau {\tau'}^{-1+\delta} \|V(\tau')\|^2_{L^2(P_u)} d\tau,
\end{aligned}$$

for any $\delta > 0$. This yields:

$$\begin{aligned}
\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_u)} d\tau' &\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)}) \\
&\quad \times \int_0^\tau {\tau'}^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau \\
&\quad + \varepsilon^2 \int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau
\end{aligned}$$

and integrating in u, we obtain:

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(S)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(S)} d\tau' \\
&\lesssim \varepsilon^2 \sup_u \left(\int_0^\tau {\tau'}^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau\right) \\
&\quad + \varepsilon^2 \int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(S)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(S)} d\tau,
\end{aligned} \tag{B.4}$$

where we used the estimate (2.30) for a and θ, and the smallness assumption (2.1) for R. Now, we have:

$$\begin{aligned}
&\int_0^\tau {\tau'}^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau' \\
&\lesssim \int_0^\tau {\tau'}^{1-2\delta} \|\Delta\!\!\!/ U(\tau')\|^2_{L^2(P_u)} d\tau' + \int_0^\tau \|\nabla\!\!\!/ U(\tau')\|^2_{L^2(P_u)} d\tau',
\end{aligned}$$

where we used the Bochner inequality for scalars (5.63). Together with the heat flow estimate (5.24), we obtain:

$$\sup_u \left(\int_0^\tau {\tau'}^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau'\right) \lesssim \|\Lambda^{3\delta} f\|^2_{L^\infty_{[-2,2]} L^2(P_u)}. \tag{B.5}$$

Also, we have:

$$\int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(S)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(S)} d\tau' \tag{B.6}$$

$$\lesssim \int_0^\tau \tau'^{\frac{1}{2}-\delta}\|\not{\Delta}U(\tau')\|^2_{L^2(S)}d\tau' + \left(\sup_\tau \tau'^{\frac{1}{2}}\|\not{\nabla}U(\tau)\|^2_{L^2(S)}\right)\left(\int_0^\tau \tau'^{-1+\delta}\right)$$
$$\lesssim \|\Lambda^{\frac{1}{2}+2\delta}f\|^2_{L^2(S)},$$

where we used in the last inequality the Bochner inequality for scalars (5.63) and a heat flow estimate. Finally, (B.4), (B.5) and (B.6) imply:

$$\sup_\tau \|V(\tau)\|_{L^2(S)} \lesssim \varepsilon\|\Lambda^{3\delta}f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \varepsilon\|\Lambda^{\frac{1}{2}+2\delta}f\|_{L^2(S)}.$$

Since $\delta > 0$ is arbitrary, this yields (B.3), which concludes the proof of the proposition.

B.2. Proof of Proposition 5.15

Proceeding as in (B.1) (B.2), we have:

$$[\nabla_{aN}, P_j]F = \int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{B.7}$$

where V is given by:

$$(\partial_\tau - \not{\Delta})V(\tau) = [\nabla_{aN}, \not{\Delta}]U(\tau)F,\ V(0) = 0. \tag{B.8}$$

In view of (B.7), we have:

$$\|[\nabla_{aN}, P_j]F\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim \int_0^\infty m_j(\tau)\|V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)}d\tau.$$

Thus, to obtain (5.73), it suffices to show:

$$\|V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim \tau^{\frac{1}{2}-\frac{\delta}{2}}\varepsilon\left(\|\not{\nabla}F\|_{L^2(S)} + \|F\|_{L^\infty_{[-2,2]}L^2(P_u)}\right). \tag{B.9}$$

From now on, we focus on proving (B.9).

In view of (B.8) and the heat flow estimate (5.5), we have:

$$\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau\int_{P_u} V(\tau')[\nabla_{aN}, \not{\Delta}]U(\tau')d\mu_u d\tau'. \tag{B.10}$$

Injecting the commutator Formula (2.22) in (B.10), integrating by parts, and using the $L^\infty(P_u)$ estimate (3.10), we obtain the following estimate:

$$\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau'$$
$$\lesssim (\|a\not{\nabla}(\theta)\|_{L^2(P_u)} + \|\not{\nabla}(a)\theta\|_{L^2(P_u)} + \|aR\|_{L^2(P_u)})\int_0^\tau \|\not{\nabla}U(\tau')\|_{L^p(P_u)}\|\not{\nabla}V(\tau')\|_{L^2(P_u)}d\tau',$$

where $2 < p < 4$ will be chosen later. Together with the Gagliardo-Nirenberg inequality (3.9), we obtain:

$$\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau'$$
$$\lesssim (\|a\not{\nabla}(\theta)\|^2_{L^2(P_u)} + \|\not{\nabla}(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)})\int_0^\tau \|\not{\nabla}^2U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}\|\not{\nabla}U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}d\tau'.$$

Taking the square root, and integrating in u, this yields:

$$\begin{aligned}(\mathrm{B.11})\quad \|V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)} &\lesssim (\|a\nabla\!\!\!/(\theta)\|_{L^2(S)} + \|\nabla\!\!\!/(a)\theta\|_{L^2(S)} + \|aR\|_{L^2(S)})\\ &\quad\times\left\|\left(\int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}\|\nabla\!\!\!/ U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}d\tau'\right)^{\frac12}\right\|_{L^2_u}\\ &\lesssim \varepsilon\left\|\left(\int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}\|\nabla\!\!\!/ U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}d\tau'\right)^{\frac12}\right\|_{L^2_u},\end{aligned}$$

where we used in the last inequality the estimate (2.30) for a and θ, and the smallness assumption (2.1) for R. Now, in view of the Bochner identity for tensors (3.28), we have:

$$\begin{aligned}&\int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}\|\nabla\!\!\!/ U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}d\tau'\\ &\lesssim \int_0^\tau \Big(\|\Delta\!\!\!/ U(\tau')\|_{L^2(P_u)} + \|K\|_{L^2(P_u)}\|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} + \|K\|^2_{L^2(P_u)}\|U(\tau')\|_{L^2(P_u)}\Big)^{2(1-\frac{2}{p})}\\ &\quad\times\|\nabla\!\!\!/ U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}d\tau'\\ &\lesssim \left(\int_0^\tau \|\Delta\!\!\!/ U(\tau')\|^2_{L^2(P_u)}d\tau'\right)^{1-\frac{2}{p}}\left(\int_0^\tau \|\nabla\!\!\!/ U(\tau')\|^2_{L^2(P_u)}d\tau'\right)^{\frac{2}{p}}\\ &\quad+\|K\|^{4(1-\frac{2}{p})}_{L^2(P_u)}\int_0^\tau(\|\nabla\!\!\!/ U(\tau')\|^2_{L^2(P_u)} + \|U(\tau')\|^2_{L^2(P_u)})d\tau'.\end{aligned}$$

Integrating in u, and using the fact that $2 < p < 4$, this yields:

$$\begin{aligned}&\int_{-2}^2\int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}\|\nabla\!\!\!/ U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}d\tau' du\\ &\qquad\lesssim \tau^{\frac{2}{p}}\left(\sup_\tau\|\nabla\!\!\!/ U(\tau')\|^2_{L^2(S)} + \int_0^\tau\|\Delta\!\!\!/ U(\tau')\|^2_{L^2(S)}d\tau'\right)\\ &\qquad\quad+\|K\|^{4(1-\frac{2}{p})}_{L^2(S)}\tau^{\frac{4}{p}-1}\Bigg(\sup_\tau(\|\nabla\!\!\!/ U(\tau')\|^2_{L^2(S)} + \|U(\tau')\|^2_{L^2(S)})\\ &\qquad\quad+\sup_u\left(\int_0^\tau(\|\nabla\!\!\!/ U(\tau')\|^2_{L^2(P_u)} + \|U(\tau')\|^2_{L^2(P_u)})d\tau'\right)\Bigg).\end{aligned}$$

Together with the estimate (2.30) for K, and the heat flow estimates (5.1) and (5.2), we obtain:

$$\begin{aligned}&\int_{-2}^2\int_0^\tau \|\nabla\!\!\!/^2 U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}\|\nabla\!\!\!/ U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}d\tau' du\\ &\qquad\lesssim \left(\tau^{\frac{2}{p}} + \tau^{\frac{4}{p}-1}\right)(\|\nabla\!\!\!/ F\|^2_{L^2(S)} + \|F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}).\end{aligned}$$

Together with (B.11), we finally obtain:

$$\|V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim \varepsilon\left(\tau^{\frac{1}{p}} + \tau^{\frac{2}{p}-\frac12}\right)(\|\nabla\!\!\!/ F\|_{L^2(S)} + \|F\|_{L^\infty_{[-2,2]}L^2(P_u)}).$$

Since $\delta > 0$, we may choose p such that:

$$2 < p < \min\left(4, \frac{4}{2-\delta}\right),$$

which yields (B.9). This concludes the proof of the proposition.

B.3. Proof of Corollary 5.16

Using the inequality (5.48), the fact that $P_j F \equiv 0$ on $u = -2$, and properties (ii) and (iii) of Theorem 5.5, we have:

(B.12)

$$\begin{aligned}
&\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\\
&\lesssim \sum_{j\geq 0} 2^j \left(\int_{-2}^{2} \|P_j F\|_{L^2(P_u)} \|\nabla_N P_j F\|_{L^2(P_u)} du + \|P_j F\|_{L^2(S)} \|\nabla\!\!\!/ P_j F\|_{L^2(S)}\right)\\
&\lesssim \sum_{j\geq 0} 2^j \left(\int_{-2}^{-2} \|P_j F\|_{L^2(P_u)} \|\nabla_N P_j F\|_{L^2(P_u)} du\right) + \sum_{j\geq 0} 2^{2j} \|P_j F\|^2_{L^2(S)}\\
&\lesssim \sum_{j\geq 0} 2^j \left(\int_{-2}^{-2} \|P_j F\|_{L^2(P_u)} \|\nabla_{aN} P_j F\|_{L^2(P_u)} du\right) + \|\nabla\!\!\!/ F\|^2_{L^2(S)},
\end{aligned}$$

where we used the estimate (2.30) for a in the last inequality. Now, we have:

$$\|\nabla_{aN} P_j F\|_{L^2(P_u)} \lesssim \|P_j(\nabla_{aN} F)\|_{L^2(P_u)} + \|[\nabla_{aN}, P_j]F\|_{L^2(P_u)},$$

which together with (B.12), and the properties (ii) and (iii) of Theorem 5.5 implies:

(B.13)

$$\begin{aligned}
&\textstyle\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\\
&\lesssim \sum_{j\geq 0} 2^j \|P_j F\|_{L^2(S)} \|P_j(\nabla_{aN} F)\|_{L^2(S)}\\
&\quad + \sum_{j\geq 0} 2^j \|P_j F\|_{L^\infty_{[-2,2]}L^2(P_u)} \|[\nabla_{aN}, P_j]F\|_{L^1_{[-2,2]}L^2(P_u)} + \|\nabla\!\!\!/ F\|^2_{L^2(S)}\\
&\lesssim \sum_{j\geq 0} 2^{2j} \|P_j F\|^2_{L^2(S)} + \sum_{j\geq 0} \|P_j(\nabla_{aN} F)\|^2_{L^2(S)}\\
&\quad + \left(\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac12} \left(\sum_{j\geq 0} 2^j \|[P_j, \nabla_{aN}]F\|^2_{L^1_{[-2,2]}L^2(P_u)}\right)^{\frac12} + \|\nabla\!\!\!/ F\|^2_{L^2(S)}\\
&\lesssim \left(\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\right)^{\frac12} \left(\sum_{j\geq 0} 2^j \|[P_j, \nabla_{aN}]F\|^2_{L^1_{[-2,2]}L^2(P_u)}\right)^{\frac12} + \|\nabla F\|^2_{L^2(S)}.
\end{aligned}$$

This yields:

(B.14) $$\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \sum_{j\geq 0} 2^j \|[P_j, \nabla_{aN}]F\|^2_{L^1_{[-2,2]}L^2(P_u)} + \|\nabla F\|^2_{L^2(S)}.$$

Now, we have in view of the commutator estimate (5.73):

$$\|[P_j, \nabla_{aN}]F\|^2_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^{-j(1-\delta)}\varepsilon(\|\nabla\!\!\!/ F\|_{L^2(S)} + \|F\|_{L^\infty_{[-2,2]}L^2(P_u)}),$$

for any $\delta > 0$. In view of Corollary 3.8 and the fact that $F \equiv 0$ on $u = -2$, we obtain:

$$\|[P_j, \nabla_{aN}]F\|^2_{L^1_{[-2,2]}L^2(P_u)} \lesssim 2^{-j(1-\delta)}\varepsilon\|F\|_{H^1(S)}.$$

Together with (B.14), this yields:

$$\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \left(1 + \sum_{j\geq 0} 2^{-j(1-2\delta)}\right) \|F\|^2_{H^1(S)}.$$

Choosing $0 < \delta < 1/2$, we obtain:

(B.15) $$\sum_{j\geq 0} 2^j \|P_j F\|^2_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|F\|^2_{H^1(S)},$$

which is the wanted estimate. This concludes the proof of the corollary.

B.4. Proof of Proposition 5.17

In view of (B.1), we have:

$$\|[\nabla_{aN}, P_j]f\|_{L^1_{[-2,2]}L^2(P_u)} \lesssim \int_0^\infty m_j(\tau)\|V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)}d\tau,$$

where V is given by:

(B.16) $$(\partial_\tau - \Delta\!\!\!/)V(\tau) = [\nabla_{aN}, \Delta\!\!\!/]U(\tau)f,\ V(0) = 0.$$

Thus, to obtain (5.75), it suffices to show:

(B.17)

$$\|\Lambda^{-\alpha}V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)} + \int_{-2}^2 \left(\int_0^\tau \|\nabla\!\!\!/\Lambda^{-\alpha}V(\tau')\|^2_{L^2(P_u)}d\tau'\right)^{\frac{1}{2}} du \lesssim \varepsilon\|\Lambda^{-\delta}F\|_{L^2(S)}.$$

Indeed, once (B.17) is obtained, one proceeds as in (5.55) (5.56) to deduce (5.75). From now on, we focus on proving (B.17).

In view of (B.16) and the heat flow estimate (5.21), we have:

$$\|\Lambda^{-\alpha}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-\alpha}V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau \int_{P_u} \Lambda^{-2\alpha}V(\tau')[\nabla_{aN}, \Delta\!\!\!/]U(\tau')d\mu_u d\tau'.$$

Injecting the commutator Formula (2.23), integrating by parts, we obtain the following estimate:

(B.18) $$\|\Lambda^{-\alpha}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-\alpha}V(\tau')\|^2_{L^2(P_u)}d\tau'$$

$$\lesssim (\|a\nabla\!\!\!/(\theta)\|_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|_{L^2(P_u)} + \|aR\|_{L^2(P_u)})$$
$$\times \int_0^\tau \|\nabla\!\!\!/ U(\tau')\|_{L^p(P_u)}\|\nabla\!\!\!/\Lambda^{-2\alpha}V(\tau')\|_{L^2(P_u)}d\tau',$$

where

$$2 < p < \frac{2}{1-\alpha}$$

will be chosen later. Now, we have in view of (5.26) and (5.18):

$$\|\nabla\!\!\!/\Lambda^{-2\alpha}V(\tau')\|_{L^2(P_u)} \lesssim \|\Lambda^{-\alpha}V(\tau')\|^{\alpha}_{L^2(P_u)}\|\nabla\!\!\!/\Lambda^{-2\alpha}V(\tau')\|^{1-\alpha}_{L^2(P_u)},$$

which together with (B.18) implies:

(B.19)

$$\|\Lambda^{-\alpha}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-\alpha}V(\tau')\|^2_{L^2(P_u)}d\tau'$$
$$\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)}) \int_0^\tau {\tau'}^{\alpha_-}\|\nabla\!\!\!/ U(\tau')\|^2_{L^p(P_u)}d\tau'.$$

The Gagliardo-Nirenberg inequality (3.9) and the Bochner inequality (5.63) imply:

$$\int_0^\tau {\tau'}^{\alpha_-}\|\nabla\!\!\!/ U(\tau')\|^2_{L^p(P_u)}d\tau'$$
$$\lesssim \int_0^\tau {\tau'}^{\alpha_-}\|\nabla\!\!\!/ U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}\|\Delta\!\!\!/ U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}d\tau'$$
$$\lesssim \int_0^\tau {\tau'}^{b}\|\nabla\!\!\!/ U(\tau')\|^2_{L^2(P_u)}d\tau' + \int_0^\tau {\tau'}^{1+b}\|\Delta\!\!\!/ U(\tau')\|^2_{L^2(P_u)}d\tau',$$

where b is given by:

(B.20)
$$b = \alpha_- - 1 + \frac{2}{p}.$$

We have $0 < b < 1$ from the choice of α and p. Thus, we obtain in view of the heat flow estimates (5.22) and (5.24):

$$\int_0^\tau {\tau'}^{\alpha_-}\|\nabla\!\!\!/ U(\tau')\|^2_{L^p(P_u)}d\tau' \lesssim \|\Lambda^{-b_-}f\|^2_{L^2(P_u)}.$$

Together with (B.19), this yields:

$$\|\Lambda^{-\alpha}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-\alpha}V(\tau')\|^2_{L^2(P_u)}d\tau'$$
$$\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)})\|\Lambda^{-b_-}f\|^2_{L^2(P_u)}.$$

Integrating in u, this yields:

(B.21)
$$\|\Lambda^{-\alpha}V(\tau)\|_{L^1_{[-2,2]}L^2(P_u)} + \int_{-2}^2\left(\int_0^\tau \|\nabla\!\!\!/\Lambda^{-\alpha}V(\tau')\|^2_{L^2(P_u)}d\tau'\right)^{\frac{1}{2}}du$$
$$\lesssim (\|a\nabla\!\!\!/(\theta)\|_{L^2(S)} + \|\nabla\!\!\!/(a)\theta\|_{L^2(S)} + \|aR\|_{L^2(S)})\|\Lambda^{-b_-}f\|_{L^2(S)}$$
$$\lesssim \varepsilon\|\Lambda^{-b_-}F\|_{L^2(S)},$$

where we used in the last inequality the estimate (2.30) for a and θ, and the smallness assumption (2.1) for R. Now, in view of the Definition (B.20) of b, and since $\delta < \alpha$, we may choose $p > 2$ close enough to 2 such that $b_- > \delta$, which together with (B.21) implies (B.17). This concludes the proof of the proposition.

B.5. Proof of Proposition 5.18

Proceeding as in (B.1) (B.2), we have:

$$[\nabla_{aN}, P_j]f = \int_0^\infty m_j(\tau)V(\tau)d\tau, \tag{B.22}$$

where V is given by:

$$(\partial_\tau - \not{\Delta})V(\tau) = [\nabla_{aN}, \not{\Delta}]U(\tau)f,\ V(0) = 0. \tag{B.23}$$

In view of (B.22), we have:

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim \int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(S)}d\tau.$$

Thus, to obtain (5.75), it suffices to show:

$$\|V(\tau)\|_{L^2(S)} \lesssim \tau^{\frac{1}{2}-\frac{\delta}{2}}\varepsilon\left(\|\not{\Delta}f\|_{L^2(S)} + \|\not{\nabla}f\|_{L^\infty_{[-2,2]}L^2(P_u)}\right). \tag{B.24}$$

From now on, we focus on proving (B.24).

In view of (B.23) and the heat flow estimate (5.5), we have:

$$\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau' \lesssim \int_0^\tau\int_{P_u} V(\tau')[\nabla_{aN}, \not{\Delta}]U(\tau')d\mu_u d\tau'.$$

Using the commutator Formula (2.23), we obtain the following estimate:

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau' \\
&\lesssim \|a\theta\|_{L^\infty_{[-2,2]}L^4(P_u)}\int_0^\tau \|\not{\nabla}^2U(\tau')\|_{L^2(P_u)}\|V(\tau')\|_{L^4(P_u)}d\tau' \\
&\quad + (\|a\not{\nabla}\theta\|_{L^2(P_u)} + \|\not{\nabla}(a)\theta\|_{L^2(P_u)} + \|aR\|_{L^2(P_u)})\int_0^\tau \|\not{\nabla}U(\tau')\|_{L^4(P_u)}\|V(\tau')\|_{L^4(P_u)}d\tau'.
\end{aligned}$$

Together with the Gagliardo-Nirenberg inequality (3.9), Proposition 3.7, and the estimate (2.30) for a and θ, we obtain:

$$\begin{aligned}
&\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau' \\
&\lesssim (\|a\not{\nabla}(\theta)\|^2_{L^2(P_u)} + \|\not{\nabla}(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)}) \\
&\quad \times \int_0^\tau \tau'^{\frac{1}{2}-\delta}\|\not{\nabla}^2U(\tau')\|_{L^2(P_u)}\|\not{\nabla}U(\tau')\|_{L^2(P_u)}d\tau + \varepsilon^2\int_0^\tau \tau'^{1-2\delta}\|\not{\nabla}^2U(\tau')\|^2_{L^2(P_u)}d\tau \\
&\quad + \frac{1}{2}\int_0^\tau \|\not{\nabla}V(\tau')\|^2_{L^2(P_u)}d\tau' + \frac{1}{2}\int_0^\tau \tau'^{-1+\delta}\|V(\tau')\|^2_{L^2(P_u)}d\tau,
\end{aligned}$$

for any $\delta > 0$. This yields:

$$\|V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(P_u)} d\tau'$$
$$\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)})$$
$$\times \int_0^\tau \tau'^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau + \varepsilon^2 \int_0^\tau \tau'^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|^2_{L^2(P_u)} d\tau$$

and integrating in u, we obtain:

$$\begin{aligned} \|V(\tau)\|^2_{L^2(S)} + \int_0^\tau \|\nabla\!\!\!/ V(\tau')\|^2_{L^2(S)} d\tau' &\lesssim \varepsilon^2 \sup_u \left(\int_0^\tau \tau'^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau \right) \\ &\quad + \varepsilon^2 \int_0^\tau \tau'^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|^2_{L^2(S)} d\tau, \end{aligned} \tag{B.25}$$

where we used the estimate (2.30) for a and θ, and the smallness assumption (2.1) for R. Now, we have:

$$\begin{aligned} \int_0^\tau \tau'^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|_{L^2(P_u)} \|\nabla\!\!\!/ U(\tau')\|_{L^2(P_u)} d\tau' &\lesssim \tau^{1-\delta} \sup_\tau \|\nabla\!\!\!/ U(\tau)\|_{L^2(P_u)} \left(\int_0^\tau \|\Delta\!\!\!/ U(\tau')\|^2_{L^2(P_u)} d\tau' \right)^{\frac{1}{2}} \\ &\lesssim \tau^{1-\delta} \|\nabla\!\!\!/ f\|^2_{L^2(P_u)}, \end{aligned} \tag{B.26}$$

where we used the Bochner inequality for scalars (5.63) and the heat flow estimate (5.2). Also, we have:

$$\begin{aligned} \int_0^\tau \tau'^{\frac{1}{2}-\delta} \|\nabla\!\!\!/^2 U(\tau')\|^2_{L^2(S)} d\tau' &\lesssim \tau^{\frac{3}{2}-\delta} \sup_\tau \|\Delta\!\!\!/ U(\tau)\|^2_{L^2(S)} \\ &\lesssim \tau^{\frac{3}{2}-\delta} \|\Delta\!\!\!/ f\|^2_{L^2(S)}, \end{aligned} \tag{B.27}$$

where we used the Bochner inequality for scalars (5.63) and a heat flow estimate. Finally, (B.25), (B.26) and (B.27) yield (B.24). This concludes the proof of the proposition.

B.6. Proof of Proposition 5.19

Let us start by proving the corollary in the case where f is a scalar function on S satisfying the same assumptions that F. We estimate $\|P_j f\|^2_{L^\infty_{[-2,2]} L^2(P_u)}$. Using the inequality (5.48) and the fact that $P_j f \equiv 0$ on $u = -2$, we have:

$$\begin{aligned} \|P_j f\|^2_{L^\infty_{[-2,2]} L^2(P_u)} &\lesssim \|P_j f\|_{L^2(S)} \|\nabla_N P_j f\|_{L^2(S)} + \|P_j f\|_{L^2(S)} \|\nabla\!\!\!/ P_j f\|_{L^2(S)} \\ &\lesssim \|P_j f\|_{L^2(S)} \|\nabla_{aN} P_j f\|_{L^2(S)} + 2^j \|P_j f\|^2_{L^2(S)}, \end{aligned} \tag{B.28}$$

where we used in the last inequality the estimate (2.30) for a, and the finite band property for P_j. Now, we have:

$$\|\nabla_{aN} P_j f\|_{L^2(P_u)} \lesssim \|P_j(\nabla_{aN} f)\|_{L^2(P_u)} + \|[\nabla_{aN}, P_j]f\|_{L^2(P_u)},$$

which together with (B.28) implies:

$$\begin{aligned}
&\|P_j f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\\
&\lesssim \|P_j f\|_{L^2(S)}\|P_j(\nabla_{aN} f)\|_{L^2(S)} + \|P_j f\|_{L^2(S)}\|[\nabla_{aN}, P_j]f\|_{L^2(S)} + 2^j\|P_j f\|^2_{L^2(S)}\\
&\lesssim \Big(2^{-(2+b)j}\|\nabla_{aN} f\|_{L^2_u H^b(P_u)} + 2^{-2j}\|[\nabla_{aN}, P_j]f\|_{L^2(S)} + 2^{-3j}\|\Delta\!\!\!/ f\|_{L^2(S)}\Big)\|\Delta\!\!\!/ f\|_{L^2(S)},
\end{aligned}$$

where we used in the last inequality the finite band property for P_j, and the definition of $H^b(P_u)$. Together with (6.43) and the commutator estimate (5.76), we obtain:

(B.29)

$$\begin{aligned}
&\|P_j f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\\
&\lesssim \Big(2^{-(2+b)j}\|\nabla_N f\|_{L^2_u H^b(P_u)} + 2^{-(3-\delta)j}(\|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\Delta\!\!\!/ f\|_{L^2(S)})\Big)\|\Delta\!\!\!/ f\|_{L^2(S)},
\end{aligned}$$

for any $\delta > 0$. Now, in view of Proposition 3.15, we have:

$$\|\nabla\!\!\!/ f\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|\Delta\!\!\!/ f\|_{L^2(S)} + \|\nabla_N f\|_{L^2(S)}. \tag{B.30}$$

Since $b \geq 0$, (B.29) and (B.30) imply:

$$\|P_j f\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim 2^{-(1+\frac{b}{2})j}(\|\nabla_N f\|_{L^2_u H^b(P_u)} + \|\nabla\!\!\!/^2 f\|_{L^2(S)}). \tag{B.31}$$

Now, we have:

$$\|f\|_{L^\infty(S)} \lesssim \sum_{j\geq 0}\|P_j f\|_{L^\infty(S)} \lesssim \sum_{j\geq 0} 2^j\|P_j f\|_{L^\infty_{[-2,2]}L^2(P_u)},$$

where we used in the last inequality the strong Bernstein inequality for scalars (5.61). Together with (B.32) and the fact that $b > 0$, we obtain:

$$\|f\|_{L^\infty(S)} \lesssim \|\nabla_N f\|_{L^2_u H^b(P_u)} + \|\nabla\!\!\!/^2 f\|_{L^2(S)}. \tag{B.32}$$

Next, we turn to the case where F is a tensor. Using (B.32) with $\frac{b}{2}$ instead for b, and with $f = |F|^2$, we obtain:

$$\begin{aligned}
\|F\|^2_{L^\infty(S)} &\lesssim \|F\cdot\nabla_N F\|_{L^2_u H^{\frac{b}{2}}(P_u)} + \|F\nabla\!\!\!/^2 F\|_{L^2(S)} + \|\nabla\!\!\!/ F\|^2_{L^4(S)}\\
&\lesssim \|F\cdot\nabla_N F\|_{L^2_u H^{\frac{b}{2}}(P_u)} + \|F\|_{L^\infty(S)}\|\nabla\!\!\!/^2 F\|_{L^2(S)}\\
&\quad + \|\nabla\!\!\!/^2 F\|^2_{L^2(S)} + \|\nabla_N F\|^2_{L^2(S)},
\end{aligned}$$

where we used in the last inequality Proposition 3.11 to estimate $\|\nabla\!\!\!/ F\|_{L^4(S)}$. This yields:

$$\|F\|^2_{L^\infty(S)} \lesssim \|F\cdot\nabla_N F\|_{L^2_u H^{\frac{b}{2}}(P_u)} + \|\nabla\!\!\!/^2 F\|^2_{L^2(S)} + \|\nabla_N F\|^2_{L^2(S)}. \tag{B.33}$$

Next, we estimate the first term in the right-hand side of (B.33). We have:

$$\|P_j(F\cdot\nabla_N F)\|_{L^2(S)} \lesssim \sum_{l\geq 0}\|P_j(F\cdot P_l\nabla_N F)\|_{L^2(S)}. \tag{B.34}$$

In the case $l>j$, the boundedness of P_j on $L^2(P_u)$ yields:

$$\begin{aligned} 2^{\frac{bj}{2}}\|P_j(F\cdot P_l\nabla_N F)\|_{L^2(S)} &\lesssim 2^{\frac{bj}{2}}\|F\cdot P_l\nabla_N F\|_{L^2(S)}\\ &\lesssim 2^{\frac{bj}{2}}\|F\|_{L^\infty(S)}\|P_l\nabla_N F\|_{L^2(S)}\\ &\lesssim 2^{\frac{bj}{2}-bl}\|F\|_{L^\infty(S)}\|\nabla_N F\|_{L^2_uH^b(P_u)}. \end{aligned} \tag{B.35}$$

In the case $l\leq j$, we use the finite band property for P_j. We have:

$$\begin{aligned} P_j(F\cdot P_l\nabla_N F) &= 2^{-2j}P_j(\Delta\!\!\!/(F\cdot P_l\nabla_N F))\\ &= 2^{-2j}P_j(\mathrm{d}\!\!\!/\mathrm{iv}(\nabla\!\!\!\!/ F\cdot P_l\nabla_N F)) + 2^{-2j}P_j(\mathrm{d}\!\!\!/\mathrm{iv}(F\cdot\nabla\!\!\!\!/ P_l\nabla_N F)). \end{aligned}$$

Together with (5.64) -note that $\nabla\!\!\!\!/ F\cdot P_l\nabla_N F$ is a 1-form—and the finite band property for P_j, we obtain:

$$\begin{aligned} &2^{\frac{bj}{2}}\|P_j(F\cdot P_l\nabla_N F)\|_{L^2(S)}\\ &\lesssim 2^{\frac{bj}{2}-\frac{j}{2}}\|\nabla\!\!\!\!/ F\cdot P_l\nabla_N F\|_{L^2_{[-2,2]}L^{\frac{4}{3}}(P_u)} + 2^{\frac{bj}{2}-j}\|F\cdot\nabla\!\!\!\!/ P_l\nabla_N F\|_{L^2(S)}\\ &\lesssim 2^{\frac{bj}{2}-\frac{j}{2}}\|\nabla\!\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)}\|P_l\nabla_N F\|_{L^2_{[-2,2]}L^4(P_u)} + 2^{\frac{bj}{2}-j}\|F\|_{L^\infty(S)}\|\nabla\!\!\!\!/ P_l\nabla_N F\|_{L^2(S)}. \end{aligned}$$

Using Bernstein and the finite band property for P_l, this yields for $l\leq j$:

$$\begin{aligned} &2^{\frac{bj}{2}}\|P_j(F\cdot P_l\nabla_N F)\|_{L^2(S)}\\ &\lesssim 2^{\frac{bj}{2}}(2^{-\frac{j}{2}+\frac{l}{2}}\|\nabla\!\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)} + 2^{-j+l}\|F\|_{L^\infty(S)})\|P_l\nabla_N F\|_{L^2(S)}\\ &\lesssim 2^{-\frac{1-b}{2}j+(\frac{1}{2}-b)l}(\|\nabla\!\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|F\|_{L^\infty(S)})\|\nabla_N F\|_{L^2_uH^b(P_u)}. \end{aligned} \tag{B.36}$$

We may assume $b<\frac{1}{2}$. Then, using (B.34), (B.35) for $l>j$, and (B.36) for $l\leq j$, we obtain:

$$2^{\frac{bj}{2}}\|P_j(F\cdot\nabla_N F)\|_{L^2(S)} \lesssim 2^{-\frac{b}{2}j}(\|\nabla\!\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|F\|_{L^\infty(S)})\|\nabla_N F\|_{L^2_uH^b(P_u)},$$

which yields:

$$\|F\cdot\nabla_N F\|_{L^2_uH^{\frac{b}{2}}(P_u)} \lesssim (\|\nabla\!\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|F\|_{L^\infty(S)})\|\nabla_N F\|_{L^2_uH^b(P_u)}.$$

Together with (B.33), we obtain:

$$\begin{aligned} &\|F\|^2_{L^\infty(S)}\\ &\lesssim (\|\nabla\!\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|F\|_{L^\infty(S)})\|\nabla_N F\|_{L^2_uH^b(P_u)} + \|\nabla\!\!\!\!/^2 F\|^2_{L^2(S)} + \|\nabla_N F\|^2_{L^2(S)} \end{aligned}$$

and thus:

$$\|F\|_{L^\infty(S)} \lesssim \|\nabla\!\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla_N F\|_{L^2_uH^b(P_u)} + \|\nabla\!\!\!\!/^2 F\|_{L^2(S)}.$$

Now, using Proposition 3.11 to estimate $\|\nabla\!\!\!/ F\|_{L^\infty_{[-2,2]}L^2(P_u)}$, we finally get:

$$\|F\|_{L^\infty(S)} \lesssim \|\nabla_N F\|_{L^2_u H^b(P_u)} + \|\nabla\!\!\!/^2 F\|_{L^2(S)}.$$

This concludes the proof of the corollary.

B.7. Proof of Proposition 5.20

In view of (B.1), we have:

$$\|[\nabla_{aN}, P_j]f\|_{L^2(S)} \lesssim \int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(S)}d\tau,$$

where V is given by:

$$(\partial_\tau - \Delta\!\!\!/)V(\tau) = [\nabla_{aN}, \Delta\!\!\!/]U(\tau)f,\ V(0) = 0. \tag{B.37}$$

Thus, to obtain (5.75), it suffices to show:

$$\int_0^\tau \|V(\tau')\|^2_{L^2(S)}d\tau' \lesssim \varepsilon\|\Lambda^{-(1-\delta)}f\|^2_{L^\infty_{[-2,2]}L^2(P_u)}. \tag{B.38}$$

Indeed, (B.38) yields:

$$\begin{aligned}\|[\nabla_{aN}, P_j]f\|_{L^2(S)} &\lesssim \int_0^\infty m_j(\tau)\|V(\tau)\|_{L^2(S)}d\tau \\ &\lesssim \left(\int_0^\infty m_j(\tau)^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^\infty \|V(\tau)\|^2_{L^2(S)}d\tau\right)^{\frac{1}{2}} \\ &\lesssim 2^j\varepsilon\|\Lambda^{-(1-\delta)}f\|_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned}$$

which is (5.77). From now on, we focus on proving (B.38).

In view of (B.37) and the heat flow estimate (5.21), we have:

$$\begin{aligned}&\|\Lambda^{-1}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-1}V(\tau')\|^2_{L^2(P_u)}d\tau' \\ &\qquad\qquad \lesssim \int_0^\tau\int_{P_u}\Lambda^{-2}V(\tau')[\nabla_{aN}, \Delta\!\!\!/]U(\tau')d\mu_u d\tau'.\end{aligned}$$

Injecting the commutator Formula (2.23), integrating by parts, we obtain the following estimate:

$$\begin{aligned}&\|\Lambda^{-1}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-1}V(\tau')\|^2_{L^2(P_u)}d\tau' \\ &\lesssim (\|a\nabla\!\!\!/(\theta)\|_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|_{L^2(P_u)} + \|aR\|_{L^2(P_u)}) \\ &\quad\times\int_0^\tau \|\nabla\!\!\!/ U(\tau')\|_{L^p(P_u)}\|\nabla\!\!\!/\Lambda^{-2}V(\tau')\|_{L^2(P_u)}d\tau',\end{aligned} \tag{B.39}$$

where

$$2 < p < 4$$

will be chosen later. Now, we have in view of (5.26):

$$\|\nabla\!\!\!/\Lambda^{-2}V(\tau')\|_{L^2(P_u)} \lesssim \|\Lambda^{-1}V(\tau')\|_{L^2(P_u)},$$

which together with (B.39) implies:

(B.40)

$$\|\Lambda^{-1}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-1}V(\tau')\|^2_{L^2(P_u)}d\tau'$$
$$\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)}) \int_0^\tau \tau'^{1-}\|\nabla\!\!\!/U(\tau')\|^2_{L^p(P_u)}d\tau'.$$

The Gagliardo-Nirenberg inequality (3.9) and the Bochner inequality (5.63) imply:

$$\begin{aligned}\int_0^\tau \tau'^{1-}\|\nabla\!\!\!/U(\tau')\|^2_{L^p(P_u)}d\tau' &\lesssim \int_0^\tau \tau'^{1-}\|\nabla\!\!\!/U(\tau')\|^{\frac{4}{p}}_{L^2(P_u)}\|\Delta\!\!\!/U(\tau')\|^{2(1-\frac{2}{p})}_{L^2(P_u)}d\tau'\\ &\lesssim \int_0^\tau \tau'^{b}\|\nabla\!\!\!/U(\tau')\|^2_{L^2(P_u)}d\tau'\\ &\quad+\int_0^\tau \tau'^{1+b}\|\Delta\!\!\!/U(\tau')\|^2_{L^2(P_u)}d\tau',\end{aligned}$$

where b is given by:

$$b = 1_- - 1 + \frac{2}{p}. \tag{B.41}$$

We have $0 < b < 1$ from the choice of p. Thus, we obtain in view of the heat flow estimates (5.22) and (5.24):

$$\int_0^\tau \tau'^{\alpha-}\|\nabla\!\!\!/U(\tau')\|^2_{L^p(P_u)}d\tau' \lesssim \|\Lambda^{-b-}f\|^2_{L^2(P_u)}.$$

Together with (B.19), this yields:

$$\|\Lambda^{-1}V(\tau)\|^2_{L^2(P_u)} + \int_0^\tau \|\nabla\!\!\!/\Lambda^{-1}V(\tau')\|^2_{L^2(P_u)}d\tau'$$
$$\lesssim (\|a\nabla\!\!\!/(\theta)\|^2_{L^2(P_u)} + \|\nabla\!\!\!/(a)\theta\|^2_{L^2(P_u)} + \|aR\|^2_{L^2(P_u)})\|\Lambda^{-b-}f\|^2_{L^2(P_u)}.$$

Integrating in u, this yields:

(B.42)

$$\begin{aligned}\|\Lambda^{-1}V(\tau)\|^2_{L^2(S)} &+ \int_0^\tau \|\nabla\!\!\!/\Lambda^{-1}V(\tau')\|^2_{L^2(S)}d\tau'\\ &\lesssim (\|a\nabla\!\!\!/(\theta)\|_{L^2(S)} + \|\nabla\!\!\!/(a)\theta\|_{L^2(S)} + \|aR\|_{L^2(S)})\|\Lambda^{-b-}f\|_{L^\infty_{[-2,2]}L^2(P_u)}\\ &\lesssim \varepsilon\|\Lambda^{-b-}f\|_{L^\infty_{[-2,2]}L^2(P_u)},\end{aligned}$$

where we used in the last inequality the estimate (2.30) for a and θ, and the smallness assumption (2.1) for R. Now, in view of the Definition (B.41) of b, and since $\delta > 0$, we may choose $p > 2$ close enough to 2 such that $b_- > 1 - \delta$, which together with (B.42) implies (B.38). This concludes the proof of the proposition.

APPENDIX C

PRODUCT ESTIMATES

In this appendix, we prove the commutator estimates stated in Section 5.4.

C.1. Proof of Proposition 5.21

We have:

$$\|P_j(F\cdot G\cdot H)\|_{L^2(P_u)} \lesssim \sum_{l\geq 0}\|P_j(F\cdot G\cdot P_lH)\|_{L^2(P_u)}. \tag{C.1}$$

We first consider the case where $l \leq j$. Since $0 < b < \frac{1}{2}$, there exists a real number p such that:

$$\frac{2}{\frac{3}{2}-b} < p < 2. \tag{C.2}$$

We have:

$$\begin{aligned}\|P_j(F\cdot G\cdot P_lH)\|_{L^2(P_u)} &= 2^{-2j}\|P_j(\not\Delta(F\cdot G\cdot P_lH))\|_{L^2(P_u)}\\ &= 2^{-2j}\|P_j(\mathrm{di\!\!/v}(\not\nabla(F\cdot G\cdot P_lH)))\|_{L^2(P_u)}.\end{aligned}$$

Since $F\cdot G\cdot H$ is a scalar, we may use (5.64), and we obtain:

$$\begin{aligned}&2^{jb}\|P_j(F\cdot G\cdot P_lH)\|_{L^2(P_u)}\\ &\lesssim 2^{jb}2^{-2j}2^{\frac{2j}{p}}\|\not\nabla(F\cdot G\cdot P_lH)\|_{L^p(P_u)}\\ &\lesssim 2^{j(b-2+\frac{2}{p})}(\|\not\nabla F\|_{L^2(P_u)}\|G\|_{L^r(P_u)}\|P_lH\|_{L^r(P_u)} + \|F\|_{L^r(P_u)}\|\not\nabla G\|_{L^2(P_u)}\|P_lH\|_{L^r(P_u)}\\ &\quad + \|F\|_{L^r(P_u)}\|G\|_{L^r(P_u)}\|\not\nabla P_lH\|_{L^2(P_u)}),\end{aligned}$$

where $4 < r < +\infty$ is given by:

$$\frac{2}{r} = \frac{1}{p} - \frac{1}{2}.$$

Together with the finite band property and Bernstein for P_l, and using the Gagliardo-Nirenberg inequality (3.9), we obtain in the case $l \leq j$:

(C.3)

$$2^{jb}\|P_j(F\cdot G\cdot P_lH)\|_{L^2(P_u)} \lesssim 2^{j(b-2+\frac{2}{p})}\|F\|_{H^1(P_u)}\|G\|_{H^1(P_u)}2^l\|P_lH\|_{L^2(P_u)}$$

$$\lesssim 2^{-\max(j,l)(\frac{3}{2}-b-\frac{2}{p})}\|F\|_{H^1(P_u)}\|G\|_{H^1(P_u)}\|H\|_{H^{\frac{1}{2}}(P_u)},$$

where we used in the last inequality the fact that $l \leq j$ and the choice of p (C.2).

Next, we consider the case $l > j$. Since $0 < b < \frac{1}{2}$, there exists a real number q such that:

$$2 < q < \frac{2}{b+\frac{1}{2}}. \tag{C.4}$$

Then, let $4 < r < +\infty$ such that:

$$\frac{2}{r} + \frac{1}{q} = \frac{1}{2}.$$

Using the boundedness of P_j on $L^2(P_u)$, Bernstein for P_l, and the Gagliardo-Nirenberg inequality (3.9), we have:

(C.5)
$$\begin{aligned}
2^{jb}\|P_j(F \cdot G \cdot P_l H)\|_{L^2(P_u)} &\lesssim 2^{jb}\|F \cdot G \cdot P_l H\|_{L^2(P_u)}\\
&\lesssim 2^{jb}\|F\|_{L^r(P_u)}\|G\|_{L^r(P_u)}\|P_l H\|_{L^q(P_u)}\\
&\lesssim 2^{jb}\|F\|_{H^1(P_u)}\|G\|_{H^1(P_u)}2^{l(1-\frac{2}{q}}\|P_l H\|_{L^2(P_u)}\\
&\lesssim 2^{-\max(j,l)(\frac{2}{q}-\frac{1}{2}-b)}\|F\|_{H^1(P_u)}\|G\|_{H^1(P_u)}\|H\|_{H^{\frac{1}{2}}(P_u)},
\end{aligned}$$

where we used in the last inequality the fact that $l > j$ and the choice of q (C.4).

Let δ given by:

$$\delta = \min\left(\frac{3}{2} - b - \frac{2}{p}, \frac{2}{q} - \frac{1}{2} - b\right).$$

Then, we have $\delta > 0$ in view of (C.2) and (C.4). Now, in view of (C.1), (C.3) and (C.5), we have:

$$\begin{aligned}
\sum_{j\geq 0} 2^{2jb}\|P_j(G \cdot G \cdot H)\|^2_{L^2(P_u)} &\lesssim \|F\|^2_{H^1(P_u)}\|G\|^2_{H^1(P_u)}\|H\|^2_{H^{\frac{1}{2}}(P_u)}\sum_{j\geq 0}\left(\sum_{l\geq 0}2^{-\delta\max(l,j)}\right)^2\\
&\lesssim \|F\|^2_{H^1(P_u)}\|G\|^2_{H^1(P_u)}\|H\|^2_{H^{\frac{1}{2}}(P_u)},
\end{aligned}$$

since $\delta > 0$. This concludes the proof of the proposition.

C.2. Proof of Proposition 5.22

We have:

$$\|P_j(G \cdot H)\|_{L^2(P_u)} \lesssim \sum_{l,m\geq 0}\|P_j(P_l G \cdot P_m H)\|_{L^2(P_u)}. \tag{C.6}$$

By symmetry, we may assume:

$$l \leq m.$$

We first consider the case where $l \leq m \leq j$. Then, we have:

$$\|P_j(P_lG\cdot P_mH)\|_{L^2(P_u)} = 2^{-2j}\|P_j(\not\Delta(P_lG\cdot P_mH))\|_{L^2(P_u)}$$
$$= 2^{-2j}\|P_j(\mathrm{d}\not{\mathrm{i}}\mathrm{v}(\not\nabla(P_lG\cdot P_mH)))\|_{L^2(P_u)}.$$

Since $G\cdot H$ is a scalar, we may use (5.64), and we obtain:

$$\begin{aligned}\|P_j(P_lG\cdot P_mH)\|_{L^2(P_u)} &\\ \lesssim 2^{-2j}2^{\frac{4j}{3}}\|\not\nabla(P_lG\cdot P_mH)\|_{L^{\frac{3}{2}}(P_u)} &\\ \lesssim 2^{-\frac{2j}{3}}\|\not\nabla P_lG\|_{L^2(P_u)}\|P_mH\|_{L^6(P_u)}+2^{-\frac{2j}{3}}\|P_lG\|_{L^6(P_u)}\|\not\nabla P_mH\|_{L^2(P_u)} &\\ \lesssim 2^{-\frac{2j}{3}}(2^{l+\frac{2m}{3}}+2^{m+\frac{2l}{3}})\|P_lG\|_{L^2(P_u)}\|P_mH\|_{L^2(P_u)} &\\ \lesssim 2^{-\frac{|j-m|}{6}-\frac{|j-l|}{6}}2^{\frac{l}{2}}\|P_lG\|_{L^2(P_u)}2^{\frac{m}{2}}\|P_mH\|_{L^2(P_u)}, &\end{aligned}\tag{C.7}$$

where we used the finite band property and Bernstein for P_l and P_m, and the fact that $l \leq m \leq j$.

Next, we consider the case where $l \leq j < m$. Then, we use the boundedness of P_j on $L^2(P_u)$ which yields:

$$\begin{aligned}\|P_j(P_lG\cdot P_mH)\|_{L^2(P_u)} &\lesssim \|P_lG\cdot P_mH\|_{L^2(P_u)}\\ &\lesssim \|P_lG\|_{L^6(P_u)}\|P_mH\|_{L^3(P_u)}\\ &\lesssim 2^{\frac{2l}{3}}\|P_lG\|_{L^2(P_u)}2^{\frac{m}{3}}\|P_mH\|_{L^2(P_u)}\\ &\lesssim 2^{-\frac{|j-m|}{12}-\frac{|j-l|}{12}}2^{\frac{l}{2}}\|P_lG\|_{L^2(P_u)}2^{\frac{m}{2}}\|P_mH\|_{L^2(P_u)},\end{aligned}\tag{C.8}$$

where we used Bernstein for P_l and P_m, and the fact that $l \leq j < m$.

Finally, we consider the case where $j < l \leq m$. Then, we use Bernstein for P_j which yields:

$$\begin{aligned}\|P_j(P_lG\cdot P_mH)\|_{L^2(P_u)} &\lesssim 2^{\frac{j}{3}}\|P_lG\cdot P_mH\|_{L^{\frac{3}{2}}(P_u)}\\ &\lesssim 2^{\frac{j}{3}}\|P_lG\|_{L^3(P_u)}\|P_mH\|_{L^3(P_u)}\\ &\lesssim 2^{\frac{j}{3}}2^{\frac{l}{3}}\|P_lG\|_{L^2(P_u)}2^{\frac{m}{3}}\|P_mH\|_{L^2(P_u)}\\ &\lesssim 2^{-\frac{|j-m|}{6}-\frac{|j-l|}{6}}2^{\frac{l}{2}}\|P_lG\|_{L^2(P_u)}2^{\frac{m}{2}}\|P_mH\|_{L^2(P_u)},\end{aligned}\tag{C.9}$$

where we used Bernstein for P_l and P_m, and the fact that $j < l \leq m$.

Finally, we have in view of (C.6), (C.7), (C.8) and (C.9):

$$\begin{aligned}\sum_{j\geq 0}\|P_j(G\cdot H)\|^2_{L^2(P_u)} &\lesssim \sum_{j\geq 0}\left(\sum_{l,m\geq 0}\|P_j(P_lG\cdot P_mH)\|_{L^2(P_u)}\right)^2\\ &\lesssim \sum_{j\geq 0}\left(\sum_{l,m\geq 0}2^{-\frac{|j-m|}{12}-\frac{|j-l|}{12}}2^{\frac{l}{2}}\|P_lG\|_{L^2(P_u)}2^{\frac{m}{2}}\|P_mH\|_{L^2(P_u)}\right)^2\end{aligned}$$

$$\lesssim \left(\sum_{l\geq 0} 2^l\|P_l G\|^2_{L^2(P_u)}\right)\left(\sum_{m\geq 0} 2^m\|P_m H\|^2_{L^2(P_u)}\right)$$
$$\lesssim \|G\|^2_{H^{\frac{1}{2}}(P_u)}\|H\|^2_{H^{\frac{1}{2}}(P_u)}.$$

This yields (5.79) which concludes the proof of the proposition.

C.3. Proof of Proposition 5.23

We have:

$$\|P_j(fh)\|_{L^2(P_u)} \lesssim \sum_{l\geq 0}\|P_j(fP_lh)\|_{L^2(P_u)}. \tag{C.10}$$

If $l \leq j$, we use the boundedness of P_j on $L^2(P_u)$ to obtain:

$$\begin{aligned}2^{-\frac{j}{2}}\|P_j(fP_lh)\|_{L^2(P_u)} &\lesssim 2^{-\frac{j}{2}}\|fP_lh\|_{L^2(P_u)}\\ &\lesssim 2^{-\frac{j}{2}}\|f\|_{L^\infty(P_u)}\|P_lh\|_{L^2(P_u)}\\ &\lesssim 2^{-\frac{|j-l|}{2}}\|f\|_{L^\infty(P_u)}2^{-\frac{l}{2}}\|P_lh\|_{L^2(P_u)},\end{aligned} \tag{C.11}$$

where we used in the last inequality the fact that $l \leq j$.

If $l > j$, we use the following identity:

$$P_j(fP_lh) = 2^{-2l}P_j(fP_l\not{\Delta}h) = 2^{-2l}P_j(\not{\mathrm{div}}(f\not\nabla P_lh)) + 2^{-2l}P_j(\not\nabla f\cdot\not\nabla P_lh).$$

Together with the finite band property for P_j, the strong Bernstein inequality (5.61) for scalars, and the finite band property for P_l, we obtain:

$$\begin{aligned}2^{-\frac{j}{2}}\|P_j(fP_lh)\|_{L^2(P_u)} &\lesssim 2^{-\frac{j}{2}-2l}(\|P_j\not{\mathrm{div}}(f\not\nabla P_lh)\|_{L^2(P_u)} + \|P_j(\not\nabla f\cdot\not\nabla P_lh)\|_{L^2(P_u)})\\ &\lesssim 2^{\frac{j}{2}-2l}(\|f\not\nabla P_lh\|_{L^2(P_u)} + \|\not\nabla f\cdot\not\nabla P_lh\|_{L^1(P_u)})\\ &\lesssim 2^{\frac{j}{2}-2l}(\|f\|_{L^\infty(P_u)} + \|\not\nabla f\|_{L^2(P_u)})\|\not\nabla P_lh\|_{L^2(P_u)}\\ &\lesssim 2^{\frac{j}{2}-l}(\|f\|_{L^\infty(P_u)} + \|\not\nabla f\|_{L^2(P_u)})\|P_lh\|_{L^2(P_u)}\\ &\lesssim 2^{-\frac{|j-l|}{2}}(\|f\|_{L^\infty(P_u)} + \|\not\nabla f\|_{L^2(P_u)})2^{-\frac{l}{2}}\|P_lh\|_{L^2(P_u)},\end{aligned} \tag{C.12}$$

where we used in the last inequality the fact that $l > j$.

Finally, (C.10), (C.11) and (C.12) imply:

$$\begin{aligned}\sum_{j\geq 0}2^{-j}\|P_j(fh)\|^2_{L^2(P_u)} &\lesssim (\|f\|^2_{L^\infty(P_u)} + \|\not\nabla f\|^2_{L^2(P_u)})\sum_{j\geq 0}\left(2^{-\frac{|j-l|}{2}}2^{-\frac{l}{2}}\|P_lh\|_{L^2(P_u)}\right)^2\\ &\lesssim (\|f\|^2_{L^\infty(P_u)} + \|\not\nabla f\|^2_{L^2(P_u)})\sum_{l\geq 0}2^{-l}\|P_lh\|^2_{L^2(P_u)}\\ &\lesssim (\|f\|^2_{L^\infty(P_u)} + \|\not\nabla f\|^2_{L^2(P_u)})\|h\|^2_{H^{-\frac{1}{2}}(P_u)}.\end{aligned}$$

This concludes the proof of the proposition.

C.4. Proof of Proposition 5.24

We estimate $\|P_j(G\cdot P_lH)\|_{L^2(P_u)}$ starting with the case where $j\geq l$. Using the boundedness of P_j on $L^2(P_u)$ of P_j, we have:

$$\begin{aligned}\|P_j(G\cdot P_lH)\|_{L^2(S)} &\lesssim \|G\|_{L^\infty_{[-2,2]}L^4(P_u)}\|P_lH\|_{L^2_{[-2,2]}L^4(P_u)}\\ &\lesssim 2^{\frac{l}{2}}\|G\|_{H^1(S)}\|P_lH\|_{L^2(S)},\end{aligned}$$

where we used in the last inequality the Proposition 3.7 and the Bernstein inequality for P_l. This yields in the case $j\geq l$:

$$2^{-\frac{j}{2}}\|P_j(G\cdot P_lH)\|_{L^2(S)}\lesssim 2^{-\frac{|l-j|}{2}}\|G\|_{H^1(S)}\|P_lH\|_{L^2(S)}. \tag{C.13}$$

Next, we consider the case where $l>j$, and we estimate $\|P_j(P_mG\cdot P_lH)\|_{L^2(P_u)}$ starting with the case where $m\geq l$. Using the sharp Bernstein inequality (5.61), we have:

$$\|P_j(P_mG\cdot P_lH)\|_{L^2(S)}\lesssim 2^j\|P_mG\|_{L^\infty_{[-2,2]}L^2(P_u)}\|P_lH\|_{L^2(S)}. \tag{C.14}$$

Finally, we consider the case where $l>j$ and $l>m$. Using the finite band property for P_l, we have:

$$\begin{aligned}\|P_j(P_mG\cdot P_lH)\|_{L^2(S)} &\lesssim 2^{-2l}\|P_j(P_mG\cdot \Delta\!\!\!/ P_lH)\|_{L^2(S)}\\ &\lesssim 2^{-2l}\|P_j(\nabla\!\!\!/ P_mG\cdot\nabla\!\!\!/ P_lH)\|_{L^2(S)}\\ &\quad + 2^{-2l}\|P_j(\mathrm{d}\!\!\!/\mathrm{iv}(\nabla\!\!\!/ P_mG\cdot P_lH))\|_{L^2(S)}.\end{aligned}$$

Using the sharp Bernstein inequality (5.61) for the first term and the estimate (5.64) with $p=4/3$ for the second term, we obtain:

$$\begin{aligned}\|P_j(P_mG\cdot P_lH)\|_{L^2(S)} &\lesssim 2^{j-2l}\|\nabla\!\!\!/ P_mG\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\nabla\!\!\!/ P_lH\|_{L^2(S)}\\ &\quad + 2^{\frac{3j}{2}-2l}\|\nabla\!\!\!/ P_mG\|_{L^\infty_{[-2,2]}L^2(P_u)}\|P_lH\|_{L^2_{[-2,2]}L^4(P_u)}\\ &\lesssim (2^{j-l+m}+2^{\frac{3j}{2}-\frac{3l}{2}+m})\|P_mG\|_{L^\infty_{[-2,2]}L^2(P_u)}\|P_lH\|_{L^2(S)},\end{aligned} \tag{C.15}$$

where we used in the last inequality Bernstein for P_l and P_m. (C.14) and (C.15) yield in the case $l>j$:

$$2^{-\frac{j}{2}}\|P_j(P_mG\cdot P_lH)\|_{L^2(S)}\lesssim 2^{-\frac{|j-l|}{4}-\frac{|j-m|}{4}}(2^{\frac{m}{2}}\|P_mG\|_{L^\infty_{[-2,2]}L^2(P_u)})\|P_lH\|_{L^2(S)}. \tag{C.16}$$

Finally, (C.13) and (C.16) imply:

$$\begin{aligned}\sum_{j\geq 0}2^{-j}\|P_j(G\cdot H)\|^2_{L^2(S)} &\lesssim \left(\|G\|^2_{H^1(S)}+\left(\sum_{m\geq 0}2^m\|P_mG\|^2_{L^\infty_{[-2,2]}L^2(P_u)}\right)\right)\\ &\quad\times\left(\sum_{l\geq 0}\|P_lH\|^2_{L^2(S)}\right)\end{aligned}$$

$$\lesssim \|G\|^2_{H^1(S)}\|H\|^2_{L^2(S)},$$

where we used in the last inequality Corollary 5.16 for G and the Bessel inequality for H. This concludes the proof of the proposition.

C.5. Proof of Lemma 5.25

We have:

$$\|P_j(F\cdot G)\|_{L^2(P_u)} \lesssim \sum_{l,m\geq 0}\|P_j(P_lF\cdot P_mG)\|_{L^2(P_u)}. \tag{C.17}$$

If $j = \max(j,l,m)$, we use the boundedness of P_j on $L^2(P_u)$ and the Bernstein inequality for P_l and P_m to obtain:

$$\begin{aligned} 2^{-j}\|P_j(P_lF\cdot P_mG)\|_{L^2(P_u)} &\lesssim 2^{-j}\|P_lF\cdot P_mG\|_{L^2(P_u)}\\ &\lesssim 2^{-j}\|P_lF\|_{L^6(P_u)}\|P_mG\|_{L^3(P_u)}\\ &\lesssim 2^{-j}2^{\frac{2l}{3}}2^{\frac{m}{3}}\|P_lF\|_{L^2(P_u)}\|P_mG\|_{L^2(P_u)}\\ &\lesssim 2^{-\frac{|l-m|}{6}}2^{\frac{l}{2}}\|P_lF\|_{L^2(P_u)}2^{-\frac{m}{2}}\|P_mG\|_{L^2(P_u)}, \end{aligned} \tag{C.18}$$

where we used in the last inequality the fact that $j=\max(j,l,m)$.

If $l=\max(j,l,m)$, we use for P_j the strong Bernstein inequality for scalars (5.61) which yields:

$$\begin{aligned} 2^{-j}\|P_j(P_lF\cdot P_mG)\|_{L^2(P_u)} &\lesssim \|P_lF\cdot P_mG\|_{L^1(P_u)}\\ &\lesssim \|P_lF\|_{L^2(P_u)}\|P_mG\|_{L^2(P_u)}\\ &\lesssim 2^{-\frac{|l-m|}{2}}2^{\frac{l}{2}}\|P_lF\|_{L^2(P_u)}2^{-\frac{m}{2}}\|P_mG\|_{L^2(P_u)}, \end{aligned} \tag{C.19}$$

where we used in the last inequality the fact that $l=\max(j,l,m)$.

If $m=\max(j,l,m)$, we use the following identity:

$$\begin{aligned} P_j(P_lF\cdot P_mG) &= 2^{-2m}P_j(P_lF\cdot \not\Delta P_mG)\\ &= 2^{-2m}(P_j(\not\Delta(P_lF\cdot P_mG)) + P_j(\not\Delta(P_lF)\cdot P_mG)\\ &\quad + P_j(\not{\mathrm{div}}(\not\nabla(P_lF)\cdot P_mG)))\\ &= 2^{-2m}(2^{2j}P_j(P_lF\cdot P_mG) + 2^{2l}P_j(P_lF\cdot P_mG)\\ &\quad + P_j(\not{\mathrm{div}}(\not\nabla(P_lF)\cdot P_mG))). \end{aligned}$$

Together with the boundedness of P_j on $L^2(P_u)$, the strong Bernstein inequality for scalars (5.61), and the estimate (5.64), we obtain:

(C.20)

$$\begin{aligned} &2^{-j}\|P_j(P_lF\cdot P_mG)\|_{L^2(P_u)}\\ &\lesssim 2^{-j-2m}(2^{2j}\|P_lF\cdot P_mG\|_{L^2(P_u)} + 2^{2l}2^j\|P_lF\cdot P_mG\|_{L^1(P_u)} + 2^{\frac{2j}{3}}\|\not\nabla(P_lF)\cdot P_mG\|_{L^{\frac{4}{3}}(P_u)}\\ &\lesssim 2^{-j-2m}(2^{2j}\|P_lF\|_{L^6(P_u)}\|P_mG\|_{L^3(P_u)} + 2^{2l+j}\|P_lF\|_{L^2(P_u)}\|P_mG\|_{L^2(P_u)} \end{aligned}$$

$$+ 2^{\frac{3j}{2}}\|\nabla\!\!\!/(P_lF)\|_{L^2(P_u)}\|P_mG\|_{L^4(P_u)})$$
$$\lesssim 2^{-j-2m}(2^{2j+\frac{2l}{3}+\frac{m}{3}} + 2^{2l+j} + 2^{\frac{3j}{2}+l+\frac{m}{2}})\|P_lF\|_{L^2(P_u)}\|P_mG\|_{L^2(P_u)}$$
$$\lesssim 2^{-\frac{|l-m|}{6}}2^{\frac{l}{2}}\|P_lF\|_{L^2(P_u)}2^{-\frac{m}{2}}\|P_mG\|_{L^2(P_u)},$$

where we used Bernstein for P_m, the finite band property and Bernstein for P_l, and the fact that $m = \max(j, l, m)$.

Finally, (C.17), (C.18), (C.19) and (C.20) imply for all $j \geq 0$:

$$\begin{aligned}2^{-j}\|P_j(F\cdot G)\|_{L^2(P_u)} &\lesssim \sum_{l,m\geq 0} 2^{-\frac{|l-m|}{6}}2^{\frac{l}{2}}\|P_lF\|_{L^2(P_u)}2^{-\frac{m}{2}}\|P_mG\|_{L^2(P_u)}\\ &\lesssim \left(\sum_{l\geq 0}2^l\|P_lF\|^2_{L^2(P_u)}\right)^{\frac{1}{2}}\left(\sum_{m\geq 0}2^{-m}\|P_mG\|^2_{L^2(P_u)}\right)^{\frac{1}{2}}\\ &\lesssim \|F\|_{H^{\frac{1}{2}}(P_u)}\|G\|_{H^{-\frac{1}{2}}(P_u)}.\end{aligned}$$

This concludes the proof of the lemma.

C.6. Proof of Lemma 5.26

We have:

$$\|P_j(\mathrm{di\!\!\!/v}(fG)\|_{L^2(P_u)} \lesssim \sum_{l\geq 0}\|P_j(\mathrm{di\!\!\!/v}(P_l(f)G))\|_{L^2(P_u)}. \tag{C.21}$$

If $l \leq j$, we used the boundedness of P_j on $L^2(P_u)$, and the strong Bernstein inequality for scalars (5.61) and the finite band property for P_l. We obtain:

$$\begin{aligned}&2^{j(b-1)}\|P_j(\mathrm{di\!\!\!/v}(P_l(f)G))\|_{L^2(P_u)}\\ &\lesssim 2^{j(b-1)}\|\mathrm{di\!\!\!/v}(P_l(f)G)\|_{L^2(P_u)}\\ &\lesssim 2^{j(b-1)}(\|\nabla\!\!\!/(P_lf)G\|_{L^2(P_u)} + \|P_lf\nabla\!\!\!/G\|_{L^2(P_u)})\\ &\lesssim 2^{j(b-1)}(\|\nabla\!\!\!/(P_lf)\|_{L^2(P_u)}\|G\|_{L^\infty(P_u)} + \|P_lf\|_{L^\infty(P_u)}\|\nabla\!\!\!/G\|_{L^2(P_u)})\\ &\lesssim 2^{j(b-1)}2^l(\|G\|_{L^\infty(P_u)} + \|\nabla\!\!\!/G\|_{L^2(P_u)})\|P_lf\|_{L^2(P_u)}\\ &\lesssim 2^{-|j-l|(1-b)}(\|G\|_{L^\infty(P_u)} + \|\nabla\!\!\!/G\|_{L^2(P_u)})2^{lb}\|P_lf\|_{L^2(P_u)},\end{aligned} \tag{C.22}$$

where we used in the last inequality the fact that $l \leq j$ and $b < 1$.

If $l > j$, we use the following identity:

$$\begin{aligned}P_j(\mathrm{di\!\!\!/v}(P_l(f)G)) &= 2^{-2l}P_j(\mathrm{di\!\!\!/v}(\Delta\!\!\!/P_l(f)G))\\ &= 2^{-2l}(P_j(\mathrm{di\!\!\!/v}(\mathrm{di\!\!\!/v}(\nabla\!\!\!/P_l(f)G))) + P_j(\mathrm{di\!\!\!/v}(\nabla\!\!\!/P_l(f)\cdot\nabla\!\!\!/G))).\end{aligned}$$

Together with the estimate (5.64) for P_j, we obtain:

$$\begin{aligned}&2^{j(b-1)}\|P_j(\mathrm{di\!\!\!/v}(P_l(f)G))\|_{L^2(P_u)}\\ &\lesssim 2^{j(b-1)-2l}(\|P_j(\mathrm{di\!\!\!/v}(\mathrm{di\!\!\!/v}(\nabla\!\!\!/P_l(f)G)))\|_{L^2(P_u)} + \|P_j(\mathrm{di\!\!\!/v}(\nabla\!\!\!/P_l(f)\cdot\nabla\!\!\!/G))\|_{L^2(P_u)})\end{aligned}$$

$$\lesssim 2^{j(b-1)-2l}(\|P_j \mathrm{d}\!\!/\mathrm{iv}\, \mathrm{d}\!\!/\mathrm{iv}\|_{\mathcal{L}(L^2(P_u))}\|\nabla\!\!\!/ P_l(f)G\|_{L^2(P_u)} + 2^{\frac{2j}{p}}\|\nabla\!\!\!/ P_l(f)\cdot\nabla\!\!\!/ G\|_{L^2(P_u)}),$$

where p satisfies:

$$1 < p < \frac{2}{1-b}, \tag{C.23}$$

which is possible since $-1 < b < 1$. Together with the Bochner inequality for scalars (5.63) and the finite band property for P_j, this yields:

$$\begin{aligned} &2^{j(b-1)}\|P_j(\mathrm{d}\!\!/\mathrm{iv}(P_l(f)G))\|_{L^2(P_u)} \\ &\quad\lesssim 2^{j(b-1)-2l}(2^{2j}\|\nabla\!\!\!/ P_l f\|_{L^2(P_u)}\|G\|_{L^\infty(P_u)} + 2^{\frac{2j}{p}}\|\nabla\!\!\!/ P_l f\|_{L^r(P_u)}\|\nabla\!\!\!/ G\|_{L^2(P_u)}),\end{aligned}$$

where r is given by:

$$\frac{1}{r}+\frac{1}{2}=\frac{1}{p}.$$

Together with the Gagliardo-Nirenberg inequality (3.9), the Bochner inequality for scalars (5.63), and the finite band property for P_l, we obtain:

(C.24)

$$\begin{aligned} &2^{j(b-1)}\|P_j(\mathrm{d}\!\!/\mathrm{iv}(P_l(f)G))\|_{L^2(P_u)} \\ &\quad\lesssim 2^{j(b-1)-2l}(2^{2j}2^l + 2^{\frac{2j}{p}}2^{l(2-\frac{2}{r})})(\|G\|_{L^\infty(P_u)} + \|\nabla\!\!\!/ G\|_{L^2(P_u)})\|P_l f\|_{L^2(P_u)} \\ &\quad\lesssim (2^{-|j-l|(1+b)} + 2^{-|j-l|(b-1+\frac{2}{p})})(\|G\|_{L^\infty(P_u)} + \|\nabla\!\!\!/ G\|_{L^2(P_u)})2^{lb}\|P_l f\|_{L^2(P_u)},\end{aligned}$$

where we used in the last inequality (C.23), the fact that $b+1>0$, and the fact that $l>j$.

Finally, (C.21), (C.23) and (C.24) imply:

$$\begin{aligned} &\sum_{j\geq 0} 2^{2(b-1)j}\|P_j(\mathrm{d}\!\!/\mathrm{iv}(P_l(f)G))\|^2_{L^2(P_u)} \\ &\lesssim \sum_{j\geq 0}\left(\sum_{l\geq 0} 2^{-|j-l|\min(1-b,1+b,b-1+\frac{2}{p})}(\|G\|_{L^\infty(P_u)} + \|\nabla\!\!\!/ G\|_{L^2(P_u)})2^{lb}\|P_l f\|_{L^2(P_u)}\right)^2 \\ &\lesssim (\|G\|^2_{L^\infty(P_u)} + \|\nabla\!\!\!/ G\|^2_{L^2(P_u)})\sum_{l\geq 0} 2^{2lb}\|P_l f\|^2_{L^2(P_u)} \\ &\lesssim (\|G\|^2_{L^\infty(P_u)} + \|\nabla\!\!\!/ G\|^2_{L^2(P_u)})\|f\|^2_{H^b(P_u)}.\end{aligned}$$

This concludes the proof of the lemma.

C.7. Proof of Lemma 5.27

We have:

$$\|P_j(\mathrm{d}\!\!/\mathrm{iv}(fG)\|_{L^2(S)} \lesssim \sum_{l\geq 0}\|P_j(\mathrm{d}\!\!/\mathrm{iv}(P_l(f)G))\|_{L^2(S)}. \tag{C.25}$$

If $l \leq j$, we use the finite band property for P_j to obtain:

$$\begin{aligned}
&2^{j(b-1)}\|P_j(\mathrm{d}\not{i}\mathrm{v}(P_l(f)G))\|_{L^2(S)}\\
&\lesssim 2^{j(b-2)}\|\nabla\!\!\!/\mathrm{d}\not{i}\mathrm{v}(P_l(f)G)\|_{L^2(S)}\\
&\lesssim 2^{j(b-2)}(\|\nabla\!\!\!/^2(P_l(f))G\|_{L^2(S)} + \|\nabla\!\!\!/(P_l(f))\mathrm{d}\not{i}\mathrm{v}(G)\|_{L^2(S)} + \|P_l(f)\nabla\!\!\!/\mathrm{d}\not{i}\mathrm{v}(G)\|_{L^2(S)}\\
&\lesssim 2^{j(b-2)}(\|\nabla\!\!\!/^2(P_l(f))\|_{L^2(S)}\|G\|_{L^\infty(S)} + \|\nabla\!\!\!/(P_l(f))\|_{L^2_{[-2,2]}L^p(P_u)}\|\mathrm{d}\not{i}\mathrm{v}(G)\|_{L^2_{[-2,2]}L^q(P_u)}\\
&\quad + \|P_l(f)\|_{L^\infty(S)}\|\nabla\!\!\!/^2(G)\|_{L^2(S)}),
\end{aligned}$$

where p and q are such that:

$$\frac{2}{p} + \frac{2}{q} = \frac{1}{2},\ 2 < q < p < +\infty.$$

Together with the Bochner inequality for scalars (5.63), the Gagliardo-Nirenberg inequality (3.9), the finite band property for P_l, and the strong Bernstein inequality for scalars (5.61), we obtain:

$$\begin{aligned}
&2^{j(b-1)}\|P_j(\mathrm{d}\not{i}\mathrm{v}(P_l(f)G))\|_{L^2(S)}\\
&\quad\lesssim 2^{j(b-2)}(2^{2l}(\|G\|_{L^\infty(S)} + \|\mathrm{d}\not{i}\mathrm{v}(G)\|_{L^2_{[-2,2]}L^q(P_u)})\|P_l f\|_{L^2(S)}\\
&\qquad + 2^l\|\nabla\!\!\!/^2(G)\|_{L^2(S)}\|P_l(f)\|_{L^\infty_{[-2,2]}L^2(P_u)})\\
&\quad\lesssim 2^{-|j-l|(2-b)}(\|G\|_{L^\infty(S)} + \|\mathrm{d}\not{i}\mathrm{v}(G)\|_{L^2_{[-2,2]}L^q(P_u)} + \|\nabla\!\!\!/^2(G)\|_{L^2(S)})\\
&\qquad \times (2^{lb}\|P_l f\|_{L^2(S)} + 2^{l(b-1)}\|P_l(f)\|_{L^\infty_{[-2,2]}L^2(P_u)}),
\end{aligned}$$

where we used in the last inequality the fact that $l \leq j$ and $b < 2$. Since this holds for any $q > 2$, we finally obtain:

$$\begin{aligned}
&2^{j(b-1)}\|P_j(\mathrm{d}\not{i}\mathrm{v}(P_l(f)G))\|_{L^2(S)}\\
&\quad\lesssim 2^{-|j-l|(2-b)}(\|G\|_{L^\infty(S)} + \|\mathrm{d}\not{i}\mathrm{v}(G)\|_{L^2_{[-2,2]}L^{2+}(P_u)} + \|\nabla\!\!\!/^2(G)\|_{L^2(S)})\\
&\qquad \times (2^{lb}\|P_l f\|_{L^2(S)} + 2^{l(b-1)}\|P_l(f)\|_{L^\infty_{[-2,2]}L^2(P_u)}).
\end{aligned} \tag{C.26}$$

If $l > j$, the finite band property for P_j yields:

$$\begin{aligned}
2^{j(b-1)}\|P_j(\mathrm{d}\not{i}\mathrm{v}(P_l(f)G))\|_{L^2(S)} &\lesssim 2^{jb}\|P_l(f)G\|_{L^2(S)}\\
&\lesssim 2^{jb}\|G\|_{L^\infty(S)}\|P_l(f)\|_{L^2(S)}\\
&\lesssim 2^{-|j-l|b}\|G\|_{L^\infty(S)}2^{lb}\|P_l(f)\|_{L^2(S)},
\end{aligned} \tag{C.27}$$

where we used in the last inequality the fact that $l > j$ and $b > 0$.

Finally, (C.25), (C.26) and (C.27) imply:

$$\begin{aligned}
&\sum_{j\geq 0} 2^{2(b-1)j}\|P_j(\mathrm{d}\not{i}\mathrm{v}(fG))\|^2_{L^2(S)}\\
&\quad\lesssim (\|G\|^2_{L^\infty(S)} + \|\mathrm{d}\not{i}\mathrm{v}(G)\|^2_{L^2_{[-2,2]}L^{2+}(P_u)} + \|\nabla\!\!\!/^2(G)\|^2_{L^2(S)})
\end{aligned}$$

$$\times \sum_{j\geq 0}\left(\sum_{l\geq 0} 2^{-|l-j|\min(2-b,b)}(2^{lb}\|P_l(f)\|_{L^2(S)} + 2^{l(b-1)}\|P_l(f)\|_{L^\infty_{[-2,2]}L^2(P_u)})\right)^2$$
$$\lesssim (\|G\|^2_{L^\infty(S)} + \|\mathrm{d}\!\!/\mathrm{iv}(G)\|^2_{L^2_{[-2,2]}L^{2+}(P_u)} + \|\nabla\!\!\!/^2(G)\|^2_{L^2(S)})$$
$$\times \sum_{l\geq 0}(2^{2lb}\|P_l(f)\|^2_{L^2(S)} + 2^{2l(b-1)}\|P_l(f)\|^2_{L^\infty_{[-2,2]}L^2(P_u)})$$
$$\lesssim (\|G\|^2_{L^\infty(S)} + \|\mathrm{d}\!\!/\mathrm{iv}(G)\|^2_{L^2_{[-2,2]}L^{2+}(P_u)} + \|\nabla\!\!\!/^2(G)\|^2_{L^2(S)})$$
$$\times (\|f\|^2_{L^2_u H^b(P_u)} + \|f\|_{L^\infty_u H^{b-1}(P_u)}).$$

This concludes the proof of the lemma.

C.8. Proof of Lemma 5.28

We have:

$$\|P_j(Fh)\|_{L^2(P_u)} \lesssim \sum_{l\geq 0}\|P_j(FP_l(h))\|_{L^2(P_u)}. \tag{C.28}$$

If $l\leq j$, we use the finite band property for P_j, and the strong Bernstein inequality for scalars (5.61) and the finite band property for P_l, which yields:

(C.29)
$$\begin{aligned}2^{jb}\|P_j(FP_l(h))\|_{L^2(P_u)} &\lesssim 2^{j(b-1)}\|\nabla\!\!\!/(FP_l(h))\|_{L^2(P_u)}\\ &\lesssim 2^{j(b-1)}(\|\nabla\!\!\!/ F\|_{L^2(P_u)}\|P_l(h)\|_{L^\infty(P_u)} + \|F\|_{L^\infty(P_u)}\|\nabla\!\!\!/ P_l(h)\|_{L^2(P_u)})\\ &\lesssim 2^{j(b-1)}(\|\nabla\!\!\!/ F\|_{L^2(P_u)}2^l\|P_l(h)\|_{L^2(P_u)} + \|F\|_{L^\infty(P_u)}2^l\|P_l(h)\|_{L^2(P_u)})\\ &\lesssim 2^{-|j-l|(1-b)}(\|\nabla\!\!\!/ F\|_{L^2(P_u)} + \|F\|_{L^\infty(P_u)})2^{lb}\|P_l(h)\|_{L^2(P_u)},\end{aligned}$$

where we used in the last inequality the fact that $l\leq j$ and $b<1$.

If $l>j$, we use the boundedness of P_j on $L^2(P_u)$ which yields:

$$\begin{aligned}2^{jb}\|P_j(FP_l(h))\|_{L^2(P_u)} &\lesssim 2^{jb}\|FP_l(h)\|_{L^2(P_u)}\\ &\lesssim 2^{jb}\|F\|_{L^\infty(P_u)}\|P_l(h)\|_{L^2(P_u)})\\ &\lesssim 2^{-|j-l|b}\|F\|_{L^\infty(P_u)}2^{lb}\|P_l(h)\|_{L^2(P_u)},\end{aligned} \tag{C.30}$$

where we used in the last inequality the fact that $l>j$ and $b>0$.

Finally, (C.28), (C.29) and (C.30) imply:

$$\sum_{j\geq 0}2^{2jb}\|P_j(FP_l(h))\|^2_{L^2(P_u)}$$
$$\lesssim (\|F\|^2_{L^\infty(P_u)} + \|\nabla\!\!\!/ F\|^2_{L^2(P_u)})\sum_{j\geq 0}\left(\sum_{l\geq 0}2^{-|j-l|b}2^{lb}\|P_l(h)\|_{L^2(P_u)}\right)^2$$

$$
\begin{aligned}
&\lesssim (\|F\|^2_{L^\infty(P_u)} + \|\nabla\!\!\!/ F\|^2_{L^2(P_u)}) \sum_{l\geq 0} 2^{2lb}\|P_l(h)\|^2_{L^2(P_u)} \\
&\lesssim (\|F\|^2_{L^\infty(P_u)} + \|\nabla\!\!\!/ F\|^2_{L^2(P_u)})\|h\|^2_{H^b(P_u)}.
\end{aligned}
$$

This concludes the proof of the lemma.

PART II

CONSTRUCTION AND CONTROL AT INITIAL TIME

CHAPTER 10

INTRODUCTION

We consider the Einstein vacuum equations,

$$\mathbf{R}_{\alpha\beta} = 0, \tag{10.1}$$

where $\mathbf{R}_{\alpha\beta}$ denotes the Ricci curvature tensor of a four dimensional Lorentzian space time $(\mathscr{M}, \mathbf{g})$. The Cauchy problem consists in finding a metric $\mathbf{g}$ satisfying (10.1) such that the metric induced by $\mathbf{g}$ on a given space-like hypersurface Σ_0 and the second fundamental form of Σ_0 are prescribed. The initial data then consists of a Riemannian three dimensional metric g_{ij} and a symmetric tensor k_{ij} on the space-like hypersurface $\Sigma_0 = \{t = 0\}$. Now, (10.1) is an overdetermined system and the initial data set (Σ_0, g, k) must satisfy the constraint equations

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \mathrm{Tr} k = 0, \\ R - |k|^2 + (\mathrm{Tr} k)^2 = 0, \end{cases} \tag{10.2}$$

where the covariant derivative ∇ is defined with respect to the metric g, R is the scalar curvature of g, and $\mathrm{Tr}k$ is the trace of k with respect to the metric g.

The fundamental problem in general relativity is to study the long term regularity and asymptotic properties of the Cauchy developments of general, asymptotically flat, initial data sets (Σ_0, g, k). As far as local regularity is concerned it is natural to ask what are the minimal regularity properties of the initial data which guarantee the existence and uniqueness of local developments. In **[17]**, we obtain the following result which solves bounded L^2 curvature conjecture proposed in **[10]**:

***Theorem 10.1* (Theorem 1.10 in [17]).** — *Let* $(\mathscr{M}, \mathbf{g})$ *an asymptotically flat solution to the Einstein vacuum Equations* (10.1) *together with a maximal foliation by space-like hypersurfaces* Σ_t *defined as level hypersurfaces of a time function* t. *Let* $r_{\mathrm{vol}}(\Sigma_t, 1)$ *the volume radius on scales* ≤ 1 *of* Σ_t [1]. *Assume that the initial slice* (Σ_0, g, k) *is such that:*

$$\|R\|_{L^2(\Sigma_0)} \leq \varepsilon,\ \|k\|_{L^2(\Sigma_0)} + \|\nabla k\|_{L^2(\Sigma_0)} \leq \varepsilon \ \textit{and}\ r_{\mathrm{vol}}(\Sigma_0, 1) \geq \frac{1}{2}.$$

1. See Remark 10.5 below for a definition.

Then, there exists a small universal constant $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then the following control holds on $0 \leq t \leq 1$:

$$\|\mathbf{R}\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \varepsilon,\ \|k\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} + \|\nabla k\|_{L^\infty_{[0,1]}L^2(\Sigma_t)} \lesssim \varepsilon \text{ and } \inf_{0\leq t\leq 1} r_{\mathrm{vol}}(\Sigma_t, 1) \geq \frac{1}{4}.$$

Remark 10.2. — While the first nontrivial improvements for well posedness for quasilinear hyperbolic systems (in spacetime dimensions greater than $1+1$), based on Strichartz estimates, were obtained in **[2]**, **[1]**, **[27]**, **[28]**, **[11]**, **[14]**, **[20]**, Theorem 10.1, is the first result in which the full nonlinear structure of the quasilinear system, not just its principal part, plays a crucial role. We note that though the result is not optimal with respect to the standard scaling of the Einstein equations, it is nevertheless critical with respect to its causal geometry, i.e., L^2 bounds on the curvature is the minimum requirement necessary to obtain lower bounds on the radius of injectivity of null hypersurfaces. We refer the reader to Section 1 in **[17]** for more motivations and historical perspectives concerning Theorem 10.1.

Remark 10.3. — The regularity assumptions on Σ_0 in Theorem 10.1—i.e., R and ∇k bounded in $L^2(\Sigma_0)$—correspond to an initial data set $(g, k) \in H^2_{\mathrm{loc}}(\Sigma_0) \times H^1_{\mathrm{loc}}(\Sigma_0)$.

Remark 10.4. — In **[17]**, our main result is stated for corresponding large data. We then reduce the proof to the small data statement of Theorem 10.1 relying on a truncation and rescaling procedure, the control of the harmonic radius of Σ_0 based on Cheeger-Gromov convergence of Riemannian manifolds together with the assumption on the lower bound of the volume radius of Σ_0, and the gluing procedure in **[7]**, **[6]**. We refer the reader to Section 2.3 in **[17]** for the details.

Remark 10.5. — We recall for the convenience of the reader the definition of the volume radius of the Riemannian manifold Σ_t. Let $B_r(p)$ denote the geodesic ball of center p and radius r. The volume radius $r_{\mathrm{vol}}(p, r)$ at a point $p \in \Sigma_t$ and scales $\leq r$ is defined by

$$r_{\mathrm{vol}}(p, r) = \inf_{r' \leq r} \frac{|B_{r'}(p)|}{r^3},$$

with $|B_r|$ the volume of B_r relative to the metric g_t on Σ_t. The volume radius $r_{\mathrm{vol}}(\Sigma_t, r)$ of Σ_t on scales $\leq r$ is the infimum of $r_{\mathrm{vol}}(p, r)$ over all points $p \in \Sigma_t$.

The proof of Theorem 10.1, obtained in the sequence of papers **[17]**, **[23]**, **[24]**, **[25]**, **[26]**, **[22]**, relies on the following ingredients [(2)]:

A *Provide a system of coordinates relative to which* (10.1) *exhibits a null structure.*
B *Prove appropriate bilinear estimates for solutions to $\square_{\mathbf{g}}\phi = 0$, on a fixed Einstein vacuum background* [(3)].

2. We also need trilinear estimates and an $L^4(\mathcal{M})$ Strichartz estimate (see the introduction in **[17]**).
3. Note that the first bilinear estimate of this type was obtained in **[12]**.

C *Construct a parametrix for solutions to the homogeneous wave equations $\square_{\mathbf{g}}\phi = 0$ on a fixed Einstein vacuum background, and obtain control of the parametrix and of its error term only using the fact that the curvature tensor is bounded in L^2.*

Steps **A** and **B** are carried out in **[17]**. In particular, the proof of the bilinear estimates rests on a representation formula for the solutions of the wave equation using the following plane wave parametrix:

$$Sf(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega,\ (t,x) \in \mathscr{M}, \tag{10.3}$$

where $u(.,.,\omega)$ is a solution to the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on $\mathscr{M}$ such that $u(0,x,\omega) \sim x.\omega$ when $|x| \to +\infty$ on Σ_0.

Remark 10.6. — Actually, (10.3) only corresponds to a half wave parametrix. The full parametrix will be derived in Section 11.1.

Remark 10.7. — The asymptotic behavior for $u(0,x,\omega)$ when $|x| \to +\infty$ will be important to generate arbitrary initial data for the wave equation (see (11.21)).

Remark 10.8. — Note that the parametrix (10.3) is invariantly defined [(4)], i.e., without reference to any coordinate system. This is crucial since coordinate systems consistent with L^2 bounds on the curvature would not be regular enough to control a parametrix.

In order to complete the proof of the bounded L^2 curvature conjecture, we need to carry out step **C** with the parametrix defined in (10.3).

Remark 10.9. — In addition to their relevance to the resolution of the bounded L^2 curvature conjecture, the methods and results of step **C** are also of independent interest. Indeed, they deal on the one hand with the control of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ at a critical level [(5)], and on the other hand with the derivation of L^2 bounds for Fourier integral operators with significantly lower differentiability assumptions both for the corresponding phase and symbol compared to classical methods (see for example **[21]** and references therein).

In view of the energy estimates for the wave equation, it suffices to control the parametrix at $t = 0$ (i.e., restricted to Σ_0)

$$Sf(0,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(0,x,\omega)} f(\lambda\omega)\lambda^2 d\lambda d\omega,\ x \in \Sigma_0 \tag{10.4}$$

4. Our choice is reminiscent of the one used in **[20]** in the context of $H^{2+\epsilon}$ solutions of quasilinear wave equations. Note however that the construction in that paper is coordinate dependent.

5. We need at least L^2 bounds on the curvature to obtain a lower bound on the radius of injectivity of the null level hypersurfaces of the solution u of the eikonal equation, which in turn is necessary to control the local regularity of u (see **[25]**).

and the error term
(10.5)

$$Ef(t,x) = \square_{\mathbf{g}} Sf(t,x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(t,x,\omega)}\square_{\mathbf{g}} u(t,x,\omega) f(\lambda\omega)\lambda^3 d\lambda d\omega,\ (t,x)\in\mathcal{M}.$$

This requires the following ingredients, the two first being related to the control of the parametrix restricted to Σ_0 (10.4), and the two others being related to the control of the error term (10.5):

- **C1** *Make an appropriate choice for the equation satisfied by $u(0,x,\omega)$ on Σ_0, and control the geometry of the foliation generated by the level surfaces of $u(0,x,\omega)$ on Σ_0.*
- **C2** *Prove that the parametrix at $t=0$ given by* (10.4) *is bounded in $\mathcal{L}(L^2(\mathbb{R}^3), L^2(S))$ using the estimates for $u(0,x,\omega)$ obtained in* **C1**.
- **C3** *Control the geometry of the foliation generated by the level hypersurfaces of u on $\mathcal{M}$.*
- **C4** *Prove that the error term* (10.5) *satisfies the estimate $\|Ef\|_{L^2(\mathcal{M})}\leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}$ using the estimates for u and $\square_{\mathbf{g}}u$ proved in* **C3**.

Concerning step **C1**, let us note that the typical choice $u(0,x,\omega)=x\cdot\omega$ in a given coordinate system would not work for us, since we don't have enough control on the regularity of a given coordinate system within our framework (6). Instead, in [23], we rely on a geometric definition for $u(0,x,\omega)$ to achieve step **C1**. In the present paper, we focus on step **C2**.

Note that the parametrix at $t=0$ given by (10.4) is a Fourier integral operator (FIO) with phase $u(0,x,\omega)$. Now, we only assume $R\in L^2(\Sigma_0)$ and $\nabla k\in L^2(\Sigma_0)$ in order to be consistent with the statement of Theorem 10.1. This severely limits the regularity we are able to obtain in step **C1** for $u(0,x,\omega)$ (see [23] and Section 11.2). Although R and k do not depend on the parameter ω, the regularity in ω we are able to obtain in step **C1** for $u(0,x,\omega)$ is very limited as well (7). In particular, we obtain for the phase $u(0,x,\omega)$ of $Sf(x,0)$ in (10.4) (8):

$$(10.6)\quad \sup_\omega\left(\|\nabla^3 u\|_{L^2(\Sigma_0)}+\|\nabla\partial_\omega u\|_{L^\infty(\Sigma_0)}+\|\nabla^2\partial_\omega u\|_{L^2(\Sigma_0)}+\|\partial_\omega^3 u\|_{L^\infty_{\mathrm{loc}}(\Sigma_0)}\right)\lesssim\varepsilon.$$

Let us note that the classical arguments for proving L^2 bounds for FIO are based either on a TT^* argument, or a T^*T argument, which requires in our setting (9) taking at least 4 derivatives of the phase in $L^\infty(\Sigma_0\times\mathbb{S}^2)$ either with respect to x for T^*T, or

6. This issue appears because we are working at the level of H^2 solutions for Einstein equations. In particular, the choice $u(0,x,\omega)=x\cdot\omega$ in a given coordinate system is used in [20] in the context of $H^{2+\epsilon}$ solutions for quasilinear wave equations.

7. This is due to the fact that our estimates are better in directions tangent to the u-foliation on Σ_0. Now, after differentiation with respect to ω, derivatives in tangential directions pick up a nonzero component along the normal direction to the u-foliation on Σ_0 (see [23] for details).

8. Actually, we have weaker bounds for the estimates where all the spatial derivatives are taken in the direction normal to the u-foliation on Σ_0 (see Section 11.2).

9. Since Σ_0 is 3-dimensional.

with respect to (λ, ω) for TT^* (see for example [**21**]). Both methods would fail by a large margin, in particular in view of the regularity (10.6) obtained for the phase of the parametrix at initial time $Sf(x, 0)$. In order to obtain the control required in step **C2** with the regularity of the phase of the FIO $Sf(x, 0)$ given by (10.6), we are forced to design a method which allows us to take advantage both of the regularity in x and ω. This is achieved using in particular the following ingredients:

— geometric integrations by parts taking full advantage of the better regularity properties in directions tangent to the level surfaces of $u(0, x, \omega)$ (10),
— the standard first and second dyadic decomposition in frequency and angle (see [**21**]), as well as another decomposition involving frequency and angle,
— after localization in frequency and angle, an estimate for the diagonal term using the TT^* argument and a change of variable tied to $u(0, x, \omega)$.

The rest of the paper is as follows. In Chapter 2, we present the full parametrix for solutions to the homogeneous wave equation $\square_{\mathbf{g}}\phi = 0$, we recall the regularity for the phase $u(0, x, \omega)$ obtained in [**23**], and we state our main results. In Chapter 3, we prove the boundedness on L^2 of a pseudodifferential operator acting on $\mathbb{R}^3$ with a rough symbol introducing the main ideas in a simple setting. In Chapter 4, we prove the boundedness on L^2 of a Fourier integral operator acting on Σ_0 with phase $u(0, x, \omega)$ and a symbol having limited regularity consistent with the one given by our parametrix. Finally, we use the results of Chapter 4 to show the existence and to control our parametrix in Chapter 5.

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10. Let us repeat that we actually obtain a weaker bound than (10.6) for the estimates where all the spatial derivatives are taken in the direction normal to the u-foliation on Σ_0 (see Section 11.2).

CHAPTER 11

MAIN RESULTS

From now on, there will be no further reference to Σ_t for $t > 0$. Since there is no confusion, we will denote Σ_0 simply by Σ in the rest of the paper.

11.1. Presentation of the parametrix

In this section, we construct a parametrix for the following homogeneous wave equation:

$$\begin{cases} \square_{\mathbf{g}}\phi = 0 \text{ on } \mathscr{M}, \\ \phi\big|_{\Sigma} = \phi_0,\ T(\phi)\big|_{\Sigma} = \phi_1, \end{cases} \tag{11.1}$$

where ϕ_0 and ϕ_1 are two given functions on Σ and T is the future oriented unit normal to Σ in $\mathscr{M}$.

We recall the plane wave representation of the solution of the flat wave equation. This corresponds to the case where $\mathbf{g}$ is the Minkowski metric. (11.1) becomes:

$$\begin{cases} \square\phi = 0 \text{ on } \mathbb{R}^{1+3}, \\ \phi(0,.) = \phi_0,\ \partial_t\phi(0,.) = \phi_1 \text{ on } \mathbb{R}^3. \end{cases} \tag{11.2}$$

The plane wave representation of the solution ϕ of (11.2) is given by:

$$\begin{aligned} &\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i(-t+x\cdot\omega)\lambda}\frac{1}{2}\left(\mathscr{F}\phi_0(\lambda\omega) + i\frac{\mathscr{F}\phi_1(\lambda\omega)}{\lambda}\right)d\lambda d\omega \\ &+\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i(t+x\cdot\omega)\lambda}\frac{1}{2}\left(\mathscr{F}\phi_0(\lambda\omega) - i\frac{\mathscr{F}\phi_1(\lambda\omega)}{\lambda}\right)d\lambda d\omega, \end{aligned} \tag{11.3}$$

where $\mathscr{F}$ denotes the Fourier transform on $\mathbb{R}^3$.

We would like to construct a parametrix in the curved case similar to (11.3). We introduce two solutions $u_\pm$ of the eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u_\pm \partial_\beta u_\pm = 0 \text{ on } \mathscr{M}, \tag{11.4}$$

such that:

$$T(u_\pm) = \mp|\nabla u_\pm| = \mp a_\pm^{-1} \text{ on } \Sigma, \tag{11.5}$$

where T is the future oriented unit normal to Σ in the space-time $\mathcal{M}$, ∇ is the gradient on Σ associated to the metric g, $|\cdot|$ is the length associated to g for vector fields on Σ, and $a_\pm$ is the lapse of $u_\pm$ on Σ. We look for a parametrix for (11.1) of the form:
(11.6)

$$S_+f_+(t,x)+S_-f_-(t,x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u_+(t,x,\omega)}f_+(\lambda\omega)\lambda^2d\lambda d\omega$$
$$+\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u_-(t,x,\omega)}f_-(\lambda\omega)\lambda^2d\lambda d\omega,\ (t,x)\in\mathcal{M}.$$

Thanks to (11.4), this parametrix generates the following error term:
(11.7)

$$E_+f_+(t,x)+E_-f_-(t,x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u_+(t,x,\omega)}\square_g u_+(t,x,\omega)f_+(\lambda\omega)\lambda^3d\lambda d\omega$$
$$+\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u_-(t,x,\omega)}\square_g u_-(t,x,\omega)f_-(\lambda\omega)\lambda^3d\lambda d\omega,\ (t,x)\in\mathcal{M}.$$

In the next two sections, we precise the parametrix (11.6) by prescribing $u_\pm$ on Σ and by making our choice for $f_\pm$ explicit.

11.1.1. Prescription of u_+ and u_- on Σ. — (11.4) and (11.5) are not enough to define $u_\pm$ in a unique manner. Indeed, we still need to prescribe $u_\pm$ on Σ. To motivate our choice, we need to introduce some geometric objects connected to $u_\pm$. Let $N_\pm$ the vector field on Σ defined by:

$$N_\pm=\frac{\nabla u_\pm}{|\nabla u_\pm|}=a_\pm\nabla u_\pm \tag{11.8}$$

and $L_\pm$ the vector field on $\mathcal{M}$ which is given on Σ by:

$$L_\pm=a_\pm\mathbf{g}^{\alpha\beta}\partial_\alpha u_\pm\partial_\beta=a_\pm(-T(u_\pm)T+\nabla u_\pm)=\pm T+N_\pm. \tag{11.9}$$

Let $P_{u_\pm}=\{x\in\Sigma/\,u_\pm(x)=u_\pm\}$ denote the level surfaces of $u_\pm$ in Σ. Since $N_\pm$ is the unit normal to $P_{u_\pm}$, the second fundamental form of $P_{u_\pm}$ in Σ is given by:

$$\theta_\pm(e_A^\pm,e_B^\pm)=g(D_{e_A^\pm}N_\pm,e_B^\pm),\ A,B=1,2, \tag{11.10}$$

where $(e_1^\pm,e_2^\pm)$ is an arbitrary orthonormal frame of $TP_{u_\pm}$. Let

$$\mathcal{H}_{u_\pm}=\{(t,x)\in\mathcal{M}/\,u_\pm(t,x)=u_\pm\}$$

denote the null level hypersurfaces of $u_\pm$ in $\mathcal{M}$. Since $L_\pm$ is null and orthogonal to $P_{u_\pm}$ in $\mathcal{H}_{u_\pm}$, the null second fundamental form $\chi_\pm$ is given on $P_{u_\pm}$ by:

$$\chi_\pm(e_A^\pm,e_B^\pm)=g(\mathbf{D}_{e_A^\pm}L_\pm,e_B^\pm),\quad A,B=1,2. \tag{11.11}$$

Taking the trace in (11.10) and (11.11), and using (11.9) and the fact that k is the second fundamental form of Σ, we obtain:

$$\mathrm{tr}\chi_\pm=\pm\mathrm{tr}k+\mathrm{tr}\theta_\pm. \tag{11.12}$$

Note that $\mathrm{Tr}k = \mathrm{tr}k + k_{NN}$, where Tr denotes the trace for 2-tensors on Σ. In addition to the constraint Equations (10.2), we choose a maximal foliation to be consistent with the statement of Theorem 10.1. This corresponds to $\mathrm{Tr}k = 0$. Together with (11.12), this yields:

$$\mathrm{tr}\chi_\pm = \mp k_{N_\pm N_\pm} + \mathrm{tr}\theta_\pm. \tag{11.13}$$

Now, an easy computation yields:

$$\square_{\mathbf{g}} u_\pm = a_\pm^{-1}\mathrm{tr}\chi_\pm, \tag{11.14}$$

so that the error term (11.7) may be rewritten:

$$\begin{aligned}
&E_+f_+(t,x) + E_-f_-(t,x)\\
&\quad= \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u_+(t,x,\omega)}a_+(t,x,\omega)^{-1}\mathrm{tr}\chi_+(t,x,\omega)f_+(\lambda\omega)\lambda^3 d\lambda d\omega\\
&\qquad+ \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u_-(t,x,\omega)}a_-(t,x,\omega)^{-1}\mathrm{tr}\chi_-(t,x,\omega)f_-(\lambda\omega)\lambda^3 d\lambda d\omega,\ (t,x)\in\mathscr{M}.
\end{aligned} \tag{11.15}$$

In view of (11.15), one has to show in particular that $\mathrm{tr}\chi_\pm$ belongs to $L^\infty(\mathscr{M})$ as part of step **C3** in order to complete step **C4**. This estimate is obtained in [**13**] using a transport equation (the Raychadhouri equation). Thus, one needs the corresponding estimate on Σ (i.e., at $t = 0$):

$$\mathrm{tr}\chi_\pm \in L^\infty(S), \tag{11.16}$$

which in view of (11.13) is equivalent to:

$$\mp k_{N_\pm N_\pm} + \mathrm{tr}\theta_\pm \in L^\infty(S). \tag{11.17}$$

Now, we construct in [**23**] a function $u(x,\omega)$ on $\Sigma\times\mathbb{S}^2$ such that

$$-k_{NN} + \mathrm{tr}\theta \in L^\infty(S). \tag{11.18}$$

Note that $-u(x,-\omega)$ satisfies:

$$k_{NN} + \mathrm{tr}\theta \in L^\infty(S). \tag{11.19}$$

Thus, in view of (11.17), (11.18) and (11.19), we initialize $u_\pm$ on Σ by:

$$u_+(0,x,\omega) = u(x,\omega) \text{ and } u_-(0,x,\omega) = -u(x,-\omega) \text{ for } (x,\omega)\in\Sigma\times\mathbb{S}^2. \tag{11.20}$$

Remark 11.1. — Note that in the particular case where $k\equiv 0$—the so-called time symmetric case-, we may take

$$u_+(0,x,\omega) = u_-(0,x,\omega) = u(x,\omega) \text{ for } (x,\omega)\in\Sigma\times\mathbb{S}^2.$$

In particular, we have $u_+(0,x,\omega) = u_-(0,x,\omega) = x\cdot\omega$ in the flat case.

11.1.2. The choice of f_+ and f_-. — Having defined $u_\pm$, we still need to define $f_\pm$ in the parametrix (11.6). According to (11.1), the half wave parametrix S_+ and S_-

should satisfy on Σ:

$$\begin{cases} S_+f_+(0,x)+S_-f_-(0,x)=\phi_0(x), \\ T(S_+f_+)(0,x)+T(S_-f_-)(0,x)=\phi_1(x). \end{cases} \tag{11.21}$$

Let us introduce the following operators acting on functions of $\mathbb{R}^3$:

$$M_\pm f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)}f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{11.22}$$

and

$$Q_\pm f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)}a(x,\pm\omega)^{-1}f(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{11.23}$$

where $a(x,\omega)=|\nabla u(x,\omega)|^{-1}$ is the lapse of u. Using (11.5), the definition of $S_\pm$ in (11.6), (11.20), the Definition (11.22) of $M_\pm$ and the Definition (11.23) of $Q_\pm$, we may rewrite (11.21) as:

$$\begin{cases} M_+f_++M_-f_-=\phi_0, \\ Q_+(\lambda f_+)-Q_-(\lambda f_-)=i\phi_1. \end{cases} \tag{11.24}$$

The goal of this paper will be to show that there exist a unique (f_+,f_-) satisfying (11.24), and that (f_+,f_-) satisfies the following estimate:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)}+\|\lambda f_-\|_{L^2(\mathbb{R}^3)}\lesssim\|\nabla\phi_0\|_{L^2(S)}+\|\phi_1\|_{L^2(S)}. \tag{11.25}$$

Remark 11.2. — In the case of the flat wave Equation (11.2), we have $(\Sigma,g)=(\mathbb{R}^3,\delta)$, $u_\pm(t,x,\omega)=\mp t+x\cdot\omega$, $u(x,\omega)=x\cdot\omega$ and $a(x,\omega)=1$. In particular, the operators $M_\pm$ and $Q_\pm$ defined respectively by (11.22) and (11.23) all coincide with the inverse Fourier transform. Then, the system (11.24) admits the following solutions:

$$f_\pm(\lambda\omega)=\frac{1}{2}\left(\mathscr{F}\phi_0(\lambda\omega)\pm i\frac{\mathscr{F}\phi_1(\lambda\omega)}{\lambda}\right),$$

which clearly satisfy the estimate (11.25).

Before stating precisely the main results of this paper, we will first recall the regularity obtained for the phase $u(x,\omega)$ constructed in [**23**].

11.2. Regularity assumptions on the phase $u(x,\omega)$

The operators $M_\pm$ and $Q_\pm$ defined respectively in (11.22) and (11.23) are Fourier integral operators with phase $\pm u(x,\pm\omega)$. The regularity assumptions on $u(x,\omega)$ will be crucial to show the existence of (f_+,f_-) satisfying (11.24) and the estimate (11.25). In this section, we state our assumptions on $u(x,\omega)$.

We define the lapse $a(x,\omega)=|\nabla u(x,\omega)|^{-1}$, and the unit vector N such that $\nabla u(x,\omega)=a(x,\omega)^{-1}N(x,\omega)$. We also define the level surfaces $P_u=\{x\,/\,u(x,\omega)=u\}$ so that N is the normal to P_u. The second fundamental form θ of P_u is defined by

$$\theta(X,Y)=g(\nabla_X N,Y), \tag{11.26}$$

with X, Y arbitrary vector fields tangent to the u-foliation P_u of Σ and where ∇ denotes the covariant differentiation with respect to g. We denote by $\mathrm{tr}\,\theta$ the trace of θ, i.e., $\mathrm{tr}\,\theta = \delta^{AB}\theta_{AB}$ where θ_{AB} are the components of θ relative to an orthonormal frame $(e_A)_{A=1,2}$ on P_u.

Let μ_u denote the area element of P_u. Then, for all integrable function f on Σ, the coarea formula implies:

$$\int_\Sigma f d\Sigma = \int_u \int_{P_u} f a d\mu_u du. \tag{11.27}$$

It is also well-known that for a scalar function f:

$$\frac{d}{du}\left(\int_{P_u} f d\mu_u\right) = \int_{P_u}\left(\frac{df}{du} + \mathrm{tr}\,\theta f\right) d\mu_u. \tag{11.28}$$

For $1 \leq p, q \leq +\infty$, we define the spaces $L^p_{[-2,2]}L^q(P_u)$ using the norm

$$\|F\|_{L^p_{[-2,2]}L^q(P_u)} = \left(\int_u \|F\|^p_{L^q(P_u)} du\right)^{1/p}.$$

We assume that $1/2 \leq a(x) \leq 2$ for all $x \in \Sigma$ (see Assumption 1 below) so that $L^p_{[-2,2]}L^p(P_u)$ coincides with $L^p(\Sigma)$ for all $1 \leq p \leq +\infty$. We denote by γ the metric induced by g on P_u, and by $\nabla\!\!\!/$ the induced covariant derivative.

We now state our assumptions for the phase $u(x,\omega)$ of our Fourier integral operators. These assumptions are compatible with the regularity obtained for the function $u(x,\omega)$ constructed in **[23]** (this construction corresponds to step **C1**). The constant $\varepsilon > 0$ below satisfies $0 < \varepsilon < 1$ and will be chosen later to be sufficiently small.

Assumption 1. Regularity with respect to x. — We have

$$\|\nabla a\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|a-1\|_{L^\infty(S)} + \|\nabla\!\!\!/\nabla a\|_{L^2(S)} + \|\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|\nabla\theta\|_{L^2(S)} \lesssim \varepsilon. \tag{11.29}$$

Assumption 2. Regularity with respect to ω. — We have

$$\|\partial_\omega a\|_{L^2(S)} + \|\nabla\partial_\omega a\|_{L^2(S)} + \|\partial_\omega\theta\|_{L^2(S)} + \|\nabla\partial_\omega\theta\|_{L^2(S)} \lesssim \varepsilon, \tag{11.30}$$

$$\|\partial^\alpha_\omega a\|_{L^\infty(S)} \lesssim 1 \text{ for some } 0 < \alpha < 1. \tag{11.31}$$

$$\|\partial_\omega N\|_{L^\infty(S)} \lesssim 1, \tag{11.32}$$

$$||N(x,\omega) - N(x,\omega')| - |\omega-\omega'|| \lesssim (\varepsilon + |\omega-\omega'|)|\omega-\omega'|,\ \forall x \in \Sigma, \omega, \omega' \in \mathbb{S}^2, \tag{11.33}$$

$$\|\nabla\partial^2_\omega N\|_{L^2(S)} \lesssim \varepsilon \tag{11.34}$$

and

$$\|\partial^3_\omega u\|_{L^\infty_{\mathrm{loc}}(\Sigma)} \lesssim 1. \tag{11.35}$$

Assumption 3. Additional regularity with respect to x. — For all $j \geq 0$, there are scalar functions a_1^j and a_2^j such that:

$$\begin{aligned}&\nabla_N a = a_1^j + a_2^j \text{ where } \|a_1^j\|_{L^2(S)} \lesssim 2^{-j/2}\varepsilon,\ \|a_2^j\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \varepsilon\\ &\text{and } \|\nabla_N a_2^j\|_{L^2(S)} + \|a_2^j\|_{L^2_{[-2,2]}L^\infty(P_u)} \lesssim 2^{j/2}\varepsilon.\end{aligned} \tag{11.36}$$

Assumption 4. Global change of variable on Σ. — Let $\omega \in \mathbb{S}^2$. Let $\phi_\omega : \Sigma \to \mathbb{R}^3$ defined by:

$$\phi_\omega(x) := u(x,\omega)\omega + \partial_\omega u(x,\omega). \tag{11.37}$$

Then ϕ_ω is a bijection, and the determinant of its Jacobian satisfies the following estimate:

$$\||\det(\text{Jac}\phi_\omega)| - 1\|_{L^\infty(S)} \lesssim \varepsilon. \tag{11.38}$$

Assumption 5. Comparison of $u(x,\omega)$ with a phase linear in ω. — Let $\nu \in \mathbb{S}^2$ and ϕ_ν the map defined in (11.37). Then, we have:

$$\begin{aligned}u(x,\omega) - \phi_\nu(x)\cdot\omega &= O(\varepsilon|\omega-\nu|^2),\\ \partial_\omega u(x,\omega) - \partial_\omega(\phi_\nu(x)\cdot\omega) &= O(\varepsilon|\omega-\nu|),\\ \partial_\omega^2 u(x,\omega) - \partial_\omega^2(\phi_\nu(x)\cdot\omega) &= O(\varepsilon).\end{aligned} \tag{11.39}$$

Assumption 6. Comparison of $N(x,-\omega)$ with $N(x,\omega)$. — For all $x \in \Sigma$ and $\omega \in \mathbb{S}^2$, we have:

$$|N(x,\omega) + N(x,-\omega)| \lesssim \varepsilon. \tag{11.40}$$

Remark 11.3. — In Assumptions 1–6, all inequalities hold for any $\omega \in \mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.

Remark 11.4. — The fact that we may take a small constant $\varepsilon > 0$ in Assumptions 1–6 is directly related to the assumptions on Σ for R and k in Theorem 10.1.

Remark 11.5. — In the case of the flat wave Equation (11.2), we have $(\Sigma, g) = (\mathbb{R}^3, \delta)$, $u(x,\omega) = x\cdot\omega$, $a = 1$, $N = \omega$ and $\phi_\omega = \text{Id}_{\mathbb{R}^3}$. Thus, Assumptions 1–6 are clearly satisfied with $\varepsilon = 0$.

Remark 11.6. — In **[23]**, the phase $u(x,\omega)$ is actually exactly equal to $x\cdot\omega$ on $|x| \geq 2$. This is made possible by exploiting the finite speed of propagation of Einstein vacuum equations (see **[23]**).

Remark 11.7. — Recall that the lapse a is at the level of one derivative of u with respect to x. Thus, we obtain from (11.29) that some components of $\nabla^3 u$ are in $L^2(S)$. Note that this is not true for all components since (11.36) does not allow us to control $\nabla_N^2 a$ in $L^2(S)$. In fact, (11.36) is only at the level of 3/2 derivatives of a with respect to N in L^2.

11.3. Main results

We first state a result of boundedness on L^2 for Fourier integral operators with phase $u(x,\omega)$.

Theorem 11.8. — *Let u be a function on $\Sigma\times\mathbb{S}^2$ satisfying Assumption 1, Assumption 2 and Assumption 4. Let U the Fourier integral operator with phase $u(x,\omega)$ and symbol $b(x,\omega)$:*

$$Uf(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u(x,\omega)}b(x,\omega)f(\lambda\omega)\lambda^2d\lambda d\omega. \tag{11.41}$$

Let $D>0$. We assume furthermore that $b(x,\omega)$ satisfies:

$$\|b\|_{L^\infty(S)}+\|\nabla b\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|\nabla\!\!\!/\nabla b\|_{L^2(S)}\lesssim D, \tag{11.42}$$

$$\|\partial_\omega b\|_{L^2(S)}+\|\nabla\partial_\omega b\|_{L^2(S)}\lesssim D \tag{11.43}$$

and

$$\begin{aligned}\nabla_N b=b_1^j+b_2^j \text{ where } \|b_1^j\|_{L^2(S)}\lesssim 2^{-\frac{j}{2}}D,\ \|b_2^j\|_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim D\\ \text{and } \|\nabla_N b_2^j\|_{L^2(S)}+\|b_2^j\|_{L^2_{[-2,2]}L^\infty(P_u)}\lesssim 2^{\frac{j}{2}}D.\end{aligned} \tag{11.44}$$

Then, U is bounded on L^2 and satisfies the estimate:

$$\|Uf\|_{L^2(S)}\lesssim D\|f\|_{L^2(\mathbb{R}^3)}. \tag{11.45}$$

Remark 11.9. — We intend to apply Theorem 11.8 to the Fourier integral operators $M_\pm$ and $Q_\pm$ introduced in Section 11.1.2 whose symbol are respectively 1 and a^{-1}. Thus, our assumptions on the regularity of the symbol $b(x,\omega)$ are consistent with the assumptions on the regularity of $a(x,\omega)$ given by Assumptions 1-3.

Recall the definition of the Fourier integral operators $M_\pm$ and $Q_\pm$ introduced in Section 11.1.2:

$$M_\pm f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{\pm i\lambda u(x,\pm\omega)}f(\lambda\omega)\lambda^2d\lambda d\omega \tag{11.46}$$

and

$$Q_\pm f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{\pm i\lambda u(x,\pm\omega)}a(x,\pm\omega)^{-1}f(\lambda\omega)\lambda^2d\lambda d\omega. \tag{11.47}$$

The following theorem is the main result of this paper and achieves step **C2**.

Theorem 11.10. — *Let u be a function on $\Sigma\times\mathbb{S}^2$ satisfying Assumptions 1-6. Then, there exist a unique (f_+,f_-) satisfying:*

$$\begin{cases} M_+f_++M_-f_-=\phi_0,\\ Q_+(\lambda f_+)-Q_-(\lambda f_-)=i\phi_1.\end{cases} \tag{11.48}$$

Furthermore, (f_+,f_-) satisfies the following estimate:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)}+\|\lambda f_-\|_{L^2(\mathbb{R}^3)}\lesssim\|\nabla\phi_0\|_{L^2(S)}+\|\phi_1\|_{L^2(S)}. \tag{11.49}$$

Remark 11.11. — In view of the definition of U, $M_\pm$ and $Q_\pm$, the estimates (11.45) and (11.49) correspond to the obtention of L^2 bounds for Fourier integral operators. Let us repeat that the classical arguments for proving L^2 bounds for Fourier operators are based either on a TT^* argument, or a T^*T argument, which requires in our setting [(1)] taking at least 4 derivatives of the phase in $L^\infty(\Sigma_0\times\mathbb{S}^2)$ either with respect to x for T^*T, or with respect to (λ,ω) for TT^* (see for example [**21**]). Both methods would fail by far within the regularity for the phase $u(x,\omega)$ given by Assumptions 1-4 and for the symbol $b(x,\omega)$ given by (11.42) (11.43) (11.44).

11.4. Boundnessness on L^2 for pseudodifferential operators acting on $\mathbb{R}^3$ with rough symbols

Theorem 11.8 yields the following result on the L^2 boundedness of pseudodifferential operators acting on $\mathbb{R}^3$ which corresponds to the case $\Sigma=\mathbb{R}^3$, $g=\delta$ and $u(x,\omega)=x\cdot\omega$.

Theorem 11.12. — *Let B the pseudodifferential operator with symbol $b(x,\omega)$:*

$$Bf(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda x\cdot\omega}b(x,\omega)\mathscr{F}f(\lambda\omega)\lambda^2d\lambda d\omega. \tag{11.50}$$

We assume furthermore that $b(x,\omega)$ satisfies:

$$\|b\|_{H^{3/2}(\mathbb{R}^3)}+\|\nabla b\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|\nabla\!\!\!/\nabla b\|_{L^2(\mathbb{R}^3)}+\|\partial_\omega b\|_{H^{1/2+\alpha}(\mathbb{R}^3)}\leq D, \tag{11.51}$$

for some constant $D>0$ and $\alpha>0$. Then, B is bounded on L^2 and satisfies the estimate:

$$\|Bf\|_{L^2(\mathbb{R}^3)}\lesssim D\|f\|_{L^2(\mathbb{R}^3)}. \tag{11.52}$$

Remark 11.13. — We do not claim that Theorem 11.12 is an improvement compared to the vast literature on boundedness on L^2 for pseudodifferential operators. Its purpose is to give a warm up for the proof of Theorem 11.8, i.e., boundedness on L^2 for Fourier integral operators on a 3 dimensional Riemannian manifold Σ.

Remark 11.14. — The assumptions (11.51) hold for any $\omega\in\mathbb{S}^2$ with the constant in the right-hand side being independent of ω. Thus, one may take the supremum in ω everywhere. To ease the notations, we do not explicitly write down this supremum.

Remark 11.15. — In the Euclidean setting, the derivative $\nabla\!\!\!/$ simply refers to derivatives in directions orthogonal to ω. Also, the space $L^\infty_{[-2,2]}L^2(P_u)$ is defined with respect to $u(x,\omega)=x\cdot\omega$ and the level surfaces of u are now planes $P_u=\{x/\,x\cdot\omega=u\}$.

Remark 11.16. — The assumptions on the symbol $b(x,\omega)$ in Theorem 11.12 are slightly different from the ones in Theorem 11.8. In particular, we do not assume that $b\in\leq\infty$ since this is a consequence of the assumption (11.51) and Sobolev

1. Since Σ_0 is 3-dimensional.

embeddings in dimension 3. Also, the assumption (11.44) follows from assumption (11.51). Indeed, let Δ_j denote the usual Littlewood Paley projections in $\mathbb{R}^3$ which localizes at frequencies of size 2^j. We may decompose $\nabla b = b_1^j + b_2^j$ with $b_1^j = \Delta_{>j}\nabla b$ and $b_j^2 = \Delta_{\leq j}\nabla b$ and we obtain (11.44) by using $\|b\|_{H^{3/2}(\mathbb{R}^3)} \leq D$. Finally, (11.51) only assumes $\|\partial_\omega b\|_{H^{1/2+\alpha}(\mathbb{R}^3)} \leq D$ while (11.43) assumes essentially that $\|\partial_\omega b\|_{H^1(\mathbb{R}^3)}$ is bounded. We may actually relax (11.43) by replacing it with the analog of $\|\partial_\omega b\|_{H^{1/2+\alpha}(\mathbb{R}^3)} \leq D$. However, this would require to discuss fractional Sobolev spaces on Σ and would complicate the exposition.

The rest of the paper is as follows. In Chapter 12, we prove Theorem 11.12. In Chapter 13, we prove Theorem 11.8. Finally, we prove Theorem 11.10 in Chapter 14.

CHAPTER 12

PROOF OF THEOREM 11.12

While the conclusion of Theorem 11.12 follows from Theorem 11.8 in the case $(\Sigma, g) = (\mathbb{R}^3, \delta)$ where δ is the euclidean metric, and $u(x,\omega) = x\cdot\omega$, it will be instructive to perform the proof first in this simple case of a pseudodifferential operator on $\mathbb{R}^3$. This will clarify the main ideas, before turning to the proof of Theorem 11.8 for Fourier integral operators on a 3 dimensional Riemannian manifold Σ in Chapter 13.

12.1. The basic computation

Since the Fourier transform is an isomorphism of $L^2(\mathbb{R}^3)$, we may remove the Fourier transform in the Definition (11.50) of B in order to ease the notations:

$$Bf(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda x\cdot\omega} b(x,\omega) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{12.1}$$

We start the proof of Theorem 11.12 with the following instructive computation:

$$\begin{aligned}
\|Bf\|_{L^2(\mathbb{R}^3)} &\leq \int_{\mathbb{S}^2}\left\| b(x,\omega)\int_0^{+\infty} e^{i\lambda x\cdot\omega} f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2(\mathbb{R}^3)} d\omega \\
&\leq \int_{\mathbb{S}^2}\|b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)}\left\|\int_0^{+\infty} e^{i\lambda x\cdot\omega} f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2_{x\cdot\omega}} d\omega \\
&\leq D\|\lambda f\|_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{12.2}$$

where we have used Plancherel with respect to λ, Cauchy-Schwarz with respect to ω and (11.51) to bound $\|b\|_{L^\infty_{[-2,2]}L^2(P_u)}$ (note that the space $L^\infty_{[-2,2]}L^2(P_u)$ is defined with respect to $u(x,\omega) = x\cdot\omega$ and the level surfaces of u are now planes $P_u = \{x/\, x\cdot\omega = u\}$). (12.2) misses the conclusion (11.52) of Theorem 11.12 by a power of λ. Now, assume for a moment that we may replace a power of λ by a derivative on $b(x,\omega)$. Then, the same computation yields:

$$
\begin{aligned}
(12.3)\qquad & \left\| \int_{\mathbb{S}^2} \int_0^{+\infty} \nabla b(x,\omega) e^{i\lambda x\cdot\omega} f(\lambda\omega)\lambda d\lambda d\omega \right\|_{L^2(\mathbb{R}^3)} \\
& \leq \int_{\mathbb{S}^2} \|\nabla b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)} \left\| \int_0^{+\infty} e^{i\lambda x\cdot\omega} f(\lambda\omega)\lambda^2 d\lambda \right\|_{L^2_{x\cdot\omega}} d\omega \\
& \leq D\|f\|_{L^2(\mathbb{R}^3)},
\end{aligned}
$$

which is (11.52). This suggests a strategy which consists in making integrations by parts to trade powers of λ against derivatives of the symbol $b(x,\omega)$.

12.2. Structure of the proof of Theorem 11.12

The proof of Theorem 11.12 proceeds in three steps. We first localize in frequencies of size $\lambda \sim 2^j$. We then localize the angle ω in patches on the sphere $\mathbb{S}^2$ of diameter $2^{-j/2}$. Finally, we estimate the diagonal terms.

12.2.1. Step 1: decomposition in frequency.

12.2.1. Step 1: decomposition in frequency. — For the first step, we introduce φ and ψ two smooth compactly supported functions on $\mathbb{R}$ such that:

$$\varphi(\lambda) + \sum_{j\geq 0} \psi(2^{-j}\lambda) = 1 \text{ for all } \lambda \in \mathbb{R}. \tag{12.4}$$

We use (12.4) to decompose Bf as follows:

$$Bf(x) = \sum_{j\geq -1} B_j f(x), \tag{12.5}$$

where for $j \geq 0$:

$$B_j f(x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda x\cdot\omega} b(x,\omega)\psi(2^{-j}\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega \tag{12.6}$$

and

$$B_{-1} f(x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda x\cdot\omega} b(x,\omega)\varphi(\lambda) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{12.7}$$

This decomposition is classical and is known as the first dyadic decomposition (see **[21]**). The goal of this first step is to prove the following proposition:

***Proposition 12.1.* —** *The decomposition* (12.5) *satisfies an almost orthogonality property:*

$$\|Bf\|^2_{L^2(\mathbb{R}^3)} \lesssim \sum_{j\geq -1} \|B_j f\|^2_{L^2(\mathbb{R}^3)} + D^2\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{12.8}$$

The proof of Proposition 12.1 is postponed to Section 12.3.

12.2.2. Step 2: decomposition in angle. — Proposition 12.1 allows us to estimate $\|B_j f\|_{L^2(\mathbb{R}^3)}$ instead of $\|Bf\|_{L^2(\mathbb{R}^3)}$. The analog of computation (12.2) for $\|B_j f\|_{L^2(\mathbb{R}^3)}$ yields:

$$\|B_j f\|_{L^2(\mathbb{R}^3)} \leq D\|\lambda\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)} \lesssim D2^j\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}, \tag{12.9}$$

which misses the wanted estimate by a power of 2^j. We thus need to perform a second dyadic decomposition (see **[21]**). We introduce a smooth partition of unity on the sphere $\mathbb{S}^2$:

$$\sum_{\nu\in\Gamma}\eta_j^\nu(\omega) = 1 \text{ for all } \omega\in\mathbb{S}^2, \tag{12.10}$$

where the support of η_j^ν is a patch on $\mathbb{S}^2$ of diameter $\sim 2^{-j/2}$. We use (12.10) to decompose $B_j f$ as follows:

$$B_j f(x) = \sum_{\nu\in\Gamma} B_j^\nu f(x), \tag{12.11}$$

where:

$$B_j^\nu f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda x\cdot\omega} b(x,\omega)\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{12.12}$$

We also define:

$$\begin{aligned} \gamma_{-1} = \|\varphi(\lambda)f\|_{L^2(\mathbb{R}^3)}, \qquad \gamma_j = \|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\\ \gamma_j^\nu = \|\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\ \nu\in\Gamma, \end{aligned} \tag{12.13}$$

which satisfy:

$$\|f\|^2_{L^2(\mathbb{R}^3)} = \sum_{j\geq -1}\gamma_j^2 = \sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2. \tag{12.14}$$

The goal of this second step is to prove the following proposition:

***Proposition 12.2.* —** *The decomposition* (12.11) *satisfies an almost orthogonality property:*

$$\|B_j f\|^2_{L^2(\mathbb{R}^3)} \lesssim \sum_{\nu\in\Gamma}\|B_j^\nu f\|^2_{L^2(\mathbb{R}^3)} + D^2\gamma_j^2. \tag{12.15}$$

The proof of Proposition 12.2 is postponed to Section 12.4.

12.2.3. Step 3: control of the diagonal term. — Proposition 12.2 allows us to estimate $\|B_j^\nu f\|_{L^2(\mathbb{R}^3)}$ instead of $\|B_j f\|_{L^2(\mathbb{R}^3)}$. The analog of computation (12.2) for $\|B_j^\nu f\|_{L^2(\mathbb{R}^3)}$ yields:

(12.16)

$$\begin{aligned}\|B_j^\nu f\|_{L^2(\mathbb{R}^3)} &\leq \int_{\mathbb{S}^2} \|b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)} \left\| \int_0^{+\infty} e^{i\lambda x\cdot\omega}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda \right\|_{L^2_{x\cdot\omega}} d\omega \\ &\leq D\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))}\|\lambda\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f\|_{L^2(\mathbb{R}^3)} \\ &\lesssim D2^{j/2}\gamma_j^\nu,\end{aligned}$$

where the term $\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))}$ comes from the fact that we apply Cauchy-Schwarz in ω. Note that we have used in (12.16) the fact that the support of η_j^ν is 2 dimensional and has diameter $2^{-j/2}$ so that:

$$\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))} \lesssim 2^{-j/2}. \tag{12.17}$$

Now, (12.16) still misses the wanted estimate by a power of $2^{j/2}$. Nevertheless, taking advantage of the regularity of $\partial_\omega b$ given by (11.51), we are able to estimate the diagonal term:

Proposition 12.3. — *The diagonal term $B_j^\nu f$ satisfies the following estimate:*

$$\|B_j^\nu f\|_{L^2(\mathbb{R}^3)} \lesssim D\gamma_j^\nu. \tag{12.18}$$

The proof of Proposition 12.3 is postponed to Section 12.5.

12.2.4. Proof of Theorem 11.12. — Proposition 12.1, 12.2 and 12.3 immediately yield the proof of Theorem 11.12. Indeed, (12.8), (12.14), (12.15) and (12.18) imply:

$$\begin{aligned}\|Bf\|^2_{L^2(\mathbb{R}^3)} &\lesssim \sum_{j\geq -1}\|B_jf\|^2_{L^2(\mathbb{R}^3)} + D^2\|f\|^2_{L^2(\mathbb{R}^3)} \\ &\lesssim \sum_{j\geq -1}\sum_{\nu\in\Gamma}\|B_j^\nu f\|^2_{L^2(\mathbb{R}^3)} + D^2\sum_{j\geq -1}\gamma_j^2 + D^2\|f\|^2_{L^2(\mathbb{R}^3)} \\ &\lesssim D^2\sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2 + D^2\sum_{j\geq -1}\gamma_j^2 + D^2\|f\|^2_{L^2(\mathbb{R}^3)} \\ &\lesssim D^2\|f\|^2_{L^2(\mathbb{R}^3)},\end{aligned} \tag{12.19}$$

which is the conclusion of Theorem 11.12. □

The remainder of Chapter 12 is dedicated to the proof of Proposition 12.1, 12.2 and 12.3.

12.3. Proof of Proposition 12.1 (almost orthogonality in frequency)

We have to prove (12.8):

$$\|Bf\|^2_{L^2(\mathbb{R}^3)} \lesssim \sum_{j\geq -1}\|B_jf\|^2_{L^2(\mathbb{R}^3)} + D^2\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{12.20}$$

This will result from the following inequality using Shur's Lemma:

$$\left|\int_{\mathbb{R}^3} B_j f(x)\overline{B_k f(x)}dx\right| \lesssim D^2 2^{-\frac{|j-k|}{2}}\gamma_j\gamma_k \text{ for } |j-k|>2. \tag{12.21}$$

12.3.1. A first integration by parts. — From now on, we focus on proving (12.21). We may assume $j \geq k+3$. We have:

(12.22)
$$\begin{aligned}\int_{\mathbb{R}^3} B_j f(x)\overline{B_k f(x)}dx = &\int_{\mathbb{S}^2}\int_0^{+\infty}\int_{\mathbb{S}^2}\int_0^{+\infty}\left(\int_{\mathbb{R}^3} e^{i\lambda x\cdot\omega - i\lambda' x\cdot\omega'} b(x,\omega)\overline{b(x,\omega')}dx\right)\\ &\times \psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2\psi(2^{-k}\lambda')\overline{f(\lambda'\omega')}(\lambda')^2 d\lambda d\omega d\lambda' d\omega'.\end{aligned}$$

We integrate by parts with respect to $\partial_{x\cdot\omega}$ in $\int_{\mathbb{R}^3} e^{i\lambda x\cdot\omega - i\lambda' x\cdot\omega'} b(x,\omega)\overline{b(x,\omega')}dx$ using the fact that:

$$e^{i\lambda x\cdot\omega - i\lambda' x\cdot\omega'} = -\frac{i}{\lambda - \lambda'\omega\cdot\omega'}\partial_{x\cdot\omega}(e^{i\lambda x\cdot\omega - i\lambda' x\cdot\omega'}). \tag{12.23}$$

We obtain:

(12.24)
$$\begin{aligned}\int_{\mathbb{R}^3} e^{i\lambda x\cdot\omega - i\lambda' x\cdot\omega'} b(x,\omega)\overline{b(x,\omega')}dx = &\, i\int_{\mathbb{R}^3} e^{i\lambda x\cdot\omega - i\lambda' x\cdot\omega'}\frac{\partial_{x\cdot\omega} b(x,\omega)\overline{b(x,\omega')}}{\lambda - \lambda'\omega\cdot\omega'}dx\\ &+ i\int_{\mathbb{R}^3} e^{i\lambda x\cdot\omega - i\lambda' x\cdot\omega'}\frac{b(x,\omega)\partial_{x\cdot\omega}\overline{b(x,\omega')}}{\lambda - \lambda'\omega\cdot\omega'}dx.\end{aligned}$$

Since $|\lambda'\omega\cdot\omega'| < \lambda$, we may expand the fractions in (12.24):

$$\frac{1}{\lambda - \lambda'\omega\cdot\omega'} = \sum_{p\geq 0}\frac{1}{\lambda}\left(\frac{\lambda'\omega\cdot\omega'}{\lambda}\right)^p. \tag{12.25}$$

For $p \in \mathbb{Z}$, We introduce the notation $F_{j,p}(x\cdot\omega)$:

$$F_{j,p}(x\cdot\omega) = \int_0^{+\infty} e^{i\lambda x\cdot\omega}\psi(2^{-j}\lambda)f(\lambda\omega)(2^{-j}\lambda)^p\lambda^2 d\lambda. \tag{12.26}$$

Together with (12.22), (12.24) and (12.25), this implies:

$$\int_{\mathbb{R}^3} B_j f(x)\overline{B_k f(x)}dx = \sum_{p\geq 0} A_p^1 + \sum_{p\geq 0} A_p^2, \tag{12.27}$$

where A_p^1 and A_p^2 are given by:

(12.28)
$$\begin{aligned}A_p^1 = &\, 2^{-j-p(j-k)}\\ &\times \int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}\partial_{x\cdot\omega} b(x,\omega)\omega^p F_{j,-p-1}(x\cdot\omega)d\omega\right)\cdot\overline{\left(\int_{\mathbb{S}^2} b(x,\omega')\omega'^p F_{k,p}(x\cdot\omega')d\omega'\right)}dx\end{aligned}$$

and

(12.29)

$$A_p^2 = 2^{-j-p(j-k)}$$
$$\times \int_{\mathbb{R}^3} \left(\int_{\mathbb{S}^2} b(x,\omega)\omega^{p+1} F_{j,-p-1}(x\cdot\omega)d\omega \right) \cdot \overline{\left(\int_{\mathbb{S}^2} \nabla b(x,\omega')\omega'^p F_{k,p}(x\cdot\omega')d\omega' \right)} dx.$$

Remark 12.4. — The expansion (12.25) allows us to rewrite $\int_{\mathbb{R}^3} B_j f(x)\overline{B_k f(x)}dx$ in the form (12.27), i.e., as a sum of terms A_p^1, A_p^2. The key point is that in each of these terms—according to (12.28) and (12.29)—one may separate the terms depending of (λ,ω) from the terms depending on (λ',ω').

12.3.2. Estimates for A_p^1 and A_p^2. — The term containing one derivative of b in (12.28) may be estimated using the basic computation (12.2):

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} \partial_{x\cdot\omega} b(x,\omega)\omega^p F_{j,-p-1}(x\cdot\omega)d\omega \right\|_{L^2(\mathbb{R}^3)} \\ &\leq \int_{\mathbb{S}^2} \|\partial_{x\cdot\omega} b(x,\omega)\omega^p\|_{L^\infty_{[-2,2]}L^2(P_u)} \|F_{j,-p-1}(x\cdot\omega)\|_{L^2_{x\cdot\omega}} d\omega \\ &\leq \|\nabla b\|_{L^\infty_{[-2,2]}L^2(P_u)} \|\psi(2^{-j}\lambda)f(\lambda\omega)(2^{-j}\lambda)^{-p-1}\lambda\|_{L^2(\mathbb{R}^3)} \\ &\leq D2^{p+1+j}\gamma_j, \end{aligned} \tag{12.30}$$

where we have used the assumption (11.51) on b and the fact that $(2^{-j}\lambda)^{-1} \leq 2$ on the support of $\psi(2^{-j}\lambda)$. In the same way, the term containing one derivative of b in (12.29) may be estimated by:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} \nabla b(x,\omega')\omega'^p F_{k,p}(x\cdot\omega')d\omega' \right\|_{L^2(\mathbb{R}^3)} \\ &\leq \int_{\mathbb{S}^2} \|\nabla b(x,\omega')\omega'^p\|_{L^\infty_{[-2,2]}L^2(P_u)} \|F_{k,p}(x\cdot\omega')\|_{L^2_{x\cdot\omega'}} d\omega' \\ &\leq \|\nabla b\|_{L^\infty_{[-2,2]}L^2(P_u)} \|\psi(2^{-k}\lambda')f(\lambda'\omega')(2^{-k}\lambda')^p\lambda'\|_{L^2(\mathbb{R}^3)} \\ &\leq D2^{p+k}\gamma_k, \end{aligned} \tag{12.31}$$

where we have used the assumption (11.51) on b and the fact that $(2^{-k}\lambda') \leq 2$ on the support of $\psi(2^{-k}\lambda')$.

Note that Proposition 12.2 together with Proposition 12.3 yields the estimate:

$$\|B_j f\|_{L^2(\mathbb{R}^3)} \lesssim D\gamma_j, \tag{12.32}$$

for any symbol b satisfying the assumptions (11.51). Now, the term containing no derivative of b in (12.28) has a symbol given by $b(x,\omega')\omega'^p$ which satisfies the assumptions (11.51) since b does. Applying (12.32), we obtain:

$$\begin{aligned}(12.33)\qquad &\left\|\int_{\mathbb{S}^2} b(x,\omega')\omega'^p F_{k,p}(x\cdot\omega')d\omega'\right\|_{L^2(\mathbb{R}^3)}\\ &\lesssim D\|\psi(2^{-k}\lambda')f(\lambda'\omega')(2^{-k}\lambda')^p\|_{L^2(\mathbb{R}^3)}\\ &\leq D2^p\gamma_k.\end{aligned}$$

In the same way, the term containing no derivative of b in (12.29) has a symbol given by $b(x,\omega)\omega^{p+1}$ which satisfies the assumptions (11.51) since b does. Applying again (12.32), we obtain:

$$\begin{aligned}(12.34)\qquad &\left\|\int_{\mathbb{S}^2} b(x,\omega)\omega^{p+1} F_{j,-p-1}(x\cdot\omega)d\omega\right\|_{L^2(\mathbb{R}^3)}\\ &\lesssim D\|\psi(2^{-j}\lambda)f(\lambda\omega)(2^{-j}\lambda)^{-p-1}\|_{L^2(\mathbb{R}^3)}\\ &\lesssim D2^{p+1}\gamma_j.\end{aligned}$$

Finally, the definition of A_p^1 (12.28) and the estimates (12.30) and (12.33) yield:

$$(12.35)\qquad |A_p^1|\lesssim D2^{2p-p(j-k)}\gamma_j\gamma_k,\ \forall p\geq 0.$$

Similarly, the definition of A_p^2 (12.29) and the estimates (12.31) and (12.34) yield:

$$(12.36)\qquad |A_p^2|\lesssim D2^{2p-(p+1)(j-k)}\gamma_j\gamma_k,\ \forall p\geq 0.$$

(12.35) and (12.36) imply:

$$(12.37)\qquad \sum_{p\geq 1}|A_p^1|+\sum_{p\geq 0}|A_p^2|\lesssim D2^{-(j-k)}\left(\sum_{p\geq 0}2^{-p(j-k-2)}\right)\gamma_j\gamma_k\lesssim D2^{-(j-k)}\gamma_j\gamma_k,$$

where we have used the assumption $j-k-2>0$. (12.27) and (12.37) will yield (12.21) provided we obtain a similar estimate for A_0^1. Now, the estimate of A_0^1 provided by (12.35) is not sufficient since it does not contain any decay in $j-k$. We will need to perform a second integration by parts for this term.

12.3.3. A more precise estimate for A_0^1. —

From (12.28) with $p=0$, we have:

$$(12.38)\qquad A_0^1=2^{-j}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}\partial_{x\cdot\omega}b(x,\omega)F_{j,0}(x\cdot\omega)d\omega\right)\overline{B_k(x)}dx.$$

Since $b(x,\omega)$ is assumed to be in $H^{3/2}(\mathbb{R}^3)$, we may only make one half integration by parts. To this end, we decompose $\partial_{x\cdot\omega}b$ as in Remark 11.16. Let Δ_j denote the usual Littlewood Paley projections in $\mathbb{R}^3$ which localizes at frequencies of size 2^j. We decompose $\partial_{x\cdot\omega}b=b_1^j+b_2^j$ with $b_1^j=\Delta_{>j}\partial_{x\cdot\omega}b$ and $b_2^j=\Delta_{\leq j}\partial_{x\cdot\omega}b$ and we obtain

$$(12.39)\qquad \|b_j^1\|_{L^2(\mathbb{R}^3)}\lesssim D2^{-\frac{j}{2}}\text{ and }\|\nabla b_2^j\|_{L^2(\mathbb{R}^3)}\lesssim D2^{\frac{j}{2}}$$

by using $\|b\|_{H^{3/2}(\mathbb{R}^3)} \leq D$. In turn, this yields a decomposition for A_0^1:

$$A_0^1 = A_{0,1}^1 + A_{0,2}^1, \tag{12.40}$$

where:

$$\begin{aligned} A_{0,1}^1 &= 2^{-j}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2} b_1^j(x,\omega)F_{j,0}(x\cdot\omega)d\omega\right)\overline{B_k(x)}dx, \\ A_{0,2}^1 &= 2^{-j}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2} b_2^j(x,\omega)F_{j,0}(x\cdot\omega)d\omega\right)\overline{B_k(x)}dx. \end{aligned} \tag{12.41}$$

We first estimate $A_{0,1}^1$. We have:

$$\begin{aligned} |A_{0,1}^1| &\leq 2^{-j}\int_{\mathbb{S}^2}\left|\int_{\mathbb{R}^3} b_1^j(x,\omega)F_{j,0}(x\cdot\omega)\overline{B_k(x)}dx\right|d\omega \\ &\leq 2^{-j}\int_{\mathbb{S}^2}\|b_1^j(.,\omega)\|_{L^2(\mathbb{R}^3)}\|F_{j,0}\|_{L^2_{x\cdot\omega}}\|B_k\|_{L^\infty_{[-2,2]}L^2(P_u)}d\omega \\ &\lesssim D2^{-\frac{3j}{2}}\int_{\mathbb{S}^2}\|F_{j,0}\|_{L^2_{x\cdot\omega}}\|B_k\|_{L^\infty_{[-2,2]}L^2(P_u)}d\omega, \end{aligned} \tag{12.42}$$

where we have used (12.39) in the last inequality. Plancherel yields:

$$\|F_{j,0}\|_{L^2_{\omega,x\cdot\omega}} \leq \|\psi(2^{-j}\lambda)f(\lambda\omega)\lambda\|_{L^2(\mathbb{R}^3)} \lesssim 2^j\gamma_j. \tag{12.43}$$

In view of (12.42), we also need to estimate $\|B_k\|_{L^\infty_{[-2,2]}L^2(P_u)}$. We have:

$$\|B_k\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|B_k\|^{\frac{1}{2}}_{H^1(\mathbb{R}^3)}\|B_k\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \lesssim D^{\frac{1}{2}}\gamma_k^{\frac{1}{2}}\|B_k\|^{\frac{1}{2}}_{H^1(\mathbb{R}^3)}, \tag{12.44}$$

where we have used a standard trace theorem for the first inequality, and (12.32) for the second inequality. We still need to estimate $\|\nabla B_k\|_{L^2(\mathbb{R}^3)}$. We have:

$$\begin{aligned} \nabla B_k(x) = &\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda x\cdot\omega}\nabla b(x,\omega)\psi(2^{-k}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega \\ &+ i2^k\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda x\cdot\omega}\omega b(x,\omega)\psi(2^{-k}\lambda)(2^{-k}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \end{aligned} \tag{12.45}$$

Using the basic computation (12.2) for the first term together with the fact that $\nabla b \in L^\infty_{[-2,2]}L^2(P_u)$, and (12.32) for the second term together with the fact that $\omega b(x,\omega)$ satisfies the assumption (11.51), we obtain:

$$\|\nabla B_k\|_{L^2(\mathbb{R}^3)} \lesssim D2^k\gamma_k. \tag{12.46}$$

Finally, (12.42), (12.43), (12.44) and (12.46) yield:

$$|A_{0,1}^1| \lesssim D2^{-\frac{j-k}{2}}\gamma_j\gamma_k. \tag{12.47}$$

12.3.4. A second integration by parts. — We now estimate the term $A^1_{0,2}$ defined in (12.41). We perform a second integration by parts relying again on (12.23). We obtain:

(12.48)
$$A^1_{0,2} = 2^{-2j}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}\partial_{x\cdot\omega}b_2^j(x,\omega)F_{j,0}(x\cdot\omega)d\omega\right)\overline{B_k(x)}dx + 2^{-2j}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}b_2^j(x,\omega)\omega F_{j,0}(x\cdot\omega)d\omega\right)\cdot\overline{\left(\int_{\mathbb{S}^2}\nabla b(x,\omega')F_{k,0}(x\cdot\omega')d\omega'\right)}dx+\cdots,$$

where we only mention the first term generated by the expansion (12.25). In fact, the other terms are estimated in the same way and generate more decay in $j-k$ similarly to the estimates (12.35), (12.36).

The first term in the right-hand side of (12.48) has the same form than $A^1_{0,1}$ defined in (12.41) where b_1^j is replaced by $2^{-j}\partial_{x\cdot\omega}b_2^j$. By (12.39), $2^{-j}\partial_{x\cdot\omega}b_2^j$ satisfies:

$$\|2^{-j}\partial_{x\cdot\omega}b_2^j\|_{L^2(\mathbb{R}^3)}\lesssim D2^{-\frac{j}{2}}.$$

Since b_j^1 and $2^{-j}\partial_{x\cdot\omega}b_2^j$ satisfy the same estimate, we obtain the analog of (12.47) for the first term in the right-hand side of (12.48):

$$\left|2^{-2j}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}\partial_{x\cdot\omega}b_2^j(x,\omega)F_{j,0}(x\cdot\omega)d\omega\right)\overline{B_k(x)}dx\right|\lesssim D2^{-\frac{j-k}{2}}\gamma_j\gamma_k. \tag{12.49}$$

We now estimate the second term in the right-hand side of (12.48). Recall that $b_2^j=\Delta_{\leq j}\partial_{x\cdot\omega}b$ so that together with the assumption (11.51), we have:

$$\|b_2^j\|_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim D. \tag{12.50}$$

We estimate the scorn term in the right-hand side of (12.48) using the assumption (11.51), the basic computation (12.2) and (12.50):

(12.51)
$$\begin{aligned}&\left|2^{-2j}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}b_2^j(x,\omega)\omega F_{j,0}(x\cdot\omega)d\omega\right)\cdot\overline{\left(\int_{\mathbb{S}^2}\nabla b(x,\omega')F_{k,0}(x\cdot\omega')d\omega'\right)}dx\right|\\
&\leq 2^{-2j}\left\|\int_{\mathbb{S}^2}b_2^j(x,\omega)\omega F_{j,0}(x\cdot\omega)d\omega\right\|_{L^2(\mathbb{R}^3)}\left\|\int_{\mathbb{S}^2}\nabla b(x,\omega')F_{k,0}(x\cdot\omega')d\omega'\right\|_{L^2(\mathbb{R}^3)}\\
&\leq 2^{-2j}\left(\int_{\mathbb{S}^2}\|b_2^j(.,\omega)\omega\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{j,0}\|_{L^2_{x\cdot\omega}}d\omega\right)\\
&\quad\times\left(\int_{\mathbb{S}^2}\|\nabla b(.,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{k,0}\|_{L^2_{x\cdot\omega}}d\omega\right)\\
&\lesssim D^2 2^{-(j-k)}\gamma_j\gamma_k.\end{aligned}$$

Finally, (12.48), (12.49) and (12.51) imply:

$$|A^1_{0,2}|\lesssim D^2 2^{-\frac{j-k}{2}}\gamma_j\gamma_k. \tag{12.52}$$

12.3.5. End of the proof of Proposition 12.1. — Since $A_0^1 = A_1^1 + A_2^1$, the estimate (12.47) of $A_{0,1}^1$ and the estimate (12.52) of $A_{0,2}^1$ yield:

$$|A_0^1| \lesssim D^2 2^{-\frac{j-k}{2}} \gamma_j \gamma_k. \tag{12.53}$$

Together with (12.27) and (12.37), this implies:

$$\left| \int_{\mathbb{R}^3} B_j f(x) \overline{B_k f(x)} dx \right| \lesssim D^2 2^{-\frac{|j-k|}{2}} \gamma_j \gamma_k \text{ for } |j-k| > 2. \tag{12.54}$$

Finally, (12.54) together with Shur's Lemma yields:

$$\|Bf\|_{L^2(\mathbb{R}^3)}^2 \lesssim \sum_{j \geq -1} \|B_j f\|_{L^2(\mathbb{R}^3)}^2 + D^2 \|f\|_{L^2(\mathbb{R}^3)}^2. \tag{12.55}$$

This concludes the proof of Proposition 12.1. □

12.4. Proof of Proposition 12.2 (almost orthogonality in angle)

We have to prove (12.15):

$$\|B_j f\|_{L^2(\mathbb{R}^3)}^2 \lesssim \sum_{\nu \in \Gamma} \|B_j^\nu f\|_{L^2(\mathbb{R}^3)}^2 + D^2 \gamma_j^2. \tag{12.56}$$

This will result from the following inequality:

$$\left| \int_{\mathbb{R}^3} B_j^\nu f(x) \overline{B^{\nu'} f(x)} dx \right| \lesssim \frac{D^2 \gamma_j^\nu \gamma_j^{\nu'}}{2^{j\alpha/2} (2^{j/2} |\nu - \nu'|)^{2-\alpha}}, \ |\nu - \nu'| \neq 0, \tag{12.57}$$

where $\alpha > 0$. Indeed, since $\mathbb{S}^2$ is 2 dimensional and $1 \leq 2^{j/2}|\nu - \nu'| \leq 2^{j/2}$ for $\nu, \nu' \in \Gamma$ and $\nu \neq \nu'$, we have:

$$\sup_\nu \sum_{\nu'} \frac{1}{2^{j\alpha/2} (2^{j/2} |\nu - \nu'|)^{2-\alpha}} \leq C_\alpha < +\infty \, \forall \alpha > 0. \tag{12.58}$$

Thus, (12.57) and (12.58) together with Shur's Lemma imply (12.56).

12.4.1. A second decomposition in frequency. — From now on, we focus on proving (12.57). Integrating by parts twice in $\int_{\mathbb{R}^3} B_j^\nu f(x) \overline{B_j^{\nu'} f(x)} dx$ would ultimately yield:

$$\left| \int_{\mathbb{R}^3} B_j^\nu f(x) \overline{B_j^{\nu'} f(x)} dx \right| \lesssim \frac{D^2 \gamma_j^\nu \gamma_j^{\nu'}}{(2^{j/2} |\nu - \nu'|)^2}, \ |\nu - \nu'| \neq 0. \tag{12.59}$$

This corresponds to the case $\alpha = 0$ in (12.58) and yields to a log-loss since we have:

$$\sup_\nu \sum_{\nu'} \frac{1}{(2^{j/2} |\nu - \nu'|)^2} \sim j. \tag{12.60}$$

To avoid this log-loss, we do a second decomposition in frequency. λ belongs to the interval $[2^{j-1}, 2^{j+1}]$ which we decompose in intervals I_k:

$$[2^{j-1}, 2^{j+1}] = \bigcup_{1 \leq k \leq |\nu - \nu'|^{-\alpha}} I_k \text{ where } \operatorname{diam}(I_k) \sim 2^j |\nu - \nu'|^\alpha. \tag{12.61}$$

Let ϕ_k a partition of unity of the interval $[2^{j-1}, 2^{j+1}]$ associated to the I_k's. We decompose $B_j^\nu f$ as follows:

$$B_j^\nu f(x) = \sum_{1\leq k\leq |\nu-\nu'|^{-\alpha}} B_j^{\nu,k} f(x), \tag{12.62}$$

where:

$$B_j^{\nu,k} f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda x\cdot\omega} b(x,\omega)\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\phi_k(\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{12.63}$$

We also define:

$$\gamma_j^{\nu,k} = \|\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\phi_k(\lambda)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\ \nu\in\Gamma, 1\leq k\leq |\nu-\nu'|^{-\alpha}, \tag{12.64}$$

which satisfy:

$$(\gamma_j^\nu)^2 = \sum_{1\leq k\leq |\nu-\nu'|^{-\alpha}} (\gamma_j^{\nu,k})^2. \tag{12.65}$$

12.4.2. The two key estimates. — We will prove the following two estimates:

$$\left|\int_{\mathbb{R}^3} B_j^{\nu,k} f(x)\overline{B_j^{\nu',k} f(x)}dx\right| \lesssim \frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}},\ |\nu-\nu'|\neq 0,\ 1\leq k\leq |\nu-\nu'|^{-\alpha} \tag{12.66}$$

and

$$\left|\int_{\mathbb{R}^3} B_j^{\nu,k} f(x)\overline{B_j^{\nu',k'} f(x)}dx\right| \lesssim \frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k'}}{|k-k'|2^{j(1-\alpha/2)/2}(2^{j/2}|\nu-\nu'|)^{1+\alpha/2}},\quad \text{for } |\nu-\nu'|\neq 0,\ 1\leq k,k'\leq |\nu-\nu'|^{-\alpha},\ k\neq k'. \tag{12.67}$$

(12.66) and (12.67) imply:

$$\begin{aligned}\left|\int_{\mathbb{R}^3} B_j^\nu f(x)\overline{B_j^{\nu'} f(x)}dx\right| &\leq \sum_{1\leq k\leq |\nu-\nu'|^{-\alpha}} \left|\int_{\mathbb{R}^3} B_j^{\nu,k} f(x)\overline{B_j^{\nu',k} f(x)}dx\right| \\ &\quad + \sum_{1\leq k\neq k'\leq |\nu-\nu'|^{-\alpha}} \left|\int_{\mathbb{R}^3} B_j^{\nu,k} f(x)\overline{B_j^{\nu',k'} f(x)}dx\right| \\ &\lesssim \sum_{1\leq k\leq |\nu-\nu'|^{-\alpha}} \frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}} \\ &\quad + \sum_{1\leq k\neq k'\leq |\nu-\nu'|^{-\alpha}} \frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k'}}{|k-k'|2^{\frac{j}{2}(1-\frac{\alpha}{2})}(2^{j/2}|\nu-\nu'|)^{1+\alpha/2}} \\ &\lesssim \frac{D^2\gamma_j^\nu\gamma_j^{\nu'}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}},\end{aligned} \tag{12.68}$$

where we have used (12.65) in the last inequality and the fact that:

$$\sup_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\sum_{1\leq k'\leq|\nu-\nu'|^{-\alpha},\,k'\neq k}\frac{1}{|k-k'|}\lesssim\alpha|\log(|\nu-\nu'|)|. \tag{12.69}$$

Since (12.68) yields the wanted estimate (12.57), we are left with proving (12.66) and (12.67).

12.4.3. Proof of (12.66). — The estimate (12.66) will result of two integrations by parts with respect to tangential derivatives. By definition of $\nabla\!\!\!/$, we have $\nabla\!\!\!/ h=\nabla h-(\nabla_\omega h)\omega$ for any function h on $\mathbb{R}^3$. In particular, we have $\nabla\!\!\!/(x\cdot\omega)=0$ and $\nabla\!\!\!/(x\cdot\omega')=\omega'-(\omega'\cdot\omega)\omega$. Now, since $|\omega'-(\omega'\cdot\omega)\omega|^2=1-(\omega'\cdot\omega)^2$, this yields:

$$e^{i\lambda x\cdot\omega-i\lambda' x\cdot\omega'}=\frac{i}{\lambda'\sqrt{1-(\omega'\cdot\omega)^2}}\nabla\!\!\!/_e(e^{i\lambda x\cdot\omega-i\lambda' x\cdot\omega'}), \tag{12.70}$$

where

$$e=\frac{\omega'-(\omega'\cdot\omega)\omega}{\sqrt{1-(\omega'\cdot\omega)^2}} \tag{12.71}$$

is a tangent vector with respect of the level surfaces of $x\cdot\omega$. Similarly, we have:

$$e^{i\lambda x\cdot\omega-i\lambda' x\cdot\omega'}=-\frac{i}{\lambda\sqrt{1-(\omega'\cdot\omega)^2}}\nabla\!\!\!/'_{e'}(e^{i\lambda x\cdot\omega-i\lambda' x\cdot\omega'}), \tag{12.72}$$

where

$$e'=\frac{\omega-(\omega\cdot\omega')\omega'}{\sqrt{1-(\omega'\cdot\omega)^2}} \tag{12.73}$$

is a tangent vector with respect of the level surfaces of $x\cdot\omega'$. For $p\in\mathbb{Z}$, We introduce the notation $F_{j,k,p}(x\cdot\omega)$:

$$F_{j,k,p}(x\cdot\omega)=\int_0^{+\infty}e^{i\lambda x\cdot\omega}\psi(2^{-j}\lambda)\phi_k(\lambda)f(\lambda\omega)(2^{-j}\lambda)^p\lambda^2d\lambda. \tag{12.74}$$

We integrate once by parts using (12.70) in $\int_{\mathbb{R}^3}B_j^{\nu,k}f(x)\overline{B_j^{\nu',k}f(x)}dx$ and we obtain:

$$\begin{aligned}&\int_{\mathbb{R}^3}B_j^{\nu,k}f(x)\overline{B_j^{\nu',k}f(x)}dx \\ &=2^{-j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{i\nabla\!\!\!/_e b(x,\omega)\overline{b(x',\omega')}}{\sqrt{1-(\omega'\cdot\omega)^2}}F_{j,k,0}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx \\ &+2^{-j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{ib(x,\omega)\overline{\nabla\!\!\!/_e b(x',\omega')}}{\sqrt{1-(\omega'\cdot\omega)^2}}F_{j,k,0}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx.\end{aligned} \tag{12.75}$$

We then integrate a second time by parts using (12.72) (so that there is at least one tangential derivative on $b(x,\omega')$):

(12.76)

$$\int_{\mathbb{R}^3}B_j^{\nu,k}f(x)\overline{B_j^{\nu',k}f(x)}dx$$

$$
\begin{aligned}
&= 2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\not\nabla_e\nabla_{e'}b(x,\omega)\overline{b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,0}(x\cdot\omega)\overline{F_{j,k,-2}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\\
&+ 2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\not\nabla_e b(x,\omega)\overline{\not\nabla_{e'}b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,0}(x\cdot\omega)\overline{F_{j,k,-2}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\\
&+ 2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla_{e'}b(x,\omega)\overline{\nabla_e b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\\
&+ 2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{b(x,\omega)\overline{\not\nabla'_{e'}\nabla_e b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx.
\end{aligned}
$$

Control of the right-hand side of (12.76). — We now estimate the four terms in (12.76). Using the fact that:

$$\omega\cdot\omega' = 1-\frac{|\omega-\omega'|^2}{2}, \tag{12.77}$$

we obtain the following expansions:

$$\frac{1}{1-(\omega\cdot\omega')^2} = \frac{1}{|\nu-\nu'|^2}\left(1+\sum_{p+q\geq 1}c^1_{p,q}\left(\frac{\omega-\nu}{|\nu-\nu'|}\right)^p\left(\frac{\omega'-\nu'}{|\nu-\nu'|}\right)^q\right) \tag{12.78}$$

and

$$e = \frac{\nu'-(\nu'\cdot\nu)\nu}{\sqrt{1-(\nu'\cdot\nu)^2}}+\sum_{p+q\geq 1}c^2_{p,q}\left(\frac{\omega-\nu}{|\nu-\nu'|}\right)^p\left(\frac{\omega'-\nu'}{|\nu-\nu'|}\right)^q, \tag{12.79}$$

where $c^1 p,q$ and $c^2_{p,q}$ are constants. The expansions (12.78) and (12.79) allow us to rewrite the four terms of $\int_{\mathbb{R}^3}B_j^{\nu,k}f(x)\overline{B_j^{\nu',k}f(x)}dx$ such that one may separate the terms depending of (λ,ω) from the terms depending on (λ',ω'). For instance, the first term in the right-hand side of (12.76) becomes:

(12.80)
$$
\begin{aligned}
&\frac{2^{-j}}{(2^{j/2}|\nu-\nu'|)^2}\sum_{p+q\geq 0}c_{p,q}\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}\not\nabla\nabla b(x,\omega)F_{j,k,0}(x\cdot\omega)\left(\frac{\omega-\nu}{|\nu-\nu'|}\right)^p\eta_j^\nu(\omega)d\omega\right)\\
&\qquad\times\left(\int_{\mathbb{S}^2}b(x,\omega')F_{j,k,-2}(x\cdot\omega')\left(\frac{\omega'-\nu'}{|\nu-\nu'|}\right)^q\eta_j^{\nu'}(\omega')d\omega'\right)dx,
\end{aligned}
$$

where $c_{p,q}$ are constants. Since we have:

$$\frac{|\omega-\nu|}{|\nu-\nu'|}\lesssim\frac{1}{2^{j/2}|\nu-\nu'|}\text{ and }\frac{|\omega'-\nu'|}{|\nu-\nu'|}\lesssim\frac{1}{2^{j/2}|\nu-\nu'|}, \tag{12.81}$$

the terms in the expansion (12.80) have more and more decay, and it is enough to consider the first one. We have:

$$
\begin{aligned}
&\frac{2^{-j}}{(2^{j/2}|\nu-\nu'|)^2}\left|\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}\nabla\!\!\!/\nabla b(x,\omega)F_{j,k,0}(x\cdot\omega)\eta_j^\nu(\omega)d\omega\right)\right.\\
&\qquad\left.\times\left(\int_{\mathbb{S}^2}b(x,\omega')F_{j,k,-2}(x\cdot\omega')\eta_j^{\nu'}(\omega')d\omega'\right)dx\right|\\
&\leq\frac{2^{-j}}{(2^{j/2}|\nu-\nu'|)^2}\left(\int_{\mathbb{S}^2}\|\nabla\!\!\!/\nabla b\|_{L^2(\mathbb{R}^3)}\|F_{j,k,0}\|_{L^\infty_{x\cdot\omega}}\eta_j^\nu(\omega)d\omega\right)\\
&\qquad\times\left\|\int_{\mathbb{S}^2}b(x,\omega')F_{j,k,-2}(x\cdot\omega')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathbb{R}^3)}.
\end{aligned}
\tag{12.82}
$$

Using the estimate for the diagonal term (12.18) yields:

$$
\left\|\int_{\mathbb{S}^2}b(x,\omega')F_{j,k,-2}(x\cdot\omega')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(\mathbb{R}^3)}\lesssim D\gamma_j^{\nu',k'}.
\tag{12.83}
$$

Using Cauchy Schwartz in λ together with the size of the support of ϕ_k yields:

$$
\|F_{j,k,0}\|_{L^\infty_{x\cdot\omega}}\lesssim 2^{3j/2}|\nu-\nu'|^{\frac{\alpha}{2}}\|\psi(2^{-j}\lambda)\phi_k(\lambda)f(\lambda\omega)\lambda\|_{L^2_\lambda}.
\tag{12.84}
$$

Finally, the assumption (11.51) on $b(x,\omega)$, the size of the support in ω, (12.82), (12.83) and (12.84) imply:

$$
\begin{aligned}
&\frac{2^{-j}}{(2^{j/2}|\nu-\nu'|)^2}\left|\int_{\mathbb{R}^3}\left(\int_{\mathbb{S}^2}\nabla\!\!\!/\nabla b(x,\omega)F_{j,k,0}(x\cdot\omega)\eta_j^\nu(\omega)d\omega\right)\right.\\
&\quad\left.\times\left(\int_{\mathbb{S}^2}b(x,\omega')F_{j,k,-2}(x\cdot\omega')\eta_j^{\nu'}(\omega')d\omega'\right)dx\right|\lesssim\frac{D^2|\nu-\nu'|^{\frac{\alpha}{2}}}{(2^{j/2}|\nu-\nu'|)^2}\gamma_j^{\nu,k}\gamma_j^{\nu',k'},
\end{aligned}
\tag{12.85}
$$

which satisfies the wanted estimate (12.66). The last term in the right-hand side of (12.76) is estimated exactly in the same way.

Control of the second term in the right-hand side of (12.76). — We still need to estimate the second and the third term in the right-hand side of (12.76). Estimating them directly would yield the estimate (12.59) and ultimately the log-loss (12.60). Thus, we need to integrate by parts once more. We first consider the second term in the right-hand side of (12.76). Integrating by parts using (12.72) yields:

(12.86)

$$
\begin{aligned}
&2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla\!\!\!/_e b(x,\omega)\overline{\nabla\!\!\!/_{e'}b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,0}(x\cdot\omega)\overline{F_{j,k,-2}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\\
&=i2^{-3j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla_{e'}\nabla\!\!\!/_e b(x,\omega)\overline{\nabla_{e'}b(x',\omega')}}{(1-(\omega'\cdot\omega)^2)^{3/2}}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-2}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx
\end{aligned}
$$

$$+ i2^{-3j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla\!\!\!/_e b(x,\omega)\overline{\nabla\!\!\!/_{e'}\nabla_{e'}b(x',\omega')}}{(1-(\omega'\cdot\omega)^2)^{3/2}}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-2}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx.$$

The two terms in the right-hand side of (12.86) are estimated in the same way, so we only consider the first one. It is estimated by:

$$\begin{aligned}
&2^{-3j}\left|\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla_{e'}\nabla\!\!\!/_e b(x,\omega)\overline{\nabla_{e'}b(x',\omega')}}{(1-(\omega'\cdot\omega)^2)^{3/2}}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-2}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\right|\\
&\leq 2^{-3j}\int_{\mathbb{S}^2\times\mathbb{S}^2}\frac{1}{(1-(\omega'\cdot\omega)^2)^{3/2}}\|\nabla\nabla\!\!\!/ b(x,\omega)\|_{L^2(\mathbb{R}^3)}\|F_{j,k,-1}\|_{L^\infty_{x\cdot\omega}}\\
&\quad\times\|\nabla b(x',\omega')\|_{L^\infty_{u'}L^2(P_{u'})}\|F_{j,k,-2}\|_{L^2_{x\cdot\omega'}}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'
\end{aligned}$$

(12.87)

$$\lesssim\frac{D^2|\nu-\nu'|^{\frac{\alpha}{2}}}{(2^{j/2}|\nu-\nu'|)^3}\gamma_j^{\nu,k}\gamma_j^{\nu',k'},$$

where we have used Plancherel to estimate $\|F_{j,k,-2}\|_{L^2_{x\cdot\omega'}}$, Cauchy-Schwartz in ω and ω', the assumption (11.51) on b, and the estimate (12.84). (12.87) satisfies the wanted estimate (12.66).

Control of the third term in the right-hand side of (12.76) *and end of the proof of* (12.66). — Finally, we consider the third term in the right-hand side of (12.76). Neither of the two terms $\nabla_{e'}b$ and $\nabla_e b$ contain tangential derivatives, so integrating by parts directly would require to control two normal derivatives of b, which is not part of the assumptions (11.51). We first remark using the Definition (12.71) of e and (12.73) of e' that:

$$e+e'=\frac{(1-\omega'\cdot\omega)(\omega+\omega')}{\sqrt{1-(\omega'\cdot\omega)^2}}, \tag{12.88}$$

which yields the estimate:

$$e+e'\lesssim|\nu-\nu'|. \tag{12.89}$$

This allows us to rewrite the third term in the right-hand side of (12.76) as:

(12.90)

$$\begin{aligned}
&2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla_{e'}b(x,\omega)\overline{\nabla_e b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\\
&\quad=2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla\!\!\!/_e b(x,\omega)\overline{\nabla_e b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\\
&\qquad+2^{-2j}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla_{e+e'}b(x,\omega)\overline{\nabla_e b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx.
\end{aligned}$$

The first term in the right-hand side of (12.90) is estimated in exactly as we proceeded for the second term in the right-hand side of (12.76) (i.e., by performing an additional integration by parts with the help of (12.72)). The second term in the right-hand side of (12.90) is estimated directly by:

$$
\begin{aligned}
&2^{-2j}\left|\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{\nabla_{e+e'}b(x,\omega)\overline{\nabla_e b(x',\omega')}}{1-(\omega'\cdot\omega)^2}F_{j,k,-1}(x\cdot\omega)\overline{F_{j,k,-1}(x\cdot\omega')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' dx\right|\\
&\leq 2^{-2j}\int_{\mathbb{S}^2\times\mathbb{S}^2}\frac{|e+e'|}{1-(\omega'\cdot\omega)^2}\|\nabla b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{j,k,-1}\|_{L^2_{x\cdot\omega}}\|\nabla b(x',\omega')\|_{L^\infty_{u'}L^2(P_{u'})}\\
&\quad\times\|F_{j,k,-1}\|_{L^2_{x\cdot\omega'}}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega'\\
&\lesssim\frac{D^2}{2^{j/2}(2^{j/2}|\nu-\nu'|)}\gamma_j^{\nu,k}\gamma_j^{\nu',k},
\end{aligned}\tag{12.91}
$$

where we have used Plancherel to estimate $\|F_{j,k,-1}\|_{L^2_{x\cdot\omega}}$ and $\|F_{j,k,-1}\|_{L^2_{x\cdot\omega'}}$, Cauchy-Schwartz in ω and ω', the assumption (11.51) on b, and the estimate (12.89). (12.91) satisfies the wanted estimate (12.66) for $0<\alpha\leq 1$. We now control all the terms in the right-hand side of (12.76) which concludes the proof of (12.66).

12.4.4. Proof of (12.67). — The estimate (12.67) will result of two integrations by parts, one with respect to the normal derivative, and one with respect to tangential derivatives. We first integrate by parts with respect to $\partial_{x\cdot\omega}$ in $\int_{\mathbb{R}^3}B_j^{\nu,k}f(x)\overline{B_j^{\nu',k'}f(x)}dx$ using (12.23). We obtain:

$$
\begin{aligned}
\int_{\mathbb{R}^3}B_j^{\nu,k}f(x)\overline{B_j^{\nu',k'}f(x)}dx = &\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{i}{\lambda-\lambda'\omega\cdot\omega'}\\
&\times\partial_{x\cdot\omega}b(x,\omega)b(x,\omega')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\\
&\times\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')\\
&\times f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega' dx\\
&+\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{i}{\lambda-\lambda'\omega\cdot\omega'}\\
&\times b(x,\omega)\partial_{x\cdot\omega}b(x,\omega')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\\
&\times\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')\\
&\times f(\lambda\omega)f(\lambda'\omega')\lambda^2\lambda'^2 d\lambda d\lambda' d\omega d\omega' dx.
\end{aligned}\tag{12.92}
$$

We then integrate a second time by parts using (12.70) for the first term in the right-hand side of (12.92), and using (12.72) for the second term in the right-hand side of

(12.92) (so that there is at least one tangential derivative on $b(x,\omega')$). We obtain:

(12.93)

$$\begin{aligned}
&\int_{\mathbb{R}^3} B_j^{\nu,k} f(x)\overline{B_j^{\nu',k'} f(x)}dx = \int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(\lambda-\lambda'\omega\cdot\omega')\lambda'\sqrt{1-(\omega\cdot\omega')^2}}\\
&\times \nabla\!\!\!/_e\partial_{x\cdot\omega}b(x,\omega)b(x,\omega')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')f(\lambda\omega)f(\lambda'\omega')\\
&\times \lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega' dx + \int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(\lambda-\lambda'\omega\cdot\omega')\lambda'\sqrt{1-(\omega\cdot\omega')^2}}\\
&\times \partial_{x\cdot\omega}b(x,\omega)\nabla_e b(x,\omega')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')f(\lambda\omega)f(\lambda'\omega')\\
&\times \lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega' dx + \int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(\lambda-\lambda'\omega\cdot\omega')\lambda\sqrt{1-(\omega\cdot\omega')^2}}\\
&\times \nabla_{e'} b(x,\omega)\partial_{x\cdot\omega}b(x,\omega')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')f(\lambda\omega)f(\lambda'\omega')\\
&\times \lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega' dx + \int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(\lambda-\lambda'\omega\cdot\omega')\lambda\sqrt{1-(\omega\cdot\omega')^2}}\\
&\times b(x,\omega)\nabla\!\!\!/_{e'}\partial_{x\cdot\omega}b(x,\omega')\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')f(\lambda\omega)f(\lambda'\omega')\\
&\times \lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega' dx.
\end{aligned}$$

Since $|\lambda - k2^j|\nu-\nu'|^\alpha| \leq 2^j|\nu-\nu'|^\alpha$ on the support of ϕ_k and

$$|\lambda' - k'2^j|\nu-\nu'|^\alpha| \leq 2^j|\nu-\nu'|^\alpha$$

on the support of $\phi_{k'}$, we have the following expansion:

$$\begin{aligned}
\frac{1}{\lambda-\lambda'\omega\cdot\omega'} &= \frac{1}{(k-k')2^j|\nu-\nu'|^\alpha}\sum_{p,q,r\geq 0} c_{p,q,r}\left(\frac{\lambda-k2^j|\nu-\nu'|^\alpha}{(k-k')2^j|\nu-\nu'|^\alpha}\right)^p\\
&\quad\times\left(\frac{\lambda'-k'2^j|\nu-\nu'|^\alpha}{(k-k')2^j|\nu-\nu'|^\alpha}\right)^q\left(\frac{\lambda'|\omega-\omega'|^2}{(k-k')2^j|\nu-\nu'|^\alpha}\right)^r.
\end{aligned}\tag{12.94}$$

For $p\in\mathbb{Z}$ and $q\in\mathbb{N}$, we introduce the notation $F_{j,k,p,q}(x\cdot\omega)$:

(12.95)

$$F_{j,k,p,q}(x\cdot\omega) = \int_0^{+\infty} e^{i\lambda x\cdot\omega}\psi(2^{-j}\lambda)\phi_k(\lambda)f(\lambda\omega)(2^{-j}\lambda)^p\left(\frac{\lambda-k2^j|\nu-\nu'|^\alpha}{2^j|\nu-\nu'|^\alpha}\right)^q\lambda^2 d\lambda.$$

(12.93), (12.94) and (12.95) yield:

$$\int_{\mathbb{R}^3} B_j^{\nu,k} f(x)\overline{B_j^{\nu',k'} f(x)}dx = \sum_{p,q,r\geq 0} c_{p,q,r}(A^{1,1}_{p,q,r}+A^{1,2}_{p,q,r}+A^{2,1}_{p,q,r}+A^{2,2}_{p,q,r}),\tag{12.96}$$

where $A^{1,1}_{p,q,r}$, $A^{1,2}_{p,q,r}$, $A^{2,1}_{p,q,r}$ and $A^{2,2}_{p,q,r}$ are given by:

(12.97)
$$A^{1,1}_{p,q,r} = \frac{1}{(k-k')^{p+q+r+1}2^{2j}|\nu-\nu'|^\alpha}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{1}{\sqrt{1-(\omega\cdot\omega')^2}} \times\left(\frac{|\omega-\omega'|^2}{|\nu-\nu'|^\alpha}\right)^r \nabla\!\!\!/_e\partial_{x\cdot\omega}b(x,\omega)F_{j,k,0,p}(x\cdot\omega)\overline{b(x,\omega')F_{j,k',r-1,q}(x\cdot\omega')}d\omega d\omega' dx,$$

(12.98)
$$A^{1,2}_{p,q,r} = \frac{1}{(k-k')^{p+q+r+1}2^{2j}|\nu-\nu'|^\alpha}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{1}{\sqrt{1-(\omega\cdot\omega')^2}} \times\left(\frac{|\omega-\omega'|^2}{|\nu-\nu'|^\alpha}\right)^r \partial_{x\cdot\omega}b(x,\omega)F_{j,k,0,p}(x\cdot\omega)\overline{\nabla_e b(x,\omega')F_{j,k',r-1,q}(x\cdot\omega')}d\omega d\omega' dx,$$

(12.99)
$$A^{2,1}_{p,q,r} = \frac{1}{(k-k')^{p+q+r+1}2^{2j}|\nu-\nu'|^\alpha}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{1}{\sqrt{1-(\omega\cdot\omega')^2}} \times\left(\frac{|\omega-\omega'|^2}{|\nu-\nu'|^\alpha}\right)^r \nabla_{e'}b(x,\omega)F_{j,k,-1,p}(x\cdot\omega)\overline{\partial_{x\cdot\omega}b(x,\omega')F_{j,k',r,q}(x\cdot\omega')}d\omega d\omega' dx$$

and

(12.100)
$$A^{2,2}_{p,q,r} = \frac{1}{(k-k')^{p+q+r+1}2^{2j}|\nu-\nu'|^\alpha}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{1}{\sqrt{1-(\omega\cdot\omega')^2}} \times\left(\frac{|\omega-\omega'|^2}{|\nu-\nu'|^\alpha}\right)^r b(x,\omega)F_{j,k,-1,p}(x\cdot\omega)\overline{\nabla\!\!\!/_{e'}\partial_{x\cdot\omega}b(x,\omega')F_{j,k',r,q}(x\cdot\omega')}d\omega d\omega' dx.$$

Control of $A^{1,1}_{p,q,r}$, $A^{1,2}_{p,q,r}$, $A^{2,1}_{p,q,r}$ and $A^{2,2}_{p,q,r}$. — We start by evaluating $A^{1,2}_{p,q,r}$. We have:

$$|A^{1,2}_{p,q,r}| \le \frac{1}{(k-k')^{p+q+r+1}2^{2j}|\nu-\nu'|^\alpha}\int_{\mathbb{R}^3\times\mathbb{S}^2\times\mathbb{S}^2}\frac{1}{\sqrt{1-(\omega\cdot\omega')^2}}\|\nabla b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)} \times\|F_{j,k,0,p}\|_{L^2_{x\cdot\omega}}\|\nabla b(x,\omega')\|_{L^\infty_{u'}L^2(P_{u'})}\|F_{j,k',r-1,q}\|_{L^2_{x\cdot\omega'}}d\omega d\omega'$$

(12.101)
$$\lesssim \frac{D^2\gamma^{\nu,k}_j\gamma^{\nu',k'}_j}{(k-k')^{p+q+r+1}2^{j/2(1-\alpha)}(2^{j/2}|\nu-\nu'|)^{1+\alpha}},$$

where we have used Plancherel to estimate $\|F_{j,k,0,p}\|_{L^2_{x\cdot\omega}}$ and $\|F_{j,k',r,q}\|_{L^2_{x\cdot\omega'}}$, Cauchy-Schwartz in ω and ω' and the assumption (11.51) on b. We control $A^{2,1}_{p,q,r}$ in the same way.

It remains to estimate $A^{1,1}_{p,q,r}$ and $A^{2,2}_{p,q,r}$. They are controlled in the the same way, so we focus on estimating $A^{1,1}_{p,q,r}$. Using the expansions (12.78) and (12.79), we obtain:

$$A^{1,1}_{p,q,r} = \sum_{l,m\ge 0} c_{p,q,r,l,m}A^{1,1}_{p,q,r,l,m}, \tag{12.102}$$

where $A^{1,1}_{p,q,r,l,m}$ are given by:

$$\begin{aligned} A^{1,1}_{p,q,r,l,m} = & \frac{1}{(k-k')^{p+q+r+1}2^{j(3/2-\alpha/2)}(2^{j/2}|\nu-\nu'|)^{1+\alpha}} \\ & \times \int_{\mathbb{R}^3} \overline{\left(\int_{\mathbb{S}^2}\left(\frac{\omega-\nu}{|\nu-\nu'|}\right)^l \nabla\!\!\!/\nabla b(x,\omega)F_{j,k,0,p}(x\cdot\omega)\eta^\nu_j(\omega)d\omega\right)} \\ & \times \left(\int_{\mathbb{S}^2}\left(\frac{\omega'-\nu'}{|\nu-\nu'|}\right)^m b(x,\omega')F_{j,k',r-1,q}(x\cdot\omega')\eta^{\nu'}_j(\omega')d\omega'\right)dx. \end{aligned} \tag{12.103}$$

The terms in the expansion (12.102) have more and more decay, and it is enough to consider the first one. We have:

$$\begin{aligned} & \frac{1}{(k-k')^{p+q+r+1}2^{j(3/2-\alpha/2)}(2^{j/2}|\nu-\nu'|)^{1+\alpha}} \\ & \quad \times \left|\int_{\mathbb{R}^3} \overline{\left(\int_{\mathbb{S}^2} \nabla\!\!\!/\nabla b(x,\omega)F_{j,k,0,p}(x\cdot\omega)\eta^\nu_j(\omega)d\omega\right)}\right. \\ & \quad \times \left.\left(\int_{\mathbb{S}^2} b(x,\omega')F_{j,k',r-1,q}(x\cdot\omega')\eta^{\nu'}_j(\omega')d\omega'\right)dx\right| \\ & \leq \frac{1}{(k-k')^{p+q+r+1}2^{j(3/2-\alpha/2)}(2^{j/2}|\nu-\nu'|)^{1+\alpha}} \\ & \quad \times \left(\int_{\mathbb{S}^2} \|\nabla\!\!\!/\nabla b(x,\omega)\|_{L^2(\mathbb{R}^3)}\|F_{j,k,0,p}\|_{L^\infty_{x\cdot\omega}}\eta^\nu_j(\omega)d\omega\right) \\ & \quad \times \left\|\int_{\mathbb{S}^2} b(x,\omega')F_{j,k',r-1,q}(x\cdot\omega')\eta^{\nu'}_j(\omega')d\omega'\right\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{12.104}$$

Using the estimate for the diagonal term (12.18) yields:

$$\left\|\int_{\mathbb{S}^2} b(x,\omega')F_{j,k',r-1,q}(x\cdot\omega')\eta^{\nu'}_j(\omega')d\omega'\right\|_{L^2(\mathbb{R}^3)} \lesssim D\gamma^{\nu',k'}_j. \tag{12.105}$$

Finally, the assumption (11.51) on $b(x,\omega)$, the size of the support in ω, the bound (12.84) on $\|F_{j,k,0,p}\|_{L^\infty_{x\cdot\omega}}$, (12.103), (12.104) and (12.105) imply:

$$|A^{1,1}_{p,q,r}| \lesssim \frac{D^2|\nu-\nu'|^\alpha\gamma^{\nu,k}_j\gamma^{\nu',k'}_j}{(k-k')^{p+q+r+1}2^{j/2(1-\alpha)}(2^{j/2}|\nu-\nu'|)^{1+\alpha}}. \tag{12.106}$$

Summing in p,q,r the estimate (12.101) and its analog for $A^{2,1}_{p,q,r}$ together with (12.106) and its analog for $A^{2,2}_{p,q,r}$, and using (12.96), we obtain the wanted estimate (12.67).

12.4.5. End of the proof of Proposition 12.2. — We have proved the estimates (12.66) and (12.67) in the two previous sections. Since (12.66) and (12.67) yield (12.57) (see Section 12.4.2), this concludes the proof of Proposition 12.2. □

12.5. Proof of Proposition 12.3 (Control of the diagonal term)

We have to prove (12.18):

$$\|B_j^\nu f\|_{L^2(\mathbb{R}^3)} \lesssim D\gamma_j^\nu. \tag{12.107}$$

Recall that B_j^ν is given by:

$$B_j^\nu f(x) = \int_{\mathbb{S}^2} b(x,\omega)F_j(x\cdot\omega)\eta_j^\nu(\omega)d\omega, \tag{12.108}$$

where $F_j(x\cdot\omega)$ is defined by:

$$F_j(x\cdot\omega) = \int_0^{+\infty} e^{i\lambda x\cdot\omega}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda. \tag{12.109}$$

We decompose B_j^ν in the sum of two terms:

$$B_j^\nu f(x) = b(x,\nu)\int_{\mathbb{S}^2} F_j(x\cdot\omega)\eta_j^\nu(\omega)d\omega + \int_{\mathbb{S}^2}(b(x,\omega)-b(x,\nu))F_j(x\cdot\omega)\eta_j^\nu(\omega)d\omega. \tag{12.110}$$

Notice that the first term in the right-hand side of (12.110) is equal to

$$b(x,\nu)\int_{\mathbb{S}^2} F_j(x\cdot\omega)\eta_j^\nu(\omega)d\omega = b(x,\nu)\mathscr{F}^{-1}(\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega))(x), \tag{12.111}$$

where $\mathscr{F}$ denotes the Fourier transform on $\mathbb{R}^3$. Now, the assumption (11.51) on b imply that $\|b\|_{L^\infty(\mathbb{R}^3)} \lesssim D$. Together with (12.111), this yields:

$$\left\| b(x,\nu)\int_{\mathbb{S}^2} F_j(x\cdot\omega)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathbb{R}^3)} \lesssim D\gamma_j^\nu. \tag{12.112}$$

We turn to the second term in the right-hand side of (12.110). We have:

$$\begin{aligned}&\left\|\int_{\mathbb{S}^2}(b(x,\omega)-b(x,\nu))F_j(x\cdot\omega)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathbb{R}^3)}\\ &\leq \int_{\mathbb{S}^2}\|b(x,\omega)-b(x,\nu)\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_j\|_{L^2_{x\cdot\omega}}\eta_j^\nu(\omega)d\omega.\end{aligned} \tag{12.113}$$

Now, $H^{1/2+\alpha}(\mathbb{R}^3)$ embeds in $L^\infty_{[-2,2]}L^2(P_u)$ for any $\alpha>0$, thus:

$$\|b(x,\omega)-b(x,\nu)\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim \|b(x,\omega)-b(x,\nu)\|_{H^{1/2+\alpha}(\mathbb{R}^3)} \lesssim |\omega-\nu|\|\partial_\omega b\|_{H^{1/2+\alpha}}. \tag{12.114}$$

Together with (12.113), this yields:

$$\begin{aligned}&\left\|\int_{\mathbb{S}^2}(b(x,\omega)-b(x,\nu))F_j(x\cdot\omega)\eta_j^\nu(\omega)d\omega\right\|_{L^2(\mathbb{R}^3)}\\ &\leq \int_{\mathbb{S}^2}|\omega-\nu|\|\partial_\omega b\|_{H^{1/2+\alpha}(\mathbb{R}^3)}\|F_j\|_{L^2_{x\cdot\omega}}\eta_j^\nu(\omega)d\omega \lesssim D\gamma_j^\nu,\end{aligned} \tag{12.115}$$

where we have used Plancherel to estimate $\|F_j\|_{L^2_{x\cdot\omega}}$, Cauchy-Schwartz in ω, the assumption (11.51) on b, and the fact that $|\omega-\nu|\lesssim 2^{-j/2}$ on the support of η_j^ν.

Finally, (12.110), (12.112) and (12.115) yield the wanted estimate (12.107) which concludes the proof of Proposition 12.3. $\square$.

CHAPTER 13

PROOF OF THEOREM 11.8 (L^2 BOUNDEDNESS FOR FOURIER INTEGRAL OPERATOR)

13.1. The basic computation

We start the proof of Theorem 11.8 with the following instructive computation:

$$\begin{aligned}\|Uf\|_{L^2(S)} &\leq \int_{\mathbb{S}^2}\left\|b(x,\omega)\int_0^{+\infty} e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2(S)} d\omega \\ &\leq \int_{\mathbb{S}^2}\|b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)}\left\|\int_0^{+\infty} e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2_u} d\omega \\ &\leq D\|\lambda f\|_{L^2(\Sigma)},\end{aligned} \tag{13.1}$$

where we have used Plancherel with respect to λ, Cauchy-Schwarz with respect to ω and (11.51) to bound $\|b\|_{L^\infty_{[-2,2]}L^2(P_u)}$. (13.1) misses the conclusion (11.45) of Theorem 11.8 by a power of λ. Now, assume for a moment that we may replace a power of λ by a derivative on $b(x,\omega)$. Then, the same computation yields:

$$\begin{aligned}&\left\|\int_{\mathbb{S}^2}\int_0^{+\infty} \nabla b(x,\omega)e^{i\lambda u}f(\lambda\omega)\lambda d\lambda d\omega\right\|_{L^2(S)} \\ &\leq \int_{\mathbb{S}^2}\|\nabla b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)}\left\|\int_0^{+\infty} e^{i\lambda u}f(\lambda\omega)\lambda^2 d\lambda\right\|_{L^2_u} d\omega \\ &\leq D\|f\|_{L^2(\Sigma)},\end{aligned} \tag{13.2}$$

which is (11.45). This suggests a strategy which consists in making integrations by parts to trade powers of λ against derivatives of the symbol $b(x,\omega)$.

13.2. Structure of the proof of Theorem 11.8

The proof of Theorem 11.8 proceeds in three steps. We first localize in frequencies of size $\lambda \sim 2^j$. We then localize the angle ω in patches on the sphere $\mathbb{S}^2$ of diameter $2^{-j/2}$. Finally, we estimate the diagonal terms.

13.2.1. Step 1: Decomposition in frequency. — For the first step, we introduce φ and ψ two smooth compactly supported functions on $\mathbb{R}$ such that:

$$\varphi(\lambda)+\sum_{j\geq 0}\psi(2^{-j}\lambda)=1 \text{ for all } \lambda\in\mathbb{R}. \tag{13.3}$$

We use (13.3) to decompose Uf as follows:

$$Uf(x)=\sum_{j\geq -1}U_jf(x), \tag{13.4}$$

where for $j\geq 0$:

$$U_jf(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u}b(x,\omega)\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2d\lambda d\omega \tag{13.5}$$

and

$$U_{-1}f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u}b(x,\omega)\varphi(\lambda)f(\lambda\omega)\lambda^2d\lambda d\omega. \tag{13.6}$$

This decomposition is classical and is known as the first dyadic decomposition (see [**21**]). The goal of this first step is to prove the following proposition:

Proposition 13.1. — *The decomposition* (13.4) *satisfies an almost orthogonality property:*

$$\|Uf\|^2_{L^2(S)}\lesssim\sum_{j\geq -1}\|U_jf\|^2_{L^2(S)}+D^2\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{13.7}$$

The proof of Proposition 13.1 is postponed to Section 13.3.

13.2.2. Step 2: Decomposition in angle. — Proposition 13.1 allows us to estimate $\|U_jf\|_{L^2(S)}$ instead of $\|Uf\|_{L^2(S)}$. The analog of computation (13.1) for $\|U_jf\|_{L^2(S)}$ yields:

$$\|U_jf\|_{L^2(S)}\leq D\|\lambda\psi(2^{-j}\lambda)f\|_{L^2(\Sigma)}\lesssim D2^j\|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)}, \tag{13.8}$$

which misses the wanted estimate by a power of 2^j. We thus need to perform a second dyadic decomposition (see [**21**]). We introduce a smooth partition of unity on the sphere $\mathbb{S}^2$:

$$\sum_{\nu\in\Gamma}\eta_j^\nu(\omega)=1 \text{ for all } \omega\in\mathbb{S}^2, \tag{13.9}$$

where the support of η_j^ν is a patch on $\mathbb{S}^2$ of diameter $\sim 2^{-j/2}$. We use (13.9) to decompose U_jf as follows:

$$U_jf(x)=\sum_{\nu\in\Gamma}U_j^\nu f(x), \tag{13.10}$$

where:

$$U_j^\nu f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{i\lambda u}b(x,\omega)\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2d\lambda d\omega. \tag{13.11}$$

We also define:

$$\gamma_{-1} = \|\varphi(\lambda)f\|_{L^2(\mathbb{R}^3)}, \qquad \gamma_j = \|\psi(2^{-j}\lambda)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,$$
$$\gamma_j^\nu = \|\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\ \nu\in\Gamma, \tag{13.12}$$

which satisfy:

$$\|f\|^2_{L^2(\mathbb{R}^3)} = \sum_{j\geq -1}\gamma_j^2 = \sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2. \tag{13.13}$$

The goal of this second step is to prove the following proposition:

Proposition 13.2. — *The decomposition* (13.10) *satisfies an almost orthogonality property:*

$$\|U_jf\|^2_{L^2(S)} \lesssim \sum_{\nu\in\Gamma}\|U_j^\nu f\|^2_{L^2(S)} + D^2\gamma_j^2. \tag{13.14}$$

The proof of Proposition 13.2 is postponed to Section 13.4.

13.2.3. Step 3: Control of the diagonal term. — Proposition 13.2 allows us to estimate $\|U_j^\nu f\|_{L^2(S)}$ instead of $\|U_jf\|_{L^2(S)}$. The analog of computation (13.1) for $\|U_j^\nu f\|_{L^2(S)}$ yields:

(13.15)
$$\begin{aligned}\|U_j^\nu f\|_{L^2(S)} &\leq \int_{\mathbb{S}^2}\|b(x,\omega)\|_{L^\infty_{[-2,2]}L^2(P_u)}\left\|\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2d\lambda\right\|_{L^2_u}d\omega\\ &\leq D\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))}\|\lambda\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f\|_{L^2(\mathbb{R}^3)}\\ &\lesssim D2^{j/2}\gamma_j^\nu,\end{aligned}$$

where the term $\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))}$ comes from the fact that we apply Cauchy-Schwarz in ω. Note that we have used in (13.15) the fact that the support of η_j^ν is 2 dimensional and has diameter $2^{-j/2}$ so that:

$$\sqrt{\mathrm{vol}(\mathrm{supp}(\eta_j^\nu))} \lesssim 2^{-j/2}. \tag{13.16}$$

Now, (13.15) still misses the wanted estimate by a power of $2^{j/2}$. Nevertheless, we are able to estimate the diagonal term:

Proposition 13.3. — *The diagonal term $U_j^\nu f$ satisfies the following estimate:*

$$\|U_j^\nu f\|_{L^2(S)} \lesssim D\gamma_j^\nu. \tag{13.17}$$

The proof of Proposition 13.3 is postponed to Section 13.5.

13.2.4. Proof of Theorem 11.8. — Proposition 13.1, 13.2 and 13.3 immediately yield the proof of Theorem 11.8. Indeed, (13.7), (13.13), (13.14) and (13.17) imply:

$$\begin{aligned}
\|Uf\|^2_{L^2(S)} &\lesssim \sum_{j\geq -1}\|U_jf\|^2_{L^2(S)} + D^2\|f\|^2_{L^2(\mathbb{R}^3)}\\
&\lesssim \sum_{j\geq -1}\sum_{\nu\in\Gamma}\|U_j^\nu f\|^2_{L^2(S)} + D^2\sum_{j\geq -1}\gamma_j^2 + D^2\|f\|^2_{L^2(\mathbb{R}^3)}\\
&\lesssim D^2\sum_{j\geq -1}\sum_{\nu\in\Gamma}(\gamma_j^\nu)^2 + D^2\sum_{j\geq -1}\gamma_j^2 + D^2\|f\|^2_{L^2(\mathbb{R}^3)}\\
&\lesssim D^2\|f\|^2_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{13.18}$$

which is the conclusion of Theorem 11.8. □

The remainder of Chapter 13 is dedicated to the proof of Propositions 13.1, 13.2 and 13.3.

13.3. Proof of Proposition 13.1 (almost orthogonality in frequency)

We have to prove (13.7):

$$\|Uf\|^2_{L^2(S)} \lesssim \sum_{j\geq -1}\|U_jf\|^2_{L^2(S)} + D^2\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{13.19}$$

This will result from the following inequality using Shur's Lemma:

$$\left|\int_\Sigma U_jf(x)\overline{U_kf(x)}d\Sigma\right| \lesssim D^2 2^{-\frac{|j-k|}{2}}\gamma_j\gamma_k \text{ for } |j-k|>2. \tag{13.20}$$

13.3.1. A first integration by parts. — From now on, we focus on proving (13.20). We may assume $j\geq k+3$. We have:

$$\begin{aligned}
\int_\Sigma U_jf(x)\overline{U_kf(x)}d\Sigma = &\int_{\mathbb{S}^2}\int_0^{+\infty}\int_{\mathbb{S}^2}\int_0^{+\infty}\left(\int_\Sigma e^{i\lambda u-i\lambda' u'}b(x,\omega)\overline{b(x,\omega')}d\Sigma\right)\\
&\times\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2\psi(2^{-k}\lambda')\overline{f(\lambda'\omega')}(\lambda')^2d\lambda d\omega d\lambda' d\omega'.
\end{aligned} \tag{13.21}$$

We integrate by parts with respect to ∂_u in $\int_\Sigma e^{i\lambda u-i\lambda' u'}b(x,\omega)\overline{b(x,\omega')}d\Sigma$ using the coarea formula (11.27) and the fact that:

$$e^{i\lambda u-i\lambda' u'} = -\frac{i}{\lambda-\lambda'\frac{a}{a'}g(N,N')}\partial_u(e^{i\lambda u-i\lambda' u'}), \tag{13.22}$$

where we use the notation u for $u(x,\omega)$, a for $a(x,\omega)$, N for $N(x,\omega)$, u' for $u(x,\omega')$, a' for $a(x,\omega')$ and N' for $N(x,\omega')$. We will also use the notation b for $b(x,\omega)$, b' for $b(x,\omega')$, θ for $\theta(x,\omega)$ and θ' for $\theta(x,\omega')$. Using (13.22), we obtain:

(13.23)

$$\int_\Sigma e^{i\lambda u-i\lambda' u'}b\overline{b'}d\Sigma = i\int_\Sigma e^{i\lambda u-i\lambda' u'}\frac{\partial_u b\overline{b'}}{\lambda-\lambda'\frac{a}{a'}g(N,N')}d\Sigma$$

$$
\begin{aligned}
&+i\int_\Sigma e^{i\lambda u-i\lambda' u'}\frac{b\partial_u\overline{b'}}{\lambda-\lambda'\frac{a}{a'}g(N,N')}d\Sigma\\
&+i\int_\Sigma e^{i\lambda u-i\lambda' u'}\frac{b\overline{b'}\operatorname{tr}\theta}{\lambda-\lambda'\frac{a}{a'}g(N,N')}d\Sigma\\
&+i\lambda'\int_\Sigma e^{i\lambda u-i\lambda' u'}\frac{b\overline{b'}(\frac{\nabla_N a}{a'}g(N,N')-\frac{a\nabla_N a'}{a'^2}g(N,N'))}{(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}d\Sigma\\
&+i\lambda'\int_\Sigma e^{i\lambda u-i\lambda' u'}\frac{b\overline{b'}\frac{a}{a'}(g(\nabla_N N,N')+g(N,\nabla_N N'))}{(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}d\Sigma,
\end{aligned}
$$

where we have used (11.28) to obtain the third term in the right-hand side of (13.23). Since $|\lambda'\frac{a}{a'}g(N,N')|<\lambda$, we may expand the fractions in (13.23):

$$
\frac{1}{\lambda-\lambda'\frac{a}{a'}g(N,N')}=\sum_{p\geq 0}\frac{1}{\lambda}\left(\frac{\lambda'\frac{a}{a'}g(N,N')}{\lambda}\right)^p \tag{13.24}
$$

and

$$
\frac{1}{(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}=\sum_{p\geq 0}\frac{p+1}{\lambda^2}\left(\frac{\lambda'\frac{a}{a'}g(N,N')}{\lambda}\right)^p. \tag{13.25}
$$

For $p\in\mathbb{Z}$, We introduce the notation $F_{j,p}(u)$:

$$
F_{j,p}(u)=\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)f(\lambda\omega)(2^{-j}\lambda)^p\lambda^2 d\lambda. \tag{13.26}
$$

Together with (13.21), (13.23) and (13.24), this implies:

$$
\int_\Sigma U_jf(x)\overline{U_kf(x)}d\Sigma=\sum_{p\geq 0}A_p^1+\sum_{p\geq 0}A_p^2+\sum_{p\geq 0}A_p^3+\sum_{p\geq 0}A_p^4, \tag{13.27}
$$

where A_p^1, A_p^2, A_p^3 and A_p^4 are given by:

(13.28)
$$
\begin{aligned}
A_p^1=2^{-j-p(j-k)}\int_\Sigma&\left(\int_{\mathbb{S}^2}(\nabla_N b+b\operatorname{tr}\theta)a^{p+1}N^pF_{j,-p-1}(u)d\omega\right)\\
&\cdot\overline{\left(\int_{\mathbb{S}^2}b'a'^{-p}N'^pF_{k,p}(u')d\omega'\right)}d\Sigma,
\end{aligned}
$$

(13.29)
$$
\begin{aligned}
A_p^2=2^{-j-p(j-k)}\int_\Sigma&\left(\int_{\mathbb{S}^2}ba^{p+1}N^{p+1}F_{j,-p-1}(u)d\omega\right)\\
&\cdot\overline{\left(\int_{\mathbb{S}^2}\nabla b'a'^{-p}N'^pF_{k,p}(u')d\omega'\right)}d\Sigma.
\end{aligned}
$$

(13.30)
$$
A_p^3=(p+1)2^{-j-(p+1)(j-k)}\int_\Sigma\left(\int_{\mathbb{S}^2}b(\nabla_N aN+a\nabla_N N)a^pN^pF_{j,-p-2}(u)d\omega\right)
$$

$$\cdot \overline{\left(\int_{\mathbb{S}^2} b'a'^{-p-1}N'^{p+1}F_{k,p+1}(u')d\omega'\right)}d\Sigma$$

and

(13.31)

$$A_p^4 = (p+1)2^{-j-(p+1)(j-k)}\int_\Sigma\left(\int_{\mathbb{S}^2} ba^{p+1}N^{p+2}F_{j,-p-2}(u)d\omega\right)$$
$$\cdot\overline{\left(\int_{\mathbb{S}^2} b'(\nabla\log(a')N'+\nabla N')a'^{-p-1}N'^pF_{k,p+1}(u')d\omega'\right)}d\Sigma.$$

Remark 13.4. — The expansion (13.24) allows us to rewrite $\int_\Sigma U_jf(x)\overline{U_kf(x)}d\Sigma$ in the form (13.27), i.e., as a sum of terms A_p^1, A_p^2, A_p^3, A_p^4. The key point is that in each of these terms—according to (13.28)-(13.31)—one may separate the terms depending of (λ,ω) from the terms depending on (λ',ω').

13.3.2. Estimates for A_p^1 and A_p^2. — Let $H(x,\omega)$ a tensor such that $\|H\|_{L^\infty_{[-2,2]}L^2(P_u)}\lesssim D$. Then proceeding as in the basic computation (13.1), we have for any $p\in\mathbb{Z}$:

(13.32)

$$\begin{aligned}\left\|\int_{\mathbb{S}^2}H(x,\omega)F_{j,p}(u)d\omega\right\|_{L^2(S)} &\le \int_{\mathbb{S}^2}\|H\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{j,p}(u)\|_{L^2_u}d\omega\\ &\le \|H\|_{L^\infty_{[-2,2]}L^2(P_u)}\|\psi(2^{-j}\lambda)f(\lambda\omega)(2^{-j}\lambda)^p\lambda\|_{L^2(\mathbb{R}^3)}\\ &\lesssim D2^{|p|+j}\gamma_j,\end{aligned}$$

where we have used the fact that $1/2\le 2^{-j}\lambda\le 2$ on the support of $\psi(2^{-j}\lambda)$. Now, Assumption 1 on the regularity of a, N, θ and assumption (11.42) on the regularity of b yield:

$$\begin{aligned}&\|(\nabla_Nb+b\operatorname{tr}\theta)a^{p+1}N^p\|_{L^\infty_{[-2,2]}L^2(P_u)}+\|\nabla b'a'^{-p}N'^p\|_{L^\infty_{u'}L^2(P_{u'})}\\ &\quad+\|b(\nabla_NaN+a\nabla_NN)a^pN^p\|_{L^\infty_{[-2,2]}L^2(P_u)}\\ &\quad+\|b'(\nabla\log(a')N'+\nabla N')a'^{-p-1}N'^p\|_{L^\infty_{u'}L^2(P_{u'})}\lesssim D,\end{aligned}\tag{13.33}$$

which together with (13.32) implies:

$$\begin{aligned}&\left\|\int_{\mathbb{S}^2}(\nabla_Nb+b\operatorname{tr}\theta)a^{p+1}N^pF_{j,-p-1}(u)d\omega\right\|_{L^2(S)}\\ &+\left\|\int_{\mathbb{S}^2}\nabla b'a'^{-p}N'^pF_{k,p}(u')d\omega'\right\|_{L^2(S)}\\ &+\left\|\int_{\mathbb{S}^2}b(\nabla_NaN+a\nabla_NN)a^pN^pF_{j,-p-2}(u)d\omega\right\|_{L^2(S)}\\ &+\left\|\int_{\mathbb{S}^2}b'(\nabla\log(a')N'+\nabla N')a'^{-p-1}N'^pF_{k,p+1}(u')d\omega'\right\|_{L^2(S)}\\ &\lesssim D2^{p+j}\gamma_j.\end{aligned}\tag{13.34}$$

Note that Proposition 13.2 together with Proposition 13.3 yields the estimate:

$$\|U_j f\|_{L^2(S)} \lesssim D\gamma_j, \tag{13.35}$$

for any symbol b satisfying the assumptions (11.42) and (11.43). Now, the terms containing no derivative in (13.28)-(13.31) have a symbol given respectively by $b'a'^{-p}N'^p$, $ba^{p+1}N^{p+1}$, $b'a'^{-p-1}N'^{p+1}$ and $ba^{p+1}N^{p+2}$. These symbols satisfies the assumptions (11.42) and (11.43) since b does, and since a, N, θ satisfy Assumption 1 and Assumption 2. Applying (13.35), we obtain:

$$\left\|\int_{\mathbb{S}^2} b'a'^{-p}N'^p F_{k,p}(u')d\omega'\right\|_{L^2(S)} + \left\|\int_{\mathbb{S}^2} b'a'^{-p-1}N'^{p+1} F_{k,p+1}(u')d\omega'\right\|_{L^2(S)} \lesssim D2^p\gamma_k \tag{13.36}$$

and

$$\left\|\int_{\mathbb{S}^2} ba^{p+1}N^{p+1} F_{j,-p-1}(u)d\omega\right\|_{L^2(S)} + \left\|\int_{\mathbb{S}^2} ba^{p+1}N^{p+2} F_{j,-p-2}(u)d\omega\right\|_{L^2(S)} \lesssim D2^p\gamma_j, \tag{13.37}$$

where we have used the fact that $1/2 \leq 2^{-j}\lambda \leq 2$ on the support of $\psi(2^{-j}\lambda)$.

Finally, the definition of $A_p^1 - A_p^4$ given by (13.28)-(13.31) and the estimates (13.34), (13.36) and (13.37) yield:

$$|A_p^1| \lesssim D2^{2p-p(j-k)}\gamma_j\gamma_k,\ \forall p \geq 0 \tag{13.38}$$

and

$$|A_p^2| + |A_p^3| + |A_p^4| \lesssim D2^{2p-(p+1)(j-k)}\gamma_j\gamma_k,\ \forall p \geq 0. \tag{13.39}$$

(13.38) and (13.39) imply:

$$\begin{aligned}\sum_{p\geq 1}|A_p^1| + \sum_{p\geq 0}(|A_p^2| + |A_p^3| + |A_p^4|) &\lesssim D2^{-(j-k)}\left(\sum_{p\geq 0}2^{-p(j-k-2)}\right)\gamma_j\gamma_k \\ &\lesssim D2^{-(j-k)}\gamma_j\gamma_k,\end{aligned} \tag{13.40}$$

where we have used the assumption $j-k-2>0$. (13.27) and (13.40) will yield (13.20) provided we obtain a similar estimate for A_0^1. Now, the estimate of A_0^1 provided by (13.38) is not sufficient since it does not contain any decay in $j-k$. We will need to perform a second integration by parts for this term.

13.3.3. A more precise estimate for A_0^1. — From (13.28) with $p=0$, we have:

$$A_0^1 = 2^{-j}\int_\Sigma\left(\int_{\mathbb{S}^2}(a\nabla_N b + b\operatorname{tr}\theta)F_{j,-1}(u)d\omega\right)\overline{U_k(x)}. \tag{13.41}$$

We decompose $\nabla_N b = b_1^j + b_2^j$ using the assumption (11.44). In turn, this yields a decomposition for A_0^1:

$$A_0^1 = A_{0,1}^1 + A_{0,2}^1, \tag{13.42}$$

where:

$$A^1_{0,1} = 2^{-j}\int_\Sigma\left(\int_{\mathbb{S}^2} ab^j_1 F_{j,0}(u)d\omega\right)\overline{U_k(x)}d\Sigma,$$
$$A^1_{0,2} = 2^{-j}\int_\Sigma\left(\int_{\mathbb{S}^2} (ab^j_2 + b\mathrm{tr}\,\theta)F_{j,0}(u)d\omega\right)\overline{U_k(x)}d\Sigma. \tag{13.43}$$

We first estimate $A^1_{0,1}$. We have:

$$\begin{aligned}|A^1_{0,1}| &\leq 2^{-j}\int_{\mathbb{S}^2}\left|\int_\Sigma ab^j_1 F_{j,0}(u)\overline{U_k(x)}d\Sigma\right|d\omega\\ &\leq 2^{-j}\int_{\mathbb{S}^2}\|b^j_1\|_{L^2(S)}\|a\|_{L^\infty(S)}\|F_{j,0}\|_{L^2_u}\|U_k\|_{L^\infty_{[-2,2]}L^2(P_u)}d\omega\\ &\lesssim D2^{-\frac{3j}{2}}\int_{\mathbb{S}^2}\|F_{j,0}\|_{L^2_u}\|U_k\|_{L^\infty_{[-2,2]}L^2(P_u)}d\omega,\end{aligned}\tag{13.44}$$

where we have used Assumption 1 on a and the assumption (11.44) on b^j_1 in the last inequality. Plancherel yields:

$$\|F_{j,0}\|_{L^2_{\omega,u}} \leq \|\psi(2^{-j}\lambda)f(\lambda\omega)\lambda\|_{L^2(\mathbb{R}^3)} \lesssim 2^j\gamma_j. \tag{13.45}$$

In view of (13.44), we also need to estimate $\|U_k\|_{L^\infty_{[-2,2]}L^2(P_u)}$. We have:

$$\|U_k\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim (\|\nabla U_k\|_{L^2(S)}+\|U_k\|_{L^2(S)})^{\frac12}\|U_k\|^{\frac12}_{L^2(S)} \lesssim D\gamma_k + D^{\frac12}\gamma_k^{\frac12}\|\nabla U_k\|^{\frac12}_{L^2(S)}, \tag{13.46}$$

where we have used the fact that $H^1(\Sigma)$ embeds in $L^\infty_{[-2,2]}L^2(P_u)$ for the first inequality (see **[23]** Corollary 3.6 for a proof only using the regularity given by Assumption 1), and (13.35) for the second inequality. We still need to estimate $\|\nabla U_k\|_{L^2(S)}$. We have:

$$\begin{aligned}\nabla U_k(x) =& \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u}\nabla b\psi(2^{-k}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega\\ &+ i2^k\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u}ba^{-1}N\psi(2^{-k}\lambda)(2^{-k}\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega.\end{aligned}\tag{13.47}$$

Using the basic computation (13.1) for the first term together with the fact that $\nabla b \in L^\infty_{[-2,2]}L^2(P_u)$, and (13.35) for the second term together with the fact that $ba^{-1}N$ satisfies the assumptions (11.42) and (11.43), we obtain:

$$\|\nabla U_k\|_{L^2(S)} \lesssim D2^k\gamma_k. \tag{13.48}$$

Finally, (13.44), (13.45), (13.46) and (13.48) yield:

$$|A^1_{0,1}| \lesssim D2^{-\frac{j-k}{2}}\gamma_j\gamma_k. \tag{13.49}$$

13.3.4. A second integration by parts. — We now estimate the term $A^1_{0,2}$ defined in (13.43). We perform a second integration by parts relying again on (13.22). We obtain:

$$\begin{aligned}A^1_{0,2} = {} & 2^{-2j}\int_\Sigma\left(\int_{\mathbb{S}^2}(\nabla_N b^j_2 a + b^j_2\nabla_N a + b^j_2 a\operatorname{tr}\theta)F_{j,0}(u)d\omega\right)\overline{U_k(x)}d\Sigma \\ & + 2^{-2j}\int_\Sigma\left(\int_{\mathbb{S}^2} b^j_2 a^2 N F_{j,0}(u)d\omega\right)\cdot\overline{\nabla U_k(x)}d\Sigma + \cdots,\end{aligned} \tag{13.50}$$

where we only mention the first term generated by the expansion (13.24). In fact, the other terms generated by (13.24) and the ones generated by (13.25) are estimated in the same way and generate more decay in $j-k$ similarly to the estimates (13.38) (13.39).

The first term in the right-hand side of (13.50) has the same form than $A^1_{0,1}$ defined in (13.43) where ab^j_1 is replaced by $2^{-j}(\nabla_N b^j_2 a + b^j_2\nabla_N a + ab^j_2\operatorname{tr}\theta)$. By Assumption 1 and (11.44), $2^{-j}(\nabla_N b^j_2 a + b^j_2\nabla_N a + ab^j_2\operatorname{tr}\theta)$ satisfies:

$$\begin{aligned}&\|2^{-j}(\nabla_N b^j_2 a + b^j_2\nabla_N a + ab^j_2\operatorname{tr}\theta)\|_{L^2(S)} \\ &\leq 2^{-j}\|\nabla_N b^j_2\|_{L^2(S)}\|a\|_{L^\infty(S)} + 2^{-j}\|b^j_2\|_{L^2_{[-2,2]}L^\infty(P_u)}\|\nabla_N a\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\quad + 2^{-j}\|a\|_{L^\infty(S)}\|b^j_2\|_{L^2_{[-2,2]}L^\infty(P_u)}\|\operatorname{tr}\theta\|_{L^\infty_{[-2,2]}L^2(P_u)} \\ &\lesssim D2^{-\frac{j}{2}}.\end{aligned} \tag{13.51}$$

Since b^1_j and $2^{-j}(\nabla_N b^j_2 a + b^j_2\nabla_N a + ab^j_2\operatorname{tr}\theta)$ satisfy the same estimate, we obtain the analog of (13.49) for the first term in the right-hand side of (13.50):

$$\left|2^{-2j}\int_\Sigma\left(\int_{\mathbb{S}^2}(\nabla_N b^j_2 a + b^j_2\nabla_N a + ab^j_2\operatorname{tr}\theta)F_{j,0}(u)d\omega\right)\overline{U_k(x)}d\Sigma\right| \lesssim D2^{-\frac{j-k}{2}}\gamma_j\gamma_k. \tag{13.52}$$

We now estimate the second term in the right-hand side of (13.50). Using Assumption 1 on a together with (11.44), we have:

$$\|b^j_2 a^2 N\|_{L^\infty_{[-2,2]}L^2(P_u)} \lesssim D. \tag{13.53}$$

We estimate the second term in the right-hand side of (13.50) using the assumption (11.42) on b, the basic computation (13.1) and (13.53):

$$\begin{aligned}&\left|2^{-2j}\int_\Sigma\left(\int_{\mathbb{S}^2} b^j_2 a^2 N F_{j,0}(u)d\omega\right)\cdot\overline{\left(\int_{\mathbb{S}^2}\nabla b' F_{k,0}(u')d\omega'\right)}d\Sigma\right| \\ &\leq 2^{-2j}\left\|\int_{\mathbb{S}^2} b^j_2 a^2 N F_{j,0}(u)d\omega\right\|_{L^2(S)}\left\|\int_{\mathbb{S}^2}\nabla b' F_{k,0}(u')d\omega'\right\|_{L^2(S)} \\ &\leq 2^{-2j}\left(\int_{\mathbb{S}^2}\|b^j_2 a^2 N\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{j,0}\|_{L^2_u}d\omega\right)\left(\int_{\mathbb{S}^2}\|\nabla b\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{k,0}\|_{L^2_u}d\omega\right) \\ &\lesssim D^2 2^{-(j-k)}\gamma_j\gamma_k.\end{aligned} \tag{13.54}$$

Finally, (13.50), (13.52) and (13.54) imply:

$$|A^1_{0,2}| \lesssim D^2 2^{-\frac{j-k}{2}} \gamma_j \gamma_k. \tag{13.55}$$

13.3.5. End of the proof of Proposition 13.1. — Since $A^1_0 = A^1_1 + A^1_2$, the estimate (13.49) of $A^1_{0,1}$ and the estimate (13.55) of $A^1_{0,2}$ yield:

$$|A^1_0| \lesssim D^2 2^{-\frac{j-k}{2}} \gamma_j \gamma_k. \tag{13.56}$$

Together with (13.27) and (13.40), this implies:

$$\left| \int_\Sigma U_j f(x) \overline{U_k f(x)} d\Sigma \right| \lesssim D^2 2^{-\frac{|j-k|}{2}} \gamma_j \gamma_k \text{ for } |j-k| > 2. \tag{13.57}$$

Finally, (13.57) together with Shur's Lemma yields:

$$\|Uf\|^2_{L^2(S)} \lesssim \sum_{j \geq -1} \|U_j f\|^2_{L^2(S)} + D^2 \|f\|^2_{L^2(\mathbb{R}^3)}. \tag{13.58}$$

This concludes the proof of Proposition 13.1. □

13.4. Proof of Proposition 13.2 (almost orthogonality in angle)

We have to prove (13.14):

$$\|U_j f\|^2_{L^2(S)} \lesssim \sum_{\nu \in \Gamma} \|U^\nu_j f\|^2_{L^2(S)} + D^2 \gamma_j^2. \tag{13.59}$$

This will result from the following inequality:

(13.60)
$$\left| \int_\Sigma U^\nu_j f(x) \overline{U^{\nu'} f(x)} d\Sigma \right| \lesssim \frac{D^2 \gamma^\nu_j \gamma^{\nu'}_j}{2^{j\alpha/2}(2^{j/2}|\nu - \nu'|)^{2-\alpha}} + \frac{D^2 \gamma^\nu_j \gamma^{\nu'}_j}{(2^{j/2}|\nu - \nu'|)^3}, \; |\nu - \nu'| \neq 0,$$

where $\alpha > 0$. Indeed, since $\mathbb{S}^2$ is 2 dimensional and $1 \leq 2^{j/2}|\nu - \nu'| \leq 2^{j/2}$ for $\nu, \nu' \in \Gamma$ and $\nu \neq \nu'$, we have:

$$\sup_\nu \sum_{\nu'} \frac{1}{(2^{j/2}|\nu - \nu'|)^3} \leq C < +\infty \tag{13.61}$$

and

$$\sup_\nu \sum_{\nu'} \frac{1}{2^{j\alpha/2}(2^{j/2}|\nu - \nu'|)^{2-\alpha}} \leq C_\alpha < +\infty \, \forall \alpha > 0. \tag{13.62}$$

Thus, (13.60), (13.61) and (13.62) together with Shur's Lemma imply (13.59).

Remark 13.5. — In **[20]**, the authors rely on a partial Fourier transform with respect to a coordinate system on P_u to prove almost orthogonality in angle for their parametrix. In our case, coordinate systems on P_u are not regular enough, which forces us to work invariantly. More precisely, we will use geometric integrations by parts in tangential directions to P_u in order to obtain (13.60).

13.4.1. A second decomposition in frequency. — From now on, we focus on proving (13.60). Integrating by parts twice in $\int_\Sigma U_j^\nu f(x)\overline{U_j^{\nu'} f(x)}d\Sigma$ would ultimately yield:

$$\left|\int_\Sigma U_j^\nu f(x)\overline{U_j^{\nu'} f(x)}d\Sigma\right| \lesssim \frac{D^2\gamma_j^\nu\gamma_j^{\nu'}}{(2^{j/2}|\nu-\nu'|)^2},\ |\nu-\nu'|\neq 0. \tag{13.63}$$

This corresponds to the case $\alpha=0$ in (13.61) and yields to a log-loss since we have:

$$\sup_\nu\sum_{\nu'}\frac{1}{(2^{j/2}|\nu-\nu'|)^2}\sim j. \tag{13.64}$$

To avoid this log-loss, we do a second decomposition in frequency. λ belongs to the interval $[2^{j-1},2^{j+1}]$ which we decompose in intervals I_k:

$$[2^{j-1},2^{j+1}]=\bigcup_{1\leq k\leq|\nu-\nu'|^{-\alpha}} I_k \text{ where } \operatorname{diam}(I_k)\sim 2^j|\nu-\nu'|^\alpha. \tag{13.65}$$

Let ϕ_k a partition of unity of the interval $[2^{j-1},2^{j+1}]$ associated to the I_k's. We decompose $U_j^\nu f$ as follows:

$$U_j^\nu f(x)=\sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}} U_j^{\nu,k}f(x), \tag{13.66}$$

where:

$$U_j^{\nu,k}f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u}b(x,\omega)\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\phi_k(\lambda)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{13.67}$$

We also define:

$$\gamma_j^{\nu,k}=\|\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\phi_k(\lambda)f\|_{L^2(\mathbb{R}^3)},\ j\geq 0,\ \nu\in\Gamma, 1\leq k\leq|\nu-\nu'|^{-\alpha}, \tag{13.68}$$

which satisfy:

$$(\gamma_j^\nu)^2=\sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}}(\gamma_j^{\nu,k})^2. \tag{13.69}$$

13.4.2. The two key estimates. — We will prove the following two estimates:

$$\left|\int_\Sigma U_j^{\nu,k} f(x)\overline{U_j^{\nu',k} f(x)}d\Sigma\right| \lesssim \frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}}+\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^3} \quad \text{for } |\nu-\nu'|\neq 0,\ 1\leq k\leq|\nu-\nu'|^{-\alpha} \tag{13.70}$$

and

$$\left|\int_\Sigma U_j^{\nu,k} f(x)\overline{U_j^{\nu',k'} f(x)}d\Sigma\right| \lesssim \frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k'}}{|k-k'|2^{j/2(1-4\alpha)}(2^{j/2}|\nu-\nu'|)^{1+4\alpha}}, \quad \text{for } |\nu-\nu'|\neq 0,\ 1\leq k,k'\leq|\nu-\nu'|^{-\alpha},\ k\neq k'. \tag{13.71}$$

(13.70) and (13.71) imply:

(13.72)
$$\begin{aligned}\left|\int_\Sigma U_j^\nu f(x)\overline{U_j^{\nu'} f(x)}d\Sigma\right| \leq & \sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\left|\int_\Sigma U_j^{\nu,k} f(x)\overline{U_j^{\nu',k} f(x)}d\Sigma\right|\\ & + \sum_{1\leq k\neq k'\leq|\nu-\nu'|^{-\alpha}}\left|\int_\Sigma U_j^{\nu,k} f(x)\overline{U_j^{\nu',k'} f(x)}d\Sigma\right|\\ \lesssim & \sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}}\\ & + \sum_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^3}\\ & + \sum_{1\leq k\neq k'\leq|\nu-\nu'|^{-\alpha}}\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k'}}{|k-k'|2^{\frac{j}{2}(1-4\alpha)}(2^{j/2}|\nu-\nu'|)^{1+4\alpha}}\\ \lesssim & \frac{D^2\gamma_j^\nu\gamma_j^{\nu'}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}}+\frac{D^2\gamma_j^\nu\gamma_j^{\nu'}}{(2^{j/2}|\nu-\nu'|)^3},\end{aligned}$$

where we have used (13.69) and the fact that we may choose $0<\alpha<1/5$, together with the fact that:

$$\sup_{1\leq k\leq|\nu-\nu'|^{-\alpha}}\sum_{1\leq k'\leq|\nu-\nu'|^{-\alpha},\, k'\neq k}\frac{1}{|k-k'|}\lesssim\alpha|\log(|\nu-\nu'|)|. \tag{13.73}$$

Since (13.72) yields the wanted estimate (13.60), we are left with proving (13.70) and (13.71).

13.4.3. Proof of (13.70). — The estimate (13.70) will result of two integrations by parts with respect to tangential derivatives. By definition of $\nabla\!\!\!/$, we have $\nabla\!\!\!/ h=\nabla h-(\nabla_N h)N$ for any function h on Σ. In particular, we have $\nabla\!\!\!/(u)=0$ and $\nabla\!\!\!/(u')=a'^{-1}N'-a'^{-1}g(N',N)N$. Now, since $|N'-(N'\cdot N)N|^2=1-(N'\cdot N)^2$, this yields:

$$e^{i\lambda u-i\lambda' u'}=\frac{i}{\lambda'(1-(N'\cdot N)^2)}\nabla_{N'-g(N,N')N}(e^{i\lambda u-i\lambda' u'}), \tag{13.74}$$

where we have used the fact that $N'-(N'\cdot N)N$ is a tangent vector with respect of the level surfaces of u. Similarly, we have:

$$e^{i\lambda u-i\lambda' u'}=-\frac{i}{\lambda(1-(N'\cdot N)^2)}\nabla_{N-g(N,N')N'}(e^{i\lambda u-i\lambda' u'}), \tag{13.75}$$

where we have used the fact that $N-(N\cdot N')N'$ is a tangent vector with respect of the level surfaces of u'. For $p\in\mathbb{Z}$, We introduce the notation $F_{j,k,p}(u)$:

$$F_{j,k,p}(u)=\int_0^{+\infty}e^{i\lambda u}\psi(2^{-j}\lambda)\phi_k(\lambda)f(\lambda\omega)(2^{-j}\lambda)^p\lambda^2d\lambda. \tag{13.76}$$

We integrate once by parts using (13.74) in $\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma$ and we obtain:

$$\begin{aligned}&\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\\ &=i2^{-j}\int_{\Sigma\times\mathbb{S}^2\times\mathbb{S}^2}\text{div}\left(\frac{(N'-(N\cdot N')N)a'b\overline{b'}}{1-(N\cdot N')^2}\right)\\ &\quad\times F_{j,k,0}(u)\overline{F_{j,k,-1}(u')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\Sigma.\end{aligned} \tag{13.77}$$

We then integrate a second time by parts using (13.75) (so that there is at least one tangential derivative for each quantity where two derivatives are taken):

$$\begin{aligned}&\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\\ &=2^{-2j}\int_{\Sigma\times\mathbb{S}^2\times\mathbb{S}^2}\text{div}\left(\frac{(N-(N\cdot N')N')a}{1-(N\cdot N')^2}\text{div}\left(\frac{(N'-(N\cdot N')N)a'b\overline{b'}}{1-(N\cdot N')^2}\right)\right)\\ &\times F_{j,k,-1}(u)\overline{F_{j,k,-1}(u')}\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')d\omega d\omega' d\Sigma.\end{aligned} \tag{13.78}$$

Computation of the right-hand side of (13.78). — We would like to compute the double divergence term in the right-hand side of (13.78). This is done in the following lemma.

Lemma 13.6. — *The double divergence term in the right-hand side of* (13.78) *is given by:*

$$\begin{aligned}&\text{div}\left(\frac{(N-(N\cdot N')N')a}{1-(N\cdot N')^2}div\left(\frac{(N'-(N\cdot N')N)a'b\overline{b'}}{1-(N\cdot N')^2}\right)\right)\\ &=\frac{1}{|N_\nu-N_{\nu'}|^2}\left(\sum_{p,q\geq 0}c_{p,q}\left(\frac{N-N_\nu}{|N_\nu-N_{\nu'}|}\right)^p\left(\frac{N'-N_{\nu'}}{|N_\nu-N_{\nu'}|}\right)^q\right)F,\end{aligned} \tag{13.79}$$

where F is a combination of terms in the following list:

$$\begin{aligned}&\frac{(\nabla\theta-\nabla\theta')aa'bb'}{|N_\nu-N_{\nu'}|},\ \frac{(\theta-\theta')\nabla(ab)a'b'}{|N_\nu-N_{\nu'}|},\ \theta\nabla(ab)a'b',\ ab\theta\nabla(a'b'),\ \nabla\!\!\!/\nabla(ab)a'b',\\ &\nabla(a)\nabla(b)a'b',\ \nabla(ab)\nabla(a'b'),\ \frac{(\theta-\theta')^2aa'bb'}{|N_\nu-N_{\nu'}|^2},\ \theta\theta'aa'bb',\ \theta^2aa'bb'.\end{aligned} \tag{13.80}$$

The proof of Lemma 13.6 is postponed to the Appendix A. The following lemma gives the structure of the terms in the list (13.80).

Lemma 13.7. — *The terms in the list* (13.80) *have the following form:*

$$H_1(x,\omega,\nu,\nu')H_2(x,\omega',\nu,\nu') + \frac{H_3(x,\omega,\nu,\nu')H_4(x,\omega',\nu,\nu')}{2^{j/2}|\nu-\nu'|}, \tag{13.81}$$

where H_1, H_3, H_4 *satisfy:*

$$\|H_1\|_{L^2(S)} + \|H_3\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|H_4\|_{L^\infty_{u'}L^2(P_{u'})} \lesssim D \tag{13.82}$$

and where H_2 *satisfies:*

$$\|H_2\|_{L^\infty(S)} + \|\partial_\omega H_2\|_{L^2(S)} + \|\nabla\partial_\omega H_2\|_{L^2(S)} \lesssim D, \tag{13.83}$$

for ω *in the support of* η_j^ν *and* ω' *in the support of* $\eta_j^{\nu'}$.

The proof of Lemma 13.7 is postponed to the Appendix B. In the rest of this section, we show how Lemma 13.6 and Lemma 13.7 yield the proof of (13.70).

End of the proof of (13.70). — Using (13.78), Lemma 13.6 and Lemma 13.7, we may rewrite $\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma$ as:

$$\begin{aligned}
&\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma \\
&= \sum_{p,q\geq 0} c_{p,q}\int_\Sigma \frac{2^{-j}}{(2^{j/2}|N_\nu-N_{\nu'}|)^2}\left(\int_{\mathbb{S}^2}\left(\frac{N-N_\nu}{|N_\nu-N_{\nu'}|}\right)^p H_1(x,\omega,\nu,\nu')F_{j,k,-1}(u)\eta_j^\nu(\omega)d\omega\right)\\
&\quad\times\left(\int_{\mathbb{S}^2}\left(\frac{N'-N_{\nu'}}{|N_\nu-N_{\nu'}|}\right)^q H_2(x,\omega',\nu,\nu')\overline{F_{j,k,-1}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\\
&\quad+\sum_{p,q\geq 0} c_{p,q}\int_\Sigma \frac{2^{-j}}{(2^{j/2}|N_\nu-N_{\nu'}|)^3}\left(\int_{\mathbb{S}^2}\left(\frac{N-N_\nu}{|N_\nu-N_{\nu'}|}\right)^p H_3(x,\omega,\nu,\nu')F_{j,k,-1}(u)\eta_j^\nu(\omega)d\omega\right)\\
&\quad\times\left(\int_{\mathbb{S}^2}\left(\frac{N'-N_{\nu'}}{|N_\nu-N_{\nu'}|}\right)^q H_4(x,\omega',\nu,\nu')\overline{F_{j,k,-1}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma.
\end{aligned} \tag{13.84}$$

We estimate the two terms in the right-hand side of (13.84) starting with the second one. We have:

$$\begin{aligned}
&\left|\int_\Sigma \frac{2^{-j}}{(2^{j/2}|N_\nu-N_{\nu'}|)^3}\left(\int_{\mathbb{S}^2}\left(\frac{N-N_\nu}{|N_\nu-N_{\nu'}|}\right)^p H_3(x,\omega,\nu,\nu')F_{j,k,-1}(u)\eta_j^\nu(\omega)d\omega\right)\right.\\
&\quad\left.\times\left(\int_{\mathbb{S}^2}\left(\frac{N'-N_{\nu'}}{|N_\nu-N_{\nu'}|}\right)^q H_4(x,\omega',\nu,\nu')\overline{F_{j,k,-1}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\right|\\
&\leq \frac{2^{-j}}{(2^{j/2}|\nu-\nu'|)^{3+p+q}}\left(\int_{\mathbb{S}^2}\|H_3\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{j,k,-1}\|_{L^2_u}\eta_j^\nu(\omega)d\omega\right)\\
&\quad\times\left(\int_{\mathbb{S}^2}\|H_4\|_{L^\infty_{u'}L^2(P_{u'})}\|F_{j,k,-1}\|_{L^2_{u'}}\eta_j^{\nu'}(\omega')d\omega'\right)\\
&\lesssim \frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^{3+p+q}},
\end{aligned} \tag{13.85}$$

where we have used Assumption 2 to estimate $|N_\nu - N_{\nu'}|$, Plancherel to estimate $\|F_{j,k,-1}\|_{L^2_u}$ and $\|F_{j,k,-1}\|_{L^2_{u'}}$, Cauchy-Schwartz in ω and ω', and the estimate (13.82) for H_3 and H_4.

We now estimate the second term in the right-hand side of (13.84). We have:

$$\begin{aligned}
&\left|\int_\Sigma \frac{2^{-j}}{(2^{j/2}|N_\nu - N_{\nu'}|)^2}\left(\int_{\mathbb{S}^2}\left(\frac{N-N_\nu}{|N_\nu - N_{\nu'}|}\right)^p H_1(x,\omega,\nu,\nu')F_{j,k,-1}(u)\eta_j^\nu(\omega)d\omega\right)\right.\\
&\quad\left.\times\left(\int_{\mathbb{S}^2}\left(\frac{N'-N_{\nu'}}{|N_\nu - N_{\nu'}|}\right)^q H_2(x,\omega',\nu,\nu')\overline{F_{j,k,-1}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\right|\\
&\leq \frac{2^{-j}}{(2^{j/2}|\nu-\nu'|)^{2+p+q}}\left(\int_{\mathbb{S}^2}\|H_1\|_{L^2(S)}\|F_{j,k,-1}\|_{L^\infty_u}\eta_j^\nu(\omega)d\omega\right)\\
&\quad\times\left\|\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q H_2(x,\omega',\nu,\nu')F_{j,k,-1}(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(S)}\\
&\lesssim \frac{D|\nu-\nu'|^\alpha\gamma_j^{\nu,k}}{(2^{j/2}|\nu-\nu'|)^{2+p+q}}
\end{aligned}$$

(13.86)

$$\times\left\|\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q H_2(x,\omega',\nu,\nu')F_{j,k,-1}(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(S)},$$

where we have used Assumption 2 to estimate $|N_\nu - N_{\nu'}|$, Cauchy-Schwartz in ω, the estimate (13.82) for H_1 and the following estimate for $\|F_{j,k,-1}\|_{L^\infty_u}$:

$$\|F_{j,k,-1}\|_{L^\infty_u} \lesssim 2^{3j/2}|\nu-\nu'|^{\frac{\alpha}{2}}\|\psi(2^{-j}\lambda)\phi_k(\lambda)f(\lambda\omega)\lambda\|_{L^2_\lambda}, \tag{13.87}$$

which follows form taking Cauchy Schwartz in λ together with the size of the support of ϕ_k. Note that the symbol $F = (2^{j/2}(N'-N_{\nu'}))^q H_2(x,\omega',\nu,\nu')$ satisfies the following assumptions:

$$\|F\|_{L^\infty(S)} \lesssim D,\ \|\partial_\omega F\|_{L^2(S)} \lesssim qD,\ \|\nabla\partial_\omega F\|_{L^2(S)} \lesssim q^2 D, \tag{13.88}$$

where we have used Assumption 2 for $\partial_\omega N$ and $\partial^2_\omega N$, and the assumption (13.83) satisfied by H_2. We will see in Section 13.5 that assumptions (13.88) on a symbol is enough to control the diagonal term in $L^2(S)$ (i.e., to obtain the estimate (13.17)). Thus, we obtain:

(13.89)

$$\left\|\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q H_2(x,\omega',\nu,\nu')F_{j,k,-1}(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(S)} \lesssim (1+q^2)D\gamma_j^{\nu',k}.$$

(13.86) and (13.89) imply:

$$\begin{aligned}
&\left|\int_\Sigma \frac{2^{-j}}{(2^{j/2}|N_\nu - N_{\nu'}|)^2}\left(\int_{\mathbb{S}^2}\left(\frac{N-N_\nu}{|N_\nu - N_{\nu'}|}\right)^p H_1(x,\omega,\nu,\nu')F_{j,k,-1}(u)\eta_j^\nu(\omega)d\omega\right)\right.\\
&\quad\left.\times\left(\int_{\mathbb{S}^2}\left(\frac{N'-N_{\nu'}}{|N_\nu - N_{\nu'}|}\right)^q H_2(x,\omega',\nu,\nu')\overline{F_{j,k,-1}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\right|
\end{aligned}$$

(13.90)
$$\lesssim \frac{(1+q^2)D^2|\nu-\nu'|^{\frac{\alpha}{2}}}{(2^{j/2}|\nu-\nu'|)^{2+p+q}}\gamma_j^{\nu,k}\gamma_j^{\nu',k}.$$

Finally, (13.84), (13.85) and (13.90) yield:

$$\begin{aligned}&\left|\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\right|\\ &\lesssim \sum_{p,q\geq 0} c_{p,q}\frac{(1+q)D^2|\nu-\nu'|^{\frac{\alpha}{2}}}{(2^{j/2}|\nu-\nu'|)^{2+p+q}}\gamma_j^{\nu,k}\gamma_j^{\nu',k}+\sum_{p,q\geq 0}\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^{3+p+q}}\\ &\lesssim \frac{D^2|\nu-\nu'|^{\frac{\alpha}{2}}}{(2^{j/2}|\nu-\nu'|)^2}\gamma_j^{\nu,k}\gamma_j^{\nu',k}+\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^3},\end{aligned}\tag{13.91}$$

which concludes the proof of estimate (13.70).

13.4.4. Proof of (13.71). — The estimate (13.71) will result of two integrations by parts, one with respect to the normal derivative, and one with respect to tangential derivatives. We have:

$$e^{i\lambda u-i\lambda' u'} = -\frac{ia}{\lambda-\lambda'\frac{a}{a'}g(N,N')}\nabla_N(e^{i\lambda u-i\lambda' u'}).\tag{13.92}$$

We integrate once by parts using (13.92) in $\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k'}f(x)}d\Sigma$. We obtain:

$$\begin{aligned}&\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k'}f(x)}d\Sigma = i\int_{\Sigma\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\\ &\times\operatorname{div}\left(\frac{aNbb'}{\lambda-\lambda'\frac{a}{a'}g(N,N')}\right)\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')\\ &\times f(\lambda\omega)f(\lambda'\omega')\lambda^2{\lambda'}^2d\lambda d\lambda' d\omega d\omega' d\Sigma.\end{aligned}\tag{13.93}$$

We then expand the divergence term in the right-hand side of (13.93):

$$\operatorname{div}\left(\frac{aNbb'}{\lambda-\lambda'\frac{a}{a'}g(N,N')}\right) = D_1+D_2,\tag{13.94}$$

where D_1 and D_2 are given by:

$$\begin{aligned}D_1 =&\ \frac{abb'\operatorname{div}(N)+\nabla_N(ab)b'}{\lambda-\lambda'\frac{a}{a'}g(N,N')}\\ &+\lambda'\frac{\nabla_N(a)a{a'}^{-1}bb'g(N,N')+\nabla_N(g(N,N'))a^2{a'}^{-1}bb'}{(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}\end{aligned}\tag{13.95}$$

and

$$D_2 = \frac{ab\nabla_N(b')}{\lambda-\lambda'\frac{a}{a'}g(N,N')}-\lambda'\frac{\nabla_N(a')a^2{a'}^{-2}bb'g(N,N')}{(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}.\tag{13.96}$$

We then integrate a second time by parts using (13.74) for D_1 and using (13.75) for D_2 (so that there is at least one tangential derivative on a, a', b, b' when two derivatives are taken). We obtain:

$$\begin{aligned}
&\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k'}f(x)}d\Sigma = \int_{\Sigma\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\lambda'}\\
&\times \operatorname{div}\left(\frac{(N'-g(N,N')N)a'}{1-g(N,N')^2}D_1\right)\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')\\
&\times f(\lambda\omega)f(\lambda'\omega')\lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega' d\Sigma + \int_{\Sigma\times\mathbb{S}^2\times\mathbb{S}^2}\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\lambda}\\
&\times \operatorname{div}\left(\frac{(N-g(N,N')N')a}{1-g(N,N')^2}D_2\right)\eta_j^\nu(\omega)\eta_j^{\nu'}(\omega')\psi(2^{-j}\lambda)\psi(2^{-j}\lambda')\phi_k(\lambda)\phi_{k'}(\lambda')\\
&\times f(\lambda\omega)f(\lambda'\omega')\lambda^2{\lambda'}^2 d\lambda d\lambda' d\omega d\omega' d\Sigma.
\end{aligned}\tag{13.97}$$

Computation of the right-hand side of (13.97). — We would like to compute the two divergence term in the right-hand side of (13.97). This is done in the following lemma.

Lemma 13.8. — *The two divergence term in the right-hand side of* (13.97) *have the following form:*

(13.98) where $F_j, j = 1,2,3$ is a combination of terms in the following list:

$$\begin{aligned}
&\frac{(\theta-\theta')\theta aa'bb'}{|N_\nu-N_{\nu'}|},\ \frac{(\theta-\theta')\nabla(ab)a'b'}{|N_\nu-N_{\nu'}|},\ \theta\nabla(ab)a'b',\ ab\theta\nabla(a'b'),\ \nabla\!\!\!/\nabla(ab)a'b',\\
&\nabla(a)\nabla(b)a'b',\ \nabla(ab)\nabla(a'b'),\ \nabla(\theta)aa'bb',\ \theta\theta'aa'bb',\ \theta^2aa'bb'.
\end{aligned}\tag{13.99}$$

The proof of Lemma 13.8 is postponed to the Appendix C. The following lemma gives the structure of the terms in the list (13.99).

Lemma 13.9. — *The terms in the list* (13.99) *have the following form:*

$$H_1(x,\omega,\nu,\nu')H_2(x,\omega',\nu,\nu') + H_3(x,\omega,\nu,\nu')H_4(x,\omega',\nu,\nu'),\tag{13.100}$$

where H_1, H_3, H_4 satisfy:

$$\|H_1\|_{L^2(S)} + \|H_3\|_{L^\infty_{[-2,2]}L^2(P_u)} + \|H_4\|_{L^\infty_{u'}L^2(P_{u'})} \lesssim D\tag{13.101}$$

and where H_2 satisfies:

$$\|H_2\|_{L^\infty(S)} + \|\partial_\omega H_2\|_{L^2(S)} + \|\nabla\partial_\omega H_2\|_{L^2(S)} \lesssim D,\tag{13.102}$$

for ω in the support of η_j^ν and ω' in the support of $\eta_j^{\nu'}$.

The proof of Lemma 13.9 follows the same line as the proof of Lemma 13.7 and is left to the reader. In the rest of this section, we show how Lemma 13.8 and Lemma 13.9 yield the proof of (13.71).

End of the proof of (13.71). — Using (13.97), Lemma 13.8 and Lemma 13.9, we may rewrite $\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma$ as:

(13.103)

$$\begin{aligned}
&\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\\
&= \sum_{l,m,n,o,p,q\geq 0}\left(\frac{c^1_{p,q,l,m,n,o}}{(k-k')|\nu-\nu'|^\alpha}+\frac{c^2_{p,q,l,m,n,o}}{(k-k')^2|\nu-\nu'|^{2\alpha}}+\frac{c^3_{p,q,l,m,n,o}}{(k-k')^3|\nu-\nu'|^{3\alpha}}\right)\\
&\times\frac{1}{(k-k')^{l+m+n+o}}\int_\Sigma\frac{2^{-3j/2}}{(2^{j/2}|N_\nu-N_{\nu'}|)^{1+p+q}}\\
&\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N-N_\nu))^p\left(\frac{a-a_\nu}{|\nu-\nu'|^\alpha}\right)^l H_1(x,\omega,\nu,\nu')F_{j,k,\sigma_1,n}(u)\eta_j^\nu(\omega)d\omega\right)\\
&\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_2(x,\omega',\nu,\nu')\overline{F_{j,k,\sigma_2,o}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\\
&+\sum_{l,m,n,o,p,q\geq 0}\left(\frac{c^1_{p,q,l,m,n,o}}{(k-k')|\nu-\nu'|^\alpha}+\frac{c^2_{p,q,l,m,n,o}}{(k-k')^2|\nu-\nu'|^{2\alpha}}+\frac{c^3_{p,q,l,m,n,o}}{(k-k')^3|\nu-\nu'|^{3\alpha}}\right)\\
&\times\frac{1}{(k-k')^{l+m+n+o}}\int_\Sigma\frac{2^{-3j/2}}{(2^{j/2}|N_\nu-N_{\nu'}|)^{1+p+q}}\\
&\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N-N_\nu))^p\left(\frac{a-a_\nu}{|\nu-\nu'|^\alpha}\right)^l H_3(x,\omega,\nu,\nu')F_{j,k,\sigma_1,n}(u)\eta_j^\nu(\omega)d\omega\right)\\
&\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_4(x,\omega',\nu,\nu')\overline{F_{j,k,\sigma_2,o}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma,
\end{aligned}$$

where $(\sigma_1,\sigma_2)=(0,-1)$ in the case of the term involving D_1, and $(\sigma_1,\sigma_2)=(-1,0)$ in the case of the term involving D_2. We estimate the two terms in the right-hand side of (13.103) starting with the second one. We have:

$$\begin{aligned}
&\Bigg|\int_\Sigma\frac{2^{-3j/2}}{(2^{j/2}|N_\nu-N_{\nu'}|)^{1+p+q}}\\
&\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N-N_\nu))^p\left(\frac{a-a_\nu}{|\nu-\nu'|^\alpha}\right)^l H_3(x,\omega,\nu,\nu')F_{j,k,\sigma_1,n}(u)\eta_j^\nu(\omega)d\omega\right)\\
&\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_4(x,\omega',\nu,\nu')\overline{F_{j,k,\sigma_2,o}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\Bigg|\\
&\leq\frac{2^{-3j/2}}{(2^{j/2}|\nu-\nu'|)^{1+p+q}}\left(\int_{\mathbb{S}^2}\|H_3\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_{j,k,\sigma_1,n}\|_{L^2_u}\eta_j^\nu(\omega)d\omega\right)\\
&\times\left(\int_{\mathbb{S}^2}\|H_4\|_{L^\infty_{u'}L^2(P_{u'})}\|F_{j,k,\sigma_2,o}\|_{L^2_{u'}}\eta_j^{\nu'}(\omega')d\omega'\right)
\end{aligned}$$

(13.104)
$$\lesssim \frac{2^{-j/2}D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^{1+p+q}},$$
where we have used Assumption 2 to estimate $|N_\nu - N_{\nu'}|$, Plancherel to estimate $\|F_{j,k,\sigma_1,n}\|_{L^2_u}$ and $\|F_{j,k,\sigma_2,o}\|_{L^2_{u'}}$, Cauchy-Schwartz in ω and ω', the estimate (13.101) for H_3 and H_4, and:

$$\frac{|a-a_\nu|}{|\nu-\nu'|^\alpha} \leq \frac{|\omega-\nu|^\alpha}{|\nu-\nu'|^\alpha}\|\partial_\omega^\alpha a\|_{L^\infty(S)} \lesssim \frac{\varepsilon}{(2^{j/2}|\nu-\nu'|)^\alpha} \lesssim 1 \tag{13.105}$$

on the support of η_j^ν thanks to Assumption 2.

We now estimate the second term in the right-hand side of (13.103). We have:

(13.106)
$$\begin{aligned}
&\Bigg|\int_\Sigma \frac{2^{-3j/2}}{(2^{j/2}|N_\nu-N_{\nu'}|)^{1+p+q}}\\
&\quad\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N-N_\nu))^p\left(\frac{a-a_\nu}{|\nu-\nu'|^\alpha}\right)^l H_1(x,\omega,\nu,\nu')F_{j,k,\sigma_1,n}(u)\eta_j^\nu(\omega)d\omega\right)\\
&\quad\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_2(x,\omega',\nu,\nu')\overline{F_{j,k,\sigma_2,o}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\Bigg|\\
&\leq \frac{2^{-3j/2}}{(2^{j/2}|\nu-\nu'|)^{1+p+q}}\left(\int_{\mathbb{S}^2}\|H_1\|_{L^2(S)}\|F_{j,k,\sigma_1,n}\|_{L^\infty_u}\eta_j^\nu(\omega)d\omega\right)\\
&\quad\times\left\|\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_2(x,\omega',\nu,\nu')F_{j,k,\sigma_2,o}(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(S)}\\
&\lesssim \frac{2^{-j/2}D\gamma_j^{\nu,k}}{(2^{j/2}|\nu-\nu'|)^{1+p+q}}\\
&\quad\times\left\|\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_2(x,\omega',\nu,\nu')F_{j,k,\sigma_2,o}(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(S)},
\end{aligned}$$
where we have used Assumption 2 to estimate $|N_\nu - N_{\nu'}|$, Cauchy-Schwartz in ω, the estimate (13.87) for $\|F_{j,k,\sigma_1,n}\|_{L^\infty_u}$, and the estimate (13.101) for H_1. Note that the symbol
$$F = (2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_2(x,\omega',\nu,\nu')$$
satisfies the following assumptions:

$$\begin{aligned}
&\|F\|_{L^\infty(S)} \lesssim D,\ \|\partial_\omega F\|_{L^2(S)} \lesssim \left(q+\frac{m}{|\nu-\nu'|^\alpha}\right)D\\
&\qquad\text{and } \|\nabla\partial_\omega F\|_{L^2(S)} \lesssim \left(q^2+\frac{m^2+mq}{|\nu-\nu'|^\alpha}\right)D,
\end{aligned} \tag{13.107}$$

where we have used Assumption 2 for $\partial_\omega N$, $\partial_\omega^2 N$, $\partial_\omega a$ and $\partial_\omega^\alpha a$, and the assumption (13.102) satisfied by H_2. We will see in Section 13.5 that assumptions (13.107) on a symbol is enough to control the diagonal term in $L^2(S)$ (i.e., to obtain the estimate (13.17)). Thus, we obtain:

$$\left\|\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_2(x,\omega',\nu,\nu')F_{j,k,\sigma_2,o}(u')\eta_j^{\nu'}(\omega')d\omega'\right\|_{L^2(S)}$$

(13.108)

$$\lesssim\left(1+q^2+\frac{m^2+mq}{|\nu-\nu'|^\alpha}\right)D\gamma_j^{\nu',k}.$$

(13.106) and (13.108) imply:

$$\Bigg|\int_\Sigma \frac{2^{-3j/2}}{(2^{j/2}|N_\nu-N_{\nu'}|)^{1+p+q}}$$
$$\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N-N_\nu))^p\left(\frac{a-a_\nu}{|\nu-\nu'|^\alpha}\right)^l H_1(x,\omega,\nu,\nu')F_{j,k,\sigma_1,n}(u)\eta_j^\nu(\omega)d\omega\right)$$
$$\times\left(\int_{\mathbb{S}^2}(2^{j/2}(N'-N_{\nu'}))^q\left(\frac{a'-a_{\nu'}}{|\nu-\nu'|^\alpha}\right)^m H_2(x,\omega',\nu,\nu')\overline{F_{j,k,\sigma_2,o}(u')}\eta_j^{\nu'}(\omega')d\omega'\right)d\Sigma\Bigg|$$

(13.109)

$$\lesssim\frac{(1+q^2+m^2)2^{-j/2}D^2|\nu-\nu'|^{-\alpha}}{(2^{j/2}|\nu-\nu'|)^{1+p+q}}\gamma_j^{\nu,k}\gamma_j^{\nu',k}.$$

Finally, (13.103), (13.104) and (13.109) yield:

(13.110)

$$\left|\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\right|$$
$$\lesssim\sum_{l,m,n,o,p,q\geq 0}\left(\frac{c^1_{p,q,l,m,n,o}}{|k-k'||\nu-\nu'|^\alpha}+\frac{c^2_{p,q,l,m,n,o}}{|k-k'|^2|\nu-\nu'|^{2\alpha}}+\frac{c^3_{p,q,l,m,n,o}}{|k-k'|^3|\nu-\nu'|^{3\alpha}}\right)$$
$$\times\frac{(1+q^2+m^2)2^{-j/2}D^2|\nu-\nu'|^{-\alpha}}{|k-k'|^{l+m+n+o}(2^{j/2}|\nu-\nu'|)^{1+p+q}}\gamma_j^{\nu,k}\gamma_j^{\nu',k}$$
$$+\sum_{l,m,n,o,p,q\geq 0}\left(\frac{c^1_{p,q,l,m,n,o}}{|k-k'||\nu-\nu'|^\alpha}+\frac{c^2_{p,q,l,m,n,o}}{|k-k'|^2|\nu-\nu'|^{2\alpha}}+\frac{c^3_{p,q,l,m,n,o}}{|k-k'|^3|\nu-\nu'|^{3\alpha}}\right)$$
$$\times\frac{2^{-j/2}D^2}{|k-k'|^{l+m+n+o}(2^{j/2}|\nu-\nu'|)^{1+p+q}}\gamma_j^{\nu,k}\gamma_j^{\nu',k}$$
$$\lesssim\frac{D^2\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{|k-k'|2^{j/2(1-4\alpha)}(2^{j/2}|\nu-\nu'|)^{1+4\alpha}},$$

which concludes the proof of estimate (13.71).

13.4.5. End of the proof of Proposition 13.2. — We have proved the estimates (13.70) and (13.71) in the two previous sections. Since (13.70) and (13.71) yield (13.60) (see Section 13.4.2), this concludes the proof of Proposition 13.2. □

13.5. Proof of Proposition 13.3 (control of the diagonal term)

We have to prove (13.17):

$$\|U_j^\nu f\|_{L^2(S)} \lesssim D\gamma_j^\nu. \tag{13.111}$$

Recall that U_j^ν is given by:

$$U_j^\nu f(x) = \int_{\mathbb{S}^2} bF_j(u)\eta_j^\nu(\omega)d\omega, \tag{13.112}$$

where $F_j(u)$ is defined by:

$$F_j(u) = \int_0^{+\infty} e^{i\lambda u}\psi(2^{-j}\lambda)f(\lambda\omega)\lambda^2 d\lambda. \tag{13.113}$$

We decompose U_j^ν in the sum of two terms:

$$U_j^\nu f(x) = b(x,\nu)\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega + \int_{\mathbb{S}^2}(b(x,\omega)-b(x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega. \tag{13.114}$$

We start with the first term. The assumption (11.42) on b implies:

$$\left\|b(x,\nu)\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(S)} \lesssim D\left\|\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(S)}. \tag{13.115}$$

The following proposition allows us to estimate the right-hand side of (13.115).

Proposition 13.10. — *The right-hand side of* (13.115) *satisfies the following bound:*

$$\left\|\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(S)} \lesssim \gamma_j^\nu. \tag{13.116}$$

The proof of Proposition 13.10 is postponed to Section 13.5.1. In the rest of this section, we show how Proposition 13.10 yields the proof of (13.111). In particular, (13.116) together with (13.115) implies the following bound for the first term in the right-hand side of (13.114):

$$\left\|b(x,\nu)\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(S)} \lesssim D\gamma_j^\nu. \tag{13.117}$$

We turn to the second term in the right-hand side of (13.114). We have:

$$\begin{aligned}
&\left\|\int_{\mathbb{S}^2}(b(x,\omega)-b(x,\nu))F_j(u)\eta_j^\nu(\omega)d\omega\right\|_{L^2(S)}\\
&\leq \int_{\mathbb{S}^2}\|b(x,\omega)-b(x,\nu)\|_{L^\infty_{[-2,2]}L^2(P_u)}\|F_j\|_{L^2_u}\eta_j^\nu(\omega)d\omega\\
&\leq \int_{\mathbb{S}^2}|\omega-\nu|(\|\partial_\omega b\|_{L^2(S)}+\|\nabla\partial_\omega b\|_{L^2(S)})\|F_j\|_{L^2_u}\eta_j^\nu(\omega)d\omega\\
&\lesssim D\gamma_j^\nu,
\end{aligned} \tag{13.118}$$

where we have used Plancherel to estimate $\|F_j\|_{L^2_u}$, Cauchy-Schwartz in ω, the fact that $H^1(\Sigma)$ embeds in $L^\infty_{[-2,2]}L^2(P_u)$ (see **[23]** Corollary 3.6 for a proof only using the regularity given by Assumption 1), the assumption (11.43) on b, and the fact that $|\omega-\nu|\lesssim 2^{-j/2}$ on the support of η_j^ν.

Finally, (13.114), (13.117) and (13.118) yield the wanted estimate (13.111) which concludes the proof of Proposition 13.3. $\square$.

13.5.1. Proof of Proposition 13.10. — Recall that $\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega$ is given by:

$$\int_{\mathbb{S}^2} F_j(u)\eta_j^\nu(\omega)d\omega = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{13.119}$$

Relying on the classical TT^* argument, (13.116) is equivalent to proving the boundedness in $L^2(S)$ of the operator whose kernel K is given by:

$$K(x,y) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,\ x,y\in\Sigma. \tag{13.120}$$

The decay satisfied by this kernel is stated in the following proposition.

Proposition 13.11. — *The kernel K defined in* (13.120) *satisfies the following decay estimate for all x,y in Σ:*

$$\begin{aligned}|K(x,y)| \lesssim\ & \frac{2^j}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\ &\times\frac{2^j}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}.\end{aligned} \tag{13.121}$$

The proof of Proposition 13.11 is postponed to Section 13.5.2. In the rest of this section, we show how (13.121) implies Proposition 13.10. According to Schur's Lemma, the operator whose kernel is K is bounded on $L^2(S)$ provided we can prove the following bound:

$$\sup_{x\in\Sigma}\int_\Sigma |K(x,y)|dy<+\infty,\ \sup_{y\in\Sigma}\int_\Sigma |K(x,y)|dx<+\infty. \tag{13.122}$$

Due to the symmetry of K in x,y, the two bounds in (13.122) are obtained in the same way. We focus on establishing the first bound. In view of (13.121), we have:

(13.123)

$$\begin{aligned}\int_\Sigma |K(x,y)|dy \lesssim \int_\Sigma\ & \frac{2^j}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\ &\times\frac{2^j}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}dy.\end{aligned}$$

Now, according to Assumption 4, there is a global change of variable on Σ $\phi_\nu:\Sigma\to\mathbb{R}^3$ defined by:

$$\phi_\nu(x) := u(x,\nu)\nu+\partial_\omega u(x,\nu), \tag{13.124}$$

such that ϕ_ν is a bijection, and the determinant of its Jacobian satisfies the following estimate:

$$\| |\det(\mathrm{Jac}\phi_\nu)| - 1\|_{L^\infty(S)} \lesssim \varepsilon. \tag{13.125}$$

Using the change of variable $y \to \underline{y} = \phi_\nu(y) - \phi_\nu(x) \in \mathbb{R}^3$ in the right-hand side of (13.123) together with (13.125), we obtain:

$$\int_\Sigma |K(x,y)|dy \lesssim \int_{\mathbb{R}^3} \frac{2^j}{(1+|2^j|\underline{y}\cdot\nu| - 2^{j/2}|\underline{y}'||)^2} \frac{2^j}{(1+2^{j/2}|\underline{y}'|)^3} d\underline{y}, \tag{13.126}$$

where $\underline{y} = \underline{y}\cdot\nu + \underline{y}'$ and $\underline{y}'\cdot\nu = 0$. Making the change of variable $\underline{y} \to z$ where z is defined by $z\cdot\nu = 2^j\underline{y}\cdot\nu$ and $z' = 2^{j/2}\underline{y}'$ in the right-hand side of (13.126), and remarking that $z\cdot\nu$ is one dimensional, and z' is two dimensional, we obtain:

$$\int_\Sigma |K(x,y)|dy \lesssim \int_{\mathbb{R}^3} \frac{dz}{(1+||z\cdot\nu| - |z'||)^2(1+|z'|)^3} \lesssim 1. \tag{13.127}$$

(13.127) implies the first bound in (13.122). K being symmetric with respect to x, y, the second bound in (13.122) is also true. Thus, the operator whose kernel is K is bounded on $L^2(S)$ which concludes the proof of Proposition 13.10. ☐

13.5.2. Proof of Proposition 13.11.

— Recall the definition of K:

$$K(x,y) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega) - i\lambda u(y,\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega, \; x,y\in\Sigma. \tag{13.128}$$

We need to prove that K satisfies the following decay estimate for all x, y in Σ:

$$\begin{aligned}|K(x,y)| \lesssim& \frac{2^j}{(1+|2^j|u(x,\nu)-u(y,\nu)| - 2^{j/2}|\partial_\omega u(x,\nu) - \partial_\omega u(y,\nu)||)^2}\\ &\times \frac{2^j}{(1+2^{j/2}|\partial_\omega u(x,\nu) - \partial_\omega u(y,\nu)|)^3}.\end{aligned} \tag{13.129}$$

Proof of (13.129). — Recall from Remark 11.6 that $u(x,\omega)$ is exactly equal to $x\cdot\omega$ in $|x|\geq 2$. Thus, we may restrict ourselves to $|x|\leq 2$ where we have in view of Assumption 2:

$$|u(x,\omega)| + |\partial_\omega u(x,\omega)| + |\partial_\omega^2 u(x,\omega)| + |\partial_\omega^3 u(x,\omega)| \lesssim 1, \; \forall x \text{ with } |x|\leq 2, \; \forall\omega\in\mathbb{S}^2. \tag{13.130}$$

We will obtain (13.129) as a consequence of the following estimate:

$$|K(x,y)| \lesssim \int_{\mathbb{S}^2}\int_0^{+\infty} \frac{1}{(1+2^j|u(x,\omega)-u(y,\omega)|)^3}\widetilde{\psi}(2^{-j}\lambda)\widetilde{\eta}_j^\nu(\omega)\lambda^2 d\lambda d\omega, \tag{13.131}$$

where $\widetilde{\psi}$ is smooth and compactly supported in $(0,+\infty)$ and $\widetilde{\eta}_j^\nu$ is bounded on $\mathbb{S}^2$ and has the same support as η_j^ν. Indeed, we have:

$$\begin{aligned} u(x,\omega) - u(y,\omega) =& u(x,\nu) - u(y,\nu) + (\partial_\omega u(x,\nu) - \partial_\omega u(y,\nu))(\omega-\nu)\\ &+ O(|\nu-\omega|^2),\\ \partial_\omega u(x,\omega) - \partial_\omega u(y,\omega) =& \partial_\omega u(x,\nu) - \partial_\omega u(y,\nu) + O(|\nu-\omega|),\end{aligned} \tag{13.132}$$

where we have used a Taylor expansion in ω together with (13.130). Using the fact that $2^{j/2}|\omega-\nu|\lesssim 1$ on the support of $\widetilde{\eta}_j^\nu$ together with (13.132), we obtain for ω in the support of $\widetilde{\eta}_j^\nu$:

$$1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||\lesssim 1+2^j|u(x,\omega)-u(y,\omega)|, \tag{13.133}$$
$$\text{and } 1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|\lesssim 1+2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|.$$

(13.131) and (13.133) imply:

$$\begin{aligned}|K(x,y)|\lesssim\ & \frac{1}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\ &\times\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}\int_{\mathbb{S}^2}\int_0^{+\infty}\widetilde{\psi}(2^{-j}\lambda)\widetilde{\eta}_j^\nu(\omega)\lambda^2 d\lambda d\omega.\end{aligned} \tag{13.134}$$

Now, we have:

$$\int_{\mathbb{S}^2}\int_0^{+\infty}\widetilde{\psi}(2^{-j}\lambda)\widetilde{\eta}_j^\nu(\omega)\lambda^2 d\lambda d\omega=\left(\int_0^{+\infty}\widetilde{\psi}(2^{-j}\lambda)\lambda^2 d\lambda\right)\left(\int_{\mathbb{S}^2}\widetilde{\eta}_j^\nu(\omega)d\omega\right)\lesssim 2^{2j}, \tag{13.135}$$

where we have used the fact that $\widetilde{\eta}_j^\nu$ is bounded on $\mathbb{S}^2$ and the fact that the support of $\widetilde{\eta}_j^\nu$ is two dimensional with a diameter of size $\sim 2^{-j/2}$. Finally, (13.134) and (13.135) imply (13.129) which is the wanted estimate.

Proof of (13.131). — To conclude the proof of Proposition 13.11, it remains to prove (13.131). This will follow by performing three integrations by parts with respect to ω and two integrations by parts with respect to λ. We start with the integrations by parts with respect to ω. Our goal is to show that $K(x,y)$ is a sum of terms of the form:

$$\begin{aligned}&\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}F(x,y,\omega,\nu)(\lambda(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega))^2)^l}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^m}\\ &\quad\times\psi(2^{-j}\lambda)\widetilde{\eta}_j^\nu(\omega)\lambda^2 d\lambda d\omega,\end{aligned} \tag{13.136}$$

where l,m are integers, where F does not depend on λ, where $\widetilde{\eta}_j^\nu$ is bounded on $\mathbb{S}^2$ and has the same support as η_j^ν and where the integrand in (13.136) satisfies:

$$\left|\frac{F(x,y,\omega,\nu)(\lambda(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega))^2)^l}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^m}\right|\lesssim\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|)^3}. \tag{13.137}$$

To this end, we use:

$$e^{i\lambda(u(x,\omega)-u(y,\omega))}=\frac{(1-i(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega))\partial_\omega)e^{i\lambda(u(x,\omega)-u(y,\omega))}}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)}. \tag{13.138}$$

We integrate by parts once in the integral (13.128) defining K using (13.138). We obtain:

(13.139)

$$\begin{aligned}K(x,y) = & \int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}}{1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\ & +\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_1}{1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\ & +\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_2}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\ & +\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_3}{1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2}\psi(2^{-j}\lambda)2^{-j/2}\partial_\omega\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,\end{aligned}$$

where A_1, A_2, A_3 are given by:

$$\begin{aligned}A_1 &= i(\partial_\omega^2 u(x,\omega)-\partial_\omega^2 u(y,\omega)),\\ A_2 &= -2i\lambda(\partial_\omega^2 u(x,\omega)-\partial_\omega^2 u(y,\omega))(\partial_\omega u(x,\omega)-u(y,\omega))^2,\\ A_3 &= i2^{j/2}(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)).\end{aligned}\tag{13.140}$$

The first term in the right-hand side of (13.139). — Integrating by parts in the first term of the right-hand side of (13.139) using (13.138) yields:

(13.141)

$$\begin{aligned}&\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}}{1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\ &=\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_0^1}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\ &\quad+\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_0^2}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^3}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\ &\quad+\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_0^3}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\psi(2^{-j}\lambda)2^{-j/2}\partial_\omega\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,\end{aligned}$$

where A_0^1, A_0^2, A_0^3 are given by:

$$\begin{aligned}A_0^1 &= 1+i(\partial_\omega^2 u(x,\omega)-\partial_\omega^2 u(y,\omega)),\\ A_0^2 &= -4i\lambda(\partial_\omega^2 u(x,\omega)-\partial_\omega^2 u(y,\omega))(\partial_\omega u(x,\omega)-u(y,\omega))^2,\\ A_0^3 &= i2^{j/2}(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)).\end{aligned}\tag{13.142}$$

(13.130) implies the following bound for the integrand in the right-hand side of (13.141):

(13.143)

$$\frac{|A_0^1|}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}+\frac{|A_0^2|}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^3}$$

$$+\frac{|A_0^3|}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\lesssim\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|)^3}.$$
(13.143)

Also, we have:

$$|\partial_\omega\eta_j^\nu(\omega)|\lesssim 2^{j/2}\text{ for all }\omega\in\mathbb{S}^2, \tag{13.144}$$

so that $2^{-j/2}\partial_\omega\eta_j^\nu$ satisfies the assumptions of $\widetilde{\eta}_j^\nu$. Thus, the first term of the right-hand side of (13.139) satisfies (13.136) (13.137).

The terms involving A_1, A_2 and A_3 in the right-hand side of (13.139). — Integrating by parts in the term involving A_1 of the right-hand side of (13.139) using (13.138) yields:

(13.145)
$$\begin{aligned}
&\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_1}{1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
&=\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_0^1A_1}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
&\quad+\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_0^2A_1}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^3}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
&\quad+\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}A_0^3A_1}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\psi(2^{-j}\lambda)2^{-j/2}\partial_\omega\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
&\quad+\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}i(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega))\partial_\omega A_1}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,
\end{aligned}$$

where A_0^1, A_0^2, A_0^3 are given by (13.142). In view of (13.143), we have the following bound for the integrand in the right-hand side of (13.145):

(13.146)
$$\begin{aligned}
&\frac{|A_0^1A_1|}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}+\frac{|A_0^2A_1|}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^3}\\
&\quad+\frac{|A_0^3A_1|}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}+\frac{|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)||\partial_\omega A_1|}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^2}\\
&\lesssim\frac{|A_1|}{(1+2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|)^3}+\frac{|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)||\partial_\omega A_1|}{(1+2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|)^3}.
\end{aligned}$$

Now, in view of the Definition (13.140) of A_1 and the estimate (13.130), we have:

$$|A_1|\lesssim 1,\text{ and }|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)||\partial_\omega A_1|\lesssim 1. \tag{13.147}$$

In view of (13.146) and (13.147) the term involving A_1 of the right-hand side of (13.139) satisfies (13.136) (13.137).

We proceed similarly for A_2 and A_3. In particular, in view of the Definition (13.140) of A_2, A_3 and the estimate (13.130), we may replace (13.147) with the following estimates:

$$\begin{aligned}|A_2| &\lesssim (2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|)^2,\\ |\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)||\partial_\omega A_2| &\lesssim (2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|)^2,\\ |A_3| &\lesssim 2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|\end{aligned}$$

and

$$|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)||\partial_\omega A_3| \lesssim 2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|.$$

Finally, the four terms in the right-hand side of (13.139) satisfy the estimates (13.136) (13.137). Thus $K(x,y)$ satisfies (13.136) (13.137).

Integration by parts with respect to λ and end of the proof of (13.131). — In order to obtain (13.131), we still need to perform two integration by parts with respect to λ in (13.136). We have:

$$e^{i\lambda(u(x,\omega)-u(y,\omega))} = \frac{(1-i2^j(u(x,\omega)-u(y,\omega))2^j\partial_\lambda)e^{i\lambda(u(x,\omega)-u(y,\omega))}}{(1+2^{2j}|u(x,\omega)-u(y,\omega)|^2)}. \tag{13.148}$$

Notice that the only term depending on λ under the integral (13.136) is:

$$\frac{\psi(2^{-j}\lambda)\lambda^{2+l}}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^m}. \tag{13.149}$$

Now, we have:

$$\begin{aligned}&2^j\partial_\lambda\left(\frac{\psi(2^{-j}\lambda)\lambda^{2+l}}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^m}\right)\\ &= \frac{\left((2+l-m)\overline{\psi}(2^{-j}\lambda)+\psi'(2^{-j}\lambda)\right)\lambda^{2+l}}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^m}\\ &\quad+\frac{m\overline{\psi}(2^{-j}\lambda)\lambda^{2+l}}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^{m+1}} \text{ where } \overline{\psi}(\lambda)=\frac{\psi(\lambda)}{\lambda}.\end{aligned} \tag{13.150}$$

Thus integrating by parts in λ in the integral (13.136) using (13.148) essentially divides the integrand by $1+2^j|u(x,\omega)-u(y,\omega)|$. In particular, after two integrations by parts using (13.148) in the integral (13.136), and together with the estimate (13.137), we obtain that $K(x,y)$ is a sum of terms of the form:

$$\begin{aligned}&\int_{\mathbb{S}^2}\int_0^{+\infty}\frac{e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}F(x,y,\omega,\nu)(\lambda(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega))^2)^l}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^m(1+2^j|u(x,\omega)-u(y,\omega)|)^2}\\ &\quad\times\widetilde{\psi}(2^{-j}\lambda)\widetilde{\eta}_j^\nu(\omega)\lambda^2 d\lambda d\omega,\end{aligned} \tag{13.151}$$

where l,m are integers, where $\widetilde{\psi}$ is smooth and compactly supported in $(0,+\infty)$, where $\widetilde{\eta}_j^\nu$ is bounded on $\mathbb{S}^2$ and has the same support as η_j^ν and where the integrand in (13.151) satisfies:

$$\left|\frac{F(x,y,\omega,\nu)(\lambda(\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega))^2)^l}{(1+\lambda|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|^2)^m(1+2^j|u(x,\omega)-u(y,\omega)|)^2}\right| \lesssim \frac{1}{(1+2^j|u(x,\omega)-u(y,\omega)|)^2(1+2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega u(y,\omega)|)^3}. \tag{13.152}$$

Finally, (13.151) and (13.152) yield (13.131) which is the wanted estimate. This concludes the proof of Proposition 13.11. □

CHAPTER 14

PROOF OF THEOREM 11.10

In order to prove Theorem 11.10, we first show that the Fourier integral operator U of Theorem 11.8 almost preserve the L^2 norm provided we make additional assumptions on its symbol. We then use this observation to prove the estimate (11.49). Finally, we conclude the proof of Theorem 11.10 by establishing the existence of (f_+, f_-) solution of the system (11.48).

14.1. A refinement of Theorem 11.8

In Theorem 11.8, we have proved that the Fourier integral operator U with phase u and symbol b is bounded on $L^2(S)$ provided u satisfies Assumption 1, Assumption 2 and Assumption 4, and the symbol b satisfies (11.42) (11.43). We now would like to prove that U satisfies the following bound from below:

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|Uf\|_{L^2(S)}, \tag{14.1}$$

provided u also satisfies Assumption 5 and under additional assumptions on the symbol b. This is the aim of the following proposition.

Proposition 14.1. — *Let u be a function on $\Sigma \times \mathbb{S}^2$ satisfying Assumption 1, Assumption 2, Assumption 4 and Assumption 5. Let U the Fourier integral operator with phase $u(x,\omega)$ and symbol $b(x,\omega)$:*

$$Uf(x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(x,\omega)} b(x,\omega) f(\lambda\omega) \lambda^2 d\lambda d\omega. \tag{14.2}$$

We assume furthermore that $b(x,\omega)$ satisfies:

$$\|\partial_\omega b\|_{L^2(S)} + \|\nabla \partial_\omega b\|_{L^2(S)} \lesssim 1, \tag{14.3}$$

$$\|b-1\|_{L^\infty(S)} + \|\nabla b\|_{L^\infty_{[-2,2]} L^2(P_u)} + \|\nabla\!\!\!/ \nabla b\|_{L^2(S)} \lesssim \varepsilon \tag{14.4}$$

and

$$\nabla_N b = b_1^j + b_2^j \text{ where } \|b_1^j\|_{L^2(S)} \lesssim 2^{-\frac{j}{2}}\varepsilon,\ \|b_2^j\|_{L^\infty_{[-2,2]} L^2(P_u)} \lesssim \varepsilon \quad \text{and } \|\nabla_N b_2^j\|_{L^2(S)} + \|b_2^j\|_{L^2_{[-2,2]} L^\infty(P_u)} \lesssim 2^{\frac{j}{2}}\varepsilon. \tag{14.5}$$

Then, U is bounded on L^2 and satisfies the estimate:

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|Uf\|_{L^2(S)}. \tag{14.6}$$

Remark 14.2. — Notice that the only difference in the assumptions with respect to Theorem 11.8 lies in the fact that u also satisfies Assumption 5 and in the constant D which has been replaced by 1 in (14.3) and by ε in (14.4) (14.5).

We now turn to the proof of Proposition 14.1. We review the three steps of Theorem 11.8—decomposition in frequency, decomposition in angle, and control of the diagonal term—indicating each time how to refine the estimates.

14.1.1. Step 1: Decomposition in frequency. — As in step 1 of the proof of Theorem 11.8, we decompose Uf in frequency:

$$Uf(x) = \sum_{j\geq -1} U_jf(x), \tag{14.7}$$

where the operators U_j are defined by (13.5) (13.6). We have:

$$\|Uf\|^2_{L^2(S)} = \sum_{|j-l|\leq 2} \int_\Sigma U_jf(x)\overline{U_lf(x)}d\Sigma + \sum_{|j-l|>2} \int_\Sigma U_jf(x)\overline{U_lf(x)}d\Sigma. \tag{14.8}$$

Now, the proof of Proposition 13.1 together with the fact that b satisfies (14.4) (14.5) immediately yields:

$$\left|\sum_{|j-l|>2} \int_\Sigma U_jf(x)\overline{U_lf(x)}d\Sigma\right| \lesssim \varepsilon\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{14.9}$$

Thus, together with (14.8), we obtain:

$$\|Uf\|^2_{L^2(S)} = \sum_{|j-l|\leq 2} \int_\Sigma U_jf(x)\overline{U_lf(x)}d\Sigma + O(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{14.10}$$

Remark 14.3. — The sum over $|j-l|\leq 2$ in the right-hand side of (14.10) corresponds to the terms such that the support of $\psi(2^{-j}\lambda)$ and the support of $\psi(2^{-j}\lambda')$ have a non empty intersection, where ψ has been introduced in (13.3).

14.1.2. Step 2: Decomposition in angle. — As in step 2 of the proof of Theorem 11.8, we decompose U_jf in angle:

$$U_jf(x) = \sum_{\nu\in\Gamma} U_j^\nu f(x), \tag{14.11}$$

where the operators U_j^ν are defined by (13.11). In order to control the diagonal term in a third step (see next section), we have to modify slightly the size of the support of our partition of unity η_j^ν on $\mathbb{S}^2$ introduced in (13.9). Let $\delta > 0$ such that:

$$0 < \sqrt{\varepsilon} \ll \delta \ll 1. \tag{14.12}$$

We now require that the support of η_j^ν is a patch on $\mathbb{S}^2$ of diameter $\sim \delta 2^{-j/2}$. We have:

$$\begin{aligned}\sum_{|j-l|\leq 2}\int_\Sigma U_jf(x)\overline{U_lf(x)}d\Sigma =& \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma\\ &+\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|>2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma.\end{aligned}\tag{14.13}$$

The proof of Proposition 13.2 yields:

$$\left|\sum_{|\nu-\nu'|>2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma\right|\lesssim \frac{\varepsilon}{\delta^2}\gamma_j\gamma_l,\tag{14.14}$$

where γ_j, γ_l have been defined in (13.12). Indeed, (14.14) follows from the equivalent of the two key estimates (13.70) (13.71). For example, let us consider the equivalent of (13.70). We obtain:

$$\begin{aligned}\left|\int_\Sigma U_j^{\nu,k}f(x)\overline{U_j^{\nu',k}f(x)}d\Sigma\right|\lesssim& \frac{\delta\varepsilon\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{2^{j\alpha/2}(2^{j/2}|\nu-\nu'|)^{2-\alpha}}+\frac{\delta\varepsilon\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^3}\\ &\text{for } |\nu-\nu'|\neq 0,\ 1\leq k\leq |\nu-\nu'|^{-\alpha},\end{aligned}\tag{14.15}$$

where ε comes from the fact that b satisfies (14.4), and δ from the fact that the square root of the volume of the support of η_j^ν now yields $\delta 2^{-j/2}$ instead of $2^{-j/2}$. The worst term in the right-hand side of (14.15) is the second one. It may be rewritten:

$$\frac{\delta\varepsilon\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}|\nu-\nu'|)^3}=\frac{\varepsilon}{\delta^2}\frac{\gamma_j^{\nu,k}\gamma_j^{\nu',k}}{(2^{j/2}\delta^{-1}|\nu-\nu'|)^3}\tag{14.16}$$

and yields the factor $\varepsilon\delta^{-2}$ in the right-hand side of (14.14).

Finally, (14.13) and (14.14) yield:

$$\begin{aligned}\sum_{|j-l|\leq 2}\int_\Sigma U_jf(x)\overline{U_lf(x)}d\Sigma =& \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma\\ &+O\left(\frac{\varepsilon}{\delta^2}\right)\|f\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}\tag{14.17}$$

Remark 14.4. — The sum over $|\nu-\nu'|\leq 2\delta 2^{-j/2}$ in the right-hand side of (14.17) corresponds to the terms such that the support of η_j^ν and the support of $\eta_j^{\nu'}$ have a non empty intersection. The number of terms in this sum only depends on the dimension of $\mathbb{S}^2$ and is therefore a universal constant.

14.1.3. Step 3: Control of the diagonal term. — The goal of this section is to estimate the term $\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma$.

A first reduction. — Remark first that the proof of Proposition 13.3 together with the fact that b satisfies (14.3) immediately yields:

$$\|U_j^\nu f\|_{L^2(S)}\lesssim \gamma_j^\nu.\tag{14.18}$$

We introduce the operator S_j^ν defined on Σ by:

$$S_j^\nu f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{14.19}$$

By Proposition 13.10, we have:

$$\|S_j^\nu f\|_{L^2(S)} \lesssim \gamma_j^\nu. \tag{14.20}$$

The estimate (13.118) together with the assumption (14.3) on b, (14.18), (14.20) and the fact that $|\omega-\nu|\lesssim \delta 2^{-j/2}$ on the support of η_j^ν yields:

$$\begin{aligned}&\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma\\ &= \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma b(x,\nu)S_j^\nu f(x)\overline{b(x,\nu')S_l^{\nu'}f(x)}d\Sigma + O(\delta)\|f\|^2_{L^2(\mathbb{R}^3)},\end{aligned} \tag{14.21}$$

which together with the assumption (14.4) on b and (14.20) implies:

$$\begin{aligned}&\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma\\ &= \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma S_j^\nu f(x)\overline{S_l^{\nu'}f(x)}d\Sigma + O(\delta+\varepsilon)\|f\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{14.22}$$

We want to estimate the term $\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma U_j^\nu f(x)\overline{U_l^{\nu'}f(x)}d\Sigma$. In view of (14.22), we may estimate instead the term

$$\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma S_j^\nu f(x)\overline{S_l^{\nu'}f(x)}d\Sigma.$$

End of the proof of Proposition 14.1. — Recall Assumption 4 which states that the map $\phi_\nu : \Sigma \to \mathbb{R}^3$ defined by:

$$\phi_\nu(x) := u(x,\nu)\nu + \partial_\omega u(x,\nu) \tag{14.23}$$

is a bijection, such that the determinant of its Jacobian satisfies the following estimate:

$$\||\det(\mathrm{Jac}\phi_\nu)|-1\|_{L^\infty(S)} \lesssim \varepsilon. \tag{14.24}$$

Let us note $\mathscr{F}^{-1}$ the inverse Fourier transform on $\mathbb{R}^3$. We introduce the operator $\widetilde{S}_j^\nu$ on Σ defined by:

(14.25)

$$\widetilde{S}_j^\nu f(x) = \mathscr{F}^{-1}(\psi(2^{-j}\cdot)\eta_j^\nu f)(\phi_\nu(x)) = \int_{\mathbb{R}^3} e^{i\lambda\phi_\nu(x)\cdot\omega}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega.$$

The following proposition shows that the term $\int_\Sigma S_j^\nu f(x)\overline{S_l^{\nu'}f(x)}d\Sigma$ is close to the term $\int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'}f(x)}d\Sigma$.

Proposition 14.5. — *We have the following bound:*

$$\|S_j^\nu f - \widetilde{S}_j^\nu f\|_{L^2(S)} \lesssim \delta^{\frac{1}{2}}\gamma_j^\nu. \tag{14.26}$$

We postponed the proof of Proposition 14.5 to the next section. Let us show how Proposition 14.5 allows us to conclude the proof of Proposition 14.1. (14.10), (14.17), (14.22) and (14.26) yield:

$$\|Uf\|^2_{L^2(S)} = \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'} f(x)}d\Sigma + O\left(\frac{\varepsilon}{\delta^2}+\delta^{\frac{1}{2}}\right)\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{14.27}$$

Making the change of variable $y=\phi_\nu(x)$ in $\int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'} f(x)}d\Sigma$ and using (14.24) and (14.25) implies:

$$\begin{aligned}
&\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_\Sigma \widetilde{S}_j^\nu f(x)\overline{\widetilde{S}_l^{\nu'} f(x)}d\Sigma\\
&= \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_{\mathbb{R}^3}\mathscr{F}^{-1}(\psi(2^{-j}\cdot)\eta_j^\nu f)(y)\overline{\mathscr{F}^{-1}(\psi(2^{-l}\cdot)\eta_j^{\nu'} f)(y)}dy\\
&\quad + O(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^3)}\\
&= \sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_{\mathbb{R}^3}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\overline{\psi(2^{-l}\lambda)\eta_j^{\nu'}(\omega)f(\lambda\omega)}dy\\
&\quad + O(\varepsilon)\|f\|^2_{L^2(\mathbb{R}^3)},
\end{aligned} \tag{14.28}$$

where we have used the fact that $\mathscr{F}^{-1}$ is an isomorphism on $L^2(\mathbb{R}^3)$ in the last equality of (14.28). Now, we have:

$$\sum_{|j-l|\leq 2}\sum_{|\nu-\nu'|\leq 2\delta 2^{-j/2}}\int_{\mathbb{R}^3}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)f(\lambda\omega)\overline{\psi(2^{-l}\lambda)\eta_j^{\nu'}(\omega)f(\lambda\omega)}dy = \|f\|^2_{L^2(\mathbb{R}^3)}, \tag{14.29}$$

which together with (14.27) and (14.28) yields:

$$\|Uf\|^2_{L^2(S)} = \|f\|^2_{L^2(\mathbb{R}^3)} + O\left(\frac{\varepsilon}{\delta^2}+\delta^{\frac{1}{2}}\right)\|f\|^2_{L^2(\mathbb{R}^3)}. \tag{14.30}$$

Choosing $\delta^{\frac{1}{2}}$ and $\varepsilon\delta^{-2}$ small enough, we deduce from (14.30):

$$\|f\|_{L^2(\mathbb{R}^3)} \lesssim \|Uf\|_{L^2(S)}, \tag{14.31}$$

which is the wanted estimate. This conclude the proof of Proposition 14.1. □

14.1.4. Proof of Proposition 14.5

Reduction to a decay estimate. — Relying on the classical TT^* argument, (14.26) is equivalent to proving the boundedness in $L^2(S)$ with a norm $O(\delta)$ of the operator whose kernel K is given by:

$$K(x,y) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega)-i\lambda u(y,\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega \tag{14.32}$$

$$+\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\phi_\nu(x)\cdot\omega-i\lambda\phi_\nu(y)\cdot\omega}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega$$
$$-\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda u(x,\omega)-i\lambda\phi_\nu(y)\cdot\omega}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega$$
$$-\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\phi_\nu(x)\cdot\omega-i\lambda u(y,\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega.$$

(14.26) then reduces to proving the following decay for the kernel K in (14.32):

$$(14.33)\qquad \begin{aligned}|K(x,y)| \lesssim \delta^{\frac{1}{2}} & \frac{2^j}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\ & \times\frac{2^j}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}.\end{aligned}$$

The proof of the fact that (14.33) implies (14.26) is identical to the proof in Section 13.5.1 of the fact that the decay estimate (13.121) implies (13.116). In fact, performing the exact same changes of variables leads to:

$$(14.34)\qquad \sup_{x\in\Sigma}\int_\Sigma |K(x,y)|dy \lesssim \delta,\ \sup_{y\in\Sigma}\int_\Sigma |K(x,y)|dx \lesssim \delta.$$

Finally, (14.34) yields (14.26) by Schur's Lemma.

Proof of the decay estimate (14.33). — The proof of (14.33) follows from the proof of Proposition 13.11 in Section 13.5.2. In fact, let us consider the following quantity A defined by:

$$(14.35)\qquad A=\int_{\mathbb{S}^2}\int_0^{+\infty} e^{i\lambda\rho(\omega)}\psi(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,$$

where ρ is a function defined on $\mathbb{S}^2$. Then applying 3 integrations by parts with respect to ω and 2 integrations by parts with respect to λ as in the proof of Proposition 13.11

yields to the following equality:

$$
\begin{aligned}
A = & \int_{\mathbb{S}^2}\int_0^{+\infty} F_0(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), \partial_\omega^2\rho(\omega), \partial_\omega^3\rho(\omega), 2^{-j}\lambda)\\
& \times \psi_0(2^{-j}\lambda)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
& + \delta^{-1}\int_{\mathbb{S}^2}\int_0^{+\infty} F_1(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), \partial_\omega^2\rho(\omega), \partial_\omega^3\rho(\omega), 2^{-j}\lambda)\\
& \times \psi_1(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
& + \delta^{-2}\int_{\mathbb{S}^2}\int_0^{+\infty} F_2(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), \partial_\omega^2\rho(\omega), 2^{-j}\lambda)\\
& \times \psi_2(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)^2\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
& + \delta^{-3}\int_{\mathbb{S}^2}\int_0^{+\infty} F_3(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), 2^{-j}\lambda)\\
& \times \psi_3(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)^3\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,
\end{aligned}
\tag{14.36}
$$

where $\psi_l, l = 0, 1, 2, 3$ is smooth and compactly supported in $(0, +\infty)$, $(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu$, $l = 0, 1, 2, 3$ is bounded on $\mathbb{S}^2$ and has the same support as η_j^ν, and $F_l, l = 0, 1, 2, 3$ are smooth function satisfying the following estimates:

$$
\begin{aligned}
& |F_0(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), \partial_\omega^2\rho(\omega), \partial_\omega^3\rho(\omega), 2^{-j}\lambda)|\\
& \quad + |F_1(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), \partial_\omega^2\rho(\omega), \partial_\omega^3\rho(\omega), 2^{-j}\lambda)|\\
& \quad + |F_2(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), \partial_\omega^2\rho(\omega), 2^{-j}\lambda)|\\
& \quad + |F_3(2^j\rho(\omega), 2^{j/2}\partial_\omega\rho(\omega), \partial_\omega\rho(\omega), 2^{-j}\lambda)|\\
& \lesssim \frac{1}{(1+2^j|\rho(\omega)|)^2(1+2^{j/2}|\partial_\omega\rho(\omega)|)^3}.
\end{aligned}
\tag{14.37}
$$

Indeed, this has been done in the proof of Proposition 13.11 for the particular case $\rho(\omega) = u(x, \omega) - u(y, \omega)$ but is easily seen to hold in the general case with the exact same proof. Applying (14.35) to the 4 terms in the right-hand side of (14.32) respectively with

$$
\begin{aligned}
& \rho_1(\omega) = u(x, \omega) - u(y, \omega),\ \rho_2(\omega) = \phi_\nu(x)\cdot\omega - \phi_\nu(y)\cdot\omega,\\
& \rho_3(\omega) = u(x, \omega) - \phi_\nu(y)\cdot\omega \text{ and } \rho_4(\omega) = \phi_\nu(y)\cdot\omega - u(y, \omega)
\end{aligned}
\tag{14.38}
$$

yields:

$$
\begin{aligned}
K(x, y) = & \sum_{q=1}^{2}\sum_{l=0}^{3}\delta^{-l}\int_{\mathbb{S}^2}\int_0^{+\infty} F_l[\rho_q]\psi_l(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
& - \sum_{q=3}^{4}\sum_{l=0}^{4}\delta^{-l}\int_{\mathbb{S}^2}\int_0^{+\infty} F_l[\rho_q]\psi_l(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega,
\end{aligned}
\tag{14.39}
$$

where $F_l[\rho_q]$ is defined for $q=1,2,3,4$ by:

(14.40)

$$\begin{aligned}
F_l[\rho_q] &= F_l(2^j\rho_q(\omega), 2^{j/2}\partial_\omega\rho_q(\omega), \partial_\omega\rho_q(\omega), \partial^2_\omega\rho_q(\omega), \partial^3_\omega\rho_q(\omega), 2^{-j}\lambda),\ l=0,1,\\
F_2[\rho_q] &= F_2(2^j\rho_q(\omega), 2^{j/2}\partial_\omega\rho_q(\omega), \partial_\omega\rho_q(\omega), \partial^2_\omega\rho_q(\omega), 2^{-j}\lambda),\\
F_3[\rho_q] &= F_3(2^j\rho_q(\omega), 2^{j/2}\partial_\omega\rho_q(\omega), \partial_\omega\rho_q(\omega), 2^{-j}\lambda).
\end{aligned}$$

We rewrite (14.39) as:

(14.41)

$$\begin{aligned}
K(x,y) = &\sum_{l=0}^{3}\delta^{-l}\int_{\mathbb{S}^2}\int_0^{+\infty}(F_l[\rho_1]-F_l[\rho_3])\psi_l(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega\\
&+\sum_{l=0}^{3}\delta^{-l}\int_{\mathbb{S}^2}\int_0^{+\infty}(F_l[\rho_2]-F_l[\rho_4])\psi_l(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu(\omega)\lambda^2 d\lambda d\omega.
\end{aligned}$$

We now estimate $F_l[\rho_1]-F_l[\rho_3]$ and $F_l[\rho_2]-F_l[\rho_4]$. One easily checks that the first order derivatives of F_l satisfy the same estimate as the estimates (14.37) satisfied by F_l. Together with Assumption 5 on $u(x,\omega)-\phi_\nu(x)\cdot\omega$, (13.132) and (13.133), we deduce the following estimates on the support of η_j^ν:

(14.42)

$$\begin{aligned}
&|F_0[\rho_1]-F_0[\rho_3]|+|F_0[\rho_2]-F_0[\rho_4]|+|F_1[\rho_1]-F_1[\rho_3]|+|F_1[\rho_2]-F_1[\rho_4]|\\
&\leq |F_0[\rho_1]|+|F_0[\rho_2]|+|F_0[\rho_3]|+|F_0[\rho_4]|+|F_1[\rho_1]|+|F_1[\rho_2]|+|F_1[\rho_3]|+|F_1[\rho_4]|\\
&\lesssim \frac{1}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\
&\quad\times\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3},
\end{aligned}$$

(14.43)

$$\begin{aligned}
&|F_2[\rho_1]-F_2[\rho_3]|+|F_2[\rho_2]-F_2[\rho_4]|\\
&\lesssim \big(2^j|u(x,\omega)-\phi_\nu(x)\cdot\omega|+2^j|u(y,\omega)-\phi_\nu(y)\cdot\omega|\\
&\quad+2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega(\phi_\nu(x)\cdot\omega)|+2^{j/2}|\partial_\omega u(y,\omega)-\partial_\omega(\phi_\nu(y)\cdot\omega)|\\
&\quad+|\partial^2_\omega u(x,\omega)-\partial^2_\omega(\phi_\nu(x)\cdot\omega)|+|\partial^2_\omega u(y,\omega)-\partial^2_\omega(\phi_\nu(y)\cdot\omega)|\big)\\
&\quad\times\frac{1}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\
&\quad\times\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}\\
&\lesssim \delta\frac{1}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\
&\quad\times\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}
\end{aligned}$$

and
(14.44)
$$
\begin{aligned}
&|F_3[\rho_1]-F_3[\rho_3]|+|F_3[\rho_2]-F_3[\rho_4]|\\
&\lesssim \big(2^j|u(x,\omega)-\phi_\nu(x)\cdot\omega|+2^j|u(y,\omega)-\phi_\nu(y)\cdot\omega|\\
&\quad +2^{j/2}|\partial_\omega u(x,\omega)-\partial_\omega(\phi_\nu(x)\cdot\omega)|+2^{j/2}|\partial_\omega u(y,\omega)-\partial_\omega(\phi_\nu(y)\cdot\omega)|\big)\\
&\quad\times\frac{1}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\
&\quad\times\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3}\\
&\lesssim \delta^2\frac{1}{(1+|2^j|u(x,\nu)-u(y,\nu)|-2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)||)^2}\\
&\quad\times\frac{1}{(1+2^{j/2}|\partial_\omega u(x,\nu)-\partial_\omega u(y,\nu)|)^3},
\end{aligned}
$$
where we have used the fact that $|\omega-\nu|\lesssim\delta 2^{-j/2}$ on the support of η_j^ν, and $\varepsilon\lesssim\delta$ in view of (14.12). Now, we have:

$$
\begin{aligned}
&\int_{\mathbb{S}^2}\int_0^{+\infty}\psi_l(2^{-j}\lambda)(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu(\omega)\lambda^2d\lambda d\omega \qquad (14.45)\\
&=\left(\int_0^{+\infty}\psi_l(2^{-j}\lambda)\lambda^2d\lambda\right)\left(\int_{\mathbb{S}^2}(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu(\omega)d\omega\right)\lesssim\delta^2 2^{2j},
\end{aligned}
$$

where we have used the fact that $(\delta 2^{-j/2}\partial_\omega)^l\eta_j^\nu(\omega)$ is bounded on $\mathbb{S}^2$ and the fact that its support is two dimensional with a diameter $\sim\delta 2^{-j/2}$. (14.41)-(14.45) immediately yield the decay estimate (14.33). Finally, as explained after (14.33), (14.33) yields (14.34) which implies (14.26). This concludes the proof of Proposition 14.5. □

Remark 14.6. — Note that Assumption 5 does not contain any estimate for the term $\partial_\omega^3 u-\partial_\omega^3(\phi_\nu(x)\cdot\omega)$. Instead, this term is estimated using Assumption 2:
$$|\partial_\omega^3 u-\partial_\omega^3(\phi_\nu(x)\cdot\omega)|\lesssim 1$$
and thus is not bounded from above by $O(\varepsilon)$ unlike the corresponding estimate for $\partial_\omega^2 u-\partial_\omega^2(\phi_\nu(x)\cdot\omega)$ in Assumption 5. As a consequence, (14.42) cannot be improved, and is responsible for the introduction of the extra smallness parameter δ in the decomposition in angle.

14.2. Proof of the estimate 11.19

Recall the definition of the Fourier integral operators $M_\pm$ and $Q_\pm$ introduced in Section 11.1.2:

$$M_\pm f(x)=\int_{\mathbb{S}^2}\int_0^{+\infty}e^{\pm i\lambda u(x,\pm\omega)}f(\lambda\omega)\lambda^2d\lambda d\omega \qquad (14.46)$$

and

$$Q_{\pm}f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)} a(x,\pm\omega)^{-1} f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{14.47}$$

Let (f_+, f_-) satisfying:

$$\begin{cases} M_+f_+ + M_-f_- = \phi_0, \\ Q_+(\lambda f_+) - Q_-(\lambda f_-) = i\phi_1. \end{cases} \tag{14.48}$$

The goal of this section is to prove that (f_+, f_-) satisfies the following estimate:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla\phi_0\|_{L^2(S)} + \|\phi_1\|_{L^2(S)}. \tag{14.49}$$

Using Proposition 14.1 in the case of a symbol $b \equiv 1$, we obtain:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} \lesssim \|M_+(\lambda f_+)\|_{L^2(S)} \text{ and } \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|M_-(\lambda f_-)\|_{L^2(S)}, \tag{14.50}$$

which yields:

$$\begin{aligned} &\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \\ &\quad \lesssim \|M_+(\lambda f_+)\|_{L^2(S)} + \|M_-(\lambda f_-)\|_{L^2(S)} \\ &\quad \lesssim \|M_+(\lambda f_+) + M_-(\lambda f_-)\|_{L^2(S)} + \|M_+(\lambda f_+) - M_-(\lambda f_-)\|_{L^2(S)}. \end{aligned} \tag{14.51}$$

We have:

$$(Q_{\pm} - M_{\pm})f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)} (a(x,\pm\omega)^{-1} - 1) f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{14.52}$$

Due to Assumption 1–3 on a, the symbol $a(x,\pm\omega)^{-1} - 1$ of $Q_{\pm} - M_{\pm}$ satisfies the assumptions (11.42)-(11.44) of Theorem 11.8 with $D = \varepsilon$. Thus, we obtain from (11.45) that:

$$\|(Q_{\pm} - M_{\pm})f\|_{L^2(S)} \lesssim \varepsilon\|f\|_{L^2(\mathbb{R}^3)}. \tag{14.53}$$

The second equation of (14.48), (14.51) and (14.53) yield:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|M_+(\lambda f_+) + M_-(\lambda f_-)\|_{L^2(S)} + \|\phi_1\|_{L^2(S)}. \tag{14.54}$$

The following lemma will allow us to bound the first term in the right-hand side of (14.54).

Lemma 14.7. — *For any (f_+, f_-), we have the following bound:*

$$\begin{aligned} \|M_+(\lambda f_+) + M_-(\lambda f_-)\|_{L^2(S)} &\lesssim \|\nabla M_+(f_+) + \nabla M_-(f_-)\|_{L^2(S)} \\ &\quad + \left(\delta + \frac{\varepsilon}{\delta^2}\right)(\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)}), \end{aligned} \tag{14.55}$$

where δ may be chosen as in (14.12).

Before proving Lemma 14.7, we first conclude the proof of the estimate (11.49). (14.12), (14.54) and (14.55) yield:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla M_+(f_+) + \nabla M_-(f_-)\|_{L^2(S)} + \|\phi_1\|_{L^2(S)}. \tag{14.56}$$

Applying ∇ to the first equation of (14.48) and using (14.56) implies:

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)} + \|\lambda f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\nabla\phi_0\|_{L^2(S)} + \|\phi_1\|_{L^2(S)}, \tag{14.57}$$

which is the wanted estimate (11.49).

Proof of Lemma 14.7. — Since $\nabla u = a^{-1}N$, we have:

$$\nabla M_\pm f(x) = \pm i\int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)}a(x,\pm\omega)^{-1}N(x,\pm\omega)\lambda f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{14.58}$$

We introduce the operator $P_\pm$ defined by:

$$P_\pm f(x) = \int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)}N(x,\pm\omega)f(\lambda\omega)\lambda^2 d\lambda d\omega. \tag{14.59}$$

Due to Assumption 1-3 on a and N, the symbol $i \pm (a(x,\pm\omega)^{-1} - 1)N_\pm$ of $\nabla M_\pm \mp iP_\pm(\lambda.)$ satisfies the assumptions (11.42)-(11.44) of Theorem 11.8 with $D = \varepsilon$. Thus, we obtain from (11.45) that:

$$\|\nabla M_\pm(f) \mp iP_\pm(\lambda f)\|_{L^2(S)} \lesssim \varepsilon\|\lambda f\|_{L^2(\mathbb{R}^3)}. \tag{14.60}$$

Thus, the proof of Lemma 14.7 reduces to the proof of the following estimate:

$$\begin{aligned}\|M_+(f_+) + M_-(f_-)\|_{L^2(S)} \lesssim{}& \|P_+(f_+) - P_-(f_-)\|_{L^2(S)}\\ &+\left(\delta + \frac{\varepsilon}{\delta^2}\right)(\|f_+\|_{L^2(\mathbb{R}^3)} + \|f_-\|_{L^2(\mathbb{R}^3)}),\end{aligned} \tag{14.61}$$

for any (f_+, f_-) in $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$. To prove (14.61), we decompose in frequency and angle as in the proof of Proposition 14.1, in order to reduce ourselves to diagonal terms.

Decomposition in frequency. — As in step 1 of the proof of Theorem 11.8, we decompose $M_\pm(f_\pm)$ and $P_\pm(f_\pm)$ in frequency:

$$M_\pm(f_\pm)(x) = \sum_{j\geq -1}(M_\pm)_j f_\pm(x) \text{ and } P_\pm(f_\pm)(x) = \sum_{j\geq -1}(P_\pm)_j f_\pm(x), \tag{14.62}$$

where the operators $(M_\pm)_j$, $(P_\pm)_j$ are defined as in (13.5) (13.6). Following step 1 of the proof of Proposition 14.1, we obtain the equivalent of (14.10):

$$\begin{aligned}&\|M_+(f_+) + M_-(f_-)\|^2_{L^2(S)}\\ &= \sum_{|j-l|\leq 2}\int_\Sigma ((M_+)_j f_+(x) + (M_-)_j f_-(x))\overline{((M_+)_l f_+(x) + (M_-)_l f_-(x))}d\Sigma\\ &\quad + O(\varepsilon)(\|f_+\|^2_{L^2(\mathbb{R}^3)} + \|f_-\|^2_{L^2(\mathbb{R}^3)})\end{aligned} \tag{14.63}$$

and

$$\begin{aligned}&\|P_+(f_+) - P_-(f_-)\|^2_{L^2(S)}\\ &= \sum_{|j-l|\leq 2}\int_\Sigma ((P_+)_j f_+(x) - (P_-)_j f_-(x))\cdot\overline{((P_+)_l f_+(x) - (P_-)_l f_-(x))}d\Sigma\\ &\quad + O(\varepsilon)(\|f_+\|^2_{L^2(\mathbb{R}^3)} + \|f_-\|^2_{L^2(\mathbb{R}^3)}).\end{aligned} \tag{14.64}$$

Decomposition in angle. — As in step 2 of the proof of Proposition 14.1, we decompose $(M_\pm)_j f_\pm$ and $(P_\pm)_j f_\pm$ in angle:

$$(M_\pm)_j f_\pm(x) = \sum_{\nu\in\Gamma}(M_\pm)_j^\nu f(x) \text{ and } (P_\pm)_j f_\pm(x) = \sum_{\nu\in\Gamma}(P_\pm)_j^\nu f(x), \tag{14.65}$$

where the operators $(M_\pm)_j^\nu$ and $(P_\pm)_j^\nu$ are defined as in (13.11) and where the support of our partition of unity η_j^ν on $\mathbb{S}^2$ is a patch of diameter $\sim \delta 2^{-j/2}$ with δ chosen as in (14.12). Following step 2 of the proof of Proposition 14.1, we obtain the equivalent of (14.17):

$$\begin{aligned}&\sum_{|j-l|\le 2}\int_\Sigma((M_+)_j f_+(x) + (M_-)_j f_-(x))\overline{((M_+)_l f_+(x) + (M_-)_l f_-(x))}d\Sigma\\ &= \sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma((M_+)_j^\nu f_+(x) + (M_-)_j^\nu f_-(x))\\ &\quad\times\overline{((M_+)_l^{\nu'} f_+(x) + (M_-)_l^{\nu'} f_-(x))}d\Sigma + O\left(\frac{\varepsilon}{\delta^2}\right)(\|f_+\|^2_{L^2(\mathbb{R}^3)} + \|f_-\|^2_{L^2(\mathbb{R}^3)})\end{aligned} \tag{14.66}$$

and

$$\begin{aligned}&\sum_{|j-l|\le 2}\int_\Sigma((P_+)_j f_+(x) - (P_-)_j f_-(x))\cdot\overline{((P_+)_l f_+(x) - (P_-)_l f_-(x))}d\Sigma\\ &= \sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma((P_+)_j^\nu f_+(x) - (P_-)_j^\nu f_-(x))\cdot\overline{((P_+)_l^{\nu'} f_+(x) - (P_-)_l^{\nu'} f_-(x))}d\Sigma\\ &\quad + O\left(\frac{\varepsilon}{\delta^2}\right)(\|f_+\|^2_{L^2(\mathbb{R}^3)} + \|f_-\|^2_{L^2(\mathbb{R}^3)}).\end{aligned} \tag{14.67}$$

End of the proof of Lemma 14.7. — (14.63) and (14.66) yield:

$$\begin{aligned}\|M_+(f_+) + M_-(f_-)\|^2_{L^2(S)} &= \sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma((M_+)_j^\nu f_+(x) + (M_-)_j^\nu f_-(x))\\ &\quad\times\overline{((M_+)_l^{\nu'} f_+(x) + (M_-)_l^{\nu'} f_-(x))}d\Sigma + O\left(\frac{\varepsilon}{\delta^2}\right)(\|f_+\|^2_{L^2(\mathbb{R}^3)} + \|f_-\|^2_{L^2(\mathbb{R}^3)})\end{aligned} \tag{14.68}$$

and (14.64) and (14.67) yield:

$$\begin{aligned}&\|P_+(f_+) - P_-(f_-)\|^2_{L^2(S)}\\ &= \sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma((P_+)_j^\nu f_+(x) - (P_-)_j^\nu f_-(x))\cdot\overline{((P_+)_l^{\nu'} f_+(x) - (P_-)_l^{\nu'} f_-(x))}d\Sigma\\ &\quad + O\left(\frac{\varepsilon}{\delta^2}\right)(\|f_+\|^2_{L^2(\mathbb{R}^3)} + \|f_-\|^2_{L^2(\mathbb{R}^3)}).\end{aligned} \tag{14.69}$$

The operator $(P_\pm)_j^\nu - N(x, \pm\nu)(M_+)_j^\nu$ has a symbol given by $N(x, \pm\omega) - N(x, \pm\nu)$. Thus, the estimate (13.118) together with Assumption 2 on $\partial_\omega N$, and the fact that $|\omega-\nu|\lesssim\delta 2^{-j/2}$ on the support of η_j^ν yields:

$$\sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma((P_+)_j^\nu f_+(x)-(P_-)_j^\nu f_-(x))\cdot\overline{((P_+)_l^{\nu'} f_+(x)-(P_-)_l^{\nu'} f_-(x))}d\Sigma$$

(14.70)

$$=\sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma(N(x,\nu)(M_+)_j^\nu f_+(x)-N(x,-\nu)(M_-)_j^\nu f_-(x))$$
$$\cdot\overline{(N(x,\nu)(M_+)_l^{\nu'} f_+(x)-N(x,-\nu)(M_-)_l^{\nu'} f_-(x))}d\Sigma+O(\delta)(\|f_+\|^2_{L^2(\mathbb{R}^3)}+\|f_-\|^2_{L^2(\mathbb{R}^3)}).$$

Now, Assumption 6 yields $|N(x,\nu)+N(x,-\nu)|\lesssim\varepsilon$ which together with (14.70) and the fact that N is a unit vector implies:

$$\sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma((P_+)_j^\nu f_+(x)-(P_-)_j^\nu f_-(x))\cdot\overline{((P_+)_l^{\nu'} f_+(x)-(P_-)_l^{\nu'} f_-(x))}d\Sigma$$
$$=\sum_{|j-l|\le 2}\sum_{|\nu-\nu'|\le 2\delta 2^{-j/2}}\int_\Sigma((M_+)_j^\nu f_+(x)+(M_-)_j^\nu f_-(x))\overline{((M_+)_l^{\nu'} f_+(x)+(M_-)_l^{\nu'} f_-(x))}d\Sigma$$

(14.71)

$$+O(\delta+\varepsilon)(\|f_+\|^2_{L^2(\mathbb{R}^3)}+\|f_-\|^2_{L^2(\mathbb{R}^3)}).$$

Finally, (14.68), (14.69) and (14.71) yield:

$$\|P_+(f_+)-P_-(f_-)\|^2_{L^2(S)}=\|M_+(f_+)+M_-(f_-)\|^2_{L^2(S)}+O\left(\delta+\frac{\varepsilon}{\delta^2}\right)(\|f_+\|^2_{L^2(\mathbb{R}^3)}+\|f_-\|^2_{L^2(\mathbb{R}^3)}),\tag{14.72}$$

which implies (14.61). As noticed at the beginning of the proof, (14.61) yields the wanted estimate (14.55). This concludes the proof of Lemma 14.7. □

14.3. Existence of (f_+, f_-)

In the previous section, we have proved the estimate (11.49):

$$\|\lambda f_+\|_{L^2(\mathbb{R}^3)}+\|\lambda f_-\|_{L^2(\mathbb{R}^3)}\lesssim\|\nabla\phi_0\|_{L^2(S)}+\|\phi_1\|_{L^2(S)}\tag{14.73}$$

for any (f_+, f_-) satisfying the following system:

$$\begin{cases} M_+f_++M_-f_-=\phi_0,\\ Q_+(\lambda f_+)-Q_-(\lambda f_-)=i\phi_1.\end{cases}\tag{14.74}$$

Notice that (14.73) implies the uniqueness of (f_+, f_-) solution of (14.74). In this section, we complete the proof of Theorem 11.10 by proving the existence of (f_+, f_-) solution of (14.74).

Recall that the phase $u(x,\omega)$ of our Fourier integral operators has been constructed in **[23]** on $\Sigma\times\mathbb{S}^2$ under the assumption that (Σ, g, k) satisfies the following bounds consistent with the assumptions on Σ for R and k in Theorem 10.1:

$$\|R\|_{L^2(S)}\le\varepsilon,\ \|\nabla k\|_{L^2(S)}\le\varepsilon.\tag{14.75}$$

(Σ, g, k) also satisfies the constraint equations:

$$\left\{\begin{array}{r}\nabla^j k_{ij} = 0,\\ R = |k|^2,\\ \mathrm{Tr}k = 0,\end{array}\right. \tag{14.76}$$

where the last equation in (14.76) comes from the fact that we work with a maximal foliation. We introduce two sets V and W:

$$V = \{(\Sigma, g, k) \text{ such that (14.75) and (14.76) are satisfied}\} \tag{14.77}$$

and

(14.78)

$$W = \{(\Sigma, g, k) \in V \text{ such that } (f_+, f_-) \text{ solution of (14.74) exist for all } (\phi_0, \phi_1)$$
$$\text{such that } \nabla\phi_0 \in L^2(S) \text{ and } \phi_1 \in L^2(S)\}.$$

In order to prove the existence of (f_+, f_-) solution of (14.74), we will show that $V = W$ by a connectedness argument. This will result from the following two lemmas.

Lemma 14.8. — *Let $N \geq 0$ an integer. Then, the set V is connected for the topology of $(g, k) \in C^q(\Sigma) \times C^{q-1}(\Sigma)$.*

Lemma 14.9. — *Let $N \geq 0$ an integer. Then, the set W is open and closed in V for the topology of $(g, k) \in C^q(\Sigma) \times C^{q-1}(\Sigma)$ provided q is chosen sufficiently large.*

Remark 14.10. — The assumptions on the regularity on (Σ, g, k) in Lemma 14.8 and 14.9 are much stronger than the ones appearing in the bounded L^2 curvature conjecture. We would like to insist on the fact that this smoothness is only assumed to obtain the existence of (f_+, f_-) solution of (14.74). On the other hand, we only rely on the control of $\|R\|_{L^2(S)}$ and $\|\nabla k\|_{L^2(S)}$ given by (14.75) to prove the estimate (14.73).

We postpone the proof of Lemma 14.8 and Lemma 14.9 respectively to Section 14.3.1 and Section 14.3.2. Let us now conclude the proof of Theorem 11.10. Note first that W is not empty. In fact, the flat initial data set $(\Sigma, g, k) = (\mathbb{R}^3, \delta, 0)$ belongs to V, where δ denotes the euclidean metric. In that case, our construction in **[23]** yields the usual Fourier phase $u(x, \omega) = x \cdot \omega$. Then, the system (14.74) reduces to:

$$\left\{\begin{array}{l}\mathscr{F}^{-1}(f_+) + \mathscr{F}^{-1}(f_-) = \phi_0,\\ \mathscr{F}^{-1}(\lambda f_+) - \mathscr{F}^{-1}(\lambda f_-) = i\phi_1,\end{array}\right. \tag{14.79}$$

which admits the solution:

$$f_\pm = \frac{1}{2}\left(\mathscr{F}(\phi_0) \pm i\frac{\mathscr{F}(\phi_1)}{\lambda}\right), \tag{14.80}$$

where $\mathscr{F}$ denotes the Fourier transform on $\mathbb{R}^3$. Thus, $(\Sigma, g, k) = (\mathbb{R}^3, \delta, 0)$ belongs to W, which implies that W is not empty. It is also open and closed in V for the topology of $(g, k) \in C^q(\Sigma) \times C^{q-1}(\Sigma)$ by Lemma 14.9 for q sufficiently large. Since

V is connected for the topology of $(g,k) \in C^q(\Sigma) \times C^{q-1}(\Sigma)$ by Lemma 14.8, this implies that $W = V$. This proves the existence of (f_+, f_-) solution of (14.74) and concludes the proof of Theorem 11.10. □

14.3.1. Proof of Lemma 14.8

The conformal method of Lichnerowicz. — We start by reviewing the conformal method of Lichnerowicz for constructing solutions to the constraint Equations (14.76) on Σ. Let $\underline{g}$ a Riemannian metric on Σ. We define the Riemannian metric g and the symmetric 2-tensor k as:

$$\begin{cases} g = \phi^4 \underline{g}, \\ k = \phi^{-2}\sigma, \end{cases} \tag{14.81}$$

where σ is a traceless symmetric 2-tensor and ϕ a conformal factor tending to 1 at infinity. Then, (g,k) defined in (14.81) satisfies the constraint equations (14.76) provided that (ϕ, σ) satisfy the following system:

$$\begin{cases} -8\Delta\phi + R\phi - |\sigma|^2\phi^{-7} = 0, \\ \operatorname{div}\sigma = 0, \end{cases} \tag{14.82}$$

where R is the scalar curvature of $\underline{g}$ and where the divergence and the Laplacian are taken with respect to $\underline{g}$.

The existence of σ. — We now turn to the question of the existence of ϕ and σ solution to (14.82). In order to exploit the smallness condition (14.75), we need an existence theory for rough solutions to the constraint Equations (14.76). We will follow the exposition in **[18]** (we refer to **[3]** for the smooth case). Let $l \in \mathbb{N}$ and $\rho \in \mathbb{R}$. We introduce the spaces $H^l_\rho(\Sigma)$ defined by:

$$H^l_\rho(\Sigma) = \left\{ h \;/\; \sum_{|\alpha| \le l} \|(1+|x|)^{-\rho-3/2+|\alpha|} h\|_{L^2(S)} < +\infty \right\}. \tag{14.83}$$

We recall first the construction of a symmetric traceless divergence free 2-tensor σ on Σ. To this end, we introduce the conformal Killing operator $\mathbb{L}$ and the vector Laplacian $\Delta_{\mathbb{L}}$:

$$\begin{cases} \mathbb{L}X = \mathcal{L}_X \underline{g} - \frac{2}{3}\operatorname{div}(X)\underline{g}, \\ \Delta_{\mathbb{L}} = \operatorname{div}(\mathbb{L}X), \end{cases} \tag{14.84}$$

where X is a vector field on Σ, $\mathcal{L}_X$ is the Lie derivative with respect to X, and the divergence is taken with respect to the metric $\underline{g}$. If S is a symmetric traceless 2-tensor, and if we can solve

$$\Delta_{\mathbb{L}} X = -\operatorname{div}(S), \tag{14.85}$$

then setting $\sigma = S + \mathbb{L}X$ yields $\operatorname{div}\sigma = 0$ which solves the second equation of (14.82). The fact that this is always possible is known as the York decomposition. In the context of a rough metric $\underline{g}$, the following result holds (see **[18]**):

$$\text{(14.86)}\quad \begin{array}{l}\text{Let } 1- < \rho < 0,\ \underline{g} \in H^2_\rho(\Sigma) \text{ and } S \in H^1_{\rho-1}(\Sigma). \text{ There is a unique } X \\ \text{solution to (14.85), and } X \text{ satisfies } \|X\|_{H^2_\rho(\Sigma)} \lesssim \|S\|_{H^1_{\rho-1}(\Sigma)}. \\ \text{This yields a solution } \sigma \text{ to } \mathrm{div}\sigma = 0 \text{ such that } \sigma = S + \mathbb{L}X \\ \text{and } \|\sigma\|_{H^1_{\rho-1}(\Sigma)} \lesssim \|S\|_{H^1_{\rho-1}(\Sigma)}.\end{array}$$

The existence of ϕ. — We then have to solve the first equation of (14.82) which is the Lichnerowicz equation. This is not an easy task in general since one has to show that $\underline{g}$ is conformally related to a metric with vanishing scalar curvature. However, we are in the particular case of small data in view of (14.75), and we will obtain the existence of ϕ by a fixed point method. Let for $-1 < \rho < 0$ and let $g \in H^2_\rho(s)$. Then, recall from **[18]** that $-\Delta$ is invertible as an operator from $H^2_\rho(\Sigma)$ to $H^0_{\rho-2}(\Sigma)$ so that the following estimate holds:

$$\|(-\Delta)^{-1}h\|_{H^2_\rho(\Sigma)} \lesssim \|h\|_{H^0_{\rho-2}(\Sigma)},\ -1 < \rho < 0. \tag{14.87}$$

This allows us to rewrite the first equation of (14.82) in the form of a fixed point for $\psi = \phi - 1$:

$$\psi = \frac{1}{8}(-\Delta)^{-1}\left(-R + |\sigma|^2 - R\psi + |\sigma|^2((1+\psi)^{-7} - 1)\right). \tag{14.88}$$

Now, we deduce from the embedding of $H^2_\rho(\Sigma)$ in $L^\infty(\Sigma)$ for $\rho < 0$ and from the properties of the spaces $H^l_\rho(\Sigma)$ with respect to the pointwise multiplication proved in **[18]** the following inequality:

(14.89)
$$\begin{aligned}\|-R + |\sigma|^2 - R\psi + |\sigma|^2((1+\psi)^{-7} - 1)\|_{H^0_{\rho-2}(\Sigma)} \lesssim{}& \|R\|_{H^0_{\rho-2}(\Sigma)}(1 + \|\psi\|_{H^2_\rho(\Sigma)}) \\ &+ \|\sigma\|^2_{H^1_{\rho-1}(\Sigma)}(1 + \vartheta(\|\psi\|_{H^2_\rho(\Sigma)})),\end{aligned}$$

where ϑ is an increasing function, and where we assume that the control $\|\psi\|_{H^2_\rho(\Sigma)} \leq 1/2$ holds. Thus, in view of (14.86), we have for $-1 < \rho < 0$ and $\|\psi\|_{H^2_\rho(\Sigma)} \leq 1/2$:

(14.90)
$$\begin{aligned}\|-R + |\sigma|^2 - R\psi + |\sigma|^2((1+\psi)^{-7} - 1)\|_{H^0_{\rho-2}(\Sigma)} \lesssim{}& \|R\|_{H^0_{\rho-2}(\Sigma)}(1 + \|\psi\|_{H^2_\rho(\Sigma)}) \\ &+ \|S\|^2_{H^1_{\rho-1}(\Sigma)}(1 + \vartheta(\|\psi\|_{H^2_\rho(\Sigma)})),\end{aligned}$$

where ϑ is an increasing function. In view of (14.90), we immediately obtain the existence of ψ solution to (14.88) provided $\|R\|_{H^0_\rho(\Sigma)} + \|S\|_{H^1_{\rho-1}(\Sigma)} \lesssim \varepsilon$ for a sufficiently small ε.

Proof of Lemma 14.8. — Let us come back to the proof of Lemma 14.8. We will prove that all solutions (Σ, g, k) of the constraint Equations (14.76) satisfying the bound (14.75) are connected to $(\mathbb{R}^3, \delta, 0)$ by a continuous path. For $0 \leq \tau \leq 1$, we introduce:

$$\underline{g}_\tau = \tau g + (1-\tau)\delta \text{ and } S_\tau = \tau k - \frac{\tau \mathrm{Tr}_\tau k}{3}\underline{g}_\tau, \tag{14.91}$$

where Tr_τ denotes the trace with respect to the metric $\underline{g}_\tau$. Let $-1<\rho<0$. From the smallness assumptions (14.75) and the Definition (14.91) of $\underline{g}_\tau$ and S_τ, we immediately obtain:

$$\|\underline{R}_\tau\|_{H^0_{\rho-2}(\Sigma)} + \|S_\tau\|_{H^1_{\rho-1}(\Sigma)} \lesssim \varepsilon, \tag{14.92}$$

where $\underline{R}_\tau$ is the scalar curvature of $\underline{g}_\tau$. In view of (14.86), (14.88) (14.90) and (14.92), we obtain the existence of (σ_τ, ϕ_τ) in $H^1_{\rho-1}(\Sigma)\times H^2_\rho(s)$ solution to:

$$\begin{cases} -8\Delta\phi_\tau + \underline{R}_\tau\phi_\tau - |\sigma_\tau|^2\phi_\tau^{-7} = 0, \\ \mathrm{div}\sigma_\tau = 0, \end{cases} \tag{14.93}$$

where $\underline{R}_\tau$ is the scalar curvature of $\underline{g}_\tau$ and where the divergence and the Laplacian are taken with respect to $\underline{g}_\tau$. Finally, setting

$$\begin{cases} g_\tau = \phi_\tau^4 \underline{g}_\tau, \\ k_\tau = \phi_\tau^{-2}\sigma_\tau, \end{cases} \tag{14.94}$$

we obtain a solution (Σ, g_τ, k_τ) to the constraint Equations (14.76) which satisfies the following bound:

$$\|g_\tau - \delta\|_{H^2_\rho(\Sigma)} + \|k_\tau\|_{H^1_{\rho-1}(\Sigma)} \lesssim \varepsilon. \tag{14.95}$$

Thus (g_τ, k_τ) satisfies the bound (14.75) so that (Σ, g_τ, k_τ) belongs to the set V defined by (14.77). Furthermore, recall from (14.88) that ϕ_τ is obtained by a fixed point argument. This implies in particular the uniqueness of (σ_τ, ϕ_τ) so that $(g_\tau, k_\tau) = (\delta, 0)$ at $\tau = 0$ and $(g_\tau, k_\tau) = (g, k)$ at $\tau = 1$. Using standard results in elliptic regularity, we also obtain that the path $\tau \to (g_\tau, k_\tau)$ is continuous for the topology of $C^q(\Sigma)\times C^{q-1}(\Sigma)$ provided $(g,k)\in C^q(\Sigma)\times C^{q-1}(\Sigma)$. Thus, all solutions (Σ, g, k) of the constraint Equations (14.76) satisfying the bound (14.75) are connected to $(\mathbb{R}^3, \delta, 0)$ by a continuous path, which concludes the proof of Lemma 14.8. □

Remark 14.11. — In general, the connectedness of the set of all solutions (Σ, g, k) of the constraint Equations (14.76) is an open problem (see **[19]** for a partial answer). Here, the smallness condition (14.75) makes the problem much easier, as the solutions are obtained by a fixed point argument in this case.

14.3.2. Proof of Lemma 14.9

The operator Λ. — We start by rewriting the system (14.74) as:

$$\begin{cases} \nabla M_+ f_+ + \nabla M_- f_- = \nabla\phi_0, \\ Q_+(\lambda f_+) - Q_-(\lambda f_-) = i\phi_1. \end{cases} \tag{14.96}$$

We define the operators $\widetilde{M}_\pm$ as:

$$\widetilde{M}_\pm f(x) = \pm\int_{\mathbb{S}^2}\int_0^{+\infty} e^{\pm i\lambda u(x,\pm\omega)} a(x,\pm\omega)^{-1} N(x,\pm\omega) f(\lambda\omega)\lambda^2 d\lambda d\omega, \tag{14.97}$$

so that (14.96) becomes:

$$\begin{cases} \widetilde{M}_+(\lambda f_+) - \widetilde{M}_-(\lambda f_-) = -i\nabla\phi_0, \\ Q_+(\lambda f_+) - Q_-(\lambda f_-) = i\phi_1. \end{cases} \tag{14.98}$$

We define the linear operator Λ as:

$$\Lambda(f_+, f_-) = (\widetilde{M}_+(f_+) - \widetilde{M}_-(f_-), Q_+(f_+) - Q_-(f_-)). \tag{14.99}$$

By Theorem 11.8 and Assumption 1-4 on u, a and N, Λ is a bounded operator from $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ to $L^2(S)^3 \times L^2(S)$. In view of (11.49), it satisfies the following estimate:

$$\|f_+\|_{L^2(\mathbb{R}^3)} + \|f_-\|_{L^2(\mathbb{R}^3)} \lesssim \|\Lambda(f_+, f_-)\|_{L^2(S)^3\times L^2(S)}. \tag{14.100}$$

Finally, in view of (14.98) and the Definition (14.99) of Λ, we may rewrite the set W as:

$$W = \{(\Sigma, g, k) \in V \text{ such that } \Lambda \text{ is surjective}\}. \tag{14.101}$$

W is closed. — We have to show that the set W given by (14.101) is both open and closed for the $C^q(\Sigma)\times C^{q-1}(\Sigma)$ topology when q is sufficiently large. Let us first show that W is closed. Let $(g_n, k_n), n \in \mathbb{N}$ a sequence in W such that it has a limit (g, k) for the $C^q(\Sigma)\times C^{q-1}(\Sigma)$ topology. Let Λ_n be the operator associated to (g_n, k_n), and Λ the operator associated to Λ. Λ_n is surjective for all $n \geq 0$, and we would like to prove that Λ is surjective. Notice first that the fact that Λ is a bounded operator from $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ to $L^2(S)^3 \times L^2(S)$ together with the estimate (14.100) implies that the image of Λ is closed in $L^2(S)^3 \times L^2(S)$. Thus, we may reduce the problem to showing that a dense subset of $L^2(S)^3 \times L^2(S)$ belongs to the image of Λ. Let us consider (ϕ_0, ϕ_1) in $C^7_c(\Sigma) \times C^6_c(\Sigma)$ which is dense in $L^2(S)^3 \times L^2(S)$. Since Λ_n is surjective, there are (f^n_+, f^n_-) such that:

$$\Lambda_n(f^n_+, f^n_-) = (\nabla\phi_0, \phi_1). \tag{14.102}$$

Differentiating (14.102) six times and using (14.100), we obtain:

$$\begin{aligned} &\|(1+\lambda^6)f^n_+\|_{L^2(\mathbb{R}^3)} + \|(1+\lambda^6)f^n_-\|_{L^2(\mathbb{R}^3)} \\ &\quad\lesssim (\|g_n\|_{C^q(\Sigma)} + \|k_n\|_{C^{q-1}(\Sigma)})(\|\phi_0\|_{C^7_c(\Sigma)} + \|\phi_1\|_{C^6_c}) \end{aligned} \tag{14.103}$$

for a sufficiently large q. We deduce from (14.103) the existence of a constant $C > 0$ independent of n such that:

$$\|(1+\lambda^6)f^n_+\|_{L^2(\mathbb{R}^3)} + \|(1+\lambda^6)f^n_-\|_{L^2(\mathbb{R}^3)} \leq C < +\infty. \tag{14.104}$$

In particular, we may assume in up to a subsequence that (f^n_+, f^n_-) converges up to a subsequence to (f_+, f_-) weakly in $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. We have:

$$\Lambda_n(f^n_+, f^n_-) - \Lambda(f_+, f_-) = (\Lambda_n - \Lambda)(f^n_+, f^n_-) + \Lambda(f^n_+ - f_+, f^n_- - f_-). \tag{14.105}$$

We will show that both terms in the right-hand side of (14.105) converge to 0 weakly in $L^2(S)^3 \times L^2(S)$. We start with the first term. For $(H, h) \in C^0_c(\Sigma)^3 \times C^0_c(\Sigma)$, we have in view of the Definition (14.99) of Λ:

(14.106)

$$\begin{aligned}&\left|\int_\Sigma ((\Lambda_n-\Lambda)(f_+^n,f_-^n),(H,h))d\Sigma\right|\\&=\left|\int_{\mathbb{S}^2}\int_0^{+\infty}((f_+^n,f_-^n),(\Lambda_n-\Lambda)^*(H,h))\lambda^2d\lambda d\omega\right|\\&\lesssim \|(H,h)\|_{C^0_c(\Sigma)^3\times C^0_c(\Sigma)}\int_{\mathbb{S}^2}\int_0^{+\infty}(|f_+^n(\lambda\omega)|+|f_-^n(\lambda\omega)|)(\lambda\|u_n(.,\omega)-u(.,\omega)\|_{L^\infty(S)}\\&\quad+\|a_n^{-1}(.,\omega)-a^{-1}(.,\omega)\|_{L^\infty(S)}+\|N_n(.,\omega)-N(.,\omega)\|_{L^\infty(S)})\lambda^2d\lambda d\omega\\&\lesssim \|(H,h)\|_{C^0_c(\Sigma)^3\times C^0_c(\Sigma)}(\|(1+\lambda^6)f_+^n\|_{L^2(\mathbb{R}^3)}+\|(1+\lambda^6)f_-^n\|_{L^2(\mathbb{R}^3)})\\&\qquad\times\sup_{\omega\in\mathbb{S}^2}(\|u_n(.,\omega)-u(.,\omega)\|_{L^\infty(S)}+\|a_n^{-1}(.,\omega)-a^{-1}(.,\omega)\|_{L^\infty(S)}\\&\quad+\|N_n(.,\omega)-N(.,\omega)\|_{L^\infty(S)}).\end{aligned}$$

Since (g_n,k_n) converges to (g,k) in $C^q(\Sigma)\times C^{q-1}(\Sigma)$, we have for q large enough:

$$\begin{aligned}\lim_{n\to+\infty}\sup_{\omega\in\mathbb{S}^2}(\|u_n(.,\omega)-u(.,\omega)\|_{L^\infty(S)}+\|a_n^{-1}(.,\omega)-a^{-1}(.,\omega)\|_{L^\infty(S)}\\+\|N_n(.,\omega)-N(.,\omega)\|_{L^\infty(S)})\\=0.\end{aligned}\tag{14.107}$$

Using (14.104), (14.106) and (14.107) implies that $(\Lambda_n-\Lambda)(f_+^n,f_-^n)$ converges weakly in $L^2(S)^3\times L^2(S)$ to 0. Also, using the fact that Λ is a bounded operator from $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ to $L^2(S)^3\times L^2(S)$ and that (f_+^n,f_-^n) converges to (f_+,f_-) weakly in $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$, we obtain that $\Lambda(f_+^n-f_+,f_-^n-f_-)$ converges weakly in $L^2(S)^3\times L^2(S)$ to 0. In view of (14.105), this implies that $\Lambda_n(f_+^n,f_-^n)$ converges weakly to $\Lambda(f_+,f_-)$ in $L^2(S)^3\times L^2(S)$. Together with (14.102), this implies

$$\Lambda(f_+,f_-)=(\nabla\phi_0,\phi_1).\tag{14.108}$$

Thus, Λ is surjective which concludes the proof of the fact that W is closed.

W is open. — To conclude the proof of Lemma 14.9, we need to prove that W is open for the $C^q(\Sigma)\times C^{q-1}(\Sigma)$ topology when q is sufficiently large. Let $(g,k)\in W$ and let Λ the operator associated to (g,k). Then Λ is surjective which together with the estimate (14.100) implies that Λ is an isomorphism from $L^2(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ to $L^2(S)^3\times L^2(S)$. In turn, this implies that $\Lambda\Lambda^*$ is an isomorphism of $L^2(S)^3\times L^2(S)$. Let $(\widetilde{g},\widetilde{k})\in W$ such that:

$$\|\widetilde{g}-g\|_{C^q(\Sigma)}+\|\widetilde{k}-k\|_{C^{q-1}(\Sigma)}\leq\delta\tag{14.109}$$

for a small constant $\delta>0$ to be chosen later, and let $\widetilde{\Lambda}$ the operator associated to $(\widetilde{g},\widetilde{k})$. Then, $\widetilde{\Lambda}$ and Λ consist of Fourier integral operators whose phase and symbol are $O(\delta)$ close to each other in the $C^q(\Sigma)$ topology. Integrating by parts several times in the kernel of $\Lambda\Lambda^*$ and $\widetilde{\Lambda}\widetilde{\Lambda}^*$, we deduce the following bound provided q is sufficiently large:

$$\|\widetilde{\Lambda}\widetilde{\Lambda}^* - \Lambda\Lambda^*\|_{\mathscr{L}(L^2(S)^3\times L^2(S))} \lesssim \delta. \tag{14.110}$$

Since the isomorphism of $L^2(S)^3 \times L^2(S)$ form an open set, we deduce from (14.110) that $\widetilde{\Lambda}\widetilde{\Lambda}^*$ is an isomorphism of $L^2(S)^3 \times L^2(S)$ for $\delta > 0$ small enough. In particular, $\widetilde{\Lambda}$ is surjective for $\delta > 0$ small enough. Therefore, $\widetilde{\Lambda} \in W$ provided $\delta > 0$ defined in (14.109) is chosen small enough. Thus, W is open. This concludes the proof of Lemma 14.9. □

APPENDIX D

PROOF OF LEMMA 13.6

We would like to compute the double divergence term in the right-hand side of (13.78):

$$\mathrm{div}\left(\frac{(N-(N\cdot N')N')a}{1-(N\cdot N')^2}\mathrm{div}\left(\frac{(N'-(N\cdot N')N)a'bb'}{1-(N\cdot N')^2}\right)\right). \tag{D.1}$$

We recall the structure equations for N:

$$\begin{cases}\nabla_A N=\theta_{AB}e_B,\\ \nabla_N N=-\nabla\!\!\!/ a.\end{cases} \tag{D.2}$$

In particular, (D.2) implies:

$$\mathrm{div}(N)=\mathrm{tr}\,\theta. \tag{D.3}$$

Using (D.3), we have:

$$\begin{aligned}&\mathrm{div}\left(\frac{(N'-(N\cdot N')N)a'bb'}{1-(N\cdot N')^2}\right)\\ &=\frac{(\mathrm{tr}\,\theta'-g(N,N')\mathrm{tr}\,\theta-\nabla_N(g(N,N')))a'bb'+\nabla_{N'-g(N,N')N}(a'bb')}{1-g(N,N')^2}\\ &\quad+\frac{2a'bb'\nabla_{N'-g(N,N')N}(g(N,N'))g(N,N')}{(1-g(N,N')^2)^2}.\end{aligned} \tag{D.4}$$

Differentiating again, we obtain:

$$\begin{aligned}&\mathrm{div}\left(\frac{(N-(N\cdot N')N')a}{1-(N\cdot N')^2}\mathrm{div}\left(\frac{(N'-(N\cdot N')N)a'bb'}{1-(N\cdot N')^2}\right)\right)\\ &=\frac{A_1}{(1-g(N,N')^2)^2}+\frac{A_2}{(1-g(N,N')^2)^3}+\frac{A_3}{(1-g(N,N')^2)^4},\end{aligned} \tag{D.5}$$

where A_1,A_2,A_3 are given by:

$$\begin{aligned}A_1=&(\nabla_{N-g(N,N')N'}\mathrm{tr}\,\theta'-g(N,N')\nabla_{N-g(N,N')N'}\mathrm{tr}\,\theta)aa'bb'\\ &-(\nabla_{N-g(N,N')N'}(g(N,N'))\mathrm{tr}\,\theta+\nabla_{N-g(N,N')N'}\nabla_N(g(N,N')))aa'bb'\\ &+(\mathrm{tr}\,\theta'-g(N,N')\mathrm{tr}\,\theta-\nabla_N(g(N,N')))a\nabla_{N-g(N,N')N'}(a'bb')\end{aligned} \tag{D.6}$$

$$
\begin{aligned}
&+ a\nabla_{N-g(N,N')N'}\nabla_{N'-g(N,N')N}(a'bb')\\
&+ ((\operatorname{tr}\theta - g(N,N')\operatorname{tr}\theta' - \nabla_{N'}(g(N,N')))a + \nabla_{N-g(N,N')N'}(a))\\
&\times ((\operatorname{tr}\theta' - g(N,N')\operatorname{tr}\theta - \nabla_N(g(N,N')))a'bb' + \nabla_{N'-g(N,N')N}(a'bb')),
\end{aligned}
$$

$$
\begin{aligned}
A_2 = {}& 2a\nabla_{N-g(N,N')N'}(g(N,N'))g(N,N')\\
&\times ((\operatorname{tr}\theta' - g(N,N')\operatorname{tr}\theta - \nabla_N(g(N,N')))a'bb' + \nabla_{N'-g(N,N')N}(a'bb'))\\
&+ 2aa'bb'\nabla_{N-g(N,N')N'}\nabla_{N'-g(N,N')N}(g(N,N'))g(N,N')\\
&+ 2aa'bb'\nabla_{N'-g(N,N')N}(g(N,N'))\nabla_{N-g(N,N')N'}(g(N,N'))\\
&+ 2a\nabla_{N'-g(N,N')N}(g(N,N'))g(N,N')\nabla_{N-g(N,N')N'}(a'bb')\\
&+ ((\operatorname{tr}\theta - g(N,N')\operatorname{tr}\theta' - \nabla_{N'}(g(N,N')))a + \nabla_{N-g(N,N')N'}(a))\\
&\times 2a'bb'\nabla_{N'-g(N,N')N}(g(N,N'))g(N,N')\\
&+ 2a\nabla_{N-g(N,N')N'}(g(N,N'))g(N,N')\\
&\times ((\operatorname{tr}\theta' - g(N,N')\operatorname{tr}\theta - \nabla_N(g(N,N')))a'bb' + \nabla_{N'-g(N,N')N}(a'bb'))
\end{aligned}
\tag{D.7}
$$

and

$$A_3 = 8aa'bb'\nabla_{N'-g(N,N')N}(g(N,N'))\nabla_{N-g(N,N')N'}(g(N,N'))g(N,N')^2. \tag{D.8}$$

Notice that $N - (N\cdot N')N'$ is tangent to $P_{u'}$ and that $N' - (N\cdot N')N$ is tangent to P_u. Notice also that $1 - g(N,N')$ is of order two in $N - N'$:

$$1 - g(N,N') = \frac{g(N-N', N-N')}{2}. \tag{D.9}$$

In view of (D.5)-(D.9), one easily checks that the double divergence (D.1) takes the wanted form (13.79) (13.80) provided that we are able to control all the terms in the following list:

(D.10)

$$
\begin{aligned}
&\frac{\nabla_N(g(N,N'))}{(1-g(N,N'))^{1/2}},\ \frac{\nabla_{N-g(N,N')N'}(N'-g(N,N')N)}{1-g(N,N')},\ \frac{\nabla_{N-g(N,N')N'}(g(N,N'))}{(1-g(N,N'))^{3/2}},\\
&\frac{\nabla_{N-g(N,N')N'}\nabla_N(g(N,N'))}{1-g(N,N')},\ \frac{\nabla_{N-g(N,N')N'}\nabla_{N'-g(N,N')N}(g(N,N'))}{(1-g(N,N'))^2}.
\end{aligned}
$$

Control of the first term of (D.10). — Using the structure equation for N (D.2), we have:

$$
\begin{aligned}
&\nabla_N(g(N,N')) \\
&= g(\nabla_N N, N') + g(N,N')g(N,\nabla_{N'}N') + g(N,\nabla_{N-g(N,N')N'}N')\\
&= -g(\nabla\!\!\!/ \log(a), N') + g(N,N')g(N, \nabla\!\!\!/' \log(a')) + \theta'(N-g(N,N')N', N-g(N,N')N')\\
&= -g(\nabla\log(a), N'-g(N,N')N) + g(N,N')g(N-g(N,N')N', \nabla\log(a'))\\
&\quad + \theta'(N-g(N,N')N', N-g(N,N')N').
\end{aligned}
\tag{D.11}
$$

Using (D.9), we have:

$$\text{(D.12)}\qquad \frac{|N-g(N,N')N'|}{(1-g(N,N'))^{1/2}}+\frac{|N'-g(N,N')N|}{(1-g(N,N'))^{1/2}}\lesssim 1.$$

In view of (D.9), (D.11) and (D.12), the term $\frac{\nabla_N(g(N,N'))}{(1-g(N,N'))^{1/2}}$ is under control and involves terms in the list (13.80).

Control of the second term of (D.10). — Using the structure equation for N (D.2), we have:

(D.13)

$$\begin{aligned}
\nabla_{N-g(N,N')N'}(N'-g(N,N')N) &= \nabla_{N-g(N,N')N'}N' - g(N,N')\nabla_{N-g(N,N')N'}N\\
&\quad - \nabla_{N-g(N,N')N'}(g(N,N'))N\\
&= \theta'(N-g(N,N')N', e_{A'})e_{A'}\\
&\quad - g(N,N')((1-g(N,N')^2)\nabla_N N - g(N,N')\nabla_{N'-g(N,N')N}N)\\
&\quad - ((1-g(N,N')^2)g(-\nabla\!\!\!/ \log(a), N')\\
&\quad - g(N,N')\theta(N'-g(N,N')N, N'-g(N,N')N)\\
&\quad + \theta'(N-g(N,N')N', N-g(N,N')N'))N\\
&= \theta'(N-g(N,N')N', e_{A'})e_{A'} + \theta(N'-g(N,N')N, e_A)e_A\\
&\quad - (1-g(N,N')^2)\theta(N'-g(N,N')N, e_A)e_A\\
&\quad + g(N,N')(1-g(N,N')^2)\nabla\!\!\!/ \log(a)\\
&\quad + (1-g(N,N')^2)\nabla_{N'-g(N,N')N}\log(a)N\\
&\quad + g(N,N')\theta(N'-g(N,N')N, N'-g(N,N')N)N\\
&\quad - \theta'(N-g(N,N')N', N-g(N,N')N')N,
\end{aligned}$$

where we have used the fact that:

$$\text{(D.14)}\qquad N-g(N,N')N' = (1-g(N,N')^2)N - g(N,N')(N'-g(N,N')N).$$

Note that the tangential components $N-(N\cdot N')N'$ and $N'-(N\cdot N')N$ satisfy:

$$\text{(D.15)}\qquad (N-g(N,N')N')+(N'-g(N,N')N) = (N+N')(1-g(N,N')),$$

so that we may divide $\theta'(N-g(N,N')N', e_{A'})e_{A'} + \theta(N'-g(N,N')N, e_A)e_A$ by $1-g(N,N')$. Thus, in view of (D.9), (D.12), (D.13) and (D.15), the term

$$\frac{\nabla_{N-g(N,N')N'}(N'-g(N,N')N)}{1-g(N,N')}$$

is under control and involves terms in the list (13.80).

Control of the third term of (D.10). — Using the structure equation for N (D.2) together with (D.14), we have:

$$
\begin{aligned}
\nabla_{N-g(N,N')N'}(g(N,N')) &= g(\nabla_{N-g(N,N')N'}N, N') + g(N, \nabla_{N-g(N,N')N'}N') \\
&= -(1-g(N,N')^2)g(\nabla\log(a), N'-g(N,N')N) \\
&\quad - g(N,N')\theta(N'-g(N,N')N, N'-g(N,N')N) \\
&\quad + \theta'(N-g(N,N')N', N-g(N,N')N').
\end{aligned} \tag{D.16}
$$

In view of (D.9), (D.12), (D.15) and (D.16), the term $\frac{\nabla_{N-g(N,N')N'}(g(N,N'))}{(1-g(N,N'))^{3/2}}$ is under control and involves terms in the list (13.80).

Control of the fourth term of (D.10). — Differentiating (D.11) with respect to $\nabla_{N-g(N,N')N'}$, we have:

$$
\begin{aligned}
\nabla_{N-g(N,N')N'}\nabla_N(g(N,N')) &= -\nabla^2\log(a)(N-g(N,N')N', N'-g(N,N')N) \\
&\quad - g(\nabla\log(a), \nabla_{N-g(N,N')N'}(N'-g(N,N')N)) \\
&\quad + \nabla_{N-g(N,N')N'}(g(N,N'))g(\nabla\log(a'), N-g(N,N')N') \\
&\quad + g(N,N')\nabla^2\log(a')(N-g(N,N')N', N-g(N,N')N') \\
&\quad + g(N,N')g(\nabla\log(a'), \nabla_{N-g(N,N')N'}(N-g(N,N')N')) \\
&\quad + \nabla_{N-g(N,N')N'}\theta'(N-g(N,N')N', N-g(N,N')N') \\
&\quad + 2\theta'(\nabla_{N-g(N,N')N'}(N-g(N,N')N'), N-g(N,N')N').
\end{aligned} \tag{D.17}
$$

In view of (D.17), we need to control the term $\frac{\nabla_{N-g(N,N')N'}(N-g(N,N')N')}{1-g(N,N')}$. This is very similar to (D.13). Using the structure equation for N (D.2), we obtain:

$$
\begin{aligned}
&\nabla_{N-g(N,N')N'}(N-g(N,N')N') \\
&= \nabla_{N-g(N,N')N'}N - g(N,N')\nabla_{N-g(N,N')N'}N' \\
&\quad - \nabla_{N-g(N,N')N'}(g(N,N'))N' \\
&= (1-g(N,N')^2)\nabla_N N - g(N,N')\nabla_{N'-g(N,N')N}N \\
&\quad - g(N,N')\theta'(N-g(N,N')N', e_{A'})e_{A'} \\
&\quad - ((1-g(N,N')^2)g(-\nabla\!\!\!/\log(a), N') \\
&\quad - g(N,N')\theta(N'-g(N,N')N, N'-g(N,N')N) \\
&\quad + \theta'(N-g(N,N')N', N-g(N,N')N'))N' \\
&= -(1-g(N,N')^2)\nabla\!\!\!/\log(a) - g(N,N')\theta(N'-g(N,N')N, e_A)e_A \\
&\quad - g(N,N')\theta'(N-g(N,N')N', e_{A'})e_{A'} \\
&\quad + (1-g(N,N')^2)\nabla_{N'-g(N,N')N}\log(a)N' \\
&\quad + g(N,N')\theta(N'-g(N,N')N, N'-g(N,N')N)N' \\
&\quad - \theta'(N-g(N,N')N', N-g(N,N')N')N'.
\end{aligned} \tag{D.18}
$$

In view of (D.9), (D.12), (D.15) and (D.18), the term $\frac{\nabla_{N-g(N,N')N'}(N-g(N,N')N')}{1-g(N,N')}$ is under control and involves terms in the list (13.80). Note also that the terms

$$\nabla^2\log(a)(N-g(N,N')N', N'-g(N,N')N)$$
$$\text{and } \nabla^2\log(a')(N-g(N,N')N', N-g(N,N')N')$$

appearing in (D.17) both contain at least one tangential derivative. Together with (D.9), (D.12), (D.15), (D.16), (D.17) and (D.18), this yields that the term $\frac{\nabla_{N-g(N,N')N'}\nabla_N(g(N,N'))}{1-g(N,N')}$ is under control and involves terms in the list (13.80).

Control of the fifth term of (D.10). — Exchanging the role of N and N' in (D.16), we obtain:

$$\begin{aligned}\nabla_{N'-g(N,N')N}(g(N,N')) = &-(1-g(N,N')^2)g(\nabla\log(a'), N-g(N,N')N')\\ &-g(N,N')\theta'(N-g(N,N')N', N-g(N,N')N')\\ &+\theta(N'-g(N,N')N, N'-g(N,N')N).\end{aligned}\tag{D.19}$$

Differentiating (D.19) with respect to $\nabla_{N-g(N,N')N'}$, we obtain:

$$\begin{aligned}&\nabla_{N-g(N,N')N'}\nabla_{N'-g(N,N')N}(g(N,N'))\\ &\quad= -(1-g(N,N')^2)\nabla^2\log(a')(N-g(N,N')N', N-g(N,N')N')\\ &\qquad-(1-g(N,N')^2)\nabla_{\nabla_{N-g(N,N')N'}(N-g(N,N')N')}\log(a')\\ &\qquad+2g(N,N')\nabla_{N-g(N,N')N'}(g(N,N'))\nabla_{N-g(N,N')N'}\log(a')\\ &\qquad-\nabla_{N-g(N,N')N'}(g(N,N'))\theta'(N-g(N,N')N', N-g(N,N')N')\\ &\qquad-g(N,N')\nabla_{N-g(N,N')N'}\theta'(N-g(N,N')N', N-g(N,N')N')\\ &\qquad-2g(N,N')\theta'(\nabla_{N-g(N,N')N'}(N-g(N,N')N'), N-g(N,N')N')\\ &\qquad+\nabla_{N-g(N,N')N'}\theta(N'-g(N,N')N, N'-g(N,N')N)\\ &\qquad+2\theta(\nabla_{N-g(N,N')N'}(N'-g(N,N')N), N'-g(N,N')N).\end{aligned}\tag{D.20}$$

Together with (D.13), (D.16) and (D.18), we get:

$$\begin{aligned}&\nabla_{N-g(N,N')N'}\nabla_{N'-g(N,N')N}(g(N,N'))\\ &= -(1-g(N,N')^2)\nabla^2\log(a')(N-g(N,N')N', N-g(N,N')N')\\ &\quad+(1-g(N,N')^2)^2 g(\nabla\log(a), \nabla\log(a'))\\ &\quad+(1-g(N,N')^2)g(N,N')\theta(N'-g(N,N')N, \nabla\log(a'))\\ &\quad+(1-g(N,N')^2)g(N,N')\theta'(N-g(N,N')N', \nabla\log(a'))\\ &\quad-(1-g(N,N')^2)^2\nabla_{N'-g(N,N')N}\log(a)\nabla_{N'}\log(a')\\ &\quad-(1-g(N,N')^2)g(N,N')\theta(N'-g(N,N')N, N'-g(N,N')N)\nabla_{N'}\log(a')\\ &\quad+(1-g(N,N')^2)\theta'(N-g(N,N')N', N-g(N,N')N')\nabla_{N'}\log(a')\\ &\quad-2g(N,N')\nabla_{N-g(N,N')N'}\log(a')(1-g(N,N')^2)\nabla_{N'-g(N,N')N}\log(a)\end{aligned}\tag{D.21}$$

$$
\begin{aligned}
&- 2g(N,N')^2\nabla_{N-g(N,N')N'}\log(a')\theta(N'-g(N,N')N, N'-g(N,N')N)\\
&+ 2g(N,N')\nabla_{N-g(N,N')N'}\log(a')\theta'(N-g(N,N')N', N-g(N,N')N')\\
&+ (1-g(N,N')^2)\nabla_{N'-g(N,N')N}\log(a)\theta'(N-g(N,N')N', N-g(N,N')N')\\
&+ g(N,N')\theta(N'-g(N,N')N, N'-g(N,N')N)\\
&\quad\times\theta'(N-g(N,N')N', N-g(N,N')N')\\
&- \theta'(N-g(N,N')N', N-g(N,N')N')^2\\
&- g(N,N')\nabla_{N-g(N,N')N'}\theta'(N-g(N,N')N', N-g(N,N')N')\\
&+ \nabla_{N-g(N,N')N'}\theta(N'-g(N,N')N, N'-g(N,N')N)\\
&+ 2g(N,N')(1-g(N,N')^2)\theta'(\nabla\!\!\!/\log(a), N-g(N,N')N')\\
&+ 2g(N,N')(1-g(N,N')^2)\theta(\nabla\!\!\!/\log(a), N'-g(N,N')N)\\
&- 2(1-g(N,N')^2)\theta(e_A, N'-g(N,N')N)^2\\
&+ 2g(N,N')^2\theta'(e_{A'}, N-g(N,N')N')\theta(N'-g(N,N')N, e_{A'})\\
&+ 2g(N,N')^2\theta'(e_{A'}, N-g(N,N')N')^2\\
&+ 2g(N,N')^2\theta'(e_A, N-g(N,N')N')\theta(N'-g(N,N')N, e_A)\\
&+ 2g(N,N')^2\theta(e_A, N'-g(N,N')N)^2.
\end{aligned}
$$

Note that the term $\nabla^2\log(a')(N-g(N,N')N', N-g(N,N')N')$ appearing in (D.21) contains at least one tangential derivative (it actually contains two tangential derivatives). Note also that the terms:

$$
\begin{aligned}
&2g(N,N')^2\theta'(e_{A'}, N-g(N,N')N')\theta(N'-g(N,N')N, e_{A'})\\
&\qquad + 2g(N,N')^2\theta'(e_{A'}, N-g(N,N')N')^2\\
&\qquad + 2g(N,N')^2\theta'(e_A, N-g(N,N')N')\theta(N'-g(N,N')N, e_A)\\
&\qquad + 2g(N,N')^2\theta(e_A, N'-g(N,N')N)^2
\end{aligned}
\tag{D.22}
$$

appearing in (D.21) may be rewritten:

$$
2g(N,N')^2(\theta'(N-g(N,N')N',.)+\theta(N'-g(N,N')N,.))^2. \tag{D.23}
$$

Together with (D.9), (D.12), (D.15) and (D.21), this yields that the term

$$
\frac{\nabla_{N-g(N,N')N'}\nabla_{N'-g(N,N')N}(g(N,N'))}{(1-g(N,N'))^2}
$$

is under control and involves terms in the list (13.80). This concludes the proof of Lemma 13.6. □

APPENDIX E

PROOF OF LEMMA 13.7

We start with the terms $\nabla\!\!\!/\nabla(ab)a'b'$, $\theta\nabla(ab)a'b'$, $\nabla(a)\nabla(b)a'b'$, $\theta^2 aa'bb'$ in the list (13.80). They all take the form (13.81) with $H_3 = H_4 = 0$, $H_2 = a'b'$ and taking respectively $H_1 = \nabla\!\!\!/\nabla(ab)$, $H_1 = \theta\nabla(ab)$, $H_1 = \nabla(a)\nabla(b)$ and $H_1 = \theta^2 ab$. Thanks to Assumption 1 and Assumption 2 on a, θ and the assumptions (11.42) (11.43) on b, the estimates (13.82) and (13.83) are satisfied.

We now consider the other terms:

$$\frac{(\nabla\theta - \nabla\theta')aa'bb'}{|N_\nu - N_{\nu'}|},\ \frac{(\theta - \theta')\nabla(ab)a'b'}{|N_\nu - N_{\nu'}|},\ ab\theta\nabla(a'b'),$$
$$\nabla(ab)\nabla(a'b'),\ \frac{(\theta - \theta')^2 aa'bb'}{|N_\nu - N_{\nu'}|^2},\ \theta\theta' aa'bb'.$$

We focus on $\frac{(\nabla\theta-\nabla\theta')aa'bb'}{|N_\nu - N_{\nu'}|}$ and $\frac{(\theta-\theta')^2 aa'bb'}{|N_\nu - N_{\nu'}|^2}$ the others being similar. For $\frac{(\nabla\theta-\nabla\theta')aa'bb'}{|N_\nu - N_{\nu'}|}$, we have:

$$\frac{(\nabla\theta - \nabla\theta')aa'bb'}{|N_\nu - N_{\nu'}|} = \frac{(\nabla\theta - \nabla\theta_\nu)aa'bb'}{|N_\nu - N_{\nu'}|} + \frac{(\nabla\theta_\nu - \nabla\theta')aa'bb'}{|N_\nu - N_{\nu'}|} \tag{E.1}$$

and the two terms in (E.1) are of the form (13.81) with $H_3 = H_4 = 0$, and respectively $H_1 = \frac{(\nabla\theta-\nabla\theta_\nu)ab}{|N_\nu - N_{\nu'}|}$, $H_2 = a'b'$ and $H_1 = \frac{(\nabla\theta_\nu-\nabla\theta')a'b'}{|N_\nu - N_{\nu'}|}$, $H_2 = ab$. Thanks to Assumption 1 and Assumption 2 on a, θ and the assumptions (11.42) (11.43) on b, the estimates (13.82) and (13.83) are satisfied. In particular, we have:

$$\left\| \frac{(\nabla\theta - \nabla\theta_\nu)ab}{|N_\nu - N_{\nu'}|} \right\|_{L^2(S)} \lesssim D \frac{\|\nabla\partial_\omega\theta\|_{L^2(S)}}{2^{j/2}|\nu - \nu'|} \lesssim D \tag{E.2}$$

and

$$\left\| \frac{(\nabla\theta_\nu - \nabla\theta')ab}{|N_\nu - N_{\nu'}|} \right\|_{L^2(S)} \lesssim D\|\nabla\partial_\omega\theta\|_{L^2(S)} \lesssim D, \tag{E.3}$$

where we have used Assumption 2 to estimate $|N_\nu - N_{\nu'}|$, the fact that $|\nu - \omega| \lesssim 2^{j/2}$ on the support of η_j^ν and the fact that $2^{j/2}|\nu - \nu'| \geq 1$. We finally consider the term $\frac{(\theta-\theta')^2aa'bb'}{|N_\nu-N_{\nu'}|^2}$. We have:

$$\frac{(\theta-\theta')^2aa'bb'}{|N_\nu-N_{\nu'}|^2}=\frac{(\theta-\theta_\nu)^2aa'bb'}{|N_\nu-N_{\nu'}|^2}+2\frac{(\theta-\theta_\nu)(\theta_\nu-\theta')aa'bb'}{|N_\nu-N_{\nu'}|^2}+\frac{(\theta_\nu-\theta')^2aa'bb'}{|N_\nu-N_{\nu'}|^2}. \tag{E.4}$$

The first and the last term in (E.4) are estimated like the term $\frac{(\nabla\theta-\nabla\theta')aa'bb'}{|N_\nu-N_{\nu'}|}$ remarking that

$$\left\|\frac{(\theta-\theta_\nu)^2ab}{|N_\nu-N_{\nu'}|^2}\right\|_{L^2(S)}\lesssim\frac{\|\partial_\omega\theta\|^2_{L^4(S)}}{(2^{j/2}|\nu-\nu'|)^2}\lesssim\|\partial_\omega\theta\|^2_{L^2(S)}+\|\nabla\partial_\omega\theta\|^2_{L^2(S)}\lesssim 1 \tag{E.5}$$

and

$$\left\|\frac{(\theta_\nu-\theta')^2ab}{|N_\nu-N_{\nu'}|^2}\right\|_{L^2(S)}\lesssim\|\partial_\omega\theta\|^2_{L^4(S)}\lesssim\|\partial_\omega\theta\|^2_{L^2(S)}+\|\nabla\partial_\omega\theta\|^2_{L^2(S)}\lesssim 1. \tag{E.6}$$

Finally, the second term in (E.4) is of the form (13.81) with $H_1=H_2=0$, $H_3=2^{j/2}(\theta-\theta_\nu)ab$ and $H_4=\frac{(\theta_\nu-\theta')a'b'}{|N_\nu-N_{\nu'}|}$.

Thanks to Assumption 1 and Assumption 2 on a,θ and the assumption (11.42) on b, the estimates (13.82) and (13.83) are satisfied. In particular, we have:

$$\begin{aligned}\|2^{j/2}(\theta-\theta_\nu)ab\|_{L^\infty_{[-2,2]}L^2(P_u)}&\lesssim D\|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^2(P_u)}\\&\lesssim D(\|\partial_\omega\theta\|_{L^2(S)}+\|\nabla\partial_\omega\theta\|_{L^2(S)})\lesssim D\end{aligned} \tag{E.7}$$

and

$$\begin{aligned}\left\|\frac{(\theta_\nu-\theta')a'b'}{|N_\nu-N_{\nu'}|}ab\right\|_{L^\infty_{[-2,2]}L^2(P_u)}&\lesssim D\|\partial_\omega\theta\|_{L^\infty_{[-2,2]}L^2(P_u)}\\&\lesssim D(\|\partial_\omega\theta\|_{L^2(S)}+\|\nabla\partial_\omega\theta\|_{L^2(S)})\lesssim D,\end{aligned} \tag{E.8}$$

where we have used the fact that $H^1(\Sigma)$ embeds in $L^\infty_{[-2,2]}L^2(P_u)$ (see **[23]** Corollary 3.6 for a proof only using the regularity given by Assumption 1). This concludes the proof of Lemma 13.7. □

APPENDIX F

PROOF OF LEMMA 13.8

We need to compute the divergence terms involving D_1 and D_2 in (13.97). We start with the term involving D_1.

F.1. The divergence term involving D_1 in (13.98)

Using the Definition (13.95) of D_1 together with the structure Equation (D.2) for N and (D.3), we obtain:

(F.1)

$$\begin{aligned}\operatorname{div}\left(\frac{(N'-g(N,N')N)a'}{1-g(N,N')^2}D_1\right) &= \frac{A_1}{(1-g(N,N')^2)(\lambda-\lambda'\frac{a}{a'}g(N,N'))}\\ &+\frac{A_2\lambda'}{(1-g(N,N')^2)(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}+\frac{A_3}{(1-g(N,N')^2)^2(\lambda-\lambda'\frac{a}{a'}g(N,N'))}\\ &+\frac{A_4\lambda'}{(1-g(N,N')^2)^2(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}+\frac{A_5\lambda'^2}{(1-g(N,N')^2)(\lambda-\lambda'\frac{a}{a'}g(N,N'))^3},\end{aligned}$$

where A_1, A_2, A_3, A_4, A_5 are given by:

(F.2)

$$\begin{aligned}A_1 &= a'\nabla_{N-g(N,N')N'}(abb')\mathrm{tr}\,\theta + aa'bb'\nabla_{N'-g(N,N')N}\mathrm{tr}\,\theta\\ &+ a'\nabla^2(ab)(N,N'-g(N,N')N)b' + \nabla_{\nabla_{N'-g(N,N')N}N}(ab)a'b'\\ &+ a'\nabla_N(ab)\nabla_{N'-g(N,N')N}(b') + (aa'bb'\mathrm{tr}\,\theta + \nabla_N(ab)a'b')\\ &\quad\times(\mathrm{tr}\,\theta' - g(N,N')\mathrm{tr}\,\theta - \nabla_N(g(N,N')) + a'^{-1}\nabla_{N'-g(N,N')N}(a')),\end{aligned}$$

(F.3)

$$\begin{aligned}A_2 &= \nabla^2 a(N,N'-g(N,N')N)abb'g(N,N') + \nabla_{\nabla_{N'-g(N,N')N}N}(a)abb'g(N,N')\\ &+ a'\nabla_N(a)\nabla_{N'-g(N,N')N}(a'^{-1}abb')g(N,N') + \nabla_N(a)abb'\nabla_N(g(N,N'))\\ &+a'\nabla_{N'-g(N,N')N}(a^2a'^{-2}bb')\nabla_N(g(N,N')) + a^2a'^{-1}bb'\nabla_{N'-g(N,N')N}\nabla_N(g(N,N'))\\ &+ (\mathrm{tr}\,\theta' - g(N,N')\mathrm{tr}\,\theta - \nabla_N(g(N,N')) + a'^{-1}\nabla_{N'-g(N,N')N}(a'))\end{aligned}$$

$$\times (a\nabla_N(a)bb'g(N,N') + a^2{a'}^{-1}\nabla_N(g(N,N'))bb') + (aa'bb'\mathrm{tr}\,\theta + \nabla_N(ab)a'b')$$
$$\times (\nabla_{N'-g(N,N')N}(a{a'}^{-1})g(N,N') + a{a'}^{-1}\nabla_{N'-g(N,N')N}(g(N,N'))),$$

(F.4)
$$A_3 = 2a'b'(ab\mathrm{tr}\,\theta + \nabla_N(ab))\nabla_{N'-g(N,N')N}(g(N,N'))g(N,N'),$$

(F.5)
$$A_4 = 2bb'(a\nabla_N(a){a'}^{-1}g(N,N') + a^2{a'}^{-2}\nabla_N(g(N,N')))$$
$$\times \nabla_{N'-g(N,N')N}(g(N,N'))g(N,N')$$

and

(F.6)
$$A_5 = 2(\nabla_{N'-g(N,N')N}(a{a'}^{-1})g(N,N') + a{a'}^{-1}\nabla_{N'-g(N,N')N}(g(N,N')))$$
$$\times (a\nabla_N(a)bb'g(N,N') + a^2{a'}^{-2}\nabla_N(g(N,N'))bb').$$

Note that the term $\nabla^2(ab)(N,N'-g(N,N')N)$ appearing in (F.2) and the term $\nabla^2(a)(N,N'-g(N,N')N)$ appearing in (F.3) contain at least one tangential derivative. In view of (F.1)-(F.6), one easily checks that the divergence term involving D_1 in (13.97) takes the wanted form (13.8) (13.99) provided that we are able to control the two following terms:

$$\frac{\nabla_N(g(N,N'))}{(1-g(N,N'))^{1/2}},\ \frac{\nabla_{N-g(N,N')N'}(g(N,N'))}{(1-g(N,N'))^{3/2}}. \tag{F.7}$$

The terms in (F.7) correspond to the first and the third term of (D.10). Thus, this control has already been proved in Appendix D.

F.2. The divergence term involving D_2 in (13.98)

Using the Definition (13.96) of D_2 together with the structure Equation (D.2) for N and (D.3), we obtain:

(F.8)
$$\begin{aligned}\mathrm{div}\left(\frac{(N-g(N,N')N')a}{1-g(N,N')^2}D_2\right) &= \frac{A_1}{(1-g(N,N')^2)(\lambda-\lambda'\frac{a}{a'}g(N,N'))}\\ &+\frac{A_2\lambda'}{(1-g(N,N')^2)(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}+\frac{A_3}{(1-g(N,N')^2)^2(\lambda-\lambda'\frac{a}{a'}g(N,N'))}\\ &+\frac{A_4\lambda'}{(1-g(N,N')^2)^2(\lambda-\lambda'\frac{a}{a'}g(N,N'))^2}+\frac{A_5{\lambda'}^2}{(1-g(N,N')^2)(\lambda-\lambda'\frac{a}{a'}g(N,N'))^3},\end{aligned}$$

where A_1, A_2, A_3, A_4, A_5 are given by:

(F.9)
$$A_1 = a\nabla_{N-g(N,N')N'}(ab)\nabla_N(b') + a^2b\nabla^2(b')(N,N-g(N,N')N')$$
$$+ a^2b\nabla_{\nabla_{N-g(N,N')N'}N}(b')$$

$$+ (a\mathrm{tr}\,\theta - ag(N,N')\mathrm{tr}\,\theta' - a\nabla_{N'}(g(N,N')) + \nabla_{N-g(N,N')N'}(a))ab\nabla_N(b'),$$

(F.10)

$$\begin{aligned}A_2 = & -\nabla_{N-g(N,N')N'}(a^2b)\nabla_N(a')g(N,N')ab'a'^{-2}\\ & - a^3a'^{-2}\nabla^2(a')(N, N-g(N,N')N')g(N,N')bb'\\ & - a^3a'^{-2}\nabla_{\nabla_{N-g(N,N')N'}N}(a')g(N,N')bb'\\ & - a^3b\nabla_N(a')\nabla_{N-g(N,N')N'}(b'a'^{-2})g(N,N')\\ & - a^3a'^{-2}\nabla_N(a')bb'\nabla_{N-g(N,N')N'}(g(N,N')) - a^3a'^{-2}\nabla_N(a')g(N,N')bb'\\ & \times (a\mathrm{tr}\,\theta - ag(N,N')\mathrm{tr}\,\theta' - a\nabla_{N'}(g(N,N')) + \nabla_{N-g(N,N')N'}(a)) + a^2b\nabla_N(b')\\ & \times (\nabla_{N-g(N,N')N'}(aa'^{-1})g(N,N') + aa'^{-1}\nabla_{N-g(N,N')N'}(g(N,N'))),\end{aligned}$$

(F.11)

$$A_3 = 2a^2b\nabla_{N-g(N,N')N'}(g(N,N'))g(N,N')\nabla_N(b'),$$

(F.12)

$$A_4 = -2a^3a'^{-2}bb'\nabla_{N-g(N,N')N'}(g(N,N'))g(N,N')^2\nabla_N(a'),$$

and

(F.13)

$$\begin{aligned}A_5 = & -2(\nabla_{N-g(N,N')N'}(aa'^{-1})g(N,N') + aa'^{-1}\nabla_{N-g(N,N')N'}(g(N,N')))\\ & a^3a'^{-2}\nabla_N(a')g(N,N')bb'.\end{aligned}$$

Note that the term $\nabla^2(b')(N, N - g(N,N')N')$ appearing in (F.9) and the term $\nabla^2(a')(N, N - g(N,N')N')$ appearing in (F.10) contain at least one tangential derivative. In view of (F.8)-(F.13), one easily checks that the divergence term involving D_2 in (13.97) takes the wanted form (13.8) (13.99) provided that we are able to control the two terms in (F.7). This control has already been proved in Appendix D. This concludes the proof of Lemma 13.8. □

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This book is dedicated to the construction and the control of a parametrix to the homogeneous wave equation $\square_{\mathbf{g}}\phi = 0$, where $\mathbf{g}$ is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor $\mathbf{R}$ of $\mathbf{g}$ is a major step of the proof of the bounded L^2 curvature conjecture proposed in Klainerman (2000), and solved jointly in Klainerman, Rodnianski & Szeftel (2015). On a more general level, this book deals with the control of the eikonal equation on a rough background, and with the derivation of L^2 bounds for Fourier integral operators on manifolds with rough phases and symbols, and as such is also of independent interest.

Cet ouvrage est dédié à la construction et au contrôle d'une paramétrix pour l'équation des ondes homogène $\square_{\mathbf{g}}\phi = 0$*, où* $\mathbf{g}$ *est une métrique peu régulière satisfaisant les équations d'Einstein dans le vide. Le contrôle d'une telle paramétrix ainsi que du terme d'erreur associé lorsque l'on suppose seulement des bornes* L^2 *sur le tenseur de courbure* $\mathbf{R}$ *de* $\mathbf{g}$ *est une étape cruciale de la preuve de la conjecture de courbure* L^2 *proposée dans Klainerman (2000), et résolue dans Klainerman, Rodnianski & Szeftel (2015). Plus généralement, cet ouvrage concerne le contrôle de l'équation eikonale sur un espace-temps peu régulier et la dérivation de bornes* L^2 *pour des opérateurs intégraux de Fourier sur des variétés avec une phase et un symbole peu réguliers, et possède de ce point de vue un intérêt propre.*

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