Mémoires
de la SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Numéro 160
Nouvelle série

# ERGODIC PROPERTIES OF SOME NEGATIVELY CURVED MANIFOLDS WITH INFINITE MEASURE

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*Vente au numéro :* 35 € ($ 52)
*Abonnement électronique :* 113 € ($ 170)
*Abonnement avec supplément papier :* 167 €, hors Europe : 197 € ($ 296)
Des conditions spéciales sont accordées aux membres de la SMF.

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ISSN papier 0249-633-X; électronique : 2275-3230

ISBN 978-2-85629-901-2

doi:10.24033/msmf.468

Directeur de la publication : Stéphane SEURET

MÉMOIRES DE LA SMF 160

# ERGODIC PROPERTIES OF SOME NEGATIVELY CURVED MANIFOLDS WITH INFINITE MEASURE

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Société Mathématique de France 2019

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**2000** ***Mathematics Subject Classification.*** **–** 58F17, 58F20, 20H10.

***Key words and phrases.*** **–** Ergodic theory, infinite measure, geodesic flow, orbital function, mixing, critical exponent, Schottky groups.

# ERGODIC PROPERTIES OF SOME NEGATIVELY CURVED MANIFOLDS WITH INFINITE MEASURE

**Pierre Vidotto**

***Abstract.*** – Let $\mathrm{M} = \mathrm{X}/\Gamma$ be a geometrically finite negatively curved manifold with fundamental group $\Gamma$ acting on X by isometries. The purpose of this book is to study the mixing property of the geodesic flow on $\mathrm{T}^1\mathrm{M}$, the asymptotic behavior as $R \longrightarrow +\infty$ of the number of closed geodesics on M of length less than $R$ and of the orbital counting function $\#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$.

These properties are well known when the Bowen-Margulis measure on $\mathrm{T}^1\mathrm{M}$ is finite. We consider here Schottky group $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ whose Bowen-Margulis measure is infinite and ergodic, such that one of the elementary factor $\Gamma_i$ is parabolic with $\delta_{\Gamma_i} = \delta_\Gamma$. We specify these ergodic properties using a symbolic coding induced by the Schottky group structure.

***Résumé.*** – Soit $\mathrm{M} = \mathrm{X}/\Gamma$ une variété géométriquement finie de courbure strictement négative et $\Gamma$ son groupe fondamental agissant par isométries sur X. Nous étudions successivement dans cet article une propriété de mélange du flot géodésique sur $\mathrm{T}^1\mathrm{M}$, le comportement quand $R \longrightarrow +\infty$ du nombre de géodésiques fermées de M de longueur plus petite que $R$ et celui de la fonction orbitale $\#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$.

Ces propriétés sont bien connues dans le cas où la mesure de Bowen-Margulis est finie sur $\mathrm{T}^1\mathrm{M}$. Nous considérons ici un groupe de Schottky $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ de mesure de Bowen-Margulis infinie et ergodique, pour lequel au moins un facteur $\Gamma_i$ est parabolique et satisfait $\delta_{\Gamma_i} = \delta_\Gamma$. Les propriétés ergodiques ci-dessus sont alors précisées, en utilisant un codage symbolique induit par la structure de groupe de Schottky de $\Gamma$.

# CONTENTS

# CHAPTER 1

# INTRODUCTION

## 1.1. Background and previous results

Let X be a connected, simply connected and complete riemannian manifold with pinched negative sectional curvature. Denote by d the distance on X induced by the riemannian structure of X and by $\Gamma$ a discrete group of isometries of $(\mathrm{X}, \mathrm{d})$, acting properly discontinuously without fixed point and let $\mathrm{M} = \mathrm{X}/\Gamma$. Fix $\mathbf{o} \in \mathrm{X}$. The study of quantities like the orbital function

$$\mathrm{N}_\Gamma(\mathbf{o}, R) := \#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$$

is strongly related to the dynamics of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $\mathrm{T}^1\mathrm{M}$ of the quotient manifold. Let us first define precisely this flow: each pair $(\mathbf{p}, \mathbf{v}) \in \mathrm{T}^1\mathrm{M}$ determines a unique geodesic $(\boldsymbol{\gamma}(t))_{t\in\mathbb{R}}$ satisfying $(\boldsymbol{\gamma}(0), \boldsymbol{\gamma}'(0)) = (\mathbf{p}, \mathbf{v})$ and for any $t \in \mathbb{R}$, the action of $g_t$ is given by $g_t(\mathbf{p}, \mathbf{v}) = (\boldsymbol{\gamma}(t), \boldsymbol{\gamma}'(t))$. It is known (see [**37**]) that the topological entropy of the geodesic flow is given by the rate of exponential growth $\delta_\Gamma$ of the orbital function, that is

$$\delta_\Gamma := \limsup_{R\longrightarrow+\infty} \frac{\ln\left(\mathrm{N}_\Gamma(\mathbf{o}, R)\right)}{R}.$$

This last quantity is also the critical exponent of the *Poincaré series* $\mathscr{P}_\Gamma$ of the group $\Gamma$ defined as follows: for any $s > 0$

$$\mathscr{P}_\Gamma(s) := \sum_{\gamma\in\Gamma} e^{-s\mathrm{d}(\mathbf{o}, \gamma\cdot\mathbf{o})}.$$

S. J. Patterson (in [**39**]) and D. Sullivan (in [**43**]) used these series to construct a family of measures $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$, the so-called Patterson-Sullivan measures. More precisely, each measure $\sigma_{\mathbf{x}}$ is fully-supported by the limit set $\Lambda_\Gamma \subset \partial\mathrm{X}$, which is defined as the set of all accumulation points of one (all) $\Gamma$-orbit(s) in the visual boundary $\partial\mathrm{X}$ of X. This set is also the smallest non-empty $\Gamma$-invariant closed subset of $\mathrm{X} \cup \partial\mathrm{X}$. It is the closure in the boundary of the set of fixed points of $\Gamma^* := \Gamma\backslash\{\mathrm{Id}\}$. A group $\Gamma$ is said to be elementary if its limit set is a finite set. S.J. Patterson and D. Sullivan described a process to associate to this family a measure $m_\Gamma$ defined on $\mathrm{T}^1\mathrm{M}$, which is invariant under the action of the geodesic flow. When the group $\Gamma$ is *divergent*, i.e., $\mathscr{P}_\Gamma(\delta_\Gamma) = +\infty$ (otherwise $\Gamma$ is said to be *convergent*), the family $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$ is unique up to a normalization, hence $m_\Gamma$ is also unique. We will focus in this book on the case of divergent groups, which allows us to speak about "the" Bowen-Margulis measure $m_\Gamma$ even when $m_\Gamma$ has infinite mass. Nevertheless, in this introduction, the assumption "$\mathrm{M} = \mathrm{X}/\Gamma$ has infinite Bowen-Margulis measure" should be in general understood as the fact that *any* invariant measure obtained from a Patterson-Sullivan density $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$ has infinite mass. We first study here a property of mixing of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ with respect to this measure. We say that the geodesic flow $(g_t)_{t\in\mathbb{R}}$ is mixing with respect to a measure $m$ with finite total mass $||m||$ on $\mathrm{T}^1\mathrm{M}$, if for any $m$-measurable sets $\mathfrak{A}, \mathfrak{B} \subset \mathrm{T}^1\mathrm{M}$, one gets

$$m\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \longrightarrow \frac{m(\mathfrak{A})m(\mathfrak{B})}{||m||} \text{ as } t \longrightarrow \pm\infty. \tag{1}$$

When the measure $m$ has infinite mass, this definition may be extended saying that the flow $(g_t)_{t\in\mathbb{R}}$ is mixing if

$$m\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \longrightarrow 0 \text{ as } t \longrightarrow \pm\infty, \text{ where } \mathfrak{A} \text{ and } \mathfrak{B} \text{ have finite measure.}$$

When the measure $m_\Gamma$ is finite, Property (1) was first proved by G. A. Hedlund in [**27**] for finite volume surfaces in constant curvature, by F. Dal'bo and M. Peigné for Schottky groups with parabolic isometries acting on Hadamard manifolds with pinched negative curvature (see [**17**]) and by M. Babillot in the general case (see [**1**]). The following result of T. Roblin [**41**] gathers all the information known in such a general content.

THEOREM (Roblin). – *If the Bowen-Margulis measure $m_\Gamma$ has finite mass $||m_\Gamma||$ (resp. infinite mass), the flow $(g_t)_{t\in\mathbb{R}}$ satisfies*

$$m_\Gamma\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \underset{t\longrightarrow\pm\infty}{\longrightarrow} \frac{m_\Gamma(\mathfrak{A})m_\Gamma(\mathfrak{B})}{||m_\Gamma||} \quad \left(\textit{resp. } m_\Gamma\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \underset{t\longrightarrow\pm\infty}{\longrightarrow} 0\right).$$

REMARK 1.1.1. – *The definition of mixing in infinite measure seems to be weak (see the third chapter of* [**41**] *about this fact). Nevertheless, our Theorem A below will furnish an asymptotic of the form*

$$m_\Gamma(\mathfrak{A} \cap g_{-t}\mathfrak{B}) \underset{t\longrightarrow\pm\infty}{\sim} \varepsilon(|t|)m_\Gamma(\mathfrak{A})m_\Gamma(\mathfrak{B}), \textit{ for an explicit function } \varepsilon,$$

*which can be understood as a mixing property, up to a renormalization.*

On the one hand, this property is interesting from the point of view of the ergodic theory. On the other hand, in the case of geometrically finite manifolds with finite measure, the property of mixing of the geodesic flow may be used to find an asymptotic of the orbital function $\mathrm{N}_\Gamma(\mathbf{o}, R)$. This idea was initially developed in G.A. Margulis' thesis [**35**] for compact manifolds with negative curvature: the mixing of the geodesic flow implies a property of equidistribution of spheres on M, which leads to the orbital counting. In the constant curvature case, other proofs of the orbital counting have been developed using spectral theory (see for instance [**34**]) or symbolic coding (see [**33**]). In [**41**], Roblin generalized the ideas of Margulis and deduced the asymptotic for the orbital counting function from the mixing of the geodesic flow. He showed the following theorem.

THEOREM (Roblin). – *Let* $\mathrm{M} = \mathrm{X}/\Gamma$ *be a complete manifold with pinched negative sectional curvatures. If the Bowen-Margulis measure* $m_\Gamma$ *has finite mass* $||m_\Gamma||$ *(resp. infinite mass), the asymptotic behavior of the orbital function* $\mathrm{N}_\Gamma(\mathbf{o}, R)$ *is given by*

$$\mathrm{N}_\Gamma(\mathbf{o}, R) \sim \frac{||\sigma_{\mathbf{o}}||^2}{||m_\Gamma||} e^{\delta_\Gamma R} \ \left(\textit{resp. } \mathrm{N}_\Gamma(\mathbf{o}, R) = o\left(e^{\delta_\Gamma R}\right)\right) \ \textit{as } R \longrightarrow +\infty,$$

*where* $||\sigma_{\mathbf{o}}||$ *is the mass of the Patterson-Sullivan measure* $\sigma_{\mathbf{o}}$*.*

We eventually focus on finding an asymptotic formula for the number of closed geodesics on M of length less than $R$, as $R$ goes to infinity. Such an asymptotic was first found by Huber in constant curvature $-1$ (see [**31**]), then by Margulis ([**35**]) for compact manifolds of variable negative curvature and extended in [**38**] to periodic orbits of axiom-A flows. This was generalized by T. Roblin in [**41**] as follows.

THEOREM ([**41**]). – *Let* $\mathrm{M} = \mathrm{X}/\Gamma$ *be a geometrically finite complete manifold with sectional curvatures less than* $-1$*, whose Bowen-Margulis measure has finite mass. For all* $R > 0$*, let* $\mathrm{N}_{\mathscr{G}}(R)$ *be the number of closed geodesics on* M *of length less than* $R$*. Then*

$$\mathrm{N}_{\mathscr{G}}(R) \sim \frac{e^{\delta_\Gamma R}}{\delta_\Gamma R} \ \textit{as } R \longrightarrow +\infty.$$

When the Bowen-Margulis measure is infinite, Roblin's method does not yield to such asymptotic. We will detail why in the next subsection.

## 1.2. Assumptions and results

In this article, we will focus on some manifolds $\mathrm{M} = \mathrm{X}/\Gamma$, where $\Gamma$ is a divergent group whose Bowen-Margulis measure $m_\Gamma$ on $\mathrm{T}^1\mathrm{M}$ has infinite mass and whose Poincaré series are controlled at infinity in a way that will be specified below. For such manifolds, we establish a speed of convergence to 0 of the quantities $m_\Gamma\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right)$ for any $\mathfrak{A}, \mathfrak{B} \subset \mathrm{T}^1\mathrm{M}$ with finite measure, an asymptotic behavior for the orbital counting function $\mathrm{N}_\Gamma(\mathbf{o}, R)$ and an asymptotic lower bound for the number of closed geodesics $\mathrm{N}_{\mathscr{G}}(R)$. The groups $\Gamma$ which we consider are exotic Schottky groups, whose construction was explained in the articles **[14]** and **[40]** and will be recalled in the second section. The main idea of these papers is the following: Let M be a geometrically finite hyperbolic manifold with dimension $N \geqslant 2$, with a cusp and whose fundamental group is a non elementary Schottky group. Theorems A and B in **[14]** ensure that $\Gamma$ is a divergent group and that $m_\Gamma$ is finite. The proofs of these results give a way to modify the metric in the cusp in order to obtain a manifold M′ isometric to a quotient $\mathrm{X}/\Gamma$ where

- the manifold X is a Hadamard manifold with pinched negative curvature;
- the group $\Gamma$ acts by isometries on X and is convergent; the measure $m_\Gamma$ is thus infinite.

The article **[40]** extends the previous construction and allows us to modify the metric in the cusp of M in such a way that the group $\Gamma$ is divergent with respect to the metric on X and the measure $m_\Gamma$ is still infinite: this construction will be discussed in more detail in Chapter 2. This article furnishes examples of manifolds $\mathrm{M} = \mathrm{X}/\Gamma$ on which our work applies.

Let X be a Hadamard manifold with pinched negative curvature between $-b^2$ and $-a^2$, where $0 < a < 1 \leqslant b$ and let $\Gamma$ be a Schottky group, i.e., $\Gamma$ is generated by elementary groups $\Gamma_1, \dots, \Gamma_{p+q}$ in Schottky position (see Subsection 2.1.2), where $p, q \geqslant 1$ and $p + q \geqslant 3$. Assume that for some $\beta \in \,]0, 1]$ and some slowly varying function $L$ (i.e., $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a measurable function such that $L(xt)/L(t)$ tends to 1 as $t \longrightarrow +\infty$, for all $x > 0$), the groups $\Gamma_1, \dots, \Gamma_{p+q}$ satisfy the following family of assumptions $(H_\beta)$:

(D) *The group* $\Gamma = \Gamma_1 \star \cdots \star \Gamma_{p+q}$ *is divergent.*

($\mathrm{P}_1$) *For any* $j \in [\![1, p]\!]$, *the group* $\Gamma_j$ *is parabolic, convergent and its critical exponent is equal to* $\delta_\Gamma$.

($\mathrm{P}_2$) *For any* $j \in [\![1, p]\!]$, *the tail of the Poincaré series at* $\delta_\Gamma$ *of the group* $\Gamma_j$ *satisfies*

$$\sum_{\alpha \in \Gamma_j \ \mid \ \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) > T} e^{-\delta_\Gamma \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})} \underset{T \longrightarrow +\infty}{\sim} C_j \frac{L(T)}{T^\beta}$$

*for some constant* $C_j > 0$.

(N) *For any* $j \in [\![p+1, p+q]\!]$, *the group* $\Gamma_j$ *satisfies the following property*

$$\sum_{\alpha \in \Gamma_j \ \mid \ \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) > T} e^{-\delta_\Gamma \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})} \underset{T \longrightarrow +\infty}{\sim} o\left(\frac{L(T)}{T^\beta}\right).$$

We *add* the following assumption to the family $(H_\beta)$:

(S) *For any* $\Delta > 0$, *there exists* $C = C_\Delta > 0$ *such that for any* $j \in [\![1, p+q]\!]$ *and any* $T > 0$ *large enough*

$$\sum_{\alpha \in \Gamma_j \ | \ T-\Delta \leqslant \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) < T+\Delta} e^{-\delta_\Gamma \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})} \leqslant C \frac{L(T)}{T^{1+\beta}}.$$

We will say that the parabolic groups $\Gamma_1, \dots, \Gamma_p$ are "influential," since their properties will determine all the dynamical properties of $\Gamma$ on $\partial \mathrm{X}$. The reader should notice that each of these subgroups $\Gamma_1, \dots, \Gamma_p$ is convergent and has the same critical exponent $\delta_\Gamma$ as the whole group $\Gamma$. The existence of at least a factor having these properties is needed to get a Schottky group with infinite Bowen-Margulis measure, see **[14]** and Chapter 2 below. On the contrary, the groups $\Gamma_{p+1}, \dots, \Gamma_{p+q}$ are said to be "non-influential" and their own critical exponent may be in particular strictly less than $\delta_\Gamma$.

Since $\Gamma$ is divergent, Hopf-Tsuji-Sullivan's theorem ensures that the geodesic flow $(g_t)_{t\in\mathbb{R}}$ is totally conservative with respect to the measure $m_\Gamma$. Under these assumptions, we first show the following theorem, which specifies the rate of mixing of the geodesic flow.

THEOREM A. – *Let* $\Gamma$ *be a Schottky group satisfying Hypotheses* $(H_\beta)$ *for some* $\beta \in ]0, 1]$ *and* $\mathfrak{A}, \mathfrak{B} \subset \mathrm{T}^1\mathrm{X}/\Gamma$ *be two* $m_\Gamma$*-measurable sets with finite measure.*

- *If* $\beta \in ]0, 1[$, *there exists a constant* $C = C_{\beta,\Gamma} > 0$ *such that*

$$m_\Gamma(\mathfrak{A} \cap g_{-t}\mathfrak{B}) \sim C \frac{m_\Gamma(\mathfrak{A}) m_\Gamma(\mathfrak{B})}{|t|^{1-\beta} L(|t|)} \ \text{ as } t \longrightarrow \pm\infty;$$

- *if* $\beta = 1$, *there exists a constant* $C = C_\Gamma > 0$ *such that*

$$m_\Gamma(\mathfrak{A} \cap g_{-t}\mathfrak{B}) \sim C \frac{m_\Gamma(\mathfrak{A}) m_\Gamma(\mathfrak{B})}{\tilde{L}(|t|)} \ \text{ as } t \longrightarrow \pm\infty,$$

  *where* $\tilde{L}(t) = \int_1^t \frac{L(x)}{x}\,\mathrm{d}x$ *for any* $t \geqslant 1$.

The proof of this result relies on the study of a coding of the limit set of $\Gamma$ and on a symbolic representation of the geodesic flow given in the fourth section.

In Chapter 7, we establish an asymptotic lower bound for the number of closed geodesics with length less than $R$. We prove

THEOREM B. – *Let* $\mathrm{M} = \mathrm{X}/\Gamma$ *be a manifold with pinched negative curvature whose fundamental group* $\Gamma$ *satisfies Hypotheses* $(H_\beta)$ *for* $\beta \in ]0, 1[$. *Then*

$$\liminf_{R \longrightarrow +\infty} \frac{\delta_\Gamma R}{\beta e^{\delta_\Gamma R}} N_{\mathscr{G}}(R) \geqslant 1.$$

We could not improve our proof to get a full asymptotic for $N_{\mathscr{G}}(R)$. This lower bound remains nevertheless surprising. Let us recall the following result proved in **[41]**. For all $R \geqslant 0$, let $\mathscr{G}_\Gamma(R)$ be the set of closed orbits for the geodesic flow on $\mathrm{T}^1\mathrm{M}$ with period less than $R$. For any closed orbit $\wp$, let $\mathscr{D}_\wp$ be the normalized Lebesgue measure along $\wp$. Then as $R \longrightarrow +\infty$,

$$\delta_\Gamma R e^{-\delta_\Gamma R} \sum_{\wp \in \mathscr{G}_\Gamma(R)} \mathscr{D}_\wp \longrightarrow \frac{m_\Gamma}{||m_\Gamma||} \tag{2}$$

in the dual of the set of continuous functions with compact support in $\mathrm{T}^1\mathrm{M}$. In particular, when $\Gamma$ is convex-cocompact ($m_\Gamma$ is thus finite in this case), the set $\mathrm{T}^1\mathrm{M}$ is compact and (2) applied with $\mathbb{1}_{\mathrm{T}^1\mathrm{X}/\Gamma}$ implies the counting result $\mathrm{N}_{\mathscr{G}}(R) \sim e^{\delta_\Gamma R}/(\delta_\Gamma R)$. When M is geometrically finite with finite Bowen-Margulis measure, Roblin shows that (2) still implies $\mathrm{N}_{\mathscr{G}}(R) \sim e^{\delta_\Gamma R}/(\delta_\Gamma R)$. When the Bowen-Margulis measure has infinite mass, Roblin shows that, still in the dual of compactly supported continuous functions of $\mathrm{T}^1\mathrm{M}$,

$$\delta_\Gamma R e^{-\delta_\Gamma R} \sum_{\wp \in \mathscr{G}_\Gamma(R)} \mathscr{D}_\wp \longrightarrow 0.$$

Therefore, we could have expected that the $\mathrm{N}_{\mathscr{G}}(R)$ would have been negligible with respect to $e^{\delta R}/R$ as $R$ goes to $+\infty$; Theorem B above contradicts this intuition.

We eventually establish the following asymptotic behavior for the orbital counting function.

Theorem C. – *Let $\Gamma$ be a Schottky group satisfying the family of assumptions $(H_\beta)$ for some $\beta \in\, ]0,1]$.*

- *If $\beta \in\, ]0,1[$, there exists $C' = C'_{\beta,\Gamma} > 0$ such that*
$$\mathrm{N}_\Gamma(\mathbf{o}, R) \sim C' \frac{e^{\delta_\Gamma R}}{R^{1-\beta} L(R)}.$$
- *If $\beta = 1$, there exists $C' = C'_\Gamma > 0$ such that*
$$\mathrm{N}_\Gamma(\mathbf{o}, R) \sim C' \frac{e^{\delta_\Gamma R}}{\tilde{L}(R)},$$
*where $\tilde{L}(t) = \int_1^t \frac{L(x)}{x}\,\mathrm{d}x$ for any $t \geqslant 1$.*

To prove Theorem C, we need to extend the coding of the points of the limit set $\Lambda_\Gamma$ to the $\Gamma$-orbit of some point $x_0 \notin \Lambda_\Gamma$ in the boundary at infinity.

The constants appearing in Theorems A and C will be specified in the proofs.

Remark 1.2.1. – *In his seminal work, T. Roblin always assumes the non-arithmeticity of the length spectrum, i.e., the set of lengths of closed geodesics in $\mathrm{X}/\Gamma$ is not contained in a discrete subgroup of $\mathbb{R}$. This assumption is satisfied in our setting, because the quotient manifold has cusps (see* **[16]***).*

REMARK 1.2.2. – *In the case of a Schottky group $\Gamma = \Gamma_1 \star \Gamma_2$ with only two factors, satisfying assumptions $(H_\beta)$ for some $\beta \in \,]0,1]$ and with at least one influential factor, the results presented above are still valid but their proofs are slightly more technical. Indeed, the transfer operator will then have two dominant eigenvalues $+1$ and $-1$ (see* [**3**] *and* [**16**]*). The proof of our result hence would have to be adapted in this case, similarly to the arguments in* [**16**]*, which we will not do here.*

Let us now explain why we present separately the additional assumption (S) in the family $(H_\beta)$. In [**18**], the authors prove a result similar to Theorem C for $\beta \in \,]1/2,1]$, without this assumption. But they can not obtain it for $\beta \leqslant 1/2$, their proof (Section 6 in [**18**]) being based on the renewal theorem of [**21**], which does not ensure any more a true limit when $\beta \leqslant 1/2$. Our arguments rely on the article [**24**] and we avoid this distinction between $\beta \leqslant 1/2$ and $\beta > 1/2$ thanks to the additional assumption (S). We do not know whether this assumption is a consequence of the first four in our geometric setting.

REMARK 1.2.3. – *We may notice that the assumption (S) is equivalent to each following statements:*

$(S')$ *For any $\Delta > 0$, there exists a constant $C = C_\Delta > 0$ such that for any $j \in [\![1, p+q]\!]$ and any $T > 0$ large enough, the following inequality is satisfied*

$$\#\{\alpha \in \Gamma_j \mid T - \Delta \leqslant \mathrm{d}(\mathbf{o}, \alpha \cdot \mathbf{o}) < T + \Delta\} \leqslant C \frac{L(T) e^{\delta_\Gamma T}}{T^{1+\beta}}.$$

$(S'')$ *There exists a constant $C > 0$ such that for any $j \in [\![1, p+q]\!]$ and any $T > 0$ large enough*

$$\#\{\alpha \in \Gamma_j \mid \mathrm{d}(\mathbf{o}, \alpha \cdot \mathbf{o}) \leqslant T\} \leqslant C \frac{L(T) e^{\delta_\Gamma T}}{T^{1+\beta}}.$$

*The equivalence between the statements (S) and $(S')$ is clear and the fact that $(S'')$ implies $(S')$ follows from definitions. We may find a proof of the equivalence between the statements $(S')$ and $(S'')$ in Proposition 2.5 of* [**19**] *(in the finite volume case). In Chapter 3, we detail another proof of the converse property involving the lemmas of Karamata and Potter.*

REMARK 1.2.4. – *In the family of assumptions $(H_\beta)$ for $\beta \in \,]0,1]$, the "influential" parabolic groups $\Gamma_1, \ldots, \Gamma_p$ are supposed to be elementary. Their rank may be larger than 2. Nevertheless, in the proofs of Proposition 4.2.15 and 6.2.1 and of Facts 8.2.4 and 8.2.5, we work with parabolic groups of rank 1 in order to simplify the notation. The arguments are always true in higher rank.*

## 1.3. Outline of the book

The second section is devoted to the construction of Schottky groups satisfying assumptions $(H_\beta)$ as explained in [**14**] and [**40**].

The third section is devoted to the presentation of some properties of stable laws with parameter $\beta \in ]0,1]$ for random variables, together with results on regularly varying functions. These will be crucial tools in the sequel due to our assumptions $(\mathrm{P}_2)$ and (N).

In the fourth section, we define a coding of the limit set of $\Gamma$ and of the geodesic flow. We then introduce a family of transfer operators associated to this coding. We finally end this part with a study of the spectrum and of the regularity of the spectral radii of these operators.

The proof of the mixing rate given in Theorem A in the case $\beta \in ]0,1[$ is presented in Chapter 5: it is based on the proof of Theorem 1.4 in [**24**]. The case $\beta = 1$ is presented in Chapter 6 and the approach is inspired by [**36**].

We establish in Chapter 7 the asymptotic lower bound for closed geodesics given in Theorem B.

In Chapter 8, we extend the previous coding of the limit set to include the orbit of a base point $x_0 \in \partial \mathrm{X} \backslash \Lambda_\Gamma$ and study the family of extended transfer operators associated to this new coding. These extended operators will be central in the proof of Theorem C.

The final section is dedicated to the proof of Theorem C, which follows the same steps as the proof of Theorem A.

*Notation.* – In this book, we will use the following notation. For two functions $f, g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, we will write $f \underset{C}{\preceq} g$ (or $f \preceq g$) if $f(R) \leqslant Cg(R)$ for a constant $C > 0$ and $R$ large enough and $f \underset{C}{\asymp} g$ (or $f \asymp g$) if $f \underset{C}{\preceq} g$ and $g \underset{C}{\preceq} f$. Similarly, for any real numbers $a$ and $b$ and $C > 0$, the notation $a \overset{C}{\sim} b$ means $|a - b| \leqslant C$.

If $A, B$ are two subsets of $E$, we denote by $A \overset{\Delta}{\times} B := \{(a,b) \in A \times B \mid a \neq b\}$.

The value of the constant $C$ which appears in the proofs may change from line to line.

## 1.4. Acknowledgements

This book comes from my PhD at the University of Nantes between 2013 and 2016 under the direction of Marc Peigné and Samuel Tapie. During that time, my research was supported by Centre Henri Lebesgue, in programm “Investissements d’avenir” – ANR-11-LABX-0020-01. I want to thank Sébastien Gouëzel for his help and all his explanations during the writing of this book. I also want to thank the referee for all their corrections and suggestions.

# CHAPTER 2

# EXOTIC SCHOTTKY GROUPS

In this section, we first recall some definitions and properties about manifolds of negative curvature. Then we give a sketch of the construction of *exotic* Schottky groups following [**14**] and [**40**].

## 2.1. Negatively curved manifolds and Schottky groups

**2.1.1. Notation. –** Let X be a Hadamard manifold of pinched negative curvature $-b^2 \leqslant \kappa \leqslant -a^2$ with $0 < a < 1 < b$, endowed with the distance d induced by the metric. We denote by $\partial \mathrm{X}$ its boundary at infinity (see [**4**]). For a point $x \in \partial \mathrm{X}$ and two points $\mathbf{x}, \mathbf{y} \in \mathrm{X}$, the *Busemann cocycle* $\mathscr{B}_x(\mathbf{x}, \mathbf{y})$ is defined as the limit of $\mathrm{d}(\mathbf{x}, \mathbf{z}) - \mathrm{d}(\mathbf{z}, \mathbf{y})$ when $\mathbf{z}$ goes to $x$. This quantity represents the algebraic distance between the horospheres centered at $x$ and passing through $\mathbf{x}$ and $\mathbf{y}$ respectively. This function satisfies the following property : for any $x \in \partial \mathrm{X}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{X}$

$$\mathscr{B}_x(\mathbf{x}, \mathbf{y}) = \mathscr{B}_x(\mathbf{x}, \mathbf{z}) + \mathscr{B}_x(\mathbf{z}, \mathbf{y}). \tag{3}$$

From now, we fix a point $\mathbf{o} \in \mathrm{X}$. The *Gromov product* of two points $x$ and $y$ of $\partial \mathrm{X}$ seen from the point $\mathbf{o}$ is given by the following formula

$$(x|y)_{\mathbf{o}} = \frac{1}{2}\Big( \mathscr{B}_x(\mathbf{o}, \mathbf{z}) + \mathscr{B}_y(\mathbf{o}, \mathbf{z}) \Big),$$

where $z$ is any point on the geodesic $(xy)$ with endpoints $x$ and $y$; this product does not depend on the point $\mathbf{z}$. The curvature being bounded from above by $-a^2$, we may find in [**8**] a proof of the fact that the quantity $\mathrm{d}_{\mathbf{o}}(x, y) := \exp(-a(x|y)_{\mathbf{o}})$ defines a distance on $\partial \mathrm{X}$, which satisfies the following "visibility" property: there exists a constant $C > 0$ depending only on the bounds of the curvature of X such that for any $x, y \in \partial \mathrm{X}$

$$\frac{1}{C}\mathrm{d}(\mathbf{o}, (xy)) \leqslant \mathrm{d}_{\mathbf{o}}(x, y) \leqslant C\mathrm{d}(\mathbf{o}, (xy)). \tag{4}$$

As a consequence of this property, we mention the following important lemma.

Lemma 2.1.1 (Triangular "quasi-equality"). – *Let $E, F \subset \mathrm{X}$ such that $\overline{E} \cap \overline{F} \cap \partial\mathrm{X} = \varnothing$. There exists a constant $C = C(E,F) > 0$ such that* $\mathrm{d}(\mathbf{o},\mathbf{x}) + \mathrm{d}(\mathbf{o},\mathbf{y}) - C \leqslant \mathrm{d}(\mathbf{x},\mathbf{y})$ *for any $\mathbf{x} \in E$ and $\mathbf{y} \in F$.*

This result can be proved for instance using the arguments given in Section 2.3 in **[42]**. This lemma thus furnishes a complement to the classical triangular inequality; many results of this book are based on it.

Denote by Isom(X) the group of orientation-preserving isometries of X. The action of $\gamma \in \mathrm{Isom}(\mathrm{X})$ can be extended to $\partial\mathrm{X}$ by homeomorphism. It follows from the previous definitions that for any $\gamma \in \mathrm{Isom}(\mathrm{X})$ and any $x, y \in \partial\mathrm{X}$

$$\mathrm{d}_{\mathbf{o}}(\gamma\cdot x, \gamma\cdot y) = \sqrt{e^{-a\,\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})}e^{-a\,\mathcal{B}_y(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})}}\,\mathrm{d}_{\mathbf{o}}(x,y). \tag{5}$$

We thus talk about "conformal action" of Isom(X) on the boundary at infinity; the conformal factor $|\gamma'(x)|_{\mathbf{o}}$ of an isometry $\gamma$ at the point $x \in \partial\mathrm{X}$ is given by the formula $|\gamma'(x)|_{\mathbf{o}} = e^{-a\,\mathcal{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})}$. From (3), we deduce that the function $b(\gamma, x) := -\frac{\log|\gamma'(x)|_{\mathbf{o}}}{a} = \mathcal{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})$ satisfies the following cocycle relation: for any $\gamma_1, \gamma_2 \in \Gamma$ and any $x \in \partial\mathrm{X}$

$$b(\gamma_1\gamma_2, x) = b(\gamma_1, \gamma_2\cdot x) + b(\gamma_2, x).$$

**2.1.2. Schottky product groups.** – Let $k \geqslant 1$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ be isometries satisfying the following property: there exist non-empty pairwise disjoint closed subsets $D_1, \ldots, D_k$ of $\partial\mathrm{X}$ such that for all $j \in [\![1,k]\!]$ and all $n \in \mathbb{Z}^*$, one has $\mathfrak{a}_j^n\cdot(\partial\mathrm{X}\backslash D_j) \subset D_j$. The isometries $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ are said to be in Schottky position. Klein's ping-pong lemma implies that the group $\Gamma$ generated by the isometries $(\mathfrak{a}_i)_{1\leqslant i\leqslant k}$ is free and acts properly discontinuously and without fixed point on X. The group $\Gamma$ is called a Schottky group. The limit set $\Lambda_\Gamma$ of such a group $\Gamma$ is a perfect nowhere dense set.

Let us give an example in the model of the Poincaré half-plan. Let $p\,:\, z \longmapsto z+1$ be a parabolic isometry; the non-trivial powers of $p$ send $[0,1]$ into $]\infty,0] \cup [1,\infty[$, hence we set $D_p = ]\infty,0] \cup [1,\infty[ \cup \{\infty\}$. On the other hand, let us conjugate the hyperbolic isometry $h\,:\, z \longmapsto 64z$ by $\gamma(z) = (3z/4+1/2)/(z+2)$. The so-obtained isometry $h'$ is hyperbolic with fixed points $1/4$ and $3/4$; its negative powers send $\mathbb{R}\cup\{\infty\}\backslash[7/52, 25/76]$ into $[7/52, 25/76]$, and its positive powers send $\mathbb{R}\cup\{\infty\}\backslash[37/52, 37/44]$ into $[37/52, 37/44]$. We finally set $D_h = [7/52, 25/76] \cup [37/52, 37/44]$ (see Figure 1 below). We extend this definition to products of groups. Fix $k \geqslant 2$. We say that $k$ groups $\Gamma_1, \ldots, \Gamma_k$ are in Schottky position if there exist non-empty pairwise disjoint closed subsets $D_1, \ldots, D_k$ of $\partial\mathrm{X}$ such that for all $j \in [\![1,k]\!]$, one has $\Gamma_j^*\cdot(\partial\mathrm{X}\backslash D_j) \subset D_j$. We may note that $D_j$ contains the limit set $\Lambda_{\Gamma_j}$ of $\Gamma_j$. Thus the group $\Gamma$ generated by $\Gamma_1, \ldots, \Gamma_k$ is the free product $\Gamma_1 \star\cdots\star \Gamma_k$; it is called

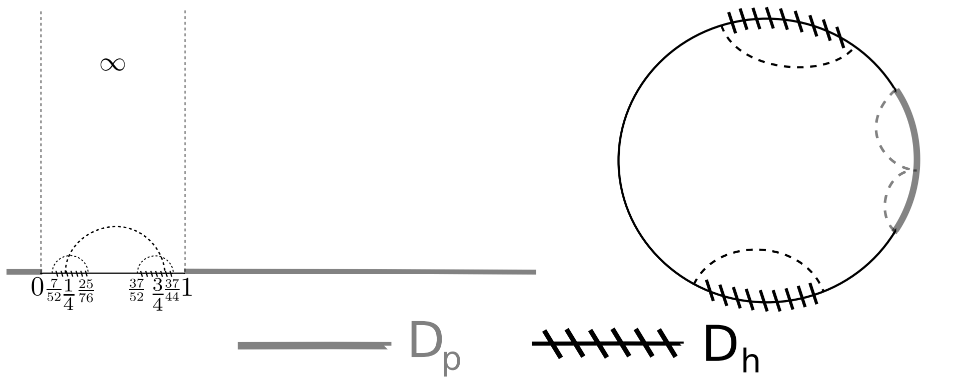


FIGURE 1. Schottky position in the half-plan or in the disk model of $\mathbb{H}^2$.

the Schottky product of the groups $\Gamma_1, \ldots, \Gamma_k$. Each group $\Gamma_i$, $1 \leqslant i \leqslant k$, is called a Schottky factor of $\Gamma$.

In the sequel, we will need to consider subsets $(\mathbf{D}_j)_{1\leqslant j\leqslant k}$ of $\overline{\mathrm{X}}$ with the same dynamical properties as the sets $(D_j)_{1\leqslant j\leqslant k}$ under the action of $\Gamma$. For each $j \in [\![1, k]\!]$, we introduce a set $\mathbf{D}_j \subset \overline{\mathrm{X}}$, which is geodesically convex and connected (resp. consists of two geodesically convex connected components) when $\Gamma_j$ is generated by a parabolic (resp. hyperbolic) isometry $\mathfrak{a}_j$, and whose intersection with $\partial\mathrm{X}$ contains $D_j$; in addition, we assume that $\Gamma_j^* \cdot \left(\overline{\mathrm{X}}\backslash\mathbf{D}_j\right) \subset \mathbf{D}_j$ and that the sets $(\mathbf{D}_j)_{1\leqslant j\leqslant k}$ are pairwise disjoint. Figure 2 illustrates the situation for the above isometries $p$ and $h'$.

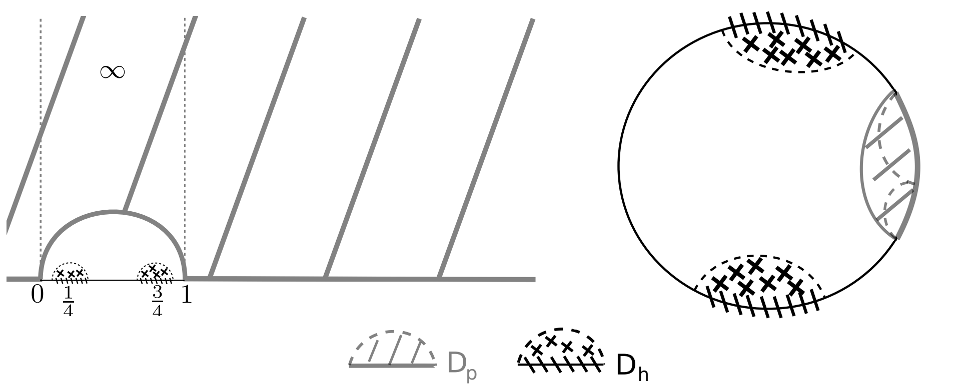


FIGURE 2. Sets $\mathbf{D}_j$.

**2.1.3. Geodesic flow and Bowen-Margulis measure. –** Using Hopf coordinates, we identify the unit tangent bundle $\mathrm{T}^1\mathrm{X}$ with the set $\partial\mathrm{X} \overset{\Delta}{\times} \partial\mathrm{X} \times \mathbb{R}$: the point $(\mathbf{x}, \mathbf{v}) \in \mathrm{T}^1\mathrm{X}$ determines a unique triple $(x^-, x^+, r)$ in $\partial\mathrm{X} \overset{\Delta}{\times} \partial\mathrm{X} \times \mathbb{R}$, where $x^-$ and $x^+$ are the endpoints of the oriented geodesic passing through $\mathbf{x}$ at time 0 with tangent vector $\mathbf{v}$ and $r = \mathscr{B}_{x^+}(\mathbf{o}, \mathbf{x})$. The group $\Gamma$ acts on $\partial\mathrm{X} \overset{\Delta}{\times} \partial\mathrm{X} \times \mathbb{R}$ as follows: for any $\gamma \in \Gamma$

$$\gamma \cdot (x^-, x^+, r) = \big(\gamma \cdot x^-, \gamma \cdot x^+, r + \mathscr{B}_{x^+}(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})\big)$$

and the action of the geodesic flow $(\tilde{g}_t)_{t\in\mathbb{R}}$ is given by

$$\tilde{g}_t(x^-, x^+, r) = (x^-, x^+, r + t)$$

for any $t \in \mathbb{R}$. These two actions commute and, quotienting $\mathrm{T}^1\mathrm{X}$ by $\Gamma$, define the action of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on $\partial\mathrm{X} \overset{\Delta}{\times} \partial\mathrm{X} \times \mathbb{R}/\Gamma \simeq \mathrm{T}^1\mathrm{M}$. By **[20]**, the *non-wandering set* of $(g_t)_{t\in\mathbb{R}}$ on $\mathrm{T}^1\mathrm{M}$ is $\Omega_\Gamma := \Lambda_\Gamma \overset{\Delta}{\times} \Lambda_\Gamma \times \mathbb{R}/\Gamma$.

By Patterson's construction (see **[39]** and **[43]** in constant curvature $-1$ and, for example, **[45]** in variable negative curvature), there exists a family $\sigma = (\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$ of finite measures on $\partial\mathrm{X}$ supported on $\Lambda_\Gamma$ and satisfying, for any $\mathbf{x}, \mathbf{x}' \in \mathrm{X}$, any $x \in \Lambda_\Gamma$ and $\gamma \in \Gamma$:

$$\frac{\mathrm{d}\sigma_{\mathbf{x}'}}{\mathrm{d}\sigma_{\mathbf{x}}}(x) = e^{-\delta_\Gamma \mathscr{B}_x(\mathbf{x}', \mathbf{x})} \text{ and } \gamma^*\sigma_{\mathbf{x}} = \sigma_{\gamma^{-1}\cdot\mathbf{x}}, \tag{6}$$

where $\gamma^*\sigma(A) = \sigma(\gamma A)$ for any Borel subset of $\partial\mathrm{X}$ and $\delta_\Gamma$ is the Poincaré exponent of $\Gamma$. As soon as $\Gamma$ is divergent and geometrically finite, the measures $\sigma_{\mathbf{x}}$ do not have atomic part (see **[14]**). As observed by Sullivan **[43]**, the Patterson measure of $\Gamma$ may be used to construct an invariant measure for the geodesic flow with support $\Omega_\Gamma$. It follows from (5) and (6) that the measure $\mu$ defined on $\Lambda_\Gamma \overset{\Delta}{\times} \Lambda_\Gamma$ by

$$\mathrm{d}\mu(y, x) = \frac{\mathrm{d}\sigma_{\mathbf{o}}(y)\,\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(y, x)^{2\delta_\Gamma/a}}$$

is $\Gamma$-invariant so that $\tilde{m}_\Gamma = \mu \otimes \mathrm{d}t$ on $\Lambda_\Gamma \overset{\Delta}{\times} \Lambda_\Gamma \times \mathbb{R}$ is both $(\tilde{g}_t)$ and $\Gamma$-invariant. It thus induces on $\Omega_\Gamma$ an invariant measure $m_\Gamma$ for $(g_t)$. When $m_\Gamma$ has finite total mass, this is the unique measure which maximizes the measure-theoretic entropy of the geodesic flow restricted to its non wandering set: $m_\Gamma$ is called the *Bowen-Margulis measure* (see **[37]**). When $m_\Gamma$ is infinite, there is no finite invariant measure which maximizes the entropy: however, we still call $m_\Gamma$ the Bowen-Margulis measure.

## 2.2. Construction of exotic Schottky groups

Let us recall the genesis of the setting in which we will work, i.e., the construction of some exotic Schottky groups introduced in **[14]** and **[40]**.

**2.2.1. Divergent group and finite Bowen-Margulis measure. –** In [**14**], Dal'bo, Otal and Peigné gave the first known example of geometrically finite manifolds $\mathrm{M} = \mathrm{X}/\Gamma$ with pinched negative curvature whose Bowen-Margulis measure has infinite mass. These examples were constructed by providing convergent Schottky groups $\Gamma$ (which hence are geometrically finite). Therefore, the Bowen-Margulis measure on $\mathrm{T}^1\mathrm{M}$ has infinite mass. In order to obtain these examples, it was first shown that the group $\Gamma$ needs to contain a parabolic subgroup $P < \Gamma$ whose critical exponent is $\delta_P = \delta_\Gamma$.

THEOREM A ([**14**]). – *Let $\Gamma$ be a geometrically finite group with parabolic transformations. If $\delta_\Gamma > \delta_P$ for any parabolic subgroup $P$, then $\Gamma$ is divergent.*

The assumption $\delta_\Gamma > \delta_P$ for any parabolic subgroup $P$ is called the "critical gap property" of the group $\Gamma$. It follows from Proposition 2 of [**14**] that if $\Gamma = \Gamma_1 \star \cdots \star \Gamma_k$ is a Schottky group, whose factors $\Gamma_i$, $1 \leqslant i \leqslant k$ are each divergent, then it has the critical gap property.

When the group $\Gamma$ is divergent, but still has parabolic elements, a necessary and sufficient condition for the finiteness of the Bowen-Margulis measure is given by the following criterion.

THEOREM B ([**14**]). – *Let $\Gamma$ be a divergent geometrically finite group containing parabolic isometries. The measure $m_\Gamma$ is finite if and only if for any parabolic subgroup $P$ of $\Gamma$, the series $\sum_{p\in P} \mathrm{d}(\mathbf{o}, p\cdot\mathbf{o})e^{-\delta_\Gamma \mathrm{d}(\mathbf{o},p\cdot\mathbf{o})}$ converges.*

We can deduce at least two things from both previous theorems. On the one hand, a geometrically finite group containing parabolic isometries satisfying the critical gap property is divergent and admits a finite measure $m_\Gamma$. On the other hand, we understand that the first step to obtaining a group with infinite measure $m_\Gamma$ involves the construction of a convergent parabolic group. This is the purpose of the next subsection, which is based on [**14**].

**2.2.2. Construction of a convergent parabolic group. –** Let us first consider the situation in constant curvature $-1$. Fix $N \geqslant 2$. We may identify $\mathbb{H}^N$ with the product $\mathbb{R}_x^{N-1} \times \mathbb{R}_y^{+*}$ endowed with the metric $(\,\mathrm{d}x^2 + \mathrm{d}y^2)/y^2$. Let $P$ an elementary parabolic group acting on $\mathbb{H}^N$. Up to a conjugacy, we may suppose that the elements of $P$ fix the point at infinity $\infty$. Denote by $\mathscr{H}$ the horoball centered at $\infty$ and passing through $\mathbf{i} := (0, 0, \ldots, 0, 1)$. The group $P$ acts by euclidean isometries on the horosphere $\partial\mathscr{H}$. By Bieberbach's theorem (see [**5**] and [**6**]), there exists a finite index abelian subgroup $Q$ of $P$ which acts by translations on a subspace $\mathbb{R}^k \subset \mathbb{R}^{N-1}$, $1 \leqslant k \leqslant N-1$. There thus exist $k$ linearly independent vectors $v_1, \ldots, v_k$ and a finite set $F \subset P$ such that any element $p \in P$ decomposes into $\tau_{v_1}^{n_1} \cdots \tau_{v_k}^{n_k} f$ for $\overline{n} := (n_1, \ldots, n_k) \in \mathbb{Z}^k$ and $f \in F$, where $\tau_{v_i}^{n_i}$ is the $n_i$-th power of the translation of vector $v_i$. In this case, the Poincaré series $\mathscr{P}_P$ of $P$ is given by: for $s > 0$

$$\mathscr{P}_P(s) = \sum_{p\in P} e^{-s\mathrm{d}(\mathbf{i},p.\mathbf{i})} = \sum_{f\in F}\sum_{\overline{n}\in\mathbb{Z}^k} e^{-s\mathrm{d}(\mathbf{i},\tau_{v_1}^{n_1}\cdots\tau_{v_k}^{n_k} f.\mathbf{i})}.$$

The quantity

$$\mathrm{d}(\mathbf{i},\tau_{v_1}^{n_1}\cdots\tau_{v_k}^{n_k} f.\mathbf{i}) - 2\ln\left(||n_1v_1+\cdots+n_kv_k||\right)$$

is bounded as $n_1^2+\cdots+n_k^2 \longrightarrow +\infty$, where $||\cdot||$ is the euclidean norm in $\mathbb{R}^{N-1}$. The previous series thus behaves like the following

$$\sum_{\substack{\overline{n}\in\mathbb{Z}^k\\ \overline{n}\neq\overline{0}}} \frac{1}{||n_1v_1+\cdots+n_kv_k||^{2s}},$$

which diverges at its critical exponent $k/2$.

In the sequel, following **[14]**, we will modify the metric in the horoball $\mathscr{H}$ in such a way that the parabolic group $P$ will still have critical exponent $\delta_P = k/2$, but its Poincaré series will converge at $k/2$. To this end, we consider another model of the hyperbolic space, which will be more suitable for understanding the action of $P$ on the horospheres. The classical upper half space model of the hyperbolic space, $\mathbb{H}^N \cong \left(\mathbb{R}^{N-1}\times\mathbb{R}_+^*, \frac{\mathrm{d}x^2\oplus\mathrm{d}y^2}{y^2}\right)$ is isometric to $\left(\mathbb{R}^{N-1}\times\mathbb{R}, e^{-2t}\,\mathrm{d}x^2\oplus\mathrm{d}t^2\right)$ via the diffeomorphism

$$\Psi\ :\ \begin{cases} \mathbb{R}^{N-1}\times\,]0,+\infty[ & \longrightarrow \mathbb{R}^N \\ \qquad (x,z) & \longmapsto (x,\ln(z)) = (x,t) \end{cases}.$$

Let us denote by $\mathscr{H}_t = \{(x,s)\mid x\in\mathbb{R}^{N-1},\ s\geqslant t\}$ the horoball of level $t$ centered at infinity in this model; one gets $\Psi(\mathscr{H}) = \mathscr{H}_0$. Fix $x,y\in\mathbb{R}^{N-1}$ and let us denote $\mathbf{x}_t = (x,t)$ and $\mathbf{y}_t = (y,t)$ for $t>0$; these two points both belong to the horosphere $\partial\mathscr{H}_t$, and the distance between them, with respect to the metric on $\partial\mathscr{H}_t$ induced by the hyperbolic metric on $\mathbb{R}^N$, is equal to $e^{-t}||x-y||$, where $||\cdot||$ is the Euclidean norm on $\mathbb{R}^{N-1}$. Therefore, on the horosphere of level $t=\ln\left(||x-y||\right)$, the distance induced on the horosphere between $x_t$ and $y_t$ is 1. Since the curve $[\mathbf{x}_0\mathbf{x}_t]\cup[\mathbf{x}_t\mathbf{y}_t]\cup[\mathbf{y}_t\mathbf{y}_0]$ is a quasi-geodesic, we can deduce from **[28]** that the quantity $\mathrm{d}(\mathbf{x}_0,\mathbf{y}_0) - 2\ln\left(||x-y||\right)$ is bounded. Let us now consider on $\mathbb{R}^{N-1}\times\mathbb{R}$ the metric $g_T = T^2(t)\,\mathrm{d}x^2 + \mathrm{d}t^2$, where $T:\ \mathbb{R}\longrightarrow\mathbb{R}^{+*}$ is chosen such that $g_T$ has pinched negative curvatures. Let us write $\mathrm{d}_T$ for the distance induced by $g_T$ on $\mathbb{R}^N$. The same argument as previously given for the hyperbolic space shows that if $\mathbf{x}_0 = (x,0)\in\mathbb{R}^{N-1}\times\mathbb{R}$, $\mathbf{y}_0 = (y,0)\in\mathbb{R}^{N-1}\times\mathbb{R}$, then $\mathrm{d}_T(\mathbf{x}_0,\mathbf{y}_0) - 2u\left(||x-y||\right)$ is bounded uniformly in $x,y\in\mathbb{R}^{N-1}$, where $u:\ \mathbb{R}^{+*}\longrightarrow\mathbb{R}$ is defined by the implicit equation $T(u(s)) = \frac{1}{s}$ for all $s>0$. When $u(s)=\ln(s)$ and $T(t) = e^{-t}$, we obtain the previous model of the hyperbolic space. The sectional curvature at $\mathbf{x}_t$ equals $K_{g_T}(t) = -\frac{T''(t)}{T(t)}$ on any plane $\langle \frac{\partial}{\partial X_i}, \frac{\partial}{\partial t} \rangle, 1 \leqslant i \leqslant N-1$, and $-K_{g_T}(t)^2$ on any plane $\langle \frac{\partial}{\partial X_i}, \frac{\partial}{\partial X_j} \rangle, 1 \leqslant i < j \leqslant N-1$ (see **[13]**, Chapter 8, Section 3). One of the steps in **[14]** Section 3 and **[40]** Section 2 is to explain how the functions $u$ and $T$ have to be chosen so that the sectional curvature remains negative and pinched on $\mathbb{R}^N$ endowed with $g_T = T^2(t)\,\mathrm{d}x^2 + \mathrm{d}t^2$. More precisely, Lemma 2.2 in **[40]** states the following.

LEMMA 2.2.1. – *Fix a constant $\kappa \in \,]0,1[$. For any $\beta \geqslant 0$, there exist $s_\beta \geqslant 1$ and a $\mathcal{C}^2$ non-decreasing function $u_\beta : \mathbb{R}^{+*} \longrightarrow \mathbb{R}$ satisfying:*

- $u_\beta(s) = \ln(s)$ *if* $s \in \,]0,1]$;
- $u_\beta(s) = \ln(s) + (1+\beta)\ln(\ln(s))$ *if* $s \geqslant s_\beta$;
- *if* $T_\beta(u_\beta(s)) = 1/s$ *for any* $s > 0$ *and* $g_\beta = T_\beta^2(t)\,\mathrm{d}x^2 + \mathrm{d}t^2$, *then* $K_{g_\beta}(u_\beta(s)) \leqslant -\kappa^2$;
- $\lim\limits_{s \longrightarrow +\infty} K_{g_\beta}(u_\beta(s)) = -1$ *and the derivatives of* $K_{g_\beta} \circ u_\beta$ *tend to* $0$ *as* $s$ *goes to* $+\infty$.

We may notice that this metric coincides with the hyperbolic one on the set $\mathbb{R}^{N-1} \times \mathbb{R}^-$; we can enlarge this area shifting the metric $g_\beta$ along the $t$-axis (see Subsection 2.2 in **[40]** and Subsection 2.2.4 of this book).

On $\left(\mathbb{R}^N, g_\beta\right)$, the group $P$ defined above still acts by isometries and its Poincaré series behaves like

$$\sum_{\substack{\overline{n} \in \mathbb{Z}^k \\ \overline{n} \neq \overline{0}}} \frac{e^{-sO(\overline{n})}}{||n_1 v_1 + \cdots + n_k v_k||^{2s} \ln\left(||n_1 v_1 + \cdots + n_k v_k||\right)^{2s(1+\beta)}}.$$

This series still admits $k/2$ as critical exponent and is convergent if and only if $\beta > 1/k - 1$. Thus, for any $k \geqslant 1$, the group $P$ is convergent when $\beta > 0$. In the next subsection, we will see how to adapt the above construction of metric $g_\beta$, $\beta > 0$, to highlight the existence of convergent parabolic group satisfying the assumptions ($\mathrm{P}_2$) and (S).

**2.2.3. A convergent parabolic group satisfying assumptions ($\mathrm{P}_2$) and (S).** – Here we fix $N = 2$, but the following construction may be adapted in higher dimension. Let $\mathbf{o}$ be the point $(0,0)$ in $\mathbb{R}^2$ and $p$ the translation of vector $(1,0)$. As mentioned previously, for these metrics $g_\beta$, $\beta > 0$, there exists $C > 0$ such that for $|n|$ sufficiently large, one gets

$$2\ln|n| + 2(1+\beta)\ln\ln|n| - C \leqslant \mathrm{d}(\mathbf{o}, p^n \cdot \mathbf{o}) \leqslant 2\ln|n| + 2(1+\beta)\ln\ln|n| + C.$$

As we saw in Subsection 2.2.2, this is enough to ensure that the parabolic group $P = \langle p \rangle$ is convergent. Nevertheless, this estimate is not precise enough to ensure that $P$ satisfies Hypotheses ($\mathrm{P}_2$) and (S). Therefore, in the sequel, we present new metrics $g_\beta$, $\beta > 0$, close to those presented in Lemma 2.2.1, for which we can specify the behavior of the bounded term as $n \longrightarrow \pm\infty$.

Let us fix $\beta > 0$. For all real $t$ greater than some $\mathfrak{a} > 0$ to be chosen later, let us set

$$T(t) = T_{\beta,L}(t) = e^{-t}\frac{t^{1+\beta}}{L(t)},$$

where $L$ is a slowly varying function on $[0, +\infty[$ with values in $\mathbb{R}^{+*}$. Without loss of generality (see Theorem 1.3.3 in [7]), we assume that $L$ is $C^\infty$ on $\mathbb{R}^+$ and, as $t \longrightarrow +\infty$, satisfies

$$\frac{L'(t)}{L(t)} = o\left(\frac{1}{t}\right) \text{ and } \left(\frac{L'}{L}\right)'(t) = o\left(\frac{1}{t^2}\right).$$

Furthermore, for any $\theta > 0$, there exist $t_\theta \geqslant 0$ and $C_\theta \geqslant 1$ such that for any $t \geqslant t_\theta$

$$\frac{1}{C_\theta t^\theta} \leqslant L(t) \leqslant C_\theta t^\theta. \tag{7}$$

Notice that, for $\mathfrak{a}$ large enough and all $t \geqslant \mathfrak{a}$,

$$-\frac{T''(t)}{T(t)} = -\left(1 - \frac{(1+\beta)}{t} + \frac{L'(t)}{L(t)}\right)^2 + \left(\frac{(1+\beta)}{t^2} + \left(\frac{L'}{L}\right)'(t)\right) < 0.$$

By Lemma 2.2.1 in [**40**], we may extend $T_{\beta,L}$ on $\mathbb{R}$ as follows.

LEMMA 2.2.2. – *There exists $\mathfrak{a} = \mathfrak{a}(\beta, L) > 0$ such that the map $T = T_{\beta,L} : \mathbb{R}^- \cup [\mathfrak{a}, +\infty[ \to \mathbb{R}^{+*}$ defined by:*

— $T(t) = e^{-t}$ *if* $t \leqslant 0$;
— $T(t) = e^{-t}\frac{t^{1+\beta}}{L(t)}$ *if* $t \geqslant \mathfrak{a}$;

*admits a decreasing and twice continuously differentiable extension on $\mathbb{R}$ such that*

$$-b^2 \leqslant K_{g_\beta}(t) = -\frac{T''(t)}{T(t)} \leqslant -a^2 < 0,$$

*where* $g_\beta = g_{\beta,L} = T^2_{\beta,L}(t)\,\mathrm{d}x^2 + \mathrm{d}t^2$.

Notice that if this property holds for some $\mathfrak{a} > 0$, it holds for any $\mathfrak{a}' \geqslant \mathfrak{a}$. *For technical reasons* (see Lemma 2.2.6), we will assume without loss of generality that $\mathfrak{a} > 4(1+\beta)$. A direct computation yields the following estimate for the function $u = u_{\beta,L}$.

LEMMA 2.2.3. – *Let $u = u_{\beta,L} : \mathbb{R}^{+*} \to \mathbb{R}$ be such $T(u(s)) = 1/s$ for all $s > 0$. Then*

$$u(s) = \ln s + (1+\beta)\ln\ln s - \ln L(\ln s) + \epsilon(s)$$

*with $\epsilon(s) \to 0$ as $s \to +\infty$.*

The group $P = \langle p \rangle$ is a parabolic subgroup of the group of isometries of $\mathbb{R}^2$ endowed with the metric $g_\beta$, fixing the point $\infty$. It follows from Lemma 2.2.3 above that, up to a bounded term, $\mathrm{d}(\mathbf{o}, p^n \cdot \mathbf{o})$ equals $2\ln|n| + 2(1+\beta)\ln\ln|n| - 2\ln L(\ln|n|)$ for $|n|$ large enough. The group $P$ still has critical exponent $1/2$ and is convergent when $\beta > 0$. The following proposition gives a precise estimate for $\mathrm{d}(\mathbf{o}, p^n \cdot \mathbf{o})$; this is the key point in proving that $P$ satisfies Assumptions $(\mathrm{P}_2)$ and (S).

PROPOSITION 2.2.4. – *The parabolic group $P = \langle p \rangle$ on $(\mathbb{R}^2, g_\beta)$ satisfies the following property: for all $n \in \mathbb{Z}\backslash\{0\}$ such that $|n|$ is large enough,*

$$\mathrm{d}(\mathbf{o}, p^n \cdot \mathbf{o}) = 2\ln|n| + 2(1+\beta)\ln\ln|n| - 2\ln L(\ln|n|) + \epsilon(n)$$

*with $\lim_{n\to\pm\infty}\epsilon(n) = 0$. In particular it is convergent with respect to $g_\beta$.*

Let $\mathscr{H} = \{(x,t) \mid t \geqslant 0\}$ be the upper half plane and $\mathscr{H}/P$ the quotient cylinder endowed with the metric $g_\beta$. We cannot estimate directly the distances $\mathrm{d}(\mathbf{o}, p^n \cdot \mathbf{o})$, since the metric $g_\beta$ is not explicit for $t \in [0, \mathfrak{a}]$. Let us introduce the point $\mathbf{a} = (0, \mathfrak{a}) \in \mathscr{H}$. The union of the three geodesic segments $[\mathbf{o}, \mathbf{a}], [\mathbf{a}, p^n \cdot \mathbf{a}]$ and $[p^n \cdot \mathbf{a}, p^n \cdot \mathbf{o}]$ is a quasi-geodesic. Since $d(\mathbf{o}, \mathbf{a}) = d(p^n \cdot \mathbf{o}, p^n \cdot \mathbf{a})$ is fixed and $d(\mathbf{a}, p^n \cdot \mathbf{a}) \to +\infty$, we have the following lemma.

LEMMA 2.2.5. – *Under the previous notation,*

$$\lim_{n\to\pm\infty} (\mathrm{d}(\mathbf{o}, p^n \cdot \mathbf{o}) - \mathrm{d}(\mathbf{a}, p^n \cdot \mathbf{a})) = 2\mathfrak{a}.$$

Therefore, Proposition 2.2.4 follows from the following lemma.

LEMMA 2.2.6. – *Assume that $\mathfrak{a} > 4(1+\beta)$. Then*

$$\mathrm{d}(\mathbf{a}, p^n \cdot \mathbf{a}) = 2\ln|n| + 2(1+\beta)\ln\ln|n| - 2\ln L(\ln|n|) - 2\mathfrak{a} + \epsilon(n)$$

*with $\lim_{n\to\pm\infty}\epsilon(n) = 0$.*

*Proof.* – Throughout this proof, we work on the upper half-plane $\mathbb{R} \times [\mathfrak{a}, +\infty[$ whose points are denoted $(x, \mathfrak{a} + t)$ with $x \in \mathbb{R}$ and $t \geqslant 0$; we set

$$\mathscr{T}(t) = T_\beta(t + \mathfrak{a}) = e^{-\mathfrak{a}-t}\frac{(t+\mathfrak{a})^{1+\beta}}{L(t+\mathfrak{a})}.$$

In these coordinates, the quotient cylinder $\mathbb{R} \times [\mathfrak{a}, +\infty[/P$ is a surface of revolution endowed with the metric $\mathscr{T}(t)^2\,\mathrm{d}x^2 + \mathrm{d}t^2$. For any $n \in \mathbb{Z}$, denote $h_n$ the maximal height at which the geodesic segment $\sigma_n = [\mathbf{a}, p^n \cdot \mathbf{a}]$ penetrates inside the upper half-plane $\mathbb{R} \times [\mathfrak{a}, +\infty[$. Note that due to the negative upper bound on the curvature,

$\lim_{n\to\pm\infty} h_n = +\infty$. The relation between $n, h_n$ and $d_n := \mathrm{d}(\mathbf{a}, p^n \cdot \mathbf{a})$ may be deduced from the Clairaut's relation ([**12**], Section 4.4 Example 5):

$$\frac{n}{2} = \mathscr{T}(h_n)\int_0^{h_n} \frac{\mathrm{d}t}{\mathscr{T}(t)\sqrt{\mathscr{T}^2(t)-\mathscr{T}^2(h_n)}} \quad \text{and} \quad d_n = 2\int_0^{h_n}\frac{\mathscr{T}(t)\,\mathrm{d}t}{\sqrt{\mathscr{T}^2(t)-\mathscr{T}^2(h_n)}}.$$

These identities may be rewritten as

$$\text{(i)}\quad \frac{n}{2} = \frac{1}{\mathscr{T}(h_n)}\int_0^{h_n}\frac{f_n^2(s)\,\mathrm{d}s}{\sqrt{1-f_n^2(s)}} \quad\text{and}\quad \text{(ii)}\quad d_n = 2h_n + 2\int_0^{h_n}\Big(\frac{1}{\sqrt{1-f_n^2(s)}} - 1\Big)\,\mathrm{d}s,$$

where $f_n(s) := \frac{\mathscr{T}(h_n)}{\mathscr{T}(h_n-s)}\mathbb{1}_{[0,h_n]}(s)$.

First, for any $s \geqslant 0$, the quantity $\frac{f_n^2(s)}{\sqrt{1-f_n^2(s)}}$ converges towards $\frac{e^{-2s}}{\sqrt{1-e^{-2s}}}$ as $n \to \pm\infty$.

LEMMA 2.2.7. – *There exists $N_0 > 0$ such that for all $n$, $|n| \geqslant N_0$ and all $s \geqslant 0$,*

$$0 \leqslant f_n(s) \leqslant f(s) := e^{-s/2}.$$

*Proof.* – To simplify notation, write $\alpha = 1 + \beta$.

Assume first $h_n/2 \leqslant s \leqslant h_n$; taking $\theta = \alpha/2$ in (7) yields

$$\begin{aligned} 0 \leqslant f_n(s) &= \left(\frac{\mathfrak{a}+h_n}{\mathfrak{a}+h_n-s}\right)^{\alpha}\frac{L(\mathfrak{a}+h_n-s)}{L(\mathfrak{a}+h_n)}e^{-s} \\ &\leqslant C_{\alpha/2}^2\frac{(\mathfrak{a}+h_n)^{3\alpha/2}}{(\mathfrak{a}+h_n-s)^{\alpha/2}}e^{-s} \\ &\leqslant \frac{C_{\alpha/2}^2}{\mathfrak{a}^{\alpha/2}}(\mathfrak{a}+h_n)^{3\alpha/2}e^{-s} \\ &\leqslant \frac{C_{\alpha/2}^2}{\mathfrak{a}^{\alpha/2}}(\mathfrak{a}+h_n)^{3\alpha/2}e^{-h_n/4}e^{-s/2} \leqslant e^{-s/2}, \end{aligned}$$

where the last inequality holds if $|n|$ is large enough, only depending on $\mathfrak{a}$ and $\alpha$.

Assume now $0 \leqslant s \leqslant h_n/2$; it holds $\frac{1}{2} \leqslant \frac{\mathfrak{a}+h_n-s}{\mathfrak{a}+h_n} \leqslant 1$ and $0 \leqslant \frac{s}{\mathfrak{a}+h_n} \leqslant \min\left(\frac{1}{2}, \frac{s}{\mathfrak{a}}\right)$. The facts that $L'(t)/L(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ and that $0 \leqslant (1-v)^{-1} \leqslant e^{2v}$ for $0 \leqslant v \leqslant 1/2$ imply that, for any $\varepsilon > 0$ and $n$ large enough, there exists $s_n \in \,]0, s[$ such that

$$\begin{aligned} 0 \leqslant f_n(s) &= \frac{L(\mathfrak{a}+h_n-s)}{L(\mathfrak{a}+h_n)}\left(1-\frac{s}{\mathfrak{a}+h_n}\right)^{-\alpha}e^{-s} \\ &\leqslant \left(1-\frac{sL'(\mathfrak{a}+h_n-s_n)}{L(\mathfrak{a}+h_n)}\right)e^{-(1-2\alpha/\mathfrak{a})s} \\ &\leqslant (1+\varepsilon s)e^{-(1-2\alpha/\mathfrak{a})s} \\ &\leqslant e^{-(1-\varepsilon-2\alpha/\mathfrak{a})s}. \end{aligned}$$

Fixing $\varepsilon > 0$ in such a way that $1-2\alpha/\mathfrak{a}-\varepsilon \geqslant 1/2$ (which is possible, since $4\alpha < \mathfrak{a}$), we obtain $0 \leqslant f_n(s) \leqslant e^{-s/2}$ for $n$ large enough. □

We have therefore

$$0 \leqslant \frac{f_n^2(s)}{\sqrt{1-f_n^2(s)}} \leqslant F(s) := \frac{f^2(s)}{\sqrt{1-f^2(s)}},$$

where the function $F$ is integrable on $\mathbb{R}^+$. By the dominated convergence theorem, equality (i) yields

$$\frac{n}{2} = \frac{1+\epsilon(n)}{\mathcal{T}(h_n)} \int_0^{+\infty} \frac{e^{-2s}}{\sqrt{1-e^{-2s}}}\, \mathrm{d}s = \frac{1+\epsilon(n)}{\mathcal{T}(h_n)}.$$

Consequently $h_n = \ln|n| + (1+\beta)\ln\ln|n| - \ln L(\ln|n|) - \log 2 - \mathfrak{a} + \epsilon(n)$ for $|n|$ large enough. Similarly

$$\lim_{n\to\pm\infty} \int_0^{h_n} \Big( \frac{1}{\sqrt{1-f_n^2(s)}} - 1 \Big)\, \mathrm{d}s = \int_0^{+\infty} \Big( \frac{1}{\sqrt{1-e^{-2s}}} - 1 \Big)\, \mathrm{d}s = \ln 2,$$

thus equality (ii) yields

$$d_n = 2\ln|n| + 2(1+\beta)\ln\ln|n| - 2\ln L(\ln|n|) - 2\mathfrak{a} + \epsilon(n) \text{ for } |n| \text{ large enough.} \qquad \square$$

The Poincaré exponent of $P$ equals $1/2$ and using Proposition 2.2.4, we obtain that its orbital function satisfies the following property:

$$\#\{p \in P \mid 0 \leqslant \mathrm{d}(\mathbf{o}, p\cdot\mathbf{o}) < T\} \sim e^{T/2} \frac{L(T)}{(T/2)^{\beta+1}} \quad \text{as} \quad T \to +\infty.$$

Hence, for any $\Delta > 0$,

$$\#\{p \in P \mid T \leqslant \mathrm{d}(\mathbf{o}, p\cdot\mathbf{o}) < T+\Delta\} \sim \frac{1}{2}\int_T^{T+\Delta} e^{t/2} \frac{L(t)}{(t/2)^{\beta+1}}\, \mathrm{d}t \quad \text{as} \quad T \to +\infty$$

and

$$\lim_{T\to+\infty} \frac{T^{\beta+1}}{L(T)} \sum_{\substack{p\in P \\ T\leqslant \mathrm{d}(\mathbf{o},p\cdot\mathbf{o})<T+\Delta}} e^{-\frac{1}{2}\mathrm{d}(\mathbf{o},p\cdot\mathbf{o})} = 2^\beta \Delta. \tag{8}$$

On the one hand, for any $T > 0$, decomposing

$$\sum_{\substack{p\in P \\ \mathrm{d}(\mathbf{o},p\cdot\mathbf{o})>T}} e^{-\frac{1}{2}\mathrm{d}(\mathbf{o},p\cdot\mathbf{o})}$$

as a sum over annuli whose size is arbitrarily small and using (8), we obtain

$$\sum_{\substack{p\in P \\ \mathrm{d}(\mathbf{o},p\cdot\mathbf{o})>T}} e^{-\frac{1}{2}\mathrm{d}(\mathbf{o},p\cdot\mathbf{o})} \sim \frac{2^\beta}{\beta}\frac{L(T)}{T^\beta}$$

which corresponds to Assumption ($P_2$). On the other hand, let us note that, as soon as, for some $C > 0$,

$$\lim_{T\to+\infty} \frac{T^{\beta+1}}{L(T)} \sum_{\substack{p\in P\\ T\leqslant \mathrm{d}(\mathbf{o},p\cdot\mathbf{o})<T+\Delta}} e^{-\frac{1}{2}d(\mathbf{o},p\cdot\mathbf{o})} = C\Delta,$$

there exists $T_0 = T_0(\Delta) > 0$ such that for all $T \geqslant T_0$, we have

$$\frac{T^{\alpha}}{L(T)} \sum_{\substack{p\in P\\ T\leqslant \mathrm{d}(\mathbf{o},p\cdot\mathbf{o})<T+\Delta}} e^{-\frac{1}{2}\mathrm{d}(\mathbf{o},p\cdot\mathbf{o})} \leqslant 2C,$$

which is precisely Hypothesis (S).

**2.2.4. A group with convergent parabolic subgroup. –** Let us now describe explicit constructions of exotic groups, i.e., non-elementary groups $\Gamma$ containing a parabolic subgroup $P$ such that $\delta_P = \delta_\Gamma$ in the context of the metrics $g_\beta$ presented above. Let $\beta > 0$ be fixed and $N \geqslant 2$. For all $\mathfrak{h} > 0$ and $t \in \mathbb{R}$, we write

$$T_{\beta,\mathfrak{h}} = \begin{cases} e^{-t} & \text{if} \quad t \leqslant \mathfrak{h}, \\ e^{-\mathfrak{h}} T_{\beta,L}(t-\mathfrak{h}) & \text{if} \quad t \geqslant \mathfrak{h}, \end{cases}$$

where $T_{\beta,L}$ was defined at the beginning of the previous subsection. Following [**40**], we introduce the metric on $\mathbb{R}^{N-1}\times\mathbb{R}$ given at all $(x,t)\in\mathbb{R}^n$ by $\mathrm{d}g_{\beta,\mathfrak{h}} = T^2_{\beta,\mathfrak{h}}(t)\,\mathrm{d}x^2 + \mathrm{d}t^2$. It is a complete $C^2$ metric, with pinched negative curvature by Lemmas 2.2.1 and 2.2.2, isometric to the hyperbolic plane on $\mathbb{R}^{N-1} \times ]-\infty,\mathfrak{h}[$. Note that $g_{\beta,0} = g_\beta$ and $g_{\beta,+\infty}$ is the hyperbolic metric on $\mathbb{H}^N$ (see Figure 3). We now construct non-

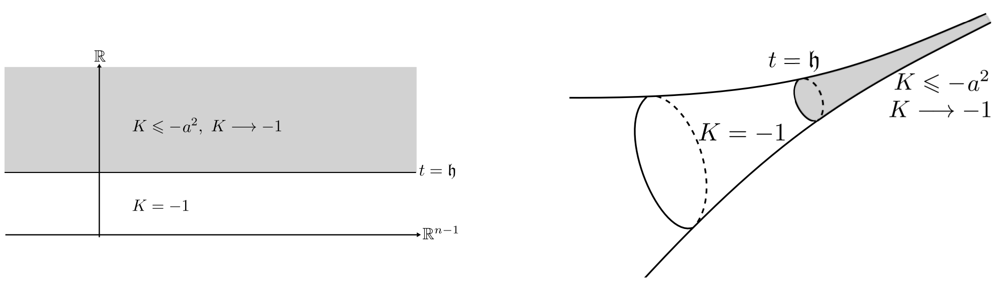


FIGURE 3. Curvature in the horoball and in the cusp in rank 1.

elementary groups containing parabolic subgroups of the above type. Let us consider a geometrically finite group $\Gamma$, acting freely, properly and discontinuously on $\mathbb{H}^N$ and containing parabolic isometries. The quotient manifold $\mathbb{H}^N/\Gamma$ thus has finitely many cusps $\mathcal{C}_1,\dots,\mathcal{C}_l$, each of them being isometric to the quotient of an horoball $\mathscr{H}_i$ by a parabolic group $P_i$ with rank $k_i \in [\![1, N-1]\!]$. By the previous discussion, each group $P_i$ acts by isometries on $\mathbb{R}^N$ endowed with the metric $T^2_{\beta,\mathfrak{h}}(t)\,\mathrm{d}x^2 + \mathrm{d}t^2$. Let us remove the cusp $\mathcal{C}_1$ and then replace it by gluing the quotient $(\mathbb{R}^{N-1} \times [\mathfrak{h}, +\infty[)/P_1$ onto $\left(\mathbb{H}^N/\Gamma\right) \setminus \mathcal{C}_1$. The previous construction ensures that $\Gamma$ still acts by isometries on the universal cover $\mathrm{X}$ of this quotient manifold endowed with the metric $\tilde{g}_{\beta,\mathfrak{h}}$, which is the hyperbolic metric on $\mathrm{X}$ except on the copies of $\mathscr{H}_1$ where it coincides with $g_{\beta,\mathfrak{h}}$.

The fact that $\beta > 0$ implies that the group $P_1$ is convergent. Let us denote by $\mathrm{d}_{\mathfrak{h}}$ the distance induced by the metric $\tilde{g}_{\beta,\mathfrak{h}}$. Following **[40]** Section 4, we will use the previous construction of manifolds with a cusp associated to a convergent parabolic group to present some discrete groups $G$ acting on the above space $\mathrm{X}$, which are convergent or divergent, according to the value of the parameter $\mathfrak{h}$. The following was proved in **[40]**.

Proposition 2.2.8. – *There exist Schottky subgroups $G$ of $\Gamma$ and a positive real $\mathfrak{h}_0$ such that:*

- *$G$ admits $1/2$ as critical exponent and is convergent on $(\mathrm{X}, \tilde{g}_{\beta,0})$;*
- *$G$ has a critical exponent $> 1/2$ and is divergent on $(\mathrm{X}, \tilde{g}_{\beta,\mathfrak{h}})$ for $\mathfrak{h} \geqslant \mathfrak{h}_0$.*

We recall here the arguments presented in **[40]** to prove the proposition. Fix $\mathbf{o} \in \mathrm{X}$. The result relies on the existence of a parabolic isometry $p \in P_1$ and a hyperbolic one $h \in \Gamma$ with no common fixed point. We may shrink the horoball $\mathscr{H}_1$ such that the closed geodesic of $\mathrm{X}/\Gamma$ obtained as the projection of the axis of $h$ does not enter the cusp $\mathscr{H}_1/P_1$ (see Figure 4).

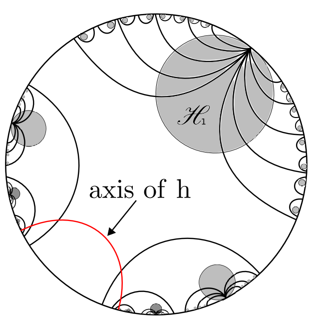


Figure 4. Isometries $p$ and $h$ and horoball $\mathscr{H}_1$.

Then this geodesic stays in the area of constant curvature $-1$ and so does its lift. The isometries $p$ and $h$ being in Schottky position, there exist non-empty disjoint closed subsets $\mathbf{D}_p$ and $\mathbf{D}_h$ of $\overline{\mathrm{X}}$ such that $p^n \cdot \left(\overline{\mathrm{X}} \setminus \mathbf{D}_p\right) \subset \mathbf{D}_p$ and $h^n \cdot \left(\overline{\mathrm{X}} \setminus \mathbf{D}_h\right) \subset \mathbf{D}_h$

for any $n \neq 0$. Write $D_\gamma = \mathbf{D}_\gamma \cap \partial \mathrm{X}$ for $\gamma \in \{p, h\}$. In this case, for any $s > 0$, the Poincaré series of the group $\langle p, h\rangle$ behaves, up to a bounded term, like

$$\sum_{l \geqslant 1} \sum_{m_i, n_i \in \mathbb{Z}^*} e^{-s \mathrm{d}_{\mathfrak{h}}(\mathbf{o}, h^{m_1} p^{n_1} \cdots h^{m_l} p^{n_l} \cdot \mathbf{o})}. \tag{9}$$

Using Lemma 2.1.1 for sets $\mathbf{D}_p$ and $\mathbf{D}_h$, we show that (9) behaves like

$$\sum_{l \geqslant 1} \left( \sum_{m \in \mathbb{Z}^*} e^{-s \mathrm{d}_{\mathfrak{h}}(\mathbf{o}, h^m \cdot \mathbf{o})} \sum_{n \in \mathbb{Z}^*} e^{-s \mathrm{d}_{\mathfrak{h}}(o, p^n \cdot \mathbf{o})} \right)^l. \tag{10}$$

With any metric $\tilde{g}_{\beta, \mathfrak{h}}$, $\mathfrak{h} > 0$, we have $\delta_{\langle p\rangle} = 1/2 \leqslant \delta_{\langle p, h\rangle}$. For $\mathfrak{h} = 0$, we get $\sum_{n \neq 0} e^{-\frac{1}{2} \mathrm{d}_0(\mathbf{o}, p^n \cdot \mathbf{o})} < +\infty$. Furthermore $\sum_{m \neq 0} e^{-\frac{1}{2} \mathrm{d}_0(\mathbf{o}, h^m \cdot \mathbf{o})} \asymp e^{-\frac{1}{2} l(h)}$ where $l(h)$ is the length of $h$. Therefore, if we replace $h$ by a sufficiently large power of $h$, we may assume that

$$\sum_{m \in \mathbb{Z}^*} e^{-\frac{1}{2} \mathrm{d}_0(\mathbf{o}, h^m \cdot \mathbf{o})} \sum_{n \in \mathbb{Z}^*} e^{-\frac{1}{2} \mathrm{d}_0(\mathbf{o}, p^n \cdot \mathbf{o})} < 1.$$

Hence $\mathscr{P}_{\langle p, h\rangle}(1/2) < +\infty$ so that $\delta_{\langle p, h\rangle} \leqslant 1/2$. Finally the subgroup $G = \langle p, h\rangle$ of $\Gamma$ is convergent with critical exponent $1/2 = \delta_{\langle p\rangle}$.

When $\mathfrak{h}$ goes to $+\infty$, the area of X with constant curvature grows up and the Poincaré series of the group $\langle p\rangle$ tends to $+\infty$ at $1/2$. Since $\sum_{m \neq 0} e^{-s \mathrm{d}_{\mathfrak{h}}(\mathbf{o}, h^m \cdot \mathbf{o})} < +\infty$ for any $s > 0$, there exist $\varepsilon_0 > 0$ and $\mathfrak{h}_0 > 0$ such that for any $\varepsilon \in ]0, \varepsilon_0[$ and $\mathfrak{h} \geqslant \mathfrak{h}_0$:

$$\sum_{m \in \mathbb{Z}^*} e^{-(1/2+\varepsilon) \mathrm{d}_{\mathfrak{h}}(\mathbf{o}, h^m \cdot \mathbf{o})} \sum_{n \in \mathbb{Z}^*} e^{-(1/2+\varepsilon) \mathrm{d}_{\mathfrak{h}}(\mathbf{o}, p^n \cdot \mathbf{o})} > 1.$$

Henceforth (10) implies that $G = \langle p, h\rangle$ satisfies $\delta_G > 1/2$ and is divergent with finite Bowen-Margulis measure by Theorems A and B of [**14**].

**2.2.5. A divergent group with infinite measure $m_\Gamma$. –** We now briefly explain the following proposition.

Proposition 2.2.9. – *There exists a unique $\mathfrak{h}_* \in ]0, \mathfrak{h}_0[$ such that the group $G = \langle p, h\rangle$ is divergent with critical exponent $\delta_G = 1/2 = \delta_{\langle p\rangle}$ with respect to the metric $\tilde{g}_{\beta, \mathfrak{h}_*}$. Moreover, G has infinite measure when $0 < \beta \leqslant 1$* [1].

This result seems to be an intermediate state between the two alternatives given in Proposition 2.2.8. It relies on the comparison between the Poincaré series $\mathscr{P}_G$ (and more precisely (9)) and the potential of a transfer operator $\mathscr{L}_{\mathfrak{h}}$ associated to the action of $G$ on $(\mathrm{X}, \tilde{g}_{\beta, \mathfrak{h}})$ (see Section 4 in [**40**]). Transfer operators are a standard tool and they will be discussed in detail in Chapter 4 in the context of the present book.

1. See Remark 2.2.10 for the case $\beta = 1$.

Recall that $G = \langle p, h\rangle$, where $\langle p\rangle$ is convergent with critical exponent $1/2$. Fix $\mathfrak{h} \in [0, \mathfrak{h}_0]$. We formally introduce the following operator: for any $\varphi \in \mathbb{L}^\infty(\Lambda_G, \mathbb{R})$, any $s > 0$ and $x \in \Lambda_G$

$$\mathscr{L}_{\mathfrak{h},s}\varphi(x) = \sum_{\gamma\in\{p,h\}} \sum_{n\in\mathbb{Z}^*} \mathbb{1}_{D_\gamma^c}(x) e^{-s\,\mathscr{B}_x^{(\mathfrak{h})}(\gamma^{-n}\cdot\mathbf{o},\mathbf{o})}\varphi(\gamma^n\cdot x),$$

where $\mathscr{B}_x^{(\mathfrak{h})}(\gamma^{-n}\cdot\mathbf{o},\mathbf{o})$ is the Busemann cocycle corresponding to the metric $\tilde{g}_{\beta,\mathfrak{h}}$. Using the fact that $p$ and $h$ are in Schottky position, we get for any $l \geqslant 1$ and $s > 0$,

$$\sum_{m_i,n_i\in\mathbb{Z}^*} e^{-s\mathrm{d}_{\mathfrak{h}}(h^{m_1}p^{n_1}\cdots h^{m_l}p^{n_l}\cdot\mathbf{o},\mathbf{o})} \asymp \left|\mathscr{L}^{2l}_{\mathfrak{h},s}\mathbb{1}_{\Lambda_G}\right|_\infty,$$

where $\mathrm{d}_{\mathfrak{h}}$ is the distance induced by the metric $\tilde{g}_{\beta,\mathfrak{h}}$ and $|\cdot|_\infty$ is the essential supremum norm on $\mathbb{L}^\infty(\Lambda_G,\mathbb{R})$. By (9), this implies that $\mathscr{P}_G(s)$ behaves like $\sum_{l\geqslant1}\left|\mathscr{L}^{2l}_{\mathfrak{h},s}\mathbb{1}_{\Lambda_G}\right|_\infty$. Since $\mathscr{L}_{\mathfrak{h},s}$ is a positive operator, its spectral radius $\rho_\infty(\mathscr{L}_{\mathfrak{h},s})$ on $\mathbb{L}^\infty(\Lambda_G,\mathbb{R})$ is given by

$$\limsup_{l\longrightarrow+\infty}\left(\left|\mathscr{L}^{l}_{\mathfrak{h},s}\mathbb{1}_{\Lambda_G}\right|_\infty\right)^{1/l}.$$

Proposition 2.2.8 thus implies that

- the series $\sum_{l\geqslant1}\left|\mathscr{L}^{2l}_{0,1/2}\mathbb{1}_{\Lambda_G}\right|_\infty$ converges and $\rho_\infty(\mathscr{L}_{0,1/2}) \leqslant 1$;
- the series $\sum_{l\geqslant1}\left|\mathscr{L}^{2l}_{\mathfrak{h},1/2}\mathbb{1}_{\Lambda_G}\right|_\infty$ diverges and $\rho_\infty(\mathscr{L}_{\mathfrak{h},1/2}) \geqslant 1$ when $\mathfrak{h} \geqslant \mathfrak{h}_0$.

It is proved in **[40]** that there is a unique $\mathfrak{h}_* \in \,]0,\mathfrak{h}_0[$ such that $\rho_\infty(\mathscr{L}_{\mathfrak{h}_*,1/2}) = 1$. The existence of such $\mathfrak{h}_*$ is based on the regularity of the map $\mathfrak{h} \longmapsto \rho_\infty(\mathscr{L}_{\mathfrak{h},1/2})$. This is hard to obtain. Generally the map $\mathscr{L} \longmapsto \rho_\infty(\mathscr{L})$ is only upper semi-continuous (**[32]**). To overcome this lack of regularity, the approach in **[40]** was to let the family of operators $\left(\mathscr{L}_{\mathfrak{h},1/2}\right)_{\mathfrak{h}\in[0,\mathfrak{h}_0]}$ act on the following subspace $\mathbb{L}_\omega(\Lambda_G)$ of $\mathbb{L}^\infty(\Lambda_G,\mathbb{R})$

$$\mathbb{L}_\omega(\Lambda_G) := \{\varphi\in\mathcal{C}(\Lambda_G) \mid |\varphi|_\omega := |\varphi|_\infty + [\varphi]_\omega < +\infty\},$$

where $[\varphi]_\omega := \sup_{\gamma\in\{p,h\}} \sup_{(x,y)\in D_\gamma \overset{\Delta}{\times} D_\gamma} \frac{|\varphi(x)-\varphi(y)|}{\mathrm{d}_{\mathbf{o}}^{(\mathfrak{h})}(x,y)^\omega}$ and $\omega \in \,]0,1[$ is suitably chosen in order to satisfy Fact 3.7 of **[40]**. Here the distance $\mathrm{d}_{\mathbf{o}}^{(\mathfrak{h})}(x,y)$ is the Gromov distance on the boundary of X seen from the point $\mathbf{o}\in$ X and associated with the metric $\tilde{g}_{\beta,\mathfrak{h}}$ (see Section 2.1). Denote by $\rho(a)$ the spectral radius of $\mathscr{L}_{\mathfrak{h},1/2}$ on this space. Peigné then showed that this spectral radius is a simple isolated eigenvalue in the spectrum of $\mathscr{L}_{\mathfrak{h},1/2}$ and equals to $\rho_\infty\left(\mathscr{L}_{a,1/2}\right)$. This property of spectral gap for each $\mathscr{L}_{\mathfrak{h},1/2}$, $\mathfrak{h}\in[0,\mathfrak{h}_0]$, is the key to obtaining the required regularity of the map $\mathfrak{h}\longmapsto\rho_\infty(\mathscr{L}_{\mathfrak{h},1/2})$.

Finally, Subsection 4.5 of **[40]** explains with detailed arguments why there exists $\mathfrak{h}_*\in[0,\mathfrak{h}_0]$ satisfying $\rho_\infty\left(\mathscr{L}_{\mathfrak{h}_*,1/2}\right) = 1$ and why $G = \langle p,h\rangle$ is divergent with critical exponent $1/2$ for the metric $\tilde{g}_{\beta,\mathfrak{h}_*}$. Since $G$ is convergent for $\tilde{g}_{\beta,0}$ and has critical exponent $> 1/2$ for $\tilde{g}_{\beta,\mathfrak{h}_0}$ (because of Proposition 2.2.8), then $\mathfrak{h}_*\in\,]0,\mathfrak{h}_0[$. It follows that we can apply the criterion of finiteness of $m_G$ given by Theorem B of [**14**] above: we obtain that the group $G$ has infinite measure if $\beta \leqslant 1$ (see the remark below). In the last section of [**40**], Peigné also proved that the parameter $\mathfrak{h}_*$ is unique in $]0, \mathfrak{h}_0[$, which achieves the proof of Proposition 2.2.9.

Remark 2.2.10. – *By Theorem B in* [**14**], *the measure $m_G$ is infinite if and only if*

$$\sum_{p \in P} \mathrm{d}(\mathbf{o}, p \cdot \mathbf{o}) e^{-\frac{1}{2}\mathrm{d}(\mathbf{o}, p\cdot\mathbf{o})} \asymp \sum_{k \geqslant 1} \sum_{\substack{p \in P \\ \mathrm{d}(\mathbf{o}, p\cdot\mathbf{o}) > k}} e^{-\frac{1}{2}\mathrm{d}(\mathbf{o}, p\cdot\mathbf{o})} \asymp \sum_{k \geqslant 1} \frac{L(k)}{k^\beta} = +\infty.$$

*If we write $\tilde{L}_\beta(t) = \int_1^t \frac{L(x)}{x^\beta}\,\mathrm{d}x$, then $m_G$ is infinite if and only if $\lim_{t \longrightarrow +\infty} \tilde{L}_\beta(t) = +\infty$. The measure is thus infinite for $\beta < 1$. When $\beta = 1$, it may not be true: for example, if we choose as slowly varying function $L = \ln^{-2}$, then using Proposition 2.2.4, one gets $\mathrm{d}(\mathbf{o}, p^n \cdot \mathbf{o}) = 2\ln(|n|) + 4\ln\ln(|n|) + 4\ln\ln\ln(|n|) + \varepsilon(n)$ with $\lim_{n \longrightarrow +\infty} \varepsilon(n) = 0$ and in this case*

$$\sum_{p \in P} \mathrm{d}(\mathbf{o}, p \cdot \mathbf{o}) e^{-\frac{1}{2}\mathrm{d}(\mathbf{o}, p\cdot\mathbf{o})} \asymp \sum_{k \geqslant 1} \frac{1}{k \ln(k)^2}.$$

*In our setting of infinite measure $m_\Gamma$, the function $\tilde{L}_\beta$ always will satisfy $\lim_{x \longrightarrow +\infty} \tilde{L}_\beta(x) = +\infty$.*

**2.2.6. Comments. –** Combining Subsection 2.2.3 and Proposition 2.2.9, we may notice that $G = \langle p, h \rangle$ is an example of group satisfying the family of assumptions $(H_\beta)$ given in the introduction (except about the number of Schottky factors: we may fix this, for instance, adding a hyperbolic generator to $G$). Indeed:

- $G$ satisfies Assumption (D) by Proposition 2.2.9;
- the subgroup $P = \langle p \rangle$ is convergent by the choice of the metric $g_{\beta, \mathfrak{h}*}$; moreover Proposition 2.2.9 and Subsection 2.2.3 ensure that $P$ satisfies Assumptions ($\mathrm{P}_1$), ($\mathrm{P}_2$) and (S);
- similarly, Assumptions (N) and (S) are satisfied by the tail of the Poincaré series of the hyperbolic group $\langle h \rangle$ at the exponent $\delta_G = 1/2$: indeed, the critical exponent of this group is 0, therefore the sums

$$\sum_{n \ \mid\ \mathrm{d}(\mathbf{o}, h^n\cdot\mathbf{o}) > T} e^{-\frac{1}{2}\mathrm{d}(\mathbf{o}, h^n\cdot\mathbf{o})} \quad \text{and} \quad \sum_{n \ \mid\ T \leqslant \mathrm{d}(\mathbf{o}, h^n\cdot\mathbf{o}) \leqslant T+\Delta} e^{-\frac{1}{2}\mathrm{d}(\mathbf{o}, h^n\cdot\mathbf{o})}$$

  behaves like $e^{-CT}$ as $T \longrightarrow +\infty$ (for a constant $C > 0$ depending on $h$): (N) and (S) follow in this case.

# CHAPTER 3

# REGULARLY VARYING FUNCTIONS AND STABLE LAWS

This chapter is devoted to the statements of some properties of regularly varying functions and their applications to the study of stable laws. We first recall some facts about regularly varying functions. We then use them to the study of the local behavior of the characteristic function of a probability law whose tail is controlled by regularly varying functions.

## 3.1. Slowly varying functions

### 3.1.1. Definitions and classical results

DEFINITION 3.1.1. – i) *A measurable function $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ varies slowly at infinity if for any $x > 0$*

$$\lim_{t \longrightarrow +\infty} \frac{L(xt)}{L(t)} = 1. \tag{11}$$

ii) *A measurable function $U : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ varies regularly with exponent $\beta \in \mathbb{R}$ if for any $t \in \mathbb{R}^+$, it satisfies $U(t) = t^\beta L(t)$ with $L$ slowly varying.*

Notice that, by Theorem 1.2.1 in [7], the convergence (11) is uniform in $x \in [a, b]$, for any compact interval $[a, b] \subset ]0, +\infty[$.

**3.1.2. Karamata's and Potter's lemmas. –** The following lemma specifies the property of integration of regularly varying functions.

LEMMA 3.1.2 (Karamata). – *Let $\beta \in \mathbb{R}$ and $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a slowly varying function.*

– *If $\beta > 1$, then*

$$\int_x^{+\infty} \frac{L(y)}{y^\beta} \, \mathrm{d}y \sim \frac{L(x)}{(\beta - 1)x^{\beta - 1}} \ \textit{as}\ x \longrightarrow +\infty.$$

– *If* $\beta \leqslant 1$*, then the function* $\tilde{L}_\beta : x \longmapsto \int_1^x \frac{L(y)}{y^\beta}\,\mathrm{d}y$ *is regularly varying with exponent* $1-\beta$*; moreover, as* $x \longrightarrow +\infty$

$$\tilde{L}_\beta(x) \sim \frac{x^{1-\beta}L(x)}{1-\beta} \text{ if } \beta < 1 \text{ and } \frac{L(x)}{\tilde{L}_1(x)} \longrightarrow 0 \text{ otherwise.}$$

REMARK 3.1.3. – *This lemma is also true in the discrete case. If* $\beta > 1$*, then* $\sum_{n=N}^{+\infty} \frac{L(n)}{n^\alpha} \sim \frac{L(N)}{(\beta-1)N^{\alpha-1}}$ *as* $N \longrightarrow +\infty$*. If* $\beta \leqslant 1$*, then, setting* $\tilde{L}_\beta(N) = \sum_{n=1}^{N} \frac{L(n)}{n^\beta}$*, one gets* $\tilde{L}_\beta(N) \sim \frac{N^{1-\beta}L(N)}{1-\beta}$ *if* $\beta < 1$ *and* $\frac{L(N)}{\tilde{L}_1(N)} \longrightarrow 0$ *otherwise, as* $N \longrightarrow +\infty$*. In the sequel, to simplify the text, we will denote* $\tilde{L}_1$ *by* $\tilde{L}$*, when there is no possible confusion.*

The following lemma controls the oscillations of a slowly varying function.

LEMMA 3.1.4 (Potter's Bound). – *If* $L$ *is a slowly varying function then for any fixed* $B > 1$ *and* $\rho > 0$*, there exists* $T = T(B, \rho)$ *such that for any* $x, y \geqslant T$

$$\frac{L(x)}{L(y)} \leqslant B \max\left(\left(\frac{x}{y}\right)^{\rho}, \left(\frac{x}{y}\right)^{-\rho}\right).$$

For more details about these lemmas, we refer to [**7**].

## 3.2. Applications

**3.2.1. Local estimates for characteristic functions. –** We now study the local behavior of the characteristic function of probability laws, which are in the domain of attraction of a stable law.

DEFINITION 3.2.1. – *Let* $\mu$ *be a probability measure on* $\mathbb{R}^+$*. The probability measure* $\mu$ *is said to be stable if for any* $a, b > 0$ *and any independent random variables* $X, Y$ *and* $Z$ *with law* $\mu$*, there exist* $c > 0$ *and* $\alpha \in \mathbb{R}$ *such that the laws of the random variables* $aX + bY$ *and* $cZ + \alpha$ *are the same.*

This notion of stable law appears in the study of limit distributions of normalized sums

$$S_n = \frac{X_1 + \cdots + X_n}{a_n} - B_n \tag{12}$$

of independent, identically distributed random variables $X_1, X_2, \ldots, X_n, \cdots$, where $a_n > 0$ and $B_n$ are suitably chosen real constants. We will focus on particular stable laws: a probability law is said to be fully asymmetric and stable with parameter $\beta \in ]0,1[$ if its characteristic function is given by $g_\beta(t) = \exp\left(-\Gamma(1-\beta)e^{i\mathrm{sign}(t)\beta\pi/2}|t|^\beta\right)$, where $\Gamma$ is the gamma function ([**23**] p.162). The density of such a distribution is a continuous function $\Psi_\beta$ supported on $]0, +\infty[$. If a probability law $\mu$ is such that a normalized sum of independent and identically distributed random variables $X_n$ with law $\mu$ converges in distribution to a stable law, we say that $\mu$ belongs to the domain of attraction of this stable law. Let us now give some information about the normalizing sequence $(a_n)_n$ appearing in sums of type (12) in the case of stable law with parameter $\beta \in ]0,1[$. Such a sequence must satisfy $a_n^\beta/L(a_n) = n$ where $L$ is a slowly varying function ([**23**], p.180). In other words, setting $A(x) = x^\beta/L(x)$, the sequence $(a_n)_n$ satisfies $A(a_n) = n$. By Proposition 1.5.12 in [**7**], there exists an increasing and regularly varying function $A^*$ with exponent $1/\beta$ such that $a_n = A^*(n)$. The function $A^*$ also satisfies $A^*(A(x)) \sim A(A^*(x)) \sim x$ as $x \longrightarrow +\infty$.

The following proposition explains the link between our setting and the class of stable laws with parameter $\beta$.

Proposition 3.2.2. – *Let $\beta \in ]0,1[$, $L$ be a slowly varying function and $\mu$ be a probability measure on $\mathbb{R}^+$. If the distribution function $F$ of the law $\mu$ satisfies $1 - F(T) \sim L(T)/T^\beta$ as $T \longrightarrow +\infty$, then $\mu$ is in the domain of attraction of a fully asymmetric stable law with parameter $\beta$.*

We deduce from this proposition that a law $\mu$ whose density $f$ satisfies the asymptotic $f(x) \sim L(x)/x^{1+\beta}$ is in the domain of attraction of a fully asymmetric stable law with parameter $\beta$. Moreover, this asymptotic control of the distribution function $F$ is the one imposed on the tail of the Poincaré series of the Schottky factors $(\Gamma_j)_j$ in assumptions $(P_2)$ and (N). As we will see in Chapter 4, it will be of interest to specify the local expansion at 0 of the characteristic function $\widehat{\mu}$ of $\mu$, which yields in our setting a local expansion of the spectral radius of a family of transfer operators. We state the following proposition, which will be a key tool in the next section.

Proposition 3.2.3. – *Let $\beta \in ]0,1]$ and $\mu$ be a probability measure on $\mathbb{R}^+$ whose distribution function $F(t) := \mu([0,T])$ satisfies*

$$1 - F(T) \asymp \frac{L(T)}{T^\beta} \quad \left(\textit{resp. } 1 - F(T) = o\left(\frac{L(T)}{T^\beta}\right)\right) \quad \textit{as } T \longrightarrow +\infty,$$

*where $L$ is a slowly varying function. For any $\kappa > 0$ and $t \in \mathbb{R}$*

1. *if $\beta < 1$, then, as $t, \kappa \longrightarrow 0$,*

$$\text{a)} \int_0^{+\infty} \left|e^{ity} - 1\right| \mu(\mathrm{d}y) \preceq |t|^\beta L\left(\frac{1}{|t|}\right) \quad \left(\textit{resp.} \ = o\left(|t|^\beta L\left(\frac{1}{|t|}\right)\right)\right);$$

$$\text{b)} \int_0^{+\infty} \left|e^{-\kappa y} - 1\right| \mu(\mathrm{d}y) \preceq \kappa^\beta L\left(\frac{1}{\kappa}\right) \quad \left(\textit{resp.} \ = o\left(\kappa^\beta L\left(\frac{1}{\kappa}\right)\right)\right).$$

2. *if* $\beta = 1$, *then, as* $t, \kappa \longrightarrow 0$,

a) $\displaystyle\int_0^{+\infty} \left|e^{ity} - 1\right| \mu(\,\mathrm{d}y) \preceq |t|\tilde{L}\left(\frac{1}{|t|}\right) \left(\textit{resp.} \ = o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right)\right)\right)$;

b) $\displaystyle\int_0^{+\infty} \left|e^{-\kappa y} - 1\right| \mu(\,\mathrm{d}y) \preceq \kappa\tilde{L}\left(\frac{1}{\kappa}\right) \left(\textit{resp.} \ = o\left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right)\right)$.

Its proof follows that of Lemma 2 in [**21**]. In the following proposition, we focus on the case $\beta = 1$. It will be useful in Chapter 6.

Proposition 3.2.4. – *Let* $\mu$ *be a probability measure on* $\mathbb{R}^+$ *with distribution function* $F$*; for any* $t \in \mathbb{R}$ *and* $\kappa > 0$*, denote by*

$$I_S := \int_0^{+\infty} e^{-\kappa y} \sin(ty)(1 - F(y))\,\mathrm{d}y$$

*and*

$$I_C := \int_0^{+\infty} e^{-\kappa y} \cos(ty)(1 - F(y))\,\mathrm{d}y.$$

*If* $F$ *satisfies* $1 - F(T) \sim L(T)/T$ *as* $T \longrightarrow +\infty$*, one gets as* $t, \kappa \longrightarrow 0$

i) $|I_S| \preceq \dfrac{|t|}{\kappa} L\left(\dfrac{1}{\kappa}\right)$;

ii) $|I_S| \preceq L\left(\dfrac{1}{|t|}\right)$;

iii) $I_C = \tilde{L}\left(\dfrac{1}{\kappa}\right)(1 + o(1)) + O\left(\dfrac{|t|}{\kappa} L\left(\dfrac{1}{\kappa}\right)\right)$;

iv) $I_C = \tilde{L}\left(\dfrac{1}{|t|}\right)(1 + o(1)) + O\left(\dfrac{\kappa}{|t|} L\left(\dfrac{1}{|t|}\right)\right)$.

The proof of this proposition is the same as that of Proposition 6.2 in [**36**].

**3.2.2. Equivalence in Remark 1.2.3.** – We now prove that Assertion $(S')$ implies $(S'')$ in Remark 1.2.3. Fix $j \in [\![1, p+q]\!]$. We thus assume that for any $\Delta > 0$, there exists $C > 0$ such that for any $T > 0$, the quantity $\#\{\alpha \in \Gamma_j \mid T - \Delta \leqslant \mathrm{d}(\mathbf{o}, \alpha \cdot \mathbf{o}) < T + \Delta\}$ is bounded from above by $Ce^{\delta_\Gamma T} L(T)/T^{1+\beta}$. We want to bound $\#\{\alpha \in \Gamma_j \mid \mathrm{d}(\mathbf{o}, \alpha \cdot \mathbf{o}) \leqslant T\}$ for any $T > 0$. A ball centered in $\mathbf{o} \in \mathrm{X}$ of radius $T$ is the disjoint union of annuli of width $2\Delta$, so it is sufficient to prove that for any $N$ large enough, one gets

$$\Sigma := \sum_{n=1}^{N} \frac{e^{\delta_\Gamma n} L(n)}{n^{1+\beta}} \preceq \frac{e^{\delta_\Gamma N} L(N)}{N^{1+\beta}}.$$

We split $\Sigma$ into $\Sigma_1 + \Sigma_2$ where

$$\Sigma_1 = \sum_{n=1}^{[N/2]} \frac{e^{\delta_\Gamma n} L(n)}{n^{1+\beta}} \text{ and } \Sigma_2 = \sum_{n=[N/2]+1}^{N} \frac{e^{\delta_\Gamma n} L(n)}{n^{1+\beta}},$$

where $[\cdot]$ denotes the integer part of $x$. By Karamata's lemma

$$\Sigma_1 \preceq e^{\delta_\Gamma N/2} \sum_{n=1}^{[N/2]} L(n) \preceq e^{\delta_\Gamma N/2} \left[\frac{N}{2}\right] L\left(\left[\frac{N}{2}\right]\right) \preceq N L(N) e^{\delta_\Gamma N/2}, \tag{13}$$

the last inequality following from Potter's lemma with $B = 1$, $\rho = 1$, $x = [N/2]$ and $y = N$. Similarly

$$\Sigma_2 = \frac{L(N)}{N^{1+\beta}} \sum_{n=[N/2]+1}^{N} e^{\delta_\Gamma n} \frac{L(n)}{L(N)} \frac{N^{1+\beta}}{n^{1+\beta}} \preceq \frac{L(N)}{N^{1+\beta}} \sum_{n=[N/2]+1}^{N} e^{\delta_\Gamma n}, \tag{14}$$

where the last inequality may be deduced from Potter's lemma with $B, \rho = 1$, $x = n$ and $y = N$. The result follows combining (13) and (14) for $N$ large enough.

# CHAPTER 4

# CODING AND TRANSFER OPERATORS

To study the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on a negatively curved manifold, it is classical to conjugate it to a suspension over a shift on a symbolic space (see for example [**9**] and [**38**] or [**2**], [**16**] and [**18**]). The first part of this section is dedicated to the construction of the coding which will be used in Chapter 5.

In a second part, we introduce a family of transfer operators associated to this coding. This will be crucial in the sequel; we will need in particular a precise control on the regularity of this family of operators and their dominant eigenvalue: this will be done in the last part of this section.

## 4.1. Coding of the limit set and of the geodesic flow

We recall now the setting. We fix $p, q \geqslant 1$ such that $p + q \geqslant 3$ and consider $p + q$ discrete elementary subgroups $\Gamma_1, \dots, \Gamma_{p+q}$ of $\mathrm{Isom(X)}$ in Schottky position with $\Gamma_1, \dots, \Gamma_p$ parabolic and let $\Gamma$ be the Schottky product of $\Gamma_1, \dots, \Gamma_{p+q}$. We also consider the families of sets $(D_j)_{1\leqslant j\leqslant p+q}$ and $(\mathbf{D}_j)_{1\leqslant j\leqslant p+q}$ introduced in Chapter 2. We will write $D := \bigcup_{1\leqslant j\leqslant p+q} D_j$; notice that $D$ is a proper subset of $\partial\mathrm{X}$. Since $\Gamma = \Gamma_1 * \cdots * \Gamma_{p+q}$, any element $\gamma$ in $\Gamma$ can be uniquely written as the product $\alpha_1 \cdots \alpha_k$ for some $\alpha_1, \dots, \alpha_k \in \bigcup_j \Gamma_j \backslash \{\mathrm{Id}\}$ with the property that no two consecutive elements $\alpha_j$ and $\alpha_{j+1}$ belong to the same subgroup $\Gamma_j, 1 \leqslant j \leqslant p+q$. The set $\mathscr{A} := \bigcup_j \Gamma_j \backslash \{\mathrm{Id}\}$ is called the *alphabet* of $\Gamma$, and $\alpha_1, \dots, \alpha_k$ will be called the *letters* of $\gamma$; the word $\alpha_1 \cdots \alpha_k$ is said $\mathscr{A}$-admissible. The *symbolic length* $|\gamma|$ of $\gamma$ is equal to the number $k$ of letters appearing in its decomposition with respect to $\mathscr{A}$. For any $n \in \mathbb{N}\backslash\{0\}$, set $\Gamma(n) := \{\gamma \in \Gamma \mid |\gamma| = n\}$. Notice that both $\mathscr{A}$ and $\Gamma(n)$ are infinite and countable. The initial and last letters of $\gamma$ play a special role, so the index of the group they belong to will be denoted by $i(\gamma)$ and $l(\gamma)$ respectively. In the sequel, the point $\mathbf{o}$ refers to a fixed point in $\mathrm{X}\backslash \bigcup_{1\leqslant j\leqslant p+q} \mathbf{D}_j$.

Let us now give some geometrical properties of the action of such a Schottky group $\Gamma$. First of all, combining Lemma 2.1.1 and the relative position of sets

$(\mathbf{D}_j)_{1\leqslant j\leqslant p+q}$, we deduce the corollaries below: the first one is a reformulation of Lemma 2.1.1 for triangles with two vertices in different sets $\mathbf{D}_j$; the second furnishes a well known improvement of the inequality $\mathscr{B}_x(\gamma^{-1}\cdot \mathbf{o},\mathbf{o}) \leqslant \mathrm{d}(\mathbf{o},\gamma^{-1}\cdot\mathbf{o})$.

PROPERTY 4.1.1. – *There exists a constant $C>0$, which depends only on the bounds of the curvature of* X *and on* $\Gamma$, *such that* $\mathrm{d}(\mathbf{o},\gamma_1\cdot\mathbf{o})+\mathrm{d}(\mathbf{o},\gamma_2\cdot\mathbf{o})-C\leqslant \mathrm{d}(\gamma_1\cdot\mathbf{o},\gamma_2\cdot\mathbf{o})$ *for any* $\gamma_1,\gamma_2\in\Gamma$ *with* $i(\gamma_1)\neq i(\gamma_2)$.

REMARK 4.1.2. – *From this property and Assumptions ($P_1$) and (N), we deduce that the sum* $\sum_{\gamma\in\Gamma(n)} e^{-\delta_\Gamma \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}$ *is finite for any* $n\geqslant 1$*: if* $n=1$, *this is a direct consequence of Assumptions ($P_1$) and (N); for* $n\geqslant 2$, *Corollary 4.1.1 implies that there exists a constant* $C>0$ *such that for any* $\gamma=\alpha_1\cdots\alpha_n$, *one gets*

$$\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\geqslant \mathrm{d}(\mathbf{o},\alpha_1.\mathbf{o})+\mathrm{d}(\mathbf{o},\alpha_2\cdot\mathbf{o})+\cdots+\mathrm{d}(\mathbf{o},\alpha_n.\mathbf{o})-nC,$$

*so that*

$$\sum_{\gamma\in\Gamma(n)} e^{-\delta_\Gamma \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}\leqslant e^{nC}\left(\sum_{\alpha\in\mathscr{A}} e^{-\delta_\Gamma \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}\right)^n<+\infty.$$

PROPERTY 4.1.3. – *There exists a constant $C>0$ such that for any $\gamma\in\Gamma$ and any* $x\in\bigcup_{j\neq l(\gamma)} D_j$

$$\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})-C\leqslant \mathscr{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})\leqslant \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}).$$

This estimate plays an important role in the proof of the following proposition, which allows us to bound from above the conformal coefficient of an isometry $\gamma\in\Gamma$.

PROPOSITION 4.1.4. – *There exist a constant $C>0$ and $r\in\,]0,1[$ such that for any* $\gamma\in\Gamma(n)$, $n\geqslant 1$, *and for any* $x\in\bigcup_{j\neq l(\gamma)} D_j$

$$|\gamma'(x)|_{\mathbf{o}}\leqslant Cr^n.$$

*Proof.* – Recall that $|\gamma'(x)|_{\mathbf{o}}=e^{-a\mathscr{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})}$. By Property 4.1.3, it is sufficient to find a constant $A>0$ such that $\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\geqslant An$ for all $\gamma\in\Gamma(n)$. Fix $n\geqslant 1$. Since the $\Gamma$-orbits accumulate at infinity, there exists an integer $l_0\geqslant 1$ such that $\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\geqslant 1+2C$ for any $g$ with symbolic length at least $l_0$ and where $C>0$ is given in Property 4.1.1. We split the transformation $\gamma\in\Gamma(n)$ into a product of transformations with length $l_0$. There are two cases:

1) if $n>2l_0$, we decompose $\gamma$ as $\gamma=\gamma_1\cdot..\gamma_k\overline{\gamma}$ where $|\gamma_i|=l_0$ for any $1\leqslant i\leqslant k:=\left[\frac{n}{l_0}\right]$ and $|\overline{\gamma}|<l_0$. Therefore, Property 4.1.1 implies

$$\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\geqslant\sum_{1\leqslant i\leqslant k}\mathrm{d}(\mathbf{o},\gamma_i\cdot\mathbf{o})+\mathrm{d}(\mathbf{o},\overline{\gamma}\cdot\mathbf{o})-2kC$$

which yields $\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\geqslant k\geqslant n/(2l_0)$;

2) if $n \leqslant 2l_0$, the discreteness of $\Gamma$ implies

$$B := \inf_{1\leqslant n\leqslant 2l_0} \inf_{\gamma\in\Gamma(n)} \mathrm{d}(\mathbf{o}, \gamma\cdot\mathbf{o}) > 0,$$

hence $\mathrm{d}(\mathbf{o}, \gamma\cdot\mathbf{o}) \geqslant B \geqslant Bn/(2l_0)$.

The result follows with $A := \min(1/(2l_0), B/(2l_0))$. □

COROLLARY 4.1.5. – *There exist a constant $C > 0$ and $r \in ]0,1[$ such that for any $n \geqslant 1$, any $\gamma \in \Gamma(n)$ and $x, y \in \Lambda_\Gamma \cap \left(\partial\mathrm{X}\backslash D_{l(\gamma)}\right)$,*

$$\mathrm{d}_{\mathbf{o}}(\gamma\cdot x, \gamma\cdot y) \leqslant Cr^n \mathrm{d}_{\mathbf{o}}(x,y).$$

**4.1.1. Coding of the limit set. –** Let $\Sigma^+_{\mathscr{A}}$ denote the set of *$\mathscr{A}$-admissible sequences* $(\alpha_n)_{n\geqslant 1}$, i.e., sequences for which each letter $\alpha_n$ belongs to the alphabet $\mathscr{A}$ and such that no two consecutive letters belong to the same subgroup $\Gamma_j, 1 \leqslant j \leqslant p+q$. *Fix a point $x_0 \in \partial\mathrm{X}\backslash D$.* Recall that *the radial limit set* consists of points $x \in \Lambda_\Gamma$, which are approached by orbit points in some $M$-neighborhood of any given ray issued from $x$, for some $M > 0$. We may find in **[3]** the following result.

PROPOSITION 4.1.6. – 1) *For any $\boldsymbol{\alpha} = (\alpha_n)_{n\geqslant 1} \in \Sigma^+_{\mathscr{A}}$, the sequence $(\alpha_1\cdots\alpha_n\cdot x_0)_{n\geqslant 1}$ converges to a point $\pi(\boldsymbol{\alpha}) \in \Lambda_\Gamma$, which does not depend on the choice of $x_0$.*

2) *The map $\pi : \Sigma^+_{\mathscr{A}} \longrightarrow \Lambda_\Gamma$ is one-to-one and $\pi\left(\Sigma^+_{\mathscr{A}}\right)$ is included in the radial limit set of $\Gamma$.*

3) *The complement of $\pi\left(\Sigma^+_{\mathscr{A}}\right)$ in $\Lambda_\Gamma$ is countable and consists of the $\Gamma$-orbit of the union of the limit sets $\Lambda_{\Gamma_j}$ (each of which being finite here). In particular $\sigma_{\mathbf{o}}\left(\Lambda_\Gamma\backslash\pi\left(\Sigma^+_{\mathscr{A}}\right)\right) = 0$, where $\sigma_{\mathbf{o}}$ is the Patterson-Sullivan measure on $\partial\mathrm{X}$.*

Let us write $\Lambda^0 := \pi\left(\Sigma^+_{\mathscr{A}}\right)$ and $\Lambda^0_j := \Lambda^0 \cap D_j = \{\pi(\boldsymbol{\alpha}) \mid \alpha_1 \in \Gamma^*_j\}$ for $1 \leqslant j \leqslant p+q$. Let us emphasize that $\Lambda^0_j$ is not equal to the limit set of the group $\Gamma_j$, but nevertheless $\Lambda_{\Gamma_j} \subset \overline{\Lambda^0_j} = \Lambda_\Gamma \cap D_j$. The sets $D_j$, $1 \leqslant j \leqslant p+q$, being disjoint, the sets $\left(\Lambda^0_j\right)_j$ have disjoint closures. The following description of $\Lambda^0$ will be useful:

(1) $\Lambda^0$ is the finite union of the sets $\Lambda^0_1, \ldots, \Lambda^0_{p+q}$;

(2) each of sets $\Lambda^0_j$ is partitioned into a countable number of subsets with disjoint closures: indeed, for any $j \in [\![1, p+q]\!]$

$$\Lambda^0_j = \bigcup_{\alpha\in\Gamma^*_j} \bigcup_{k\neq j} \alpha\cdot\Lambda^0_k.$$

The *shift* $\Theta$ on the symbolic space $\Sigma^+_{\mathscr{A}}$ is defined by:

$$\forall \boldsymbol{\alpha} \in \Sigma^+_{\mathscr{A}},\ \Theta(\boldsymbol{\alpha}) = (\alpha_{n+1})_{n\geqslant 1}.$$

This operator $\Theta$ induces a transformation $\mathrm{T} : \Lambda^0 \to \Lambda^0$ whose action is defined for all $x = \pi(\boldsymbol{\alpha})$ by

$$\mathrm{T}x = \alpha_1^{-1} \cdot x.$$

As a consequence of Corollary 4.1.5, the map T is expanding on $\Lambda^0$ (see Corollary II.4 in [**16**]).

**4.1.2. Coding of the geodesic flow. –** In Subsection 2.1.3, we recalled how to define the action of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on $\partial \mathrm{X} \overset{\Delta}{\times} \partial \mathrm{X} \times \mathbb{R}/\Gamma \simeq \mathrm{T}^1\mathrm{M}$. We propose here a coding of the geodesic flow on a $(g_t)$-invariant subset $\Omega^0$ of $\Omega_\Gamma$ defined by $\Omega^0 := \Lambda^0 \overset{\Delta}{\times} \Lambda^0 \times \mathbb{R}/\Gamma$. We first conjugate the action of $\Gamma$ on $\Lambda^0 \overset{\Delta}{\times} \Lambda^0 \times \mathbb{R}$ with the action of a single transformation. Observe that the subset $\mathscr{D}^0 := \bigcup_{k\neq j} \Lambda^0_k \times \Lambda^0_j$ of $\Lambda^0 \overset{\Delta}{\times} \Lambda^0$ is in one-to-one correspondence with the symbolic space $\Sigma_{\mathscr{A}}$ of bi-infinite $\mathscr{A}$-admissible sequences $(\alpha_n)_{n\in\mathbb{Z}}$. Moreover the shift on $\Sigma_{\mathscr{A}}$ induces a transformation still denoted by T on this set $\mathscr{D}^0$ whose action is given by

$$\mathrm{T}(y, x) = (\alpha_1^{-1} \cdot y, \alpha_1^{-1} \cdot x) \text{ if } x = \pi(\boldsymbol{\alpha}).$$

In [**2**] it is proved that the action of $\Gamma$ on $\Lambda^0 \overset{\Delta}{\times} \Lambda^0$ is orbit-equivalent with the action of T on $\mathscr{D}^0$. Similarly, the action of $\Gamma$ on the space $\Lambda^0 \overset{\Delta}{\times} \Lambda^0 \times \mathbb{R}$ is orbit-equivalent to the action of the transformation $\mathrm{T}_{\mathfrak{r}}$ on $\mathscr{D}^0 \times \mathbb{R}$ given by

$$\mathrm{T}_{\mathfrak{r}}(y, x, r) = (\mathrm{T}(y, x), r - \mathfrak{r}(x)), \tag{15}$$

where the *roof function* $\mathfrak{r}$ is given by $\mathfrak{r}(x) = -\mathscr{B}_x(\alpha_1.\mathbf{o}, \mathbf{o})$ when $x = \pi(\boldsymbol{\alpha})$. Let us write $S_k\mathfrak{r}(x) = \mathfrak{r}(x) + \mathfrak{r}(\mathrm{T}x) + \cdots + \mathfrak{r}(\mathrm{T}^{k-1}x)$ for all $k \in \mathbb{N}\backslash\{0\}$ and all $x \in \Lambda^0$. For all $k \geqslant 1$, one gets

$$\mathrm{T}^k_{\mathfrak{r}}(y, x, r) = \left(\mathrm{T}^k(y, x), r - S_k\mathfrak{r}(x)\right). \tag{16}$$

When $\mathfrak{r} > 0$, a fundamental domain $\mathscr{D}^0_{\mathfrak{r}}$ for the action of $\mathrm{T}_{\mathfrak{r}}$ on $\mathscr{D}^0 \times \mathbb{R}$ is given by

$$\mathscr{D}^0_{\mathfrak{r}} = \left\{(y, x, r) \in \mathscr{D}^0 \times \mathbb{R} \mid 0 \leqslant r < \mathfrak{r}(x)\right\} \text{ (see Figure 5).}$$

In general, the function $\mathfrak{r}$ is not positive; nevertheless we have the following property.

Lemma 4.1.7. – *The roof function $\mathfrak{r}$ satisfies:*

- *$\mathfrak{r}$ is uniformly bounded from below by $-C$, where the constant $C > 0$ depends only on X and $\Gamma$;*
- *there exists $k_0 \geqslant 1$ such that $S_k\mathfrak{r}(x) > 0$ for any $k \geqslant k_0$ and $x \in \Lambda^0$.*

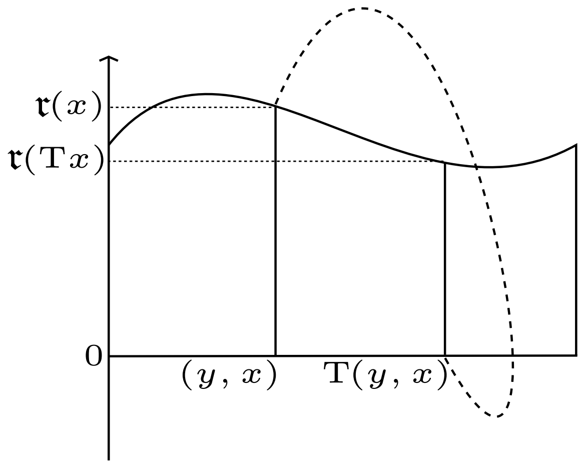


FIGURE 5. Action of $\mathrm{T}_{\mathfrak{r}}$ when $\mathfrak{r} > 0$.

*Proof.* – Let $x \in \Lambda^0$ and $\boldsymbol{\alpha} \in \Sigma^+_{\mathscr{A}}$ such that $x = \pi(\boldsymbol{\alpha})$. Since $\mathfrak{r}(x) = \mathscr{B}_{\alpha_1^{-1}\cdot x}\left(\alpha_1^{-1}\cdot \mathbf{o}, \mathbf{o}\right)$, Property 4.1.3 implies $\mathfrak{r}(x) \geqslant \mathrm{d}(\mathbf{o}, \alpha_1^{-1}\cdot \mathbf{o}) - C$, which proves the first assertion. Concerning the second one, let us notice that

$$\begin{aligned} S_k\mathfrak{r}(x) &= \mathscr{B}_{\alpha_1^{-1}\cdot x}(\alpha_1^{-1}\cdot \mathbf{o}, \mathbf{o}) + \mathscr{B}_{\alpha_2^{-1}\alpha_1^{-1}\cdot x}(\alpha_2^{-1}\cdot \mathbf{o}, \mathbf{o}) + \cdots + \mathscr{B}_{\alpha_k^{-1}\cdots\alpha_1^{-1}\cdot x}(\alpha_k^{-1}\cdot \mathbf{o}, \mathbf{o}) \\ &= \mathscr{B}_{\alpha_k^{-1}\cdots\alpha_1^{-1}\cdot x}(\alpha_k^{-1}\cdots\alpha_1^{-1}\cdot \mathbf{o}, \mathbf{o}) \geqslant \mathrm{d}(\mathbf{o}, \alpha_k^{-1}\cdots\alpha_1^{-1}\cdot \mathbf{o}) - C. \end{aligned}$$

Since the group $\Gamma$ is discrete, the sums $S_k\mathfrak{r}(x)$ are positive for $k$ large enough, uniformly in $x \in \Lambda^0$. □

Using this lemma, a classical argument in Ergodic Theory allows us to describe an explicit fundamental domain for the action of $\mathrm{T}_{\mathfrak{r}}$.

PROPOSITION 4.1.8. – *The function* $\mathfrak{r}$ *is* cohomologous *to a positive function* $\mathfrak{R}$, *i.e., there exists a measurable function* $f : \Lambda^0 \longrightarrow \mathbb{R}$ *such that* $\mathfrak{r} = \mathfrak{R} + f - f\circ \mathrm{T}$.

*Proof.* – By Lemma 4.1.7, there exists $k_0 \geqslant 1$ such that for any $k \geqslant k_0$ and any $x \in \Lambda^0$, one gets $S_k\mathfrak{r}(x) > 0$. Denote by $\varepsilon = 1/k_0$ and let us introduce $a_i = 1 - i\varepsilon$ for any $i \in [\![0, k_0]\!]$. We notice that $a_0 = 1$, $a_{k_0} = 0$ and $a_i - a_{i-1} = -\varepsilon$ for any $i \in [\![1, k_0]\!]$. Fix $x \in \Lambda^0$ and set

$$f(x) := \sum_{i=0}^{k_0-1} a_i \mathfrak{r}\left(\mathrm{T}^i x\right).$$

Therefore

$$\begin{aligned} f(x) - f(\mathrm{T}x) &= \sum_{i=0}^{k_0-1} a_i \mathfrak{r}\left(\mathrm{T}^i x\right) - \sum_{i=0}^{k_0-1} a_i \mathfrak{r}\left(\mathrm{T}^{i+1} x\right) \\ &= a_0 \mathfrak{r}(x) - a_{k_0} \mathfrak{r}\left(\mathrm{T}^{k_0} x\right) + \sum_{i=1}^{k_0} a_i \mathfrak{r}\left(\mathrm{T}^i x\right) - \sum_{i=1}^{k_0} a_{i-1} \mathfrak{r}\left(\mathrm{T}^i x\right), \end{aligned}$$

which yields

$$f(x) - f(\mathrm{T}x) = \mathfrak{r}(x) - \varepsilon \sum_{i=1}^{k_0} \mathfrak{r}\left(\mathrm{T}^i x\right).$$

Let us denote by $\mathfrak{R}(x) = \varepsilon \sum_{i=1}^{k_0} \mathfrak{r}\left(\mathrm{T}^i x\right) = \frac{1}{k_0} S_{k_0} \mathfrak{r}(\mathrm{T}x)$; Lemma 4.1.7 implies that the function $\mathfrak{R}$ is positive, which ends the proof of the lemma. □

The set

$$\mathscr{D}^0_{f,\mathfrak{R}} = \left\{(y,x,r) \in \mathscr{D}^0 \times \mathbb{R} \mid f(x) \leqslant r < f(x) + \mathfrak{R}(x)\right\}$$

is a fundamental domain for the action of $\mathrm{T}_\mathfrak{r}$ on $\mathscr{D}^0 \times \mathbb{R}$: indeed

$$\mathrm{T}_\mathfrak{r}(y, x, f(x) + \mathfrak{R}(x)) = (\mathrm{T}(y,x), f(x) + \mathfrak{R}(x) - \mathfrak{r}(x)) = (\mathrm{T}(y,x), f(\mathrm{T}x))$$

(see Figure 6).

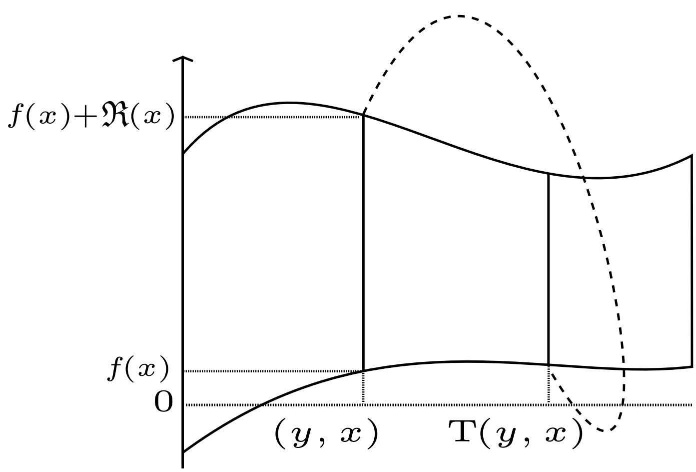


FIGURE 6. Domain $\mathscr{D}^0_{f,\mathfrak{R}}$.

Let $\tilde{\phi}_t$ denote the transformation, whose action on triples $(y,x,r) \in \mathscr{D}^0 \times \mathbb{R}$ is given by translation of $t$ on the third coordinate. The actions of $\left(\tilde{\phi}_t\right)_t$ and $\mathrm{T}_\mathfrak{r}$ commute and define a *special flow* $(\phi_t)_t$ on $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r}\rangle$. Identifying $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r}\rangle$ with $\mathscr{D}^0_{f,\mathfrak{R}}$, for any $(y,x,r) \in \mathscr{D}^0_{f,\mathfrak{R}}$ and $t > 0$, one gets from (16)

$$\phi_t\,(y,x,r) = (y,x,r+t) = \big(\mathrm{T}^k(y,x), r+t-S_k\mathfrak{r}(x)\big) = \mathrm{T}^k_\mathfrak{r}\,(y,x,r+t)\,, \tag{17}$$

where $k \in \mathbb{Z}$ is the unique integer such that $f(x) \leqslant r+t-S_k\mathfrak{r}(x) < f(x)+\mathfrak{R}(x)$. We finally deduce the following lemma.

LEMMA 4.1.9. – *We have:*

i) *The spaces* $\Lambda^0 \overset{\Delta}{\times} \Lambda^0 \times \mathbb{R}/\Gamma$ *and* $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r}\rangle$ *are in one-to-one correspondence.*
ii) *The geodesic flow on* $\Omega^0$ *is conjugated to the special flow on* $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r}\rangle$.

This lemma implies in particular that there is a one-to-one correspondence between the primitive periodic orbits of the geodesic flow on $\mathrm{T}^1\mathrm{X}/\Gamma$ and the primitive periodic orbits of the special flow $(\phi_t)_t$ on $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r}\rangle$. This correspondence allows us to characterize periodic orbits of $(g_t)_t$. Let $\ell$[1] $> 0$ and $(y,x,f(x)) \in \mathscr{D}^0_{f,\mathfrak{R}}$ a $\phi_\ell$-periodic triple: the equality $\phi_\ell\,(y,x,f(x)) = (y,x,f(x))$ may be written

$$(y,x,f(x)+\ell) \sim (y,x,f(x)).$$

Therefore $\ell \geqslant \mathfrak{R}(x)$ and there exists an integer $k \geqslant 1$ satisfying

$$f(x) \leqslant f(x)+\ell-S_k\mathfrak{r}(x) < f(x)+\mathfrak{R}(x).$$

The unique representative of $(y,x,f(x)+\ell)$ in $\mathscr{D}^0_{f,\mathfrak{R}}$ is given by

$$\big(\mathrm{T}^k y, \mathrm{T}^k x, f(x)+\ell-S_k\mathfrak{r}(x)\big)\,;$$

it follows that

$$\mathrm{T}^k(y,x) = (y,x) \text{ and } \ell = S_k\mathfrak{r}(x).$$

These equalities determine $k$ periodic pairs for $\mathrm{T}^k$

$$(y,x), (\mathrm{T}y, \mathrm{T}x)\,, \dots, \big(\mathrm{T}^{k-1}y, \mathrm{T}^{k-1}x\big)$$

in $\mathscr{D}^0$ and the length $\ell$ of the orbit is given by $\ell = S_k\mathfrak{r}(x)$.

Furthermore, the closed geodesics on $\mathrm{X}/\Gamma$ are in one-to-one correspondence with the periodic orbits of the geodesic flow $(g_t)_t$ on $\mathrm{T}^1\mathrm{X}/\Gamma$. Let $\wp$ be a closed geodesic. If it is not the projection of the axis of a hyperbolic isometry of some Schottky factor $\Gamma_j$, $j \in [\![p+1, p+q]\!]$, a lift of $\wp$ in $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r}\rangle$ corresponds to a periodic orbit for the special flow. There thus exist $(y,x) \in \mathscr{D}^0$ and $k \geqslant 2$ such that

$$\mathrm{T}^k(y,x) = (y,x) \text{ and } \ell(\wp) = S_k\mathfrak{r}(x),$$

where $\ell(\wp)$ is the length of $\wp$. The pair $(y,x)$ is associated to a $k$-periodic two-sided admissible sequence $(\alpha_n)_{n\in\mathbb{Z}}$ of $\Sigma_\mathscr{A}$: the point $x$ is the attractive fixed point of the

1. In the sequel, this character will always stand for a length: length of a periodic orbit or of a closed geodesic.

hyperbolic isometry $\alpha_1\alpha_2\cdots\alpha_k$ and the closed geodesic $\wp$ is the projection of the axis of this isometry.

We may notice that the pairs $(y,x), \mathrm{T}(y,x), \dots, \mathrm{T}^{k-1}(y,x)$ lead to the same orbit of the special flow, and thus define the same closed geodesic. We conclude that the closed geodesics which do not correspond to hyperbolic generators of $\Gamma$ are in one-to-one correspondence with the orbits of $\mathrm{T}^k$-periodic pairs of $\mathscr{D}^0$, $k \geqslant 2$.

**4.1.3. The dynamical system** $(\Lambda^0, \mathrm{T}, \nu)$. – Recall that by the identification $\mathrm{T}^1\mathrm{X} \simeq \partial\mathrm{X} \overset{\Delta}{\times} \partial\mathrm{X} \times \mathbb{R}$ given by Hopf coordinates (see Subsection 2.1.3), the Bowen-Margulis measure $m_\Gamma$ on $\mathrm{T}^1\mathrm{X}/\Gamma$ is given by the quotient under the action of $\Gamma$ of the measure $\tilde{m}_\Gamma$ defined by

$$\mathrm{d}\tilde{m}_\Gamma(y,x,t) = \frac{\mathrm{d}\sigma_{\mathbf{o}}(y)\,\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(y,x)^{2\delta_\Gamma/a}}\,\mathrm{d}t = \mathrm{d}\mu(y,x)\,\mathrm{d}t,$$

where $\sigma_{\mathbf{o}}$ is the Patterson-Sullivan measure seen from $\mathbf{o}$, $\mathrm{d}_{\mathbf{o}}$ the Gromov distance on $\partial\mathrm{X}$ and $a > 0$ is such that the curvatures of X are less than $-a^2$. Notice that under our assumptions, the measure $m_\Gamma$ has infinite total mass and $m_\Gamma\left(\mathrm{T}^1\mathrm{M}\backslash\Omega^0\right) = 0$. Up to multiplying the Patterson density $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$ by a constant, we may assume that $\mu\left(\mathscr{D}^0\right) = 1$.

The dynamical system $\left(\Omega^0, (g_t)_{t\in\mathbb{R}}, m_\Gamma\right)$ is conjugated to the special flow $\left(\mathscr{D}^0 \times \mathbb{R}/\langle\mathrm{T}_{\mathfrak{r}}\rangle, (\phi_t)_{t\in\mathbb{R}}, \overline{m}_\Gamma\right)$ where $\overline{m}_\Gamma$ denotes the projection of $(\tilde{m}_\Gamma)|_{\mathscr{D}^0\times\mathbb{R}}$ to $\mathscr{D}^0 \times \mathbb{R}/\langle\mathrm{T}_{\mathfrak{r}}\rangle$ under the action of $\mathrm{T}_{\mathfrak{r}}$.

We have $\mathscr{D}^0 \subset \Lambda^0 \times \Lambda^0$ and may consider the measure $\nu$ on $\Lambda^0$, obtained as the projection of $\mu|_{\mathscr{D}^0}$ on the second coordinates. This measure $\nu$ is absolutely continuous with respect to the Patterson-Sullivan measure $\sigma_{\mathbf{o}}$.

Proposition 4.1.10. – *The map $h : \Lambda^0 \longrightarrow \mathbb{R}_+^*$ defined by: for all $j \in \{1,\dots,p+q\}$ and $x \in \Lambda_j^0$*

$$h(x) = \int_{\bigcup_{l\neq j}\Lambda_l^0} \frac{\mathrm{d}\sigma_{\mathbf{o}}(y)}{\mathrm{d}_{\mathbf{o}}(x,y)^{2\delta_\Gamma/a}}, \tag{18}$$

*is the density of $\nu$ with respect to $\sigma_{\mathbf{o}}$; moreover, the measure $\nu$ is* T*-invariant.*

*Proof.* – Let $\varphi : \Lambda^0 \longrightarrow \mathbb{R}$ be a Borel function. Denote by $p$ the projection

$$p : \left\{\begin{array}{ll} \mathscr{D}^0 & \longrightarrow \Lambda^0 \\ (x,y) & \longmapsto y \end{array}\right. .$$

We write

$$\begin{aligned}\nu(\varphi) = \int_{\Lambda^0} \varphi(x)\nu(\,\mathrm{d}x) &= \sum_{j=1}^{p+q} \int_{\Lambda_j^0} \int_{\Lambda^0\setminus\Lambda_j^0} \varphi(x)\,\mathrm{d}\mu(x,y)\\ &= \sum_{j=1}^{p+q} \int_{\Lambda_j^0} \int_{\Lambda^0\setminus\Lambda_j^0} \varphi(x)\frac{\mathrm{d}\sigma_{\mathbf{o}}(x)\,\mathrm{d}\sigma_{\mathbf{o}}(y)}{\mathrm{d}_{\mathbf{o}}(x,y)^{2\delta_\Gamma/a}}\\ &= \sum_{j=1}^{p+q} \int_{\Lambda_j^0} \varphi(x)h(x)\,\mathrm{d}\sigma_{\mathbf{o}}(x) = \int_{\Lambda^0} h(x)\varphi(x)\,\mathrm{d}\sigma_{\mathbf{o}}(x).\end{aligned}$$

Since the measure $\mu$ is $\Gamma$-invariant, the measure $\mu|_{\mathscr{D}^0}$ is T-invariant on $\mathscr{D}^0$: indeed, the family $(\alpha\Lambda_i \times \beta\Lambda_j)_{i\neq j,\alpha,\beta\in\Gamma}$ is a partition of $\mathscr{D}^0$ and the action of T on $\mathscr{D}^0$ is given by the action of an isometry $\gamma \in \Gamma$ on each atom of this partition. □

REMARK 4.1.11. – *We may extend the function $h$ on $\Lambda_\Gamma$ setting for any $j \in [\![1, p+q]\!]$ and $x \in \overline{\Lambda_j^0}$*

$$h(x) = \int\limits_{\bigcup_{l\neq j}\Lambda_l^0} \frac{\mathrm{d}\sigma_{\mathbf{o}}(y)}{\mathrm{d}_{\mathbf{o}}(x,y)^{2\delta_\Gamma/a}}.$$

## 4.2. Transfer operators

Let us now introduce a family of transfer operators $(\mathscr{L}_z)$ associated to the transformation T. The effectiveness of such operators in the study of hyperbolic flows has already been widely illustrated in the literature, see for instance [**10**], [**38**] or [**46**]. These operators use the non-injectivity of the shift on $\Sigma^+_{\mathscr{A}}$ to describe the dynamics of T on $\Lambda^0$. To define these transfer operators, we will associate a weight to each point $x \in \Lambda^0$ to take into account the number of its antecedents under the action of T. The operators $\mathscr{L}_z$ may thus be seen as transition kernels ruled by the action of inverse branches of T on $\Lambda^0$. Hence, for $n$ large enough, the study of $\mathrm{T}^n$ is strongly related to the behavior of a Markov chain on $\Lambda^0$ (see [**15**] and [**30**]).

We first present the family of transfer operators which we will consider and the spaces on which they act. We then study their relationship with the dynamical system $(\Lambda^0, \mathrm{T}, \nu)$ presented in the previous subsection. Eventually, we study the spectral properties and the regularity of the family of operators.

**4.2.1. Definition and first properties.** – Let us introduce the family $(\mathcal{L}_z)$ of *transfer operators* defined formally for a parameter $z \in \mathbb{C}$ and a function $\varphi : \Lambda_\Gamma \longrightarrow \mathbb{C}$ by

$$\mathcal{L}_z\varphi(x) = \sum_{\mathrm{T}y=x} e^{-z\mathfrak{r}(y)}\varphi(y).$$

These operators are associated with the roof-function $\mathfrak{r}$ defined in (15) and allow us to describe the dynamical system $(\Lambda^0, \mathrm{T}, \nu)$ from an analytic viewpoint. We first check that, for any $z$ with $\mathrm{Re}\,(z) \geqslant \delta_\Gamma$, the operators $\mathcal{L}_z$ act on the space $\mathcal{C}(\Lambda_\Gamma)$ of continuous functions from $\Lambda_\Gamma$ to $\mathbb{C}$ equipped with the norm of uniform convergence $|\cdot|_\infty$. Next, to get some critical gap property for their spectrum, we will consider their restriction to some subspace of $(\mathcal{C}(\Lambda_\Gamma), |\cdot|_\infty)$.

In order to lighten the text, we will denote by $\Lambda$ the limit set $\Lambda_\Gamma$, by $\Lambda_j$ the set $\overline{\Lambda_j^0} = D_j \cap \Lambda_\Gamma$ for any $j \in [\![1, p+q]\!]$ and $\delta = \delta_\Gamma$; similarly, the quantity $b(\gamma, x)$ will stand for $\mathcal{B}_x(\gamma^{-1} \cdot \mathbf{o}, \mathbf{o})$ for any $x \in \partial\mathrm{X}$ and $\gamma \in \Gamma$.

Fix $z \in \mathbb{C}$ and $l \in [\![1, p+q]\!]$. If $x$ belongs to $\Lambda_l^0$, its pre-images by T are the points $y = \alpha \cdot x$ with $\alpha \in \bigcup_{j \neq l} \Gamma_j^*$ and $\mathfrak{r}(y) = b(\alpha, x)$. Consequently for any bounded Borel function $\varphi : \Lambda \longrightarrow \mathbb{C}$

$$\mathcal{L}_z\varphi(x) = \sum_{\substack{\alpha \in \Gamma_j^* \\ j \neq l}} e^{-zb(\alpha,x)}\varphi(\alpha \cdot x) = \sum_{j=1}^{p+q} \mathbb{1}_{\Lambda_j^c}(x) \sum_{\alpha \in \Gamma_j^*} e^{-zb(\alpha,x)}\varphi(\alpha \cdot x). \tag{19}$$

Combining Assumptions ($\mathrm{P}_2$) and (N) with Property 4.1.3, we notice that this quantity is finite for $\mathrm{Re}\,(z) \geqslant \delta$ and defines a continuous function on $\Lambda^0$. Since the convergence of the series appearing in (19) is normal on $\Lambda$ when $\mathrm{Re}\,(z) \geqslant \delta$, the function $\mathcal{L}_z\varphi$ may be continuously extended on $\Lambda$. The operator $\mathcal{L}_\delta$ is positive on $\Lambda$.

LEMMA 4.2.1. – *The function $h$ defined in Proposition 4.1.10 satisfies $\mathcal{L}_\delta h = h$. Moreover, for any $\varphi, \psi$ in $\mathcal{C}(\Lambda)$*

$$\int_\Lambda \varphi(x)\psi(\mathrm{T}x)\sigma_{\mathbf{o}}(\,\mathrm{d}x) = \int_\Lambda \mathcal{L}_\delta\varphi(x)\psi(x)\sigma_{\mathbf{o}}(\,\mathrm{d}x).$$

*In particular, the measure $\sigma_{\mathbf{o}}$ is $\mathcal{L}_\delta$-invariant.*

*Proof.* – We first prove that $\mathscr{L}_\delta h = h$. Let $j \in \{1, \dots, p+q\}$ and $x \in \Lambda_j$. Using (19) and the definition of the function $h$, we obtain

$$\mathscr{L}_\delta h(x) = \sum_{l \neq j} \sum_{\alpha \in \Gamma_l^*} e^{-\delta b(\alpha, x)} h(\alpha \cdot x) = \sum_{l \neq j} \sum_{\alpha \in \Gamma_l^*} e^{-\delta b(\alpha, x)} \int\limits_{\bigcup_{k \neq l} \Lambda_k^0} \frac{\mathrm{d}\sigma_{\mathbf{o}}(y)}{\mathrm{d}_{\mathbf{o}}(\alpha \cdot x, y)^{2\delta/a}}$$

$$= \sum_{l \neq j} \sum_{\alpha \in \Gamma_l^*} e^{-\delta b(\alpha, x)} \int\limits_{\bigcup_{k \neq l} \Lambda_k^0} \frac{\mathrm{d}\sigma_{\mathbf{o}}(\alpha\alpha^{-1} \cdot y)}{\mathrm{d}_{\mathbf{o}}(\alpha \cdot x, \alpha\alpha^{-1} \cdot y)^{2\delta/a}}.$$

From (6), we deduce $\mathrm{d}\sigma_{\mathbf{o}}(\alpha\alpha^{-1} \cdot y) = e^{-\delta \mathscr{B}_{\alpha^{-1} \cdot y}(\alpha^{-1} \cdot \mathbf{o}, \mathbf{o})} \, \mathrm{d}\sigma_{\mathbf{o}}(\alpha^{-1} \cdot y)$ and the equality (5) implies

$$\mathrm{d}_{\mathbf{o}}(\alpha \cdot x, \alpha\alpha^{-1} \cdot y)^{2\delta/a} = e^{-\delta b(\alpha, x)} e^{-\delta \mathscr{B}_{\alpha^{-1} \cdot y}(\alpha^{-1} \cdot \mathbf{o}, \mathbf{o})} \mathrm{d}_{\mathbf{o}}(x, \alpha^{-1} \cdot y)^{2\delta/a}.$$

It thus follows that

$$\mathscr{L}_\delta h(x) = \sum_{l \neq j} \sum_{\alpha \in \Gamma_l^*} \int\limits_{\bigcup_{k \neq l} \Lambda_k^0} \frac{\mathrm{d}\sigma_{\mathbf{o}}(\alpha^{-1} \cdot y)}{\mathrm{d}_{\mathbf{o}}(x, \alpha^{-1} \cdot y)^{2\delta/a}} = \sum_{l \neq j} \int\limits_{\Lambda_l^0} \frac{\mathrm{d}\sigma_{\mathbf{o}}(z)}{\mathrm{d}_{\mathbf{o}}(x, z)^{2\delta/a}}.$$

In conclusion, for any $j \in \{1, \dots, p+q\}$ and any $x \in \Lambda_j$, we have

$$\mathscr{L}_\delta h(x) = \int\limits_{\bigcup_{l \neq j} \Lambda_l^0} \frac{\mathrm{d}\sigma_{\mathbf{o}}(z)}{\mathrm{d}_{\mathbf{o}}(x, z)^{2\delta/a}} = h(x).$$

Now, using Property (6) of the family of measures $(\sigma_{\mathbf{x}})_{\mathbf{x} \in \mathrm{X}}$, we obtain

$$\int\limits_{\Lambda} \varphi(x)\psi(\mathrm{T}x)\sigma_{\mathbf{o}}(\,\mathrm{d}x) = \sum_{j=1}^{p+q} \sum_{\alpha \in \Gamma_j^*} \int\limits_{\alpha \cdot (\Lambda \setminus \Lambda_j)} \varphi(x)\psi(\alpha^{-1} \cdot x)\sigma_{\mathbf{o}}(\,\mathrm{d}x)$$

$$= \sum_{j=1}^{p+q} \sum_{\alpha \in \Gamma_j^*} \int\limits_{\Lambda} \mathbb{1}_{\Lambda_j^c}(y)\varphi(\alpha \cdot y)\psi(y) e^{-\delta \mathscr{B}_y(\alpha^{-1} \cdot \mathbf{o}, \mathbf{o})} \sigma_{\mathbf{o}}(\,\mathrm{d}y)$$

$$= \int\limits_{\Lambda} \mathscr{L}_\delta \varphi(y)\psi(y)\sigma_{\mathbf{o}}(\,\mathrm{d}y).$$

The equality $\sigma_{\mathbf{o}} \mathscr{L}_\delta = \sigma_{\mathbf{o}}$ follows with $\psi = \mathbb{1}_\Lambda$. □

Let us now introduce the normalized operator $P := \frac{1}{h} \mathscr{L}_\delta\,(h\cdot)$. It is a positive Markov operator, (i.e., $P\mathbb{1}_\Lambda = \mathbb{1}_\Lambda$). By the previous lemma, we deduce that $P$ satisfies: for any $\varphi, \psi$ in $\mathscr{C}(\Lambda)$

$$\int\limits_{\Lambda} \varphi(x)\psi(\mathrm{T}x)\nu(\,\mathrm{d}x) = \int\limits_{\Lambda} P\varphi(x)\psi(x)\nu(\,\mathrm{d}x), \tag{20}$$

where $\nu = h\sigma_{\mathbf{o}}$. A similar property will be useful in the proof of Theorem A for an extension $\tilde{P}$ of the operator $P$ defined as follows: for all continuous functions $\varphi : \Lambda \longrightarrow \mathbb{R}$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support, any $x \in \Lambda$ and $t \in \mathbb{R}$

$$\tilde{P}(\varphi \otimes u)(x,t) = \sum_{j=1}^{p+q} \mathbb{1}_{\Lambda_j^c}(x) \sum_{\alpha \in \Gamma_j^*} e^{-\delta b(\alpha,x)} \frac{h(\alpha \cdot x)\varphi(\alpha \cdot x)}{h(x)} u(t + b(\alpha,x)). \tag{21}$$

By density, the operator $\tilde{P}$ extends continuously to the space of continuous maps with compact support on $\Lambda \times \mathbb{R}$.

LEMMA 4.2.2. – *The operator $\tilde{P}$ is the adjoint of the transformation $(x,s) \longmapsto (\mathrm{T}x, s - \mathfrak{r}(x))$ with respect to the measure $\nu \otimes \mathrm{d}s$, i.e., for all continuous maps $\Phi, \Psi$ on $\Lambda \times \mathbb{R}$ with compact support*

$$\int_{\Lambda \times \mathbb{R}} \Phi(x,s)\Psi(\mathrm{T}x, s - \mathfrak{r}(x)) \,\mathrm{d}\nu(x)\,\mathrm{d}s = \int_{\Lambda \times \mathbb{R}} \tilde{P}\Phi(x,s)\Psi(x,s)\,\mathrm{d}\nu(x)\,\mathrm{d}s.$$

Let us introduce the following *weight functions*: for any $z \in \mathbb{C}$ and $\gamma \in \Gamma$, let

$$w_z(\gamma,x) = \begin{cases} 0 & \text{if } x \in \Lambda_{l(\gamma)}, \\ e^{-zb(\gamma,x)} & \text{if } x \in \Lambda_j,\ j \neq l(\gamma). \end{cases}$$

These weight functions satisfy the following cocycle relation: if $\alpha_1, \alpha_2 \in \mathscr{A}$ do not belong to the same group $\Gamma_j$, then

$$w_z(\alpha_1\alpha_2, x) = w_z(\alpha_1, \alpha_2 \cdot x) w_z(\alpha_2, x). \tag{22}$$

This implies that for any $k \geqslant 1$, the $k$-th iterate of $\mathscr{L}_z$ is given by: for any $\varphi \in (\mathcal{C}(\Lambda), |\cdot|_\infty)$, for any $z \in \mathbb{C}$ with $\mathrm{Re}\,(z) \geqslant \delta$ and for any $x \in \Lambda$,

$$\mathscr{L}_z^k \varphi(x) = \sum_{\mathrm{T}^k y = x} e^{-zS_k\mathfrak{r}(y)}\varphi(y) = \sum_{\gamma \in \Gamma(k)} w_z(\gamma,x)\varphi(\gamma \cdot x).$$

We will need to control the regularity of the family of functions $(x \mapsto b(\gamma,x))_{\gamma \in \Gamma}$. The following proposition is proven in [**2**].

PROPOSITION 4.2.3. – *Let $E \subset \partial \mathrm{X}$ and $F \subset \mathrm{X}$ be two sets with disjoint closures in $\mathrm{X} \cup \partial\mathrm{X}$. Then the family $(x \longmapsto \mathscr{B}_x(\mathbf{p},\mathbf{o}))_{\mathbf{p} \in F}$ is equi-Lipschitz continuous on $(E, \mathrm{d}_{\mathbf{o}})$.*

Let us now consider the restriction of the operator $\mathscr{L}_z$ to the subspace of Lipschitz functions from $\Lambda$ to $\mathbb{C}$, defined by

$$\mathrm{Lip}(\Lambda) = \{\varphi \in \mathcal{C}(\Lambda) \mid ||\varphi|| = |\varphi|_\infty + [\varphi] < +\infty\} \subset \mathcal{C}(\Lambda),$$

where $[\varphi] = \sup\limits_{1 \leqslant j \leqslant p+q} \sup\limits_{(x,y) \in \Lambda_j \overset{\Delta}{\times} \Lambda_j} \frac{|\varphi(x) - \varphi(y)|}{\mathrm{d}_{\mathbf{o}}(x,y)}$. The space $(\mathrm{Lip}(\Lambda), ||\cdot||)$ is a $\mathbb{C}$-Banach space; it follows from Ascoli's theorem that the canonical one-to-one map from

$(\mathrm{Lip}(\Lambda), ||\cdot||)$ into $(\mathscr{C}(\Lambda), |\cdot|_\infty)$ is compact. One may readily check that the function $h$ belongs to $\mathrm{Lip}\,(\Lambda)$. The following proposition may be proved as Lemma 2.1 in **[2]**.

PROPOSITION 4.2.4. – *For all $z \in \mathbb{C}$, the weight $w_z(\gamma,\cdot)$ belongs to $(\mathrm{Lip}(\Lambda), ||\cdot||)$ and there exists a constant $C = C(z) > 0$ such that for any $\gamma$ in $\Gamma^*$,*

$$||w_z(\gamma,\cdot)|| \leqslant Ce^{-\mathrm{Re}(z)\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}.$$

This yields the following.

COROLLARY 4.2.5. – *The operator $\mathscr{L}_z$ is bounded on $\mathrm{Lip}(\Lambda)$ when $\mathrm{Re}\,(z) \geqslant \delta$.*

From Lemma 4.2.1 and from Corollary 4.2.5, we deduce that 1 is an eigenvalue of $\mathscr{L}_\delta$ on $\mathrm{Lip}\,(\Lambda)$. We need more information about the spectrum of $\mathscr{L}_\delta$ on this space. Since the operator $\mathscr{L}_\delta$ is positive, the spectral radius $\rho_\infty(\delta)$ of $\mathscr{L}_\delta$ on $(\mathscr{C}(\Lambda), |\cdot|_\infty)$ is given by

$$\rho_\infty(\delta) = \limsup_{n\longrightarrow+\infty} \left|\mathscr{L}_\delta^n \mathbb{1}_\Lambda\right|_\infty^{1/n}.$$

Since the function $h$ is continuous and positive on $\Lambda$, we have

$$\left|\mathscr{L}_\delta^n \mathbb{1}_\Lambda\right|_\infty \asymp \left|\mathscr{L}_\delta^n h\right|_\infty = |h|_\infty,$$

hence $\rho_\infty(\delta) = 1$. Denote now by $\rho(\delta)$ the spectral radius of $\mathscr{L}_\delta$ on $\mathrm{Lip}\,(\Lambda)$. The following proposition gives more details about the spectrum of $\mathscr{L}_\delta$ on $\mathrm{Lip}\,(\Lambda)$. Its proof relies on the notion of quasi-compacity.

DEFINITION 4.2.6. – *Let $(\mathscr{B}, ||\cdot||)$ be a Banach space and $Q$ a bounded operator on $\mathscr{B}$ with spectral radius $\rho(Q)$. The operator $Q$ is said to be quasi-compact if $\mathscr{B}$ may be split into $Q$-stable subspaces $\mathscr{B} = F \oplus H$, where $F$ has finite dimension and $Q|_F$ admits only eigenvalues of modulus $\rho(Q)$, whereas $\rho\left(Q|_H\right) < \rho(Q)$.*

This notion is stable under small perturbation (**[29]**): we will use this fact in the sequel.

PROPOSITION 4.2.7. – *The operator $\mathscr{L}_\delta$ is quasi-compact. The spectral radius $\rho(\delta)$ is a simple and isolated eigenvalue in the spectrum of $\mathscr{L}_\delta$, satisfying $\rho(\delta) = \rho_\infty(\delta) = 1$; this is the unique eigenvalue with modulus $\rho(\delta)$. Moreover, the rest of the spectrum of $\mathscr{L}_\delta$ is included in a disk of radius $< 1$.*

The proof of this proposition is identical to the proof of Proposition III.4 in **[3]**. We will give a full proof of an analogous proposition for an extension of these transfer operators in Chapter 8: see Proposition 8.3.2. We can thus write

$$\mathscr{L}_\delta = \Pi_\delta + R_\delta,$$

where $\Pi_\delta\ :\ \mathrm{Lip}\,(\Lambda) \longrightarrow \mathbb{C}h$ is the spectral projection on the eigenspace associated to 1 and $R_\delta = \mathscr{L}_\delta - \Pi_\delta$ satisfies $\Pi_\delta R_\delta = R_\delta \Pi_\delta = 0$ and has a spectral radius $< 1$.

There thus exists a linear form $\sigma_\delta : \mathrm{Lip}(\Lambda) \longrightarrow \mathbb{C}$ such that $\Pi_\delta(\cdot) = \sigma_\delta(\cdot)h$. It follows that $\Pi_\delta(\mathscr{L}_\delta\varphi) = \sigma_\delta(\mathscr{L}_\delta\varphi)h$ for any $\varphi \in \mathrm{Lip}(\Lambda)$ on the one hand and $\Pi_\delta(\mathscr{L}_\delta\varphi) = \mathscr{L}_\delta\Pi_\delta(\varphi) = \sigma_\delta(\varphi)h$ on the other hand, which implies that the measure $\sigma_\delta$ is $\mathscr{L}_\delta$-invariant.

REMARK 4.2.8. – *The measure $\sigma_\delta$ corresponds to the Patterson measure $\sigma_{\mathbf{o}}$. Indeed, for any $\varphi \in \mathrm{Lip}(\Lambda)$ and $k \geqslant 1$,*

$$\sigma_{\mathbf{o}}(\varphi) = \sigma_{\mathbf{o}}\left(\mathscr{L}_\delta^k\varphi\right) = \sigma_{\mathbf{o}}(\sigma_\delta(\varphi)h) + \sigma_{\mathbf{o}}\left(R_\delta^k\varphi\right).$$

*By definition of $h$, one gets*

$$\sigma_{\mathbf{o}}(\varphi) = \sigma_\delta(\varphi) + \sigma_{\mathbf{o}}\left(R_\delta^k\varphi\right) \longrightarrow \sigma_\delta(\varphi) \ \text{as } k \longrightarrow +\infty.$$

*The remark thus follows from the density of the space* $\mathrm{Lip}(\Lambda)$ *in* $\mathrm{L}^1(\Lambda)$.

**4.2.2. Study of perturbations of $\mathscr{L}_\delta$. –** In this subsection, we extend the previous spectral gap property to small perturbations of $\mathscr{L}_\delta$ given by $z \longmapsto \mathscr{L}_z$ for $z \in \mathbb{C}$ with $\mathrm{Re}(z) \geqslant \delta$. We first prove the following.

PROPOSITION 4.2.9. – *Under assumptions $(H_\beta)$, for any compact subset $K$ of $\mathbb{R}$, there exists a constant $C = C_K > 0$ such that for any $s, t \in K$ and $\kappa$ small enough*

1) *if $\beta \in ]0,1[$*

   a. $||\mathscr{L}_{\delta+it} - \mathscr{L}_{\delta+is}|| \leqslant C|s-t|^\beta L\left(\frac{1}{|s-t|}\right)$;

   b. $||\mathscr{L}_{\delta+\kappa+it} - \mathscr{L}_{\delta+it}|| \leqslant C\kappa^\beta L\left(\frac{1}{\kappa}\right)$;

2) *if $\beta = 1$*

   a. $||\mathscr{L}_{\delta+it} - \mathscr{L}_{\delta+is}|| \leqslant C|s-t|\tilde{L}\left(\frac{1}{|s-t|}\right)$;

   b. $||\mathscr{L}_{\delta+\kappa+it} - \mathscr{L}_{\delta+it}|| \leqslant C\kappa\tilde{L}\left(\frac{1}{\kappa}\right)$,

   *where* $\tilde{L}(x) = \int_1^x \frac{L(y)}{y}\,\mathrm{d}y$.

*Proof.* – We only give the proof of assertion 1.a, the arguments being similar for the others. Let $\varphi \in \mathrm{Lip}(\Lambda)$ : it is sufficient to check that

$$||\mathscr{L}_{\delta+it}\varphi - \mathscr{L}_{\delta+is}\varphi|| \leqslant C|s-t|^\beta L\left(\frac{1}{|s-t|}\right)||\varphi||.$$

Proposition 3.2.3 is the key point in obtaining such estimates. For any $1 \leqslant j \leqslant p+q$ and $x \in \Lambda \backslash \Lambda_j$, we define the following measure

$$\mu_j^x = \frac{1}{M_j(x)} \sum_{\alpha \in \Gamma_j^*} e^{-\delta b(\alpha,x)} \mathrm{D}_{b(\alpha,x)},$$

where $\mathrm{D}_{b(\alpha,x)}$ is the Dirac mass at $b(\alpha,x)$ and $M_j(x) := \sum_{\alpha \in \Gamma_j^*} e^{-\delta b(\alpha,x)}$. These measures are supported on $[-C,+\infty[$ where $C > 0$ is the constant appearing in Property 4.1.3. We also deduce from this property that

$$e^{-\delta C} \sum_{\alpha \in \Gamma_j^*} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})} \leqslant M_j(x) \leqslant e^{\delta C} \sum_{\alpha \in \Gamma_j^*} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})},$$

so that

$$\frac{e^{-2\delta C}}{\sum_{\alpha \in \Gamma_j^*} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}} \sum_{\substack{\alpha \in \Gamma_j^* \\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o}) > T+C}} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})} \leqslant 1 - \mu_j^x\left([-C,T]\right)$$

and

$$1 - \mu_j^x\left([-C,T]\right) \leqslant \frac{e^{2\delta C}}{\sum_{\alpha \in \Gamma_j^*} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}} \sum_{\substack{\alpha \in \Gamma_j \\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o}) > T-C}} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}.$$

Hence for $T$ large enough, one gets

$$\frac{e^{-2\delta C}}{\sum_{\alpha \in \Gamma_j^*} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}} \frac{L(T)}{T^\beta} \leqslant 1 - \mu_j^x\left([-C,T]\right) \leqslant \frac{e^{2\delta C}}{\sum_{\alpha \in \Gamma_j^*} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}} \frac{L(T)}{T^\beta}$$

for $j \in [\![1,p]\!]$ and

$$1 - \mu_j^x\left([-C,T]\right) = o_j(T) \frac{L(T)}{T^\beta}$$

for $j \in [\![p+1,p+q]\!]$. From Assertion 1.$a$) of Proposition 3.2.3, we deduce that

$$\int_0^{+\infty} \left|e^{ity} - 1\right| \mu_j^x(\mathrm{d}y) \preceq |t|^\beta L\left(\frac{1}{|t|}\right)$$

for $j \in [\![1,p]\!]$, whereas for $j \in [\![p+1,p+q]\!]$, we have

$$\int_0^{+\infty} \left|e^{ity} - 1\right| \mu_j^x(\mathrm{d}y) = o\left(|t|^\beta L\left(\frac{1}{|t|}\right)\right),$$

uniformly in $x \in \Lambda \backslash \Lambda_j$. These estimates may also be written as

$$\sum_{\alpha \in \Gamma_j^*} \left|e^{itb(\alpha,x)} - 1\right| e^{-\delta b(\alpha,x)} \asymp |t|^\beta L\left(\frac{1}{|t|}\right) \tag{23}$$

for $j \in [\![1,p]\!]$ and

$$\sum_{\alpha \in \Gamma_j^*} \left|e^{itb(\alpha,x)} - 1\right| e^{-\delta b(\alpha,x)} = |t|^\beta L\left(\frac{1}{|t|}\right) o\left(\frac{1}{|t|}\right) \tag{24}$$

for $j \in [\![p+1, p+q]\!]$. Therefore, for any $j \in [\![1, p+q]\!]$ and $x \in \Lambda \backslash \Lambda_j$

$$\sum_{\alpha \in \Gamma_j^*} |w_{\delta+it}(\alpha,x)\varphi(\alpha\cdot x) - w_{\delta+is}(\alpha,x)\varphi(\alpha\cdot x)| \leqslant \left(\sum_{\alpha\in\Gamma_j} \left|e^{i(t-s)b(\alpha,x)} - 1\right| e^{-\delta b(\alpha,x)}\right) |\varphi|_\infty$$
$$\preceq |t-s|^\beta L\left(\frac{1}{|t-s|}\right) ||\varphi||,$$

which finally implies

$$|\mathscr{L}_{\delta+it}\varphi - \mathscr{L}_{\delta+is}\varphi|_\infty \preceq |t-s|^\beta L\left(\frac{1}{|t-s|}\right)||\varphi||.$$

In order to control the Lipschitz coefficient of the function $x \longmapsto \mathscr{L}_{\delta+it}\varphi(x) - \mathscr{L}_{\delta+is}\varphi(x)$, we first notice that

$$[\mathscr{L}_{\delta+it}\varphi - \mathscr{L}_{\delta+is}\varphi] \leqslant \sup_{j\in[\![1,p+q]\!]} \sup_{(x,y)\in\Lambda_j \overset{\Delta}{\times} \Lambda_j} \sum_{\substack{\alpha\in\Gamma_l \\ l\neq j}} \frac{A_j(l,\alpha,x,y)}{\mathrm{d}_{\mathbf{o}}(x,y)}, \tag{25}$$

where

$$A_j(l,\alpha,x,y) := |w_{\delta+it}(\alpha,x)\varphi(\alpha\cdot x) - w_{\delta+is}(\alpha,x)\varphi(\alpha\cdot x)$$
$$- (w_{\delta+it}(\alpha,y)\varphi(\alpha\cdot y) - w_{\delta+is}(\alpha,y)\varphi(\alpha\cdot y))|,$$

for any $j, l \in [\![1, p+q]\!]$, $l \neq j$, any $(x,y) \in \Lambda_j \overset{\Delta}{\times} \Lambda_j$ and $\alpha \in \Gamma_l$. We observe that

$$A_j(l,\alpha,x,y) \leqslant |w_{\delta+it}(\alpha,x) - w_{\delta+is}(\alpha,x) - (w_{\delta+it}(\alpha,y) - w_{\delta+is}(\alpha,y))|\,|\varphi(\alpha\cdot x)|$$
$$+ |w_{\delta+it}(\alpha,y) - w_{\delta+is}(\alpha,y)|\,|\varphi(\alpha\cdot y) - \varphi(\alpha\cdot x)|$$
$$\leqslant B_j(l,\alpha,x,y) + C_j(l,\alpha,x,y),$$

where

$$B_j(l,\alpha,x,y) = |w_{\delta+it}(\alpha,x) - w_{\delta+is}(\alpha,x) - (w_{\delta+it}(\alpha,y) - w_{\delta+is}(\alpha,y))|\,|\varphi(\alpha\cdot x)|$$

and

$$C_j(l,\alpha,x,y) = |w_{\delta+it}(\alpha,y) - w_{\delta+is}(\alpha,y)|\,|\varphi(\alpha\cdot y) - \varphi(\alpha\cdot x)|.$$

On the one hand

$$\begin{aligned} B_j(l,\alpha,x,y) &\leqslant e^{-\delta b(\alpha,x)}\left|e^{i(t-s)b(\alpha,x)}-e^{i(t-s)b(\alpha,y)}\right| ||\varphi|| \\ &\quad + \left|e^{-(\delta+it)b(\alpha,x)}-e^{-(\delta+it)b(\alpha,y)}\right|\left|e^{i(t-s)b(\alpha,y)}-1\right| ||\varphi|| \\ &\leqslant e^{-\delta b(\alpha,x)}\left|e^{i(t-s)(b(\alpha,x)-b(\alpha,y))}-1\right| ||\varphi|| \\ &\quad + e^{-\delta b(\alpha,y)}\left|e^{(\delta+it)(b(\alpha,y)-b(\alpha,x))}-1\right|\left|e^{i(t-s)b(\alpha,y)}-1\right| ||\varphi|| \\ &\preceq \left(e^{-\delta b(\alpha,x)}\,[b(\alpha,\cdot)]\,|t-s|\mathrm{d}_{\mathbf{o}}(x,y)\right) ||\varphi|| \\ &\quad + \left(e^{-\delta b(\alpha,y)}|\delta+it|\,[b(\alpha,\cdot)]\left|e^{i(t-s)b(\alpha,y)}-1\right|\mathrm{d}_{\mathbf{o}}(x,y)\right) ||\varphi||. \end{aligned}$$

Since the sequence $([b(\gamma,\cdot)])_{\gamma\in\Gamma_j}$ is bounded and $t\in K$, we deduce that

$$\frac{B_j(l,\alpha,x,y)}{\mathrm{d}_{\mathbf{o}}(x,y)} \preceq \left(e^{-\delta b(\alpha,x)}|t-s|\right)||\varphi|| + \left(e^{-\delta b(\alpha,y)}\left|e^{i(t-s)b(\alpha,y)}-1\right|\right)||\varphi||.$$

On the other hand

$$\frac{C_j(l,\alpha,x,y)}{\mathrm{d}_{\mathbf{o}}(x,y)} \preceq e^{-\delta b(\alpha,y)}\left|e^{i(t-s)b(\alpha,y)}-1\right| ||\varphi||,$$

so that for any $j\in[\![1,p+q]\!]$ and $x,y\in\Lambda_j$, one gets

$$\begin{aligned} \sum_{\alpha\in\Gamma_l,l\neq j}\frac{A_j(l,\alpha,x,y)}{\mathrm{d}_{\mathbf{o}}(x,y)} &\leqslant \sum_{\alpha\in\Gamma_l,l\neq j}\frac{B_j(l,\alpha,x,y)+C_j(l,\alpha,x,y)}{\mathrm{d}_{\mathbf{o}}(x,y)} \\ &\preceq \sum_{\alpha\in\Gamma_l,l\neq j}\left(e^{-\delta b(\alpha,x)}|t-s| + e^{-\delta b(\alpha,y)}\left|e^{i(t-s)b(\alpha,y)}-1\right|\right)||\varphi||. \end{aligned}$$

We deduce from (23) and (24) that

$$\sum_{\alpha\in\Gamma_l,l\neq j}\frac{A_j(l,\alpha,x,y)}{\mathrm{d}_{\mathbf{o}}(x,y)} \preceq |t-s|^\beta L\left(\frac{1}{|t-s|}\right)||\varphi||$$

and this last estimate combined with (25) yields

$$[\mathscr{L}_{\delta+it}\mathbb{1}_\Lambda - \mathscr{L}_{\delta+is}\mathbb{1}_\Lambda] \preceq |t-s|^\beta L\left(\frac{1}{|t-s|}\right)||\varphi||. \qquad \square$$

Combining the previous proposition with the proof of Proposition 2.2 in **[2]**, we deduce the following corollary.

Corollary 4.2.10. – *The map $z\longmapsto\mathscr{L}_z$ is continuous on $\{z\in\mathbb{C}\mid \mathrm{Re}\,(z)\geqslant\delta\}$.*

Let us now show the existence of a simple dominant eigenvalue for $\mathscr{L}_z$ and isolated in its spectrum. Denote by $\rho(z)$ the spectral radius of $\mathscr{L}_z$ on Lip $(\Lambda)$ and by $|x+iy|_\infty = \max(|x|,|y|)$ for any $x,y\in\mathbb{R}$.

PROPOSITION 4.2.11 ([2] p.92). – *There exist* $\varepsilon > 0$ *and* $\rho_\varepsilon \in ]0,1[$ *such that for all* $z \in \mathbb{C}$ *satisfying* $|z-\delta|_\infty < \varepsilon$ *and* $\operatorname{Re}(z) \geqslant \delta$*, one gets:*

- $\rho(z) > \rho_\varepsilon$;
- $\mathcal{L}_z$ *has a unique eigenvalue* $\lambda_z$ *with modulus* $\rho(z)$;
- *this eigenvalue is simple and* $\lim_{z \longrightarrow \delta} \lambda_z = 1$;
- *the rest of the spectrum is included in a disk of radius* $\rho_\varepsilon$.

*Furthermore for any* $A > 0$*, there exists* $\rho_A < 1$ *such that* $\rho(z) < \rho_A$ *as soon as* $z \in \mathbb{C}$ *satisfies* $|z-\delta|_\infty \geqslant \varepsilon$, $\operatorname{Re}(z) \geqslant \delta$ *and* $|\operatorname{Im}(z)| \leqslant A$*. Finally, if* $z \in \mathbb{C}$ *satisfies* $\operatorname{Re}(z) \geqslant \delta$*, then* $\rho(z) \leqslant 1$ *with equality if and only if* $z = \delta$.

REMARK 4.2.12. – *In the proof of Propositions A.1, A.2, B.1 and C.1, we will use Potter's lemma and thus choose* $\varepsilon > 0$ *small enough in such a way that*

$$\frac{L(x)}{L(y)} \leqslant \max\left(\frac{y}{x}, \frac{x}{y}\right)^{\beta/2}$$

*for any* $x, y \geqslant 1/\varepsilon$.

We denote by $h_z$ the unique eigenfunction of $\mathcal{L}_z$ associated to $\lambda_z$ satisfying $\sigma_{\mathbf{o}}(h_z) = 1$; let $\Pi_z : \operatorname{Lip}(\Lambda) \longrightarrow \mathbb{C}h_z$ denote the spectral projection associated to $\lambda_z$. There exists a unique linear form $\sigma_z : \operatorname{Lip}(\Lambda) \longrightarrow \mathbb{C}$ such that $\Pi_z(\cdot) = \sigma_z(\cdot)h_z$ and $\sigma_z(h_z) = 1$. We set $R_z := \mathcal{L}_z - \Pi_z$. By perturbation theory, the maps $z \longmapsto \lambda_z$, $z \longmapsto h_z$ and $z \longmapsto \Pi_z$ have the same regularity as $z \longmapsto \mathcal{L}_z$ (see the proof of Lemma 2.5 in [2]).

**4.2.3. Regularity of the dominant eigenvalue. –** In proof of Theorems A and B, we will have to deal with quantities like

$$\sum_{\mathrm{T}^k y = x} e^{-\delta S_k \mathfrak{r}(y)} \varphi(S_k \mathfrak{r}(y) - R), \tag{26}$$

for functions $\varphi$ with a $\mathcal{C}^\infty$ Fourier transform and with compact support. The inverse Fourier transform leads us to write

$$\sum_{\mathrm{T}^k y = x} e^{-\delta S_k \mathfrak{r}(y)} \varphi(S_k \mathfrak{r}(y) - R) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itR} \mathcal{L}^k_{\delta+it} \left(\mathbb{1}_\Lambda(x)\right) \widehat{\varphi}(t) \, \mathrm{d}t.$$

The study of the behavior as $R \longrightarrow +\infty$ of the quantity (26) thus involves a precise knowledge of the behavior of the function $t \longmapsto \lambda_{\delta+it}$ in a neighborhood of 0. We first establish a result concerning some probability measures depending on the Schottky factors $\Gamma_i$, $1 \leqslant i \leqslant p+q$. In the proof, we will need the following result obtained in [21].

PROPOSITION 4.2.13 ([**21**]). – *Let $\nu$ be a probability measure on $\mathbb{R}^+$ such that there exist $\beta \in ]0,1]$ and a slowly varying function $L$ satisfying $\nu([T,+\infty[) \sim CL(T)/T^\beta$ as $T \longrightarrow +\infty$. Then the characteristic function $\widehat{\nu}(t) := \int_0^{+\infty} e^{itx}\,\mathrm{d}\nu(x)$ has the following behavior in a neighborhood of $0$:*

– *if $\beta \in ]0,1[$*

$$\widehat{\nu}(t) = 1 - Ce^{-i\mathrm{sign}(t)\beta\pi/2}\Gamma(1-\beta)|t|^\beta L\left(\frac{1}{|t|}\right)(1+o(1));$$

– *if $\beta = 1$*

$$* \ \widehat{\nu}(t) = 1 + iC|t|\tilde{L}\left(\frac{1}{|t|}\right)(1+o(1));$$

$$* \ \mathrm{Re}\,(1-\widehat{\nu}(t)) = \frac{\pi}{2}C|t|L\left(\frac{1}{|t|}\right)(1+o(1)),$$

*where $\tilde{L}(t) = \int_1^t \frac{L(x)}{x}\,\mathrm{d}x$.*

In our setting, we obtain the following local expansions.

PROPOSITION 4.2.14. – *Let $j \in [\![1,p+q]\!]$ and $x \in \Lambda\backslash\Lambda_j$. Denote by $N_j(x) = \sum_{\alpha\in\Gamma_j^*} h(\alpha\cdot x)e^{-\delta b(\alpha,x)}$ and let us introduce the following probability measure $\nu_j^x$ on $[-C,+\infty[$*

$$\nu_j^x := \frac{1}{N_j(x)}\sum_{\alpha\in\Gamma_j^*} h(\alpha\cdot x)e^{-\delta b(\alpha,x)}D_{b(\alpha,x)},$$

*where $D_{b(\alpha,x)}$ is the Dirac mass at $b(\alpha,x) \in \mathbb{R}$. This measure satisfies one of the following two assertions.*

1) *For $j \in [\![1,p]\!]$ and $t \longrightarrow 0$,*
   – *if $\beta \in ]0,1[$*

$$\widehat{\nu_j^x}(t) = 1 - \left(\frac{C_j}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}\right)e^{-i\mathrm{sign}(t)\beta\pi/2}\Gamma(1-\beta)|t|^\beta L\left(\frac{1}{|t|}\right)(1+o(1));$$

   – *if $\beta = 1$*

$$* \ \widehat{\nu_j^x}(t) = 1 + \left(\frac{C_j}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}\right)i|t|\tilde{L}\left(\frac{1}{|t|}\right)(1+o(1));$$

$$* \ \mathrm{Re}\left(1-\widehat{\nu_j^x}(t)\right) = \frac{\pi}{2}\left(\frac{C_j}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}\right)|t|L\left(\frac{1}{|t|}\right)(1+o(1)),$$

   *where $x_j$ is the fixed point of the parabolic group $\Gamma_j$ and the $(C_j)_{1\leqslant j\leqslant p}$ are the constants appearing in Assumption ($P_2$);*

2) *For $j \in [\![p+1,p+q]\!]$, there exists a function $f_j : \Lambda \longrightarrow \mathbb{R}$ satisfying $\sigma_{\mathbf{o}}(\mathbb{1}_{\Lambda_j^c}f_j) < +\infty$ such that for any $t \longrightarrow 0$, one has*

– *if* $\beta \in ]0,1[$

$$\widehat{\nu_j^x}(t) = 1 - f_j(x) o\left(|t|^\beta L\left(\frac{1}{|t|}\right)\right);$$

– *if* $\beta = 1$

$$* \widehat{\nu_j^x}(t) = 1 + f_j(x) o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right)\right);$$

$$* \operatorname{Re}\left(1 - \widehat{\nu_j^x}(t)\right) = f_j(x) o\left(|t| L\left(\frac{1}{|t|}\right)\right).$$

*Proof.* – We just give the proof when $\beta \in ]0,1[$. We first consider $j \in [\![1,p]\!]$. Fix $x \in \Lambda \backslash \Lambda_j$. By Property 4.1.3, there exist two constants $m, M > 0$ such that

$$m \sum_{\alpha \in \Gamma_j} e^{-\delta \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})} \leqslant N_j(x) \leqslant M \sum_{\alpha \in \Gamma_j} e^{-\delta \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})}. \tag{27}$$

On the other hand, Assumption ($P_2$) gives

$$\sum_{\substack{\alpha \in \Gamma_j \\ \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) > T}} e^{-\delta \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})} \sim C_j \frac{L(T)}{T^\beta}.$$

We want to show that the distribution function $F_j^x$ of $\nu_j^x$ satisfies

$$1 - F_j^x(T) \sim \left(\frac{C_j}{N_j(x)} h(x_j) e^{2\delta (x_j|x)_\mathbf{o}}\right) \frac{L(T)}{T^\beta} \text{ uniformly in } x \notin \Lambda_j. \tag{28}$$

Fix $\varepsilon > 0$. There exists $T_0 \gg 1$ such that for any $T \geqslant T_0$, $x \in \Lambda \backslash \Lambda_j$ and $\alpha \in \Gamma_j$ satisfying $\mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) > T$

i) $-\varepsilon \leqslant b(\alpha, x) - \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) + 2\,(x_j|x)_\mathbf{o} \leqslant \varepsilon$ (Lemma 6.7 in **[18]**);

ii) $(1-\varepsilon) h(x_j) \leqslant h(\alpha\cdot x) \leqslant (1+\varepsilon) h(x_j)$;

iii) $(1-\varepsilon) C_j \dfrac{L(T)}{T^\beta} \leqslant \displaystyle\sum_{\substack{\alpha \in \Gamma_j \\ \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) > T}} e^{-\delta \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})} \leqslant (1+\varepsilon) C_j \dfrac{L(T)}{T^\beta}$;

hence

$$(1-\varepsilon)^2 e^{-\delta\varepsilon} \frac{C_j}{N_j(x)} h(x_j) e^{2\delta (x_j|x)_\mathbf{o}} \frac{L\left(T + \varepsilon + 2\,(x_j|x)_\mathbf{o}\right)}{\left(T + \varepsilon + 2\,(x_j|x)_\mathbf{o}\right)^\beta} \leqslant 1 - F_j^x(T),$$

and

$$1 - F_j^x(T) \leqslant (1+\varepsilon)^2 e^{\delta\varepsilon} \frac{C_j}{N_j(x)} h(x_j) e^{2\delta (x_j|x)_\mathbf{o}} \frac{L\left(T - \varepsilon + 2\,(x_j|x)_\mathbf{o}\right)}{\left(T - \varepsilon + 2\,(x_j|x)_\mathbf{o}\right)^\beta}.$$

Since $(x_j|x)_\mathbf{o} \asymp \mathrm{d}(\mathbf{o}, (x_j x))$, this quantity is bounded uniformly in $x \in \Lambda \backslash \Lambda_j$; moreover, the functions

$$T \longmapsto \frac{L(T+t)}{L(T)} \text{ and } T \longmapsto \frac{T+t}{T}$$

tend to 1 uniformly when $t$ lives in any compact subset of $\mathbb{R}$. Hence

$$(1-\varepsilon)^3 e^{-\delta\varepsilon}\left(\frac{C_j}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}\right)\frac{L(T)}{T^\beta} \leqslant 1-F_j^x(T)$$

and

$$1-F_j^x(T) \leqslant (1+\varepsilon)^3 e^{\delta\varepsilon}\left(\frac{C_j}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}\right)\frac{L(T)}{T^\beta},$$

for $T$ large enough. Therefore (28) is true for $j \in [\![1,p]\!]$. The measures $\nu_j^x$, $1 \leqslant j \leqslant p$, satisfy (28); Assertion 1) of Proposition 4.2.14 thus follows from Proposition 4.2.13.

Now fix $j \in [\![p+1,p+q]\!]$. When $\Gamma_j$ is parabolic, the previous arguments still work, but Assumption (N) imposes

$$\sum_{\substack{\alpha\in\Gamma_j\\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})>T}} e^{-\delta \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})} = o\left(\frac{L(T)}{T^\beta}\right),$$

which implies

$$1-F_j^x(T) = \left(\frac{1}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}\right)\frac{L(T)}{T^\beta}o(T)$$

with $\lim_{T\longrightarrow+\infty} o(T) = 0$ uniformly in $x \notin \Lambda_j$. The second part of the result follows from **[21]**, with $f_j(x)$ given in that case by

$$f_j(x) = \frac{1}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}.$$

Inequality (27) yields

$$\begin{aligned}\sigma_{\mathbf{o}}\left(\mathbb{1}_{\Lambda_j^c}f_j\right) &= \int_{\Lambda\setminus\Lambda_j}\frac{1}{N_j(x)}h(x_j)e^{2\delta(x_j|x)_{\mathbf{o}}}\,\mathrm{d}\sigma_{\mathbf{o}}(x)\\ &\preceq \frac{1}{\sum_{\alpha\in\Gamma_j}e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}}\int_{\Lambda\setminus\Lambda_j}e^{2\delta(x_j|x)_{\mathbf{o}}}\,\mathrm{d}\sigma_{\mathbf{o}}(x)\\ &\preceq \int_{\Lambda\setminus\Lambda_j}\frac{\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(x_j,x)^{2\delta/a}} < +\infty.\end{aligned}$$

When $\Gamma_j$ is generated by an hyperbolic isometry, with attractive (resp. repulsive) fixed point $x_j^+$ (resp. $x_j^-$), we write

$$\sum_{\substack{\alpha\in\Gamma_j\\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})>T}} e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})} \asymp \sum_{\substack{n\in\mathbb{Z}^*\\ |n|l_j\geq T}} e^{-\delta l_j n} \asymp e^{-\delta T},$$

where $l_j$ is the length of the axis of the generator of $\Gamma_j$. The arguments are the same as for the non-influential parabolics. In that case, the function $f_j(x)$ is given by

$$f_j(x) = \frac{1}{2N_j(x)}\left(h(x_j^+)e^{2\delta\left(x_j^+|x\right)_{\mathbf{o}}} + h(x_j^-)e^{2\delta\left(x_j^-|x\right)_{\mathbf{o}}}\right).$$

The quantity $\sigma_{\mathbf{o}}\left(\mathbb{1}_{\Lambda_j^c} f_j\right)$ is finite for the same reasons. This ends the proof of Proposition 4.2.14. □

The following proposition specifies the local behavior in 0 of the function $t \longmapsto \lambda_{\delta+it}$.

PROPOSITION 4.2.15. – *There exists a constant $E_\Gamma > 0$ such that for any $t$ small enough*

– *if* $\beta \in ]0,1[$

$$\lambda_{\delta+it} = 1 - E_\Gamma \Gamma(1-\beta) e^{+i\mathrm{sign}(t)\beta\pi/2} |t|^\beta L\left(\frac{1}{|t|}\right)(1+o(1));$$

– *if* $\beta = 1$

$$* \ \lambda_{\delta+it} = 1 - E_\Gamma \mathrm{sign}(t) i |t| \tilde{L}\left(\frac{1}{|t|}\right)(1+o(1));$$

$$* \ \mathrm{Re}\,(1-\lambda_{\delta+it}) = \frac{\pi}{2} E_\Gamma |t| L\left(\frac{1}{|t|}\right)(1+o(1)).$$

*Proof.* – As previously, we only detail the proof for $\beta \in ]0,1[$. We first write

$$\lambda_{\delta+it} = \sigma_{\mathbf{o}}(\mathscr{L}_{\delta+it} h_{\delta+it}) = \sigma_{\mathbf{o}}(\mathscr{L}_{\delta+it} h) + \sigma_{\mathbf{o}}\left((\mathscr{L}_{\delta+it} - \mathscr{L}_\delta)(h_{\delta+it} - h)\right).$$

By Proposition 4.2.9, the second term of the right hand side is bounded from above by $\sigma_{\mathbf{o}}(\Lambda)\left(|t|^\beta L\left(\frac{1}{|t|}\right)\right)^2$ and it remains to specify the behavior of the first one near 0. We write

$$\begin{aligned}\sigma_{\mathbf{o}}(\mathscr{L}_{\delta+it} h) &= 1 + \sigma_{\mathbf{o}}(\mathscr{L}_{\delta+it} h) - 1 = 1 + \sigma_{\mathbf{o}}(\mathscr{L}_{\delta+it} h) - \sigma_{\mathbf{o}}(h) \\ &= 1 + \sigma_{\mathbf{o}}\left((\mathscr{L}_{\delta+it} - \mathscr{L}_\delta) h\right) = 1 + \sum_{j=1}^{p+q} S_j,\end{aligned}$$

where

$$\begin{aligned}S_j :&= \sum_{\alpha \in \Gamma_j^*} \int_{\Lambda \setminus \Lambda_j} h(\alpha \cdot x) e^{-\delta b(\alpha,x)} (e^{-itb(\alpha,x)} - 1)\, \mathrm{d}\sigma_{\mathbf{o}}(x) \\ &= \int_{\Lambda \setminus \Lambda_j} N_j(x) \left(\widehat{\nu_j^x}(-t) - 1\right) \mathrm{d}\sigma_{\mathbf{o}}(x),\end{aligned}$$

for $j \in [\![1, p+q]\!]$. If follows from 4.2.14 that $S_j = o\left(|t|^\beta L\left(\frac{1}{|t|}\right)\right)$ for $j \in [\![p+1, p+q]\!]$, whereas for $j \in [\![1, p]\!]$

$$S_j = -C_j h(x_j)\left(\int_{\Lambda\setminus\Lambda_j} e^{2\delta(x_j|x)_{\mathbf{o}}}\,\mathrm{d}\sigma_{\mathbf{o}}(x)\right) e^{i\mathrm{sign}(t)\beta\pi/2}\Gamma(1-\beta)|t|^\beta L\left(\frac{1}{|t|}\right)(1+o(1))$$

$$= -C_j\left(\int_{\Lambda\setminus\Lambda_j}\frac{\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(x,x_j)^{2\delta/a}}\right)^2 e^{i\mathrm{sign}(t)\beta\pi/2}\Gamma(1-\beta)|t|^\beta L\left(\frac{1}{|t|}\right)(1+o(1)).$$

The result follows with the constant $E_\Gamma$ given by

$$E_\Gamma = \sum_{1\leqslant j\leqslant p} C_j\left(\int_{\Lambda\setminus\Lambda_j}\frac{\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(x,x_j)^{2\delta/a}}\right)^2 \square$$

**4.2.4. The resolvant operator when $\beta = 1$. –** In the proof of Theorem A for $\beta = 1$, we will use the operator $Q_z = (\mathrm{Id} - \mathscr{L}_z)^{-1}$ for $z \in \mathbb{C}$ such that $\mathrm{Re}\,(z) \geqslant \delta$. The following properties come from **[36]**.

PROPOSITION 4.2.16. – *There exist $\varepsilon > 0$ and $C > 0$ such that $\|Q_z - (1-\lambda_z)^{-1}\Pi_z\| \leqslant C$ when $|z-\delta|_\infty < \varepsilon$ and $\|Q_z\| \leqslant C$ for $z$ such that $|z-\delta|_\infty \geqslant \varepsilon$. Moreover, for any $t$ close enough to* 0

$$Q_{\delta+it} = \frac{1}{E_\Gamma \mathrm{sign}(t) i|t|\tilde{L}\left(\frac{1}{|t|}\right)}(1+o(1))\Pi_0 + O(1).$$

*Proof*. – Let $z \in \mathbb{C}$ such that $\mathrm{Re}\,(z) \geqslant \delta$, $z \neq \delta$ and $|z-\delta|_\infty < \varepsilon$, where $\varepsilon$ is chosen as in Proposition 4.2.11. Writing $\mathscr{L}_z = \lambda_z\Pi_z + R_z = \lambda_z\Pi_z + \mathscr{L}_z(\mathrm{Id}-\Pi_z)$, one gets

$$Q_z = (\mathrm{Id}-\mathscr{L}_z)^{-1} = (1-\lambda_z)^{-1}\Pi_z + (\mathrm{Id}-\mathscr{L}_z)^{-1}(\mathrm{Id}-\Pi_z).$$

First, Proposition 4.2.11 implies $\left\|(\mathrm{Id}-\mathscr{L}_z)^{-1}(\mathrm{Id}-\Pi_z)\right\| \leqslant C$ for $z$ such that $|z-\delta|_\infty < \varepsilon$ and $\left\|(\mathrm{Id}-\mathscr{L}_z)^{-1}\right\| \leqslant C$ when $z$ is far enough to $\delta$. Next, for $t$ close enough to 0, we get

$$Q_{\delta+it} = (1-\lambda_{\delta+it})^{-1}\Pi_{\delta+it} + O(1)$$
$$= (1-\lambda_{\delta+it})^{-1}\Pi_\delta + (1-\lambda_{\delta+it})^{-1}(\Pi_{\delta+it}-\Pi_\delta) + O(1).$$

The regularity of the function $t \longmapsto \mathscr{L}_{\delta+it}$ given in Proposition 4.2.9 and the local expansion of $\lambda_{\delta+it}$ given in (4.2.15) imply that the second term is a $O(1)$. Finally

$$Q_{\delta+it} = (1-\lambda_{\delta+it})^{-1}\Pi_\delta + O(1)$$

and the result follows from (4.2.15). $\square$

COROLLARY 4.2.17. – *The function* $t \longmapsto \operatorname{Re}(Q_{\delta+it})$ *is integrable in a neighborhood of* $0$.

*Proof.* – By the previous proposition, we split $\operatorname{Re}(Q_{\delta+it})$ into

$$\operatorname{Re}\left(Q_{\delta+it} - (1-\lambda_{\delta+it})^{-1}\Pi_\delta\right) + \operatorname{Re}\left((1-\lambda_{\delta+it})^{-1}\right)\Pi_\delta.$$

The first part is bounded by a constant $C > 0$. Furthermore

$$\operatorname{Re}\left((1-\lambda_{\delta+it})^{-1}\right) = \frac{\operatorname{Re}(1-\lambda_{\delta+it})}{|1-\lambda_{\delta+it}|^2}.$$

The local expansions (4.2.15) yield

$$\operatorname{Re}\left((1-\lambda_{\delta+it})^{-1}\right) = \frac{\pi}{2E_\Gamma}\frac{L\left(\frac{1}{|t|}\right)}{|t|\tilde{L}\left(\frac{1}{|t|}\right)^2}(1+o(1)).$$

This function is integrable near 0: indeed for any $\varepsilon > 0$

$$\int_{-\varepsilon}^{\varepsilon} \frac{L\left(\frac{1}{|t|}\right)}{|t|\tilde{L}\left(\frac{1}{|t|}\right)^2}\,\mathrm{d}t = 2\int_0^{\varepsilon}\frac{L\left(\frac{1}{t}\right)}{t\tilde{L}\left(\frac{1}{t}\right)^2}\,\mathrm{d}t = \int_{1/\varepsilon}^{+\infty}\frac{L(y)}{y\tilde{L}(y)^2}\,\mathrm{d}y$$

$$= \left[\frac{-1}{\tilde{L}(y)^2}\right]_{1/\varepsilon}^{+\infty} = \frac{1}{\tilde{L}(1/\varepsilon)} < +\infty,$$

because $\lim_{x\longrightarrow+\infty}\tilde{L}(x) = +\infty$. □

# CHAPTER 5

# THEOREM A: MIXING FOR $\beta \in ]0,1[$

This section is devoted to the mixing properties of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $\mathrm{T}^1\mathrm{M}$ of the quotient manifold $\mathrm{M} = \mathrm{X}/\Gamma$. We specify here the speed of convergence to 0 of $m_\Gamma(\mathfrak{A} \cap g_{-t}\mathfrak{B})$ as $t \longrightarrow \pm\infty$.

Since the group $\Gamma$ is divergent, the Hopf-Tsuji-Sullivan theorem ([**41**]) ensures that the geodesic flow is totally conservative: thus we do not need to formulate additional assumptions about the sets $\mathfrak{A}$ and $\mathfrak{B}$ to avoid the examples constructed by Hajan and Kakutani ([**26**]).

In this section we prove Theorem A.

Theorem A. – *Let $\Gamma$ be a Schottky group satisfying the assumptions $(H_\beta)$ for some $\beta \in ]0,1[$. Let $\mathfrak{A}, \mathfrak{B} \subset \mathrm{T}^1\mathrm{X}/\Gamma$ be two $m_\Gamma$-measurable subsets of finite measure. Then, as $t \longrightarrow \pm\infty$*

$$m_\Gamma(\mathfrak{A} \cap g_{-t}\mathfrak{B}) \sim \frac{\sin(\beta\pi)}{\pi E_\Gamma}\frac{m_\Gamma(\mathfrak{A})m_\Gamma(\mathfrak{B})}{|t|^{1-\beta}L(|t|)},$$

*where*

$$E_\Gamma = \sum_{1\leqslant j\leqslant p} C_j \left(\int_{\Lambda\setminus\Lambda_j} \frac{\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(x,x_j)^{2\delta/a}}\right)^2. \tag{29}$$

For any $m_\Gamma$-integrable function $f$, we set $m_\Gamma(f) = \int_{\mathrm{T}^1\mathrm{X}/\Gamma} f\,\mathrm{d}m_\Gamma$. The previous theorem may be reformulated as follows.

Theorem A. – *For any functions $A$,$B$ in $\mathbb{L}^1\left(\mathrm{T}^1\mathrm{X}/\Gamma, m_\Gamma\right) \cap \mathbb{L}^2\left(\mathrm{T}^1\mathrm{X}/\Gamma, m_\Gamma\right)$, as $R \longrightarrow \pm\infty$*

$$m_\Gamma(A \times B \circ g_R) \sim \frac{\sin(\beta\pi)}{\pi E_\Gamma}\frac{m_\Gamma(A)m_\Gamma(B)}{|R|^{1-\beta}L(|R|)}.$$

From now on, denote $R \in \mathbb{R}$ the parameter of the flow to emphasize the similarity of the proof of Theorem A with Theorem C. For $A, B \in \mathbb{L}^1(\mathrm{T}^1\mathrm{X}/\Gamma, m_\Gamma) \cap \mathbb{L}^2(\mathrm{T}^1\mathrm{X}/\Gamma, m_\Gamma)$, we set

$$\begin{aligned} M\,(R; A, B) &:= m_\Gamma(A \times B \circ g_R) \\ &= \int_\Omega A([x_-, x_+, s]) B\,(g_R([x_-, x_+, s]))\,\mathrm{d}m_\Gamma([x_-, x_+, s]), \end{aligned}$$

where $[x_-, x_+, s]$ stands for the $\Gamma$-orbit of $(x_-, x_+, s)$. In the next subsection, we will express the quantity $M\,(R; A, B)$ in terms of the iterates of a transfer operator, via the coding described in Chapter 4.

## 5.1. Study of $M\,(R; A, B)$

The spaces $\Omega^0 = \Lambda^0 \overset{\Delta}{\times} \Lambda^0 \times \mathbb{R}/\Gamma$ and $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r} \rangle$ are in one-to-one correspondence and the geodesic flow on $\Omega^0$ is conjugated to the special flow $(\phi_R)_{R \in \mathbb{R}}$ defined on $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r} \rangle$. If we denote by $\mathfrak{c}$ the bijection between $\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r} \rangle$ and $\Omega^0$, we may write

$$M\,(R; A, B) = \int_{\mathscr{D}^0 \times \mathbb{R}/\langle \mathrm{T}_\mathfrak{r} \rangle} A([[x_-, x_+, s]]) B\,(\phi_R([[x_-, x_+, s]]))\,\mathrm{d}\overline{m}_\Gamma([[x_-, x_+, s]]),$$

where $[[x_-, x_+, s]]$ is the $\langle \mathrm{T}_\mathfrak{r} \rangle$-orbit of $(x_-, x_+, s)$ and $A$ and $B$ are identified with $A \circ \mathfrak{c}$ and $B \circ \mathfrak{c}$ respectively. The following strategy was inspired of **[25]**. Let $S \subset \mathscr{D}^0 \times \mathbb{R}$ be a fundamental domain for the action of $\langle \mathrm{T}_\mathfrak{r} \rangle$. The vector space generated by the functions $\varphi \otimes u$, where $\varphi$ is Lipschitz on $\mathscr{D}^0$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous with compact support, is dense in $\mathbb{L}^1(S, \bar{\nu} \otimes \mathrm{d}s) \cap \mathbb{L}^2(S, \bar{\nu} \otimes \mathrm{d}s)$; we thus assume that $A = \varphi \otimes u$ and $B = \psi \otimes v$, with $\varphi, \psi, u, v$ as above. From the definition of $\mathrm{T}_\mathfrak{r}$ and the fact that $S$ is a fundamental domain, we deduce that for any $(y, x, s) \in S$ and $R \in \mathbb{R}$, there exists a unique integer $k = k\,(R, x_-, x_+, s) \in \mathbb{Z}$ such that $\mathrm{T}_\mathfrak{r}^k(x_-, x_+, s + R) \in S$. Hence, for any $(y, x, s) \in S$ and $R \in \mathbb{R}$

$$\psi \otimes v\left(\tilde{\phi}_R(x_-, x_+, s)\right) = \sum_{k \in \mathbb{Z}} \psi \otimes v(\mathrm{T}_\mathfrak{r}^k(x_-, x_+, s + R)).$$

In the sequel, we will write $\bar{\nu} := \mu|_{\mathscr{D}^0}$, so that $(\tilde{m}_\Gamma)|_{\mathscr{D}^0 \times \mathbb{R}} = \bar{\nu} \otimes \mathrm{d}s$ (see Subsection 4.1.3). We decompose $M(R; \varphi \otimes u, \psi \otimes v)$ into $M^+(R; \varphi \otimes u, \psi \otimes v)$ $+M^-(R; \varphi \otimes u, \psi \otimes v)$ where

$$\begin{aligned} &M^+(R; \varphi \otimes u, \psi \otimes v) \\ &\quad = \sum_{k \geqslant 0} \int_{\mathscr{D}^0 \times \mathbb{R}} \varphi(x_-, x_+) u(s) \psi \otimes v(\mathrm{T}_\mathfrak{r}^k(x_-, x_+, s + R))\,\mathrm{d}\bar{\nu}(x_-, x_+)\,\mathrm{d}s \end{aligned}$$

and

$$M^-(R;\varphi\otimes u,\psi\otimes v) = \sum_{k\geqslant 1}\int_{\mathscr{D}^0\times\mathbb{R}} \varphi(x_-,x_+)u(s)\psi\otimes v(\mathrm{T}_{\mathfrak{r}}^{-k}(x_-,x_+,s+R))\,\mathrm{d}\bar\nu(x_-,x_+)\,\mathrm{d}s.$$

We first prove the following

LEMMA 5.1.1. – 1) *$R^{1-\beta}L(R)M^-(R;\varphi\otimes u,\psi\otimes v)=0$ for $R$ large enough;*
2) *$R^{1-\beta}L(R)M^+(R;\varphi\otimes u,\psi\otimes v)=0$ for $-R$ large enough.*

*Proof.* – Since the measure $\bar\nu\otimes\,\mathrm{d}s$ is $\mathrm{T}_{\mathfrak{r}}$-invariant, we write

$$M^-(R;\varphi\otimes u,\psi\otimes v) = \sum_{k\geqslant 1}\int_{\mathscr{D}^0\times\mathbb{R}} \varphi\otimes u\left(\mathrm{T}_{\mathfrak{r}}^{k}(x_-,x_+,s)\right)\psi\otimes v(x_-,x_+,s+R)\,\mathrm{d}\bar\nu(x_-,x_+)\,\mathrm{d}s.$$

Recall that the coding of $\mathscr{D}^0$ identifies the pair $(x_-,x_+)$ with a two-sided sequence $(\alpha_n)_{n\in\mathbb{Z}}$. By a classical density argument in Ergodic Theory, it is sufficient to prove that $R^{1-\beta}L(R)M^-(R;\varphi\otimes u,\psi\otimes v)=0$ for functions $\varphi,\psi\,:\,\mathscr{D}^0\longrightarrow\mathbb{R}$ only depending on $(\alpha_n)_{n\geqslant -q}$ for some $q\geqslant 0$. Using the $\mathrm{T}_{\mathfrak{r}}$-invariance of $\bar\nu\otimes\,\mathrm{d}s$, one will impose $q=0$ in the sequel. Hence

$$M^-(R;\varphi\otimes u,\psi\otimes v) = \sum_{k\geqslant 1}\int_{\mathscr{D}^0\times\mathbb{R}} \varphi\otimes u\left(\mathrm{T}_{\mathfrak{r}}^{k}(p(x_-,x_+),s)\right)\psi\otimes v(p(x_-,x_+),s+R)\,\mathrm{d}\bar\nu(x_-,x_+)\,\mathrm{d}s,$$

where $p:\mathscr{D}^0\longrightarrow\Lambda$ is the projection on the second coordinate. Finally

$$M^-(R;\varphi\otimes u,\psi\otimes v)=\sum_{k\geqslant 1}\int_{\Lambda\times\mathbb{R}}\varphi\otimes u\left(\mathrm{T}^k x,s-S_k\mathfrak{r}(x)\right)\psi(x)v(s+R)\,\mathrm{d}\nu(x)\,\mathrm{d}s.$$

Recall the definition of the operator $\tilde P$ given in (21): for any $x\in\Lambda$ and $t\in\mathbb{R}$

$$\tilde P(\psi\otimes v)(x,t)=\sum_{j=1}^{p+q}\mathbb{1}_{\Lambda_j^c}(x)\sum_{\alpha\in\Gamma_j^*}e^{-\delta b(\alpha,x)}\frac{h\psi(\alpha\cdot x)}{h(x)}v(t+b(\alpha,x)).$$

By Lemma 4.2.2, this operator is the adjoint of the transformation $(x,s)\longmapsto(\mathrm{T}x,s-\mathfrak{r}(x))$ with respect to the measure $\nu\otimes\,\mathrm{d}s$. Since the supports of $u$ and $v$ are compact, setting $R_0=\max\mathrm{supp}\ v+C-\min\mathrm{supp}\ u+1$, one gets for all $R\geqslant R_0$ and for any $k\geqslant 1$

$$\tilde P^k\left(\psi\otimes v\right)(x,s+R)=0.$$

Therefore, when $R \geqslant R_0$

$$M^-(R;\varphi\otimes u,\psi\otimes v) = \sum_{k\geqslant 1}\int_{\Lambda\times\mathbb{R}} \varphi\otimes u\,(x,s)\,\tilde{P}^k\,(\psi\otimes v)\,(x,s+R)\,\mathrm{d}\nu(x)\,\mathrm{d}s = 0,$$

which proves 1). The argument is similar for 2). □

In other words, we have

- when $R \longrightarrow +\infty$

$$\begin{aligned} M(R;\varphi\otimes u,\psi\otimes v) &= M^+(R;\varphi\otimes u,\psi\otimes v)\\ &= \sum_{k\geqslant 0}\int_{\Lambda\times\mathbb{R}} \tilde{P}^k\,(\varphi\otimes u)\,(x,s-R)\psi\otimes v\,(x,s)\;\mathrm{d}\nu(x)\,\mathrm{d}s,\end{aligned}$$

  with the convention $\tilde{P}^0\,(\varphi\otimes u) = \varphi\otimes u$;
- when $R \longrightarrow -\infty$

$$\begin{aligned} M(R;\varphi\otimes u,\psi\otimes v) &= M^-(R;\varphi\otimes u,\psi\otimes v)\\ &= \sum_{k\geqslant 1}\int_{\Lambda\times\mathbb{R}} \varphi\otimes u\,(x,s)\,\tilde{P}^k\,(\psi\otimes v)\,(x,s+R)\,\mathrm{d}\nu(x)\,\mathrm{d}s.\end{aligned}$$

The investigation of an asymptotic for $M(R;\varphi\otimes u,\psi\otimes v)$ thus relies on similar arguments in each of the cases $R \longrightarrow +\infty$ and $R \longrightarrow -\infty$; in the sequel, we just explain how to obtain the asymptotic for $R \longrightarrow +\infty$ and assume $R \geqslant R_0$ to ensure $M(R;\varphi\otimes u,\psi\otimes v) = M^+(R;\varphi\otimes u,\psi\otimes v)$. From now on, we omit the symbol $+$. The following subsection is devoted to the proof of the asymptotic for $M(R;\varphi\otimes u,\psi\otimes v)$ when $\beta \in ]0,1[$.

## 5.2. Theorem A for $\beta \in ]0,1[$

We have

$$M(R;\varphi\otimes u,\psi\otimes v) = \sum_{k\geqslant 0} M_k(R;\varphi\otimes u,\psi\otimes v),$$

where, for any $k \geqslant 0$,

$$M_k(R;\varphi\otimes u,\psi\otimes v) = \int_{\Lambda\times\mathbb{R}} \tilde{P}^k\,(\varphi\otimes u)\,(x,s-R)\psi\otimes v\,(x,s)\;\mathrm{d}\nu(x)\,\mathrm{d}s.$$

We follow the steps of the proof of Theorem 1.4 in **[24]**. Let $(a_k)_{k\geqslant 1}$ satisfying $kL(a_k) = a_k^\beta$, where $L$ is the slowly varying function given in the family of assumptions $(H_\beta)$. We postpone the proofs of the following propositions to Subsections 5.2.2 and 5.2.3.

Proposition A.1. – *Let $\varphi, \psi : \Lambda \longrightarrow \mathbb{R}$ be two Lipschitz functions and $u, v : \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions with compact support. Uniformly in $K \geqslant 2$ and $R \in [0, Ka_k]$, we have, as $k \longrightarrow +\infty$,*

$$M_k(R; \varphi \otimes u, \psi \otimes v) = \frac{1}{e_\Gamma a_k} \left( \Psi_\beta \left( \frac{R}{e_\Gamma a_k} \right) m_\Gamma(\varphi \otimes u) m_\Gamma(\psi \otimes v) + o_k(1) \right),$$

*where $\Psi_\beta$ is the density of the fully asymmetric stable law with parameter $\beta$ and $e_\Gamma = E_\Gamma^{1/\beta}$.*

Proposition A.2. – *Let $\varphi, \psi : \Lambda \longrightarrow \mathbb{R}$ be two Lipschitz functions and $u, v : \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions with compact support. There exists a constant $C > 0$ depending on $u$ such that, for any $K \geqslant 2$, when $R \geqslant Ka_k$, we have*

$$|M_k(R; \varphi \otimes u, \psi \otimes v)| \leqslant Ck \frac{L(R)}{R^{1+\beta}} \, ||\varphi \otimes u||_\infty \, ||\psi \otimes v||_\infty .$$

We now explain how they imply Theorem A.

**5.2.1. Asymptotic for $M(R; \varphi \otimes u, \psi \otimes v)$. –** Using Proposition A.1, we decompose $M(R; \varphi \otimes u, \psi \otimes v)$ as

$$M^1(R; \varphi \otimes u, \psi \otimes v) + M^2(R; \varphi \otimes u, \psi \otimes v) + M^3(R; \varphi \otimes u, \psi \otimes v),$$

where

$$M^1(R; \varphi \otimes u, \psi \otimes v) := m_\Gamma(\varphi \otimes u) m_\Gamma(\psi \otimes v) \sum_{k | R < Ka_k} \frac{1}{e_\Gamma a_k} \Psi_\beta \left( \frac{R}{e_\Gamma a_k} \right),$$

$$M^2(R; \varphi \otimes u, \psi \otimes v) := \sum_{k | R < Ka_k} \frac{o_k(1)}{e_\Gamma a_k},$$

$$M^3(R; \varphi \otimes u, \psi \otimes v) := \sum_{k | R \geqslant Ka_k} M_k(R; \varphi \otimes u, \psi \otimes v).$$

a) *Contribution of $M^1(R; \varphi \otimes u, \psi \otimes v)$*. Following **[24]**, we introduce the measure $\mu_R = \sum_{0 < R/a_k \leqslant K} D_{R/a_k}$ on $\mathbb{R}$ so that

(30)
$$\sum_{k | R < Ka_k} \frac{1}{e_\Gamma a_k} \Psi_\beta \left( \frac{R}{e_\Gamma a_k} \right) = \frac{1}{R} \int_0^K \frac{z}{e_\Gamma} \Psi_\beta \left( \frac{z}{e_\Gamma} \right) \mathrm{d}\mu_R(z).$$

The definition of $\mu_R$ implies

$$\mu_R([x, y]) = \sum_{k | x \leqslant R/a_k \leqslant y} 1 = \sum_{k | a_k \in [R/y, R/x]} 1$$

for any $[x, y] \subset ]0, K]$. Recall some properties of functions $A(t) = t^\beta / L(t)$ and of its pseudo-inverse $A^*$ given in Chapter 3. First of all $A(a_n) = n$. Moreover, the function $A^*$ is a regularly varying function with exponent $1/\beta$ which satisfies $a_n = A^*(n)$.

Lemma 5.2.1. – *For $R$ large enough*

$$\left\{k | A\left(\frac{R}{y}\right) \leqslant k \leqslant A\left(\frac{R}{x}-1\right)\right\} \subset \left\{k | \frac{R}{y} \leqslant a_k \leqslant \frac{R}{x}\right\}$$

$$\text{and } \left\{k | \frac{R}{y} \leqslant a_k \leqslant \frac{R}{x}\right\} \subset \left\{k | A\left(\frac{R}{y}\right) \leqslant k \leqslant A\left(\frac{R}{x}\right)\right\}.$$

*Proof.* – Since the function $A^*$ is increasing and satisfies $A^*(A(t)) \geqslant t$ and $A^*(A(t-1)) \leqslant t$ for any $t \gg 1$ (see [**44**]), we obtain

– $A(R/y) \leqslant k$ implies $R/y \leqslant A^*(k)$;
– $k \leqslant A(R/x-1)$ implies $A^*(k) \leqslant R/x$. □

We deduce from this lemma that

$$\sum_{A(R/y) \leqslant k \leqslant A(R/x-1)} 1 \leqslant \mu_R([x,y]) \leqslant \sum_{A(R/y) \leqslant k \leqslant A(R/x)} 1,$$

which yields

$$A\left(\frac{R}{x}-1\right) - A\left(\frac{R}{y}\right) \leqslant \mu_R([x,y]) \leqslant A\left(\frac{R}{x}\right) - A\left(\frac{R}{y}\right);$$

hence

$$\mu_R([x,y]) \sim A\left(\frac{R}{x}\right) - A\left(\frac{R}{y}\right) \sim \frac{R^\beta}{L(R)}\left(x^{-\beta} - y^{-\beta}\right)$$

and finally

$$R^{-\beta} L(R) \mu_R([x,y]) \sim \int_x^y \beta z^{-\beta-1}\, \mathrm{d}z. \tag{31}$$

We now want to control quantities of the form

$$\sum_{k | R < a_k \varepsilon} \frac{1}{e_\Gamma a_k} \Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)$$

for $\varepsilon > 0$ small enough. For any $R \geqslant 1$ and $\varepsilon > 0$, we write

$$\left| R^{1-\beta} L(R) \sum_{k | R < a_k \varepsilon} \frac{1}{e_\Gamma a_k} \Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right) \right| \leq R^{1-\beta} L(R) \sum_{k | R < \varepsilon a_k} \frac{1}{a_k}. \tag{32}$$

Let $N = N(\varepsilon, R)$ denote the first integer such that $R < \varepsilon a_k$: $N$ is increasing in $R$. Karamata's lemma then implies

$$\left| \sum_{k | R < \varepsilon a_k} \frac{1}{a_k} \right| = \sum_{k \geqslant N} \frac{1}{a_k} \sim \frac{N}{a_N}.$$

From $a_N^\beta = N L(a_N)$, we deduce $N/a_N = a_N^{\beta-1}/L(a_N)$, so that

$$R^{1-\beta} L(R) \frac{N}{a_N} \leqslant \varepsilon^{1-\beta} \frac{L(R)}{L(a_N)} \leqslant \varepsilon^{1-\beta} \max\left(\frac{R}{a_N}, \frac{a_N}{R}\right)^{(1-\beta)/2}, \tag{33}$$

where the last inequality is a consequence of Potter's lemma with $B = 1$, $\rho = (1-\beta)/2$, $x = R$ and $y = a_N$. It follows from the definition of $N$ that $R/a_N < \varepsilon$ and $\varepsilon a_{N-1} \leqslant R$; hence

$$\frac{a_N}{R} = \frac{a_{N-1}}{R}\frac{a_N}{a_{N-1}} \leqslant \frac{1}{\varepsilon}\frac{a_N}{a_{N-1}} \preceq \frac{1}{\varepsilon}$$

for $N$ large enough. Finally, this last estimate combined with (33) yields

$$R^{1-\beta}L(R)\frac{N}{a_N} \preceq \varepsilon^{(1-\beta)/2}.$$

By (32), for any arbitrarily small $\eta > 0$, there exists $\varepsilon_\eta > 0$ such that, as $\varepsilon < \varepsilon_\eta$

$$\left| R^{1-\beta}L(R) \sum_{k|R<a_k\varepsilon} \frac{1}{e_\Gamma a_k}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)\right| \preceq \eta. \tag{34}$$

Properties (30), (31) and (34) imply, for $R$ large enough

$$\begin{aligned} R^{1-\beta}L(R) \sum_{k|R<Ka_k} &\frac{1}{e_\Gamma a_k}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right) \\ &= O(\eta) + R^{1-\beta}L(R)\int_{\varepsilon_\eta}^{K} \frac{z}{e_\Gamma}\Psi_\beta\left(\frac{z}{e_\Gamma}\right)\mathrm{d}\mu_R(z) \\ &= O(\eta) + \left(\frac{\beta}{e_\Gamma}\int_{\varepsilon_\eta}^{K} z^{-\beta}\Psi_\beta\left(\frac{z}{e_\Gamma}\right)\mathrm{d}z\right)(1+o(1)), \end{aligned}$$

with $\lim_{R\longrightarrow+\infty} o(1) = 0$. From the integrability of $z \longmapsto z^{-\beta}\Psi_\beta(z)$ on $[0,+\infty[$ (see [**47**]), we deduce that

$$\left|\int_0^{\varepsilon_\eta} z^{-\beta}\Psi\left(\frac{z}{e_\Gamma}\right)\mathrm{d}z\right| \leqslant \eta$$

provided $\varepsilon_\eta$ is sufficiently small; hence as $R \longrightarrow +\infty$

$$R^{1-\beta}L(R) \sum_{k|R<Ka_k} \frac{1}{e_\Gamma a_k}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right) = \frac{\beta}{e_\Gamma}\int_0^K z^{-\beta}\Psi_\beta\left(\frac{z}{e_\Gamma}\right)\mathrm{d}z + O(\eta). \tag{35}$$

It thus follows from the definition of $M^1(R;\varphi\otimes u,\psi\otimes v)$ and from (35) that, as $R \longrightarrow +\infty$

$$R^{1-\beta}L(R)M^1(R;\varphi\otimes u,\psi\otimes v) \sim \frac{\beta m_\Gamma(\varphi\otimes u)m_\Gamma(\psi\otimes v)}{E_\Gamma}\int_0^{K/e_\Gamma} z^{-\beta}\Psi_\beta(z)\,\mathrm{d}z \tag{36}$$

with $E_\Gamma = e_\Gamma^\beta$.

b) *Contribution of* $M^2(R;\varphi\otimes u,\psi\otimes v)$. Let $N = N(K,R)$ be the smallest integer such that $Ka_N > R$: the function $R \longmapsto N(K,R)$ is increasing in $R$. Let $\varepsilon > 0$; for $R$ large enough and any $k \geqslant N$, we get $|o_k(1)| \leqslant \varepsilon$. Karamata's lemma thus implies

$$\left| \sum_{k|R<Ka_k} \frac{o_k(1)}{e_\Gamma a_k} \right| \leqslant \sum_{k\geqslant N} \frac{\varepsilon}{a_k} \sim \varepsilon \frac{N}{a_N}.$$

Following the same steps as in a) for the negligible parts of $M^1(R;\varphi\otimes u,\psi\otimes v)$, we deduce from $N/a_N = a_n^{\beta-1}/L(a_N)$ that

$$R^{1-\beta}L(R)\frac{N}{a_N} \leqslant K^{1-\beta}\frac{L(R)}{L(a_N)}$$

and Potter's lemma with $B = \rho = 1$ and $x = R$ and $y = a_N$ yields

$$R^{1-\beta}L(R)\frac{N}{a_N} \preceq K^{2-\beta},$$

so that as $R \longrightarrow +\infty$

(37) $$R^{1-\beta}L(R)M^2\left(R;\varphi\otimes u,\psi\otimes v\right) = o_K(1).$$

c) *Contribution of* $M^3(R;\varphi\otimes u,\psi\otimes v)$. We write

$$\left|M^3(R;\varphi\otimes u,\psi\otimes v)\right| \leqslant \sum_{k|R\geqslant Ka_k} \left|M_k(R;\varphi\otimes u,\psi\otimes v)\right|.$$

It follows from Proposition A.2 that

$$M^3(R;\varphi\otimes u,\psi\otimes v) \leqslant ||\varphi\otimes u||_\infty\, ||\psi\otimes v||_\infty\, C\frac{L(R)}{R^{1+\beta}} \sum_{k|R\geqslant Ka_k} k,$$

and from the definition of $A$, we deduce that

$$\sum_{k|R\geqslant Ka_k} k = \sum_{k|k\leqslant A(R/K)} k \leqslant A\left(\frac{R}{K}\right)^2,$$

therefore

$$\left|M^3(R;\varphi\otimes u,\psi\otimes v)\right| \leqslant ||\varphi\otimes u||_\infty\, ||\psi\otimes v||_\infty\, C\frac{L(R)}{R^{1+\beta}}\frac{R^{2\beta}}{K^{2\beta}}\frac{1}{L\left(R/K\right)^2}.$$

Potter's lemma implies $L(R)/L(R/K) \leqslant K^{\beta/2}$ for $R$ large enough, so that

(38) $$R^{1-\beta}L(R)M^3(R;\varphi\otimes u,\psi\otimes v) \leqslant C\,||\varphi\otimes u||_\infty\, ||\psi\otimes v||_\infty\, K^{-\beta}.$$

Combining (36), (37) and (38), it follows that

$$\begin{aligned} R^{1-\beta}L(R)M(R;\varphi\otimes u,\psi\otimes v) \\ &= \frac{\beta m_\Gamma(\varphi\otimes u)m_\Gamma(\psi\otimes v)}{E_\Gamma}\int_0^{K/e_\Gamma} z^{-\beta}\Psi_\beta(z)\,\mathrm{d}z(1+o(1)) \\ &\quad + o_K(1)+O(K^{-\beta}), \end{aligned}$$

with $\lim_{R\longrightarrow+\infty} o(1)=0$ and $\lim_{R\longrightarrow+\infty} o_K(1)=0$ for any fixed $K$. Then letting $R\longrightarrow+\infty$, we obtain

$$R^{1-\beta}L(R)M(R;\varphi\otimes u,\psi\otimes v)\sim \frac{\beta m_\Gamma(\varphi\otimes u)m_\Gamma(\psi\otimes v)}{E_\Gamma}\int_0^{K/e_\Gamma} z^{-\beta}\Psi_\beta(z)\,\mathrm{d}z+O(K^{-\beta}).$$

Letting $K\longrightarrow+\infty$ and using $\int_0^{+\infty} z^{-\beta}\Psi_\beta(z)\,\mathrm{d}z=\frac{\sin(\beta\pi)}{\beta\pi}$ (see [**47**]), this achieves the proof of Theorem A in the case $\beta\in]0,1[$.

**5.2.2. Proof of Proposition A.1. –** We want to prove the following local limit theorem.

PROPOSITION A.1. – *Let $\varphi,\psi : \Lambda\longrightarrow\mathbb{R}$ be two Lipschitz functions and $u,v : \mathbb{R}\longrightarrow\mathbb{R}$ be two continuous functions with compact support. Uniformly in $K\geqslant 2$ and $R\in[0,Ka_k]$, one gets as $k\longrightarrow+\infty$*

$$M_k(R;\varphi\otimes u,\psi\otimes v)=\frac{1}{e_\Gamma a_k}\left(\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)m_\Gamma(\varphi\otimes u)m_\Gamma(\psi\otimes v)+o_k(1)\right),$$

*where $\Psi_\beta$ is the density of the fully asymmetric stable law with parameter $\beta$ and $e_\Gamma=E_\Gamma^{1/\beta}$.*

Let us fix $K\geqslant 2$ and $R\gg 1$. For all $k\in\mathbb{N}$ such that $Ka_k>R$,

$$M_k(R;\varphi\otimes u,\psi\otimes v)=\int_{\Lambda\times\mathbb{R}}\tilde{P}^k\,(\varphi\otimes u)\,(x,s-R)\psi\otimes v\,(x,s)\,\nu(\,\mathrm{d}x)\,\mathrm{d}s.$$

We have to prove that the following sequence of measures

$$\left(a_kM_k(R;\varphi\otimes\bullet,\psi\otimes v)-\frac{1}{e_\Gamma}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)\left(\nu(\varphi)\int_{\mathbb{R}}\bullet(x)\,\mathrm{d}x\right)m_\Gamma(\psi\otimes v)\right)_{k|Ka_k>R},$$

weakly converges to 0 as $k\longrightarrow+\infty$, uniformly in $K$ and $R$. Using a classical argument from Probability theory (see Theorem 10.7 p. 218 in [**11**]), it is sufficient to show that

$$a_kM_k(R;\varphi\otimes u,\psi\otimes v)-\frac{1}{e_\Gamma}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)m_\Gamma(\psi\otimes v)\nu(\varphi)\widehat{u}(0)\longrightarrow 0,$$

for functions $u : \mathbb{R}\longrightarrow\mathbb{R}$ such that $u$ and $|u|$ have a $\mathcal{C}^\infty$ Fourier transform with compact support. More precisely, let us introduce the following definition.

Definition 5.2.2. – *Let $\mathscr{U}$ be the set of test functions $u$ of the form $u(x) = e^{itx}u_0(x)$ where $t \in \mathbb{R}$ and $u_0$ belongs to the set of positive integrable function from $\mathbb{R}$ to $\mathbb{R}$, whose Fourier transform is $\mathcal{C}^\infty$ and with compact support.*

We first notice that $a_k M_k(R; \varphi\otimes u, \psi\otimes v)$ is finite for $u \in \mathscr{U}$. By the Fourier inverse formula and the definition of $\tilde{P}$ given in Lemma 4.2.2, we write

$$\begin{aligned}\tilde{P}^k(\varphi\otimes u)(x, s-R) &= \sum_{\gamma\in\Gamma(k)} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)} \frac{h\varphi(\gamma\cdot x)}{h(x)} u(s-R+b(\gamma,x)) \\ &= \frac{1}{2\pi h(x)} \int_{\mathbb{R}} e^{it(R-s)} \left(\mathscr{L}^k_{\delta+it} h\varphi\right)(x)\widehat{u}(t)\,\mathrm{d}t.\end{aligned}$$

This quantity is bounded from above by $||h\varphi||_\infty ||\widehat{u}||_1$, up to a multiplicative constant. Since the function $\psi\otimes v$ is integrable on $\Lambda\times\mathbb{R}$, the quantity $M_k(R; \varphi\otimes u, \psi\otimes v)$ is finite for any $u \in \mathscr{U}$.

Proposition A.1 is a consequence of the following lemma, combined with the Lebesgue dominated convergence theorem

Lemma 5.2.3. – *For all $u \in \mathscr{U}$, as $k \longrightarrow +\infty$,*

$$\frac{a_k}{2\pi h(x)} \int_{\mathbb{R}} e^{it(R-s)} \left(\mathscr{L}^k_{\delta+it} h\varphi\right)(x)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{e_\Gamma}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)\nu(\varphi)\widehat{u}(0) \longrightarrow 0,$$

*uniformly in $x \in \Lambda$, $s \in \operatorname{supp} v$, $R$ and $K$.*

*Proof.* – Fix $\varepsilon > 0$ according to Proposition 4.2.11. By the Fourier inverse formula applied to $\Psi_\beta$, we may write

$$\frac{a_k}{2\pi h(x)} \int_{\mathbb{R}} e^{it(R-s)} \left(\mathscr{L}^k_{\delta+it} h\varphi\right)(x)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{e_\Gamma}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)\nu(\varphi)\widehat{u}(0) = K_1(k) + K_2(k),$$

where

$$K_1(k) = \frac{a_k}{2\pi h(x)} \int_{[-\varepsilon,\varepsilon]^c} e^{it(R-s)} \left(\mathscr{L}^k_{\delta+it} h\varphi\right)(x)\widehat{u}(t)\,\mathrm{d}t$$

and

$$\begin{aligned}K_2(k) &= \frac{a_k}{2\pi h(x)} \int_{[-\varepsilon,\varepsilon]} e^{it(R-s)} \left(\mathscr{L}^k_{\delta+it} h\varphi\right)(x)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR/a_k} g_\beta(e_\Gamma t)\nu(\varphi)\widehat{u}(0)\,\mathrm{d}t \\ &= \frac{1}{2\pi h(x)} \int_{-\varepsilon a_k}^{\varepsilon a_k} e^{it(R-s)/a_k} \left(\mathscr{L}^k_{\delta+it/a_k} h\varphi\right)(x)\widehat{u}\left(\frac{t}{a_k}\right)\mathrm{d}t \\ &\quad - \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR/a_k} g_\beta(e_\Gamma t)\nu(\varphi)\widehat{u}(0)\,\mathrm{d}t.\end{aligned}$$

There exists $\rho \in ]0,1[$ such that $||\mathscr{L}_{\delta+it}|| \leqslant \rho$ for any $t \in (\operatorname{supp}\widehat{u}) \cap (\mathbb{R}\backslash[-\varepsilon,\varepsilon])$. Hence $|K_1(k)| \leq a_k\rho^k$, which goes to 0 uniformly in $x$, $s$, $R$ and $K$ as $k \longrightarrow +\infty$.

Let us deal with $K_2(k)$. Using the spectral decomposition of $\mathcal{L}_{\delta+it/a_k}$ and Proposition 4.2.11, we write for any $t \in [-\varepsilon a_k, \varepsilon a_k]$

$$\mathcal{L}^k_{\delta+it/a_k}(h\varphi) = \lambda^k_{\delta+it/a_k}\Pi_{\delta+it/a_k}(h\varphi) + R^k_{\delta+it/a_k}(h\varphi),$$

where the spectral radius of $R_{\delta+it/a_k}$ is smaller than $\rho_\varepsilon$, with $\rho_\varepsilon < 1$. The quantity $K_2(k)$ may be split into $L_1(k) + L_2(k) + L_3(k)$ where

$$L_1(k) = \frac{1}{2\pi h(x)} \int_{-\varepsilon a_k}^{\varepsilon a_k} e^{it(R-s)/a_k} R^k_{\delta+it/a_k}(h\varphi)(x)\widehat{u}\left(\frac{t}{a_k}\right) \mathrm{d}t,$$

$$L_2(k) = \frac{1}{2\pi h(x)} \int_{-\varepsilon a_k}^{\varepsilon a_k} e^{it(R-s)/a_k} \lambda^k_{\delta+it/a_k}\left(\Pi_{\delta+it/a_k}(h\varphi)(x) - \Pi_\delta(h\varphi)(x)\right)\widehat{u}\left(\frac{t}{a_k}\right) \mathrm{d}t$$

and

$$\begin{aligned} L_3(k) = {} & \frac{1}{2\pi h(x)} \int_{-\varepsilon a_k}^{\varepsilon a_k} e^{it(R-s)/a_k} \lambda^k_{\delta+it/a_k}\Pi_\delta(h\varphi)(x)\widehat{u}\left(\frac{t}{a_k}\right) \mathrm{d}t \\ & - \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR/a_k} g_\beta(e_\Gamma t)\nu(\varphi)\widehat{u}(0)\,\mathrm{d}t. \end{aligned}$$

First $|L_1(k)| \preceq a_k\rho_\varepsilon^k$, which goes to 0 uniformly in $x$, $s$, $R$ and $K$ as $k \longrightarrow +\infty$. The characteristic function $g_\beta$ of a stable law with parameter $\beta$ (see Chapter 3) is given by $g_\beta(t) = \exp\left(-\Gamma(1-\beta)e^{i\mathrm{sign}(t)\beta\pi/2}|t|^\beta\right)$. We may notice that $|g_\beta(t)| \leqslant \exp\left(-(1-\beta)\Gamma(1-\beta)|t|^\beta\right)$ for any $t \in \mathbb{R}$, which ensures that $g_\beta$ is integrable on $\mathbb{R}$. Moreover, Proposition 4.2.15 implies that for any $t$ close to 0, the dominant eigenvalue $\lambda_{\delta+it}$ satisfies

$$\lambda_{\delta+it} = \exp\left(-\Gamma(1-\beta)e^{i\mathrm{sign}(t)\beta\pi/2}|e_\Gamma t|^\beta L\left(\frac{1}{|t|}\right)(1+o(1))\right).$$

The regularity of $t \longmapsto \Pi_{\delta+it}$ implies that the integrand of $L_2(k)$ goes to 0 uniformly in $x$, $s \in \mathrm{supp}\ v$, $R$ and $K$; thus, it is sufficient to bound it by an integrable function. By Proposition 4.2.9, we obtain

$$\left|\Pi_{\delta+it/a_k}(h\varphi)(x) - \Pi_\delta(h\varphi)(x)\right| \leqslant \left|\left|\Pi_{\delta+it/a_k} - \Pi_\delta\right|\right| \, ||h\varphi|| \preceq \frac{|t|^\beta}{a_k^\beta} L\left(\frac{a_k}{|t|}\right),$$

with

$$\frac{L\left(a_k/|t|\right)}{L(a_k)} \leqslant \max\left(\frac{1}{|t|}, |t|\right)^{\beta/2}$$

for any $k$ large enough and uniformly in $t \in [-\varepsilon a_k, \varepsilon a_k]$, where $\varepsilon > 0$ is chosen small enough according to Remark 4.2.12. Therefore

$$\left|\Pi_{\delta+it/a_k}(h\varphi)(x) - \Pi_\delta(h\varphi)(x)\right| \leqslant \begin{cases} \frac{L(a_k)}{a_k^\beta}|t|^{\beta/2} \text{ if } |t| \leqslant 1 \\ \frac{L(a_k)}{a_k^\beta}|t|^{3\beta/2} \text{ if } |t| > 1 \end{cases}.$$

Similarly, the inequalities

$$\left|\lambda^k_{\delta+it/a_k}\right| \leqslant \exp\left(-(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^\beta \frac{k}{a_k^\beta} L(a_k)\frac{L\left(a_k/|t|\right)}{L(a_k)}(1+o(1))\right)$$

and

$$\min\left(\frac{1}{|t|}, |t|\right)^{\beta/2} \leqslant \frac{L\left(a_k/|t|\right)}{L(a_k)}$$

yield

$$\left|\lambda^k_{\delta+it/a_k}\right| \leqslant \begin{cases} \exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{3\beta/2}\right) \text{if } |t| \leqslant 1 \\ \exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{\beta/2}\right) \text{ if } |t| > 1. \end{cases}$$

Finally, for $k$ large enough, the integrand of $L_2(k)$ may be bounded from above by the function

$$l(t) := \begin{cases} |t|^{\beta/2}\exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{3\beta/2}\right) \text{if } |t| \leqslant 1 \\ |t|^{3\beta/2}\exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{\beta/2}\right) \text{ if } |t| > 1, \end{cases}$$

up to a multiplicative constant.

On the other hand, since $\Pi_\delta(h\varphi) = \nu(\varphi)h$, we decompose $L_3(k)$ into $M_1(k) + M_2(k) + M_3(k)$ where

$$M_1(k) = \frac{\nu(\varphi)\widehat{u}(0)}{2\pi}\int_{[-\varepsilon a_k, \varepsilon a_k]^c} e^{itR/a_k} g_\beta(e_\Gamma t)\,\mathrm{d}t,$$

$$M_2(k) = \frac{\nu(\varphi)}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{itR/a_k} g_\beta(e_\Gamma t)\left(\widehat{u}(0) - \widehat{u}\left(\frac{t}{a_k}\right)\right)\mathrm{d}t$$

and

$$M_3(k) = \frac{\nu(\varphi)}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{itR/a_k}\left(e^{-its/a_k}\lambda^k_{\delta+it/a_k} - g_\beta(e_\Gamma t)\right)\widehat{u}\left(\frac{t}{a_k}\right)\mathrm{d}t.$$

The term $M_1(k)$ goes to 0 uniformly in $x$, $s$, $R$ and $K$ and so does $M_2(k)$, thanks to the mean value relation applied to $\widehat{u}$ on $[-\varepsilon, \varepsilon]$ combined with the Lebesgue dominated convergence theorem. Similarly, the integrand of $M_3(k)$ goes to 0 uniformly in $s \in \operatorname{supp} v$, and we bound $\left|e^{-its/a_k}\lambda^k_{\delta+it/a_k} - g_\beta(e_\Gamma t)\right|$ from above by $k(t) := \exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{3\beta/2}\right) + |g_\beta(e_\Gamma t)|$ if $|t| \leqslant 1$ and by $j(t) := \exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{\beta/2}\right) + |g_\beta(e_\Gamma t)|$ otherwise. This achieves the proof of Lemma 5.2.3. □

**5.2.3. Proof of Proposition A.2.** – We now give a control of the non-influential terms $M_k$ appearing in the proof of Theorem A. Let us fix $k \in \mathbb{N}$ such that $Ka_k \leqslant R$. Once again, we recall that

$$M_k(R;\varphi \otimes u, \psi \otimes v) = \int_{\Lambda \times \mathbb{R}} \tilde{P}^k (\varphi \otimes u)(x, s-R)\psi \otimes v\,(x,s)\,\nu(\,\mathrm{d}x)\,\mathrm{d}s,$$

where $\tilde{P}$ is given in (21). To show Proposition A.2., it is sufficient to check that

$$\left|\tilde{P}^k (\varphi \otimes u)(x, s-R)\right| \leqslant Ck\frac{L(R)}{R^{1+\beta}}\,||\varphi \otimes u||_\infty$$

uniformly in $x \in \Lambda$ and $s \in \operatorname{supp} v$. We write

$$\begin{aligned}\left|\tilde{P}^k (\varphi \otimes u)(x, s-R)\right| &\leqslant \frac{1}{h(x)} \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)}\,|(h\varphi)(\gamma \cdot x)u(s-R+b(\gamma,x))| \\ &\preceq ||\varphi \otimes u||_\infty \sum_{\substack{\gamma \in \Gamma(k) \\ b(\gamma,x) \overset{M}{\sim} R-s}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)},\end{aligned}$$

where $a \overset{M}{\sim} b$ means $|a-b| \leqslant M$ and the parameter $M$ satisfies $\operatorname{supp} u \subset [-M, M]$. This notation also emphasizes that the only $\gamma$ which really appear in the above sum are the ones such that $b(\gamma,x)$ has the same order than $R-s$. Therefore we only have to show that

$$\sum_{\substack{\gamma \in \Gamma(k) \\ b(\gamma,x) \overset{M}{\sim} R-s}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)} \leqslant Ck\frac{L(R)}{R^{1+\beta}}, \tag{39}$$

where $C$ depends on the support of $u$. The proof is inspired of that of Theorem 1.6 in **[24]**. We will need the two following lemmas.

LEMMA 5.2.4. – *There exists a constant $C > 0$ such that for any $k \geqslant 1$ and $x \in \Lambda$*

$$\sum_{\gamma \in \Gamma(k)} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)} \leqslant C.$$

*Proof.* – From properties of the function $h$, we derive

$$\sum_{\gamma \in \Gamma(k)} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)} \preceq \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)} h(\gamma \cdot x) \preceq \mathscr{L}_\delta^k h(x) \leqslant |h|_\infty. \qquad \square$$

LEMMA 5.2.5. – *Let $\Delta > 0$. There exists a constant $C_\Delta > 0$ such that for any $k \geqslant 1$, any $x \in \Lambda$ and $\zeta \in [a_k/2, \infty]$*

$$\sum_{\substack{\gamma = \alpha_1 \cdots \alpha_k \\ R-\Delta \leqslant b(\gamma,x) \leqslant R+\Delta \\ \forall i,\ \mathrm{d}(\mathbf{o}, \alpha_i \cdot \mathbf{o}) \leqslant \zeta}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma,x)} \leqslant C_\Delta \frac{e^{-R/\zeta}}{a_k}.$$

*Proof.* – Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a positive function whose Fourier transform has compact support. We write

$$\sum_{\substack{\gamma=\alpha_1\cdots\alpha_k \\ R-\Delta\leqslant b(\gamma,x)\leqslant R+\Delta \\ \forall i,\ \mathrm{d}(\mathbf{o},\alpha_i\cdot\mathbf{o})\leqslant\zeta}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x)e^{-\delta b(\gamma,x)} \leqslant \frac{e^{-R/\zeta}}{\min\limits_{[-\Delta,\Delta]} f} \sum_{\substack{\gamma=\alpha_1\cdots\alpha_k \\ R-\Delta\leqslant b(\gamma,x)\leqslant R+\Delta \\ \forall i,\ \mathrm{d}(\mathbf{o},\alpha_i\cdot\mathbf{o})\leqslant\zeta}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x)e^{R/\zeta}e^{-\delta b(\gamma,x)}f(R-b(\gamma,x)).$$

The inequality $R-\Delta \leqslant b(\gamma,x) \leqslant R+\Delta$ implies that the sum on the right hand side may be bounded from above by

$$e^{\Delta/\zeta} \sum_{\substack{\gamma=\alpha_1\cdots\alpha_k \\ R-\Delta\leqslant b(\gamma,x)\leqslant R+\Delta \\ \forall i,\ \mathrm{d}(\mathbf{o},\alpha_i\cdot\mathbf{o})\leqslant\zeta}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x)e^{-(\delta-1/\zeta)b(\gamma,x)}f(R-b(\gamma,x)).$$

Since $\zeta$ is large, the quantity $e^{\Delta/\zeta}$ is close to 1. From the Fourier inverse formula, it follows

$$\sum_{\substack{\gamma=\alpha_1\cdots\alpha_k \\ R-\Delta\leqslant b(\gamma,x)\leqslant R+\Delta \\ \forall i,\ \mathrm{d}(\mathbf{o},\alpha_i\cdot\mathbf{o})\leqslant\zeta}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x)e^{-\delta b(\gamma,x)} \preceq \frac{e^{-R/\zeta}}{2\pi}\int_{\mathbb{R}} e^{itR}\left(\mathscr{L}^k_{\delta-1/\zeta+it,\zeta}\mathbb{1}_\Lambda\right)(x)\widehat{f}(t)\,\mathrm{d}t,$$

where $\mathscr{L}_{\delta-1/\zeta+it,\zeta}$ is defined, for any $\varphi \in \mathrm{Lip}\,(\Lambda)$ and $x \in \Lambda$, by

$$\mathscr{L}_{\delta-1/\zeta+it,\zeta}\,(\varphi)\,(x) = \sum_{\substack{\alpha\in\mathscr{A} \\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})\leqslant\zeta}} \mathbb{1}_{\Lambda^c_{l(\alpha)}} e^{-(\delta-1/\zeta+it)b(\alpha,x)}\varphi(\alpha\cdot x).$$

It remains to show that the integral $\int_{\mathbb{R}} e^{itR}\left(\mathscr{L}^k_{\delta-1/\zeta+it,\zeta}\mathbb{1}_\Lambda\right)(x)\widehat{f}(t)\,\mathrm{d}t$ is $\leqslant C/a_k$. Let us split it into $I_1+I_2$ where

$$I_1 := \int_{[-\varepsilon,\varepsilon]^c} e^{itR}\left(\mathscr{L}^k_{\delta-1/\zeta+it,\zeta}\mathbb{1}_\Lambda\right)(x)\widehat{f}\,(t)\ \mathrm{d}t$$

and

$$I_2 := \int_{-\varepsilon}^{\varepsilon} e^{itR}\left(\mathscr{L}^k_{\delta-1/\zeta+it,\zeta}\mathbb{1}_\Lambda\right)(x)\widehat{f}\,(t)\ \mathrm{d}t$$

for the $\varepsilon$ given in Proposition 4.2.11. We may first notice that $\mathscr{L}_{\delta-1/\zeta+it,\zeta}$ is a continuous perturbation of $\mathscr{L}_{\delta+it}$ for any $t \in \mathbb{R}$. Indeed

$$\begin{aligned}\left\|\mathscr{L}_{\delta-1/\zeta+it,\zeta} - \mathscr{L}_{\delta+it}\right\| &\leqslant \sum_{\substack{\alpha\in\mathscr{A}\\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})\leqslant\zeta}} \left\|w_{\delta-1/\zeta+it}(\alpha,\cdot) - w_{\delta+it}(\alpha,\cdot)\right\| + \sum_{\substack{\alpha\in\mathscr{A}\\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})>\zeta}} \left\|w_{\delta+it}(\alpha,\cdot)\right\| \\ &\leqslant \sum_{\substack{\alpha\in\mathscr{A}\\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})\leqslant\zeta}} \left\|\mathbb{1}_{\Lambda^c_{l(\alpha)}} e^{-(\delta+it)b(\alpha,\cdot)} \left(e^{b(\alpha,\cdot)/\zeta} - 1\right)\right\| + C\frac{L(\zeta)}{\zeta^\beta}.\end{aligned}$$

The two inequalities

$$\left\|\mathbb{1}_{\Lambda^c_{l(\alpha)}} e^{-(\delta+it)b(\alpha,\cdot)}\right\| \preceq e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})} \text{ and } \left\|e^{b(\alpha,\cdot)/\zeta} - 1\right\| \preceq \frac{1}{\zeta}\left(1 + \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})\right) e^{\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})/\zeta}$$

yield

$$\left\|\mathscr{L}_{\delta-1/\zeta+it,\zeta} - \mathscr{L}_{\delta+it}\right\| \preceq \frac{L(\zeta)}{\zeta^\beta}.$$

Potter's lemma thus implies that for $k$ large enough and $\zeta \geqslant a_k/2$

$$\left\|\mathscr{L}_{\delta-1/\zeta+it,\zeta} - \mathscr{L}_{\delta+it}\right\| \leqslant C a_k^{-\beta} L(a_k) \leqslant \frac{C}{k}. \tag{40}$$

Combining (40) and the fact that $\rho(\delta+it) < 1$ for $|t| \in \operatorname{supp} \widehat{f}\backslash[-\varepsilon,\varepsilon]$, it follows that there exists $\rho \in ]0,1[$ such that

$$|I_1| \leqslant C\int_{|t|\geqslant\varepsilon} \left\|\mathscr{L}^k_{\delta-1/\zeta,\zeta}\right\| \widehat{f}(t)\,\mathrm{d}t \leqslant \frac{C}{a_k} a_k\rho^k \int_{|t|\geqslant\varepsilon} \widehat{f}(t)\,\mathrm{d}t \preceq \frac{1}{a_k},$$

since the sequence $(a_k\rho^k)$ converges to 0.

Moreover, from Proposition 4.2.11 and (40), we deduce that for $k$ large enough and $t$ close to 0, the operator $\mathscr{L}_{\delta-1/\zeta+it,\zeta}$ admits a unique dominant eigenvalue $\lambda_{t,1/\zeta}$ close to 1, isolated in the spectrum of $\mathscr{L}_{\delta-1/\zeta+it,\zeta}$ and satisfying $|\lambda_{t,1/\zeta} - \lambda_{\delta+it}| \leqslant C/k$. To estimate $I_2$, it is thus sufficient to check that

$$J := \int_{-\varepsilon}^{\varepsilon} \left\|\mathscr{L}^k_{\delta+1/\zeta+it,\zeta}\right\| \mathrm{d}t \leqslant \frac{C}{a_k}. \tag{41}$$

The integral $J$ may be split into $J_1 + J_2$ where

$$J_1 = \int_{-C_1/a_k}^{C_1/a_k} \left\|\mathscr{L}^k_{\delta+1/\zeta+it,\zeta}\right\| \mathrm{d}t$$

and

$$J_2 = \int_{[-\varepsilon,\varepsilon]\backslash[-C_1/a_k,C_1/a_k]} \left\|\mathscr{L}^k_{\delta+1/\zeta+it,\zeta}\right\| \mathrm{d}t,$$

for a constant $C_1 > 0$, which will be chosen later. For $J_1$, the inequality $|\lambda_{\delta+it}| \leqslant 1$ yields $|\lambda_{t,1/\zeta}| \leqslant 1 + \frac{C}{k}$; hence $\left\|\mathcal{L}^k_{\delta+1/\zeta+it,\zeta}\right\| \leqslant C$ and $J_1$ is thus bounded from above by $CC_1/a_k$ for some $C_1 > 0$.

From Proposition 4.2.15, we deduce the existence of a constant $c > 0$ such that for any $|t| \in [C_1/a_k, \varepsilon]$,

$$|\lambda_{t,1/\zeta}| \leqslant |\lambda_t| + \frac{C}{k} \leqslant 1 - c|t|^\beta L\left(\frac{1}{|t|}\right) + \frac{C}{k}.$$

Moreover for $k$ large enough

$$\frac{1}{k} \sim a_k^{-\beta} L\left(a_k\right) \leqslant C_1^{\beta/2} a_k^{-\beta} L\left(\frac{a_k}{C_1}\right) \leqslant \frac{|t|^\beta}{C_1^{\beta/2}} L\left(\frac{1}{|t|}\right).$$

Therefore, for $C_1$ large enough

$$|\lambda_{t,1/\zeta}| \leqslant 1 - c'|t|^\beta L\left(\frac{1}{|t|}\right),$$

where $c' > 0$; hence

$$J_2 \leqslant C \int_{C_1/a_k}^{\varepsilon} \left(1 - c't^\beta L\left(\frac{1}{t}\right)\right)^k \mathrm{d}t \leqslant C \int_{C_1/a_k}^{\varepsilon} e^{-kc't^\beta L(1/t)}\, \mathrm{d}t.$$

Setting $u = ta_k$, it follows from Remark 4.2.12 that if $\varepsilon > 0$ is small enough, then $L\left(a_k/u\right)/L(a_k) \leqslant \max\left(u^{+\beta/2}, u^{-\beta/2}\right)$, which yields

$$\begin{aligned}\int_{C_1/a_k}^{\varepsilon} e^{-kc't^\beta L(1/t)}\, \mathrm{d}t &= \frac{1}{a_k} \int_{C_1}^{\varepsilon a_k} e^{-kc'|u|^\beta |a_k|^{-\beta} L(a_k/u)}\, \mathrm{d}u \\ &\leqslant \frac{1}{a_k} \int_{C_1}^{\varepsilon a_k} e^{-kc'|u|^{\beta\pm\beta/2} |a_k|^{-\beta} L(a_k)}\, \mathrm{d}u.\end{aligned}$$

One achieves the proof of Lemma 5.2.5 noticing that $a_k^\beta = kL(a_k)$ so that

$$J_2 \leqslant \frac{C}{a_k} \int_{C_1}^{\varepsilon a_k} e^{-c'u^{\beta\pm\beta/2}}\, \mathrm{d}u \leqslant \frac{C}{a_k} \int_{C_1}^{\infty} e^{-c'u^{\beta\pm\beta/2}}\, \mathrm{d}u. \qquad \square$$

Let us now deal with the proof of Proposition A.2. *For any $j \in [\![1, p+q]\!]$, we fix $y_j \in \Lambda\backslash\Lambda_j$ and we denote by $\Gamma(k,j)$ the set of isometries $\gamma \in \Gamma$ with symbolic length $|\gamma| = k$ such that $l(\gamma)$ equals $j$.* Let us introduce the following notation.

1. For any $t > 0$ and $\Delta > 0$, let

$$\mathscr{A}(t, \Delta) := \{\gamma \in \Gamma \mid t - \Delta \leqslant \mathrm{d}(\mathbf{o}, \gamma\cdot\mathbf{o}) < t + \Delta\};$$

2. for any $l \in \mathbb{N}^*$, $c, e \in \mathbb{R}^{+*}$, let

$$Q(l, c, e) = \sum_{j=1}^{p+q} \sum_{\substack{\gamma \in \Gamma(l,j) \\ \gamma \in \mathscr{A}(c,e)}} e^{-\delta b(\gamma, y_j)};$$

3. let $\Omega_r(j_1, j_2, j_3) \subset \Gamma(k)$ be the set defined for any $r \in [\![1, k]\!]$ and any $j_1, j_2, j_3 \in [\![1, p+q]\!]$, $j_1 \neq j_2$ and $j_2 \neq j_3$ by

– $\Omega_1(j_1, j_2, j_3) := \{\alpha_1 \cdots \alpha_k \mid \alpha_1 \in \Gamma^*_{j_2}, \alpha_k \in \Gamma^*_{j_3}\}$;
– $\Omega_r(j_1, j_2, j_3) := \{\alpha_1 \cdots \alpha_k \mid \alpha_{r-1} \in \Gamma^*_{j_1}, \alpha_r \in \Gamma^*_{j_2}, \alpha_{r+1} \in \Gamma^*_{j_3}\}$, if $2 \leqslant r \leqslant k-1$;
– $\Omega_k(j_1, j_2, j_3) := \{\alpha_1 \cdots \alpha_k \mid \alpha_{k-1} \in \Gamma^*_{j_1}, \alpha_k \in \Gamma^*_{j_2}\}$;

4. if $\gamma = \alpha_1 \cdots \alpha_k \in \Gamma(k)$, $k \geqslant 1$, we write $\gamma_{(0)} = \gamma^{(k+1)} = \mathrm{Id}$ and for any $j \in [\![1, k]\!]$, we write $\gamma_{(j)} = \alpha_1 \cdots \alpha_j$ and $\gamma^{(j)} = \alpha_j \cdots \alpha_k$.

Since $Ka_k \leqslant R$, we may write $R = wa_k$ for some $w \geqslant 1$; we introduce the following truncation level: $\zeta = w^\theta a_k/2 \in [a_k/2, R/2]$, for $\theta \in ]0, 1[$ close to 1, which will be specified in the proof. We use it to split the set

$$\mathfrak{J} = \{(\alpha_1, \ldots, \alpha_k) \mid \alpha_1 \cdots \alpha_k \text{ admissible}, \ b(\gamma, x) \overset{M}{\sim} R - s\}, \tag{42}$$

into $\mathfrak{J} = \mathfrak{J}_1 \cup \mathfrak{J}_2 \cup \mathfrak{J}_3 \cup \mathfrak{J}_4$, where

$$\mathfrak{J}_1 = \{(\alpha_1, \ldots, \alpha_k) \in J | \ \exists r, \ \mathrm{d}(\mathbf{o}, \alpha_r \cdot \mathbf{o}) \geqslant \frac{R}{2}\};$$
$$\mathfrak{J}_2 = \{(\alpha_1, \ldots, \alpha_k) \in J \mid \forall j, \ \mathrm{d}(\mathbf{o}, \alpha_j \cdot \mathbf{o}) < \frac{R}{2}; \ \exists r < t, \ \mathrm{d}(\mathbf{o}, \alpha_r \cdot \mathbf{o}), \mathrm{d}(\mathbf{o}, \alpha_t \cdot \mathbf{o}) \geqslant \zeta\};$$
$$\mathfrak{J}_3 = \{(\alpha_1, \ldots, \alpha_k) \in J \mid \forall j, \ \mathrm{d}(\mathbf{o}, \alpha_j \cdot \mathbf{o}) < \frac{R}{2}; \ \exists! r, \ \mathrm{d}(\mathbf{o}, \alpha_r \cdot \mathbf{o}) \geqslant \zeta\};$$
$$\mathfrak{J}_4 = \{(\alpha_1, \ldots, \alpha_k) \in J \mid \forall j, \ \mathrm{d}(\mathbf{o}, \alpha_j \cdot \mathbf{o}) < \zeta\}.$$

We are going to prove that there exists a constant $C > 0$ such that for any $i \in \{1, 2, 3, 4\}$

$$\Sigma_i := \sum_{\substack{\gamma = \alpha_1 \cdots \alpha_k \\ (\alpha_1, \ldots, \alpha_k) \in \mathfrak{J}_i}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma, x)} \leqslant Ck \frac{L(R)}{R^{1+\beta}}.$$

*Contribution of* $\Sigma_1$. By definition of $\mathfrak{J}_1$, if $\gamma = \alpha_1 \cdots \alpha_k$ with $(\alpha_1, \ldots, \alpha_k) \in \mathfrak{J}_1$, there exists $r \in [\![1,k]\!]$ such that $\mathrm{d}(\mathbf{o},\alpha_r\cdot\mathbf{o}) \geqslant R/2$. The cocycle property of $b(\gamma,x)$ furnishes

$$\begin{aligned}\Sigma_1 &= \sum_{r=1}^{k} \sum_{\substack{\gamma=\alpha_1\cdots\alpha_k \\ b(\gamma,x)\overset{M}{\sim}R-s \\ \mathrm{d}(\mathbf{o},\alpha_r\cdot\mathbf{o})\geqslant R/2}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma_{(r-1)},\gamma^{(r)}\cdot x)} e^{-\delta b(\alpha_r,\gamma^{(r+1)}\cdot x)} e^{-\delta b(\gamma^{(r+1)},x)} \\ &= \sum_{r=1}^{k} \sum_{\substack{j_1,j_2,j_3 \\ j_1\neq j_2, j_2\neq j_3}} \sum_{\substack{\gamma\in\Omega_r(j_1,j_2,j_3) \\ b(\gamma,x)\overset{M}{\sim}R-s}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma_{(r-1)},\gamma^{(r)}\cdot x)} e^{-\delta b(\alpha_r,\gamma^{(r+1)}\cdot x)} e^{-\delta b(\gamma^{(r+1)},x)}.\end{aligned}$$

Proposition 4.2.3 implies the existence of $D>0$ such that

$$\begin{cases} \left|b(\gamma_{(r-1)},\gamma^{(r)}\cdot x) - b(\gamma_{(r-1)},y_{j_1})\right| \leqslant D, \\ \left|b(\alpha_r,\gamma^{(r+1)}\cdot x) - b(\alpha_r,y_{j_2})\right| \leqslant D, \\ \left|b(\gamma^{(r+1)},x) - b(\gamma^{(r+1)},y_{j_3})\right| \leqslant D, \end{cases}$$

for any $r \in [\![1,k]\!]$, any $j_1,j_2,j_3 \in [\![1,p+q]\!]$, $j_1\neq j_2$, $j_2\neq j_3$ and $\gamma\in\Omega_r(j_1,j_2,j_3)$. Hence

$$\Sigma_1 \preceq \sum_{r=1}^{k} \sum_{\substack{j_1,j_2,j_2 \\ j_1\neq j_2, j_2\neq j_3}} \sum_{\substack{\gamma\in\Omega_r(j_1,j_2,j_3) \\ b(\gamma,x)\overset{M}{\sim}R-s}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma_{(r-1)},y_{j_1})} e^{-\delta b(\alpha_r,y_{j_2})} e^{-\delta b(\gamma^{(r+1)},y_{j_3})}.$$

Combining $b(\gamma,x) \overset{M}{\sim} R-s$ with the cocycle property of $b(\gamma,x)$, we obtain

$$b(\gamma_{(r-1)},\gamma^{(r)}\cdot x) + b(\alpha_r,\gamma^{(r+1)}\cdot x) + b(\gamma^{(r+1)},x) \overset{M}{\sim} R-s,$$

then

$$b(\gamma_{(r-1)},y_{j_1}) + b(\alpha_r,y_{j_2}) + b(\gamma^{(r+1)},y_{j_3}) \overset{M+3D}{\sim} R-s.$$

By Property 4.1.3, the condition $\mathrm{d}(\mathbf{o},\alpha_r\cdot\mathbf{o}) \geqslant R/2$ yields

$$b(\alpha_r,y_{j_2}) \geqslant \frac{R}{2} - C.$$

Therefore $b(\gamma_{(r-1)},y_{j_1}) + b(\gamma^{(r+1)},y_{j_3}) \leqslant R/2 - s + M + 3D + C$. Write $\Delta = M+3D+C$. Let $m,n\in\mathbb{N}^*$ such that $b(\gamma_{(r-1)},y_{j_1}) \in [(m-1)\Delta,(m+1)\Delta]$ and $b(\gamma^{(r+1)},y_{j_3}) \in [(n-1)\Delta,(n+1)\Delta]$ for $m\leqslant N$ and $n\leqslant N-m$, where

$$N := \left[\frac{R/2-s+\Delta}{2\Delta}\right] + 1.$$

Using the previous notation, we may bound $\Sigma_1$ from above by

$$\sum_{r=1}^{k} \sum_{m+n\leqslant N} \Big( Q(r-1, m\Delta, 2\Delta+2C) \\ Q(1, R-s-(m+n)\Delta, 3\Delta+2C) Q(k-r, n\Delta, 2\Delta+2C) \Big).$$

For $0 \leqslant n, m \leqslant N$ such that $n+m \leqslant N$, Property 4.1.3 implies

$$\begin{aligned} Q(1, R-s-(m+n)\Delta, 3\Delta+2C) &= \sum_{j_2=1}^{p+q} \sum_{\substack{\alpha\in\Gamma_{j_2}^* \\ \alpha\in \mathscr{A}(R-s-(m+n)\Delta, 3\Delta+2C)}} e^{-\delta b(\alpha, y_{j_2})} \\ &\preceq \sum_{\substack{|\alpha|=1 \\ \alpha\in \mathscr{A}(R-s-(m+n)\Delta, 3\Delta+2C)}} e^{-\delta \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})}. \end{aligned}$$

Combining Assumption (S) of the family $(H_\beta)$ and Potter's lemma, we obtain

$$\begin{aligned} Q(1, R-s-(m+n)\Delta, 3\Delta+2C) &\leqslant \sup_{t\geqslant R/2-s} \sum_{\substack{|\alpha|=1 \\ \alpha\in \mathscr{A}(t, 3\Delta+2C)}} e^{-\delta \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})} \\ &\preceq \sup_{t\geqslant R/2-s} \frac{L(t)}{t^{1+\beta}} \preceq \frac{L(R)}{R^{1+\beta}}. \end{aligned}$$

Thus $R^{1+\beta}L(R)^{-1}\Sigma_1$ may be bounded up to a multiplicative constant by

$$\sum_{r=1}^{k} \Bigg( \sum_{j_1=1}^{p+q} \sum_{\gamma_1\in\Gamma(r-1)} \mathbb{1}_{\Lambda^c_{l(\gamma_1)}}(y_{j_1}) e^{-\delta b(\gamma_1, y_{j_1})} \Bigg) \Bigg( \sum_{j_3=1}^{p+q} \sum_{\gamma_2\in\Gamma(k-r)} \mathbb{1}_{\Lambda^c_{l(\gamma_2)}}(y_{j_3}) e^{-\delta b(\gamma_2, y_{j_3})} \Bigg)$$

and Lemma 5.2.4 finally gives $\Sigma_1 \preceq kR^{-1-\beta}L(R)$.

*Contribution of* $\Sigma_2$. If $\gamma = \alpha_1 \cdots \alpha_k$ with $(\alpha_1, \ldots, \alpha_k) \in \mathfrak{J}_2$, there exist $r < t$ in $[\![1, k]\!]$ such that $\mathrm{d}(\mathbf{o}, \alpha_r\cdot\mathbf{o}), \mathrm{d}(\mathbf{o}, \alpha_t\cdot\mathbf{o}) > \zeta$. We decompose $\Sigma_2$ as $\Sigma_1$ according to the values of $r$ and $t$, which leads us to the following upper bound for $\Sigma_2$

$$\sum_{r<t} \sum_{m+n+l\leqslant N} \Bigg( Q(r-1, m\Delta, 2\Delta+2C) Q(1, R-\zeta-s-(m+n+l)\Delta, 3\Delta+2C) \\ Q(r-t-1, n\Delta, 2\Delta+2C) Q(k-t, l\Delta, 2\Delta+2C) \sum_{j=1}^{p+q} \sum_{\substack{\alpha\in\Gamma_j^* \\ \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o})\geqslant\zeta}} e^{-\delta b(\alpha, y_j)} \Bigg),$$

where $\Delta = M + 5D + 2C$ and $N = \left[\frac{R-2\zeta-s+\Delta}{2\Delta}\right] + 1$. We write

$$Q(1, R-\zeta-s-(m+n+l)\Delta, 3\Delta+2C) = \sum_{j_2=1}^{p+q} \sum_{\substack{\alpha\in\Gamma^*_{j_2} \\ \alpha\in\mathscr{A}(R-\zeta-s-(m+n+l)\Delta, 3\Delta+2C)}} e^{-\delta b(\alpha, y_{j_2})} \preceq \frac{L(\zeta)}{\zeta^{1+\beta}}.$$

Assumptions ($P_2$) and (N) combined with Property 4.1.3 imply

$$\sum_{j=1}^{p+q} \sum_{\substack{\alpha\in\Gamma^*_j \\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})\geqslant\zeta}} e^{-\delta b(\alpha,y_j)} \preceq \frac{L(\zeta)}{\zeta^{\beta}}.$$

We bound

$$\sum_{m+n+l\leqslant N} Q(r-1, m\Delta, 2\Delta+2C)Q(r-t-1, n\Delta, 2\Delta+2C)Q(k-t, l\Delta, 2\Delta+2C)$$

from above by $C^3$ using Lemma 5.2.4. Summing over $r < t$, we obtain

$$\Sigma_2 \preceq k^2\zeta^{-2\beta-1}L(\zeta)^2.$$

Since $k \sim a_k^{\beta}/L(a_k)$, $2\zeta/a_k = w^{\theta}$ and $R/\zeta = 2w^{1-\theta}$, the last inequality may be reformulated as follows

$$\Sigma_2 \preceq k\frac{a_k^{\beta}}{L(a_k)}\frac{L(\zeta)}{\zeta^{\beta}}\frac{L(\zeta)}{\zeta^{\beta+1}}\frac{R^{\beta+1}}{L(R)}R^{-\beta-1}L(R).$$

By Potter's lemma, for $\varepsilon > 0$, one gets

$\dfrac{L(\zeta)}{L(a_k)} \leqslant w^{\varepsilon}$ and $\dfrac{L(\zeta)}{L(R)} \leqslant w^{\varepsilon}$; therefore $\Sigma_2 \preceq k\cdot w^{-\beta\theta+\varepsilon}\cdot w^{(1-\theta)(\beta+1)+\varepsilon}R^{-\beta-1}L(R)$.

If $\beta\theta > (1-\theta)(\beta+1)$ (*i.e.*, $\theta > (1+\beta)/(1+2\beta)$), the power of $w$ may be chosen negative for $\varepsilon$ small enough. Finally $\Sigma_2 \preceq kR^{-1-\beta}L(R)$.

*Contribution of* $\Sigma_3$. If $\gamma = \alpha_1\cdots\alpha_k$ with $(\alpha_1,\ldots,\alpha_k) \in \mathfrak{J}_3$, there exists a unique integer $r \in [\![1,k]\!]$ such that $\mathrm{d}(\mathbf{o}, \alpha_r\cdot\mathbf{o}) > \zeta$. We deal separately with the cases $w \leqslant k$ and $w > k$.

When $w \leqslant k$, either $r \leqslant k/w$ or $r \geqslant k+1-k/w$, or $r \in [k/w, k+1-k/w]$.

a) If $r \leqslant k/w$ or $r \geqslant k+1-k/w$, we bound

$$\sum_{\substack{\gamma=\alpha_1\cdots\alpha_k \\ b(\gamma,x)\overset{M}{\sim}R-s \\ \mathrm{d}(\mathbf{o},\alpha_r\cdot\mathbf{o})\geqslant R/2}} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x)e^{-\delta b(\gamma_{(r-1)},\gamma^{(r)}\cdot x)}e^{-\delta b(\alpha_r,\gamma^{(r+1)}\cdot x)}e^{-\delta b(\gamma^{(r+1)},x)} \tag{43}$$

from above by

$$\sum_{m+n\leqslant N}\Big(Q(r-1,m\Delta,2\Delta+2C)Q(1,R-s-(m+n)\Delta,3\Delta+2C)$$
$$Q(k-r,n\Delta,2\Delta+2C)\Big),$$

where $\Delta = M+3D+C$ and $N = \left[\frac{R-\zeta-s+\Delta}{2\Delta}\right]+1$. As for $\Sigma_1$, for any $m,n$ such that $m+n\leqslant N$, we bound $Q(1,R-s-(m+n)\Delta,3\Delta+2C)$ from above by $C\zeta^{-1-\beta}L(\zeta)$. Moreover Lemma 5.2.4 allows us to bound from above the quantity

$$\sum_{m+n\leqslant N}\left(Q(r-1,m\Delta,2\Delta+2C)Q(k-r,n\Delta,2\Delta+2C)\right).$$

There are at most $2k/w$ such terms; their contribution is thus less than

$$C\frac{k}{w}\zeta^{-\beta-1}L(\zeta)| = Ckw^{-1}\frac{L(\zeta)}{\zeta^{\beta+1}}\frac{R^{\beta+1}}{L(R)}R^{-\beta-1}L(R)$$
$$\preceq kw^{-1}w^{(1-\theta)(\beta+1)+\varepsilon}R^{-\beta-1}L(R),$$

since

$$\frac{R^{\beta+1}}{\zeta^{\beta+1}} = 2^{\beta+1}w^{(1-\theta)(\beta+1)} \text{ and } \frac{L(\zeta)}{L(R)}\leqslant \max\left(2w^{1-\theta},\frac{1}{2}w^{\theta-1}\right)^{\varepsilon/(1-\theta)} = 2^{\varepsilon/(1-\theta)}w^{\varepsilon}.$$

For $\theta$ close enough to 1 (*i.e.*, $\theta > \beta/(1+\beta)$ here), the power of $w$ is negative and the contribution is finally $\preceq kR^{-\beta-1}L(R)$.

b) Assume now that $r\in[k/w,k+1-k/w]$. The condition

$$R-s-M\leqslant b(\gamma,x)\leqslant R-s+M$$

and the cocycle property of $b(\gamma,x)$ both imply

$$b(\gamma_{(r-1)},y_{j_1})+b(\alpha_r,y_{j_2})+b(\gamma^{(r+1)},y_{j_3}) \overset{M+3D}{\sim} R-s$$

for any $r\in[\![1,k]\!]$, any $(j_1,j_2,j_3)\in[\![1,p+q]\!]$, $j_1\neq j_2$, $j_2\neq j_3$ and any $\gamma\in\Omega_r(j_1,j_2,j_3)$. Fix $r$, $j_1$, $j_2$, $j_3$ and $\gamma\in\Omega_r(j_1,j_2,j_3)$ as above. From $\mathrm{d}(\mathbf{o},\alpha_r\cdot\mathbf{o}) < R/2$, we deduce $b(\alpha_r,y_{j_2}) < R/2+C$, hence

$$b(\gamma_{(r-1)},y_{j_1})+b(\gamma^{(r+1)},y_{j_3})\geqslant\frac{R}{2}-s-M-3D-C.$$

This last upper bound yields

$$1)\ b(\gamma_{(r-1)},y_{j_1})\geqslant\frac{R}{4}-\frac{s}{2}-\frac{M+3D+C}{2}$$

or

$$2)\ b(\gamma^{(r+1)},y_{j_3})\geqslant\frac{R}{4}-\frac{s}{2}-\frac{M+3D+C}{2}.$$

We only detail the control of the sum in the case 1); the other case may be treated similarly. Set $\Delta = M + 3D$ and let $m, n, l \in \mathbb{N}^*$ such that

$$\begin{cases} b(\gamma_{(r-1)}, y_{j_1}) \in [(m-1)\Delta, (m+1)\Delta], \\ b(\alpha_r, y_{j_2}) \in [(n-1)\Delta, (n+1)\Delta], \\ b(\gamma^{(r+1)}, y_{j_3}) \in [(l-1)\Delta, (l+1)\Delta], \end{cases}$$

for $m \leqslant N$, $n \leqslant N - m$ and $l = N - m - n$, where $N = [R - s/\Delta] + 1$. The sum (43) for $r \in [k/w, k+1-k/w]$ may thus be bounded from above by

$$\sum_{m+n+l=N} \Bigg( Q(r-1, m\Delta, 2\Delta + 2C) \sum_{j_2=1}^{p+q} \sum_{\substack{\alpha \in \mathscr{A}(l\Delta, 3\Delta+2C) \cap \Gamma^*_{j_2} \\ \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) > \zeta}} e^{-\delta b(\alpha, y_{j_2})} \\ Q(k-r, n\Delta, 2\Delta + 2C) \Bigg),$$

which is smaller than

$$(\star) \quad \sup_m \left(Q(r-1, m\Delta, 2\Delta + 2C)\right) \times \left( \sum_{j_2=1}^{p+q} \sum_{\substack{\alpha \in \Gamma^*_{j_2} \\ \mathrm{d}(\mathbf{o}, \alpha\cdot\mathbf{o}) > \zeta}} e^{-\delta b(\alpha, y_{j_2})} \right) \\ \times \left( \sum_{j_3=1}^{p+q} \sum \big|_{|\gamma_2| = k-r} \mathbb{1}_{\Lambda^c_{l(\gamma_2)}}(y_{j_3}) e^{-\delta b(\gamma_2, y_{j_3})} \right),$$

where the supremum is taken over $m \in \mathbb{N}^*$ such that $m\Delta \geqslant \frac{R}{4} - \frac{s}{2} - \frac{3}{2}(\Delta + C)$. We combined Lemma 5.2.5 and the fact that $s$ lies in a compact subset of $\mathbb{R}$ to control the first factor, Assumptions ($\mathrm{P}_2$) and (N) for the second factor and Lemma 5.2.4 for the third one, which allows us to bound the quantity $(\star)$ from above by

$$C \frac{e^{-R/(4\zeta)}}{a_{r-1}} \zeta^{-\beta} L(\zeta),$$

where $C$ only depends on the support of $\varphi$. Using the regular variation of $(a_r)_r$, we obtain $a_r/a_{r-1} \leqslant C$ for some $C > 0$ and the quantity $(\star)$ is bounded from above, up to a multiplicative constant, by

$$e^{-R/(4\zeta)} \zeta^{-\beta} \frac{L(\zeta)}{a_r}.$$

Since $k/w \leqslant r \leqslant k+1-k/w$ and $(a_i)_i$ is regularly varying with exponent $1/\beta$, Potter's lemma implies $a_r \geqslant a_k/w^{1/\beta+\varepsilon}$; combining with the equalities $a_k = R/w$ and $R/\zeta = 2w^{1-\theta}$, this yields to an upper bound of the form

$$Ce^{-w^{1-\theta}/2} w^{C'} R^{-\beta-1} L(R) \quad \left(\text{with } C' = \frac{1}{\beta} + (1-\theta)(1+\beta) + \varepsilon\right),$$

which is smaller than $CR^{-\beta-1}L(R)$ since $\theta < 1$. Since the integer $r$ takes at most $k$ values, the result follows for $\Sigma_3$.

When $k \leqslant w$, then $r \in [k/w, k+1-k/w]$ and the proof is the same as for the case b).

*Contribution of* $\Sigma_4$. By Lemma 5.2.5

$$\Sigma_4 \leqslant C\frac{e^{-R/\zeta}}{a_k},$$

with $C\frac{e^{-R/\zeta}}{a_k} = C\frac{e^{-2w^{1-\theta}}}{a_k}$ and $kR^{-\beta-1}L(R) \sim \frac{a_k^\beta}{L(a_k)}(wa_k)^{-\beta-1}L(wa_k) = \frac{w^{-\beta-1\pm\varepsilon}}{a_k}$. This completes the proof for $\Sigma_4$ and the proof of Proposition A.2.

# CHAPTER 6

## THEOREM A: MIXING FOR $\beta = 1$

This section is devoted to the proof of Theorem A when $\beta = 1$. Let $\Gamma$ be a Schottky group satisfying the family of assumptions $(H_\beta)$ for $\beta = 1$. The arguments here are slightly different from the case $\beta \in\, ]0,1[$. Indeed, recall that Karamata's lemma asserts that $\tilde{L}$ is a slowly varying function, which additionally satisfies

$$\lim_{x \longrightarrow +\infty} \frac{L(x)}{\tilde{L}(x)} = 0,$$

and the fact that the Bowen-Margulis measure $m_\Gamma$ is infinite implies $\lim\limits_{x \longrightarrow +\infty} \tilde{L}(x) = +\infty$. The proofs in this section are inspired by that of Theorem 2.1 of [**36**] in the case $\beta = 1$.

The argument of Subsection 5.1 applies verbatim in our setting. Therefore, still writing $M(R; A, B) = m_\Gamma(A.B \circ g_R)$, we have to prove that, as $R \longrightarrow \pm\infty$,

$$M(R; \varphi \otimes u, \psi \otimes v) \sim \frac{1}{E_\Gamma} \frac{m_\Gamma(\varphi \otimes u) m_\Gamma(\psi \otimes v)}{\tilde{L}(|R|)},$$

where $\varphi, \psi : \mathscr{D}^0 \longrightarrow \mathbb{R}$ are Lipschitz functions on $\Lambda$ and $u, v : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions on $\mathbb{R}$ with compact support. The arguments presented in Subsection 5.1 allow us to treat only the case $R \longrightarrow +\infty$; Lemma 5.1.1 implies in particular, for $R$ large enough

$$\begin{aligned} M(R; \varphi \otimes u, \psi \otimes v) &= M^+(R; \varphi \otimes u, \psi \otimes v) \\ &= \sum_{k \geqslant 0} \int_{\Lambda \times \mathbb{R}} \tilde{P}^k (\varphi \otimes u)(x, s - R) \psi \otimes v\,(x, s)\, \nu(\,\mathrm{d}x)\,\mathrm{d}s. \end{aligned}$$

In the sequel, we will need to consider the integral near 0 of the function $t \longmapsto Q_{\delta + it}$, where $Q_z = (\mathrm{Id} - \mathscr{L}_z)^{-1}$ for any $z \in \mathbb{C}$ with $\mathrm{Re}\,(z) \geqslant \delta$. However, Proposition 4.2.16 ensures that $t \longmapsto Q_{\delta+it}$ is not integrable in 0. To overcome this problem, we proceed as in the proof of Theorem 6.1 of [18] and introduce a symmetrized version $\left(\tilde{P}^{\mathrm{sym}}\right)^k (\varphi \otimes u)(x, s-R)$ of $\tilde{P}^k (\varphi \otimes u)(x, s-R)$, defined as follows: for any $k \geqslant 1$

$$\left(\tilde{P}^{\mathrm{sym}}\right)^k (\varphi \otimes u)(x, s-R) = \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\delta b(\gamma, x)} \frac{h\varphi(\gamma \cdot x)}{h(x)} \left[u(s-R+b(\gamma,x)) + u(s-R-b(\gamma,x))\right].$$

Thus we study the term $M^{\mathrm{sym}}(R; \varphi \otimes u, \psi \otimes v)$ defined by

$$M^{\mathrm{sym}}(R; \varphi \otimes u, \psi \otimes v) = \sum_{k \geqslant 0} \int_{\Lambda \times \mathbb{R}} \left(\tilde{P}^{\mathrm{sym}}\right)^k (\varphi \otimes u)(x, s-R) \psi \otimes v\,(x,s)\, \nu(\mathrm{d}x)\, \mathrm{d}s$$

with the convention $\left(\tilde{P}^{\mathrm{sym}}\right)^0 (\varphi \otimes u) = \varphi \otimes u$. Since $u$ has compact support, for any $s \in \operatorname{supp} v$ and for any $k \in \mathbb{N}$, we have as $R \longrightarrow +\infty$

$$\left(\tilde{P}^{\mathrm{sym}}\right)^k (\varphi \otimes u)(x, s-R) = \tilde{P}^k (\varphi \otimes u)(x, s-R).$$

It is thus sufficient to prove that as $R \longrightarrow +\infty$

$$M^{\mathrm{sym}}(R; \varphi \otimes u, \psi \otimes v) \sim \frac{1}{E_\Gamma} \frac{m_\Gamma(\varphi \otimes u) m_\Gamma(\psi \otimes v)}{\tilde{L}(R)}. \tag{44}$$

Remark 6.0.1. – *If we fix $\varphi, \psi$ (resp. the function $v$) in the space of Lipschitz functions (resp. in the space of continuous functions on $\mathbb{R}$ with compact support), statement* (44) *is equivalent to the weak convergence to $m_\Gamma(\varphi \otimes \bullet) m_\Gamma(\psi \otimes v)/E_\Gamma$ of the sequence of measures $\left(\tilde{L}(R) M^{\mathrm{sym}}(R; \varphi \otimes \bullet, \psi \otimes v)\right)_R$. By the argument already mentioned in the proof of Proposition A.1 (p. 49), it is thus sufficient to prove that $\tilde{L}(R) M^{\mathrm{sym}}(R; \varphi \otimes u, \psi \otimes v)$ is finite and converges to $m_\Gamma(\varphi \otimes u) m_\Gamma(\psi \otimes v)/E_\Gamma$ for any function $u : \mathbb{R} \longrightarrow \mathbb{R}$ in the set of test functions $\mathscr{U}$.*

## 6.1. Proof of (44)

Let $u \in \mathscr{U}$. We first introduce the following quantity: for $\xi > \delta$, $k \geqslant 1$, $x \in \Lambda$ and $s \in \mathbb{R}$

$$\left(\tilde{P}^{\mathrm{sym}}_\xi\right)^k (\varphi \otimes u)(x, s-R) := \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{\Lambda^c_{l(\gamma)}}(x) e^{-\xi b(\gamma, x)} \frac{h\varphi(\gamma \cdot x)}{h(x)} \left(u(s-R+b(\gamma,x)) + u(s-R-b(\gamma,x))\right).$$

Similarly, we set

(45)
$$M_\xi^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v) := \sum_{k\geqslant 0}\int_{\Lambda\times\mathbb{R}}\left(\tilde{P}_\xi^{\mathrm{sym}}\right)^k(\varphi\otimes u)\,(x,s-R)\psi\otimes v\,(x,s)\,\nu(\,\mathrm{d}x)\,\mathrm{d}s$$

with the convention $\left(\tilde{P}_\xi^{\mathrm{sym}}\right)^0(\varphi\otimes u)=\varphi\otimes u$. The convergence of the Poincaré series of $\Gamma$ at $\xi>\delta$ yields

$$M_\xi^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v) = \int_{\Lambda\times\mathbb{R}}\left(\sum_{k\geqslant 0}\left(\tilde{P}_\xi^{\mathrm{sym}}\right)^k(\varphi\otimes u)\,(x,s-R)\right)\psi(x)v(s)\nu(\,\mathrm{d}x)\,\mathrm{d}s;$$

moreover

$$\begin{aligned}\sum_{k\geqslant 0}&\left(\tilde{P}_\xi^{\mathrm{sym}}\right)^k(\varphi\otimes u)\,(x,s-R)\\ &= \frac{1}{2\pi h(x)}\sum_{k\geqslant 0}\int_{\mathbb{R}} e^{it(R-s)}\left(\mathscr{L}^k_{\xi+it}+\mathscr{L}^k_{\xi-it}\right)(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t\\ &= \frac{1}{2\pi h(x)}\int_{\mathbb{R}} e^{it(R-s)}\left[\sum_{k\geqslant 0}\left(\mathscr{L}^k_{\xi+it}+\mathscr{L}^k_{\xi-it}\right)(h\varphi)(x)\right]\widehat{u}(t)\,\mathrm{d}t\end{aligned}$$

for any $x\in\Lambda$ and $s\in\mathbb{R}$, so that the term $M_\xi^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v)$ equals

$$\int_{\Lambda\times\mathbb{R}}\left(\frac{1}{\pi h(x)}\int_{\mathbb{R}} e^{it(R-s)}\mathrm{Re}\,(Q_{\xi+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t\right)\psi(x)v(s)\nu(\,\mathrm{d}x)\,\mathrm{d}s.$$

To show (44), we have to understand how to relate the quantities $M_\xi^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v)$ and $M^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v)$. This is the purpose of the following proposition.

Proposition 6.1.1. – *We have*

$$\begin{aligned}\lim_{\xi\searrow\delta} &M_\xi^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v)\\ &= \int_{\Lambda\times\mathbb{R}}\left(\frac{1}{\pi h(x)}\int_{\mathbb{R}} e^{it(R-s)}\mathrm{Re}\,(Q_{\delta+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t\right)\psi(x)v(s)\nu(\,\mathrm{d}x)\,\mathrm{d}s\\ &= M^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v).\end{aligned}$$

*Proof.* – This result relies on the two following remarks:

1) $\lim_{\xi\searrow\delta} M_\xi^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v)$ exists and is equal to

$$\int_{\Lambda\times\mathbb{R}}\left(\frac{1}{\pi h(x)}\int_{\mathbb{R}} e^{it(R-s)}\mathrm{Re}\,(Q_{\delta+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t\right)v(x)\psi(s)\nu(\,\mathrm{d}x)\,\mathrm{d}s;$$

2) this limit also equals $M^{\mathrm{sym}}(R;\varphi\otimes u,\psi\otimes v)$.

To prove the first point, it is sufficient to check that

$$(46)\qquad \int_{\mathbb{R}} e^{it(R-s)}\mathrm{Re}\,(Q_{\xi+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t \underset{\xi\searrow\delta}{\longrightarrow} \int_{\mathbb{R}} e^{it(R-s)}\mathrm{Re}\,(Q_{\delta+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t$$

uniformly in $x \in \Lambda$ and $s \in \operatorname{supp} v$. We postpone the proof to the next Subsection 6.2. Let us show how the second assertion follows from the first one. First, assume that the functions $\varphi, \psi, u$ and $v$ are positive. From (45), the monotone convergence theorem implies that $M_\xi^{\mathrm{sym}}(R; \varphi\otimes u, \psi\otimes v)$ converges to $M^{\mathrm{sym}}(R; \varphi\otimes u, \psi\otimes v)$. Combining the uniqueness of the limit for $M_\xi^{\mathrm{sym}}(R; \varphi\otimes u, \psi\otimes v)$ and Corollary 4.2.17, we deduce that $M^{\mathrm{sym}}(R; \varphi\otimes u, \psi\otimes v)$ is finite for any positive $\varphi, \psi, u$ and $v$. We use the Lebesgue dominated convergence theorem to prove that it is also the case for arbitrary functions $\varphi, \psi, u$ and $v$: indeed, we bound

$$\int_{\Lambda\times\mathbb{R}} \left(\tilde{P}_\xi^{\mathrm{sym}}\right)^k (\varphi\otimes u)(x, s-R)\psi\otimes v(x,s)\nu(\,\mathrm{d}x)\,\mathrm{d}s$$

from above by

$$(47)\qquad \int_{\Lambda\times\mathbb{R}} \left(\tilde{P}^{\mathrm{sym}}\right)^k (|\varphi|\otimes|u|)(x, s-R)|\psi|\otimes|v|(x,s)\nu(\,\mathrm{d}x)\,\mathrm{d}s,$$

and notice that $|u| \in \mathscr{U}$, it follows from the finiteness of $M^{\mathrm{sym}}(R; |\varphi|\otimes|u|, |\psi|\otimes|v|)$ that the sum of the family of terms (47) does exist. ☐

To complete the proof of (44), it is thus sufficient to show the following proposition.

Proposition 6.1.2. – *Uniformly in $x \in \Lambda$ and $s \in \operatorname{supp} v$, as $R \longrightarrow +\infty$*

$$I := \frac{1}{\pi h(x)} \int_{\mathbb{R}} e^{it(R-s)}\mathrm{Re}\,(Q_{\delta+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t \sim \frac{m_\Gamma(\varphi\otimes u)}{E_\Gamma\tilde{L}(R)}.$$

*Proof.* – Let $A > 0$. We split $I$ into $I_1 + I_2$ where

$$I_1 = \frac{1}{\pi h(x)} \int_{|t|>A/(R-s)} e^{it(R-s)}\mathrm{Re}\,(Q_{\delta+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t$$

and

$$I_2 = \frac{1}{\pi h(x)} \int_{|t|\leqslant A/(R-s)} e^{it(R-s)}\mathrm{Re}\,(Q_{\delta+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t.$$

We first deal with $I_1$. We decompose this integral according the sign of $t$; we only give the arguments for

$$J = \frac{1}{\pi h(x)} \int_{t>A/(R-s)} e^{it(R-s)}\mathrm{Re}\,(Q_{\delta+it})\,(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t.$$

Setting $t = y - \pi/(R-s)$ in $J$, we may write

$$J = -\frac{1}{\pi h(x)} \int_{y>(A+\pi)/(R-s)} e^{iy(R-s)}\mathrm{Re}\,\left(Q_{\delta+i(y-\pi/(R-s))}\right)(h\varphi)(x)\widehat{u}\left(y - \frac{\pi}{R-s}\right)\mathrm{d}y,$$

hence

$$\begin{aligned}2J =& \frac{1}{\pi h(x)}\int_{A/(R-s)}^{(A+\pi)/(R-s)} e^{it(R-s)}\mathrm{Re}\left(Q_{\delta+it}\right)(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t\\ &+\frac{1}{\pi h(x)}\int_{t>(A+\pi)/(R-s)} e^{it(R-s)}\mathrm{Re}\left(Q_{\delta+i(t-\pi/(R-s))}\right)(h\varphi)(x)\left(\widehat{u}(t)-\widehat{u}\left(t-\frac{\pi}{R-s}\right)\right)\mathrm{d}t\\ &+\frac{1}{\pi h(x)}\int_{t>(A+\pi)/(R-s)} e^{it(R-s)}\left(\mathrm{Re}\left(Q_{\delta+it}\right)-\mathrm{Re}\left(Q_{\delta+i(t-\pi/(R-s))}\right)\right)(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t\\ &=: K_1+K_2+K_3.\end{aligned}$$

Let us first deal with $K_1$. By Corollary 4.2.17, for any $t$ close to 0

$$|\mathrm{Re}\left(Q_{\delta+it}\right)(h\varphi)(x)| \preceq \frac{L\left(1/|t|\right)}{|t|\tilde{L}\left(1/|t|\right)^2};$$

therefore

$$\begin{aligned}|K_1| &\preceq \int_{A/(R-s)}^{(A+\pi)/(R-s)} \frac{L\left(1/t\right)}{t\tilde{L}\left(1/t\right)^2}\,\mathrm{d}t\\ &\preceq \frac{1}{\tilde{L}(R)}\frac{L(R)}{\tilde{L}(R)}\int_A^{A+\pi}\frac{1}{t}\frac{L\left((R-s)/t\right)}{L(R)}\frac{\tilde{L}(R)^2}{\tilde{L}\left((R-s)/t\right)^2}\,\mathrm{d}t.\end{aligned}$$

Potter's lemma combined with the fact that $s$ belongs to a compact subset of $\mathbb{R}$ thus implies

$$|K_1| \preceq \frac{1}{\tilde{L}(R)}\frac{L(R)}{\tilde{L}(R)}\int_A^{A+\pi}\frac{\mathrm{d}t}{t^{1/4}} \preceq \frac{1}{\tilde{L}(R)}\frac{L(R)}{\tilde{L}(R)}\left(A+\pi\right)^{3/4}. \tag{48}$$

Similarly

$$|K_2| \preceq \frac{1}{\tilde{L}(R)}\frac{L(R)}{\tilde{L}(R)}. \tag{49}$$

Concerning $|K_3|$, we get

$$|K_3| \leqslant \frac{1}{\pi h(x)}\int_{t>(A+\pi)/(R-s)} ||Q_{\delta+it}||\left||Q_{\delta+i(t-\pi/(R-s))}\right||\left||\mathscr{L}_{\delta+it}-\mathscr{L}_{\delta+i(t-\pi/(R-s))}\right||\,|\widehat{u}(t)|\,\mathrm{d}t.$$

There exists $M>0$ such that the support of $u$ is included in $[-M,M]$; we thus deduce from Propositions 4.2.9 and 4.2.16 that

$$\begin{aligned}|K_3| \preceq& \frac{\tilde{L}\left((R-s)/\pi\right)}{R-s}\int_{(A+\pi)/(R-s)}^{M}\frac{1}{t\tilde{L}\left(1/t\right)}\left(t-\pi/(R-s)\right)^{-1}\tilde{L}\left(\frac{1}{t-\pi/(R-s)}\right)^{-1}\mathrm{d}t\\ \preceq& \frac{\tilde{L}(R)}{R}\int_{A/(R-s)}^{M-\pi/(R-s)}\frac{1}{t\tilde{L}\left(1/t\right)}\left(t+\pi/(R-s)\right)^{-1}\tilde{L}\left(\frac{1}{t+\pi/(R-s)}\right)^{-1}\mathrm{d}t.\end{aligned}$$

Noticing that $\frac{1}{t} = \frac{R-s}{t(R-s)}$, Potter's lemma implies that as $R \longrightarrow +\infty$

$$\tilde{L}\left(\frac{1}{t}\right) \preceq \tilde{L}\left(\frac{1}{t+\pi/(R-s)}\right),$$

hence

$$\begin{aligned}|K_3| \preceq & \frac{\tilde{L}(R)}{R}\int_{A/R}^{M} \frac{1}{t^2\tilde{L}\left(1/t\right)^2}\,\mathrm{d}t \preceq \tilde{L}(R)\int_{A}^{RM} \frac{1}{t^2\tilde{L}\left(R/t\right)^2}\,\mathrm{d}t \\ \preceq & \frac{1}{\tilde{L}(R)}\int_{A}^{RM} \frac{\tilde{L}(R)^2}{t^2\tilde{L}\left(R/t\right)^2}\,\mathrm{d}t \preceq \frac{1}{\tilde{L}(R)}\int_{A}^{RM} \frac{1}{t^{3/2}}\,\mathrm{d}t,\end{aligned}$$

which yields

$$|K_3| \preceq \frac{1}{\tilde{L}(R)}\frac{1}{\sqrt{A}}. \tag{50}$$

Combining (48), (49) and (50), we deduce $\lim_{A\longrightarrow+\infty}\lim_{R\longrightarrow+\infty}\tilde{L}(R)|J| = 0$ uniformly in $x \in \Lambda$ and $s \in \operatorname{supp} v$. Therefore

$$\lim_{A\longrightarrow+\infty}\lim_{R\longrightarrow+\infty}\tilde{L}(R)|I_1| = 0 \tag{51}$$

uniformly in $x \in \Lambda$ and $s \in \operatorname{supp} v$. We now explain why the contribution of $I_2$ is dominant. We write

$$\begin{aligned}I_2 = & \frac{1}{\pi h(x)}\int_{|t|\leqslant A/(R-s)} e^{it(R-s)}\left[\operatorname{Re}\left(Q_{\delta+it}\right) - \operatorname{Re}\left(\left(1-\lambda_{\delta+it}\right)^{-1}\right)\Pi_\delta\right](h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t \\ & + \frac{1}{\pi h(x)}\int_{|t|\leqslant A/(R-s)} e^{it(R-s)}\left[\operatorname{Re}\left(\left(1-\lambda_{\delta+it}\right)^{-1}\right)\Pi_\delta\right](h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t \\ & =: K_1 + K_2.\end{aligned}$$

It follows from Proposition 4.2.16 that $|K_1| \leqslant 2A/R$ for $R$ large enough. We split $K_2$ into $L_1 + L_2$ where

$$L_1 = \frac{1}{\pi h(x)}\int_{|t|\leqslant A/(R-s)} \left(e^{it(R-s)} - 1\right)\left[\operatorname{Re}\left(\left(1-\lambda_{\delta+it}\right)^{-1}\right)\Pi_\delta\right](h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t$$

and

$$L_2 = \frac{1}{\pi h(x)}\int_{|t|\leqslant A/(R-s)} \left[\operatorname{Re}\left(\left(1-\lambda_{\delta+it}\right)^{-1}\right)\Pi_\delta\right](h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t.$$

The equality $\Pi_\delta(h\varphi)(x) = \sigma_{\mathbf{o}}(h\varphi)h(x) = \nu(\varphi)h(x)$ yields

$$L_1 = \frac{\nu(\varphi)}{\pi}\int_{|t|\leqslant A/(R-s)} \left(e^{it(R-s)} - 1\right)\operatorname{Re}\left(\left(1-\lambda_{\delta+it}\right)^{-1}\right)\widehat{u}(t)\,\mathrm{d}t.$$

and the local expansion of $\mathrm{Re}\left((1-\lambda_{\delta+it})^{-1}\right)$ given in (4.2.15) thus implies

$$\begin{aligned}\tilde{L}(R)L_1 = & \ \tilde{L}(R)\frac{\nu(\varphi)}{E_\Gamma}\int_{|y|\leqslant A}\frac{(e^{iy}-1)}{y}\frac{L\left((R-s)/y\right)}{\tilde{L}\left((R-s)/y\right)^2}\widehat{u}\left(\frac{y}{R-s}\right)\mathrm{d}y \\ & \sim \frac{\nu(\varphi)}{E_\Gamma}\frac{L(R-s)}{\tilde{L}(R-s)}\int_{|y|\leqslant A}\frac{(e^{iy}-1)}{y}\frac{L\left((R-s)/y\right)}{L(R-s)}\frac{\tilde{L}(R-s)^2}{\tilde{L}\left((R-s)/y\right)^2}\widehat{u}\left(\frac{y}{R-s}\right)\mathrm{d}y, \\ & \leq A\frac{L(R)}{\tilde{L}(R)}.\end{aligned}$$

It remains now to deal with $L_2$; we decompose it according the sign of $t$. We only give the arguments for $t>0$, i.e., we control

$$M = \frac{1}{\pi h(x)}\int_0^{A/(R-s)}\left[\mathrm{Re}\left((1-\lambda_{\delta+it})^{-1}\right)\Pi_\delta\right](h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t.$$

We follow **[36]** and denote by $H$ the function defined as follows:

$$\mathrm{Re}\left((1-\lambda_{\delta+it})^{-1}\right) = \frac{\pi}{2}\frac{1}{E_\Gamma}\frac{L\left(1/t\right)}{t\tilde{L}^2\left(1/t\right)}(1+H(t)),$$

where $H(t)=o(1)$ when $t$ is close to 0. We obtain

$$\begin{aligned}M &= \frac{\nu(\varphi)}{\pi}\int_0^{A/(R-s)}\mathrm{Re}\left((1-\lambda_{\delta+it})^{-1}\right)\widehat{u}(t)\,\mathrm{d}t \\ &= \frac{\nu(\varphi)}{2E_\Gamma}\int_0^{A/(R-s)}\frac{L\left(1/t\right)}{t\tilde{L}^2\left(1/t\right)}\widehat{u}(t)\,\mathrm{d}t + \frac{\nu(\varphi)}{2E_\Gamma}\int_0^{A/(R-s)}\frac{L\left(1/t\right)}{t\tilde{L}^2\left(1/t\right)}H(t)\widehat{u}(t)\,\mathrm{d}t \\ &= \frac{\nu(\varphi)}{2E_\Gamma}\int_0^{A/(R-s)}\frac{L\left(1/t\right)}{t\tilde{L}^2\left(1/t\right)}\widehat{u}(t)\,\mathrm{d}t + O\left(\sup_{0\leqslant t\leqslant A/(R-s)}|H(t)|\int_0^{A/(R-s)}\frac{L\left(1/t\right)}{t\tilde{L}^2\left(1/t\right)}\,\mathrm{d}t\right).\end{aligned}$$

Setting $y=1/t$, it follows from an integration by parts that

$$\int_0^{A/(R-s)}\frac{L\left(1/t\right)}{t\tilde{L}^2\left(1/t\right)}\widehat{u}(t)\,\mathrm{d}t = \frac{\widehat{u}\left(A/(R-s)\right)}{\tilde{L}\left((R-s)/A\right)} + \int_{(R-s)/A}^{+\infty}\widehat{u}'\left(\frac{1}{y}\right)\tilde{L}(y)^{-1}\frac{\mathrm{d}y}{y^2}.$$

Using the properties of slowly varying functions, we deduce that the second term of the right hand side is negligible with respect to the first term as $R\longrightarrow+\infty$. The regularity of $\widehat{u}$ yields

$$\lim_{R\longrightarrow+\infty}\widehat{u}\left(\frac{A}{R-s}\right) = \widehat{u}(0)$$

uniformly in $s\in\mathrm{supp}\, v$. Finally

$$\tilde{L}(R)M \underset{R\longrightarrow+\infty}{\sim} \frac{\nu(\varphi)\widehat{u}(0)}{2E_\Gamma};$$

hence

$$\tilde{L}(R)L_2 \underset{R\longrightarrow+\infty}{\sim} \frac{1}{E_\Gamma}m_\Gamma(\varphi\otimes u).$$

Combining this result with (51), we finally obtain

$$\lim_{R\longrightarrow+\infty} \frac{\tilde{L}(R)}{\pi h(x)} \int_{\mathbb{R}} e^{it(R-s)} \mathrm{Re}\left(Q_{\delta+it}\right)(h\varphi)(x)\widehat{u}(t)\,\mathrm{d}t = \frac{m_\Gamma(\varphi\otimes u)}{E_\Gamma}$$

uniformly in $x \in \Lambda$ and $s \in \operatorname{supp} v$. This completes the proof of Proposition 6.1.2. □

## 6.2. Proof of Proposition 6.1.1

We follow the steps of Section 6 of [**36**]. To show the convergence, we use the local expansion of $(1-\lambda_{\xi+it})^{-1}$ near 0 to bound the function $t \longmapsto \mathrm{Re}\left(Q_{\xi+it}\right)(h\varphi)(x)$ from above by an integrable function. Let us write $\xi = \delta + \kappa$ where $\kappa > 0$.

PROPOSITION 6.2.1. – *There exist finite measures $\nu_j$ on $\mathbb{R}^+$ with masses $P_j$, $j \in [\![1,p]\!]$ such that*

$$1-\lambda_{\delta+\kappa+it} = \sum_{j=1}^{p} P_j\left[\left(\kappa I_{C_j}+tI_{S_j}\right) - i\left(\kappa I_{S_j} - tI_{C_j}\right)\right] + o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right) + \kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right),$$

*where, for any $j \in [\![1,p]\!]$, $I_{S_j}$ and $I_{C_j}$ are defined as in Proposition 3.2.4 by*

$$I_{S_j} = \int_0^{+\infty} e^{-\kappa y}\sin(ty)\left(1-\nu_j([0,y])\right)\,\mathrm{d}y$$

*and*

$$I_{C_j} = \int_0^{+\infty} e^{-\kappa u}\cos(ty)\left(1-\nu_j([0,y])\right)\,\mathrm{d}y.$$

*Proof.* – The steps are the same as for the local expansion of $\lambda_{\delta+it}$ (in 4.2.15). For $z = \delta+\kappa+it$, we write

$$\lambda_z = \sigma_{\mathbf{o}}(\mathcal{L}_z h_z) = \sigma_{\mathbf{o}}(\mathcal{L}_z h) + \sigma_{\mathbf{o}}\left((\mathcal{L}_z - \mathcal{L}_\delta)(h_z - h)\right).$$

By Proposition 4.2.9, the contribution of the second term is $\preceq \left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right) + |t|\tilde{L}\left(\frac{1}{|t|}\right)\right)^2$. The first term may be decomposed as

$$\begin{aligned}\sigma_{\mathbf{o}}(\mathcal{L}_{\delta+\kappa+it}h) &= 1 + \sigma_{\mathbf{o}}(\mathcal{L}_{\delta+\kappa+it}h) - 1 = 1 + \sigma_{\mathbf{o}}(\mathcal{L}_{\delta+\kappa+it}h) - \sigma_{\mathbf{o}}(\mathcal{L}_\delta h)\\ &= 1 + \sum_{j=1}^{p+q} S_j,\end{aligned}$$

where

$$S_j := \sum_{\alpha\in\Gamma_j^*} \int_{\Lambda\setminus\Lambda_j} h(\alpha\cdot x)e^{-\delta b(\alpha,x)}\left(e^{-(it+\kappa)b(\alpha,x)}-1\right)\mathrm{d}\sigma_{\mathbf{o}}(x).$$

Using the notation of Proposition 4.2.14, we obtain

$$S_j = \int_{\Lambda\setminus\Lambda_j} M_j(x)\left(\widehat{\mu_j^x}(-t+i\kappa)-1\right)\mathrm{d}\sigma_{\mathbf{o}}(x)$$

and we deduce from Proposition 3.2.3 that the contribution of this quantity is $o\left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right)+o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right)\right)$ for $j\in[\![p+1,p+q]\!]$.

Let $j\in[\![1,p]\!]$ and $x_j$ be the fixed point of the elementary parabolic group $\Gamma_j$ and set $\Delta_\alpha(x):=b(\alpha,x)-\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})$ for any $\alpha\in\Gamma_j^*$ and $x\in\Lambda\backslash\Lambda_j$. The sequence $(\Delta_\alpha(x))_{\alpha\in\Gamma_j}$ converges to $-2\,(x|x_j)_{\mathbf{o}}$ as $\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})\longrightarrow+\infty$, uniformly in $x\notin\Lambda_j$. We thus split $S_j$ into $S_{j1}+S_{j2}$ where

$$S_{j1}:=\sum_{\alpha\in\Gamma_j^*}e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}\int_{\Lambda\backslash\Lambda_j}h(\alpha\cdot x)e^{-\delta\Delta_\alpha(x)}\left(e^{-(it+\kappa)b(\alpha,x)}-e^{-(it+\kappa)\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}\right)\mathrm{d}\sigma_{\mathbf{o}}(x)$$

and

$$S_{j2}:=\sum_{\alpha\in\Gamma_j^*}e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}\left(e^{-(it+\kappa)\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}-1\right)\int_{\Lambda\backslash\Lambda_j}h(\alpha\cdot x)e^{-\delta\Delta_\alpha(x)}\,\mathrm{d}\sigma_{\mathbf{o}}(x).$$

Hence

$$|S_{j1}|\leqslant Ce^{\delta C}(|t|+\kappa)\sum_{\alpha\in\Gamma_j^*}e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}=O(|t|+\kappa)=o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right)+\kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right),$$

since $\lim_{x\longrightarrow+\infty}\tilde{L}(x)=+\infty$. We now decomposed $S_{j2}$ as $S_{j2}=P_j\widehat{\nu_j}(-t+i\kappa)-P_j$ where

$$\nu_j=\frac{1}{P_j}\sum_{\alpha\in\Gamma_j}e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}\left(\int_{\Lambda\backslash\Lambda_j}h(\alpha\cdot x)e^{-\delta\Delta_\alpha(x)}\,\mathrm{d}\sigma_{\mathbf{o}}(x)\right)D_{\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}.$$

The normalizing coefficient $P_j$ is given by

$$P_j=\sum_{\alpha\in\Gamma_j}e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}\left(\int_{\Lambda\backslash\Lambda_j}h(\alpha\cdot x)e^{-\delta\Delta_\alpha(x)}\,\mathrm{d}\sigma_{\mathbf{o}}(x)\right).$$

It is finite by Assumption ($\mathrm{P}_1$) and Property 4.1.3 implies $|\Delta_\alpha(x)|\leqslant C$ uniformly in $x\notin\Lambda_j$. In addition, these measures satisfy

(52)

$$1-\nu_j([0,T])=\frac{1}{P_j}\sum_{\alpha\ |\ \mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})>T}e^{-\delta\mathrm{d}(\mathbf{o},\alpha\cdot\mathbf{o})}\left(\int_{\Lambda\backslash\Lambda_j}h(\alpha\cdot x)e^{-\delta\Delta_\alpha(x)}\,\mathrm{d}\sigma_{\mathbf{o}}(x)\right)\sim\frac{\mathcal{C}_j}{P_j}\frac{L(T)}{T^\beta},$$

as $T\longrightarrow+\infty$, with

$$\mathcal{C}_j=\lim_{\alpha\longrightarrow+\infty}\int_{\Lambda\backslash\Lambda_j}h(\alpha\cdot x)e^{-\delta\Delta_\alpha(x)}\sigma_{\mathbf{o}}(\,\mathrm{d}x)=h(x_{\mathfrak{a}_j})\int_{\Lambda\backslash\Lambda_j}\frac{\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(x,x_{\mathfrak{a}_j})^{2\delta/a}}.$$

Therefore

$$\begin{aligned}\frac{S_{j2}}{P_j} &= \int_0^{+\infty} \left(e^{i(-t+i\kappa)y} - 1\right) \mathrm{d}\nu_j(y) \\ &= -(\kappa + it)\int_0^{+\infty} e^{i(-t+i\kappa)y}(1-\nu_j[0,y])\,\mathrm{d}y \\ &= (-\kappa I_{Cj} - tI_{Sj}) + i\,(\kappa I_{Sj} - tI_{Cj})\,,\end{aligned}$$

where

$$I_{Cj} = \int_0^{+\infty} e^{-\kappa y}\cos(ty)(1-\nu_j[0,y])\,\mathrm{d}y$$

and

$$I_{Sj} = \int_0^{+\infty} e^{-\kappa y}\sin(ty)(1-\nu_j[0,y])\,\mathrm{d}y.$$

Finally

$$1 - \lambda_{\delta+\kappa+it} = \sum_{j=1}^{p} P_j\left[(\kappa I_{Cj} + tI_{Sj}) - i\,(\kappa I_{Sj} - tI_{Cj})\right] + o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right) + \kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right).$$

□

As a consequence of Propositions 3.2.4 and 6.2.1, we may state the following proposition.

Proposition 6.2.2. – *Let $\kappa > 0$ and $\varepsilon > 0$ be as in Proposition 4.2.11. For any $t \in \mathbb{R}$ such that $\max(\kappa, |t|) < \varepsilon$, the dominant eigenvalue $\lambda_{\delta+\kappa+it}$ satisfies*

1) $\left|1 - \lambda_{\delta+\kappa+it}\right|^{-1} \leq \left|\mathrm{Re}\,(1 - \lambda_{\delta+\kappa+it})\right|^{-1} \leq \dfrac{1}{\sum_{j=1}^{p} \mathcal{C}_j}\dfrac{1}{\kappa\tilde{L}\,(1/\kappa)}$ *when* $|t| \leqslant \kappa$;

2) $\left|1 - \lambda_{\delta+\kappa+it}\right|^{-1} \leq \dfrac{1}{\sum_{j=1}^{p} \mathcal{C}_j}\dfrac{1}{(\kappa + |t|)\tilde{L}\,(1/|t|)}$ *when* $|t| > \kappa$;

3) $\left|\mathrm{Re}\left((1 - \lambda_{\delta+\kappa+it})^{-1}\right)\right| \leq \dfrac{1}{\sum_{j=1}^{p} \mathcal{C}_j}\left(\dfrac{\kappa}{(\kappa^2 + |t|^2)\tilde{L}\,(1/|t|)} + \dfrac{|t|L\,(1/|t|)}{(\kappa^2 + |t|^2)\tilde{L}\,(1/|t|)^2}\right)$

*when* $|t| > \kappa$.

*Proof.* – From Proposition 6.2.1, we deduce

$$\mathrm{Re}\,(1 - \lambda_{\delta+\kappa+it}) = \sum_{j=1}^{p} P_j\left[\kappa I_{C_j} + tI_{S_j}\right] + o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right)\right) + o\left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right)$$

and

$$\mathrm{Im}\,(\lambda_{\delta+\kappa+it}) = -\mathrm{Im}\,(1 - \lambda_{\delta+\kappa+it}) = \sum_{j=1}^{p} P_j\left[\kappa I_{S_j} - tI_{C_j}\right] + o\left(|t|\tilde{L}\left(\frac{1}{|t|}\right)\right) + o\left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right).$$

1) Combining (52) and assertions i) and iii) of Proposition 3.2.4, we obtain

$$\mathrm{Re}\,(1-\lambda_{\delta+\kappa+it}) \sim \sum_{j=1}^{p} \mathcal{C}_j \left[\kappa\tilde{L}\left(\frac{1}{\kappa}\right)(1+o(1)) + \kappa O\left(\frac{|t|}{\kappa}L\left(\frac{1}{\kappa}\right)\right) + tI_{S_j}\right].$$

Since $|t| \leqslant \kappa$, it follows that

$$\kappa\frac{|t|}{\kappa}L\left(\frac{1}{\kappa}\right) = o\left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right),$$

and Karamata's lemma yields

$$\left|I_{S_j}\right| \preceq \frac{|t|^2}{\kappa}L\left(\frac{1}{\kappa}\right) \preceq \kappa L\left(\frac{1}{\kappa}\right) = o\left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right)\right),$$

hence

$$\mathrm{Re}\,(1-\lambda_{\delta+\kappa+it}) \sim \left(\sum_{j=1}^{p}\mathcal{C}_j\right)\kappa\tilde{L}\left(\frac{1}{\kappa}\right) \text{ as } t \longrightarrow 0.$$

The first assertion follows.

2) We use properties ii) and iv) of Proposition 3.2.4. In this case

$$\mathrm{Re}\,(1-\lambda_{\delta+\kappa+it}) \sim \sum_{j=1}^{p} \mathcal{C}_j \left[\kappa\tilde{L}\left(\frac{1}{|t|}\right)(1+o(1)) + \kappa O\left(\frac{\kappa}{|t|}L\left(\frac{1}{|t|}\right)\right) + tI_{S_j}\right].$$

The inequalities

$$\left|tI_{S_j}\right| \preceq |t|L\left(\frac{1}{|t|}\right) = O\left(|t|L\left(\frac{1}{|t|}\right)\right)$$

and

$$\frac{\kappa^2}{|t|}L\left(\frac{1}{|t|}\right) \leqslant |t|L\left(\frac{1}{|t|}\right) = O\left(|t|L\left(\frac{1}{|t|}\right)\right),$$

imply that, as $t \longrightarrow 0$

(53) $$\mathrm{Re}\,(1-\lambda_{\delta+\kappa+it}) \sim \left(\sum_{j=1}^{p}\mathcal{C}_j\right)\left(\kappa\tilde{L}\left(\frac{1}{|t|}\right) + O\left(|t|L\left(\frac{1}{|t|}\right)\right)\right).$$

Similarly

$$\mathrm{Im}\,(\lambda_{\delta+\kappa+it}) = \sum_{j=1}^{p} \mathcal{C}_j \left[-t\tilde{L}\left(\frac{1}{|t|}\right)(1+o(1)) - tO\left(\frac{\kappa}{|t|}L\left(\frac{1}{|t|}\right)\right) + \kappa I_{S_j}\right]$$

and Karamata's lemma implies

(54) $$\mathrm{Im}\,(\lambda_{\delta+\kappa+it}) \sim -\left(\sum_{j=1}^{p}\mathcal{C}_j\right)t\tilde{L}\left(\frac{1}{|t|}\right).$$

Since $|1-\lambda_{\delta+\kappa+it}| \geqslant \frac{1}{2}\left(|\mathrm{Re}\,(1-\lambda_{\delta+\kappa+it})| + |\mathrm{Im}\,(1-\lambda_{\delta+\kappa+it})|\right)$, the estimates (53) and (54) combined with Karamata's lemma furnish

$$|1-\lambda_{\delta+\kappa+it}| \geq \left(\sum_{j=1}^{p} \mathcal{C}_j\right)(\kappa + |t|)\,\tilde{L}\left(\frac{1}{|t|}\right).$$

The second assertion follows.

3) It is sufficient to notice that

$$\mathrm{Re}\left((1-\lambda_{\delta+\kappa+it})^{-1}\right) = \frac{\mathrm{Re}\,(1-\lambda_{\delta+\kappa+it})}{|1-\lambda_{\delta+\kappa+it}|^2}$$

and to use (53) combined with 2). □

The previous proposition leads us to the following property.

Proposition 6.2.3. – *There exists $\kappa_0 > 0$ such that for any $0 < \kappa < \kappa_0$ and any compact subset $K$ of $\mathbb{R}$, the function $t \longmapsto \mathrm{Re}\,(Q_{\delta+\kappa+it})\,(h\varphi)(x)$ is bounded by $b_\kappa(t)$ defined by*

$$b_\kappa \;:\; t \longmapsto \frac{\kappa \tilde{L}\,(1/\kappa)}{\kappa + |t|} + \frac{1}{\kappa \tilde{L}\,(1/\kappa)}\mathbb{1}_{|t|\leqslant\kappa} + \frac{\kappa}{(\kappa^2+t^2)\tilde{L}\,(1/|t|)} + \frac{L\,(1/|t|)}{|t|\tilde{L}\,(1/|t|)^2},$$

*uniformly in $x \in \Lambda$ and $t \in K$.*

*Proof.* – Let $\kappa \in [0,1]$ and $t \in \mathbb{R}$ such that $\max(\kappa, |t|) < \varepsilon$, where $\varepsilon$ is given in Proposition 4.2.11. Using the arguments presented in the proof of Proposition 4.2.16, we obtain

$$Q_{\delta+\kappa+it} = (1-\lambda_{\delta+\kappa+it})^{-1}\,\Pi_\delta + (1-\lambda_{\delta+\kappa+it})^{-1}\,(\Pi_{\delta+\kappa+it} - \Pi_\delta) + O(1). \tag{55}$$

By assertions 1) and 3) of the previous proposition, we deduce

$$\begin{aligned}
\mathrm{Re}\left((1-\lambda_{\delta+\kappa+it})^{-1}\right) &\leqslant \left|\mathrm{Re}\left((1-\lambda_{\delta+\kappa+it})^{-1}\right)\right| \mathbb{1}_{|t|>\kappa} + \left|\mathrm{Re}\left((1-\lambda_{\delta+\kappa+it})^{-1}\right)\right| \mathbb{1}_{|t|\leqslant\kappa} \\
&\preceq \left[\frac{\kappa}{(\kappa^2+|t|^2)\tilde{L}\,(1/|t|)} + \frac{|t|\tilde{L}\,(1/|t|)}{(\kappa^2+|t|^2)\tilde{L}\,(1/|t|)^2}\right]\mathbb{1}_{]\kappa,\varepsilon]}(|t|) \\
&\quad + \frac{1}{\kappa\tilde{L}\,(1/\kappa)}\mathbb{1}_{|t|\leqslant\kappa} \\
&\preceq \frac{\kappa}{(\kappa^2+|t|^2)\tilde{L}\,(1/|t|)} + \frac{\tilde{L}\,(1/|t|)}{|t|\tilde{L}\,(1/|t|)^2} + \frac{1}{\kappa\tilde{L}\,(1/\kappa)}\mathbb{1}_{|t|\leqslant\kappa}.
\end{aligned}$$

The functions $t \longmapsto \Pi_{\delta+it}$ and $\kappa \longmapsto \Pi_{\delta+\kappa+it}$ inherit the regularity of the functions $t \longmapsto \mathcal{L}_{\delta+it}$ and $\kappa \longmapsto \mathcal{L}_{\delta+\kappa+it}$, therefore Proposition 6.2.2 combined with Potter's lemma yields to the following estimate of the second term in (55)

$$\begin{aligned}
&\left| \left(1-\lambda_{\delta+\kappa+it}\right)^{-1}\left(\Pi_{\delta+\kappa+it}-\Pi_\delta\right)\right| \\
&\preceq \left(\frac{1}{\kappa\tilde{L}\left(1/\kappa\right)}\mathbb{1}_{|t|\leqslant\kappa}+\frac{1}{(\kappa+|t|)\tilde{L}\left(1/|t|\right)}\mathbb{1}_{]\kappa,\varepsilon]}(|t|)\right)\left(\kappa\tilde{L}\left(\frac{1}{\kappa}\right)+|t|\tilde{L}\left(\frac{1}{|t|}\right)\right)\\
&\preceq 1+\frac{1}{\kappa\tilde{L}\left(1/\kappa\right)}\mathbb{1}_{|t|\leqslant\kappa}+\frac{\kappa\tilde{L}\left(1/\kappa\right)}{\kappa+|t|}\\
&\preceq \frac{1}{\kappa\tilde{L}\left(1/\kappa\right)}\mathbb{1}_{|t|\leqslant\kappa}+\frac{\kappa\tilde{L}\left(1/\kappa\right)}{\kappa+|t|}
\end{aligned}$$

since $1=o\left(\frac{1}{\kappa\tilde{L}(1/\kappa)}\right)$ as $\kappa\searrow 0$. □

We now end the proof of (46). For any $x\in\Lambda$ and $t\neq 0$, the sequence $\left(\mathrm{Re}\left(Q_{\delta+\kappa+it}\right)(h\varphi)(x)\right)_{\kappa>0}$ tends to $\mathrm{Re}\left(Q_{\delta+it}\right)(h\varphi)(x)$ as $\kappa\searrow 0$. Moreover, the previous proposition implies that for any $\kappa\searrow 0$ and any $t$ in the compact support of $\widehat{\varphi}$

$$\left|\mathrm{Re}\left(Q_{\delta+\kappa+it}\right)(h\varphi)(x)\right|\leqslant h_\kappa(t)+\frac{L\left(1/|t|\right)}{|t|\tilde{L}\left(1/|t|\right)^2},$$

where

$$h_\kappa(t)=\frac{\kappa\tilde{L}\left(1/\kappa\right)}{\kappa+|t|}+\frac{1}{\kappa\tilde{L}\left(1/\kappa\right)}1|_{t|\leqslant\kappa}+\frac{\kappa}{(\kappa^2+t^2)\tilde{L}\left(1/|t|\right)}. \tag{56}$$

Corollary 4.2.17 implies that the function $t\longmapsto\frac{L(1/|t|)}{|t|\tilde{L}(1/|t|)^2}$ is integrable on any compact. Since the support of $\widehat{\varphi}$ is compact, it is sufficient to prove that $\lim\limits_{\kappa\searrow 0}\int_{-M}^{M}h_\kappa(t)\widehat{\varphi}(t)\,\mathrm{d}t=0$ for any $M>0$. We write, on the one hand,

$$\begin{aligned}
\int_{-M}^{M}\frac{\kappa\tilde{L}\left(1/\kappa\right)}{\kappa+|t|}\,\mathrm{d}t &\preceq \kappa\tilde{L}\left(\frac{1}{\kappa}\right)\int_0^M\frac{\mathrm{d}t}{\kappa+t}\\
&\preceq \kappa\tilde{L}\left(\frac{1}{\kappa}\right)\left(\ln(\kappa+M)-\ln(\kappa)\right),
\end{aligned}$$

which goes to 0 as $\kappa\searrow 0$. On the other hand,

$$\begin{aligned}
\int_{-M}^{M}\frac{1}{\kappa\tilde{L}\left(1/\kappa\right)}\mathbb{1}_{[0,\kappa]}(|t|)\,\mathrm{d}t &\preceq \frac{1}{\kappa\tilde{L}\left(1/\kappa\right)}\int_0^\kappa\mathrm{d}t\\
&\preceq \frac{1}{\tilde{L}\left(1/\kappa\right)},
\end{aligned}$$

which goes to 0 as $\kappa \searrow 0$. The control of the integral of the last term of (56) relies on the following trick used in [**36**]. Let $A \in [0, M]$; we may write

$$\begin{aligned}
\int_{-M}^{M} \frac{\kappa}{(\kappa^2 + |t|^2)\tilde{L}\left(1/|t|\right)} \,\mathrm{d}t &\leq \int_{0}^{M} \frac{\kappa}{(\kappa^2 + t^2)\tilde{L}\left(1/t\right)} \,\mathrm{d}t \\
&\leq \int_{0}^{A} \frac{\kappa}{(\kappa^2 + t^2)\tilde{L}\left(1/t\right)} \,\mathrm{d}t + \int_{A}^{M} \frac{\kappa}{(\kappa^2 + t^2)\tilde{L}\left(1/t\right)} \,\mathrm{d}t \\
&\leq \frac{A}{\kappa \tilde{L}\left(\frac{1}{A}\right)} + \frac{\kappa}{A}.
\end{aligned}$$

We may choose $A = A(\kappa)$ such that $\kappa = \frac{A}{\sqrt{\tilde{L}(1/A)}}$. From properties of slowly varying functions, we deduce that $A(\kappa) \longrightarrow 0$ as $\kappa \longrightarrow 0$. For such $A$, we get

$$\int_{-M}^{M} \frac{\kappa}{(\kappa^2 + |t|^2)\tilde{L}\left(1/|t|\right)} \,\mathrm{d}t \leq \frac{1}{\sqrt{\tilde{L}\left(1/A\right)}} \xrightarrow[\kappa \searrow 0]{} 0.$$

# CHAPTER 7

# THEOREM B: CLOSED GEODESICS FOR $\beta \in \,]0,1[$

In this section, we will assume $\beta \in \,]0,1[$ and we will establish an asymptotic lower bound for the number of closed geodesics of $\mathrm{X}/\Gamma$ with length $\leqslant R$. Let $\mathscr{G}$ denote the set of all closed geodesics of $\mathrm{X}/\Gamma$ and set $\mathrm{N}_{\mathscr{G}}(R) := \#\{\wp \in \mathscr{G} \mid \ell(\wp) \leqslant R\}$. A closed geodesic in $\mathrm{X}/\Gamma$ is the projection to $\mathrm{X}/\Gamma$ of the axis of an hyperbolic isometry $\gamma \in \Gamma$. The contribution of geodesics with length $\leqslant R$ associated to powers of generators of some subgroup $\Gamma_j, p+1 \leqslant j \leqslant p+q$ is polynomial in $R$ and thus negligible with respect to $e^{\delta R}/R$. Therefore, in the sequel, we will only consider the hyperbolic isometries $\gamma$ of $\Gamma$ with symbolic length $\geqslant 2$. Up to a conjugacy, we may assume that $i(\gamma) \neq l(\gamma)$. Let $\tilde{\mathrm{N}}_{\mathscr{G}}(R)$ denote the number of closed geodesics $\wp$ with length $\leqslant R$ for which the associated isometry $\gamma_\wp$ is hyperbolic and has symbolic length $\geqslant 2$. Theorem B may be reformulated as follows.

Theorem B. – *Let $\Gamma$ be a Schottky group satisfying the assumptions $(H_\beta)$ for some $\beta \in \,]0,1[$. The number of closed geodesics in $\mathrm{X}/\Gamma$ with length $\leqslant R$ satisfies*

$$\liminf_{R \longrightarrow +\infty} \frac{\delta R}{\beta e^{\delta R}} \tilde{\mathrm{N}}_{\mathscr{G}}(R) \geqslant 1.$$

## 7.1. Proof of Theorem B

The coding of the geodesic flow given in Chapter 4 allows us to write $\tilde{\mathrm{N}}_{\mathscr{G}}(R)$ as

$$\tilde{\mathrm{N}}_{\mathscr{G}}(R) = \sum_{\wp \in \mathscr{G},\ |\gamma_\wp| \geqslant 2} \mathbb{1}_{\ell(\wp) \leqslant R} = \sum_{k \geqslant 2} \frac{1}{k} \sum_{\mathrm{T}^k x = x} \mathbb{1}_{[0,R]}(S_k \mathfrak{r}(x)).$$

Following [33] and [15], the idea is to approximate $\tilde{\mathrm{N}}_{\mathscr{G}}(R)$ by quantities of the form

$$\sum_{k \geqslant 2} \frac{1}{k} \sum_{\mathrm{T}^k y = x} \mathbb{1}_{[0,R]}(S_k \mathfrak{r}(y))$$

for suitable points $x$. Any point $x \in \Lambda^0$ may be obtained as a limit $\lim\limits_{k \longrightarrow +\infty} \alpha_1 \cdots \alpha_k \cdot x_0$ for a unique $\mathscr{A}$-admissible sequence $(\alpha_k)_{k \geqslant 1}$. We will call $\alpha_1, \ldots, \alpha_k$ the "first $k$ letters of $x$". For all $\gamma \in \Gamma$, denote by $\Lambda^0_\gamma = \gamma\cdot\left(\Lambda^0\backslash\Lambda^0_{l(\gamma)}\right)$ and fix a point $x^\gamma \in \Lambda^0_\gamma$. For technical reasons, we assume that each point $x^\gamma$ is non-periodic. We may also notice that for any $n \geqslant 1$, the family $\left(\Lambda^0_\gamma\right)_{\gamma\in\Gamma(n)}$ is a partition of $\Lambda^0$: the larger $n$ is, the finer is this partition. Fix $n \in \mathbb{N}$, $\gamma \in \Gamma(n)$ and a point $y \in \Lambda^0_\gamma$ such that $\mathrm{T}^k y = x^\gamma$ for some $k \geqslant 2$. There exists a unique $k$-periodic point $x \in \Lambda^0_\gamma$ such that the points $x$ and $y$ have the same $k$ first letters. From the assumptions on $y$, we deduce that these two points have in fact the same $k+n$ letters. Thus there exist a finite $\mathscr{A}$-admissible sequence $(\alpha_1, \ldots, \alpha_{k+n})$ and points $x', y' \in \Lambda\backslash\Lambda^0_{l(\alpha_{k+n})}$ such that $x = \alpha_1\cdots\alpha_{k+n}\cdot x'$ and $y = \alpha_1\cdots\alpha_{k+n}\cdot y'$ and we get

$$\begin{aligned}|S_k\mathfrak{r}(y) - S_k\mathfrak{r}(x)| &\leqslant |\mathfrak{r}(y)-\mathfrak{r}(x)| + \cdots + \left|\mathfrak{r}(\mathrm{T}^{k-1}y)-\mathfrak{r}(\mathrm{T}^{k-1}x)\right|\\ &\leqslant \left|b(\alpha_1\cdots\alpha_{k+n},y') - b(\alpha_1\cdots\alpha_{k+n},x')\right| + \cdots\\ &\quad + \left|b(\alpha_k\cdots\alpha_{k+n},y') - b(\alpha_k\cdots\alpha_{k+n},x')\right|\\ &\preceq r^{k+n} + \cdots + r^{n+1} \preceq \frac{r^{n+1}}{1-r} =: \varepsilon_n,\end{aligned}$$

for $r \in ]0,1[$ given in Corollary 4.1.5. It follows that

$$\sum_{\gamma\in\Gamma(n)}\sum_{k\geqslant 2}\frac{1}{k}\sum_{\mathrm{T}^k y = x^\gamma}\mathbb{1}_{\Lambda_\gamma}(y)\mathbb{1}_{[0,R-\varepsilon_n]}(S_k\mathfrak{r}(y)) \leqslant \tilde{\mathrm{N}}_{\mathscr{G}}(R). \tag{57}$$

Theorem B will thus be a consequence of the following proposition.

Proposition 7.1.1. – *For any function $\varphi \in \mathrm{Lip}\,(\Lambda)$ and uniformly in $x \in \Lambda^0$, we have as $R \longrightarrow +\infty$*

$$\sum_{k\geqslant 2}\frac{1}{k}\sum_{\mathrm{T}^k y = x}\varphi(y)\mathbb{1}_{[0,R]}(S_k\mathfrak{r}(y)) \sim \beta\sigma_{\mathbf{o}}(\varphi)\,h(x)\frac{e^{\delta R}}{\delta R}.$$

*Proof.* – The quantity $N(\varphi,x,R) := \sum_{k\geqslant 2}k^{-1}\sum_{\mathrm{T}^k y = x}\varphi(y)\mathbb{1}_{[0,R]}(S_k\mathfrak{r}(y))$ may be written as

$$N(\varphi,x,R) = \sum_{k\geqslant 2}\frac{1}{k}\sum_{\mathrm{T}^k y = x}\varphi(y)e^{-\delta S_k\mathfrak{r}(y)}u_R(S_k\mathfrak{r}(y)),$$

where $u_R(t) = e^{\delta t}\mathbb{1}_{[0,R]}(t)$. Let us fix $\eta > 0$ and set $P = \left[\frac{R}{\eta}\right]$. From

$$\sum_{p=0}^{P-1} e^{\delta\eta p}\mathbb{1}_{[\eta p,\eta(p+1)[}(t) \leqslant u_R(t),$$

we deduce

$$\sum_{p=0}^{P-1} e^{\delta\eta p}\sum_{k\geqslant 2}\frac{1}{k}\sum_{\mathrm{T}^k y = x}\varphi(y)e^{-\delta S_k\mathfrak{r}(y)}\mathbb{1}_{[0,\eta[}(S_k\mathfrak{r}(y)-\eta p) \leqslant N(\varphi,x,R).$$

Let us first state the following proposition, whose proof is postponed to Subsection 7.2.

Proposition 7.1.2. – *For any functions* $\varphi \in \mathrm{Lip}\,(\Lambda)$ *and* $u : \mathbb{R} \longrightarrow \mathbb{R}$ *with compact support, one has*

$$\lim_{p\longrightarrow+\infty}\sup_{x\in\Lambda^0}\left|p\sum_{k\geqslant 2}\frac{1}{k}\sum_{\mathrm{T}^k y=x}\varphi(y)e^{-\delta S_k\mathfrak{r}(y)}u\left(S_k\mathfrak{r}(y)-\eta p\right)-\beta\sigma_{\mathbf{o}}\left(\varphi\right)h(x)\frac{\widehat{u}(0)}{\eta}\right|=0.$$

We also need the following lemma (see Lemma 3.2.1 p.136 in [**15**]).

Lemma 7.1.3. – *Let* $(V_p(x))_{p\geqslant 0}$ *a sequence of positive functions defined on a compact set* $K$ *and* $(v_p)_{p\geqslant 0}$ *a divergent positive series. If* $\lim_{p\longrightarrow+\infty}\sup_{x\in K}\left|\frac{V_p(x)}{v_p}-1\right|=0$*, then*

$$\lim_{N\longrightarrow+\infty}\sup_{x\in K}\left|\frac{\sum_{p=0}^N V_p(x)}{\sum_{p=0}^N v_p}-1\right|=0.$$

Proposition 7.1.2 and Lemma 7.1.3 imply that, as $P\longrightarrow+\infty$

$$\sum_{p=1}^{P}e^{\delta\eta p}\sum_{k\geqslant 2}\frac{1}{k}\sum_{\mathrm{T}^k y=x}\varphi(y)e^{-\delta S_k\mathfrak{r}(y)}\mathbb{1}_{[0,\eta[}\left(S_k\mathfrak{r}(y)-\eta p\right)\sim\left(\sum_{p=1}^{P}\frac{e^{\delta\eta p}}{p}\right)\beta\sigma_{\mathbf{o}}\left(\varphi\right)h(x)$$

uniformly in $x\in\Lambda^0$. Since $\sum_{p=1}^P\frac{e^{\delta\eta p}}{p}\sim\frac{e^{\delta\eta P}}{\delta\eta P}$ and $\eta$ is arbitrary chosen, we obtain as $R\longrightarrow+\infty$

$$N\left(\varphi,x,R\right)\sim\beta\sigma_{\mathbf{o}}\left(\varphi\right)h(x)\frac{e^{\delta R}}{\delta R}$$

uniformly in $x\in\Lambda^0$. □

Theorem B is a direct consequence of Proposition 7.1.1; combining (57) and Fatou's lemma yields

$$e^{-\delta\varepsilon_n}\sum_{\gamma\in\Gamma(n)}\sigma_{\mathbf{o}}\left(\mathbb{1}_{\Lambda_\gamma}\right)h(x^\gamma)\leqslant\liminf_{R\longrightarrow+\infty}\frac{\delta R}{\beta e^{\delta R}}\tilde{\mathrm{N}}_{\mathscr{G}}(R).$$

Theorem B follows by noticing that

$$\lim_{n\longrightarrow+\infty}e^{-\delta\varepsilon_n}\sum_{\gamma\in\Gamma(n)}\sigma_{\mathbf{o}}\left(\mathbb{1}_{\Lambda_\gamma}\right)h(x^\gamma)=\int_{\Lambda^0}h(x)\,\mathrm{d}\sigma_{\mathbf{o}}(x)=1.$$

Remark 7.1.4. – *We may also have an upper bound of* $\tilde{\mathrm{N}}_{\mathscr{G}}(R)$ *of the form*

$$\sum_{\gamma\in\Gamma(n)}\sum_{k\geqslant 2}\frac{1}{k}\sum_{\mathrm{T}^k y=x^\gamma}\mathbb{1}_{\Lambda_\gamma}(y)\mathbb{1}_{[0,R+\varepsilon_n]}(S_k\mathfrak{r}(y))$$

*similar to the lower bound in* (57)*. Unfortunately, we do not find a domination by an integrable function in order to use Proposition 7.1.1.*

## 7.2. Proposition 7.1.2

The proof is inspired from that of Theorem 1.4 of **[24]**. Let us consider a sequence $(a_k)_{k\geqslant 1}$ of non-negative real numbers such that $a_k^\beta = kL(a_k)$, where $L$ is the slowly varying function given in assumptions $(H_\beta)$. Let $\varphi \in \mathrm{Lip}(\Lambda)$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ a function with compact support. For any $k \geqslant 1$, let us denote

$$Z_k(\varphi, u, x, p) = \sum_{\mathrm{T}^k y = x} \varphi(y) e^{-\delta S_k \mathfrak{r}(y)} u\left(S_k \mathfrak{r}(y) - \eta p\right).$$

We first admit the two following propositions.

Proposition B.1. – *Let $\varphi \in \mathrm{Lip}(\Lambda)$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ continuous with compact support. For all $\eta > 0$, uniformly in $K \geqslant 2$, $p \in [0, Ka_k]$ and $x \in \Lambda^0$, as $k \longrightarrow +\infty$*

$$Z_k(\varphi, u, x, p) = \frac{1}{e_\Gamma a_k}\left(\Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right)\sigma_{\mathbf{o}}(\varphi) h(x)\widehat{u}(0) + o_k(1)\right),$$

*where $\Psi_\beta$ stands for the density of the fully asymmetric stable law with parameter $\beta$ and $e_\Gamma$ equals to $E_\Gamma^{1/\beta}$.*

Proposition B.2. – *Let $\varphi \in \mathrm{Lip}(\Lambda)$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ continuous with compact support. There exists a constant $C > 0$, which depends only on $\eta$ and on the support of $\varphi$ such that, for any $K \geqslant 2$, when $p \geqslant Ka_k$, we have*

$$|Z_k(\varphi, u, x, p)| \leqslant Ck\frac{L(p)}{p^{1+\beta}}|\varphi|_\infty |u|_\infty.$$

We postpone the proof of these propositions to Subsections 7.2.1 and 7.2.2 and first explain how Proposition 7.1.2 follows. Let us set for any $x \in \Lambda$ and $p \geqslant 1$

$$D(x;p) = \left| p\sum_{k\geqslant 2}\frac{1}{k}Z_k(\varphi, u, x, p) - \beta\sigma_{\mathbf{o}}(\varphi) h(x)\frac{\widehat{u}(0)}{\eta}\right|,$$

where $\eta > 0$ is fixed as in the proof of Proposition 7.1.1. By Proposition B.1, the quantity $D(x;p)$ is bounded from above by $D^1(x;p) + D^2(x;p) + D^3(x;p) + D^4(x;p)$ where

$$D^1(x;p) = \left| p\sum_{k|a_k/K \leqslant p < Ka_k}\frac{1}{e_\Gamma k a_k}\Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right)\sigma_{\mathbf{o}}(\varphi) h(x)\widehat{u}(0) - \beta\sigma_{\mathbf{o}}(\varphi) h(x)\frac{\widehat{u}(0)}{\eta}\right|,$$

$$D^2(x;p) = \left| p \sum_{k|p<a_k/K} \frac{1}{e_\Gamma k a_k} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \sigma_{\mathbf{o}}(\varphi)\, h(x)\widehat{u}(0) \right|,$$

$$D^3(x;p) = \left| p \sum_{k|p<Ka_k} \frac{o_k(1)}{e_\Gamma k a_k} \right|$$

$$\text{and } D^4(x;p) = \left| p \sum_{k|p\geqslant Ka_k} \frac{1}{k} Z_k(\varphi, u, x, p) \right|.$$

a) *Study of* $D^1(x;p)$. There exists $C>0$ which depends only on $\varphi$ and $u$ such that

$$D^1(x;p) \leqslant C \left| \frac{p}{e_\Gamma} \sum_{k \ | \ a_k/K \leqslant p < Ka_k} \frac{1}{k a_k} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) - \frac{\beta}{\eta} \right|.$$

Since $k = a_k^\beta / L(a_k)$, it follows that

$$\begin{aligned} \sum_{k|a_k/K\leqslant p<Ka_k} \frac{1}{k a_k} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) &= \frac{L(p)}{p^{1+\beta}} \sum_{k|a_k/K\leqslant p<Ka_k} \frac{L(a_k)}{L(p)} \frac{p^{1+\beta}}{a_k^{1+\beta}} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \\ &\sim \frac{L(p)}{p^{1+\beta}} \sum_{k|a_k/K\leqslant p<Ka_k} \frac{p^{1+\beta}}{a_k^{1+\beta}} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \end{aligned} \tag{58}$$

uniformly in $p$ such that $a_k/K \leqslant p < Ka_k$. Using the measure $\mu_p = \sum_k D_{p/a_k}$ defined on the interval $[1/K, K]$, we may rewrite the right hand side of (58) as

$$\frac{L(p)}{p^{1+\beta}} \sum_{k|a_k/K\leqslant p<Ka_k} \frac{p^{1+\beta}}{a_k^{1+\beta}} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) = \frac{L(p)}{p^{1+\beta}} \int_{1/K}^{K} z^{\beta+1} \Psi_\beta\left(\frac{\eta}{e_\Gamma} z\right) \mathrm{d}\mu_p(z) \tag{59}$$

and from the arguments given in the proof of Theorem A in a) Section 5.2.1, we deduce that the sequence of measures $\left(p^{-1-\beta} L(p)\mu_p\right)_p$ weakly converges on $[1/K, K]$ to the measure $\beta x^{-1-\beta}\,\mathrm{d}x$. Combining (58) and (59), we get

$$D^1(x;p) \leqslant C\beta \left| \frac{1}{e_\Gamma} \left( \int_{1/K}^{K} \Psi_\beta\left(\frac{\eta}{e_\Gamma} z\right) \mathrm{d}z \right) (1+o(1)) - \frac{1}{\eta} \right|$$

where $\lim\limits_{K\longrightarrow+\infty} \lim\limits_{p\longrightarrow+\infty} o(1) = 0$. Finally

$$D^1(x;p) \leqslant C\frac{\beta}{\eta} \left| \left( \int_{\eta/(Ke_\Gamma)}^{\eta K/e_\Gamma} \Psi_\beta(z)\, \mathrm{d}z \right) (1+o(1)) - 1 \right|$$

which implies as $K \longrightarrow +\infty$, since $\Psi_\beta$ is a probability density,

$$\lim_{K\longrightarrow+\infty} \lim_{p\longrightarrow+\infty} \sup_{x\in\Lambda^0} D^1(x;p) = 0.$$

b) *Study of* $D^2(x;p)$. Recall that $k = A(a_k)$ and $A$ is increasing. The inequality $pK < a_k$ thus gives $A(pK) \leqslant k$ and yields

$$D^2(x;p) \leqslant \frac{p}{e_\Gamma A(pK)} \left| \sum_{k|p<a_k/K} \frac{1}{a_k} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \right| \leqslant C\frac{p^{1-\beta}}{K^\beta} L(pK) \left| \sum_{k|p<a_k/K} \frac{1}{a_k} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \right|.$$

From the weak convergence on $]0,K]$ of the sequence of measures $\left(p^{-1-\beta}L(p)\mu_p\right)_p$ to the measure $\beta x^{-1-\beta}\,\mathrm{d}x$, we deduce

$$\begin{aligned} \frac{p^{1-\beta}}{K^\beta} L(pK) \sum_{k|p<a_k/K} \frac{1}{a_k} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) &= \frac{p^{-\beta}}{K^\beta} L(pK) \int_0^{1/K} z\Psi_\beta\left(\frac{\eta}{e_\Gamma} z\right) \mathrm{d}\mu_p(z) \\ &= \frac{\beta}{K^\beta} \frac{L(pK)}{L(p)} \left( \int_0^{1/K} z\Psi_\beta\left(\frac{\eta}{e_\Gamma} z\right) \mathrm{d}z \right) (1+o(1)), \end{aligned}$$

where $\lim_{p\longrightarrow+\infty} o(1) = 0$; hence

$$\lim_{K\longrightarrow+\infty} \lim_{p\longrightarrow+\infty} \sup_{x\in\Lambda^0} D^2(x;p) = 0.$$

c) *Study of* $D^3(x;p)$. Fix $\varepsilon > 0$ and let $N = N(p)$ be the smallest integer such that $Ka_N > p$. The map $p \longmapsto N(p)$ is increasing. For $p$ large enough and any $k \geqslant N(p)$, one gets $|o_k(1)| \leqslant \varepsilon$. Karamata's lemma implies

$$D^3(x;p) \leqslant \frac{\varepsilon}{e_\Gamma} p \sum_{k\geqslant N} \frac{1}{ka_k} \leqslant \frac{\varepsilon}{e_\Gamma} \frac{p}{a_N}.$$

Noticing that $1/a_N = a_N^{\beta-1}/a_N^\beta$ with $a_{N-1} \leqslant p/K$ and $p < Ka_N$ and using asymptotic properties of regularly varying functions, we deduce

$$\frac{1}{a_N} = \frac{a_N^{\beta-1}}{a_{N-1}^{\beta-1}} \frac{a_{N-1}^{\beta-1}}{a_N^\beta} \preceq \frac{p^{\beta-1}}{K^{\beta-1}} \frac{K^\beta}{p^\beta} = \frac{K}{p}.$$

Finally $\sup_{x\in\Lambda^0} D^3(x;p) \preceq \varepsilon K$ and thus $\sup_{x\in\Lambda^0} D^3(x;p) = o_K(1)$ where $\lim_{p\longrightarrow+\infty} o_K(1) = 0$ for any fixed $K$; hence $\lim_{K\longrightarrow+\infty} \lim_{p\longrightarrow+\infty} \sup_{x\in\Lambda^0} D^3(x;p) = 0$.

d) *Study of* $D^4(x;p)$. By Proposition B.2, we bound $D^4(x;p)$ from above by $Cp^{-\beta}L(p)\sum_{k|Ka_k\leqslant p} 1$ where the constant $C$ only depends on $\varphi$ and $u$. Using the same arguments as in b), the inequality $a_k \leqslant p/K$ implies $k \leqslant (p/K)^\beta L(p/K)^{-1}$; therefore

$$D^4(x;p) \leqslant CK^{-\beta} \frac{L(p)}{L(p/K)}.$$

Potter's lemma with $B = 1$, $\rho = \beta/2$, $x = p$ and $y = p/K$ finally implies $D^4(x;p) \leqslant CK^{-\beta/2}$, so that $\lim_{K\longrightarrow+\infty} \lim_{p\longrightarrow+\infty} \sup_{x\in\Lambda^0} D^4(x;p) = 0$.

**7.2.1. Proof of Proposition B.1. –** We follow the proof of Proposition A.1 in Subsection 5.2.2.

Let us fix $p \gg 1$ and consider the integers $k$ such that $Ka_k > p$, where $K \geqslant 2$ is fixed. For any such $k \in \mathbb{N}$, any $\varphi \in \mathrm{Lip}\,(\Lambda)$ and $u : \mathbb{R} \longrightarrow \mathbb{R}$ continuous with compact support, we write

$$Z_k\left(\varphi, u, x, p\right) = \sum_{\mathrm{T}^k y = x} \varphi(y) e^{-\delta S_k \mathfrak{r}(y)} u\left(S_k \mathfrak{r}(y) - \eta p\right).$$

We want to show that, as $k \longrightarrow +\infty$

$$a_k Z_k\left(\varphi, u, x, p\right) - \frac{1}{e_\Gamma} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \sigma_{\mathbf{o}}\left(\varphi\right) h(x) \widehat{u}(0) \longrightarrow 0 \tag{60}$$

uniformly in $K$, $p$ and $x \in \Lambda^0$, which means that the sequence of measures

$$\left(a_k Z_k\left(\varphi, \bullet, x, p\right) - \frac{1}{e_\Gamma} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \sigma_{\mathbf{o}}\left(\varphi\right) h(x) \int_{\mathbb{R}} \bullet(y)\,\mathrm{d}y\right)_{k | K a_k > p}$$

converges weakly to 0 as $k \longrightarrow +\infty$, uniformly in $K$, $p$ and $x \in \Lambda^0$. Using the argument given in the proof of Proposition A.1 (p. 49), it is sufficient to check the following.

Property 7.2.1. – *For any function $u \in \mathscr{U}$, the quantity*

$$a_k Z_k\left(\varphi, u, x, p\right) - \frac{1}{e_\Gamma} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \sigma_{\mathbf{o}}\left(\varphi\right) h(x) \int_{\mathbb{R}} u(y)\,\mathrm{d}y$$

*is finite and converges to 0 as $k \longrightarrow +\infty$.*

Fix $u \in \mathscr{U}$. The Fourier inverse formula furnishes

$$Z_k\left(\varphi, u, x, p\right) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\eta p} \mathscr{L}^k_{\delta+it} \varphi(x) \widehat{u}(t)\,\mathrm{d}t;$$

we thus deduce $|Z_k\left(\varphi, u, x, p\right)| \preceq |\varphi|_\infty ||\widehat{u}||_1 < +\infty$. To prove Property 7.2.1, let us fix $\varepsilon > 0$ satisfying the conclusion of Proposition 4.2.11 and let us decompose

$$a_k Z_k\left(\varphi, u, x, p\right) - \frac{1}{e_\Gamma} \Psi_\beta\left(\frac{\eta p}{e_\Gamma a_k}\right) \sigma_{\mathbf{o}}\left(\varphi\right) h(x) \widehat{u}(0)$$

as $K_1(k) + K_2(k)$ where

$$K_1(k) := \frac{a_k}{2\pi} \int\limits_{[-\varepsilon,\varepsilon]^c} e^{it\eta p} \mathscr{L}^k_{\delta+it} \varphi(x) \widehat{u}(t)\,\mathrm{d}t$$

and

$$K_2(k) := \frac{a_k}{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{it\eta p}\mathscr{L}^k_{\delta+it}\varphi(x)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\eta p/a_k} g_\beta(e_\Gamma t)\sigma_{\mathbf{o}}(\varphi)\,h(x)\widehat{u}(0)\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{is\eta p/a_k}\mathscr{L}^k_{\delta+is/a_k}\varphi(x)\widehat{u}\left(\frac{s}{a_k}\right)\mathrm{d}s$$
$$- \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\eta p/a_k} g_\beta(e_\Gamma t)\sigma_{\mathbf{o}}(\varphi)\,h(x)\widehat{u}(0)\,\mathrm{d}t.$$

Proposition 4.2.11 combined with the fact that $\widehat{u}$ has a compact support implies the existence of $\rho \in ]0,1[$ such that $||\mathscr{L}^k_{\delta+it}|| \preceq \rho^k$ for any $t \in [-\varepsilon,\varepsilon]^c \cap \operatorname{supp}\widehat{u}$; hence $|K_1(k)| \preceq ||\widehat{u}||_\infty\,|\varphi|_\infty\,\rho^k a_k \longrightarrow 0$ as $k \longrightarrow +\infty$, uniformly in $K$, $p$ and $x \in \Lambda^0$.

We now deal with $K_2(k)$. The spectral decomposition of $\mathscr{L}_{\delta+is/a_k}$ furnishes

$$\mathscr{L}^k_{\delta+is/a_k}\varphi = \lambda^k_{\delta+is/a_k}\Pi_{\delta+is/a_k}\varphi + R^k_{\delta+is/a_k}\varphi,$$

where the spectral radius of $R_{\delta+is/a_k}$ is smaller than $\rho_\varepsilon < 1$. We split $K_2(k)$ into $L_1(k) + L_2(k) + L_3(k)$ where

$$L_1(k) = \frac{1}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{is\eta p/a_k} R^k_{\delta+is/a_k}\varphi(x)\widehat{u}\left(\frac{s}{a_k}\right)\mathrm{d}s,$$
$$L_2(k) = \frac{1}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{is\eta p/a_k}\lambda^k_{\delta+is/a_k}\left(\Pi_{\delta+is/a_k}\varphi(x) - \Pi_\delta\varphi(x)\right)\widehat{u}\left(\frac{s}{a_k}\right)\mathrm{d}s$$

and

$$L_3(k) = \frac{\sigma_{\mathbf{o}}(\varphi)\,h(x)}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{is\eta p/a_k}\lambda^k_{\delta+is/a_k}\widehat{u}\left(\frac{s}{a_k}\right)\mathrm{d}s$$
$$- \frac{\sigma_{\mathbf{o}}(\varphi)\,h(x)}{2\pi}\int_{\mathbb{R}} e^{it\eta p/a_k} g_\beta(e_\Gamma t)\widehat{u}(0)\,\mathrm{d}t.$$

First $|L_1(k)| \preceq a_k\rho_\varepsilon^k\,|\widehat{u}|_\infty\,||\varphi||$, hence $L_1(k)$ tends to 0 uniformly in $K$, $p$ and $x \in \Lambda^0$ as $k \longrightarrow +\infty$. We need the Lebesgue dominated convergence theorem for $L_2(k)$; as for the quantity $L_2(k)$ appearing in the proof of Proposition A.1, the local expansion of $\lambda_{\delta+it}$ given in Proposition 4.2.15 implies that the integrand of $L_2(k)$ is bounded from above up to a multiplicative constant by

$$l(t) = \begin{cases} |t|^{\beta/2}\exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{3\beta/2}\right) & \text{if } |t| \leqslant 1 \\ |t|^{3\beta/2}\exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{\beta/2}\right) & \text{if } |t| > 1. \end{cases}$$

The term $L_3(k)$ may be treated similarly as the quantity $L_3(k)$ in the proof of Proposition A.1.

**7.2.2. Proof of Proposition B.2. –** The quantity $Z_k(\varphi, u, x, p)$ may be written as

$$Z_k(\varphi, u, x, p) = \sum_{\gamma \in \Gamma(k)} \mathbb{1}_{\Lambda_{l(\gamma)}}(x)\varphi(\gamma \cdot x)e^{-\delta b(\gamma, x)} u\left(b(\gamma, x) - \eta p\right).$$

Thus there exists a constant $M > 0$ such that

$$|Z_k(\varphi, u, x, p)| \leqslant |\varphi|_\infty |u|_\infty \sum_{\substack{\gamma \in \Gamma(k) \\ b(\gamma,x) \overset{M}{\sim} \eta p}} \mathbb{1}_{\Lambda_{l(\gamma)}}(x)e^{-\delta b(\gamma, x)}.$$

It is thus sufficient to prove

$$\sum_{\substack{\gamma \in \Gamma(k) \\ b(\gamma,x) \overset{M}{\sim} \eta p}} \mathbb{1}_{\Lambda_{l(\gamma)}}(x)e^{-\delta b(\gamma, x)} \leqslant Ck\frac{L(p)}{p^{1+\beta}},$$

where $C$ only depends on $\eta$ and on the support of $\varphi$; this inequality is a consequence of (39).

# CHAPTER 8

# EXTENDED CODING

In the sequel, we aim to find an asymptotic for the orbital function $\mathrm{N}_\Gamma(\mathbf{o}, R)$ of the group $\Gamma$ defined by

$$\mathrm{N}_\Gamma(\mathbf{o}, R) = \#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}.$$

The idea is to formalize the fact that $\mathrm{N}(\mathbf{o}, \Gamma)$ "behaves" like a sum of iterates of some extension of the transfer operators. Unfortunately, the coding presented in Chapter 4 does not take into account the finite words $\gamma = \alpha_1 \cdots \alpha_k$. Adapting Lalley's approach (in [**33**]), we first extend this coding to finite sequences $(\alpha_1, \ldots, \alpha_k)$ and then study the corresponding transfer operators: in Subsection 8.3, we extend Propositions 4.2.7, 4.2.9, 4.2.11 and 4.2.15 to this new coding. As in the case of mixing, these results are essential for the proof. Then, the proof of Theorem C follows almost line by line that of Theorem A using this new coding.

## 8.1. Extension of the coding to finite sequences

In Chapter 4, we fixed a base point $\mathbf{o}$ in $\mathrm{X}\backslash\bigcup_{1\leqslant j\leqslant p+q}\mathbf{D}_j$ and a base point $x_0$ in $\partial\mathrm{X}\backslash D$, which is not related to $\mathbf{o}$. Now, let $\tilde{\Lambda}^0$ denote $\Lambda^0 \cup \Gamma \cdot x_0$ and introduce the following symbolic space (called "extended symbolic space")

$$\tilde{\Sigma}^+ := \Sigma^+ \cup \{\varnothing\} \cup \Gamma^*,$$

where we denote respectively by $\varnothing$ and $\Gamma^*$ the empty sequence and the set $\Gamma\backslash\{\mathrm{Id}\}$. The set $\tilde{\Sigma}^+$ is in one-to-one correspondence with the subset $\tilde{\Lambda}^0$ of $\partial\mathrm{X}$:

- the point $x_0$ corresponds to the empty sequence;
- the point $\alpha_1 \cdots \alpha_k \cdot x_0$ corresponds to the admissible finite sequence $(\alpha_1, \ldots, \alpha_k)$;
- the infinite sequence $(\alpha_k)_k$ corresponds to the limit point $x := \lim_{k\longrightarrow+\infty} \alpha_1 \cdots \alpha_k \cdot x_0$.

Analogously to Subsection 4.1.1, we will use the following description of $\tilde{\Lambda}^0\backslash\{x_0\}$:

(1) the set $\tilde{\Lambda}^0\backslash\{x_0\}$ is the disjoint union of sets $(\tilde{\Lambda}^0_j)_j$, where

$$\tilde{\Lambda}^0_j := \Lambda^0_j \cup \{g\cdot x_0 \mid g \text{ has first letter in } \Gamma_j\} = \tilde{\Lambda}^0 \cap D_j;$$

(2) each subset $\tilde{\Lambda}^0_j$ is partitioned into a countable number of subsets with disjoint closures: indeed, for any $j \in [\![1, p+q]\!]$

$$\tilde{\Lambda}^0_j = \bigcup_{\alpha\in\Gamma_j^*} \alpha\cdot\left(\{x_0\}\cup\bigcup_{l\neq j}\tilde{\Lambda}^0_l\right).$$

Now we extend the cocycle $b(\gamma, x) = \mathscr{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o})$ on $\tilde{\Lambda}^0$ in way as to decompose the distance $\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})$ for any $\gamma\in\Gamma$ as a sum of terms expressed using the cocycle. For any $\gamma, g\in\Gamma$, let us set

$$\tilde{b}(\gamma, g\cdot x_0) := \mathrm{d}(\gamma^{-1}\cdot\mathbf{o}, g\cdot\mathbf{o}) - \mathrm{d}(\mathbf{o}, g\cdot\mathbf{o}).$$

The function $x = g\cdot x_0 \in \Gamma\cdot x_0 \longmapsto \tilde{b}(\gamma, x)$ satisfies the three following properties:

i) when the sequence of points $(g\cdot\mathbf{o})_g$ tends to a point $x\in\Lambda$, so does the sequence $(g\cdot x_0)_g$ and the quantity $\mathrm{d}(\gamma^{-1}\cdot\mathbf{o}, g\cdot\mathbf{o}) - \mathrm{d}(\mathbf{o}, g\cdot\mathbf{o})$ converges to $\mathscr{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o}) = b(\gamma, x)$;

ii) there exists a constant $C > 0$ depending only on X and $\Gamma$ such that for any $g\in\Gamma$ with $i(g)\neq l(\gamma)$, one gets

$$\left|\left(\mathrm{d}(\gamma^{-1}\cdot\mathbf{o}, g\cdot\mathbf{o}) - \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\right) - \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\right| \leqslant C,$$

hence $\left|\tilde{b}(\gamma, g\cdot x_0) - \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\right| \leqslant C$;

iii) $\tilde{b}(\gamma g, x_0) = \tilde{b}(\gamma, g\cdot x_0) + \tilde{b}(g, x_0)$ for any $\gamma, g\in\Gamma$.

The function $\tilde{b}(\gamma,\cdot)$ is a cocycle, which extends continuously $b$ to $\tilde{\Lambda}^0$; in particular, if $\gamma$ decomposes into $\gamma = \alpha_1\cdots\alpha_k$, then

$$\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}) = \tilde{b}(\alpha_1, \alpha_2\cdots\alpha_k\cdot x_0) + \tilde{b}(\alpha_2, \alpha_3\cdots\alpha_k\cdot x_0) + \cdots + \tilde{b}(\alpha_k, x_0). \tag{61}$$

In the sequel, we thus consider the following "extended cocycle," also denoted by $\tilde{b}$ and defined by: for any $\gamma\in\Gamma$ and $x\in\tilde{\Lambda}^0$

$$\tilde{b}(\gamma, x) = \begin{cases} \mathscr{B}_x(\gamma^{-1}\cdot\mathbf{o},\mathbf{o}) \text{ if } x\in\Lambda^0 \\ \mathrm{d}(\gamma^{-1}\cdot\mathbf{o}, g\cdot\mathbf{o}) - \mathrm{d}(\mathbf{o}, g\cdot\mathbf{o}) \text{ if } x = g\cdot x_0 \end{cases}. \tag{62}$$

The map T previously defined on $\Lambda^0$, may be naturally extended to $\Gamma\cdot x_0$ as follows:

– we use the convention $\mathrm{T}x_0 = x_0$;
– we set $\mathrm{T}(\gamma\cdot x_0) := \alpha_1^{-1}\cdot(\gamma\cdot x_0)$ for any $\gamma\in\Gamma$ with first letter $\alpha_1\in\mathscr{A}$.

We can similarly extend the roof function $\mathfrak{r}$ to the discrete orbit $\Gamma\cdot x_0$ setting

– $\mathfrak{r}(x_0) = 0$;
– $\mathfrak{r}(\gamma\cdot x_0) = \tilde{b}(\alpha_1, \alpha_2\cdots\alpha_k\cdot x_0)$ for any $\gamma = \alpha_1\cdots\alpha_k$ in $\Gamma^*$.

Then for $y = \gamma \cdot x_0$ the relation (61) may be rewritten as

$$\mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) = \mathfrak{r}(y) + \mathfrak{r}(\mathrm{T}y) + \cdots + \mathfrak{r}(\mathrm{T}^{k-1}y) = S_k\mathfrak{r}(y). \tag{63}$$

## 8.2. Regularity of the extended cocycle

To simplify the notation, from now on we write $\tilde{\Lambda} = \overline{\tilde{\Lambda}^0} = \Lambda \cup \Gamma \cdot x_0$ and $\tilde{\Lambda}_j := \overline{\tilde{\Lambda}_j^0} = \tilde{\Lambda} \cap D_j$ for any $j \in [\![1, p+q]\!]$. In the sequel, we will need the definition of a Gromov-$\kappa$-hyperbolic space.

Definition 8.2.1. – *Let* $(\mathrm{X}, \mathrm{d})$ *be a geodesic metric space and* $\kappa > 0$. *For any geodesic triangle* $\Delta \subset \mathrm{X}$, *let* $c_1$, $c_2$ *and* $c_3$ *denote its three sides. The space* $(\mathrm{X}, \mathrm{d})$ *is said to be a Gromov-*$\kappa$*-hyperbolic space if for any geodesic triangle* $\Delta \subset \mathrm{X}$, *for any* $(i, j, k) \subset \{1, 2, 3\}$, $i \neq j, i \neq k, j \neq k$, *we have : for any* $\mathbf{x} \in c_i$, $\mathrm{d}(\mathbf{x}, c_j \cup c_k) \leqslant \kappa$.

Such spaces have a lot of properties (see **[22]**). For instance, they present a very good control in the converse triangular inequality.

In this subsection, we aim to show the following.

Proposition 8.2.2. – *There exists a constant* $C = C(x_0) > 0$ *such that for any* $j \in [\![1, p+q]\!]$, $\gamma \in \Gamma$ *with* $l(\gamma) = j$ *and* $x, y \in \tilde{\Lambda}_l$, $l \neq j$,

$$\left|\tilde{b}(\gamma, x) - \tilde{b}(\gamma, y)\right| \leqslant C\mathrm{d}_{\mathbf{o}}(x, y).$$

The proof is long and technical. It is sufficient to prove this result for $\gamma \in \Gamma$ with symbolic length 1. Indeed, assume that $\gamma = \alpha_1 \cdots \alpha_k$ for $k \geqslant 2$ and denote by $\gamma^{(j)} = \alpha_j \cdots \alpha_k$ for any $j \in [\![2, k-1]\!]$. Using the cocycle property of $\tilde{b}$, we obtain for any $x, y \in \tilde{\Lambda}\backslash\tilde{\Lambda}_j$

$$\begin{aligned}\left|\tilde{b}(\gamma, x) - \tilde{b}(\gamma, y)\right| &\leqslant \left|\tilde{b}(\alpha_1, \gamma^{(2)} \cdot x) - \tilde{b}(\alpha_1, \gamma^{(2)} \cdot y)\right| + \left|\tilde{b}(\gamma^{(2)}, x) - \tilde{b}(\gamma^{(2)}, y)\right| \\ &\leqslant \left|\tilde{b}(\alpha_1, \gamma^{(2)} \cdot x) - \tilde{b}(\alpha_1, \gamma^{(2)} \cdot y)\right| + \left|\tilde{b}(\alpha_2, \gamma^{(3)} \cdot x) - \tilde{b}(\alpha_2, \gamma^{(3)} \cdot y)\right| \\ &\quad + \cdots + \left|\tilde{b}(\alpha_k, x) - \tilde{b}(\alpha_k, y)\right|.\end{aligned}$$

If Proposition 8.2.2 holds for element of $\Gamma$ with symbolic length 1, we get

$$\left|\tilde{b}(\gamma, x) - \tilde{b}(\gamma, y)\right| \leqslant C(x_0)\left(\mathrm{d}_{\mathbf{o}}(\gamma^{(2)} \cdot x, \gamma^{(2)} \cdot y) + \cdots + \mathrm{d}_{\mathbf{o}}(x, y)\right),$$

where $C(x_0)$ depends only on $x_0$. By Corollary 4.1.5, there exist $r \in ]0, 1[$ and $C > 0$ such that $\mathrm{d}_{\mathbf{o}}(\gamma^{(2)} \cdot x, \gamma^{(2)} \cdot y) \leqslant C.r^{k-1}\mathrm{d}_{\mathbf{o}}(x, y), \ldots, \mathrm{d}_{\mathbf{o}}(\alpha_k \cdot x, \alpha_k \cdot y) \leqslant C.r\mathrm{d}_{\mathbf{o}}(x, y)$, so that

$$\left|\tilde{b}(\gamma, x) - \tilde{b}(\gamma, y)\right| \leqslant CC(x_0)\frac{1}{1-r}\mathrm{d}_{\mathbf{o}}(x, y),$$

which proves that the inequality is still valid when $|\gamma| \geqslant 2$.

Let us fix for this subsection $\alpha \in \Gamma_j^*$, $j \in [\![1, p+q]\!]$, and $x, y \in \tilde{\Lambda}_l$ for some $l \neq j$. There are three cases to consider:

(a) the points $x$ and $y$ both belong to $\Lambda$;
(b) $x \in \Lambda$ and $y \in \Gamma \cdot x_0$;
(c) the points $x$ and $y$ belong to $\Gamma \cdot x_0$.

The different cases are treated in the next three subsections.

*Case (a): the points* $\mathbf{x}$ *and* $\mathbf{y}$ *both belong to* $\Lambda$. – The statement follows from Proposition 4.2.3.

*Case (b):* $\mathbf{x}\in \Lambda$ *and* $\mathbf{y}\in \Gamma \cdot x_0$. – Set $y = g \cdot x_0$ for some $g \in \Gamma$ with $i(g) = l$. The statement may be thus reformulated as follows.

*Proposition 8.2.2 in case* (b). – *There exists a constant* $C = C(x_0) > 0$ *such that for any* $j \in [\![1, p+q]\!]$, $\alpha \in \Gamma_j^*$, $g \in \Gamma$ *with* $i(g) \neq j$ *and* $x \in \Lambda_{i(g)}$, *the following inequality holds*

$$\left| \mathscr{B}_x(\alpha^{-1} \cdot \mathbf{o}, \mathbf{o}) - \tilde{b}(\alpha, g \cdot x_0) \right| \leqslant C \mathrm{d}_{\mathbf{o}}(x, g \cdot x_0).$$

*Proof of the case* (b). – The proof is split into two steps. In the first step (1), we assume that $x \in g \cdot \left(\Lambda \backslash \Lambda_{l(g)}\right)$ and then we prove the case (b) without this assumption in step (2).

(1) Denote by $V(x, \mathbf{o}, t)$ the subset of points $y \in \partial \mathrm{X}$ whose projection $\tilde{\mathbf{y}}$ on the geodesic ray $[\mathbf{o}x)$ satisfies $\mathrm{d}(\tilde{\mathbf{y}}, \mathbf{o}) \geqslant t$; the $\mathrm{d}_{\mathbf{o}}$-diameter of such a set is $\preceq e^{-at}$. The set $\tilde{V}(x, \mathbf{o}, t)$ stands for a connected and geodesically convex subset of $\overline{\mathrm{X}}$ whose intersection with $\partial \mathrm{X}$ equals to $V(x, \mathbf{o}, t)$. Recall that the space X is a Gromov-$\kappa$-hyperbolic space for some $\kappa = \kappa(a) > 0$. The following properties are proved in [**42**].

Proposition 8.2.3. – *Let* $\kappa > 0$ *such that* X *is a Gromov-*$\kappa$*-hyperbolic space.*

1. *Let* $\mathbf{p} \in \mathrm{X}$, $x \in \partial \mathrm{X}$ *and* $t \geqslant 1$. *For any* $y \in V(x, \mathbf{p}, t + 7\kappa)$

$$V(x, \mathbf{p}, t + 6\kappa) \subset V(y, \mathbf{p}, t) \subset V(x, \mathbf{p}, t - 6\kappa).$$

2. *For any* $D > 0$, *denote by* $K_2 = 2D + 4\kappa$. *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be in* X *such that* $\mathrm{d}(\mathbf{p}, \mathbf{q}) \leqslant D$, $x \in \partial \mathrm{X}$ *and* $t \geqslant K_2$. *Hence*

$$V(x, \mathbf{p}, t + K_2) \subset V(x, \mathbf{q}, t) \subset V(x, \mathbf{p}, t - K_2).$$

Let $y \in \Lambda \backslash \Lambda_{l(g)}$ such that $x = g \cdot y$. The conformality Equation (5) implies

$$\mathrm{d}_{\mathbf{o}}(x, g \cdot x_0) = \mathrm{d}_{\mathbf{o}}(g \cdot y, g \cdot x_0) = \sqrt{|g'(y)|_{\mathbf{o}} |g'(x_0)|_{\mathbf{o}}} \mathrm{d}_{\mathbf{o}}(y, x_0).$$

Since $y \in \Lambda \backslash D_{l(g)}$, Property 4.1.3 implies

$$\mathrm{d}_{\mathbf{o}}(x, g \cdot x_0) \asymp e^{-a\mathrm{d}(\mathbf{o}, g \cdot \mathbf{o})} \mathrm{d}_{\mathbf{o}}(y, x_0) \asymp e^{-a\mathrm{d}(\mathbf{o}, g \cdot \mathbf{o})}.$$

We now estimate $|\mathscr{B}_x(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})-\tilde{b}(\alpha,g\cdot x_0)|$. Denote by $\xi'$ (resp. $\xi''$) the endpoint of the geodesic ray starting from $\mathbf{o}$ (resp. from $\alpha^{-1}\cdot\mathbf{o}$) and passing through $g\cdot\mathbf{o}$. From the definition of Busemann functions (see Figure 7), we derive

$$\mathscr{B}_{\xi'}(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})\leqslant \mathrm{d}(\alpha^{-1}\cdot\mathbf{o},g\cdot\mathbf{o})-\mathrm{d}(\mathbf{o},g\cdot\mathbf{o})\leqslant \mathscr{B}_{\xi''}(\alpha^{-1}\cdot\mathbf{o},\mathbf{o}). \tag{64}$$

The points $\xi'$ and $\xi''$ belong to the path-connected set $V(\xi',\mathbf{o},\mathrm{d}(\mathbf{o},g\cdot\mathbf{o}))$ which

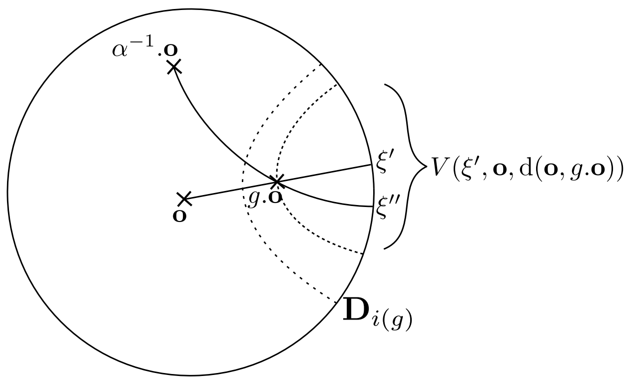


FIGURE 7. Estimate of $\mathrm{d}(\alpha^{-1}\cdot\mathbf{o},\gamma\cdot\mathbf{o})-\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})$.

is path-connected. From Proposition 4.2.3 and (64), we deduce the existence of $\xi\in D_{i(g)}$ such that $\tilde{b}(\alpha,g\cdot x_0)=\mathscr{B}_\xi(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})$. Proposition 4.2.3 also implies

$$|\mathscr{B}_x(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})-\tilde{b}(\alpha,g\cdot x_0)|=|\mathscr{B}_x(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})-\mathscr{B}_\xi(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})|\leqslant C\mathrm{d}_{\mathbf{o}}(x,\xi)$$

for some constant $C>0$. Since $\xi\in V(\xi',\mathbf{o},\mathrm{d}(\mathbf{o},g\cdot\mathbf{o}))$, we derive $\mathrm{d}_{\mathbf{o}}(\xi',\xi)\preceq e^{-a\mathrm{d}(\mathbf{o},g\cdot\mathbf{o})}$. In order to obtain a similar conclusion for $\mathrm{d}_{\mathbf{o}}(x,\xi')$, we have two different cases to consider.

(i) *Assume first that* $i(g)\neq l(g)$. In this case, the isometry $g$ is hyperbolic with attractive (resp. repulsive) fixed point $x_g^+$ (resp. $x_g^-$); notice that $x_g^+,y\in\Lambda\backslash\Lambda_{l(g)}$, while $x_g^-\in\Lambda_{l(g)}$. There thus exist $E>0$ and a point $\tilde{\mathbf{o}}\in(x_g^-x_g^+)$ such that $\mathrm{d}(\mathbf{o},\tilde{\mathbf{o}})\leqslant E$ and the projection of $y$ on the axis $(x_g^-x_g^+)$ belongs to the geodesic ray $[\tilde{\mathbf{o}}x_g^+)$ (see Figure 8). Denote by $L=\mathrm{d}(\tilde{\mathbf{o}},g.\tilde{\mathbf{o}})$ the length of the axis of $g$. Proposition 8.2.3.2 with $\mathbf{q}=\tilde{\mathbf{o}}$, $\mathbf{p}=\mathbf{o}$, $D=E$ and $t=L-E-3\kappa$ yields

$$V(x_g^+,\tilde{\mathbf{y}},L-E-3\kappa)\subset V(x_g^+,\tilde{\mathbf{o}},L-E-3\kappa)\subset V(x_g^+,\mathbf{o},L-3E-7\kappa). \tag{65}$$

It follows from $g.\tilde{\mathbf{o}}\in B(\gamma\cdot\mathbf{o},E)$ and $B(g\cdot\mathbf{o},E)\subset\tilde{V}(\xi',\mathbf{o},\mathrm{d}(\mathbf{o},g\cdot\mathbf{o})-E)$, that $[g.\tilde{\mathbf{o}},x_g^+)\subset\tilde{V}(\xi',\mathbf{o},\mathrm{d}(\mathbf{o},g\cdot\mathbf{o})-E)$, hence $x_g^+\in V(\xi',\mathbf{o},\mathrm{d}(\mathbf{o},g\cdot\mathbf{o})-$

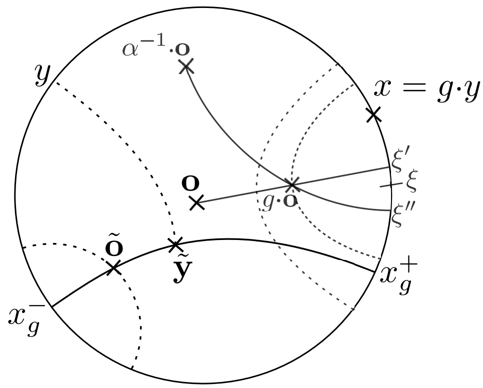


FIGURE 8. Points $\tilde{\mathbf{o}}$ and $\tilde{y}$.

$E$). The triangular inequality implies $L - 2E \leqslant \mathrm{d}(\mathbf{o}, g\cdot\mathbf{o})$, so that $x_g^+ \in V(\xi', \mathbf{o}, L - 3E)$. By Proposition 8.2.3.1 with $t = L - 3E - 7\kappa$, we get

$$V(x_g^+, \mathbf{o}, L - 3E - 7\kappa) \subset V(\xi', \mathbf{o}, L - 3E - 13\kappa), \tag{66}$$

so that combining (65) and (66)

$$V(x_g^+, \tilde{\mathbf{y}}, L - E) \subset V(\xi', \mathbf{o}, L - 3E - 13\kappa).$$

Since $x \in V(x_g^+, \tilde{\mathbf{y}}, L - E)$, we have $x \in V(\xi', \mathbf{o}, L - 3E - 13\kappa)$; moreover, the triangular inequality implies $\mathrm{d}(\mathbf{o}, g\cdot\mathbf{o}) - 2E \leqslant L$, hence $x \in V(\xi', \mathbf{o}, \mathrm{d}(\mathbf{o}, g\cdot\mathbf{o}) - 5E - 13\kappa)$. Finally $\mathrm{d}_{\mathbf{o}}(x, \xi') \leqslant K'_\Gamma e^{-a\mathrm{d}(\mathbf{o}, g\cdot\mathbf{o})}$, so that $\mathrm{d}_{\mathbf{o}}(x, \xi) \preceq e^{-a\mathrm{d}(\mathbf{o}, g\cdot\mathbf{o})}$: this completes the proof of part (1) when $i(g) \neq l(g)$.

(ii) *When $i(g) = l(g)$:* one applies the previous arguments with $\alpha g$ instead of $g$. We thus obtain $\mathrm{d}_{\mathbf{o}}(\alpha\cdot x, \alpha\cdot\xi') \preceq e^{-a\mathrm{d}(\mathbf{o}, \alpha g\cdot\mathbf{o})}$. Using (5) and Lemma 2.1.1, we finally obtained $\mathrm{d}_{\mathbf{o}}(x, \xi') \preceq e^{-a\mathrm{d}(\mathbf{o}, g\cdot\mathbf{o})}$.

(2) We now prove the case (b) without additional assumption on the position of $x$ in $\Lambda_{i(g)}$.

FACT 8.2.4. – *There exists a constant $C = C(x_0) > 0$ such that, for any $g \in \Gamma$ and $x \in \Lambda_{i(g)}$, there exists $z \in g\cdot\left(\Lambda\backslash\Lambda_{l(g)}\right)$ such that $\mathrm{d}_{\mathbf{o}}(z, x) \leqslant C\mathrm{d}_{\mathbf{o}}(x, g\cdot x_0)$ and $\mathrm{d}_{\mathbf{o}}(z, g\cdot x_0) \leqslant C\mathrm{d}_{\mathbf{o}}(x, g\cdot x_0)$.*

*Proof of Fact 8.2.4..* – Since $x \in \Lambda_{i(g)}$, there exists $g' \in \Gamma$ with $i(g') = i(g)$ and $x' \in \Lambda$ such that $x = g'.x'$. Let $k$ be the first index $\leqslant \min(|g'|, |g|)$ for which the $k$-th letters of $g'$ and $g$ are different. There are two cases to consider.

(i) *Both $k$-th letters of $g$ and $g'$ do not belong to the same Schottky factor* $\Gamma_j, 1 \leqslant j \leqslant p + q$; in this case, we may write $g = \alpha_1\cdots\alpha_{k-1}\alpha_k g_1$ and $g' = \alpha_1\cdots\alpha_{k-1}\alpha'_k g'_1$ where $\alpha_k$ and $\alpha'_k$ do not belong to the same factor.

Fix $w \in \Lambda \backslash \Lambda_{l(g)}$ and a point $u$ in the boundary $\partial_{g_1}^{x_0}$ of the connected component of $D_{i(g_1)}$ containing $g_1 \cdot x_0$ (see Figure 9). It follows that

$$\mathrm{d}_{\mathbf{o}}(g \cdot x_0, \alpha_1 \cdots \alpha_k \cdot u) = e^{-\frac{a}{2}\mathscr{B}_{g_1 \cdot x_0}(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})} e^{-\frac{a}{2}\mathscr{B}_u(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})} \mathrm{d}_{\mathbf{o}}(g_1 \cdot x_0, u).$$

Setting $m = \min_{j \in [\![1, p+q]\!]} \min_{\substack{\gamma \in \Gamma \\ i(\gamma) = j}} \min_{u \in \partial_\gamma^{x_0}} \mathrm{d}_{\mathbf{o}}(\gamma \cdot x_0, u) > 0$, we obtain

$$m e^{-\frac{a}{2}\mathscr{B}_{g_1 \cdot x_0}(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})} e^{-\frac{a}{2}\mathscr{B}_u(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})} \leqslant \mathrm{d}_{\mathbf{o}}(g \cdot x_0, \alpha_1 \cdots \alpha_k \cdot u).$$

There also exists a constant $M > 0$ (which actually is the $\mathrm{d}_{\mathbf{o}}$-diameter of $\partial \mathrm{X}$) such that

$$\mathrm{d}_{\mathbf{o}}(g \cdot x_0, g \cdot w) \leqslant M e^{-\frac{a}{2}\mathscr{B}_{g_1 \cdot x_0}(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})} e^{-\frac{a}{2}\mathscr{B}_{g_1 \cdot w}(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})}.$$

This yields

$$\frac{\mathrm{d}_{\mathbf{o}}(g \cdot x_0, g \cdot w)}{\mathrm{d}_{\mathbf{o}}(g \cdot x_0, \alpha_1 \cdots \alpha_k \cdot u)} \leqslant \frac{M}{m} e^{-\frac{a}{2}\left(\mathscr{B}_{g_1 \cdot w}(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o}) - \mathscr{B}_u(\alpha_k^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})\right)} \preceq 1,$$

where the last estimate is uniform in $u \in \partial_{g_1}^{x_0}$ and follows from Property 4.1.3. Since this upper bound is true for any $u \in \partial_{g_1}^{x_0}$, we deduce from the compactness of $\partial_{g_1}^{x_0}$ that $\mathrm{d}_{\mathbf{o}}(g \cdot x_0, g \cdot w) \leqslant C'(x_0) \mathrm{d}_{\mathbf{o}}(g \cdot x_0, \alpha_1 \cdots \alpha_k \cdot \partial_{g_1}^{x_0})$. Since $x \notin \alpha_1 \cdots \alpha_k \cdot D_{i(g_1)}$, we obtain $\mathrm{d}_{\mathbf{o}}(g \cdot x_0, g \cdot w) \leqslant C'(x_0) \mathrm{d}_{\mathbf{o}}(x, g \cdot x_0)$. The triangular inequality yields $\mathrm{d}_{\mathbf{o}}(g \cdot w, x) \leqslant \mathrm{d}_{\mathbf{o}}(x, g \cdot x_0) + \mathrm{d}(g \cdot x_0, g \cdot w)$ and the result follows with $z = g \cdot w$ and $C = 1 + C'(x_0)$.

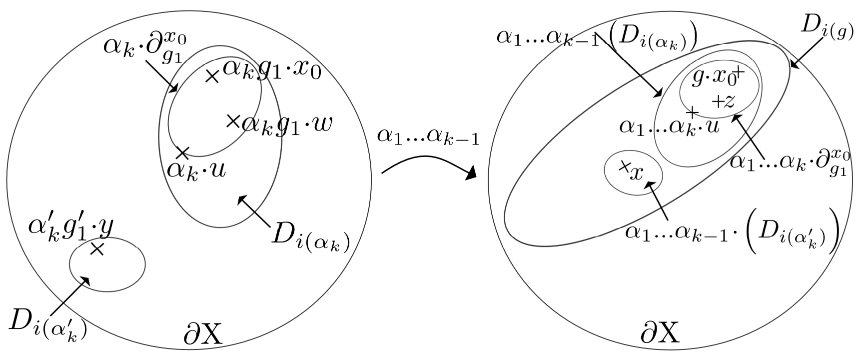


FIGURE 9. Action of $\alpha_1 \cdots \alpha_{k-1}$.

(ii) *Both k-th letters belong to the same Schottky factor* $\Gamma_j, 1 \leqslant j \leqslant p+q$: there thus exist $\beta \in \mathscr{A}$ and $n > n' \in \mathbb{N}^*$ such that $g = \alpha_1 \cdots \alpha_{k-1} \beta^n g_1$ and $g' = \alpha_1 \cdots \alpha_{k-1} \beta^{n'} g'_1$. Assume that $\beta$ generates $\Gamma_l$, $l \in [\![1, p+q]\!]$. Fix $u \in \partial_{g_1}^{x_0}$ and $w \in \Lambda \backslash \Lambda_{l(g)}$. Similarly to (i), we get

$$m e^{-\frac{a}{2}\mathscr{B}_{g_1 \cdot x_0}(\beta^{-n} \alpha_{k-1}^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})} e^{-\frac{a}{2}\mathscr{B}_u(\beta^{-n} \alpha_{k-1}^{-1} \cdots \alpha_1^{-1} \cdot \mathbf{o}, \mathbf{o})} \leqslant \mathrm{d}_{\mathbf{o}}(g \cdot x_0, \alpha_1 \cdots \alpha_{k-1} \beta^n \cdot u)$$

and

$$\mathrm{d}_{\mathbf{o}}(g\cdot x_0, g\cdot w) \leqslant M e^{-\frac{a}{2}\mathscr{B}_{g_1\cdot x_0}(\beta^{-n}\alpha_{k-1}^{-1}\cdots\alpha_1^{-1}\cdot\mathbf{o},\mathbf{o})} e^{-\frac{a}{2}\mathscr{B}_{g_1\cdot w}(\beta^{-n}\alpha_{k-1}^{-1}\cdots\alpha_1^{-1}\cdot\mathbf{o},\mathbf{o})}.$$

Using Property 4.1.3 and noticing that the previous estimates are satisfied for any $u \in \partial_{g_1}^{x_0}$, we deduce the existence of a constant $C'(x_0) > 0$ such that $\mathrm{d}_{\mathbf{o}}(g\cdot x_0, g\cdot w) \leqslant C'(x_0)\mathrm{d}_{\mathbf{o}}(g\cdot x_0, \alpha_1\cdots\alpha_{k-1}\beta^n\cdot\partial_{g_1}^{x_0})$. Since $x \in \alpha_1\cdots\alpha_{k-1}\beta^{n'}\cdot D_{i(g_1')}$ and $\alpha_1\cdots\alpha_{k-1}\beta^{n'}\cdot D_{i(g_1')} \cap \alpha_1\cdots\alpha_{k-1}\beta^n\cdot\partial_{g_1}^{x_0} = \varnothing$, we deduce $\mathrm{d}_{\mathbf{o}}(g\cdot w, g\cdot x_0) \leqslant C'(x_0)\mathrm{d}_{\mathbf{o}}(x, g\cdot x_0)$. We set $z = g\cdot w$. This completes the proof of Fact 8.2.4. □

Let us now come back to the proof of the part (2) of case (b). Fix $x \in \Lambda_{i(g)}$ and a point $z \in g\cdot\left(\Lambda\backslash\Lambda_{l(g)}\right)$ satisfying the conclusions of Fact 8.2.4. We bound $\left|\mathscr{B}_x(\alpha^{-1}\cdot\mathbf{o},\mathbf{o}) - \tilde{b}(\alpha, g\cdot x_0)\right|$ from above by

$$\left|\mathscr{B}_x(\alpha^{-1}\cdot\mathbf{o},\mathbf{o}) - \mathscr{B}_z(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})\right| + \left|\mathscr{B}_z(\alpha^{-1}\cdot\mathbf{o},\mathbf{o}) - \tilde{b}(\alpha, g\cdot x_0)\right|. \tag{67}$$

Proposition 4.2.3 and part (1) of case (b) applied at $z$ thus imply that there exists $C = C(x_0) > 0$ such that the bound (67) is smaller than $C\mathrm{d}_{\mathbf{o}}(x, g\cdot x_0)$, which concludes the proof of case (b). □

*Case* (c)*: the points* $\mathbf{x}$ *and* $\mathbf{y}$ *belong to* $\boldsymbol{\Gamma}\cdot\boldsymbol{x_0}$. – Fix $g_1, g_2 \in \Gamma$ such that $i(g_1) = i(g_2) = l$, $x = g_1\cdot x_0$ and $y = g_2\cdot x_0$. Set $g_1 = g\beta_1$ and $g_2 = g\beta_2$.

(1) Assume first that $i(\beta_1) \neq i(\beta_2)$. Let $z_1 \in \Lambda\backslash\Lambda_{l(g_1)}$ and $z_2 \in \Lambda\backslash\Lambda_{l(g_2)}$. From Proposition 4.2.3 and from case (b) above, we deduce that there exists a constant $C' = C'(x_0) > 0$ such that

$$\begin{aligned}\left|\tilde{b}(\alpha, g_1\cdot x_0) - \tilde{b}(\alpha, g_2\cdot x_0)\right| &\leqslant \left|\tilde{b}(\alpha, g_1\cdot x_0) - \mathscr{B}_{g_1\cdot z_1}(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})\right| \\ &\quad + \left|\mathscr{B}_{g_1\cdot z_1}(\alpha^{-1}\cdot\mathbf{o},\mathbf{o}) - \mathscr{B}_{g_2\cdot z_2}(\alpha^{-1}\cdot\mathbf{o},\mathbf{o})\right| \\ &\quad + \left|\mathscr{B}_{g_2\cdot z_2}(\alpha^{-1}\cdot\mathbf{o},\mathbf{o}) - \tilde{b}(\alpha, g_2\cdot x_0)\right| \\ &\leqslant C'\left(\mathrm{d}_{\mathbf{o}}(g_1\cdot x_0, g_1\cdot z) + \mathrm{d}_{\mathbf{o}}(g_1\cdot z_1, g_2\cdot z_2)\right. \\ &\quad \left.\mathrm{d}_{\mathbf{o}}(g_2\cdot z_2, g_2\cdot x_0)\right).\end{aligned}$$

Combining Property (5) and Property 4.1.3, it follows that

$$\begin{aligned}\left|\tilde{b}(\alpha, g_1\cdot x_0) - \tilde{b}(\alpha, g_2\cdot x_0)\right| &\preceq e^{-a\mathrm{d}(\mathbf{o}, g_1\cdot\mathbf{o})}\mathrm{d}_{\mathbf{o}}(x_0, z_1) + e^{-a\mathrm{d}(\mathbf{o}, g\cdot\mathbf{o})}\mathrm{d}_{\mathbf{o}}(\beta_1\cdot z_1, \beta_2\cdot z_2) \\ &\quad + e^{-a\mathrm{d}(\mathbf{o}, g_2\cdot\mathbf{o})}\mathrm{d}_{\mathbf{o}}(z_2, x_0) \\ &\preceq \left(e^{-a\mathrm{d}(\mathbf{o}, g_1\cdot\mathbf{o})} + e^{-a\mathrm{d}(\mathbf{o}, g\cdot\mathbf{o})} + e^{-a\mathrm{d}(\mathbf{o}, g_2\cdot\mathbf{o})}\right).\end{aligned}$$

Lemma 2.1.1 implies that, for $i \in \{1, 2\}$,

$$\mathrm{d}(\mathbf{o}, g\cdot\mathbf{o}) + \mathrm{d}(\mathbf{o}, \beta_i\cdot\mathbf{o}) - C \leqslant \mathrm{d}(\mathbf{o}, g_i.\mathbf{o}) \leqslant \mathrm{d}(\mathbf{o}, g\cdot\mathbf{o}) + \mathrm{d}(\mathbf{o}, \beta_i\cdot\mathbf{o}) + C,$$

hence

$$\left|\tilde{b}(\alpha, g_1 \cdot x_0) - \tilde{b}(\alpha, g_2 \cdot x_0)\right| \leq e^{-a\mathrm{d}(\mathbf{o}, g \cdot \mathbf{o})}.$$

Since $\min_{(x,y) \in \bigcup_{l \neq j} \tilde{\Lambda}_l \times \tilde{\Lambda}_j} \mathrm{d}_{\mathbf{o}}(x, y) > 0$, this yields

$$\left|\tilde{b}(\alpha, g_1 \cdot x_0) - \tilde{b}(\alpha, g_2 \cdot x_0)\right| \leq e^{-a\mathrm{d}(\mathbf{o}, g \cdot \mathbf{o})} \mathrm{d}_{\mathbf{o}}(\beta_1 \cdot x_0, \beta_2 \cdot x_0).$$

We conclude part (1) of case (c) using (5) again.

(2) If $i(\beta_1) = i(\beta_2)) = l \in [\![1, p+q]\!]$: there exist $\beta \in \Gamma_l^*$, $n_1, n_2 \in \mathbb{Z}^*$, $n_1 \neq n_2$ and $\beta_{1,1}, \beta_{2,1} \in \Gamma$ such that $\beta_1 = \beta^{n_1}\beta_{1,1}$ and $\beta_2 = \beta^{n_2}\beta_{2,1}$. We need the following fact.

Fact 8.2.5. – *There exists $z \in \Lambda_l$ such that $\mathrm{d}_{\mathbf{o}}(\beta_1 \cdot x_0, z) \leqslant C\mathrm{d}(\beta_1 \cdot x_0, \beta_2 \cdot x_0)$ and $\mathrm{d}_{\mathbf{o}}(\beta_2 \cdot x_0, z) \leqslant C\mathrm{d}(\beta_1 \cdot x_0, \beta_2 \cdot x_0)$ for a constant $C = C(x_0) > 0$.*

*Proof of Fact 8.2.5.* – Fix $w \in \Lambda_{i(\beta_{2,1})}$ and $u \in \partial^{x_0}_{\beta_{2,1}}$. As in the proof of Fact 8.2.4, we may write

$$m e^{-\frac{a}{2}\mathscr{B}_{\beta_{2,1}\cdot x_0}(\beta^{-n_2}\cdot\mathbf{o}, \mathbf{o})} e^{-\frac{a}{2}\mathscr{B}_u(\beta^{-n_2}\cdot\mathbf{o}, \mathbf{o})} \leqslant \mathrm{d}_{\mathbf{o}}(\beta^{n_2}\beta_{2,1} \cdot x_0, \beta^{n_2} \cdot u)$$

and

$$\mathrm{d}_{\mathbf{o}}(\beta^{n_2}\beta_{2,1} \cdot x_0, \beta^{n_2} \cdot w) \leqslant M e^{-\frac{a}{2}\mathscr{B}_{\beta_{2,1}\cdot x_0}(\beta^{-n_2}\cdot\mathbf{o}, \mathbf{o})} e^{-\frac{a}{2}\mathscr{B}_w(\beta^{-n_2}\cdot\mathbf{o}, \mathbf{o})}.$$

Setting $z = \beta^{n_2} \cdot w$, Property 4.1.3 and the previous estimates yield

$$\mathrm{d}_{\mathbf{o}}(\beta^{n_2}\beta_{2,1} \cdot x_0, z) \leqslant C'\mathrm{d}_{\mathbf{o}}(\beta^{n_2}\beta_{2,1} \cdot x_0, \beta^{n_2} \cdot \partial^{x_0}_{\beta_{2,1}})$$

for $C' = C'(x_0) > 0$. But $\beta^{n_1}\beta_{1,1} \cdot x_0 \in \beta^{n_1} \cdot (\partial \mathrm{X} \backslash D_l)$ and $\beta^{n_1} \cdot (\partial \mathrm{X} \backslash D_l) \cap \beta^{n_2} \cdot \partial^{x_0}_{\beta_{2,1}} = \varnothing$, therefore

$$\mathrm{d}_{\mathbf{o}}(\beta^{n_2}\beta_{2,1} \cdot x_0, z) \leqslant C'\mathrm{d}_{\mathbf{o}}(\beta^{n_2}\beta_{2,1} \cdot x_0, \beta^{n_1}\beta_{1,1} \cdot x_0).$$

Fact 8.2.5 follows with $C = 1 + C'$. □

We now complete the proof of part (2) of case (c). Let $z$ be a point of $\Lambda_l$ satisfying the conclusion of Fact 8.2.5 above. Let us split $\left|\tilde{b}(\alpha, g_1 \cdot x_0) - \tilde{b}(\alpha, g_2 \cdot x_0)\right|$ into

$$\left|\tilde{b}(\alpha, g_1 \cdot x_0) - \tilde{b}(\alpha, g \cdot z)\right| + \left|\tilde{b}(\alpha, g \cdot z) - \tilde{b}(\alpha, g_2 \cdot x_0)\right|;$$

The case (c) with $x = g_1 \cdot x_0$ and $y = g_2 \cdot x_0$ follows using case (b), the conformality equality (5) and the definition of $z$.

## 8.3. The extended transfer operator and its spectral properties

We now introduce the transfer operator associated to the roof function $\tilde{b}$. Recall that $\tilde{\Lambda} = \Lambda \cup \Gamma \cdot x_0$ and $\tilde{\Lambda}_j := \overline{\tilde{\Lambda}_j^0}$ for any $j \in [\![1, p+q]\!]$.

For any $z \in \mathbb{C}$ and any function $\varphi \in \mathcal{C}(\tilde{\Lambda})$, we formally define the operator $\tilde{\mathcal{L}}_z$ as follows: for any $x \in \tilde{\Lambda}^0$

$$\tilde{\mathcal{L}}_z \varphi(x) = \sum_{j=1}^{p+q} \sum_{\alpha \in \Gamma_j^*} \mathbb{1}_{\tilde{\Lambda}_j^c}(x) e^{-z\tilde{b}(\alpha, x)} \varphi(\alpha \cdot x).$$

Assumptions ($P_1$) and (N) combined with Property 4.1.3 imply that the previous sums are finite for $\mathrm{Re}\,(z) \geqslant \delta$. The normal convergence of these series for $\mathrm{Re}\,(z) \geqslant \delta$ and the fact that $\varphi \in \mathcal{C}(\tilde{\Lambda})$ also imply that $\tilde{\mathcal{L}}_z \varphi$ may be continuously extended on $\tilde{\Lambda}$; in particular, the operator $\tilde{\mathcal{L}}_\delta$ acts on $\mathcal{C}(\tilde{\Lambda})$. Denote by $\tilde{\rho}_\infty$ its spectral radius on this set. The operator $\tilde{\mathcal{L}}_\delta$ is positive, hence

$$\tilde{\rho}_\infty = \limsup_{k \longrightarrow +\infty} \left| (\tilde{\mathcal{L}}_\delta)^k \mathbb{1}_{\tilde{\Lambda}} \right|_\infty^{1/k}.$$

To obtain a spectral gap, we will study the action of $\tilde{\mathcal{L}}_\delta$ on the space $\mathrm{Lip}(\tilde{\Lambda})$, defined by

$$\mathrm{Lip}(\tilde{\Lambda}) := \left\{ \varphi \in \mathcal{C}(\tilde{\Lambda}) \mid ||\varphi|| := |\varphi|_\infty + [\varphi] < +\infty \right\},$$

where

$$[\varphi] = \sup_{1 \leqslant j \leqslant p+q} \sup_{(x,y) \in \tilde{\Lambda}_j \overset{\Delta}{\times} \tilde{\Lambda}_j} \frac{|\varphi(x) - \varphi(y)|}{\mathrm{d}_{\mathbf{o}}(x,y)}.$$

The following proposition implies that $\tilde{\mathcal{L}}_z$ is bounded on $\mathrm{Lip}(\tilde{\Lambda})$ for $z \in \mathbb{C}$ such that $\mathrm{Re}\,(z) \geqslant \delta$.

PROPOSITION 8.3.1. – *Each weight* $\tilde{w}_z(\gamma, \cdot) := e^{-z\tilde{b}(\gamma, \cdot)} \mathbb{1}_{\tilde{\Lambda}^c_{l(\gamma)}}$ *belongs to* $\mathrm{Lip}(\tilde{\Lambda})$ *and there exists a constant* $C = C(z) > 0$ *such that for any* $\gamma$ *in* $\Gamma^*$*, we have*

$$||\tilde{w}_z(\gamma, \cdot)|| \leqslant C e^{-\mathrm{Re}(z) \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o})}.$$

The proof is similar to that of Proposition 4.2.4 using Proposition 8.2.2. Let $\tilde{\rho}$ denote the spectral radius of $\tilde{\mathcal{L}}_\delta$ on $\left(\mathrm{Lip}(\tilde{\Lambda}), || \cdot ||\right)$.

PROPOSITION 8.3.2. – *The operator* $\tilde{\mathcal{L}}_\delta$ *is quasi-compact on* $\mathrm{Lip}(\tilde{\Lambda})$ *and* $\tilde{\rho}$ *is a simple and isolated eigenvalue in its spectrum, associated to a positive eigenfunction; this is the unique eigenvalue with modulus* $\tilde{\rho}$*. Furthermore* $\tilde{\rho} = \tilde{\rho}_\infty = 1$.

*Proof.* – As in the proof of Proposition 4.2.7, the first step is to prove that $\tilde{\mathcal{L}}_\delta$ is quasi-compact on $\mathrm{Lip}(\tilde{\Lambda})$. By **[29]**, it is sufficient to prove that $\tilde{\mathcal{L}}_\delta$ satisfies the following property.

Definition 8.3.3 (Property DF(s) for Doeblin and Fortet). – *The operator* $\tilde{\mathcal{L}}_\delta$ *satisfies the DF(s)-property on* $(\mathrm{Lip}(\tilde{\Lambda}), ||\cdot||)$ *if*

i) $\tilde{\mathcal{L}}_\delta$ *is compact from* $(\mathrm{Lip}(\tilde{\Lambda}), ||\cdot||)$ *to* $(\mathrm{Lip}(\tilde{\Lambda}), |\cdot|_\infty)$;

ii) *for any* $k \in \mathbb{N}\backslash\{0\}$, *there exist positive reals* $S_k, s_k$ *such that*

$$\left|\left|\left(\tilde{\mathcal{L}}_\delta\right)^k \varphi\right|\right| \leqslant S_k\, |\varphi|_\infty + s_k\, ||\varphi|| \text{ for any } \varphi \in \mathrm{Lip}(\tilde{\Lambda}) \text{ and } \liminf_k (s_k)^{1/k} = s < \tilde{\rho}.$$

We follow once again the steps of proofs given in **[2]** and **[40]**. The set $\tilde{\Lambda} = \Lambda \cup \Gamma\cdot x_0$ being compact, Ascoli's result implies that the inclusion $\left(\mathrm{Lip}(\tilde{\Lambda}), ||\cdot||\right) \hookrightarrow \left(\mathrm{Lip}(\tilde{\Lambda}), |\cdot|_\infty\right)$ is compact; hence $\tilde{\mathcal{L}}_\delta$ is compact from $\left(\mathrm{Lip}(\tilde{\Lambda}), ||\cdot||\right)$ to $\left(\mathrm{Lip}(\tilde{\Lambda}), |\cdot|_\infty\right)$. Let us show the existence of two sequences $(s_k)_{k\geqslant 1}$ and $(S_k)_{k\geqslant 1}$ satisfying property ii) above. Let $\varphi \in \mathrm{Lip}(\tilde{\Lambda})$, $k \geqslant 1$ and $j \in [\![1, p+q]\!]$. For any $x \in \tilde{\Lambda}_j$,

$$\left|(\tilde{\mathcal{L}}_\delta)^k \varphi(x)\right| \leqslant \left(\sum\nolimits_{|\gamma|=k} ||\tilde{w}_\delta(\gamma,\cdot)||\right) \cdot |\varphi|_\infty. \tag{68}$$

For any $(x,y) \in \tilde{\Lambda}_j \overset{\Delta}{\times} \tilde{\Lambda}_j$, we bound $\left|(\tilde{\mathcal{L}}_\delta)^k\varphi(x) - (\tilde{\mathcal{L}}_\delta)^k\varphi(y)\right|$ from above by $K_1 + K_2$ where

$$K_1 := \sum_{\gamma\in\Gamma(k), l(\gamma)\neq j} e^{-\delta\tilde{b}(\gamma,x)}\, |\varphi(\gamma\cdot x) - \varphi(\gamma\cdot y)|$$

and

$$K_2 := \sum_{\gamma\in\Gamma(k), l(\gamma)\neq j} \left|e^{-\delta\tilde{b}(\gamma,y)} - e^{-\delta\tilde{b}(\gamma,x)}\right| |\varphi(\gamma\cdot y)|\,.$$

The points $x$ and $y$ belong to $\tilde{\Lambda}_j$ and $\gamma$ satisfies $l(\gamma) \neq j$; hence, by Corollary 4.1.5, there exists $r < 1$ such that

$$K_1 \leqslant C r^k \left(\sum_{\gamma\in\Gamma(k), l(\gamma)\neq j} e^{-\delta\tilde{b}(\gamma,x)}\right)[\varphi]\, \mathrm{d}_{\mathbf{o}}(x,y),$$

hence

$$K_1 \leqslant C r^k \left|(\tilde{\mathcal{L}}_\delta)^k \mathbb{1}_{\tilde{\Lambda}}\right|_\infty ||\varphi|| \cdot \mathrm{d}_{\mathbf{o}}(x,y). \tag{69}$$

The second term $K_2$ is bounded from above by

$$\left(\sum_{\gamma\in\Gamma(k)} ||\tilde{w}_\delta(\gamma,\cdot)||\right)|\varphi|_\infty \cdot \mathrm{d}_{\mathbf{o}}(x,y). \tag{70}$$

Consequently

$$\left|\left|(\tilde{\mathcal{L}}_\delta)^k\varphi\right|\right| \leqslant \left(C r^k \left|(\tilde{\mathcal{L}}_\delta)^k \mathbb{1}_{\tilde{\Lambda}}\right|_\infty\right) ||\varphi|| + 2\left(\sum_{\gamma\in\Gamma(k)} ||\tilde{w}_\delta(\gamma,\cdot)||\right)|\varphi|_\infty\,.$$

Set $s_k := Cr^k \left|(\tilde{\mathcal{L}}_\delta)^k \mathbb{1}_{\tilde{\Lambda}_\Gamma}\right|_\infty$ and $S_k = 2\sum_{\gamma\in\Gamma(k)} ||\tilde{w}_\delta(\gamma,\cdot)||$. Finally $\tilde{\mathcal{L}}_\delta$ satisfies the DF(s)-property and is thus quasi-compact on $\mathrm{Lip}(\tilde{\Lambda})$.

In the second step, we prove that $\tilde{\rho} = \tilde{\rho}_\infty = 1$. The equality $\tilde{\rho} = \tilde{\rho}_\infty$ follows from the arguments presented in the proof of Proposition III.4 in [**3**]. Let us now show $\tilde{\rho} = 1$. We know by Proposition 4.2.7 that the spectral radius of $\mathcal{L}_\delta$ equals 1 and is a simple and isolated eigenvalue associated to the positive eigenfunction $h$ defined in Chapter 4. Let $\varphi \in \mathrm{Lip}(\tilde{\Lambda})$ be an eigenfunction associated to an eigenvalue $\lambda$ with modulus $\tilde{\rho}$. The function $\varphi|_\Lambda$ belongs to $\mathrm{Lip}(\Lambda)$ and satisfies that for any $x \in \Lambda$

$$\begin{aligned}\mathcal{L}_\delta\varphi|_\Lambda(x) &= \sum_{\alpha\in\mathcal{A}} \mathbb{1}_{\Lambda^c_{l(\alpha)}}(x)e^{-\delta b(\alpha,x)}\varphi|_\Lambda(\alpha\cdot x)\\ &= \left(\tilde{\mathcal{L}}_\delta\varphi\right)|_\Lambda(x)\\ &= \lambda\varphi|_\Lambda(x).\end{aligned}$$

Therefore $\tilde{\rho} = |\lambda| \leqslant 1$. On the other hand $\tilde{\rho} = \limsup_n \left|\left(\tilde{\mathcal{L}}_\delta\right)^n \mathbb{1}_{\tilde{\Lambda}}\right|_\infty^{1/n}$. Fix $\varepsilon > 0$; there exists a subsequence $(n_k)_k$ such that $\left(\tilde{\mathcal{L}}_\delta\right)^{n_k} \mathbb{1}_{\tilde{\Lambda}}(x) \preceq (\tilde{\rho})^{n_k}(1+\varepsilon)^{n_k}$ for any $k \geqslant 1$ and $x \in \Lambda$. By the definition of $\tilde{\mathcal{L}}_\delta$, we obtain that for any $k \geqslant 1$ and $x \in \Lambda$

$$h(x) = \mathcal{L}_\delta^{n_k} h(x) \asymp \mathcal{L}_\delta^{n_k}\mathbb{1}_\Lambda(x) \preceq (\tilde{\rho})^{n_k}(1+\varepsilon)^{n_k}. \tag{71}$$

Letting $k \longrightarrow +\infty$, we obtain $\tilde{\rho}(1+\varepsilon) \geqslant 1$ for any $\varepsilon > 0$. Finally $\tilde{\rho} = 1$.

Let $e^{i\theta}$ be an eigenvalue with modulus 1 for $\tilde{\mathcal{L}}_\delta$: there exists $\varphi \in \mathrm{Lip}\left(\tilde{\Lambda}\right)$ such that $\tilde{\mathcal{L}}_\delta\varphi = e^{i\theta}\varphi$. As before, for any $x \in \Lambda$, we get

$$\mathcal{L}_\delta\varphi|_\Lambda(x) = \left(\tilde{\mathcal{L}}_\delta\varphi\right)|_\Lambda(x) = e^{i\theta}\varphi|_\Lambda(x).$$

Since 1 is the unique eigenvalue with modulus 1 of $\mathcal{L}_\delta$, it follows that $\theta \in 2\pi\mathbb{Z}$; hence 1 is also the unique eigenvalue of $\tilde{\mathcal{L}}_\delta$ with modulus 1.

The next step is devoted to the proof of the existence of a positive eigenfunction for 1. Since the operator $\tilde{\mathcal{L}}_\delta$ is positive, there exists a non-negative function $\phi$ such that $\tilde{\mathcal{L}}_\delta\phi = \phi$. Let us show that $\phi$ is positive. Assume that $\phi$ vanishes at $x \in \tilde{\Lambda}$. There are two cases.

– If $x \in \Lambda$, there exists $j \in [\![1, p+q]\!]$ such that $x \in \Lambda_j$. For any $k \geqslant 1$, we have

$$\left(\tilde{\mathcal{L}}_\delta\right)^k \phi(x) = \sum_{\substack{\gamma\in\Gamma(k)\\ l(\gamma)\neq j}} e^{-\delta\tilde{b}(\gamma,x)}\phi(\gamma\cdot x) = \phi(x) = 0. \tag{72}$$

Therefore $\phi(\gamma\cdot x) = 0$ for any $k \geqslant 1$ and any $\gamma \in \Gamma(k)$ with $l(\gamma) \neq j$. By minimality of the action of $\Gamma$ on $\Lambda$ and continuity of $\phi$ on $\Lambda$, we derive that $\phi$ vanishes at every point of $\Lambda$. It remains to show that we may draw the same conclusion for $\Gamma \cdot x_0$. Equation (72) ensures that it is sufficient to prove that $\phi(x_0) = 0$, or equivalently, that the sequence $\left(\sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}\phi(\gamma\cdot x_0)\right)_k$ tends to 0. Let $k \geqslant 1$. Corollary 4.1.5 implies that for any $|\gamma| = k$ and any limit point $y \in \Lambda^c_{l(\gamma)}$

$$\phi(\gamma\cdot x_0) = |\phi(\gamma\cdot x_0)| = |\phi(\gamma\cdot x_0) - \phi(\gamma\cdot y)| \leqslant [\phi]\, \mathrm{d}_{\mathbf{o}}(\gamma\cdot x_0, \gamma\cdot y) \leqslant [\phi]\, r^k \mathrm{d}_{\mathbf{o}}(x_0, y).$$

Lemma 5.2.4 combined with Property 4.1.3 implies that there exists a constant $C > 0$ such that $\sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})} \leq C$, so that $\sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}\phi(\gamma\cdot x_0) \leq r^k$, which tends to 0 as $k \longrightarrow +\infty$.

– Otherwise $x \in \Gamma\cdot x_0$. It follows $\phi(\gamma\cdot x) = 0$ for any $x \in \Lambda$ and any $\gamma \in \Gamma$ such that $x \notin \tilde{\Lambda}_{l(\gamma)}$. By continuity of $\phi$ and by the definition of the limit set, the function $\phi$ vanishes on $\Lambda$ and the above arguments ensure that this is true everywhere.

The next step deals with the dimension of the eigenspace associated to 1. To prove that it is equal to 1, we introduce the normalized operator $\tilde{P}$ defined for any $\varphi \in \mathrm{Lip}(\tilde{\Lambda})$ by

$$\tilde{P}\varphi = \frac{1}{\phi}\tilde{\mathscr{L}}_\delta(\varphi\phi),\ ^{(1)}$$

where $\phi$ is a positive eigenfunction associated to 1. This operator is positive, bounded on the space $\mathrm{Lip}(\tilde{\Lambda})$ and quasi-compact; it also satisfies $\tilde{P}(\mathbb{1}_{\tilde{\Lambda}}) = \mathbb{1}_{\tilde{\Lambda}}$ and its spectral radius is 1. Let us prove that the eigenspace associated to 1 for $\tilde{P}$ has dimension 1. Let $f$ be an eigenfunction associated to 1 for $\tilde{P}$.

(i) Assume first that the maximum $|f|_\infty$ of $|f|$ is obtained at a point $x \in \Lambda_j$, $j \in [\![1, p+q]\!]$. Hence

$$|f(x)| = |\tilde{P}f(x)| \leqslant \tilde{P}|f|(x) \leqslant |f(x)|. \tag{73}$$

Using the iterates of $\tilde{P}$ and a convexity argument, we obtain $|f(\gamma\cdot x)| = |f(x)|$ for any $\gamma \in \Gamma$ with $l(\gamma) \neq j$. By minimality of the action of $\Gamma$ on $\Lambda$, the function $|f|$ is constant on $\Lambda$. To check that $|f|$ is also constant on $\Gamma\cdot x_0$, we rewrite (72) as

$$1 - \frac{|f(x_0)|}{M} = \frac{1}{\phi(x_0)}\sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}\left(1 - \frac{|f(\gamma\cdot x_0)|}{M}\right)\phi(\gamma\cdot x_0),$$

hence

$$\left|1 - \frac{|f(x_0)|}{M}\right| \leqslant \frac{1}{\phi(x_0)}\sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}\frac{|M - |f(\gamma\cdot x_0)||}{|M|}\phi(\gamma\cdot x_0). \tag{74}$$

For any $\gamma \in \Gamma(k)$ and any $z \in \Lambda^c_{l(\gamma)}$ one gets $M = |f(\gamma\cdot z)|$. As previously, we bound from above the right hand side of (74) by $Cr^k$, which goes to 0

1. Contrary to Lemma 4.2.2, in this section the symbol ˜ does not imply an action on $\tilde{\Lambda} \times \mathbb{R}$.

as $k \longrightarrow +\infty$. Finally $|f(x_0)| = M$. Applying (73) to the iterates of $\tilde{P}$ and using a convexity argument once again, we deduce that $|f|$ is constant on $\Gamma \cdot x_0$, and finally $|f|$ is constant on $\tilde{\Lambda}$.

(ii) If $|f|$ reaches its maximum in $\Gamma \cdot x_0$, it follows from the minimality of the action of $\Gamma$ on the boundary that $|f|$ is constant on $\Lambda$, hence on $\tilde{\Lambda}$ as above.

Repeating the same argument, we show that $f$ is constant on $\tilde{\Lambda}$, which ends the proof of the proposition. □

Let us now choose an eigenfunction $\tilde{h}$ of $\tilde{\mathcal{L}}_\delta$ associated to the eigenvalue 1 satisfying $\sigma_{\mathbf{o}}\left(\tilde{h}\right) = 1$. There exists $\tilde{\Pi}_\delta \;:\; \mathrm{Lip}(\tilde{\Lambda}) \longrightarrow \mathbb{C}\tilde{h}$ such that for any $\varphi \in \mathrm{Lip}(\tilde{\Lambda})$,

$$\tilde{\mathcal{L}}_\delta \varphi = \tilde{\Pi}_\delta\left(\varphi\right) + \tilde{R}_\delta\left(\varphi\right),$$

where $\tilde{R}_\delta$ satisfies $\tilde{R}_\delta \tilde{\Pi}_\delta = \tilde{\Pi}_\delta \tilde{R}_\delta = 0$ and has spectral radius $< 1$. There exists a linear form $\tilde{\sigma} \;:\; \mathrm{Lip}(\tilde{\Lambda}) \longrightarrow \mathbb{C}$ satisfying $\tilde{\sigma}\left(\tilde{h}\right) = 1$ such that for any $\varphi \in \mathrm{Lip}(\tilde{\Lambda})$,

$$\tilde{\mathcal{L}}_\delta \varphi = \tilde{\sigma}(\varphi)\tilde{h} + \tilde{R}_\delta \varphi. \tag{75}$$

It follows that, for any $k \geqslant 1$,

$$\left(\tilde{\mathcal{L}}_\delta\right)^k \varphi = \tilde{\sigma}\left(\varphi\right)\tilde{h} + \left(\tilde{R}_\delta\right)^k \varphi = \tilde{\sigma}\left(\tilde{\mathcal{L}}_\delta \varphi\right)\tilde{h} + \left(\tilde{R}_\delta\right)^{k-1} \tilde{\mathcal{L}}_\delta \varphi.$$

Letting $k \longrightarrow +\infty$, we deduce that $\tilde{\sigma}$ is $\tilde{\mathcal{L}}_\delta$-invariant. To identify $\tilde{\sigma}$, let us write, for any $\varphi \in \mathrm{Lip}(\tilde{\Lambda})$ and $k \geqslant 1$

$$\mathcal{L}_\delta^k \varphi|_\Lambda = \left(\left(\tilde{\mathcal{L}}_\delta\right)^k \varphi\right)|_\Lambda = \tilde{\sigma}\left(\varphi\right)\tilde{h}|_\Lambda + \left(\tilde{R}_\delta\right)^k \varphi|_\Lambda.$$

The equality $\sigma_{\mathbf{o}}\left(\varphi\right) = \sigma_{\mathbf{o}}\left(\varphi|_\Lambda\right) = \sigma_{\mathbf{o}}\left(\mathcal{L}_\delta^k \varphi|_\Lambda\right)$, readily implies

$$\sigma_{\mathbf{o}}\left(\varphi\right) = \tilde{\sigma}\left(\varphi\right) + \sigma_{\mathbf{o}}\left(\left(\tilde{R}_\delta\right)^k \varphi|_\Lambda\right)$$

and letting $k \longrightarrow +\infty$, we obtain $\sigma_{\mathbf{o}}\left(\varphi\right) = \tilde{\sigma}\left(\varphi\right)$. Finally $\tilde{\sigma} = \sigma_{\mathbf{o}}$ and $\tilde{h}$ extends $h$ on $\tilde{\Lambda}$.

Remark 8.3.4. – *One gets* $\tilde{h}(x_0) = \lim_{k \longrightarrow +\infty} \sum_{\gamma \in \Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o})}$*; this quantity does not depend on* $x_0 \in \partial \mathrm{X}$.

Let us now check that the function $z \longmapsto \tilde{\mathcal{L}}_z$ is a continuous perturbation of $\tilde{\mathcal{L}}_\delta$ for $\mathrm{Re}\,(z) \geqslant \delta$ using the following proposition, whose proof is the same as that of Proposition 4.2.9.

Proposition 8.3.5. – *Under the assumptions* $(H_\beta)$*, for any compact* $K$ *of* $\mathbb{R}$*, there exists a constant* $C = C_K > 0$ *such that for any* $s, t \in K$ *and* $\kappa > 0$

1) *if* $\beta \in ]0,1[$

$$\text{a. } ||\tilde{\mathcal{L}}_{\delta+it} - \tilde{\mathcal{L}}_{\delta+is}|| \leqslant C|s-t|^{\beta} L\left(\frac{1}{|s-t|}\right),$$

$$\text{b. } ||\tilde{\mathcal{L}}_{\delta+\kappa+it} - \tilde{\mathcal{L}}_{\delta+it}|| \leqslant C\kappa^{\beta} L\left(\frac{1}{\kappa}\right);$$

2) *if* $\beta = 1$

$$\text{a. } ||\tilde{\mathcal{L}}_{\delta+it} - \tilde{\mathcal{L}}_{\delta+is}|| \leqslant C|s-t|\tilde{L}\left(\frac{1}{|s-t|}\right),$$

$$\text{b. } ||\tilde{\mathcal{L}}_{\delta+\kappa+it} - \tilde{\mathcal{L}}_{\delta+it}|| \leqslant C\kappa\tilde{L}\left(\frac{1}{\kappa}\right),$$

*where* $\tilde{L}(x) = \int_1^x \frac{L(y)}{y}\,\mathrm{d}y$.

Now let $\tilde{\rho}(z)$ denote the spectral radius of $\tilde{\mathcal{L}}_z$ on $\left(\mathrm{Lip}(\tilde{\Lambda}), ||\cdot||\right)$ and recall that $|x+iy|_\infty = \max(|x|,|y|)$ for any real numbers $x$ and $y$. As in **[2]**, we may state the following proposition.

Proposition 8.3.6. – *There exist* $\varepsilon > 0$ *and* $\rho_\varepsilon \in ]0,1[$ *such that for any* $z \in \mathbb{C}$ *satisfying* $|z-\delta|_\infty < \varepsilon$ *and* $\mathrm{Re}\,(z) \geqslant \delta$

- $\tilde{\rho}(z) > \rho_\varepsilon$;
- $\tilde{\mathcal{L}}_z$ *has a unique eigenvalue* $\tilde{\lambda}_z$ *with modulus* $\tilde{\rho}(z)$;
- *this eigenvalue is simple and close to* $1$;
- *the remainder of the spectrum is included in a disk of radius* $\rho_\varepsilon$.

*Moreover, for any* $A > 0$*, there exists* $\rho_A < 1$ *such that* $\tilde{\rho}(z) < \rho_A$ *as soon as* $z \in \mathbb{C}$ *satisfies* $|z-\delta|_\infty \geqslant \varepsilon$, $\mathrm{Re}\,(z) \geqslant \delta$ *and* $|\mathrm{Im}\,(z)| \leqslant A$. *Finally* $\tilde{\rho}(z) = 1$ *if and only if* $z = \delta$.

The following proposition specifies the local behavior of the dominant eigenvalue $\tilde{\lambda}_{\delta+it}$; its proof is verbatim that of Proposition 4.2.15.

Proposition 8.3.7. – *For the constant* $E_\Gamma > 0$ *given in Proposition 4.2.15 and* $t \longrightarrow 0$

- *if* $\beta \in ]0,1[$

$$\tilde{\lambda}_{\delta+it} = 1 - E_\Gamma \Gamma(1-\beta) e^{+i\mathrm{sign}(t)\beta\pi/2} |t|^{\beta} L\left(\frac{1}{|t|}\right)(1+o(1));$$

– *if* $\beta = 1$

- $\tilde{\lambda}_{\delta+it} = 1 - E_\Gamma \mathrm{sign}(t) i|t|\tilde{L}\left(\frac{1}{|t|}\right)(1+o(1));$
- $\mathrm{Re}\left(1-\tilde{\lambda}_{\delta+it}\right) = \frac{\pi}{2}E_\Gamma |t| L\left(\frac{1}{|t|}\right)(1+o(1)).$

Remark 8.3.8. – *The fact that the constant* $E_\Gamma$ *is the one of Proposition 4.2.15 relies on the equivalent* $\tilde{\lambda}_{\delta+it} \sim \sigma_{\mathbf{o}}\left(\left(\tilde{\mathscr{L}}_{\delta+it} - \tilde{\mathscr{L}}_\delta\right)\tilde{h}\right)$ *and the equality* $\sigma_{\mathbf{o}}(\Gamma \cdot x_0) = 0$. *Therefore*

$$\sigma_{\mathbf{o}}\left(\left(\tilde{\mathscr{L}}_{\delta+it} - \tilde{\mathscr{L}}_\delta\right)\tilde{h}\right) = \sigma_{\mathbf{o}}\left(\left(\mathscr{L}_{\delta+it} - \mathscr{L}_\delta\right)h\right).$$

In the proof of Theorem C (Chapter 9) for $\beta = 1$, we will use $\tilde{Q}_z = \left(\mathrm{Id} - \tilde{\mathscr{L}}_z\right)^{-1}$ for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \geqslant \delta$. The following properties are the same as Properties 4.2.16 and 4.2.17 proved in Chapter 4 concerning $Q_z = (\mathrm{Id} - \mathscr{L}_z)^{-1}$.

Proposition 8.3.9. – *There exist* $\varepsilon > 0$ *and* $C > 0$ *such that* $\left\|\tilde{Q}_z - (1-\tilde{\lambda}_z)^{-1}\tilde{\Pi}_z\right\| \leqslant C$ *for* $z$ *such that* $|z-\delta|_\infty < \varepsilon$ *and* $\left\|\tilde{Q}_z\right\| \leqslant C$ *for* $z$ *such that* $|z-\delta|_\infty \geqslant \varepsilon$. *Moreover, as* $t \longrightarrow 0$,

$$\tilde{Q}_{\delta+it} = \frac{1}{E_\Gamma \mathrm{sign}(t) i|t|\tilde{L}\left(\frac{1}{|t|}\right)}(1+o(1))\Pi_0 + O(1).$$

As a direct consequence, we obtain the following property.

Corollary 8.3.10. – *The function* $t \longmapsto \mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)$ *is integrable at* 0.

# CHAPTER 9

# THEOREM C: ASYMPTOTIC OF THE ORBITAL COUNTING FUNCTION

Let us restate Theorem C.

THEOREM C. – *Let $\Gamma$ be a Schottky group satisfying the assumptions $(H_\beta)$ for some $\beta \in \,]0,1]$. Then, as $R \longrightarrow +\infty$,*

- $\mathrm{N}_\Gamma(\mathbf{o}, R) \sim C\dfrac{e^{\delta R}}{R^{1-\beta}L(R)}$ *with* $C = \dfrac{\sin(\beta\pi)}{\pi}\dfrac{\tilde{h}(x_0)}{\delta E_\Gamma}$ *if* $\beta \in \,]0,1[$;
- $\mathrm{N}_\Gamma(\mathbf{o}, R) \sim C'\dfrac{e^{\delta R}}{\tilde{L}(R)}$ *with* $C' = \dfrac{\tilde{h}(x_0)}{\delta E_\Gamma}$ *if* $\beta = 1$,

*where* $\tilde{h}(x_0) = \lim\limits_{k\longrightarrow+\infty} \sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}$ *and* $E_\Gamma = \sum_{1\leqslant j\leqslant p} C_j \left(\int\limits_{\Lambda\setminus\Lambda_j} \frac{\mathrm{d}\sigma_{\mathbf{o}}(x)}{\mathrm{d}_{\mathbf{o}}(x,x_j)^{2\delta/a}}\right)^2$.

We write

$$\begin{aligned}\mathrm{N}_\Gamma(\mathbf{o}, R) &= e^{\delta R}\sum_{\gamma\in\Gamma^*} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}u(\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}) - R) + 1\\ &= e^{\delta R}\sum_{k\geqslant 1}\sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}u(\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}) - R) + 1\\ &\sim e^{\delta R}W(R,u),\end{aligned}$$

where $u(x) = e^{\delta x}\mathbb{1}_{\mathbb{R}^-}(x)$ and $W(R,u) = \sum_{k\geqslant 1} W_k(R,u) = \sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}u(\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}) - R)$. To prove Theorem C, it is thus sufficient to prove the following

Proposition 9.0.1. – *For any function $u : \mathbb{R} \longrightarrow \mathbb{R}$ piecewise continuous and satisfying $\int_{\mathbb{R}} u(x)\,\mathrm{d}x > 0$, as $R \longrightarrow +\infty$,*

- $W(R,u) \sim \dfrac{C}{R^{1-\beta}L(R)} \displaystyle\int_{\mathbb{R}} u(x)\,\mathrm{d}x$ *if* $\beta \in ]0,1[$;
- $W(R,u) \sim \dfrac{C'}{\tilde{L}(R)} \displaystyle\int_{\mathbb{R}} u(x)\,\mathrm{d}x$ *if* $\beta = 1$,

*where the constants $C, C'$ are specified in the above Theorem C.*

*Proof of Theorem C.* – Let us explain how Theorem C follows from Proposition 9.0.1 for $\beta \in ]0,1[$. Up to a finite number of terms, we have

$$\mathrm{N}_\Gamma(\mathbf{o},R) = \#\{\gamma \in \Gamma \mid 1 \leqslant \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}) \leqslant R\}.$$

Fix $\varepsilon > 0$. Then, for all $R > 0$, one gets $m(R) \leqslant \mathrm{N}_\Gamma(\mathbf{o},R) \leqslant M(R)$ where

$$m(R) = \sum_{q=0}^{[R/\varepsilon]-2} \#\{\gamma \in \Gamma \mid 1+q\varepsilon \leqslant \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}) < 1+(q+1)\varepsilon\}$$

and

$$M(R) = \sum_{q=0}^{[R/\varepsilon]-1} \#\{\gamma \in \Gamma \mid 1+q\varepsilon \leqslant \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o}) < 1+(q+1)\varepsilon\}.$$

Applying Proposition 9.0.1 with $u : t \longmapsto e^{\delta t}\mathbb{1}_{[0,\varepsilon]}(t)$ yields

$$m(R) \sim C\frac{e^{\delta\varepsilon}-1}{\delta\varepsilon} \sum_{q=0}^{[R/\varepsilon]-2} \varepsilon \frac{e^{\delta(1+q\varepsilon)}}{(1+q\varepsilon)^{1-\beta}L(1+q\varepsilon)} \underset{\varepsilon\longrightarrow 0}{\longrightarrow} C\int_1^R \frac{e^{\delta x}}{x^{1-\beta}L(x)}\,\mathrm{d}x$$

and

$$M(R) \sim C\frac{e^{\delta\varepsilon}-1}{\delta\varepsilon} \sum_{q=0}^{[R/\varepsilon]-1} \varepsilon \frac{e^{\delta(1+q\varepsilon)}}{(1+q\varepsilon)^{1-\beta}L(1+q\varepsilon)} \underset{\varepsilon\longrightarrow 0}{\longrightarrow} C\int_1^R \frac{e^{\delta x}}{x^{1-\beta}L(x)}\,\mathrm{d}x.$$

Finally $\mathrm{N}_\Gamma(\mathbf{o},R) \sim C\frac{e^{\delta R}}{\delta R^{1-\beta}L(R)}$. The same proof holds for $\beta = 1$ using the second estimate of Proposition 9.0.1. □

## 9.1. Proposition 9.0.1 for $\beta \in ]0,1[$

We follow step by step the proof of Theorem A. Let $K \geqslant 2$ and $(a_k)_{k\geqslant 1}$ be a sequence satisfying $kL(a_k) = a_k^\beta$. We first state the following propositions.

PROPOSITION C.1. – *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function with compact support. Uniformly in $K \geqslant 2$ and $R \in [0, Ka_k]$, we have, as $k \longrightarrow +\infty$*

$$W_k(R,u) = \frac{1}{e_\Gamma a_k}\left(C_0 \Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)\widehat{u}(0) + o_k(1)\right),$$

*where $\Psi_\beta$ is the density of the fully asymmetric stable law with parameter $\beta$, $e_\Gamma = E_\Gamma^{1/\beta}$ and $C_0 = \tilde{h}(x_0) = \lim_{k\to+\infty} \sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}$.*

PROPOSITION C.2. – *Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function with compact support. There exists $C > 0$ which only depends on $u$ such that, for any $K \geqslant 2$, when $R \geqslant Ka_k$, we have*

$$|W_k(R,u)| \leqslant Ck\frac{L(R)}{R^{1+\beta}}|u|_\infty.$$

Let us now explain how Proposition 9.0.1 follows from Propositions C.1 and C.2. Let $u : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with compact support. We want to estimate $W(R,u) = \sum_{k\geqslant 1} W_k(R,u)$. By Proposition C.1, we may decompose this quantity into $W^1(R,u) + W^2(R,u) + W^3(R,u)$ where

$$W^1(R,u) := \frac{C_0\widehat{u}(0)}{e_\Gamma}\sum_{k|R<Ka_k}\frac{1}{a_k}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right),$$

$$W^2(R,u) := \frac{1}{e_\Gamma}\sum_{k|R<Ka_k}\frac{o_k(1)}{a_k}$$

$$\text{and } W^3(R,u) := \sum_{k|R\geqslant Ka_k} W_k(R,u).$$

a) *Contribution of $W^1(R,u)$*. Following [**24**], we introduce the measure $\mu_R = \sum_{0<R/a_k\leqslant K} D_{R/a_k}$ defined on the interval $]0,K]$. One has

$$\sum_{R<Ka_k}\frac{1}{e_\Gamma a_k}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right) = \frac{1}{R}\int_0^K \frac{z}{e_\Gamma}\Psi_\beta\left(\frac{z}{e_\Gamma}\right)\mathrm{d}\mu_R(z).$$

When $R \longrightarrow +\infty$, this measure satisfies

$$R^{-\beta}L(R)\mu_R([x,y]) \sim \int_x^y \beta z^{-\beta-1}\,\mathrm{d}z,$$

so that

$$R^{1-\beta}L(R)W^1(R,u) \sim \frac{C_0\beta\widehat{u}(0)}{e_\Gamma}\int_0^K z^{-\beta}\Psi_\beta\left(\frac{z}{e_\Gamma}\right)\mathrm{d}z.$$

(76) $$\sim \frac{\beta\widehat{u}(0)}{E_\Gamma}\int_0^{K/e_\Gamma} z^{-\beta}\Psi_\beta(z)\,\mathrm{d}z \text{ with } E_\Gamma = e_\Gamma^\beta.$$

b) *Contribution of* $W^2(R,u)$. This quantity is similar to $M^2(R;\varphi\otimes u,\psi\otimes v)$ appearing in the proof of Theorem A; it satisfies

$$R^{1-\beta}L(R)W^2(R,u)=o_K(1) \text{ where } \lim_{R\longrightarrow+\infty} o_K(1)=0 \text{ for any fixed } K. \tag{77}$$

c) *Contribution of* $W^3(R,u)$. As for $M^3(R;\varphi\otimes u,\psi\otimes v)$ appearing in the proof of Theorem A, we get

$$R^{1-\beta}L(R)W^3(R,u)\leqslant CK^{-\beta}. \tag{78}$$

Combining (76), (77) and (78), we obtain

$$R^{1-\beta}L(R)W(R,u)=\frac{\beta\widehat{u}(0)C_0}{E_\Gamma}\int_0^{K/e_\Gamma} z^{-\beta}\psi(z)\,\mathrm{d}z(1+o(1))+o_K(1)+O(K^{-\beta}),$$

with $\lim_{R\longrightarrow+\infty} o(1)=0$. Letting $R\longrightarrow+\infty$ and choosing $K$ large enough yield

$$R^{1-\beta}L(R)W(R,u)\sim\frac{\beta\widehat{u}(0)C_0}{E_\Gamma}\int_0^{+\infty} z^{-\beta}\Psi_\beta(z)\,\mathrm{d}z$$

and Proposition 9.0.1 follows with $C=\frac{C_0}{E_\Gamma}\frac{\sin(\beta\pi)}{\pi}$.

**9.1.1. Proof of Proposition C.1.** – Let $K\geqslant 2$ and $R>0$ be fixed and choose $k\in\mathbb{N}$ such that $Ka_k\geqslant R$. It is sufficient to prove that, as $k\longrightarrow+\infty$,

$$a_kW_k(R,u)-\frac{\tilde{h}(x_0)}{e_\Gamma}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)\widehat{u}(0)\longrightarrow 0, \tag{79}$$

for all $u\in\mathscr{U}$, where the set of test functions $\mathscr{U}$ was defined in 5.2.2. Let us first show that the quantity $W_k(R,u)$ is finite for any $u\in\mathscr{U}$. The Fourier inverse formula implies

$$\begin{aligned}|W_k(R,u)|\leqslant&\frac{1}{2\pi}\int_{\mathbb{R}}\left|e^{itR}\left(\tilde{\mathscr{L}}_{\delta+it}\right)^k\mathbb{1}_{\tilde{\Lambda}}(x_0)\widehat{u}(t)\,\mathrm{d}t\right|\\ \leqslant&\frac{1}{2\pi}\left(\tilde{\mathscr{L}}_{\delta}\right)^k\mathbb{1}_{\tilde{\Lambda}}(x_0)\int_{\mathbb{R}}\widehat{u}(t)\,\mathrm{d}t\\ \preceq&\,||\widehat{u}||_1\,\tilde{h}(x_0)<+\infty.\end{aligned}$$

Let us now prove (79). To simplify notation, we will omit the symbol ˜ until the end of the current subsection. The Fourier inverse formula allows us to split

$$a_kW_k(R,u)-\frac{h(x_0)}{e_\Gamma}\Psi_\beta\left(\frac{R}{e_\Gamma a_k}\right)\widehat{u}(0)$$

into $K_1(k)+K_2(k)$ where

$$K_1(k):=\frac{a_k}{2\pi}\int_{[-\varepsilon,\varepsilon]^c} e^{itR}\mathscr{L}^k_{\delta+it}\mathbb{1}_{\tilde{\Lambda}}(x_0)\widehat{u}(t)\,\mathrm{d}t$$

and

$$K_2(k) := \frac{a_k}{2\pi}\int_{-\varepsilon}^{\varepsilon} e^{itR}\mathscr{L}_{\delta+it}^k \mathbb{1}_{\tilde{\Lambda}}(x_0)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR/a_k} g_\beta(e_\Gamma t)\widehat{u}(0)h(x_0)\,\mathrm{d}t$$
$$= \frac{1}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{itR/a_k}\mathscr{L}_{\delta+it/a_k}^k \mathbb{1}_{\tilde{\Lambda}}(x_0)\widehat{u}\left(\frac{t}{a_k}\right)\mathrm{d}t - \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR/a_k} g_\beta(e_\Gamma t)\widehat{u}(0)h(x_0)\,\mathrm{d}t,$$

where $\varepsilon > 0$ satisfies the conclusions of Proposition 8.3.6. The spectral properties of $\mathscr{L}_z$ given in the latter proposition and the fact that $\widehat{u}$ has compact support imply that $||\mathscr{L}_{\delta+it}^k|| \leq \rho^k$, for $0 < \rho < 1$ which only depends on the support of $u$. Therefore $|K_1(k)| \leq ||\widehat{u}||_\infty \rho^k a_k \longrightarrow 0$ as $k \longrightarrow +\infty$, uniformly in $K$ and $R$.

We now deal with $K_2(k)$. The spectral decomposition of $\mathscr{L}_{\delta+it/a_k}$ yields

$$\mathscr{L}_{\delta+it/a_k}^k \mathbb{1}_{\tilde{\Lambda}} = \lambda_{\delta+it/a_k}^k \Pi_{\delta+it/a_k}\mathbb{1}_{\tilde{\Lambda}} + R_{\delta+it/a_k}^k \mathbb{1}_{\tilde{\Lambda}} x_0$$

with $\mathrm{spec}(R_{\delta+it/a_k}) \subset B(0,\rho_\varepsilon)$ and $0 < \rho_\varepsilon < 1$. We decompose $K_2(k)$ as $L_1(k) + L_2(k) + L_3(k)$ where

$$L_1(k) = \frac{1}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{itR/a_k} R_{\delta+it/a_k}^k \mathbb{1}_{\tilde{\Lambda}}(x_0)\widehat{u}\left(\frac{t}{a_k}\right)\mathrm{d}t,$$

$$L_2(k) = \frac{1}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{itR/a_k}\lambda_{\delta+it/a_k}^k \left(\Pi_{\delta+it/a_k}(\mathbb{1}_{\tilde{\Lambda}})(x_0) - \Pi_\delta(\mathbb{1}_{\tilde{\Lambda}})(x_0)\right)\widehat{u}\left(\frac{t}{a_k}\right)\mathrm{d}t$$

and

$$L_3(k) = \frac{h(x_0)}{2\pi}\int_{-\varepsilon a_k}^{\varepsilon a_k} e^{itR/a_k}\lambda_{\delta+it/a_k}^k \widehat{u}\left(\frac{t}{a_k}\right)\mathrm{d}t - \frac{h(x_0)}{2\pi}\int_{\mathbb{R}} e^{itR/a_k} g_\beta(e_\Gamma t)\widehat{u}(0)\,\mathrm{d}t.$$

First $|L_1(k)| \leq a_k\rho_\varepsilon^k ||\widehat{u}||_\infty \longrightarrow 0$ as $k \longrightarrow +\infty$, uniformly in $K$ and $R$. We use the Lebesgue dominated convergence theorem for $L_2(k)$: as for the integral $L_2(k)$ in the proof of Proposition A.1, choosing $\varepsilon > 0$ small enough so that it satisfies Remark 4.2.12 and using the local expansion of $\lambda_{\delta+it}$ given in Proposition 8.3.7, we may bound from above the integrand of $L_2(k)$ up to a multiplicative constant by

$$l(t) = \begin{cases} |t|^{\beta/2}\exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{3\beta/2}\right) \text{if } |t| \leqslant 1 \\ |t|^{3\beta/2}\exp\left(-\frac{1}{4}(1-\beta)\Gamma(1-\beta)|e_\Gamma t|^{\beta/2}\right) \text{ if } |t| > 1 \end{cases}.$$

The term $L_3(k)$ may be treated similarly to $L_3(k)$ appearing in the proof of Proposition A.1.

**9.1.2. Proof of Proposition C.2.** – It suffices to check that there exist constants $C$, $M > 0$ such that

$$\sum_{\substack{\gamma\in\Gamma(k)\\ \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})\overset{M}{\sim}R}} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})} \leqslant Ck\frac{L(R)}{R^{1+\beta}}. \tag{80}$$

Property 4.1.3 implies that this estimate is a consequence of Proposition A.2.

## 9.2. Proposition 9.0.1 for $\beta = 1$

As in the proof of Theorem A for $\beta = 1$, we will need to symmetrize the quantity $W(R,u)$. We thus define

$$W^{\mathrm{sym}}(R,u) := \sum_{k\geqslant 1}\sum_{\gamma\in\Gamma(k)} e^{-\delta \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}\left(u(\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})-R)+u(-\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})-R)\right).$$

We may notice that $W^{\mathrm{sym}}(R,u) = W(R,u)$ for $R$ large enough, because the function $u$ has compact support. We first study

$$W^{\mathrm{sym}}_{\xi}(R,u) := \sum_{k\geqslant 1}\sum_{\gamma\in\Gamma(k)} e^{-\xi \mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}\left(u(\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})-R)+u(-\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})-R)\right)$$

for $\xi > \delta$. From now on, we assume again that $u$ belongs to the set $\mathscr{U}$ introduced in Definition 5.2.2. The fact that $W^{\mathrm{sym}}_{\xi}(R,u)$ is finite for any $u\in\mathscr{U}$ will be a consequence of the following discussion. For now, let us notice that the Fourier inverse formula combined with the convergence of the Poincaré series of $\Gamma$ at $\xi > \delta$ implies

$$W^{\mathrm{sym}}_{\xi}(R,u) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itR}\sum_{k\geqslant 1}\left(\left(\tilde{\mathscr{L}}_{\xi+it}\right)^k + \left(\tilde{\mathscr{L}}_{\xi-it}\right)^k\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t$$

which is finite for any $u\in\mathscr{U}$. Therefore

$$W^{\mathrm{sym}}_{\xi}(R,u) = \frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\xi+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\widehat{u}(t)\,\mathrm{d}t.$$

We want

$$W^{\mathrm{sym}}(R,u) = \frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\widehat{u}(t)\,\mathrm{d}t. \tag{81}$$

Notice that, as in Proposition 6.1.1,

$$\frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\xi+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t \underset{\xi\searrow\delta}{\longrightarrow} \frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t.$$

Moreover, for a positive function $u$, the monotone convergence theorem implies that $W^{\mathrm{sym}}_{\xi}(R,u)$ tends to $W^{\mathrm{sym}}(R,u)$. Since $t\longmapsto \mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)$ is integrable in a neighborhood of 0, we deduce that $W^{\mathrm{sym}}(R,u)$ is finite for any positive $u$ (which also implies that it is finite for any $u \in \mathscr{U}$). Finally

$$W^{\mathrm{sym}}(R,u) = \frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t - \frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\widehat{u}(t)\,\mathrm{d}t.$$

By Riemann-Lebesgue's lemma, the term $\int_{\mathbb{R}} e^{itR}\widehat{u}(t)\,\mathrm{d}t$ is negligible with respect to $1/\tilde{L}(R)$. As previously, let us fix $A > 0$. We split

$$\frac{1}{\pi}\int_{\mathbb{R}} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t$$

into $I_1 + I_2$ where

$$I_1 = \frac{1}{\pi}\int_{|t|>A/R} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t$$

and

$$I_2 = \frac{1}{\pi}\int_{|t|\leqslant A/R} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t.$$

We may decompose $I_1$ according to the sign of $t$. Let $J$ be

$$\frac{1}{\pi}\int_{t>A/R} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t.$$

Setting $t = y - \pi/R$ in $J$, it follows that

$$J = -\frac{1}{\pi}\int_{y>(A+\pi)/R} e^{iyR}\mathrm{Re}\left(\tilde{Q}_{\delta+i(y-\pi/R)}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}\left(y - \frac{\pi}{R}\right)\,\mathrm{d}y.$$

Hence

$$\begin{aligned}
2J &= \frac{1}{\pi}\int_{A/R}^{(A+\pi)/R} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t \\
&\quad + \frac{1}{\pi}\int_{t>(A+\pi)/R} e^{itR}\mathrm{Re}\left(\tilde{Q}_{\delta+i(t-\pi/R)}\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\left(\widehat{u}(t) - \widehat{u}\left(t-\frac{\pi}{R}\right)\right)\,\mathrm{d}t \\
&\quad + \frac{1}{\pi}\int_{t>(A+\pi)/R} e^{itR}\left(\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right) - \mathrm{Re}\left(\tilde{Q}_{\delta+i(t-\pi/R)}\right)\right)(\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t \\
&=: K_1 + K_2 + K_3.
\end{aligned}$$

As in Proposition 6.1.2, one has $|K_1| \preceq \frac{1}{\tilde{L}(R)}\frac{L(R)}{\tilde{L}(R)}(A+\pi)^{3/4}$ and $|K_2| \preceq \frac{1}{\tilde{L}(R)}\frac{L(R)}{\tilde{L}(R)}$. The arguments presented to control $K_3$ in the proof of Proposition 6.1.2 give

$$|K_3| \preceq \frac{1}{\tilde{L}(R)}\frac{1}{\sqrt{A}}.$$

Therefore $\lim_{A\longrightarrow+\infty}\lim_{R\longrightarrow+\infty}\tilde{L}(R)|J|=0$, so $\lim_{A\longrightarrow+\infty}\lim_{R\longrightarrow+\infty}\tilde{L}(R)|I_1|=0$. We rewrite the integral $I_2$ as

$$\begin{aligned}I_2 &= \frac{1}{\pi}\int_{|t|\leqslant A/R} e^{itR}\left[\mathrm{Re}\left(\tilde{Q}_{\delta+it}\right)-\mathrm{Re}\left(\left(1-\tilde{\lambda}_{\delta+it}\right)^{-1}\right)\tilde{\Pi}_\delta\right](\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t\\ &\quad+\int_{|t|\leqslant A/R} e^{itR}\left[\mathrm{Re}\left(\left(1-\tilde{\lambda}_{\delta+it}\right)^{-1}\right)\tilde{\Pi}_\delta\right](\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t\\ &=: L_1+L_2.\end{aligned}$$

By Proposition 8.3.9, for $R$ large enough, we have $|L_1|\leqslant 2A/R$. The integral $L_2$ may be split into $M_1+M_2$ as follows

$$\begin{aligned}L_2 &= \int_{|t|\leqslant A/R}(e^{itR}-1)\left[\mathrm{Re}\left(\left(1-\tilde{\lambda}_{\delta+it}\right)^{-1}\right)\tilde{\Pi}_\delta\right](\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t\\ &\quad+\int_{|t|\leqslant A/R}\left[\mathrm{Re}\left(\left(1-\tilde{\lambda}_{\delta+it}\right)\right)^{-1}\tilde{\Pi}_\delta\right](\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t\\ &= M_1+M_2,\end{aligned}$$

with, as in Proposition 6.1.2,

$$\tilde{L}(R)M_1\le A\frac{L(R)}{\tilde{L}(R)}.$$

To study the integral $M_2$, it suffices to deal with the case $0\leqslant t\leqslant A/R$. Denote by

$$N=\int_0^{A/R}\left[\mathrm{Re}\left(\left(1-\tilde{\lambda}_{\delta+it}\right)^{-1}\right)\tilde{\Pi}_\delta\right](\mathbb{1}_{\tilde{\Lambda}})(x_0)\widehat{u}(t)\,\mathrm{d}t.$$

We obtain

$$\tilde{L}(R)N\sim\frac{\tilde{h}(x_0)}{2E_\Gamma}\widehat{u}(0),$$

hence

$$\lim_{R\longrightarrow+\infty}\tilde{L}(R)M_2=C_\Gamma\widehat{u}(0),\ \text{ with } C_\Gamma=\frac{\lim_{k\longrightarrow+\infty}\sum_{\gamma\in\Gamma(k)}e^{-\delta\mathrm{d}(\mathbf{o},\gamma\cdot\mathbf{o})}}{E_\Gamma}.$$

This concludes the proof of Theorem C when $\beta=1$.

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Let M $=$ X$/\Gamma$ be a geometrically finite negatively curved manifold with fundamental group $\Gamma$ acting on X by isometries. The purpose of this book is to study the mixing property of the geodesic flow on $\mathrm{T}^1$M, the asymptotic behavior as $R \longrightarrow +\infty$ of the number of closed geodesics on M of length less than $R$ and of the orbital counting function $\#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$.

These properties are well known when the Bowen-Margulis measure on $\mathrm{T}^1$M is finite. We consider here Schottky group $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ whose Bowen-Margulis measure is infinite and ergodic, such that one of the elementary factor $\Gamma_i$ is parabolic with $\delta_{\Gamma_i} = \delta_\Gamma$. We specify these ergodic properties using a symbolic coding induced by the Schottky group structure.

Soit M $=$ X$/\Gamma$ une variété géométriquement finie de courbure strictement négative et $\Gamma$ son groupe fondamental agissant par isométries sur X. Nous étudions successivement dans cet article une propriété de mélange du flot géodésique sur $\mathrm{T}^1$M, le comportement quand $R \longrightarrow +\infty$ du nombre de géodésiques fermées de M de longueur plus petite que $R$ et celui de la fonction orbitale $\#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$.

Ces propriétés sont bien connues dans le cas où la mesure de Bowen-Margulis est finie sur $\mathrm{T}^1$M. Nous considérons ici un groupe de Schottky $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ de mesure de Bowen-Margulis infinie et ergodique, pour lequel au moins un facteur $\Gamma_i$ est parabolique et satisfait $\delta_{\Gamma_i} = \delta_\Gamma$. Les propriétés ergodiques ci-dessus sont alors précisées, en utilisant un codage symbolique induit par la structure de groupe de Schottky de $\Gamma$.