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**DISCRETE GEOMETRY
AND ISOTROPIC SURFACES**

François JAUBERTEAU,
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DISCRETE GEOMETRY AND ISOTROPIC SURFACES

François Jauberteau, Yann Rollin, Samuel Tapie

Abstract. – We consider smooth isotropic immersions from the 2-dimensional torus into \mathbb{R}^{2n} , for $n \geq 2$. When $n = 2$ the image of such map is an immersed Lagrangian torus of \mathbb{R}^4 . We prove that such isotropic immersions can be approximated by arbitrarily \mathcal{C}^0 -close piecewise linear isotropic maps. If $n \geq 3$ the piecewise linear isotropic maps can be chosen so that they are piecewise linear isotropic immersions as well.

The proofs are obtained using analogies with an infinite dimensional moment map geometry due to Donaldson. As a byproduct of these considerations, we introduce a numerical flow in finite dimension, whose limits provide, from an experimental perspective, many examples of piecewise linear Lagrangian tori in \mathbb{R}^4 . The DMMF program, which is freely available, is based on the Euler method and shows the evolution equation of discrete surfaces in real time, as a movie.

Résumé (Géométrie discrète et surfaces isotropes). – Nous considérons des immersions lisses et isotropes du tore de dimension 2 vers \mathbb{R}^{2n} , pour $n \geq 2$. Quand $n = 2$ l'image d'une telle application est un tore lagrangien immergé de \mathbb{R}^4 . Nous démontrons que de telles immersions isotropes peuvent être approximées au sens \mathcal{C}^0 , par des applications linéaires par morceaux et isotropes arbitrairement proches. Si $n \geq 3$, il est possible des choisir des applications linéaires par morceaux qui sont de plus des immersions.

Les démonstrations reposent sur des analogies avec une géométrie et une application moment en dimension infinie introduites par Donaldson. Nous en déduisons un flot en dimension finie, dont les limites, du point de vue expérimental, produisent de nombreux exemples de tores lagrangiens linéaires par morceaux de \mathbb{R}^4 . Le programme libre DMMF, basé sur la méthode d'Euler, montre l'équation d'évolution sous forme de film.

CONTENTS

| | |
|---|----|
| 1. Introduction | 1 |
| 1.1. Original motivations and background | 1 |
| 1.2. Statement of results | 2 |
| 1.2.1. Piecewise linear isotropic maps | 3 |
| 1.2.3. Isotropic quadrangular meshes | 4 |
| 1.2.5. Flow for quadrangular meshes | 5 |
| 1.3. Open questions | 6 |
| 1.4. Comments on the proofs—future works | 7 |
| 1.5. Organization of the paper | 8 |
| 1.6. Acknowledgements | 8 |
| 2. Dreaming of the smooth setting | 9 |
| 2.1. Donaldson’s moment map geometry | 9 |
| 2.2. A moment map flow | 12 |
| 2.2.5. Laplacian and related operators | 13 |
| 2.3. Variations of the moment map | 14 |
| 2.3.1. General variations | 14 |
| 2.3.3. Variations at an immersion | 15 |
| 2.4. Application | 16 |
| 2.4.3. Outreach | 18 |
| 3. Discrete analysis | 19 |
| 3.1. Conformally flat metric | 19 |
| 3.2. Square lattice and checkers board | 20 |
| 3.3. Quadrangulations | 22 |
| 3.3.1. Quadrangulations of the plane | 22 |
| 3.3.2. Quadrangulations of the torus | 23 |
| 3.3.3. Alternate quadrangulation of the plane | 23 |
| 3.4. Checkers graph | 23 |
| 3.4.1. Splitting of the ckeckers graph | 24 |
| 3.4.2. Checkers graph of the quotient | 24 |
| 3.4.3. Alternate checkers graph on the plane | 24 |
| 3.5. Examples | 25 |
| 3.6. A splitting for discrete functions | 26 |

| | |
|---|-----------|
| 3.7. Discrete Hölder norms | 27 |
| 3.7.1. \mathcal{C}^0 -norm | 27 |
| 3.7.2. Finite differences | 27 |
| 3.7.4. Definition of Hölder norms | 29 |
| 3.7.5. Weak Hölder norms | 29 |
| 3.7.7. Quotient and alternate quadrangulations | 30 |
| 3.8. Convergence of discrete functions | 30 |
| 3.8.1. Definition of converging sequences | 30 |
| 3.8.6. Continuity and limits of discrete functions | 31 |
| 3.8.9. Samples and convergence of discrete functions | 32 |
| 3.9. Precompactness | 34 |
| 3.10. Higher order convergence | 35 |
| 3.11. Examples of discrete convergence | 37 |
| 3.11.1. Samples of continuously differentiable functions | 37 |
| 3.11.3. Discrete tangent vector fields | 37 |
| 4. Perturbation theory for isotropic meshes | 39 |
| 4.1. Isotropic quadrangular meshes | 39 |
| 4.1.2. Equation for isotropic quadrilaterals | 40 |
| 4.1.4. Diagonals notation | 40 |
| 4.1.5. Equation for isotropic mesh | 41 |
| 4.2. Shear action on meshes | 42 |
| 4.3. Meshes obtained by sampling | 43 |
| 4.4. Almost isotropic samples | 44 |
| 4.5. Inner products | 44 |
| 4.5.1. The case of function | 44 |
| 4.5.3. The case of vector fields | 45 |
| 4.6. Linearized equations | 46 |
| 4.6.1. Other diagonal notations | 46 |
| 4.6.4. Computation of the discrete Laplacian | 47 |
| 4.7. Coefficients of the discrete Laplacian | 50 |
| 4.7.1. Splitting of the Laplacian | 50 |
| 4.7.3. Finite difference operators and discrete Laplacian | 51 |
| 5. Limit operator | 55 |
| 5.1. Computation of the limit operator | 55 |
| 5.2. Kernel of the limit operator | 57 |
| 5.3. Degenerate families of quadrangulations | 59 |
| 5.4. Schauder Estimates | 60 |
| 5.5. Spectral gap | 63 |
| 5.6. Modified construction in the degenerate case | 67 |
| 5.6.1. The setup | 68 |
| 5.6.2. The double cover | 68 |

| | |
|---|------------|
| 5.6.3. Meshes and operators for the modified construction | 69 |
| 5.6.4. Spectral gap for the degenerate case | 69 |
| 6. Fixed point theorem | 71 |
| 6.1. Fixed point equation | 71 |
| 6.2. Contracting map | 73 |
| 6.3. Fixed point principle | 74 |
| 6.4. Proof of Theorem C | 76 |
| 6.5. The degenerate case | 76 |
| 7. From quadrangulations to triangulations | 79 |
| 7.1. From quadrilaterals to pyramids | 79 |
| 7.1.5. Optimal apex | 81 |
| 7.2. \mathcal{C}^0 -estimates for optimal pyramids | 84 |
| 7.3. Many quadrilaterals and pyramids | 85 |
| 7.3.1. Triangulations obtained by refinement | 85 |
| 7.3.2. Optimal triangulation of isotropic quadrangular mesh | 86 |
| 7.4. Approximation by isotropic triangular mesh | 87 |
| 7.5. Piecewise linear immersions | 89 |
| 7.5.2. Perturbed meshes without flat faces | 89 |
| 7.5.6. Further perturbations by moving apexes | 91 |
| 7.6. Proof of Theorem A | 93 |
| 8. Discrete moment map flow | 95 |
| 8.1. Definition of the finite dimensional flow | 95 |
| 8.2. Implementation of the discrete flow | 97 |
| 8.2.1. Particular lattices | 97 |
| 8.2.2. The Euler method | 97 |
| 8.2.3. Visualization | 98 |
| 8.2.4. The DMMF code | 98 |
| Bibliography | 101 |

CHAPTER 1

INTRODUCTION

1.1. Original motivations and background

Lagrangian submanifolds are natural objects, arising in the context of *Hamiltonian mechanics* and *dynamical systems*. Their prominent role in symplectic topology and geometry should not come as a surprise. In spite of tremendous efforts, the classification of Lagrangian submanifolds, up to *Hamiltonian isotopy*, is generally an open problem: for instance, Lagrangian tori of the Euclidean symplectic space \mathbb{R}^4 are not classified up to Hamiltonian isotopy. Lagrangian submanifolds are also key objects of various gauge theories. For example, the *Lagrangian Floer theory* is defined by counting pseudoholomorphic disks with boundary contained in some prescribed Lagrangian submanifolds. Many examples of *smooth* Lagrangian submanifolds are known. They are easy to construct and to deform. In a nutshell, Lagrangian submanifolds are typical, rather *flexible* objects, from symplectic topology.

An elementary construction of Lagrangian submanifold is provided by considering the 0-section of the cotangent bundle of a smooth manifold T^*L , endowed with its natural symplectic structure $\omega = d\lambda$, where λ is the *Liouville form*. More generally, it is well known that any section of T^*L given by a closed 1-form is a Lagrangian submanifold. Furthermore, such Lagrangian submanifolds are Hamiltonian isotopic to the 0-section if, and only if, the corresponding 1-form is exact. These examples provide a large class of Lagrangian submanifolds which admit as many Hamiltonian deformations as smooth function on L modulo constants.

By the *Lagrangian neighborhood theorem*, every Lagrangian submanifold L of a symplectic manifold admits a neighborhood symplectomorphic to a neighborhood of the 0-section of T^*L . It follows that the local Hamiltonian deformations discussed above (in the case of T^*L) also provide deformations for Lagrangian submanifolds of *any* symplectic manifold.

The geometric notion of *stationary Lagrangians submanifolds* was introduced by Oh [8, 9] in order to seek canonical representatives, in a given isotopy class of Lagrangian submanifolds. Stationary Lagrangian submanifolds can be thought of as analogs of *minimal submanifolds* in the framework of symplectic geometry. Stationary Lagrangians are expected to be canonical in some sense, and Oh conjectured for

instance that Clifford tori of $\mathbb{C}\mathbb{P}^2$ should minimize the volume in their Hamiltonian isotopy class.

As in the case of minimal surfaces, one can define various modified versions of the *mean curvature flow*, which are expected to converge toward stationary Lagrangian submanifolds in a given isotopy class. In an attempt to implement numerical versions of these flows [5], we ended up facing theoretical problems of a *discrete geometric nature*. Indeed, from a numerical point of view, surfaces are usually understood as some type of *mesh* and their mathematical counterpart is *discrete geometry* and sometimes *piecewise linear geometry*. Two obstacles arose in order to provide a sound numerical simulation of geometric flows for Lagrangian submanifolds, namely:

1. To the best of our knowledge, discrete Lagrangian surfaces of \mathbb{R}^4 and more generally discrete isotropic surfaces of \mathbb{R}^{2n} are poorly understood, in fact hardly studied. We had no available examples of discrete Lagrangian tori in \mathbb{R}^4 in our toolbox, save some discrete analogs of product or Chekanov tori (cf. Section 3.5). Furthermore, we had no deformation theory that we could rely upon contrarily to the smooth case. Implementing a geometric evolution equation for discrete Lagrangian surfaces, with so few examples to start the flow was not an enticing project.
2. As far as a program is based on a numerical implementation, using floating point numbers, it is not natural to check if a symplectic form vanishes exactly along a plane. It only makes sense to test if the symplectic density is rather small, which means that we have an approximate solution of our problem. From an experimental point of view, we dread our numerical flow would exhibit some spurious drift of the symplectic density. We feared such instabilities may jeopardize our numerical simulations for flowing Lagrangian submanifolds.

These issues led us to consider an auxiliary flow. Ideally, the auxiliary flow should attract *any* discrete surface toward Lagrangian discrete surfaces. The utility of the auxiliary flow would be 2-fold: its limits would provide examples of Lagrangian discrete surfaces for our experiments. It may also be used to prevent instabilities of evolution equation along the moduli space of discrete Lagrangian surfaces.

These questions are part of a larger ongoing project. They have not been fully investigated yet but stirred many questions of a discrete differential geometric nature, in the context of symplectic geometry. This paper delivers a few answers to some of the simplest questions arising, as a spin-off to our initial motivations.

1.2. Statement of results

We consider smooth maps $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, where Σ is a surface and $n \geq 2$. The Euclidean space \mathbb{R}^{2n} is endowed with its standard symplectic form ω . A map ℓ is said to be *isotropic* if $\ell^*\omega = 0$. Lagrangian tori of \mathbb{R}^4 are the submanifolds obtained as the image of Σ by ℓ , in the particular case where $2n = 4$, Σ is diffeomorphic to a torus and ℓ is an isotropic embedding.

In this paper, we construct approximations of smooth isotropic immersions of the torus in \mathbb{R}^{2n} by *piecewise linear isotropic maps*. The idea is to consider a discretization of the torus by a square grid and approximate the smooth map by a quadrangular mesh. This mesh is almost isotropic, in a suitable sense. A perturbative argument shows that there exists a nearby isotropic quadrangular mesh, which is used to build a piecewise linear map. We provide a more precise statement of the above claims in the rest of the introduction.

1.2.1. Piecewise linear isotropic maps. – We recall some usual definitions before stating one of our main results. A *triangulation* of \mathbb{R}^2 is a locally finite *simplicial complex* that covers \mathbb{R}^2 entirely. In this paper, points, line segments, triangles of triangulations are understood as geometrical Euclidean objects of the plane. Similarly, we shall consider triangulations of quotients of \mathbb{R}^2 by a lattice Γ , obtained by quotient of Γ -invariant triangulations of \mathbb{R}^2 .

A *piecewise linear map* $f : \mathbb{R}^2 \rightarrow \mathbb{R}^m$ is a continuous map such that, for some triangulation of \mathbb{R}^2 , the restriction of f to any triangle is an affine map to \mathbb{R}^m .

We consider smooth isotropic immersions $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, where Σ is diffeomorphic to a 2-dimensional torus and $n \geq 2$. The Euclidean metric g of \mathbb{R}^{2n} induces a conformal structure on Σ . The uniformization theorem implies that the conformal structure of Σ actually comes from a quotient of \mathbb{R}^2 , with its canonical conformal structure, by a lattice. Thus, we have a conformal covering map

$$p : \mathbb{R}^2 \rightarrow \Sigma,$$

with group of deck transformations Γ , a lattice of \mathbb{R}^2 .

A triangulation (resp. quadrangulation) of Σ is called an *Euclidean triangulation* (resp. *quadrangulation*) of Σ if the boundary of every face lifts to an Euclidean triangle (resp. quadrilateral) of \mathbb{R}^2 via p .

Similarly, a function $f : \Sigma \rightarrow \mathbb{R}^m$ is a *piecewise linear map* if it lifts to a piecewise linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^m$ via p . Given a piecewise linear map $\hat{\ell} : \Sigma \rightarrow \mathbb{R}^{2n}$, the pull-back of the symplectic form ω of \mathbb{R}^{2n} makes sense on each triangle of the triangulation subordinate to $\hat{\ell}$. We say that $\hat{\ell}$ is a *isotropic piecewise linear map* if the pull back of ω vanishes along each face of the triangulation. A piecewise linear map which is locally injective is called a *piecewise linear immersion*.

The main result of this paper can be stated as follows:

THEOREM A. – *Let $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ be a smooth isotropic immersion, where Σ is a surface diffeomorphic to a compact torus and $n \geq 2$. Then, for every $\varepsilon > 0$, there exists a piecewise linear isotropic map $\hat{\ell} : \Sigma \rightarrow \mathbb{R}^{2n}$ such that for every $x \in \Sigma$, we have*

$$\|\ell(x) - \hat{\ell}(x)\| \leq \varepsilon.$$

Furthermore, if $n \geq 3$, we may assume that $\hat{\ell}$ is an immersion. If $n = 2$, we may assume that $\hat{\ell}$ is an immersion away from a finite union of embedded circles in Σ .

Loosely stated, Theorem A says that every isotropic immersion ℓ of a torus into \mathbb{R}^{2n} can be approximated by a piece linear map arbitrarily \mathcal{C}^0 -close to ℓ . If $n \geq 3$ the last statement of the theorem provides the following corollary:

COROLLARY B. – *Let n be an integer such that $n \geq 3$. Let Σ be a smoothly immersed surface in \mathbb{R}^{2n} , which is isotropic and diffeomorphic to a compact torus. Then, there exist piecewise linear immersed surfaces in \mathbb{R}^{2n} , which are isotropic, homeomorphic to a compact torus and arbitrarily close to Σ with respect to the Hausdorff distance.*

REMARK 1.2.2. – Our technique does not allow to get much better results than a rather rough \mathcal{C}^0 -closedness between ℓ and its approximation $\hat{\ell}$. The best evidence for this weakness is the existence of a certain *shear action* on the space of isotropic quadrangular meshes (cf. §4.2). It would be most interesting to understand whether these limitations are inherent to the techniques we employed here, or if there are geometric obstructions to get better estimates.

1.2.3. Isotropic quadrangular meshes. – The main tool to prove Theorem A relies on *quadrangulations* of Σ and *quadrangular meshes*. Quadrangulations of \mathbb{R}^2 are particular CW-complex decompositions of \mathbb{R}^2 , where edges are line segments of \mathbb{R}^2 and the boundary of every face is an Euclidean quadrilateral. Nevertheless, the precise general definition of quadrangulations is unimportant for our purpose. Indeed, we shall only work with particular standard quadrangulations $\mathcal{Q}_N(\mathbb{R}^2)$ of \mathbb{R}^2 , pictured as a regular grid with step size N^{-1} tiled by Euclidean squares.

Particular Euclidean quadrangulations of $\mathcal{Q}_N(\Sigma)$, are defined at §3.3. They are obtained as quotients of $\mathcal{Q}_N(\mathbb{R}^2)$ by certain lattices Γ_N of \mathbb{R}^2 . The associated moduli space of *quadrangular meshes* is by definition

$$\mathcal{M}_N = \mathcal{C}^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}.$$

A mesh $\tau \in \mathcal{M}_N$ is an object that associates \mathbb{R}^{2n} -coordinates to every vertex of the quadrangulation $\mathcal{Q}_N(\Sigma)$.

We would like to say that any quadrilateral of \mathbb{R}^{2n} contained in an isotropic plane is an *isotropic quadrilateral*. However, quadrilaterals are generally not contained in a 2-dimensional plane. The above attempt of definition can be extended via the Stokes theorem for every non flat quadrilateral: a quadrilateral of \mathbb{R}^{2n} is called an *isotropic quadrilateral*, if the integral of the Liouville form λ along the quadrilateral — that is four oriented line segments — vanishes (cf. §4.1).

REMARK 1.2.4. – In particular, for any compact embedded oriented surface S of \mathbb{R}^{2n} with boundary given by an isotropic quadrilateral, we have $\int_S \omega = 0$ by Stokes theorem.

By extension, we say that a mesh $\tau \in \mathcal{M}_N$ is isotropic if the quadrilateral in \mathbb{R}^{2n} associated to each face of $\mathcal{Q}_n(\Sigma)$ via τ is isotropic. The main strategy for proving Theorem A involves the following approximation result:

THEOREM C. – *Given an isotropic immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, there exists a family of isotropic quadrangular meshes $\rho_N \in \mathcal{M}_N$ defined for every N sufficiently large, with the following property: for every $\varepsilon > 0$, there exists $N_0 > 0$ such that for every $N \geq N_0$ and every vertex v of $\mathcal{Q}_N(\Sigma)$, we have*

$$\|\rho_N(v) - \ell(v)\| \leq \varepsilon.$$

An isotropic quadrilateral of \mathbb{R}^{2n} is always the base of an isotropic pyramid in \mathbb{R}^{2n} (cf. § 7.1), which is easily found as the solution of a linear system. This remark allows to pass from an isotropic quadrangular mesh to an isotropic triangular mesh. Together with Theorem C this provides essentially the proof of Theorem A.

1.2.5. Flow for quadrangular meshes. – Our approach for proving Theorem C has been inspired to a large extent by the beautiful *moment map geometry* introduced by Donaldson [4]. We shall provide a careful presentation of this infinite dimensional geometry at § 2, and merely state a few facts in this introduction: the moduli space of maps

$$\mathcal{M} = \{f : \Sigma \rightarrow \mathbb{R}^{2n}\},$$

from a surface Σ endowed with a volume form σ into \mathbb{R}^{2n} admits a natural formal Kähler structure, with a formal Hamiltonian action of $\text{Ham}(\Sigma, \sigma)$. The moment map of the action is given by

$$\mu(f) = \frac{f^*\omega}{\sigma}.$$

Zeros of the moment map are precisely isotropic maps. It is tempting to make an analogy with the Kempf-Ness theorem, which holds in the finite dimensional setting. We may conjecture that a map f admits an isotropic representative in its *complexified orbit* provided some type of algebro-geometric hypothesis of stability. Furthermore, one can also define a *moment map flow*, which is naturally defined in the context of a Kähler manifold endowed with a Hamiltonian group action. Such flow is essentially the downward gradient of the function $\|\mu\|^2$ on the moduli space, which is expected, in favorable circumstances, to converge toward a zero of the moment map in a prescribed orbit.

REMARK 1.2.6. – We shall not state any significant results aside the description of this geometric framework. For instance, it is an open question whether the moment map flow exists for short time in this context, which is part of a broader ongoing program.

In an attempt to define a finite dimensional analog of this infinite dimensional moment map picture, we define a flow analogous to the moment map flow on the moduli space of meshes \mathcal{M}_N , called the discrete moment map flow. This flow is now just an ODE and its behavior can readily be explored from a numerical perspective, using the Euler method. We provide a computer program called DMMF, available on the homepage

<http://www.math.sciences.univ-nantes.fr/~rollin/index.php?page=flow>,

which is a numerical simulation of the discrete moment map flow. From an experimental point of view, the flow seems to be converging quickly toward isotropic quadrangular meshes, for any initial quadrangular mesh (cf. § 8).

1.3. Open questions

Theorem A is a fundamental tool for the discrete geometry of isotropic tori, since it provides a vast class of examples of piecewise linear objects by approximation of the smooth ones. Here is a list of questions that arise immediately in this new territory of discrete symplectic geometry:

1. Is there a converse to Theorem A or Corollary B? Given a piecewise linear isotropic surface in \mathbb{R}^{2n} , is it possible to find a nearby smooth isotropic surface?
2. More generally, to what extent does the moduli space of piecewise linear Lagrangian submanifolds retain the properties of the moduli space of smooth Lagrangian submanifolds? In spite of groundbreaking progress in symplectic topology, the classification of Lagrangian submanifolds up to Hamiltonian isotopy remains open. It is known that there exists several types of Lagrangian tori in \mathbb{R}^4 , which are not Hamiltonian isotopic: namely, product tori and Chekanov tori [1]. On the other hand, Luttinger found infinitely many obstructions in [7] to the existence of certain type of knotted Lagrangian tori in \mathbb{R}^4 . In particular spin knots provide knotted tori in \mathbb{R}^4 which cannot be isotopic to Lagrangian tori according to Luttinger's theorem. This thread of ideas led to the conjecture that product and Chekanov tori are the only classes of Lagrangian tori in \mathbb{R}^4 , up to Hamiltonian isotopy. Although the result was claimed before, the conjecture is still open for the time being [2]. However it was proved by Dimitroglou Rizell, Goodman and Ivrii that all embedded Lagrangian tori of \mathbb{R}^4 are isotopic through Lagrangian isotopies [3]. Perhaps an interesting approach to tackle such conjecture, and more generally any questions involving some type of h -principle, would be to recast the question in the finite dimensional framework of piecewise linear Lagrangian tori of \mathbb{R}^4 .
3. The moment map framework, in an infinite dimensional context, presented at § 2, has been a great endeavor for proving our main results and introducing a finite dimensional version of the moment map flow. However, only a faint shadow of the moment map geometry is recovered in the finite dimensional world. More precisely, there exists a finite dimensional analog μ'_N of the moment map μ on \mathcal{M}_N . But it is not clear whether μ'_N is actually a moment map and for which group action on \mathcal{M}_N . It would be most interesting to define a finite dimensional analog of the group $\text{Ham}(\Sigma, \sigma)$, and try to make sense of the Kempf-Ness theorem in this setting.

1.4. Comments on the proofs—future works

The proofs given in this paper come with a strong differential geometric flavor, involving *uniformization theorem* for Riemann surfaces, *discrete analysis*, *discrete elliptic operators*, *discrete Schauder estimates*, *Riemannian geometry*, *discrete spectral gap theorem*, *Gauß curvature* and its discrete analogs.

Many of the techniques employed here may be adapted to more general settings. Working with tori seems to be a key fact however: indeed, Fourier and discrete Fourier transforms are well adapted for the analysis on tori and their quadrangulations and do not seem to extend easily. The discrete Schauder estimates derived by Thomée [10], which are a crucial ingredient of our fixed point principle, are proved using Fourier transforms.

Although geometric analysis is quite often a powerful tool for proving topological theorems, symplectic topologists may still expect some *more flexible constructions*. Boldly stated, *there may be a shorter proof based on more conventional techniques of symplectic topology*, stemming from some local ansatz, some jiggling lemma or in the spirit of the *h-principle*. Such proofs might be shorter, more elementary and, perhaps, lead to some stronger regularity results. These statements are difficult to disprove, especially since our attempts to deliver alternate proofs of, say Theorem A, along these lines failed so far.

One of the original motivation for this work was to get some effective constructions, even for *rough* PL surfaces, that is when N is quite small. It is unlikely that any of the *flexible constructions* could shed some light on this case. On the contrary, one of the outcome of this paper is a new flow for quadrangular meshes (cf. § 8) that provides large families of PL isotropic surfaces. Many questions about this finite dimensional flow remain open, and we would like to tackle them in future research. For instance, the completeness of the finite dimensional flow is unclear at the moment, although this is expected, based on numerical evidence.

Another open problem concerns the naturality of our finite dimensional flow as a *good approximation* of the infinite dimensional flow as N goes to infinity: let $\tau_N(t) \in \mathcal{M}_N$ be families of solutions of the finite dimensional flow (8.2), for every $N > 0$, and $f_t : \Sigma \rightarrow \mathbb{R}^{2n}$, a solution of the infinite dimensional flow (2.2). Assume that these flows are defined on the same interval $t \in [0, T]$, and that the initial conditions $\tau_N(0)$ converge towards f_0 in a suitable sense. Is it true that $\tau_N(t)$ converges uniformly towards f_t in a suitable sense? Under some strong regularity assumptions of f_t , a scheme of proof of such a convergence result, would depend of the following ingredients:

- Open problem : show that the sequence of finite dimensional evolution equations converge in a suitable sense to a *nice* evolution equation in the smooth setting, perhaps, in some sense, a parabolic equation. Such result could be understood as an analog of the study of the limit operator Ξ carried out in this paper 4.7.1.

- Open problem : relying on the Schauder discrete estimates, show that for suitable norms (perhaps weak discrete Hölder norms), the finite dimensional flow has some type of regularizing properties similar to parabolic flows. The answer to this question could be seen as an analog of the spectral gap Theorem 5.5.2.

At the moment of writing, we have no interpretation of the smooth moment map flow as some type parabolic flow and the above questions remain widely open.

1.5. Organization of the paper

Section §2 of this paper is a presentation of the moment map geometry of a certain infinite dimensional moduli spaces introduced by Donaldson. Finite dimensional analogs of this geometry are used in the rest of the paper. For instance a discrete version of the moment map flow is introduced at §8 and implemented on a computer, in order to obtain examples of Lagrangian piecewise linear surfaces from an experimental point of view. In §3, we introduce suitable spaces of discrete functions on tori, together with the analysis suited for implementing the fixed point principle. This section contains the definition of quadrangulations, discrete functions, discrete Hölder norms, together with the relevant notions of convergence, culminating with a type of Ascoli-Arzelà theorem (cf. Theorem 3.10.6). The equations for Lagrangian quadrangular meshes are introduced at §4, where their linearization is also computed. As the dimension of the discrete problem goes to infinity, we show that the finite dimensional linearized problem converges toward a smooth differential operator at §5. Some uniform estimates on the spectrum of the finite dimensional linearized problem are derived as a corollary. The proof of Theorem C is completed at §6, using the fixed point principle. The proof of Theorem A follows and is completed at §7 after introducing some generic perturbations in order to obtain piecewise linear immersions.

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CHAPTER 2

DREAMING OF THE SMOOTH SETTING

The main results of this work (for instance Theorem C), are of discrete geometric nature. Yet the main ideas of our proof were obtained via an analogy with the moment map geometry of the space of maps, from the tori endowed with a volume form, into \mathbb{R}^{2n} . This section is independent of the others, but we think it is important to show how smooth and discrete geometry analogies can be used to unravel unexpected ideas.

2.1. Donaldson's moment map geometry

The moment map geometry presented here was coined by Donaldson, although our specific setting is not emphasized in [4]. All the notions of moduli spaces shall be discussed from a purely *formal perspective*. With some additional effort, it may be possible to define infinite dimensional manifolds structures on moduli spaces of interest, by using suitable Sobolev or Hölder spaces.

Let M be a smooth manifold endowed with a Kähler structure (M, J, ω, g) . The Kähler structure of M is given by an integrable almost complex structure J , a Kähler form ω and the corresponding Kähler metric g . Recall that the Kähler form is related to the metric via the usual formula

$$\omega(v_1, v_2) = g(Jv_1, v_2), \text{ for all } v_1, v_2 \in T_m M.$$

The reader may keep in mind the simplest example provided by $M = \mathbb{R}^{2n} \simeq \mathbb{C}^n$ with its induced Euclidean Kähler structure. In this case, $\omega = d\lambda$, where λ is the Liouville form, which implies that the cohomology class $[\omega] \in H^2(M, \mathbb{R})$ vanishes.

Let Σ be a closed surface with orientation induced by a volume form σ . In real dimension 2, a volume form is also a symplectic form. Thus, the symplectic surface (Σ, σ) admits an infinite dimensional Lie group of Hamiltonian transformations denoted

$$\mathcal{G} = \text{Ham}(\Sigma, \sigma).$$

One can consider the moduli space of smooth maps

$$\mathcal{M} = \{f : \Sigma \rightarrow M \mid f^*[\omega] = 0\};$$

notice that in the case of $M = \mathbb{R}^{2n}$ endowed with the standard symplectic form, the condition $f^*[\omega] = 0$ is satisfied by every smooth map.

The tangent space $T_f\mathcal{M}$ is identified with the space of tangent vector fields V along f , which is the space of smooth map $V : \Sigma \rightarrow TM$ such that $V(x) \in T_{f(x)}M$. There is an obvious right-action of $\text{Ham}(\Sigma, \sigma)$ on \mathcal{M} by precomposition.

As pointed out by Donaldson, the geometry of the target space induces a formal Kähler structure on \mathcal{M} denoted $(\mathcal{M}, \mathfrak{g}, \Omega, \mathfrak{J})$ given by

$$(\mathfrak{J}V)|_x = JV_x, \quad \mathfrak{g}(V, V') = \int_{\Sigma} g(V, V')\sigma, \quad \Omega(V, V') = \int_{\Sigma} \omega(V, V')\sigma,$$

for any pair of tangent vector fields V, V' along $f : \Sigma \rightarrow \mathbb{R}^4$. By definition, the action of $\text{Ham}(\Sigma, \sigma)$ preserves the Kähler structure of \mathcal{M} .

The canonical L^2 -inner product on Σ , given by

$$\langle\langle h, h' \rangle\rangle = \int_{\Sigma} hh'\sigma,$$

allows to define the space of smooth functions orthogonal to constants $C_0^\infty(\Sigma)$, which in turn, be identified to the Lie algebra $\text{Lie}(\mathcal{G})$ of $\mathcal{G} = \text{Ham}(\Sigma, \sigma)$ via the map $h \mapsto X_h$. Here X_h is the Hamiltonian vector field with respect to the symplectic form σ , which satisfies

$$\iota_{X_h}\sigma = dh.$$

The L^2 -inner product $\langle\langle h, h' \rangle\rangle$ also provides an isomorphism between the Lie algebra of $\text{Ham}(\Sigma, \sigma)$ and its dual. The Lie algebra and its dual will be identified throughout this section without any further warning. Since $\text{Ham}(\Sigma, \sigma)$ acts on \mathcal{M} , any element of the Lie algebra $h \in \text{Lie}(\mathcal{G}) \simeq C_0^\infty(\Sigma)$ induces a *fundamental vector field* Y_h on \mathcal{M} defined by

$$Y_h(f) = f_*X_h \in T_f\mathcal{M}.$$

For $f \in \mathcal{M}$, we have $f^*[\omega] = 0$, hence

$$\int_{\Sigma} \frac{f^*\omega}{\sigma} \sigma = \int_{\Sigma} f^*\omega = 0,$$

so that we may consider the map

$$(2.1) \quad \mu : \begin{cases} \mathcal{M} & \longrightarrow & C_0^\infty(\Sigma) \\ f & \longmapsto & \mu(f) = \frac{f^*\omega}{\sigma} \end{cases}$$

By definition, we have the obvious property

$$\mu(f) = 0 \Leftrightarrow f^*\omega = 0 \Leftrightarrow f \text{ is an isotropic map.}$$

But we have much more than an equation:

PROPOSITION 2.1.1 (Donaldson). – *The action of $\text{Ham}(\Sigma, \sigma)$ on \mathcal{M} is formally Hamiltonian and admits μ as a moment map. More precisely:*

1. $\mu : \mathcal{M} \rightarrow C_0^\infty(\Sigma)$ is $\text{Ham}(\Sigma, \sigma)$ -equivariant in the sense that for every $f \in \mathcal{M}$ and $u \in \text{Ham}(\Sigma, \sigma)$

$$\langle\langle \mu(f \circ u), h \circ u \rangle\rangle = \langle\langle \mu(f), h \rangle\rangle;$$

2. for every variation V of f

$$\langle\langle D\mu|_f \cdot V, h \rangle\rangle = -\iota_{Y_h(f)}\Omega(V),$$

where D denotes the differentiation of functions on \mathcal{M} .

Proof. – Only the second property requires some explanation. We pick a smooth family of maps $f_t : \Sigma \rightarrow M$ such that $\frac{\partial}{\partial t}f_t|_{t=0} = V$ and $f_0 = f$. The family is understood as a smooth map

$$F : I \times \Sigma \rightarrow M,$$

where I is a neighborhood of 0 in \mathbb{R} and $F(t, x) = f_t(x)$. We denote by $j_0 : \Sigma \hookrightarrow I \times \Sigma$ the canonical embedding given by $j_0(x) = (0, x)$. Notice that by definition $F \circ j_0 = f$. Then

$$\begin{aligned} \langle\langle D\mu|_f \cdot V, h \rangle\rangle &= \frac{\partial}{\partial t}\Big|_{t=0} \int_{\Sigma} h f_t^* \omega \\ &= \int_{\Sigma} h j_0^* \mathcal{L}_{\partial_t} \cdot F^* \omega \\ &= \int_{\Sigma} h j_0^* (d\iota_{\partial_t} F^* \omega + \iota_{\partial_t} dF^* \omega), \end{aligned}$$

where the last line comes from the Cartan formula. The symplectic form is closed, hence $dF^* \omega = F^* d\omega = 0$. In addition $F^* \partial_t$ agrees with V along $\{0\} \times \Sigma$, so that $j_0^* d\iota_{\partial_t} F^* \omega = dj_0^* \iota_{\partial_t} F^* \omega = d\omega(V, (F \circ j_0)^* \cdot) = df^* \iota_V \omega$. It follows that

$$\begin{aligned} \langle\langle D\mu|_f \cdot V, h \rangle\rangle &= \int_{\Sigma} h df^* \iota_V \omega \\ &= - \int_{\Sigma} dh \wedge f^* \iota_V \omega \\ &= - \int_{\Sigma} \iota_{X_h} \sigma \wedge f^* \iota_V \omega. \end{aligned}$$

The interior product is an antiderivation. In particular

$$\iota_{X_h}(\sigma \wedge f^* \iota_V \omega) = (\iota_{X_h} \sigma) \wedge f^* \iota_V \omega + (\iota_{X_h} f^* \iota_V \omega) \sigma.$$

The LHS of the above identities must vanish since this is the case for a 3-form over a surface, and we obtain the identity

$$\begin{aligned} \langle\langle D\mu|_f \cdot V, h \rangle\rangle &= \int_{\Sigma} (\iota_{X_h} f^* \iota_V \omega) \sigma \\ &= \int_{\Sigma} \omega(V, Y_h(f)) \sigma \\ &= \Omega(V, Y_h(f)), \end{aligned}$$

which proves the proposition. \square

2.2. A moment map flow

From this point, gauge theorists may dream of generalizations of the Kempf-Ness Theorem, which is only known to hold in the finite dimensional setting. The Kempf-Ness theorem asserts that the existence of a zero of the moment map in a given complexified orbit of the group action is equivalent to an algebraic property of stability, in the sense of geometric invariant theory. Under the hypothesis of stability, the zeroes of the moment map are unique up to the action of the real group. Unfortunately the Kempf-Ness Theorem does not apply immediately in the infinite dimensional setting and the conjectural isomorphism

$$\mathcal{M} // \mathcal{G}^{\mathbb{C}} \simeq \mu^{-1}(0) / \mathcal{G},$$

where the LHS is some type of GIT quotient, is out of reach for the moment. To start with, the complexification of \mathcal{G} is not even well defined and the quotient $\mathcal{M} // \mathcal{G}^{\mathbb{C}}$ does not make sense. Nevertheless, a significant number of this thread of ideas may be implemented. For instance, we may define a natural *moment map flow*.

DEFINITION 2.2.1. – *Let $f_t \in \mathcal{M}$ be a family of maps, for t in an open interval of \mathbb{R} . We say that the family f_t is solution of the moment map flow if*

$$(2.2) \quad \frac{df}{dt} = \mathfrak{J}Y_{\mu(f)}(f).$$

REMARK 2.2.2. – In the finite dimensional setting, such a moment map flow preserves the complexified orbits and converges to a zero of the moment map under a suitable assumption of stability. It is natural to conjecture that this flow should converges generically to an isotropic map in a prescribed complexified orbit. We shall not tackle this problem here and only prove some very down to earth properties of the flow.

By construction, we have

$$\begin{aligned} \mathfrak{g}(\mathfrak{J}Y_{\mu(f)}(f), V) &= \Omega(Y_{\mu(f)}, V) \\ &= -\langle\langle D\mu|_f \cdot V, \mu(f) \rangle\rangle \\ &= -\frac{1}{2}D(\|\mu\|^2)|_f \cdot V, \end{aligned}$$

so that

$$\mathfrak{J}Y_{\mu(\cdot)} = -\frac{1}{2}\text{grad}\|\mu\|^2,$$

which proves the following lemma:

LEMMA 2.2.3. – *Smooth maps $f : \Sigma \rightarrow M$ are zeroes of the moment map if, and only if they are isotropic. In addition, the moment map flow is a downward gradient flow of the functional $f \mapsto \|\mu(f)\|^2$ on \mathcal{M} . More precisely, the evolution equation of the moment map flow can be written*

$$\frac{df}{dt} = -\frac{1}{2}\text{grad}\|\mu(f)\|^2,$$

where grad is the gradient of a function on \mathcal{M} endowed with its Riemannian metric \mathfrak{g} .

As a corollary, we see that the functional should decrease along flow lines:

COROLLARY 2.2.4. – *If f_t is a smooth family of maps solution of the moment map flow, then*

$$\frac{d}{dt} \|\mu(f_t)\|^2 < 0$$

unless f_t is isotropic, in which case $\frac{d}{dt} \|\mu(f_t)\|^2 = 0$ and the flow is stationary.

Proof. – Assume that f_t is not isotropic. In particular there exists $x \in \Sigma$ such that the differential of $\mu(f_t)$ does not vanish at x . Otherwise $\mu(f_t)$ would be constant. But the fact that ω is exact would force $\mu(f_t) = 0$. By definition $X_{\mu(f_t)}$ is a non vanishing vector field at x since it is the symplectic dual of $d\mu(f_t)$. It follows that $Y_{\mu(f_t)}$ does not vanish hence

$$\begin{aligned} \frac{d}{dt} \|\mu(f_t)\|^2 &= -2\mathfrak{g}(\mathfrak{J}Y_{\mu(f_t)}, \mathfrak{J}Y_{\mu(f_t)}) \\ &= -2\mathfrak{g}(Y_{\mu(f_t)}, Y_{\mu(f_t)}) < 0. \end{aligned} \quad \square$$

2.2.5. Laplacian and related operators. – For each vector field V tangent to f , we define the operator

$$\delta_f : T_f \mathcal{M} \rightarrow C_0^\infty(\Sigma)$$

by

$$(2.3) \quad \delta_f V = -D\mu|_f \cdot JV.$$

We see that the adjoint δ_f^* of δ_f satisfies

$$\begin{aligned} \mathfrak{g}(\delta_f^* h, V) &= \langle\langle \delta_f V, h \rangle\rangle \\ &= -\langle\langle D\mu|_f \cdot JV, h \rangle\rangle \\ &= \Omega(Y_h(f), JV) \\ &= \mathfrak{g}(Y_h(f), V), \end{aligned}$$

so that

$$(2.4) \quad \delta_f^* h = Y_h(f).$$

For each $f \in \mathcal{M}$, we may define a natural Laplacian

$$(2.5) \quad \Delta_f = \delta_f \delta_f^*$$

acting on smooth functions on Σ .

REMARK 2.2.6. – It seems likely that the moment map flow of Definition 2.2.1 can be interpreted as a parabolic flow, once a suitable analytical framework and gauge condition have been setup. Although we shall not prove anything about short time existence of the moment map flow in this work, we provide at least a heuristic evidence. In the next section, we compute the variation of the moment map and show that the variation of $\mu(f)$, when f is deformed in the direction of the complexified action $\mathfrak{J}Y_h$, is expressed as a Laplacian of h . However, the systematic study of the moment map

flow in the smooth setting is not our purpose here, and we shall return to this question in a sequel to this paper [6].

2.3. Variations of the moment map

The operator $f \mapsto \mu(f)$ is a first order differential operator. This section is devoted to calculate its linearization.

2.3.1. General variations. – Let $f_s : \Sigma \rightarrow M$ be a smooth family of maps, with parameter $s \in I$, where I is an open interval of \mathbb{R} . We use the notation,

$$V_s = \frac{\partial f_s}{\partial s},$$

for the infinitesimal variation $V_s \in T_{f_s} \mathcal{M}$ of the family f_s .

We consider the map $F : I \times \Sigma \rightarrow M$ given by $F(s, x) = f_s(x)$ and the canonical injection $j_{s_0} : \Sigma \hookrightarrow I \times \Sigma$, defined by $j_{s_0}(x) = (s_0, x)$ for some $s_0 \in I$. We compute, using the Cartan formula

$$\begin{aligned} \left. \frac{\partial f_s^* \omega}{\partial s} \right|_{s=s_0} &= j_{s_0}^* \frac{\partial}{\partial s} \cdot F^* \omega \\ &= j_{s_0}^* (d \circ \iota_{\partial_s} + \iota_{\partial_s} \circ d) F^* \omega \\ &= j_{s_0}^* d \circ \iota_{\partial_s} F^* \omega \\ &= j_{s_0}^* d F^* \iota_{V_s} \omega \\ &= d f_{s_0}^* \iota_{V_{s_0}} \omega, \end{aligned}$$

where we have used the fact that $d\omega = 0$, that d commutes with pullbacks and that $F \circ j_{s_0} = f_{s_0}$.

The form

$$\alpha_{s_0} = f_{s_0}^* \iota_{V_{s_0}} \omega$$

is called the *Maslov form* of the deformation f_s at $s = s_0$. The above computation shows that

$$\frac{\partial f_s^* \omega}{\partial s} = d\alpha_s,$$

which reads

$$\frac{\partial \mu(f_s)}{\partial s} = \delta \alpha_s,$$

where δ is the operator given by

$$(2.6) \quad \delta \gamma = \frac{d\gamma}{\sigma},$$

for every 1-form γ on Σ .

Thus we have proved the following result:

LEMMA 2.3.2. – *Let $f : \Sigma \rightarrow M$ be a smooth map and $V \in T_f \mathcal{M}$ be an infinitesimal variation of f . Then*

$$D\mu|_f \cdot V = \delta\alpha_V,$$

where $\alpha_V = f^* \iota_V \omega$ is the Maslov form of the deformation and δ is the operator defined by (2.6).

2.3.3. Variations at an immersion. – We assume now that $f : \Sigma \rightarrow M$ is a smooth immersion. In particular the pullback $g_\Sigma = f^*g$ is a Riemannian metric on Σ . The volume form vol_Σ may not agree with σ , but the 2-forms are related by a conformal factor

$$\text{vol}_\Sigma = \theta\sigma,$$

where $\theta : \Sigma \rightarrow \mathbb{R}$ is a positive smooth function. We introduce a conformal metric g_σ that satisfies the equation

$$g_\Sigma = \theta g_\sigma,$$

and the Hodge operator acting on forms of Σ , associated to the metric g_σ is denoted $*_\sigma$.

LEMMA 2.3.4. – *Assume that $f : \Sigma \rightarrow M$ is a smooth immersion. Then Σ has an induced Riemannian metric g_Σ . Let g_σ be the Riemannian metric with volume form σ , conformal to g_Σ . Let Y_h be the fundamental vector field on \mathcal{M} associated to the Hamiltonian function h . Then*

$$\Delta_f h = \delta_f \delta_f^* h = -D\mu|_f \cdot \mathfrak{J}Y_h(f) = d^{*\sigma} \theta dh = \theta \Delta_\sigma h - g_\sigma(d\theta, dh).$$

where Δ_σ is the Laplacian associated to the Riemannian metric g_σ and θ is the conformal factor such that $g_\Sigma = \theta g_\sigma$.

REMARK 2.3.5. – In particular, if vol_Σ agrees with σ , then $\theta = 1$, $g_\sigma = g_\Sigma$ and the above formula says that

$$\Delta_f h = \Delta_\Sigma h.$$

Proof. – Let $f_s \in \mathcal{M}$, be a smooth family of maps for $s \in I = (-\varepsilon, \varepsilon)$, with the property that $f_0 = f$ and $V_0 = JY_h = Jf_* X_h$. Then $\frac{\partial \mu(f_s)}{\partial s} = \delta\alpha_s$ by Lemma 2.3.2. But $\alpha_0(U) = f^* \omega(V_0, U) = \omega(Jf_* X_h, f_* U) = -g(f_* X_h, f_* U) = -g_\Sigma(X_h, U)$. It follows that $\alpha_0(U) = -\theta g_\sigma(X_h, U) = -\theta \sigma(X_h, *_\sigma U) = \theta *_\sigma dh$. Then we conclude that

$$\frac{\partial \mu(f_s)}{\partial s} \Big|_{s=0} = *_\sigma d\theta *_\sigma dh = -d^{*\sigma} \theta dh = -\theta \Delta_\sigma h + g_\sigma(d\theta, dh),$$

which proves the lemma. □

The next lemma shows that Δ_f is essentially an isomorphism.

LEMMA 2.3.6. – *The operator $h \mapsto d^{*\sigma} \theta dh$ is an elliptic operator of order 2, which is an isomorphism modulo constants.*

Proof. – The fact that the operator is elliptic of order 2 follows from the formula

$$d^{*\sigma}\theta dh = \theta\Delta_\sigma h - g_\sigma(d\theta, dh).$$

The operator is selfadjoint since

$$\langle\langle d^{*\sigma}\theta dh, h' \rangle\rangle = \langle\langle \theta dh, dh' \rangle\rangle = \langle\langle h, d^{*\sigma}\theta dh' \rangle\rangle.$$

If h belongs to the kernel of the operator, then

$$0 = \langle\langle d^{*\sigma}\theta dh, h \rangle\rangle = \langle\langle \theta dh, dh \rangle\rangle,$$

which implies that h is constant. Because the operator is selfadjoint, the orthogonal of its image is identified to the kernel. So the operator is an isomorphism when restricted to functions which are L^2 -orthogonal to constants. \square

2.4. Application

We know that \mathcal{M} is acted on by $\mathcal{G} = \text{Ham}(\Sigma, \sigma)$. The \mathcal{G} -orbit of $f \in \mathcal{M}$ is denoted \mathcal{O}_f . The group of Hamiltonian transformations does not admit a natural complexification. Nevertheless, it is possible to make sense of its complexified orbits.

The space of vector fields Y_u defined by $Y_u(f) = f_*X_u$ over \mathcal{M} defines an integrable distribution $\mathcal{D} \subset T\mathcal{M}$ which is the tangent space to \mathcal{G} -orbits. We can consider the complexified distribution of the tangent bundle to \mathcal{M}

$$\mathcal{D}^{\mathbb{C}} = \mathcal{D} + \mathfrak{J}\mathcal{D}$$

given by vector fields of the form $Y_u + \mathfrak{J}Y_v$. The fact that \mathcal{G} preserves the complex structure \mathfrak{J} of \mathcal{M} implies that the distribution is formally integrable into a holomorphic foliation. A leaf of the foliation, obtained by integrating the distribution, is referred to as a *complexified orbit of \mathcal{G}* . The complexified orbit of a element $f \in \mathcal{M}$ is denoted $\mathcal{O}_f^{\mathbb{C}}$.

We are now assuming for simplicity that M is the Kähler manifolds \mathbb{R}^{2n} identified to \mathbb{C}^n . In this case, given $f \in \mathcal{M}$, we may consider a type of exponential map given by

$$\exp_f(u + iv) = f + Y_u(f) + \mathfrak{J}Y_v(f),$$

for $u, v \in C_0^\infty(\Sigma)$. This type of exponential map does not come from a Lie group exponential map. However, $\exp_f(u + iv)$ provides perturbations of f in directions tangent to the complexified orbit $\mathcal{O}_f^{\mathbb{C}}$ at f .

We have now all the tools necessary to prove the following result:

THEOREM 2.4.1. – *We choose $M = \mathbb{R}^{2n}$ for the construction of \mathcal{M} . Let $\ell \in \mathcal{M}$ be a smooth isotropic immersion. If $f \in \mathcal{M}$ is sufficiently close to ℓ in $\mathcal{C}^{1,\alpha}$ -norm, there exists a nearby perturbation of the form $\tilde{\ell} = \exp_f(ih)$, where h is a $\mathcal{C}^{2,\alpha}$ function on Σ , such that $\tilde{\ell}$ is an isotropic immersion.*

Proof. – We denote by $\mathcal{C}^{k,\alpha}$, for some Hölder parameter $\alpha > 0$, the usual Hölder spaces. The moduli space \mathcal{M} is replaced with $\mathcal{M}^{k,\alpha}$ which consists of $\mathcal{C}^{k,\alpha}$ -maps $f : \Sigma \rightarrow \mathbb{R}^{2n}$. Since $\mathcal{M}^{k,\alpha}$ is an affine space modeled on $\mathcal{C}^{k,\alpha}$, it is naturally endowed with an infinite dimensional manifold structure. In particular, the map \exp_f defines a smooth map

$$\exp : \mathcal{C}^{k+1,\alpha}(\Sigma, \mathbb{C}) \times \mathcal{M}^{k,\alpha} \longrightarrow \mathcal{M}^{k,\alpha}.$$

given by $(h, f) \mapsto \exp_f(h)$.

We denote by $\mathcal{C}_0^{k,\alpha}(\Sigma)$ the subspace of $\mathcal{C}^{k,\alpha}(\Sigma)$ that consists of real valued functions $h : \Sigma \rightarrow \mathbb{R}$ such that $\int_{\Sigma} h \sigma = 0$ (i.e., functions orthogonal to constants for the inner product $\langle\langle \cdot, \cdot \rangle\rangle$). We consider the map

$$Z : \begin{cases} \mathcal{C}_0^{2,\alpha}(\Sigma) \times \mathcal{M}^{1,\alpha} & \longrightarrow \mathcal{M}^{0,\alpha} \\ (h, f) & \longmapsto \exp_f(-ih), \end{cases}$$

whose differential at $(0, \ell)$ satisfies

$$\frac{\partial Z}{\partial h} \Big|_{(0,\ell)} \cdot \dot{h} = -\mathfrak{J}Y_h(\ell)$$

by definition of the exponential map. In particular

$$(2.7) \quad \frac{\partial(\mu \circ Z)}{\partial h} \Big|_{(0,\ell)} \cdot \dot{h} = -D\mu|_{\ell} \cdot \mathfrak{J}Y_h(\ell) = \delta_{\ell} \delta_{\ell}^* h = \Delta_{\ell} h$$

by Lemma 2.3.4. This operator is an isomorphism modulo constants by Lemma 2.3.6. We consider the map

$$F : \mathcal{C}_0^{2,\alpha}(\Sigma) \times \mathcal{M}^{1,\alpha} \rightarrow \mathcal{C}_0^{0,\alpha}(\Sigma)$$

given by $F = \mu \circ Z$. We have proved that the differential

$$\frac{\partial F}{\partial h} \Big|_{(0,\ell)} : \mathcal{C}_0^{2,\alpha}(\Sigma) \longrightarrow \mathcal{C}_0^{0,\alpha}(\Sigma)$$

is an isomorphism. The rest of the proof follows from the implicit function theorem: for every $f \in \mathcal{M}^{1,\alpha}$ sufficiently close to ℓ in $\mathcal{C}^{1,\alpha}$ -norm, there exists a unique $\tilde{h} = h(f) \in \mathcal{C}_0^{2,\alpha}(\Sigma)$ in a small neighborhood of the origin, such that

$$F(\tilde{h}, f) = 0.$$

By definition $\exp_f(i\tilde{h}) = \tilde{\ell}$ satisfies $\mu(\tilde{\ell}) = 0$. By assumption ℓ is smooth. If f is also smooth, elliptic regularity and standard bootstrapping argument shows that \tilde{h} , and in turn $\tilde{\ell}$, must be smooth as well. This proves the theorem. \square

REMARK 2.4.2. – In Section 4, we will develop a perturbation theory on the space of quadrangular meshes \mathcal{M}_N that mimics Theorem 2.4.1. We shall define an analog δ_{τ} of the operator δ_f in the context of discrete geometry (cf. Formula (4.6)). The operator δ_{τ} , and more precisely its adjoint δ_{τ}^* , could be used to define an analog of Hamiltonian vector fields in the context of discrete differential geometry, in view of Formula (2.4). This could be relied upon to define a discrete analog of the gauge group action $\mathcal{G} = \text{Ham}(\Sigma, \sigma)$. This idea will be explored in a sequel to this work [6].

2.4.3. Outreach. – In §4 we shall define finite dimensional analogs of the infinite dimensional moment map picture presented in the current section. This will provide the incomplete dictionary below, where the right column is conjecturally a finite dimensional approximation of the left column:

| | |
|---|--|
| Infinite dimensional case | finite dimensional case |
| Area form σ on Σ | Quadrangulation $\mathcal{Q}_N(\Sigma)$ |
| $\mathcal{M} = \{f : \Sigma \rightarrow \mathbb{R}^{2n}\}$ | $\mathcal{M}_N = C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$ |
| Canonical Kähler structure | Canonical Kähler structure |
| Ham(Σ, σ)-action | ??? |
| Fundamental V.F $Y_h(f) = \delta_f^* h$ | $\delta_\tau^* \phi$ |
| A moment map $\mu : \mathcal{M} \rightarrow C^\infty(\Sigma)$ | $\mu_N^r : \mathcal{M}_N \rightarrow C^2(\mathcal{Q}_N(\Sigma))$ |
| The moment map flow (2.2) | The discrete flow (8.2) |

Many aspects of the above dictionary remain unclear. First, the finite dimensional picture does not come with a Lie group action that would, in some sense, approximate Ham(Σ, σ). In particular μ_N^r is not a moment map and $C^2(\mathcal{Q}_N(\Sigma))$ is not interpreted as a Lie algebra. The flows are defined on both sides and we would like to compare them as N goes to infinity. Unfortunately, we do not even know whether the infinite dimensional flow exists for short time. The discrete flow is an ODE, but it is not completely understood at this stage. For N fixed, does the flow converge, or does it blowup? Does a sequence of flow converge to the moment map flow as N goes to infinity? Can we use the above sketch of correspondence to make sense of some type of Kempf-Ness theorem in the infinite dimensional setting?

All these gripping questions are postponed to a later work. In this paper, we focus on the discrete flow on \mathcal{M}_N , for a given N , and merely provide a computer simulation of the discrete flow at §8.

CHAPTER 3

DISCRETE ANALYSIS

In this section, we consider a real surface Σ , diffeomorphic to a torus. We denote by g the canonical Euclidean metric of \mathbb{R}^{2n} and J the standard complex structure deduced from the identification $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. The standard symplectic form of \mathbb{R}^{2n} is given by $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ and $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ is an *isotropic immersion*.

3.1. Conformally flat metric

Every Riemannian metric on a surface diffeomorphic to a torus is *conformally flat*. In particular, Σ carries a pullback Riemannian metric

$$g_\Sigma = \ell^*g,$$

which must be conformally flat. In other words, there exists a covering map

$$(3.1) \quad p : \mathbb{R}^2 \rightarrow \Sigma$$

with deck transformations given by a lattice $\Gamma \subset \mathbb{R}^2$. The Euclidean metric g_{euc} of \mathbb{R}^2 descends as a flat metric g_σ on Σ . In addition there exists a positive smooth function $\theta : \Sigma \rightarrow (0, +\infty)$, known as the *conformal factor*, such that

$$g_\Sigma = \theta g_\sigma.$$

The projection p , which descends to the quotient \mathbb{R}^2/Γ , provides a preferred diffeomorphism

$$(3.2) \quad \Phi : \mathbb{R}^2/\Gamma \rightarrow \Sigma,$$

which is also an isometry from $(\mathbb{R}^2/\Gamma, g_{\text{euc}})$ to (Σ, g_σ) .

3.2. Square lattice and checkers board

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the canonical basis of \mathbb{R}^2 . The basis (e_1, e_2) is orthonormal with respect to the canonical Euclidean metric g_{euc} of \mathbb{R}^2 and it is positively oriented, by convention.

For every positive integer N , we introduce the lattice $\Lambda_N \subset \mathbb{R}^2$ spanned by e_1/N and e_2/N :

$$\Lambda_N = \mathbb{Z} \cdot \frac{e_1}{N} \oplus \mathbb{Z} \cdot \frac{e_2}{N} \subset \mathbb{R}^2.$$

The lattice Λ_N provides the familiar picture of a square grid in \mathbb{R}^2 with step size N^{-1} . The lattice Γ , introduced at § 3.1, admits a basis (γ_1, γ_2) , compatible with the canonical orientation of \mathbb{R}^2 . The lattice Γ is generally not a sublattice of Λ_N . Indeed, the components of the vectors $\gamma_1, \gamma_2 \in \mathbb{R}^2$ may not be rational. This fact will cause a technical catch for constructing quadrangulations of Σ . Luckily this difficulty is easily overcome as we shall explain below. The *checkers board* sublattice $\Lambda_N^{\text{ch}} \subset \Lambda_N$ is spanned by the vectors $\frac{e_1+e_2}{N}$ and $\frac{e_2-e_1}{N}$:

$$\Lambda_N^{\text{ch}} = \mathbb{Z} \cdot \frac{e_1 + e_2}{N} \oplus \mathbb{Z} \cdot \frac{e_2 - e_1}{N} \subset \Lambda_N.$$

The elements of $\Lambda_N \subset \mathbb{R}^2$ may be thought of as the positions of a standard checkers board game. Then Λ_N^{ch} acts on Λ_N by translations. These translations are spanned by diagonal motions, as in some kind of checkers game. One can easily see that the quotient $\Lambda_N/\Lambda_N^{\text{ch}}$ is isomorphic to \mathbb{Z}_2 which is isomorphic to the equivalence classes of the usual black and white positions of the checkers board game.

For each $N > 0$ and $i = 1, 2$, we choose $\gamma_i^N \in \Lambda_N^{\text{ch}}$ which is a best approximation of γ_i in Λ_N^{ch} , for the Euclidean distance in \mathbb{R}^2 . By definition, γ_1^N and γ_2^N are linearly independent for all sufficiently large N . We define the lattice Γ_N , at least for sufficiently large N , as

$$\Gamma_N = \mathbb{Z} \cdot \gamma_1^N \oplus \mathbb{Z} \cdot \gamma_2^N \subset \Lambda_N^{\text{ch}} \subset \Lambda_N.$$

We summarize our construction in Figure 1. The red and blue bullets represent the elements of Λ_N , where the red bullets are in Λ_N^{ch} . We draw the generators γ_i of Γ and their best approximations, in red, by elements γ_i^N of Λ_N^{ch} :

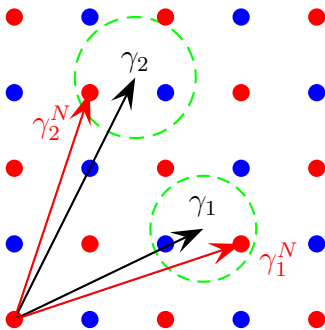


FIGURE 1. Construction of Γ_N

By construction, Γ_N is a sublattice of Λ_N^{ch} ; this choice has been designed so that the checkers graph splits into two connected components precisely (cf. § 3.4). Furthermore, the lattices Γ_N converge towards Γ , in a sense to be made more precise now: the linear transformation U_N of \mathbb{R}^2 defined by

$$U_N(\gamma_i^N) = \gamma_i$$

identifies the lattices Γ_N and Γ by an automorphism of \mathbb{R}^2 . Using an operator norm for linear transformations of \mathbb{R}^2 , we have

$$(3.3) \quad \|U_N - \text{id}|_{\mathbb{R}^2}\| = \mathcal{O}(N^{-1}).$$

In conclusion U_N converges towards the identity and $U_N(\Gamma_N) = \Gamma$, which is understood as Γ_N converges towards Γ .

By construction, U_N descends to the quotient as a (locally linear) diffeomorphism

$$u_N : \mathbb{R}^2/\Gamma_N \rightarrow \mathbb{R}^2/\Gamma.$$

The linear transformation U_N may not belong to the orthogonal group. Therefore neither U_N nor u_N are isometries. But, they are isometries in the limit, since U_N converges to the identity. This fact will be sufficient for our purpose. The quotients \mathbb{R}^2/Γ and \mathbb{R}^2/Γ_N are canonically identified to Σ via the diffeomorphisms

$$\mathbb{R}^2/\Gamma_N \xrightarrow{u_N} \mathbb{R}^2/\Gamma \xrightarrow{\Phi} \Sigma.$$

There are now several competing covering maps: we defined $p : \mathbb{R}^2 \rightarrow \Sigma$ at (3.1), but we may also consider the covering maps

$$(3.4) \quad p_N = p \circ U_N : \mathbb{R}^2 \rightarrow \Sigma.$$

The group of deck transformation of p is Γ , whereas the group of deck transformations of p_N is Γ_N . There are also several flat metrics descending on Σ via p and p_N . The first g_σ is induced by the Euclidean metric and the diffeomorphism $\Phi : \mathbb{R}^2/\Gamma \rightarrow \Sigma$. The other flat metrics g_σ^N are induced by the Euclidean metric and the diffeomorphisms

$$(3.5) \quad \Phi_N : \mathbb{R}^2/\Gamma_N \rightarrow \Sigma$$

induces by (3.4). According to (3.3) we have

$$g_\sigma^N = g_\sigma + \mathcal{O}(N^{-1}).$$

3.3. Quadrangulations

Instead of linear triangulations, we shall work with particular *linear quadrangulations* $\mathcal{Q}_N(\Sigma)$ of Σ . The current section is devoted to the definition of these CW-complexes.

3.3.1. Quadrangulations of the plane. – For $k, l \in \mathbb{Z}$, the points of \mathbb{R}^2 given by

$$\mathbf{v}_{kl} = \frac{k}{N}e_1 + \frac{l}{N}e_2,$$

are the elements of the lattice $\Lambda_N \subset \mathbb{R}^2$. The elements of the lattice Λ_N are also the vertices of a nice quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ of the plane \mathbb{R}^2 , pictured as the usual square grid with step N^{-1} . More precisely, the quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ is a particular CW-complex decomposition of \mathbb{R}^2 , characterized by the following properties:

- The edges $\mathbf{e}_{1,kl}$ and $\mathbf{e}_{2,kl}$ of the quadrangulation are the oriented line segments of \mathbb{R}^2 with oriented boundary

$$\partial\mathbf{e}_{1,kl} = \mathbf{v}_{k+1,l} - \mathbf{v}_{kl} \text{ and } \partial\mathbf{e}_{2,kl} = \mathbf{v}_{k,l+1} - \mathbf{v}_{kl}.$$

- The faces \mathbf{f}_{kl} of the quadrangulation are oriented squares of \mathbb{R}^2 with oriented boundary

$$\partial\mathbf{f}_{kl} = \mathbf{e}_{1,k,l} + \mathbf{e}_{2,k+1,l} - \mathbf{e}_{1,k,l+1} - \mathbf{e}_{2,kl}.$$

Figure 2 shows the familiar picture of the plane tiled by squares together with the notations introduced above.

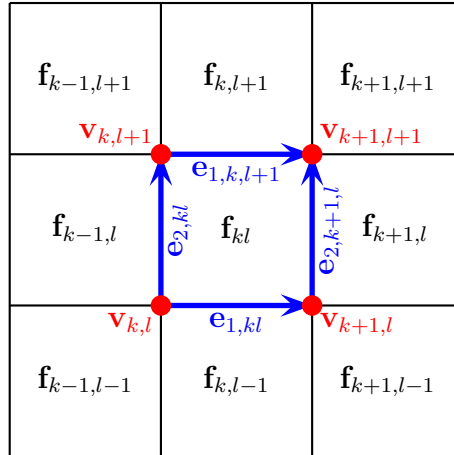


FIGURE 2. Quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$

3.3.2. Quadrangulations of the torus. – The lattice Λ_N acts on itself, by translation. It follows that Λ_N also acts naturally on the vertices, on the edges and on the faces of the quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ by translation. Since $\Gamma_N \subset \Lambda_N^{\text{ch}} \subset \Lambda_N$, the lattice Γ_N acts on $\mathcal{Q}_N(\mathbb{R}^2)$ as well. Thus, the quadrangulation descends to a quadrangulation $\mathcal{Q}_N(\Sigma)$ of the quotient Σ , via the covering map $p_N : \mathbb{R}^2 \rightarrow \Sigma$. When this is clear from the context, the vertices, edges and faces of $\mathcal{Q}_N(\Sigma)$ will still be denoted \mathbf{v}_{kl} , $\mathbf{e}_{1,kl}$, $\mathbf{e}_{2,kl}$ and \mathbf{f}_{kl} .

3.3.3. Alternate quadrangulation of the plane. – Our construction involves the various diffeomorphisms $\Phi : \mathbb{R}^2/\Gamma \rightarrow \Sigma$ and $\Phi_N : \mathbb{R}^2/\Gamma_N \rightarrow \Sigma$. For the purpose of analysis and, more specifically, the notion of convergence, it is convenient to identify Σ with a single reference quotient, say \mathbb{R}^2/Γ using Φ .

Lifting $\mathcal{Q}(\Sigma)$ via the covering map $p : \mathbb{R}^2 \rightarrow \Sigma$ provides a quadrangulation different from $\mathcal{Q}_N(\mathbb{R}^2)$. We denote by $\hat{\mathcal{Q}}_N(\mathbb{R}^2)$ the quadrangulation obtained as the image of $\mathcal{Q}_N(\mathbb{R}^2)$ by the isomorphism $U_N : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. We also denote by $\hat{\Lambda}_N$ and $\hat{\Lambda}_N^{\text{ch}}$ the images of Λ_N and Λ_N^{ch} by U_N . By definition, Γ is a sublattice of $\hat{\Lambda}_N^{\text{ch}}$ and we have a sequence of canonical inclusions

$$\Gamma \subset \hat{\Lambda}_N^{\text{ch}} \subset \hat{\Lambda}_N,$$

which is nothing else but the image of the inclusions

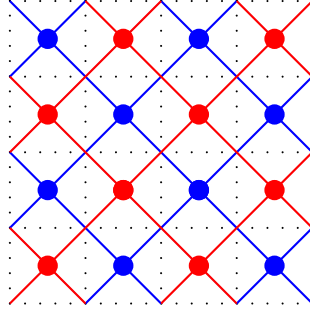
$$\Gamma_N \subset \Lambda_N^{\text{ch}} \subset \Lambda_N$$

by U_N . By construction, the quadrangulation $\hat{\mathcal{Q}}_N(\mathbb{R}^2)$ has vertices given by the elements of the lattice $\hat{\Lambda}_N$. Furthermore $\hat{\mathcal{Q}}_N(\mathbb{R}^2)$ descends to the quotient via the covering map $p : \mathbb{R}^2 \rightarrow \Sigma$ into a quadrangulation that coincides with $\mathcal{Q}_N(\Sigma)$.

3.4. Checkers graph

We associate a graph $\mathcal{G}_N(\mathbb{R}^2)$ to the quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$, called the *checkers graph* of $\mathcal{Q}_N(\mathbb{R}^2)$. Combinatorially, the vertices \mathbf{z}_{kl} of $\mathcal{G}_N(\mathbb{R}^2)$ correspond to faces \mathbf{f}_{kl} of $\mathcal{Q}_N(\mathbb{R}^2)$. However a vertex \mathbf{z}_{kl} of the graph $\mathcal{G}_N(\mathbb{R}^2)$ shall be thought of as the barycenter of the face \mathbf{f}_{kl} of $\mathcal{Q}_N(\mathbb{R}^2)$, understood as a square of \mathbb{R}^2 . The fact that vertices of the graph correspond to points in \mathbb{R}^2 will be most helpful for defining the notion of convergence at § 3.8. Two barycenters are connected by an edge if, and only if, they belong to faces having exactly one vertex in common. For instance the faces \mathbf{f}_{kl} and $\mathbf{f}_{k+1,l+1}$ of $\mathcal{Q}_N(\mathbb{R}^2)$ have exactly one vertex in common. An edge between two vertices of $\mathcal{G}_N(\mathbb{R}^2)$ is the segment of straight line of \mathbb{R}^2 between the two vertices.

Figure 3 shows the quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ using dashed lines and the corresponding checkers graph $\mathcal{G}_N(\mathbb{R}^2)$. The graph has two connected components painted with colors red and blue. The bullets correspond to vertices of the graph.

FIGURE 3. Graph $\mathcal{G}_N(\mathbb{R}^2)$

3.4.1. Splitting of the checkers graph. – The graph $\mathcal{G}_N(\mathbb{R}^2)$ splits into two connected components denoted

$$\mathcal{G}_N(\mathbb{R}^2) = \mathcal{G}_N^+(\mathbb{R}^2) \cup \mathcal{G}_N^-(\mathbb{R}^2),$$

where $\mathcal{G}_N^+(\mathbb{R}^2)$ contains the vertex \mathbf{z}_{00} corresponding to the face \mathbf{f}_{00} , by convention.

The lattice Λ_N acts by translation on $\mathcal{Q}_N(\mathbb{R}^2)$ and on $\mathcal{G}_N(\mathbb{R}^2)$. The action on the vertices of $\mathcal{G}_N(\mathbb{R}^2)$ (or, equivalently the faces of $\mathcal{Q}_N(\mathbb{R}^2)$) is transitive. However the sublattice Λ_N^{ch} does not act transitively: in fact it preserves the connected components of the graph $\mathcal{G}_N(\mathbb{R}^2)$ and acts transitively on each component. The quotient $\Lambda_N/\Lambda_N^{\text{ch}} \simeq \mathbb{Z}_2$ is the residual action of the lattice Λ_N on the connected components of $\mathcal{G}_N(\mathbb{R}^2)$.

3.4.2. Checkers graph of the quotient. – By construction $\Gamma_N \subset \Lambda_N^{\text{ch}}$, so that the action of Λ_N preserves the connected components of $\mathcal{G}_N(\mathbb{R}^2)$. It follows that the graphs $\mathcal{G}_N(\mathbb{R}^2)$, $\mathcal{G}_N^+(\mathbb{R}^2)$ and $\mathcal{G}_N^-(\mathbb{R}^2)$ descend as graphs $\mathcal{G}_N(\Sigma)$, $\mathcal{G}_N^+(\Sigma)$ and $\mathcal{G}_N^-(\Sigma)$ on the quotient $\Sigma \simeq \mathbb{R}^2/\Gamma_N$ via the covering map $p_N : \mathbb{R}^2 \rightarrow \Sigma$. Furthermore, the graph $\mathcal{G}_N(\Sigma)$ splits into two connected components $\mathcal{G}_N^+(\Sigma)$ and $\mathcal{G}_N^-(\Sigma)$:

$$\mathcal{G}_N(\Sigma) = \mathcal{G}_N^+(\Sigma) \cup \mathcal{G}_N^-(\Sigma).$$

3.4.3. Alternate checkers graph on the plane. – A discussion similar to the case of the quadrangulations $\mathcal{Q}_N(\mathbb{R}^2)$ and $\hat{\mathcal{Q}}_N(\mathbb{R}^2)$ occurs here (cf. §3.3.3). We introduce the checkers graphs $\hat{\mathcal{G}}_N(\mathbb{R}^2)$, $\hat{\mathcal{G}}_N^+(\mathbb{R}^2)$ and $\hat{\mathcal{G}}_N^-(\mathbb{R}^2)$ obtained as the image of $\mathcal{G}_N(\mathbb{R}^2)$, $\mathcal{G}_N^+(\mathbb{R}^2)$ and $\mathcal{G}_N^-(\mathbb{R}^2)$ by U_N . Similarly to the non-hat version, these graphs can be also understood as the checkers graphs of $\hat{\mathcal{Q}}_N(\mathbb{R}^2)$. They descend via the covering map $p : \mathbb{R}^2 \rightarrow \Sigma$ where we recover $\mathcal{G}_N(\Sigma)$, $\mathcal{G}_N^+(\Sigma)$ and $\mathcal{G}_N^-(\Sigma)$. If the vertices of the checkers graph $\mathcal{G}_N(\mathbb{R}^2)$ are the barycenters \mathbf{z}_{kl} of the faces f_{kl} , their images by U_N , denoted $\hat{\mathbf{z}}_{kl}$, are the vertices of $\hat{\mathcal{G}}_N(\mathbb{R}^2)$.

3.5. Examples

The lattices of \mathbb{R}^2 defined at § 3.2 come with canonical inclusions

$$\Lambda_1 \subset \cdots \Lambda_N \subset \Lambda_{N+1} \subset \cdots$$

and

$$\Lambda_1^{\text{ch}} \subset \cdots \Lambda_N^{\text{ch}} \subset \Lambda_{N+1}^{\text{ch}} \subset \cdots .$$

If Γ is a sublattice of Λ_1^{ch} , then its approximations Γ_N coincide with Γ , which makes the construction of $\mathcal{Q}_N(\Sigma)$ somewhat simpler. For example, we may consider the lattice

$$\Gamma' = \mathbb{Z}(e_1 + e_2) \oplus \mathbb{Z}(e_2 - e_1) \subset \Lambda_1^{\text{ch}},$$

or the lattice

$$\Gamma'' = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 = \Lambda_1.$$

In the latter case, $\Gamma'' \subset \Lambda_N^{\text{ch}}$ if and only if N is even and we shall only consider $\mathcal{Q}_N(\Sigma'')$ when N is even. The quotients $\Sigma' = \mathbb{R}^2/\Gamma'$ and $\Sigma'' = \mathbb{R}^2/\Gamma''$ are conformally isomorphic but the quadrangulations $\mathcal{Q}_N(\Sigma')$ and $\mathcal{Q}_N(\Sigma'')$ are not isomorphic through a conformal mapping.

Let $\ell_1 : \mathbb{S}^1 \rightarrow \mathbb{C}$ and $\ell_2 : \mathbb{S}^1 \rightarrow \mathbb{C}$ be two smooth embeddings of the circle into the complex plane \mathbb{C} . This provides an embedding of the torus

$$\begin{aligned} \ell : \mathbb{S}^1 \times \mathbb{S}^1 &\longrightarrow \mathbb{C}^2 \simeq \mathbb{R}^4 \\ (\varphi_1, \varphi_2) &\longmapsto (\ell_1(\varphi_1), \ell_2(\varphi_2)), \end{aligned}$$

which is isotropic since both maps ℓ_i are. The image of ℓ is usually called a *product Lagrangian torus of \mathbb{R}^4* .

The map ℓ can be approximated by a piecewise linear maps. The idea is to approximate the two embedded circles by polygons of \mathbb{C} . We obtain a product of two polygons approximating the product torus. More precisely, we define

$$\ell_i^N : (N^{-1}\mathbb{Z})/\mathbb{Z} \rightarrow \mathbb{C}$$

by $\ell_i^N(v) = \ell_i(v)$. The map ℓ_i^N can be extended as a piecewise linear map denoted

$$\ell_i^N : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{C}$$

as well. If N is sufficiently large, the maps ℓ_i^N are piecewise linear embeddings. For the same reasons as before, the product embedding

$$\ell_N : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^4$$

defined by $\ell_N(\varphi_1, \varphi_2) = (\ell_1^N(\varphi_1), \ell_2^N(\varphi_2))$ is isotropic and it is a piecewise linear isotropic approximation of ℓ . Notice that the maps ℓ_N can be recovered only from the \mathbb{R}^4 -coordinates of the vertices of the points in Λ_N/\mathbb{Z}^2 . These vertices are by definition the vertices of the quadrangulation $\mathcal{Q}_N(\Sigma'')$, modulo the isomorphism

$$\mathbb{S}^1 \times \mathbb{S}^1 \simeq \mathbb{R}^2/\Gamma'' = \Sigma'',$$

where $\Gamma'' = \Lambda_1$ is the standard lattice described above. Notice that each face of the quadrangulation is mapped to a quadrilateral of \mathbb{R}^4 contained in a Lagrangian plane.

REMARK 3.5.1. – The piecewise linear isotropic embeddings ℓ_N of the torus described above were essentially the only examples known at the beginning of this research project. If ℓ is any smooth isotropic map, one can construct samples ℓ_N as above (cf. § 4.3). Strictly speaking, these samples are quadrangular meshes. In general these samples are not exactly isotropic. From this point of view, the product examples described above are very special, because in this case the samples are isotropic. In general, one needs a suitable perturbation theory so that they become isotropic, which is the technical task of this paper.

3.6. A splitting for discrete functions

The vector space of discrete functions on the faces of the quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ is denoted $C^2(\mathcal{Q}_N(\mathbb{R}^2))$. A discrete function ϕ is defined by its values on faces given by

$$\phi_{kl} = \phi(\mathbf{f}_{kl}) = \langle \phi, \mathbf{f}_{kl} \rangle.$$

In the above notation, $\langle \cdot, \cdot \rangle$ is the duality bracket, and a discrete function is understood as a linear form on the vector space $C_2(\mathcal{Q}_N(\mathbb{R}^2))$ spanned by the faces of the quadrangulation.

By construction there is a canonical identification between the faces \mathbf{f}_{kl} of $\mathcal{Q}_N(\mathbb{R}^2)$ and the vertices of the graph \mathbf{z}_{kl} of $\mathcal{G}_N(\mathbb{R}^2)$. Therefore, a discrete function ϕ can be understood, either as a function on faces \mathbf{f}_{kl} of $\mathcal{Q}_N(\mathbb{R}^2)$, or as a function on vertices \mathbf{z}_{kl} of $\mathcal{G}_N(\mathbb{R}^2)$. The above identification leads to an isomorphism of discrete functions

$$(3.6) \quad C^2(\mathcal{Q}_N(\mathbb{R}^2)) \simeq C^0(\mathcal{G}_N(\mathbb{R}^2)) = C^0(\mathcal{G}_N^+(\mathbb{R}^2)) \oplus C^0(\mathcal{G}_N^-(\mathbb{R}^2)).$$

The same decomposition holds for the *hat* version of these objects and we have a canonical isomorphism

$$(3.7) \quad C^2(\hat{\mathcal{Q}}_N(\mathbb{R}^2)) \simeq C^0(\hat{\mathcal{G}}_N(\mathbb{R}^2)) = C^0(\hat{\mathcal{G}}_N^+(\mathbb{R}^2)) \oplus C^0(\hat{\mathcal{G}}_N^-(\mathbb{R}^2)).$$

The isomorphism (3.6) descends to the quotient Σ via p_N and may be expressed as an isomorphism

$$(3.8) \quad C^2(\mathcal{Q}_N(\Sigma)) \simeq C^0(\mathcal{G}_N(\Sigma)) = C^0(\mathcal{G}_N^+(\Sigma)) \oplus C^0(\mathcal{G}_N^-(\Sigma)).$$

Any discrete function ϕ , in one of the three kind of spaces $C^0(\mathcal{G}_N(\cdot))$ as above, admits a unique decomposition according to the splittings (3.6), (3.7) or (3.8)

$$\phi = \phi^+ + \phi^-,$$

where $\phi^\pm \in C^0(\mathcal{G}_N^\pm(\cdot))$.

The induced splitting of $C^2(\mathcal{Q}_N(\cdot))$ via the isomorphisms (3.6), (3.7) or (3.8) is also denoted

$$(3.9) \quad C^2(\mathcal{Q}_N(\cdot)) = C_+^2(\mathcal{Q}_N(\cdot)) \oplus C_-^2(\mathcal{Q}_N(\cdot)).$$

When the discrete function ϕ is regarded as a constant function on faces of the quadrangulation, we also write $\phi = \phi^+ + \phi^-$ according to the above splitting.

CONVENTION 3.6.1. – *In the sequel we shall use a shorthand in order to make statements that hold either for cocycles of the graph $\mathcal{G}_N^+(\Sigma)$, or for cocycles of the graph $\mathcal{G}_N^-(\Sigma)$. For this purpose, we will use the notation $\mathcal{G}_N^\pm(\Sigma)$ and the convention below:*

For every statement using the symbols \pm and \mp , the reader should either

- *replace all symbols \pm (resp \mp) consistently with $+$ (resp. $-$), or*
- *replace all symbols \pm (resp \mp) consistently with $-$ (resp. $+$).*

3.7. Discrete Hölder norms

In this section we define particular norms on the space $C^2(\mathcal{Q}_N(\mathbb{R}^2))$ (or equivalently, on the space $C^0(\mathcal{G}_N(\mathbb{R}^2))$, which is a discrete analog of the Hölder norm. The norms are defined first on each component of the splitting (3.9) (or (3.6)).

3.7.1. \mathcal{C}^0 -norm. – Given $\phi \in C^0(\mathcal{G}_N^+(\mathbb{R}^2))$ we define its \mathcal{C}^0 -norm by

$$(3.10) \quad \|\phi\|_{\mathcal{C}^0} = \sup_{\mathbf{z} \in \mathfrak{C}_0(\mathcal{G}_N^+(\mathbb{R}^2))} |\langle \phi, \mathbf{z} \rangle|.$$

We define a similar norm on $C^0(\mathcal{G}_N^-(\mathbb{R}^2))$ (resp. $C^0(\mathcal{G}_N(\mathbb{R}^2))$) by taking the sup on vertices of $\mathcal{G}_N^-(\mathbb{R}^2)$ (resp. $\mathcal{G}_N(\mathbb{R}^2)$). We deduce a norm, with the same notation $\|\cdot\|_{\mathcal{C}^0}$ on $C^2_\pm(\mathcal{Q}_N(\mathbb{R}^2))$ via the isomorphisms (3.6) and (3.9). These quantities may be infinite. Later we shall restrict to periodic functions, which are bounded and have a well defined \mathcal{C}^0 -norm.

3.7.2. Finite differences. – The canonical basis (e_1, e_2) with canonical coordinates (x, y) of \mathbb{R}^2 is not the best for our situation. Most of the times, we shall rotate the plane \mathbb{R}^2 by an angle $\pi/4$. For this purpose we introduce the rotated orthonormal basis (e'_1, e'_2) of \mathbb{R}^2 given by

$$(3.11) \quad e'_1 = \frac{e_1 + e_2}{\sqrt{2}}, \quad e'_2 = \frac{e_2 - e_1}{\sqrt{2}}.$$

The coordinates (u, v) with respect to the basis (e'_1, e'_2) are deduced from the canonical coordinates $(x, y) \in \mathbb{R}^2$ by the formula

$$(3.12) \quad u = \frac{x + y}{\sqrt{2}}, \quad v = \frac{y - x}{\sqrt{2}}.$$

We define finite differences of $\phi \in C^0(\mathcal{G}_N^\pm(\mathbb{R}^2))$, which are discrete analogs of the partial derivatives of a function on \mathbb{R}^2 , with respect to u or v . These differences are denoted

$$\frac{\partial \phi}{\partial u}, \quad \frac{\partial \phi}{\partial \bar{u}}, \quad \frac{\partial \phi}{\partial v} \quad \text{and} \quad \frac{\partial \phi}{\partial \bar{v}} \in C^0(\mathcal{G}_N^\pm(\mathbb{R}^2)),$$

where the forward or retrograde arrows indicate forward or retrograde differences, defined as follows: for $\phi \in C^0(\mathcal{G}^+(\mathbb{R}^2))$, we write $\phi_{kl} = \langle \phi, \mathbf{z}_{kl} \rangle$ for $\mathbf{z}_{kl} \in \mathfrak{C}_0(\mathcal{G}_N^+(\mathbb{R}^2))$ and put

$$(3.13) \quad \left\langle \frac{\partial \phi}{\partial \vec{u}}, \mathbf{z}_{kl} \right\rangle = \frac{N}{\sqrt{2}}(\phi_{kl} - \phi_{k-1, l-1}),$$

$$(3.14) \quad \left\langle \frac{\partial \phi}{\partial \vec{u}}, \mathbf{z}_{kl} \right\rangle = \frac{N}{\sqrt{2}}(\phi_{k+1, l+1} - \phi_{kl}),$$

$$(3.15) \quad \left\langle \frac{\partial \phi}{\partial \vec{v}}, \mathbf{z}_{kl} \right\rangle = \frac{N}{\sqrt{2}}(\phi_{kl} - \phi_{k+1, l-1}) \quad \text{and}$$

$$(3.16) \quad \left\langle \frac{\partial \phi}{\partial \vec{v}}, \mathbf{z}_{kl} \right\rangle = \frac{N}{\sqrt{2}}(\phi_{k-1, l+1} - \phi_{kl}).$$

The finite differences are defined with the same formulae if $\phi \in C^0(\mathcal{G}^-(\mathbb{R}^2))$. Since all the indices involved in the above formulae correspond to vertices in connected component of \mathbf{z}_{kl} in $\mathcal{G}_N(\mathbb{R}^2)$, the finite differences $\frac{\partial}{\partial \vec{u}}$, $\frac{\partial}{\partial \vec{u}}$, $\frac{\partial}{\partial \vec{v}}$ and $\frac{\partial}{\partial \vec{v}}$ define endomorphisms

$$C^0(\mathcal{G}_N^+(\mathbb{R}^2)) \oplus C^0(\mathcal{G}_N^-(\mathbb{R}^2)) \longrightarrow C^0(\mathcal{G}_N^+(\mathbb{R}^2)) \oplus C^0(\mathcal{G}_N^-(\mathbb{R}^2)),$$

which respect the above splitting.

Finite differences can also be expressed using the translations of Λ_N^{ch} acting on functions. If T_u, T_v are the translations acting on $\mathcal{G}_N(\mathbb{R}^2)$, given respectively by the vectors $\frac{e_1 + e_2}{N}$ and $\frac{e_2 - e_1}{N}$, then

$$(3.17) \quad \frac{\partial \phi}{\partial \vec{u}} = \frac{N}{\sqrt{2}}(\phi \circ T_u - \phi), \quad \frac{\partial \phi}{\partial \vec{u}} = \frac{N}{\sqrt{2}}(\phi - \phi \circ T_u^{-1})$$

and

$$(3.18) \quad \frac{\partial \phi}{\partial \vec{v}} = \frac{N}{\sqrt{2}}(\phi \circ T_v - \phi), \quad \frac{\partial \phi}{\partial \vec{v}} = \frac{N}{\sqrt{2}}(\phi - \phi \circ T_v^{-1}).$$

As an immediate consequence of (3.17), we have

$$(3.19) \quad \frac{\partial \phi}{\partial \vec{u}} = \frac{\partial \phi}{\partial \vec{u}} \circ T_u,$$

so that the functions $\frac{\partial \phi}{\partial \vec{u}}$ and $\frac{\partial \phi}{\partial \vec{u}}$ have the same \mathcal{C}^0 -norm. The same holds for the v -coordinate since by (3.18)

$$(3.20) \quad \frac{\partial \phi}{\partial \vec{v}} = \frac{\partial \phi}{\partial \vec{v}} \circ T_v,$$

so that finite differences $\frac{\partial \phi}{\partial \vec{u}}$ and $\frac{\partial \phi}{\partial \vec{v}}$ have the same \mathcal{C}^0 -norm.

NOTATION 3.7.3. – *As far as we are concerned with the \mathcal{C}^0 -norms of finite differences, we could drop the arrow notation over u or v , since the forward of retrograde differences have the same norms.*

3.7.4. Definition of Hölder norms. – For $\phi \in C^0(\mathcal{G}_N^+(\mathbb{R}^2))$, we may define its \mathcal{C}^1 -norm as

$$\|\phi\|_{\mathcal{C}^1} = \|\phi\|_{\mathcal{C}^0} + \left\| \frac{\partial \phi}{\partial u} \right\|_{\mathcal{C}^0} + \left\| \frac{\partial \phi}{\partial v} \right\|_{\mathcal{C}^0}$$

and its \mathcal{C}^2 -norm by

$$\|\phi\|_{\mathcal{C}^2} = \|\phi\|_{\mathcal{C}^1} + \left\| \frac{\partial^2 \phi}{\partial u^2} \right\|_{\mathcal{C}^0} + \left\| \frac{\partial^2 \phi}{\partial v^2} \right\|_{\mathcal{C}^0} + \left\| \frac{\partial^2 \phi}{\partial u \partial v} \right\|_{\mathcal{C}^0}.$$

More generally we can define a \mathcal{C}^k -norm on $C^0(\mathcal{G}^+(\mathbb{R}^2))$ by induction. Similarly we define a \mathcal{C}^k -norm on $C^0(\mathcal{G}^-(\mathbb{R}^2))$.

For a positive Hölder constant $\alpha \in (0, 1)$, we define the $\mathcal{C}^{0,\alpha}$ -Hölder norm of $\phi \in C^0(\mathcal{G}^+(\mathbb{R}^2))$ by

$$(3.21) \quad \|\phi\|_{\mathcal{C}^{0,\alpha}} = \|\phi\|_{\mathcal{C}^0} + \sup_{\substack{\mathbf{z}_{kl}, \mathbf{z}_{mn} \in \mathcal{C}_0(\mathcal{G}_N^+(\mathbb{R}^2))_q \\ \mathbf{z}_{kl} \neq \mathbf{z}_{mn}}} \frac{|\phi_{kl} - \phi_{mn}|}{\|\mathbf{z}_{kl} - \mathbf{z}_{mn}\|^\alpha},$$

where $\|\mathbf{z}_{kl} - \mathbf{z}_{mn}\|$ is the Euclidean distance between \mathbf{z}_{kl} and \mathbf{z}_{mn} in \mathbb{R}^2 . The $\mathcal{C}^{1,\alpha}$ -Hölder norm of $\phi \in C^0(\mathcal{G}^+(\mathbb{R}^2))$ is defined by

$$\|\phi\|_{\mathcal{C}^{1,\alpha}} = \|\phi\|_{\mathcal{C}^0} + \left\| \frac{\partial \phi}{\partial u} \right\|_{\mathcal{C}^{0,\alpha}} + \left\| \frac{\partial \phi}{\partial v} \right\|_{\mathcal{C}^{0,\alpha}},$$

and its $\mathcal{C}^{2,\alpha}$ -Hölder norm is defined by

$$\|\phi\|_{\mathcal{C}^{2,\alpha}} = \|\phi\|_{\mathcal{C}^1} + \left\| \frac{\partial^2 \phi}{\partial u^2} \right\|_{\mathcal{C}^{0,\alpha}} + \left\| \frac{\partial^2 \phi}{\partial v^2} \right\|_{\mathcal{C}^{0,\alpha}} + \left\| \frac{\partial^2 \phi}{\partial u \partial v} \right\|_{\mathcal{C}^{0,\alpha}}.$$

More generally, we can define a $\mathcal{C}^{k,\alpha}$ -Hölder norm by induction on $C^0(\mathcal{G}^+(\mathbb{R}^2))$, in an obvious way. We define a \mathcal{C}^k and a $\mathcal{C}^{k,\alpha}$ -Hölder norm on $C^0(\mathcal{G}^-(\mathbb{R}^2))$ by taking the sup of the above formulae on vertices of $\mathcal{G}_N^-(\mathbb{R}^2)$ instead.

3.7.5. Weak Hölder norms. – For $\phi \in C^2(\mathcal{Q}_N(\mathbb{R}^2)) \simeq C^0(\mathcal{G}_N(\mathbb{R}^2))$ we use the direct sum decomposition $\phi = \phi^+ + \phi^-$ of (3.6) or (3.9). We define the *weak* $\mathcal{C}_w^{k,\alpha}$ -norm of ϕ by

$$\|\phi\|_{\mathcal{C}_w^{k,\alpha}} = \|\phi^+\|_{\mathcal{C}^{k,\alpha}} + \|\phi^-\|_{\mathcal{C}^{k,\alpha}},$$

where the Hölder norms of each components ϕ^\pm are defined in the previous section. Similarly, the *weak* \mathcal{C}_w^k -norm of ϕ is defined by

$$\|\phi\|_{\mathcal{C}_w^k} = \|\phi^+\|_{\mathcal{C}^k} + \|\phi^-\|_{\mathcal{C}^k}.$$

REMARK 3.7.6. – As you may have noticed, the discrete $\mathcal{C}_w^{k,\alpha}$ -Hölder norms or \mathcal{C}_w^k -norms defined above on $C^0(\mathcal{G}_N(\mathbb{R}^2))$ are called *weak*. Indeed, only the variations of ϕ in the *diagonal directions* spanned by the vectors $\frac{e_1 + e_2}{2}$ and $\frac{e_2 - e_1}{2}$ are taken into account. It turns out that these weak norms are the one appropriate to set up the fixed point principle, as explained in § 6.

In the sequel, we shall drop the term *weak* for the sake of brevity. However, the reader should bear in mind that these norms may allow some unexpected behavior when N goes to infinity (cf. Example 3.8.4).

3.7.7. Quotient and alternate quadrangulations. – The alternate versions of the quadrangulation $\hat{\mathcal{Q}}_N(\mathbb{R}^2)$ and checkers graph $\hat{\mathcal{G}}_N(\mathbb{R}^2)$ are canonically isomorphic to the non-hat versions $\mathcal{Q}_N(\mathbb{R}^2)$ and $\mathcal{G}_N(\mathbb{R}^2)$. Thus, we have an isomorphism

$$C^2(\mathcal{Q}_N(\mathbb{R}^2)) \simeq C^2(\hat{\mathcal{Q}}_N(\mathbb{R}^2)).$$

This isomorphism allows to define \mathcal{C}_w^k and $\mathcal{C}_w^{k,\alpha}$ -norms on $C^2(\hat{\mathcal{Q}}_N(\mathbb{R}^2))$. A function $\phi \in C^2(\mathcal{Q}_N(\Sigma))$ admits a lift $\phi_N = \phi \circ p_N \in C^2(\mathcal{Q}_N(\mathbb{R}^2))$. We define the norms of ϕ as the norms of its lift:

$$\|\phi\|_{\mathcal{C}_w^{k,\alpha}} = \|\phi_N\|_{\mathcal{C}_w^{k,\alpha}}, \quad \|\phi\|_{\mathcal{C}_w^k} = \|\phi_N\|_{\mathcal{C}_w^k}.$$

REMARK 3.7.8. – The discrete functions on Σ have finite Hölder norm since they are bounded, and so are their finite differences.

3.8. Convergence of discrete functions

In this section, a suitable notion of convergence for a sequence of discrete functions is introduced. This concept will be the cornerstone of a version of the Ascoli-Arzelà compactness theorem. It will be an essential tool to obtain spectral gap results at § 5.5.

3.8.1. Definition of converging sequences

DEFINITION 3.8.2. – Let $(N_k)_{k \in \mathbb{N}}$ be an increasing sequence of positive integers. Let $\psi_{N_k} \in C^0(\hat{\mathcal{G}}_{N_k}^\pm(\mathbb{R}^2))$ be a sequence of discrete functions and $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function defined on the plane.

Assume that for every point $w \in \mathbb{R}^2$ and $\varepsilon > 0$, there exists $\delta > 0$ and an integer $k_0 > 0$, such that for every integer $k \geq k_0$ and vertex $\mathbf{z} \in \mathfrak{C}_0(\hat{\mathcal{G}}_{N_k}^\pm(\mathbb{R}^2))$ with the property that $\|w - \mathbf{z}\| \leq \delta$, we have $|\phi(w) - \psi_{N_k}(\mathbf{z})| \leq \varepsilon$.

Then we say that the sequence of discrete functions (ψ_{N_k}) converges toward the function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$. This property is denoted by

$$\psi_{N_k} \rightarrow \phi \quad \text{or} \quad \lim \psi_{N_k} = \phi.$$

If $\psi_{N_k} \in C^0(\hat{\mathcal{G}}_{N_k}(\mathbb{R}^2))$ is a sequence of discrete functions with associated decomposition $\psi_{N_k} = \psi_{N_k}^+ + \psi_{N_k}^-$ and with the property that the components converge to functions ϕ^+ and ϕ^- , in the sense of the above definition, i.e.,

$$\psi_{N_k}^+ \rightarrow \phi^+ \quad \text{and} \quad \psi_{N_k}^- \rightarrow \phi^-,$$

we say that ψ_{N_k} converges toward the pair of functions (ϕ^+, ϕ^-) . This property is denoted by

$$\psi_{N_k} \rightarrow (\phi^+, \phi^-) \quad \text{or} \quad \lim \psi_{N_k} = (\phi^+, \phi^-).$$

REMARK 3.8.3. – The above definition may also be stated in a somewhat slicker way: we say that a sequence $\psi_{N_k} \in C^0(\hat{\mathcal{G}}_{N_k}^+(\mathbb{R}^2))$ converges toward a function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ if, at every point w of the plane, ψ_{N_k} takes arbitrarily close values to $\phi(w)$, for every k sufficiently large and for all vertices of $\hat{\mathcal{G}}_{N_k}^+(\mathbb{R}^2)$ in a sufficiently small neighborhood of w .

EXAMPLE 3.8.4. – The splitting of discrete functions into their positive and negative components leads to some unusual type converging sequences in the sense of Definition 3.8.2. For example, we may define a sequence of *discrete comb functions* as follows. We define $\psi_N^\pm \in C^0(\hat{\mathcal{G}}_N^\pm(\mathbb{R}^2))$ as a constant function each connected component of the graph, equal to ± 1 at each vertex of $\mathcal{G}_N^\pm(\mathbb{R}^2)$. Let $\mathbf{1} : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the constant function equal to 1 at every point of the plane. Then $\lim \psi_N^+ = \mathbf{1}$ whereas $\lim \psi_N^- = -\mathbf{1}$. If $\psi_N := \psi_N^+ + \psi_N^-$, then ψ_N converges and

$$\lim \psi_N = (\mathbf{1}, -\mathbf{1}).$$

Typically, the sequence ψ_N is uniformly bounded in weak $\mathcal{C}_w^{0,\alpha}$ -norm. Our notion of convergence is designed to state a version of the Ascoli-Arzelà theorem in this setting.

The notion of convergence of discrete functions is extended to $C^2(\mathcal{Q}_N(\Sigma))$ as follows:

DEFINITION 3.8.5. – Let $\psi_{N_k} \in C^0(\mathcal{G}_{N_k}^\pm(\Sigma))$ be a sequence of discrete functions and $\phi : \Sigma \rightarrow \mathbb{R}$ be a function defined on Σ . Let $\hat{\psi}_{N_k} = \psi_{N_k} \circ p \in C^0(\hat{\mathcal{G}}_{N_k}^\pm(\mathbb{R}^2))$ be the lift of ψ_{N_k} via the canonical projection p and $\hat{\phi} = \phi \circ p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the lift of ϕ . We say that (ψ_{N_k}) converges to ϕ if $(\hat{\psi}_{N_k}) \in C^0(\hat{\mathcal{G}}_{N_k}^\pm(\mathbb{R}^2))$ converges to $\hat{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}$ in the sense Definition 3.8.2. This property is denoted by

$$\psi_N \rightarrow \phi \text{ or } \lim \psi_N = \phi.$$

If $\psi_{N_k} \in C^0(\mathcal{G}_{N_k}(\Sigma))$ is a sequence of discrete functions with associated decomposition $\psi_{N_k} = \psi_{N_k}^+ + \psi_{N_k}^-$ and with the property that both components converge to some functions $\phi^+ : \Sigma \rightarrow \mathbb{R}$ and $\phi^- : \Sigma \rightarrow \mathbb{R}$ in the sense of the above definition, we say that ψ_{N_k} converges toward the pair of functions (ϕ^+, ϕ^-) and denote this by

$$\psi_{N_k} \rightarrow (\phi^+, \phi^-) \text{ or } \lim \psi_{N_k} = (\phi^+, \phi^-).$$

3.8.6. Continuity and limits of discrete functions. – Our notion of convergence for discrete function is intimately related to the uniform convergence, in the case of continuous functions. Indeed, we have the following result:

PROPOSITION 3.8.7. – Let $\psi_{N_k} \in C^0(\hat{\mathcal{G}}_{N_k}^\pm(\mathbb{R}^2))$ be a sequence of discrete functions converging toward $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$. Then ϕ must be continuous.

Proof. – The proof goes by contradiction: assume that $\psi_{N_k} \in C^0(\hat{\mathcal{G}}_N^\pm(\mathbb{R}^2))$ is a sequence converging toward a discontinuous function ϕ . Then there exists $\varepsilon_0 > 0$, $w \in \mathbb{R}^2$ and a sequence of points $w_k \in \mathbb{R}^2$ such that $\lim_{k \rightarrow \infty} w_k = w$ and $|\phi(w_k) - \phi(w)| \geq \varepsilon_0$ for all k .

From the definition of convergence of discrete functions, we can extract a sequence N'_k from N_k and vertices \mathbf{z}_k of $\hat{\mathcal{G}}_{N'_k}^\pm(\mathbb{R}^2)$ such that $|w_k - \mathbf{z}_k| \rightarrow 0$ and $|\psi_{N'_k}(\mathbf{z}_k) - \phi(w_k)| \rightarrow 0$ as $k \rightarrow \infty$.

By construction $\lim \mathbf{z}_k = w$. Furthermore

$$|\phi(w_k) - \phi(w)| \leq |\phi(w_k) - \psi_{N'_k}(\mathbf{z}_k)| + |\psi_{N'_k}(\mathbf{z}_k) - \phi(w)|.$$

The LHS is bounded below by $\varepsilon_0 > 0$. The first term of the RHS converges to 0 by definition of the sequences. The second term of the RHS converges to zero, by definition of the convergence of a sequence of discrete functions. This is a contradiction, hence $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous. \square

COROLLARY 3.8.8. – *Let $\psi_{N_k} \in C^0(\mathcal{G}_N^\pm(\Sigma))$ be a sequence of discrete functions converging toward $\phi : \Sigma \rightarrow \mathbb{R}$. Then ϕ is continuous.*

Proof. – We use the covering map $p : \mathbb{R}^2 \rightarrow \Sigma$ and apply Proposition 3.8.7 to the lift of the functions. \square

3.8.9. Samples and convergence of discrete functions

DEFINITION 3.8.10. – *If $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is any real function, we define its samples $\phi_N^\pm \in C^0(\hat{\mathcal{G}}_N^\pm(\mathbb{R}^2))$ by*

$$\langle \phi_N^\pm, \hat{\mathbf{z}}_{kl} \rangle := \phi(\hat{\mathbf{z}}_{kl}),$$

for every $\hat{\mathbf{z}}_{kl} \in \mathfrak{C}_0(\hat{\mathcal{G}}_N^\pm(\mathbb{R}^2))$. We define similarly the samples $\phi_N^\pm \in C^0(\mathcal{G}_N^\pm(\Sigma))$ of a real function $\phi : \Sigma \rightarrow \mathbb{R}$. Let $\hat{\phi} = \phi \circ p : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the lift of ϕ via the projection p . Its samples $\hat{\phi}_N^\pm \in C^0(\hat{\mathcal{G}}_N^\pm(\mathbb{R}^2))$, as defined above, descend to discrete functions $\phi_N^\pm \in C^0(\mathcal{G}_N^\pm(\Sigma))$ on the quotient, referred to as the samples of ϕ .

The convergence defined in Definition 3.8.2 is uniform in the sense of the following lemma:

PROPOSITION 3.8.11. – *Let $\psi_{N_k}^\pm \in C^0(\mathcal{G}_{N_k}^\pm(\Sigma))$ be a sequence of discrete functions converging to $\phi : \Sigma \rightarrow \mathbb{R}$ and $\phi_{N_k}^\pm \in C^0(\mathcal{G}_{N_k}^\pm(\Sigma))$ be the samples of ϕ . Then*

$$\lim_{k \rightarrow \infty} \|\phi_{N_k}^\pm - \psi_{N_k}^\pm\|_{C^0} = 0.$$

Proof. – Since $\phi : \Sigma \rightarrow \mathbb{R}$ is a limit of a sequence of discrete functions, it is continuous by Corollary 3.8.8. The surface Σ is compact, hence ϕ is uniformly continuous by Heine theorem. We denote by $\hat{\psi}_{N_k}^\pm$ and $\hat{\phi}$ the canonical lifts of $\psi_{N_k}^\pm \in C^0(\hat{\mathcal{G}}_N^\pm(\mathbb{R}^2))$ and ϕ via the projection $p : \mathbb{R}^2 \rightarrow \Sigma$. Since ϕ is uniformly continuous, so is $\hat{\phi}$. Let ε , be a

positive real number. By uniform continuity, there exists $\delta > 0$ such that for every $w, w' \in \mathbb{R}^2$

$$(3.22) \quad \|w - w'\| \leq \delta \Rightarrow |\hat{\phi}(w) - \hat{\phi}(w')| \leq \varepsilon.$$

By definition of the convergence of discrete functions, for each $w \in \mathbb{R}^2$, we may choose an integer $k(w) \geq 0$ and a real number $\eta(w) > 0$ such that for all $k \geq k(w)$ and $\hat{\mathbf{z}} \in \mathfrak{C}_0(\hat{\mathcal{G}}_{N_k}^\pm(\mathbb{R}^2))$ we have

$$(3.23) \quad \|\hat{\mathbf{z}} - w\| \leq \eta(w) \Rightarrow |\hat{\psi}_{N_k}^\pm(\hat{\mathbf{z}}) - \hat{\phi}(w)| \leq \varepsilon.$$

For each $w \in \mathbb{R}^2$, put

$$\delta(w) = \min(\delta, \eta(w)).$$

The family of open Euclidean balls $B(w, \delta(w))$, centered at $w \in \mathbb{R}^2$ with radius $\delta(w)$, provides an open cover of \mathbb{R}^2 . Their images $U_w = p(B(w, \delta(w)))$, by the canonical projection $p : \mathbb{R}^2 \rightarrow \Sigma$, provide an open cover of the compact surface Σ . Hence we can extract a finite cover $U_i = U_{w_i}$ of Σ , for a finite collection of points $\{w_i \in \mathbb{R}^2, 1 \leq i \leq d\}$. We put $k_0 = \max\{k(w_i)_{1 \leq i \leq d}\}$ and consider $k \geq k_0$.

Every $\mathbf{z} \in \mathfrak{C}_0(\mathcal{G}_{N_k}^\pm(\Sigma))$ is an element of one of the open sets U_i . Hence \mathbf{z} admits a lift $\hat{\mathbf{z}} \in \mathfrak{C}_0(\hat{\mathcal{G}}_{N_k}^\pm(\mathbb{R}^2))$ contained in one of the balls $B(w_i, \delta(w_i))$. In particular

$$|\psi_{N_k}^\pm(\mathbf{z}) - \phi_{N_k}^\pm(\mathbf{z})| = |\hat{\psi}_{N_k}^\pm(\hat{\mathbf{z}}) - \hat{\phi}_{N_k}^\pm(\hat{\mathbf{z}})| \leq |\hat{\psi}_{N_k}^\pm(\hat{\mathbf{z}}) - \hat{\phi}(w_i)| + |\hat{\phi}(w_i) - \hat{\phi}_{N_k}^\pm(\hat{\mathbf{z}})|.$$

The first term of the RHS is bounded above by ε by (3.23). By definition $\hat{\phi}_{N_k}^\pm(\hat{\mathbf{z}}) = \hat{\phi}(\hat{\mathbf{z}})$, hence the second term of the RHS is bounded above by ε thanks to (3.22). In conclusion

$$|\psi_{N_k}^\pm(\mathbf{z}) - \phi_{N_k}^\pm(\mathbf{z})| \leq 2\varepsilon,$$

which shows that

$$\|\psi_{N_k}^\pm - \phi_{N_k}^\pm\|_{\mathcal{C}^0} \leq 2\varepsilon,$$

for $k \geq k_0$. □

We also have a sort of converse for Proposition 3.8.11:

PROPOSITION 3.8.12. – *Let $\psi_{N_k}^\pm \in C^0(\mathcal{G}_{N_k}^\pm(\Sigma))$ be a sequence of discrete functions and $\phi : \Sigma \rightarrow \mathbb{R}$ a continuous function such that*

$$\lim_{k \rightarrow \infty} \|\phi_{N_k}^\pm - \psi_{N_k}^\pm\|_{\mathcal{C}^0} = 0,$$

where $\phi_{N_k}^\pm \in C^0(\mathcal{G}_{N_k}^\pm(\Sigma))$ are the samples of ϕ . Then

$$\lim \psi_{N_k}^\pm = \phi.$$

Proof. – The compactness of Σ implies the uniform continuity of ϕ , which is a key argument in a proof closely related to the one of Proposition 3.8.11. The details are left to the interested reader. □

Proposition 3.8.12 has the following immediate corollary, which shows that samples of a function are natural approximations:

COROLLARY 3.8.13. – *Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a continuous function, and $\phi_N^\pm \in C^0(\mathcal{G}_N^\pm(\Sigma))$ its samples. Then*

$$\lim \phi_N^\pm = \phi.$$

3.9. Precompactness

We denote by $\|\cdot\|_{\mathcal{C}^{0,\alpha}}$ the usual Hölder norm on the space of function $\phi : \Sigma \rightarrow \mathbb{R}$, defined with respect to the Riemannian metric g_σ , for instance. The corresponding Hölder space is denoted $\mathcal{C}^{0,\alpha}(\Sigma)$. We may now state a version of the Ascoli-Arzelà theorem adapted to our setting:

THEOREM 3.9.1 (Ascoli-Arzelà, first version). – *Let $\psi_{N_k}^\pm$ be a sequence of discrete function in $C^0(\mathcal{G}_{N_k}^\pm(\Sigma))$, which are uniformly bounded in $\mathcal{C}^{0,\alpha}$ -norm. In other words, there exists a constant $c > 0$ with the property that*

$$\|\psi_{N_k}^\pm\|_{\mathcal{C}^{0,\alpha}} \leq c,$$

for all $k \in \mathbb{N}$. Then there exists a subsequence N'_k of N_k and a function $\phi^\pm : \Sigma \rightarrow \mathbb{R}$ in $\mathcal{C}^{0,\alpha}(\Sigma)$, such that

$$\lim \psi_{N'_k}^\pm = \phi^\pm.$$

Proof. – Let $\psi_{N_k} \in C^0(\mathcal{G}_{N_k}^+(\Sigma))$ be a sequence of discrete functions bounded in Hölder norm, as in the theorem.

We start by choosing a countable dense set $Q = \{q_n \in \Sigma, n \in \mathbb{N}\}$ of Σ ; for instance the projection by $p : \mathbb{R}^2 \rightarrow \Sigma$ of the points of rational coordinates in \mathbb{R}^2 is a possible choice. For each q_n , we choose a lift \hat{q}_n such that $p(\hat{q}_n) = q_n$. For each n we choose a sequence $\hat{\mathbf{z}}_{N_k}^n \in \mathcal{C}_0(\hat{\mathcal{G}}_{N_k}^+(\mathbb{R}^2))$ such that that

$$\lim_{k \rightarrow \infty} \hat{\mathbf{z}}_{N_k}^n = \hat{q}_n.$$

We denote by $\hat{\psi}_{N_k} = \psi_{N_k} \circ p \in C^0(\hat{\mathcal{G}}_{N_k}^+(\mathbb{R}^2))$ the canonical lift of ψ_{N_k} . By assumption, the uniform estimate on the Hölder norms provides a uniform bound $|\hat{\psi}_{N_k}(\hat{\mathbf{z}}_{N_k}^n)| \leq c$. Hence we can choose a subsequence N_k^0 of integers such that $\hat{\psi}_{N_k^0}(\hat{\mathbf{z}}_{N_k^0}^0)$ converges as $k \rightarrow \infty$.

By extracting a subsequence N_k^1 of N_k^0 , we may assume that $\hat{\psi}_{N_k^1}(\hat{\mathbf{z}}_{N_k^1}^1)$ converges for $n = 0$ or 1, as $k \rightarrow \infty$. Extracting subsequences inductively provides family of subsequences N_k^m , indexed by m , such that $\hat{\psi}_{N_k^m}(\hat{\mathbf{z}}_{N_k^m}^n)$ converges for fixed $0 \leq n \leq m$ as $k \rightarrow \infty$. Finally, using the diagonal subsequence $M_k = N_k^k$, we find a subsequence ψ_{M_k} such that $\hat{\psi}_{M_k}(\hat{\mathbf{z}}_{M_k}^n)$ converges for every $n \in \mathbb{N}$, as $k \rightarrow \infty$.

The function

$$\phi : Q \rightarrow \mathbb{R}$$

is defined on the countable dense subset $Q \subset \Sigma$ by

$$\phi(q_n) = \lim_{k \rightarrow \infty} \hat{\psi}_{M_k}(\hat{z}_{M_k}^n).$$

Since the ψ_{N_k} are uniformly bounded with respect to the discrete $\mathcal{C}^{0,\alpha}$ -norms, it follows that the function $\phi : Q \rightarrow \mathbb{R}$ is bounded with respect to the usual $\mathcal{C}^{0,\alpha}$ -norm. In particular ϕ is uniformly continuous on Q , hence it admits a unique continuous extension $\phi : \Sigma \rightarrow \mathbb{R}$ which turns out to be in $\mathcal{C}^{0,\alpha}(\Sigma)$ as well. One can readily check, using the uniform Hölder-norm estimates, that the construction of the function ϕ is independent of the choice of sequences \hat{z}_N^n . Furthermore the uniform Hölder estimates imply that

$$\lim_{k \rightarrow \infty} \|\psi_{M_k} - \phi_{M_k}\|_{\mathcal{C}^0} = 0,$$

where $\phi_{M_k} \in C^0(\mathcal{C}_{M_k}^+(\Sigma))$ are the samples of ϕ . This implies by Proposition 3.8.12 that

$$\lim \psi_{M_k} = \phi. \quad \square$$

3.10. Higher order convergence

We are interested in stronger convergence of discrete functions, taking into account higher order finite differences. We start by stating the following elementary results:

LEMMA 3.10.1. – *Let $\psi_{N_k} \in C^0(\mathcal{C}_{N_k}^\pm(\Sigma))$ be a sequence of discrete functions. The finite differences $\frac{\partial \psi_{N_k}}{\partial \vec{u}}$ (resp. $\frac{\partial \psi_{N_k}}{\partial \vec{v}}$) converge if, and only if, the finite differences $\frac{\partial \psi_{N_k}}{\partial \vec{u}}$ (resp. $\frac{\partial \psi_{N_k}}{\partial \vec{v}}$) converge. If they converge, they have the same limits:*

$$\lim \frac{\partial \psi_{N_k}}{\partial \vec{u}} = \lim \frac{\partial \psi_{N_k}}{\partial \vec{u}}, \quad \lim \frac{\partial \psi_{N_k}}{\partial \vec{v}} = \lim \frac{\partial \psi_{N_k}}{\partial \vec{v}}.$$

Proof. – This follows from the fact that finite differences in the forward and backward directions are related by the translations T_u , or T_v , spanning the lattice Λ_N^{ch} , thanks to Formulae (3.19) and (3.20). □

REMARK 3.10.2. – According to the above lemma, one can talk about the convergence of the finite differences of a sequence of discrete functions without specifying on the forward or retrograde directions.

PROPOSITION 3.10.3. – *Let $\psi_{N_k} \in C^0(\mathcal{C}_{N_k}^\pm(\Sigma))$ be a converging sequence of discrete functions such that its first order finite differences converge as well towards the limits*

$$\phi = \lim \psi_{N_k}, \phi_u = \lim \frac{\partial \psi_{N_k}}{\partial \vec{u}} \text{ and } \phi_v = \lim \frac{\partial \psi_{N_k}}{\partial \vec{v}}.$$

Then, the limit $\phi : \Sigma \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 with partial derivatives given by

$$\frac{\partial \phi}{\partial u} = \phi_u, \quad \frac{\partial \phi}{\partial v} = \phi_v.$$

Proof. – One can readily show that ϕ is a primitive function of ϕ_u (resp. ϕ_v) in the u -direction (resp. v -direction) using Riemann sums. The limits ϕ_u and ϕ_v are continuous by Lemma 3.8.7 and it follows that ϕ is continuously differentiable. \square

Lemma 3.10.1 and Proposition 3.10.3 motivate the following definition:

DEFINITION 3.10.4. – *If a sequence of discrete functions $\psi_{N_j} \in C^0(\mathcal{G}_{N_j}^\pm(\Sigma))$ converges together with its finite differences, up to order k , we say that the sequence (ψ_{N_j}) converges in the \mathcal{C}^k -sense toward the function $\phi = \lim \psi_{N_j}$. We denote this property by*

$$\psi_{N_j} \xrightarrow{\mathcal{C}^k} \phi.$$

If $\psi_{N_j} \in C^0(\mathcal{G}_{N_j}(\Sigma))$ is a sequence of discrete functions with decompositions $\psi_{N_j} = \psi_{N_j}^+ + \psi_{N_j}^-$ and $\phi^+, \phi^- : \Sigma \rightarrow \mathbb{R}$ are functions such that

$$\psi_{N_j}^+ \xrightarrow{\mathcal{C}^k} \phi^+, \text{ and } \psi_{N_j}^- \xrightarrow{\mathcal{C}^k} \phi^-,$$

we say that ψ_{N_j} converges in the weak \mathcal{C}^k -sense toward the pair of functions (ϕ^+, ϕ^-) . This property is denoted

$$\psi_{N_j} \xrightarrow{\mathcal{C}_w^k} (\phi^+, \phi^-).$$

This definition and Propositions 3.10.3 leads to the following corollary:

PROPOSITION 3.10.5. – *If $\psi_{N_j} \in C^0(\mathcal{G}_{N_j}^\pm(\Sigma))$ converges in the \mathcal{C}^k sense, the limit $\phi = \lim \psi_{N_j}$ is of class \mathcal{C}^k . Furthermore the finite differences of ψ_{N_j} converge, up to order k toward the corresponding partial derivatives of ϕ .*

We may now state an improved version of the Ascoli-Arzelà theorem in the \mathcal{C}^k setting:

THEOREM 3.10.6 (Ascoli-Arzelà, second version). – *Let ψ_{N_j} be a sequence of discrete function in $C^0(\mathcal{G}_{N_j}^\pm(\Sigma))$, which are uniformly bounded in $\mathcal{C}^{k,\alpha}$ -norm for some $k \geq 0$, in the sense that there exists a constant $c > 0$ with the property that*

$$\|\psi_{N_j}\|_{\mathcal{C}^{k,\alpha}} \leq c \quad \text{for all } j \geq 0.$$

Then there exists a subsequence N'_j of N_j and a function $\phi : \Sigma \rightarrow \mathbb{R}$ with $\phi \in \mathcal{C}^{k,\alpha}(\Sigma)$, such that

$$\psi_{N'_j} \xrightarrow{\mathcal{C}^{k,\alpha}} \phi.$$

Proof. – We give a sketch of proof in the case $k = 1$. By assumption, the ψ_{N_j} are uniformly bounded in $\mathcal{C}^{1,\alpha}$ -norms. Thus the finite differences of order 1 are bounded in $\mathcal{C}^{0,\alpha}$ -norm:

$$\left\| \frac{\partial \psi_{N_j}}{\partial \vec{u}} \right\|_{\mathcal{C}^{0,\alpha}} \leq c, \quad \left\| \frac{\partial \psi_{N_j}}{\partial \vec{v}} \right\|_{\mathcal{C}^{0,\alpha}} \leq c.$$

and we may apply Theorem 3.9.1 to the first order finite differences. After passing to suitable subsequences, we may assume that

$$\frac{\partial \psi_{N_j}}{\partial \vec{u}} \xrightarrow{\mathcal{C}^0} \phi_u, \quad \frac{\partial \psi_{N_j}}{\partial \vec{v}} \xrightarrow{\mathcal{C}^0} \phi_v,$$

where $\phi_u, \phi_v \in \mathcal{C}^{0,\alpha}(\Sigma)$. Since ψ_N is bounded in $\mathcal{C}^{1,1}$ -norm, we may apply Ascoli-Arzelà again and assume, up to further extraction, that

$$\lim \psi_{N_j} = \phi,$$

for some continuous function ϕ . The rest of the proof follows from Proposition 3.10.3. The general case is proved by induction on k . \square

3.11. Examples of discrete convergence

We present two examples of converging sequences of discrete functions that will turn out to be useful.

3.11.1. Samples of continuously differentiable functions. – Corollary 3.8.13 extends to stronger \mathcal{C}^k -convergence as follows:

PROPOSITION 3.11.2. – *Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a function of class \mathcal{C}^k , and $\phi_N^\pm \in \mathcal{C}^0(\mathcal{Q}_N^\pm(\Sigma))$ its samples. Then*

$$\phi_N^\pm \xrightarrow{\mathcal{C}^k} \phi.$$

Proof. – The Taylor formula insures that finite differences of ϕ_N^\pm converge uniformly to the corresponding partial derivative of ϕ . It follows by Proposition 3.8.12 that, up to order k , the finite differences of ϕ_N^\pm converge in the sense of Definition 3.8.5, which proves the proposition. \square

3.11.3. Discrete tangent vector fields. – We may consider discrete functions with values in \mathbb{R}^m , or more precisely \mathbb{R}^{2n} , rather than real valued functions. It is an easy exercise to check that all the notions of convergence of discrete functions, Hölder norms, introduced before trivially extend to this setting.

Given a smooth immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, we shall define a sample $\tau_N \in \mathcal{C}^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$ of ℓ at § 4.1. We will show that the discrete tangent vector fields associated to the diagonals of the sample τ_N converge in Proposition 4.3.1.

CHAPTER 4

PERTURBATION THEORY FOR ISOTROPIC MESHES

We keep on using the notations of the previous section. Recall that $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ is a smooth isotropic immersion and Σ a surface diffeomorphic to a torus. The surface is endowed with the pullback metric g_Σ and the flat metric g_σ related by a conformal factor $g_\Sigma = \theta g_\sigma$. There is also a family of flat metrics g_σ^N induced by the diffeomorphism $\Phi_N : \mathbb{R}^2/\Gamma_N \rightarrow \Sigma$. We construct the various versions of quadrangulations and the checkers graphs as in § 3.

4.1. Isotropic quadrangular meshes

A quadrangular mesh

$$\tau \in \mathcal{M}_N = C^0(Q_N(\Sigma)) \otimes \mathbb{R}^{2n}$$

associates \mathbb{R}^{2n} -coordinates to each vertex of $Q_N(\Sigma)$. One can define a unique piecewise linear map

$$\ell_\tau : \Sigma_N^1 \rightarrow \mathbb{R}^{2n}$$

from the 1-skeleton Σ_N^1 of the quadrangulation $Q_N(\Sigma)$ into \mathbb{R}^{2n} , which agrees with τ at vertices. Contrarily to the case of a triangulation, there is generally no piecewise linear extension to the 2-skeleton, that is Σ . Indeed, quadrilaterals of \mathbb{R}^{2n} may not be planar. There are several options to construct extensions of $\ell_\tau : \Sigma_N^1 \rightarrow \mathbb{R}^{2n}$ to Σ , but this is not a fundamental issue as we shall see.

DEFINITION 4.1.1. – *An Euclidean quadrilateral of \mathbb{R}^{2n} is said to be isotropic if the integral of the Liouville form λ along the quadrilateral vanishes. Similarly, a mesh $\tau \in \mathcal{M}_N$ is called isotropic if the quadrilaterals of \mathbb{R}^{2n} associated to each face of $Q_N(\Sigma)$ via τ are isotropic in the above sense. The space $\mathcal{L}_N \subset \mathcal{M}_N$ is the set of all isotropic quadrangular meshes $\tau \in \mathcal{M}_N$.*

4.1.2. Equation for isotropic quadrilaterals. – An oriented quadrilateral of \mathbb{R}^{2n} can be given by 4 ordered vertices (A_0, A_1, A_2, A_3) . We introduce the diagonals of the quadrilateral

$$(4.1) \quad D_0 = \overrightarrow{A_0A_2}, \quad D_1 = \overrightarrow{A_1A_3}.$$

Then we have the following result, which shows that the equation for an isotropic quadrilateral is quadratic:

LEMMA 4.1.3. – *The integral of the Liouville form λ along an oriented quadrilateral (A_0, \dots, A_3) of \mathbb{R}^{2n} is given by*

$$\frac{1}{2}\omega(D_0, D_1),$$

where D_i are the diagonals of the quadrilateral defined by (4.1).

Proof. – We construct a pyramid \mathcal{P} with base the quadrilateral \mathcal{Q} and with apex located at the origin $O \in \mathbb{R}^{2n}$, for instance. By Stokes Theorem

$$\int_{\mathcal{Q}} \lambda = \int_{\mathcal{P}} \omega.$$

The integral of the RHS is the sum of the symplectic areas of the four triangles (OA_iA_{i+1}) , for i considered as an index modulo 4. Hence the integral of the Liouville form is given by

$$\frac{1}{2} \sum_{i=0}^3 \omega(\overrightarrow{OA_i}, \overrightarrow{OA_{i+1}}) = \frac{1}{2}\omega(D_0, D_1). \quad \square$$

4.1.4. Diagonals notation. – For $\tau \in \mathcal{M}_N$, we consider the lifts $\tilde{\tau} = \tau \circ p_N \in C^0(\mathcal{Q}_N(\mathbb{R}^2))$. Using the notations of § 3.3.1, we define the diagonals

$$D_\tau^u, D_\tau^v \in C^2(\mathcal{Q}_N(\mathbb{R}^2)) \otimes \mathbb{R}^{2n}$$

by

$$D_\tau^u(\mathbf{f}_{kl}) = \tilde{\tau}(\mathbf{v}_{k+1, l+1}) - \tilde{\tau}(\mathbf{v}_{kl})$$

and

$$D_\tau^v(\mathbf{f}_{kl}) = \tilde{\tau}(\mathbf{v}_{k, l+1}) - \tilde{\tau}(\mathbf{v}_{k+1, l}).$$

Then, D_τ^u and D_τ^v descend to the quotient Σ and provide discrete vector fields denoted in the same way

$$D_\tau^u, D_\tau^v \in C^2(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n} \simeq C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}.$$

By definition D_τ^u and D_τ^v represent certain diagonals of each face of the quadrangular mesh τ . It is also convenient to introduce the renormalized discrete vector fields

$$\mathcal{U}_\tau = \frac{N}{\sqrt{2}} D_\tau^u \quad \text{and} \quad \mathcal{V}_\tau = \frac{N}{\sqrt{2}} D_\tau^v.$$

4.1.5. Equation for isotropic mesh. – The problem of finding isotropic meshes can be formulated using a suitable equation. Each $\tau \in \mathcal{M}_N$ and each face $\mathbf{f} \in \mathfrak{C}_2(Q_N)$ defines Euclidean quadrilateral in \mathbb{R}^{2n} , given by the \mathbb{R}^{2n} -coordinates of ordered vertices of \mathbf{f} . Such a quadrilateral has a symplectic area defined by the integral of the Liouville form λ along the quadrilateral. We can pack this data into a map

$$\mu_N : \mathcal{M}_N \longrightarrow C^2(Q_N(\Sigma)),$$

such that $\langle \mu_N(\tau), \mathbf{f} \rangle$ is the symplectic area of the corresponding quadrilateral. The space of isotropic meshes \mathcal{L}_N is by definition the set of solutions of the equation $\mu_N = 0$. In other words

$$\mathcal{M}_N \supset \mathcal{L}_N = \mu_N^{-1}(0).$$

For analytical reasons, it will be convenient to introduce a renormalized version of μ_N , defined by

$$\mu_N^r = N^2 \mu_N.$$

REMARK 4.1.6. – Given $\mathbf{f} \in \mathfrak{C}_2(Q_N(\Sigma))$, the real number $\mu_N^r(\mathbf{f})$ is the ratio between the symplectic area $\langle \mu_N(\tau), \mathbf{f} \rangle$ and the Euclidean area of \mathbf{f} with respect to the metric g_σ^N , which is

$$\text{Area}(\mathbf{f}, g_\sigma^N) = \frac{1}{N^2}.$$

In this sense μ_N^r can be regarded as a discrete version of the moment map $\mu(\ell) = \frac{\ell^* \omega}{\sigma}$ introduced at § 2 and $\langle \mu_N^r(\tau), \mathbf{f} \rangle$ as the symplectic density of the face \mathbf{f} with respect to τ .

The space of isotropic meshes \mathcal{L}_N is the zero set of μ_N^r . This subspace is defined by a system of quadratic polynomials as shown by the following lemma.

LEMMA 4.1.7. – *The map $\mu_N : \mathcal{M}_N \rightarrow C^2(Q_N(\Sigma))$ is quadratic. More precisely, we have*

$$(4.2) \quad \langle \mu_N(\tau), \mathbf{f} \rangle = \frac{1}{2} \omega(D_\tau^u(\mathbf{f}), D_\tau^v(\mathbf{f}))$$

and

$$(4.3) \quad \langle \mu_N^r(\tau), \mathbf{f} \rangle = \omega(\mathcal{U}_\tau(\mathbf{f}), \mathcal{V}_\tau(\mathbf{f})).$$

Proof. – This is an immediated consequence of Lemma 4.1.3. □

DEFINITION 4.1.8. – *Since $\mu_N : \mathcal{M}_N \rightarrow C^2(Q_N(\Sigma))$ is a quadratic map, it is associated to a unique symmetric bilinear map*

$$\Psi_N : \mathcal{M}_N \times \mathcal{M}_N \rightarrow C^2(Q_N(\Sigma)).$$

Similarly, Ψ_N^r is the symmetric bilinear map associated to the quadratic map μ_N^r .

4.2. Shear action on meshes

The space \mathcal{M}_N admits an obvious action induced by the translations of \mathbb{R}^{2n} , which preserves the subspace of isotropic meshes \mathcal{L}_N . However, translations belong to a larger group acting on \mathcal{M}_N , defined below, preserving isotropic meshes.

The space of vertices of $\mathcal{Q}_N(\mathbb{R}^2)$ admits a splitting similar to faces. Indeed, Λ_N^{ch} acts on the vertices, with exactly two orbits denoted $\mathfrak{C}_0^+(\mathbb{R}^2)$ and $\mathfrak{C}_0^-(\mathbb{R}^2)$, with the convention that $\mathbf{v}_{00} \in \mathfrak{C}_0^+(\mathbb{R}^2)$. This splitting descends to the quotient via $p_N : \mathbb{R}^2 \rightarrow \Sigma$, where we have two sets of vertices (cf. Figure 1 for a picture)

$$\mathfrak{C}_0(\Sigma) = \mathfrak{C}_0^+(\Sigma) \cup \mathfrak{C}_0^-(\Sigma).$$

For any mesh $\tau \in \mathcal{M}_N$ and vector $T = (T_+, T_-) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$, we define the action of T on τ by

$$\langle T \cdot \tau, \mathbf{v} \rangle = \begin{cases} \langle \tau, \mathbf{v} \rangle + T_+ & \text{if } \mathbf{v} \in \mathfrak{C}_0^+(\Sigma), \\ \langle \tau, \mathbf{v} \rangle + T_- & \text{if } \mathbf{v} \in \mathfrak{C}_0^-(\Sigma). \end{cases}$$

The above action of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ on \mathcal{M}_N is called the *shear action*. If $T_+ = T_-$ the action of T is the usual action by translations mentioned earlier. However, the shear action $\tau \mapsto T \cdot \tau$ by a vector $T = (T_+, 0)$ pulls apart positive and negative vertices of τ . But the shear action preserves isotropic meshes:

PROPOSITION 4.2.1. – *The space of isotropic meshes $\mathcal{L}_N \subset \mathcal{M}_N$ is invariant under the shear action.*

Proof. – The diagonals of the quadrilaterals associated to some mesh τ are invariant under the shear action. In particular, any isotropic mesh remains isotropic under the shear action by Lemma 4.1.3. \square

REMARK 4.2.2. – The shear symmetry shows that the space of isotropic quadrangular meshes $\mathcal{L}_N = (\mu_N^r)^{-1}(0)$ does not become more regular as $N \rightarrow \infty$ in a naive sense. Intuitively, if τ is isotropic and close to a smooth immersed surface (in some C^1 -sense), the isotropic mesh $(T_+, 0) \cdot \tau$ now looks wild (cf. Figure 1), even more so as the step size of the quadrangulation goes to 0. This explains why Schauder estimates for discrete elliptic operators involve only weak Hölder norms introduced at §3.7. In turn Theorem A and Theorem C are only stated with \mathcal{C}^0 -norms.

On the contrary, it could be argued that shear symmetry could be used to improve regularity, rather than destroying it. It is possible to obtain good strong \mathcal{C}^1 -estimates in Theorem C at one quadrilateral of the quadrangular mesh ρ_N , modulo the shear action. Unfortunately, it seems unlikely that one could pass in general from such a local to a global strong \mathcal{C}^1 -estimate.

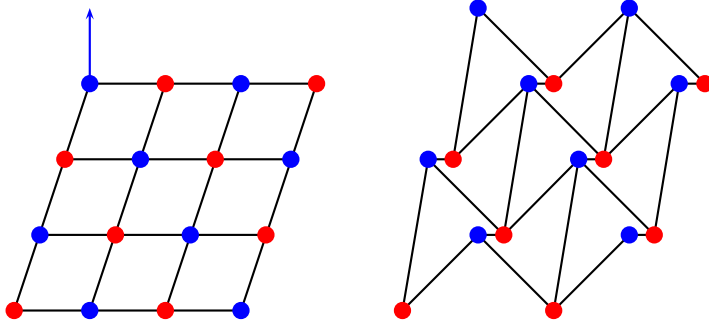


FIGURE 1. Shear action on the blue vertices of a mesh

REMARK 4.2.3. – We will make seldom mention of the shear action. But this action will be crucial at §7 to get more generic isotropic quadrangular meshes.

4.3. Meshes obtained by sampling

Given a smooth immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, we construct a canonical sequence of approximations of ℓ by quadrangular meshes

$$\tau_N \in \mathcal{M}_N.$$

The map $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ can be restricted to the vertices of $\mathcal{Q}_N(\Sigma)$. Hence we may define an element $\tau_N \in \mathcal{M}_N$, called a *sample of ℓ* , by

$$\tau_N(\mathbf{v}) = \ell(\mathbf{v}),$$

for each $\mathbf{v} \in \mathfrak{C}_0(\mathcal{Q}_N(\Sigma))$.

We would like to discuss more precisely the nature of the convergence of τ_N towards ℓ in the spirit of §3. This is possible at the cost of extending all the analysis introduced at §3 for discrete functions on faces of $\mathcal{Q}_N(\Sigma)$ to the case of functions defined at vertices. Instead of carrying this uncomplicated but lengthy work, we will adopt a more straightforward approach here.

For the special case $\tau = \tau_N$, where the meshes τ_N are the samples of an immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, the diagonals $D_\tau^u, D_\tau^v, \mathcal{U}_\tau$ and \mathcal{V}_τ are denoted $D_N^u, D_N^v, \mathcal{U}_N$ and \mathcal{V}_N instead. Then we have the following result

PROPOSITION 4.3.1. – *The sequence of discrete vector fields \mathcal{U}_N^\pm and $\mathcal{V}_N^\pm \in C^0(\mathcal{Q}_N^\pm(\Sigma)) \otimes \mathbb{R}^{2n}$ converge in the C^k -sense, for every k . Furthermore*

$$\mathcal{U}_N^\pm \xrightarrow{C^k} \frac{\partial \ell}{\partial u} \quad \text{and} \quad \mathcal{V}_N^\pm \xrightarrow{C^k} \frac{\partial \ell}{\partial v}.$$

More precisely, if we denote by \mathcal{U}'_N (resp. \mathcal{V}'_N) the samples of $\frac{\partial \ell}{\partial u}$ (resp. $\frac{\partial \ell}{\partial v}$) then

$$\|\mathcal{U}_N - \mathcal{U}'_N\|_{C_w^k} = \mathcal{O}(N^{-1}) \quad \text{and} \quad \|\mathcal{V}_N - \mathcal{V}'_N\|_{C_w^k} = \mathcal{O}(N^{-1}).$$

4.4. Almost isotropic samples

The defect of the samples τ_N to be isotropic is given by the sequence of discrete functions

$$(4.4) \quad \eta_N = \mu_N^r(\tau_N) \in C^2(\mathcal{Q}_N(\Sigma)).$$

The error η_N is small as N goes to infinity in the sense of the following proposition:

PROPOSITION 4.4.1. – *Let $\ell : \Sigma \rightarrow \mathbb{R}$ be a smooth isotropic immersion and $\tau_N \in \mathcal{M}_N$ be the sequence of samples of ℓ with respect to the quadrangulations $\mathcal{Q}_N(\Sigma)$. Let $\eta_N = \mu^r(\tau_N) \in C^2(\mathcal{Q}_N(\Sigma))$ be the isotropic defect of τ_N . Then for every integer $k \geq 0$, we have*

$$\|\eta_N\|_{C_w^k} = \mathcal{O}(N^{-1}).$$

Proof. – For each face $f \in \mathfrak{C}_2(\mathcal{Q}_N(\Sigma))$, the quantity $\eta_N(f)$ is given by

$$\eta_N(f) = \frac{N^2}{2} \omega(D_N^u(f), D_N^v(f)) = \omega(\mathcal{U}_N(f), \mathcal{V}_N(f)).$$

The formula for discrete differences of a quadratic form and the C^k -convergence of Proposition 4.3.1 proves the proposition. \square

4.5. Inner products

The tangent vectors to the space of meshes \mathcal{M}_N and the space of discrete functions come equipped with canonical inner products, which are crucial for the analysis.

4.5.1. The case of function. – The space $C^2(\mathcal{Q}_N(\Sigma))$ of discrete functions comes equipped with an Euclidean inner product which is a discrete version of the L^2 -inner product for smooth functions. The space $C_2(\mathcal{Q}_N(\Sigma))$ admits a canonical basis, given by the set of faces $\mathbf{f} \in \mathfrak{C}_2(\mathcal{Q}_N(\Sigma))$. Thus, we have a corresponding dual basis \mathbf{f}^* of $C^2(\mathcal{Q}_N(\Sigma))$ defined by

$$\langle \mathbf{f}^*, \mathbf{f}' \rangle = \begin{cases} 1 & \text{if } \mathbf{f} = \mathbf{f}' \\ 0 & \text{otherwise,} \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket.

Recall that the area of a face \mathbf{f} of $\mathcal{Q}_N(\Sigma)$, with respect to the Riemannian metric g_σ^N , is equal to N^{-2} . The 1-form \mathbf{f}^* is understood as a constant function equal to 1 on the face \mathbf{f} and 0 on other faces. This intuition gives an interpretation of the duality bracket

$$\langle \cdot, \cdot \rangle : C^2(\mathcal{Q}_N(\Sigma)) \times C_2(\mathcal{Q}_N(\Sigma)) \rightarrow \mathbb{R}$$

as the *pointwise* evaluation of functions on face. This leads to a discrete analog

$$\langle\langle \cdot, \cdot \rangle\rangle : C^2(\mathcal{Q}_N(\Sigma)) \times C^2(\mathcal{Q}_N(\Sigma)) \rightarrow \mathbb{R}$$

of the L^2 -inner product defined by

$$\langle\langle \mathbf{f}_1^*, \mathbf{f}_2^* \rangle\rangle = \begin{cases} 0 & \text{if } \mathbf{f}_1 \neq \mathbf{f}_2, \\ \frac{1}{N^2} & \text{if } \mathbf{f}_1 = \mathbf{f}_2. \end{cases}$$

The corresponding Euclidean norm on $C^2(\mathcal{Q}_N(\Sigma))$ is simply denoted $\|\cdot\|$. Notice that the splitting of

$$C^2(\mathcal{Q}_N(\Sigma)) \simeq C^0(\mathcal{G}_N(\Sigma)) = C^0(\mathcal{G}_N^+(\Sigma)) \oplus C^0(\mathcal{G}_N^-(\Sigma))$$

is orthogonal for $\langle\langle \cdot, \cdot \rangle\rangle$. By construction, we have the following result:

PROPOSITION 4.5.2. – *Let $\psi_{N_k}^\pm \in C^0(\mathcal{G}_{N_k}^\pm(\Sigma))$ be a converging sequence of discrete functions with $\lim \psi_{N_k}^\pm = \phi^\pm$. Then*

$$\lim \|\psi_{N_k}^\pm\|^2 = \frac{1}{2} \|\phi^\pm\|_{L^2}^2,$$

where $\|\phi^\pm\|_{L^2}$ is the L^2 -norm of ϕ^\pm with respect to the Riemannian flat metric g_σ . In particular if both sequences converge and $\phi^+ = \phi^- = \phi$, then $\lim \|\psi_{N_k}\|^2 = \|\phi\|_{L^2}^2$.

Proof. – Let ϕ_N^\pm be the sequence of samples of ϕ^\pm . Then $\|\phi_N^\pm\|^2$ is understood as a Riemann sum for the integral $\|\phi^\pm\|_{L^2}^2$. Compared to a usual Riemann sum, we are throwing away half of the faces of the subdivision, and we have

$$\lim \|\phi_N^\pm\|^2 = \frac{1}{2} \|\phi^\pm\|_{L^2}^2.$$

Using the \mathcal{C}^0 -convergence of ψ_N^\pm and Proposition 3.8.11, we deduce that

$$\lim \|\phi_N^\pm - \psi_N^\pm\|^2 = 0,$$

and the proposition follows by the triangle inequality. \square

4.5.3. The case of vector fields. – The space $T_\tau \mathcal{M}_N$ consists of tangent vectors $V \in C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$. Here V is understood as a family of vectors, given at each vertex \mathbf{v} of $\mathcal{Q}_N(\Sigma)$ by $V(\mathbf{v}) = \langle V, \mathbf{v} \rangle \in \mathbb{R}^{2n}$. We deduce an Euclidean inner product on $C^0(\mathcal{Q}_N) \otimes \mathbb{R}^4$, defined by

$$\langle\langle V, V' \rangle\rangle = \frac{1}{N^2} \sum_{\mathbf{v} \in \mathcal{C}_0(\mathcal{Q}_N(\Sigma))} g(V(\mathbf{v}), V'(\mathbf{v})).$$

The corresponding Euclidean norm is also denoted $\|\cdot\|$.

4.6. Linearized equations

Recall that the moduli space of quadrangular meshes \mathcal{M}_N is in fact the vector space $C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$. So for $\tau \in \mathcal{M}_N$, the tangent space at τ is identified to

$$T_\tau \mathcal{M}_N = C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n} = \mathcal{M}_N.$$

Hence a tangent vector at τ is identified to a family of vectors of \mathbb{R}^{2n} defined at each vertex of the quadrangulation.

The differential of $\mu_N^r : \mathcal{M}_N \rightarrow C^2(\mathcal{Q}_N(\Sigma))$ at τ , which is a linear map denoted

$$D\mu_N^r|_\tau : T_\tau \mathcal{M}_N \rightarrow C^2(\mathcal{Q}_N(\Sigma)),$$

is readily computed. Formally, we have

$$D\mu_N^r|_\tau \cdot V = 2\Psi_N^r(\tau, V),$$

where Ψ_N^r is the symmetric bilinear map associated to the quadratic map μ_N^r . For a more explicit formula, we merely need to compute the variation of the symplectic area of a quadrilateral in \mathbb{R}^{2n} , which is being deformed by moving its vertices. Let $V \in T_\tau \mathcal{M}_N$ be a discrete vector field. We define a path of quadrangulations by

$$\tau_t = \tau + tV, \text{ for } t \in \mathbb{R}.$$

We would like to express the variation of μ_N^r along τ_t . In order to state a result, we need some additional notations.

4.6.1. Other diagonal notations. – We introduced the diagonals D_τ^u and D_τ^v at § 4.1.4. We need now a slightly different indexing in order to have a simple expression of the differential of μ_N^r . We denote by \mathbf{f}_{kl} for $k, l \in \mathbb{Z}$, the faces of $\mathcal{Q}_N(\mathbb{R}^2)$. Their image under the projection p_N are still denoted $\mathbf{f}_{kl} \in \mathfrak{C}_2(\mathcal{Q}_N(\Sigma))$. Similarly, we denote by \mathbf{v}_{kl} the vertices of $\mathcal{Q}_N(\mathbb{R}^2)$ and their image by p_N as vertices of $\mathcal{Q}_N(\Sigma)$. Let $V \in T_\tau \mathcal{M}_N = C^0(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$ be a vector given as family of vectors

$$V_{kl} = \langle V, \mathbf{v}_{kl} \rangle \in \mathbb{R}^{2n}.$$

We define a deformation of the quadrangulation τ by $\tau_t = \tau + tV$, or in coordinates

$$\langle \tau_t, \mathbf{v}_{kl} \rangle = \langle \tau, \mathbf{v}_{kl} \rangle + tV_{kl}.$$

Let $\tau \in \mathcal{M}_N$, $\mathbf{f} \in \mathfrak{C}_2(\mathcal{Q}_N(\mathbb{R}^2))$ and $\mathbf{v} \in \mathfrak{C}_0(\mathcal{Q}_N(\mathbb{R}^2))$ be one of the vertices of \mathbf{f} . We enumerate the vertices $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ of \mathbf{f} consistently with the orientation and such that $\mathbf{v}_0 = \mathbf{v}$. The diagonals are defined by

$$D_{\mathbf{v}, \mathbf{f}}^\tau = \tau(\mathbf{v}_3) - \tau(\mathbf{v}_1) \in \mathbb{R}^{2n}$$

and if \mathbf{v} is not a vertex of \mathbf{f} , we put $D_{\mathbf{v}, \mathbf{f}}^\tau = 0$. Figure 2 shows a diagrammatic representation of the above construction, with orientation conventions.

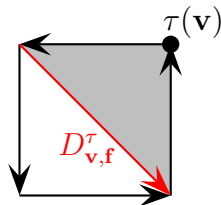


FIGURE 2. A face \mathbf{f} of a mesh τ with one diagonal and orientations

NOTATION 4.6.2. – The vector $D_{\mathbf{v},\mathbf{f}}^\tau \in \mathbb{R}^{2n}$ is called the diagonal opposite to \mathbf{v} of the face \mathbf{f} with respect to τ .

With these notations, we have the following expression for the variation of the symplectic area:

LEMMA 4.6.3. – We have

$$\frac{d}{dt} \langle \mu_N(\tau_t), \mathbf{f} \rangle |_{t=0} = -\frac{1}{2} \sum_{\mathbf{v} \in \mathfrak{C}_0(\mathcal{Q}_N(\Sigma))} \omega(V(\mathbf{v}), D_{\mathbf{v},\mathbf{f}}^\tau).$$

Proof. – We use the ordered vertices (A_0, A_1, A_2, A_3) of an oriented quadrilateral in \mathbb{R}^{2n} and consider a variation $(A_0^t, A_1^t, A_2^t, A_3^t) = (A_0, A_1, A_2, A_3) + t(V_0, V_1, V_2, V_3)$. We denote by D_0^t and D_1^t the diagonals of the deformed quadrilateral. By Lemma 4.1.3, its symplectic area is

$$\frac{1}{2} \omega(D_0^t, D_1^t).$$

Hence, the variation of symplectic area at $t = 0$ is given by

$$\frac{1}{2} (-\omega(V_0, D_1) + \omega(V_1, D_0) + \omega(V_2, D_1) - \omega(V_3, D_0)).$$

Using our conventions for the diagonals of quadrilaterals, this proves the lemma. \square

4.6.4. Computation of the discrete Laplacian. – Any discrete vector field $V \in T_\tau \mathcal{M}_N$ is given by a family of vectors

$$V_v = \langle V, v \rangle \in \mathbb{R}^{2n}.$$

The almost complex structure J of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ induces a canonical action on $T_\tau \mathcal{M}_N = C^2(\mathcal{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}$ that can be expressed as

$$(JV)_v := J(V_v).$$

Recall that the Euclidean metric g and the symplectic form ω of \mathbb{R}^{2n} are related by the formula

$$\forall u_1, u_2 \in \mathbb{R}^{2n}, \quad \omega(u_1, u_2) = g(Ju_1, u_2).$$

According to the Lemma 4.6.3, the differential of μ_N at τ_N satisfies

$$\langle D\mu_N|_\tau \cdot V, \mathbf{f} \rangle = -\frac{1}{2} \sum_{\mathbf{v}} \omega(V(\mathbf{v}), D_{\mathbf{v},\mathbf{f}}^\tau).$$

hence

$$\langle D\mu_N|_\tau \cdot JV, \mathbf{f} \rangle = \frac{1}{2} \sum_{\mathbf{v}} g(V(\mathbf{v}), D_{\mathbf{v}, \mathbf{f}}^\tau).$$

In turn, we have

$$(4.5) \quad \langle D\mu_N^r|_\tau \cdot JV, \mathbf{f} \rangle = \frac{N^2}{2} \sum_{\mathbf{v}} g(V(\mathbf{v}), D_{\mathbf{v}, \mathbf{f}}^\tau).$$

We introduce the operator (notice the analogy with Formula 2.3)

$$(4.6) \quad \boxed{\delta_\tau = -D\mu_N^r|_\tau \circ J,}$$

so that Formula (4.5) reads

$$(4.7) \quad \langle \delta_\tau V, \sum_{\mathbf{f}} \phi(\mathbf{f}) \mathbf{f} \rangle = \frac{N^2}{2} \sum_{\mathbf{v}, \mathbf{f}} \phi(\mathbf{f}) g(V(\mathbf{v}), D_{\mathbf{v}, \mathbf{f}}^\tau).$$

or, equivalently

$$(4.8) \quad \delta_\tau V = \frac{N^2}{2} \sum_{\mathbf{v}, \mathbf{f}} g(V(\mathbf{v}), D_{\mathbf{v}, \mathbf{f}}^\tau) \mathbf{f}^*.$$

With the above conventions

$$(4.9) \quad \langle \delta_\tau V, \sum_{\mathbf{f}} \phi(\mathbf{f}) \mathbf{f} \rangle = N^2 \langle\langle \delta_\tau V, \sum_{\mathbf{f}} \phi(\mathbf{f}) \mathbf{f}^* \rangle\rangle$$

and it follows from Formulae (4.7) and (4.9) that

$$\langle\langle \delta_\tau V, \sum_{\mathbf{f}} \phi(\mathbf{f}) \mathbf{f}^* \rangle\rangle = \frac{1}{2} \sum_{\mathbf{v}, \mathbf{f}} \phi(\mathbf{f}) g(V(\mathbf{v}), D_{\mathbf{v}, \mathbf{f}}^\tau).$$

We deduce that the ajoint δ_τ^* of δ_τ for the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ satisfies

$$\begin{aligned} \langle\langle V, \delta_\tau^* \sum_{\mathbf{f}} \phi(\mathbf{f}) \mathbf{f}^* \rangle\rangle &= \frac{1}{2} \sum_{\mathbf{v}} \frac{1}{N^2} g \left(V(\mathbf{v}), \sum_{\mathbf{f}} N^2 \phi(\mathbf{f}) D_{\mathbf{v}, \mathbf{f}}^\tau \right) \\ &= \langle\langle V, \frac{N^2}{2} \sum_{\mathbf{v}, \mathbf{f}} \phi(\mathbf{f}) D_{\mathbf{v}, \mathbf{f}}^\tau \mathbf{v}^* \rangle\rangle, \end{aligned}$$

which proves the following lemma

LEMMA 4.6.5. – *The operator*

$$\delta_\tau : T_{\tau N} \mathcal{M}_N \rightarrow C^2(Q_N(\Sigma))$$

is given by

$$(4.10) \quad \boxed{\delta_\tau V = \frac{N^2}{2} \sum_{\mathbf{v}, \mathbf{f}} g(V(\mathbf{v}), D_{\mathbf{v}, \mathbf{f}}^\tau) \mathbf{f}^*,}$$

whereas its adjoint

$$\delta_\tau^* : C^2(Q_N(\Sigma)) \rightarrow T_{\tau N} \mathcal{M}_N$$

is given by

$$(4.11) \quad \delta_\tau^* \phi = \frac{N^2}{2} \sum_{\mathbf{v}, \mathbf{f}} \phi(\mathbf{f}) D_{\mathbf{v}, \mathbf{f}}^\tau \otimes \mathbf{v}^*.$$

REMARK 4.6.6. – The operator $\delta_\tau = -D\mu_N^r|_\tau \circ J$ is the finite dimensional version of $\delta_f = -D\mu|_f \circ \mathfrak{J}$ considered in the smooth setting (cf. § 2.2.5). In the smooth setting, the adjoint δ_f^* allows to recover the Hamiltonian infinitesimal action of the gauge group $\mathcal{G} = \text{Ham}(\Sigma, \sigma)$ on \mathcal{M} according to the identity (2.4). In the finite dimensional approximation, there is no clear group action on \mathcal{M}_N for which μ_N^r would be the corresponding moment map. However, the vector fields $V_\phi(\tau) = \delta_\tau^* \phi$ define infinitesimal isometric Hamiltonian action which should play the role of finite dimensional approximations of $\text{Ham}(\Sigma, \sigma)$.

The kernel of δ_τ^* contains the constants discrete functions, but might contain other function as well. This is not the case generically, according to the proposition below

PROPOSITION 4.6.7. – *Let $\tau \in \mathcal{M}_N$ be a generic quadrangular mesh in the following sense: for every vertex \mathbf{v} of the quadrangulation $\mathcal{Q}_N(\Sigma)$, the four possibly non vanishing diagonals $D_{\mathbf{v}, \mathbf{f}}^\tau$, where \mathbf{f} is a face that contains the vertex \mathbf{v} span a 3-dimensional subspace of \mathbb{R}^{2n} . Then the kernel of δ_τ^* reduces to constant discrete functions.*

Proof. – The equation $\delta_\tau^* \phi = 0$ provides a linear system of rank 3 with four variables associated to each vertex. This imply that ϕ must be locally constant around each vertex and it follows that ϕ is constant. \square

DEFINITION 4.6.8. – *Given a quadrangular mesh $\tau \in \mathcal{M}_N$, we define the discrete Laplacian $\Delta_\tau : C^2(\mathcal{Q}_N(\Sigma)) \rightarrow C^2(\mathcal{Q}_N(\Sigma))$ associated to the mesh τ by*

$$\Delta_\tau = \delta_\tau \delta_\tau^*.$$

Given a smooth isotropic immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ and its samples $\tau_N \in \mathcal{M}_N$, the associated operators to δ_{τ_N} , $\delta_{\tau_N}^$ and Δ_{τ_N} are denoted δ_N , δ_N^* and Δ_N , for simplicity.*

REMARK 4.6.9. – Notice the analogy between the operator Δ_f defined by Formula (2.5) and Δ_τ . The operator Δ_τ will play a central role in the perturbation theory of quadrangular meshes, as Δ_f did for smooth isotropic immersions. The reader should already be aware that Δ_τ is *not* the classical Laplacian associated to the mesh τ , as will become clear from the sequel.

By Formula (4.11) we have

$$\delta_\tau^* \mathbf{f}^* = \frac{N^2}{2} \sum_{\mathbf{v}} D_{\mathbf{v}, \mathbf{f}} \mathbf{v}^*,$$

By Formula (4.10) we have

$$\Delta_\tau \mathbf{f}^* = \frac{N^4}{4} \sum_{\mathbf{v}, \mathbf{f}_2} g(D_{\mathbf{v}, \mathbf{f}}^\tau, D_{\mathbf{v}, \mathbf{f}_2}^\tau) \mathbf{f}_2^*.$$

We obtain the following result

PROPOSITION 4.6.10. – For $\phi \in C^2(Q_N(\Sigma))$, we have

$$\Delta_\tau \phi = \frac{N^4}{4} \sum_{\mathbf{v}, \mathbf{f}, \mathbf{f}_2} \phi(\mathbf{f}) g(D_{\mathbf{v}, \mathbf{f}}^\tau, D_{\mathbf{v}, \mathbf{f}_2}^\tau) \mathbf{f}_2^*.$$

4.7. Coefficients of the discrete Laplacian

The discrete Laplacian Δ_N is an endomorphism of $C^2(Q_N(\Sigma))$ whose coefficients are explicitly given by Proposition 4.6.10. When dealing with τ_N , we use the notation $D_{\mathbf{v}, \mathbf{f}} := D_{\mathbf{v}, \mathbf{f}}^{\tau_N}$ for simplicity. We introduce the coefficients

$$\beta_{\mathbf{f}_1 \mathbf{f}_2} = \frac{N^4}{4} \sum_{\mathbf{v}} g(D_{\mathbf{v}, \mathbf{f}_1}, D_{\mathbf{v}, \mathbf{f}_2}).$$

By Proposition 4.6.10 we have

$$\Delta_N \mathbf{f}_1^* = \sum_{\mathbf{f}_1, \mathbf{f}_2} \beta_{\mathbf{f}_1 \mathbf{f}_2} \mathbf{f}_2^*.$$

4.7.1. Splitting of the Laplacian. – The matrix $(\beta_{\mathbf{f} \mathbf{f}'})$ is obviously symmetric in \mathbf{f} and \mathbf{f}' , which is not surprising since Δ_N is selfadjoint by definition. The matrix is sparse in the sense that most of the coefficients $\beta_{\mathbf{f} \mathbf{f}'}$ vanish. There are three types of possibly non vanishing coefficients:

- (1) $\mathbf{f} = \mathbf{f}'$.
- (2) \mathbf{f} and \mathbf{f}' have only one vertex in common.
- (3) \mathbf{f} and \mathbf{f}' have exactly one edge (and two vertices) in common.

Using the above observation, we may write the operator Δ_N as a sum

$$\Delta_N = \Delta_N^E + \Delta_N^I.$$

Here

$$\Delta_N^E \mathbf{f}^* = \frac{N^4}{4} \sum_{\mathbf{v}, \mathbf{f}_2 \in E_{12}(\mathbf{f})} g(D_{\mathbf{v}, \mathbf{f}}, D_{\mathbf{v}, \mathbf{f}_2}) \mathbf{f}_2^*,$$

where $E_{12}(\mathbf{f})$ is the set of faces \mathbf{f}_2 such that the pair $(\mathbf{f}, \mathbf{f}_2)$ is of type (1) or (2), and

$$\Delta_N^I \mathbf{f}^* = \frac{N^4}{4} \sum_{\mathbf{v}, \mathbf{f}_2 \in E_3(\mathbf{f})} g(D_{\mathbf{v}, \mathbf{f}}, D_{\mathbf{v}, \mathbf{f}_2}) \mathbf{f}_2^*,$$

where $E_3(\mathbf{f})$ is the set of faces \mathbf{f}_2 such that the pair $(\mathbf{f}, \mathbf{f}_2)$ is of type (3). By definition, we have the following lemma

LEMMA 4.7.2. – *The operator Δ_N^E preserves the components of the direct sum decomposition $C_+^2(\mathcal{Q}_N(\Sigma)) \oplus C_-^2(\mathcal{Q}_N(\Sigma))$, whereas Δ_N^I exchanges the components. Accordingly, there have a block decomposition of the discrete Laplacian*

$$\Delta_N = \left(\begin{array}{c|c} \Delta_N^E & \Delta_N^I \\ \hline \Delta_N^I & \Delta_N^E \end{array} \right).$$

4.7.3. Finite difference operators and discrete Laplacian. – The smooth Laplacian (2.5) is related to a twisted Riemannian Laplacian by Lemma 2.3.4. The goal of this section is to find a similar expression for the discrete Laplacian $\Delta_N\phi$, using finite difference operators.

The strategy is to compute $\langle \Delta_N\phi, \mathbf{f} \rangle$ at some face \mathbf{f} of $\mathcal{Q}_N(\Sigma)$. For this purpose, we will use the notations \mathbf{f}_{kl} and \mathbf{v}_{ij} for faces and vertices of $\mathcal{Q}_N(\mathbb{R}^2)$, considered as vertices and faces of $\mathcal{Q}_N(\Sigma)$ (cf. § 3.3.1 and § 4.1.4). The values of a discrete function are denoted

$$\phi_{kl} = \langle \phi, \mathbf{f}_{kl} \rangle,$$

and the diagonals D_{ijkl} are obtained as $D_{\mathbf{v}_{ij}\mathbf{f}_{kl}}$, with the convention that $D_{ijkl} = 0$ if \mathbf{v}_{ij} is not a vertex of the face \mathbf{f}_{kl} in $\mathcal{Q}_N(\mathbb{R}^2)$. The coefficients β_{klmn} are denoted $\beta_{\mathbf{f}_{kl}\mathbf{f}_{mn}}$ and we choose the integers k, l so that $\mathbf{f} = \mathbf{f}_{kl}$. The coefficients $\beta_{\mathbf{f}\mathbf{f}'}$ vanishes unless $\mathbf{f} = \mathbf{f}'$ or \mathbf{f} and \mathbf{f}' are contiguous faces. In such case we may choose a unique pair of integers (m, n) such that $\mathbf{f}' = \mathbf{f}_{mn}$ with $m \in \{k - 1, k, k + 1\}$ and $n \in \{l - 1, l, l + 1\}$. Under these conditions (cf. § 4.7.1)

1. \mathbf{f}_{kl} and \mathbf{f}_{mn} are of type (1) if $(k, l) = (m, n)$,
2. \mathbf{f}_{kl} and \mathbf{f}_{mn} are of type (2) if $(m, n) = (k \pm 1, l \pm 1)$ or $(k \pm 1, l \mp 1)$,
3. \mathbf{f}_{kl} and \mathbf{f}_{mn} are of type (3) if $(m, n) = (k \pm 1, l)$ or $(k, l \pm 1)$.

For the first type of coefficients, we find

$$\beta_{klkl} = \frac{N^4}{4} \sum_{ij} \|D_{ijkl}\|^2,$$

where we may take the sum over all pairs of indices $i, j \in \mathbb{Z}$. For the second type of coefficients, we have

$$\beta_{klmn} = \frac{N^4}{4} g(D_{ijkl}, D_{ijmn}),$$

where \mathbf{v}_{ij} is the common vertex of \mathbf{f}_{kl} and \mathbf{f}_{mn} in $\mathcal{Q}_N(\mathbb{R}^2)$. In the third case there are two common vertices \mathbf{v}_{ij} and $\mathbf{v}_{i'j'}$ which belong to the same edge. Then

$$\beta_{klmn} = \frac{N^4}{4} g(D_{ijkl}, D_{ijmn}) + \frac{N^4}{4} g(D_{i'j'kl}, D_{i'j'mn}).$$

For simplicity of notations, we also use the notations D_{kl}^u and D_{kl}^v for the diagonal (cf. § 4.1.4), which differ only by a sign. We start our computations with the

operator Δ_N^E :

$$\begin{aligned} 4N^{-4}\langle \Delta_N^E \phi, \mathbf{f}_{kl} \rangle &= -\phi_{k-1,l-1}g(D_{k-1,l-1}^v, D_{kl}^v) - \phi_{k+1,l+1}g(D_{k+1,l+1}^v, D_{kl}^v) \\ &\quad - \phi_{k-1,l+1}g(D_{k-1,l+1}^u, D_{kl}^u) - \phi_{k+1,l-1}g(D_{k+1,l-1}^u, D_{kl}^u) \\ &\quad + 2\phi_{kl}(g(D_{kl}^u, D_{kl}^u) + g(D_{kl}^v, D_{kl}^v)). \end{aligned}$$

One can write

$$\begin{aligned} \phi_{k+1,l+1}g(D_{k+1,l+1}^v, D_{kl}^v) &= \phi_{k+1,l+1}g(D_{k,l}^v, D_{kl}^v) \\ &\quad + \phi_{kl}g(D_{k+1,l+1}^v - D_{kl}^v, D_{kl}^v) \\ &\quad + (\phi_{k+1,l+1} - \phi_{k,l})g(D_{k+1,l+1}^v - D_{kl}^v, D_{kl}^v) \end{aligned}$$

and similar Leibnitz type decomposition for the other terms. Thus, we obtain accordingly

$$\begin{aligned} 4N^{-4}\langle \Delta_N^E \phi, \mathbf{f}_{kl} \rangle &= (-\phi_{k-1,l-1} - \phi_{k+1,l+1} + 2\phi_{kl})g(D_{kl}^v, D_{kl}^v) \\ &\quad + (-\phi_{k-1,l+1} - \phi_{k+1,l-1} + 2\phi_{kl})g(D_{kl}^u, D_{kl}^u) \\ &\quad - \phi_{kl}g(D_{k-1,l-1}^v + D_{k+1,l+1}^v - 2D_{kl}^v, D_{kl}^v) \\ &\quad - \phi_{kl}g(D_{k+1,l-1}^u + D_{k-1,l+1}^u - 2D_{kl}^u, D_{kl}^u) \\ &\quad - (\phi_{k-1,l-1} - \phi_{kl})g(D_{k-1,l-1}^v - D_{kl}^v, D_{kl}^v) \\ &\quad - (\phi_{k+1,l+1} - \phi_{kl})g(D_{k+1,l+1}^v - D_{kl}^v, D_{kl}^v) \\ &\quad - (\phi_{k-1,l+1} - \phi_{kl})g(D_{k-1,l+1}^u - D_{kl}^u, D_{kl}^u) \\ &\quad - (\phi_{k+1,l-1} - \phi_{kl})g(D_{k+1,l-1}^u - D_{kl}^u, D_{kl}^u). \end{aligned}$$

We gather the RHS into a sum of three operators: first we define $\hat{\Delta}_N^E$. This operator will turn out to be a discrete version of the Riemannian Laplace-Beltrami operator on Σ :

$$\begin{aligned} 4N^{-4}\langle \hat{\Delta}_N^E \phi, \mathbf{f}_{kl} \rangle &= (-\phi_{k-1,l-1} - \phi_{k+1,l+1} + 2\phi_{kl})g(D_{kl}^v, D_{kl}^v) \\ &\quad + (-\phi_{k-1,l+1} - \phi_{k+1,l-1} + 2\phi_{kl})g(D_{kl}^u, D_{kl}^u), \end{aligned}$$

then we define the operator K_N^E , which is some kind of discrete curvature operator by

$$\begin{aligned} 4N^{-4}\langle K_N^E \phi, \mathbf{f}_{kl} \rangle &= -\phi_{kl}g(D_{k-1,l-1}^v + D_{k+1,l+1}^v - 2D_{kl}^v, D_{kl}^v) \\ &\quad - \phi_{kl}g(D_{k+1,l-1}^u + D_{k-1,l+1}^u - 2D_{kl}^u, D_{kl}^u). \end{aligned}$$

The last four lines can be rearranged into an operator Γ_N^E given by

$$\begin{aligned} 4N^{-4}\langle \Gamma_N^E \phi, \mathbf{f}_{kl} \rangle &= -\frac{1}{2}(\phi_{k-1,l-1} - \phi_{kl})(|D_{k-1,l-1}^v|_g^2 - |D_{kl}^v|_g^2) \\ &\quad - \frac{1}{2}(\phi_{k+1,l+1} - \phi_{kl})(|D_{k+1,l+1}^v|_g^2 - |D_{kl}^v|_g^2) \\ &\quad - \frac{1}{2}(\phi_{k-1,l+1} - \phi_{kl})(|D_{k-1,l+1}^u|_g^2 - |D_{kl}^u|_g^2) \\ &\quad - \frac{1}{2}(\phi_{k+1,l-1} - \phi_{kl})(|D_{k+1,l-1}^u|_g^2 - |D_{kl}^u|_g^2) \end{aligned}$$

plus an operator

$$\begin{aligned} 4N^{-4}\langle \mathcal{E}_N^E \phi, \mathbf{f}_{kl} \rangle &= -\frac{1}{2}(\phi_{k-1,l-1} - \phi_{kl})|D_{k-1,l-1}^v - D_{kl}^v|_g^2 \\ &\quad - \frac{1}{2}(\phi_{k+1,l+1} - \phi_{kl})|D_{k+1,l+1}^v - D_{kl}^v|_g^2 \\ &\quad - \frac{1}{2}(\phi_{k-1,l+1} - \phi_{kl})|D_{k-1,l+1}^u - D_{kl}^u|_g^2 \\ &\quad - \frac{1}{2}(\phi_{k+1,l-1} - \phi_{kl})|D_{k+1,l-1}^u - D_{kl}^u|_g^2. \end{aligned}$$

So, we have a decomposition

$$\Delta_N^E = \hat{\Delta}_N^E + K_N^E + \Gamma_N^E + \mathcal{E}_N^E.$$

Similar computations can be carried out for Δ_N^I .

$$\begin{aligned} 4N^{-4}\langle \Delta_N^I \phi, \mathbf{f}_{kl} \rangle &= \phi_{k+1,l}(-g(D_{k+1,l}^v, D_{kl}^u) - g(D_{k+1,l}^u, D_{kl}^v)) \\ &\quad \phi_{k,l+1}(g(D_{k,l+1}^v, D_{kl}^u) + g(D_{k,l+1}^u, D_{kl}^v)) \\ &\quad \phi_{k-1,l}(-g(D_{k-1,l}^v, D_{kl}^u) - g(D_{k-1,l}^u, D_{kl}^v)) \\ &\quad \phi_{k,l-1}(+g(D_{k,l-1}^v, D_{kl}^u) + g(D_{k,l-1}^u, D_{kl}^v)). \end{aligned}$$

We introduce the averaging operator $\phi \mapsto \bar{\phi}$ defined by

$$\bar{\phi}_{kl} = \langle \bar{\phi}, \mathbf{f}_{kl} \rangle = \frac{1}{4}(\phi_{k+1,l} + \phi_{k-1,l} + \phi_{k,l+1} + \phi_{k,l-1}),$$

and we write each term above under the form

$$\phi_{k+1,l}g(D_{k+1,l}^v, D_{kl}^u) = (\bar{\phi}_{kl} + (\phi_{k+1,l} - \bar{\phi}_{kl}))g(D_{k,l}^v + (D_{k+1,l}^v - D_{k,l}^v), D_{kl}^u).$$

Expanding these expressions leads to

$$\begin{aligned} 4N^{-4}\langle \Delta_N^I \phi, \mathbf{f}_{kl} \rangle &= \bar{\phi}_{kl}g((D_{k,l+1}^v - D_{k+1,l}^v) - (D_{k-1,l}^v - D_{k,l-1}^v), D_{kl}^u) \\ &\quad + \bar{\phi}_{kl}g((D_{k,l+1}^u - D_{k+1,l}^u) - (D_{k-1,l}^u - D_{k,l-1}^u), D_{kl}^v) \\ &\quad + 2((\phi_{k,l+1} - \phi_{k-1,l}) - (\phi_{k+1,l} - \phi_{k,l-1}))g(D_{kl}^v, D_{kl}^u) \\ &\quad + (\phi_{k+1,l} - \bar{\phi}_{kl})(-g(D_{k+1,l}^v - D_{kl}^v, D_{kl}^u) - g(D_{k+1,l}^u - D_{kl}^u, D_{kl}^v)) \\ &\quad + (\phi_{k,l+1} - \bar{\phi}_{kl})(g(D_{k,l+1}^v - D_{kl}^v, D_{kl}^u) + g(D_{k,l+1}^u - D_{kl}^u, D_{kl}^v)) \\ &\quad + (\phi_{k-1,l} - \bar{\phi}_{kl})(-g(D_{k-1,l}^v - D_{kl}^v, D_{kl}^u) - g(D_{k-1,l}^u - D_{kl}^u, D_{kl}^v)) \\ &\quad + (\phi_{k,l-1} - \bar{\phi}_{kl})(g(D_{k,l-1}^v - D_{kl}^v, D_{kl}^u) + g(D_{k,l-1}^u - D_{kl}^u, D_{kl}^v)). \end{aligned}$$

The first two lines can be expressed using Chasles relation as an operator

$$\begin{aligned} 4N^{-4}\langle K_N^I \phi, \mathbf{f}_{kl} \rangle &= \bar{\phi}_{kl}g(D_{k-1,l+1}^u + D_{k+1,l-1}^u - 2D_{kl}^u, D_{kl}^v) \\ &\quad + \bar{\phi}_{kl}g(D_{k+1,l+1}^v + D_{k-1,l-1}^v - 2D_{kl}^v, D_{kl}^u). \end{aligned}$$

In particular, we see that if $\phi_{kl} = \bar{\phi}_{kl}$, then $\langle (K_{\tau_0}^I + K_{\tau_0}^E)\phi, \mathbf{f}_{kl} \rangle = 0$. We decompose Δ^I as a sum

$$\Delta_N^I = K_N^I + \mathcal{E}_N^I.$$

All the above operators may be expressed in terms of finite differences. First we define analog $\theta_N^u, \theta_N^v \in C^2(\mathcal{Q}_N(\Sigma))$ of the conformal factor θ by

$$\theta_N^u = \|\mathcal{U}_N\|_g^2, \text{ and } \theta_N^v = \|\mathcal{V}_N\|_g^2$$

and a discrete analog of the Gauß curvature plus an energy term $\kappa_N \in C^2(\mathcal{Q}_N(\Sigma))$ given by

$$\kappa_N = -g \left(\frac{\partial^2}{\partial \vec{u} \partial \vec{u}} \mathcal{V}_N, \mathcal{V}_N \right) - g \left(\frac{\partial^2}{\partial \vec{v} \partial \vec{v}} \mathcal{U}_N, \mathcal{U}_N \right).$$

PROPOSITION 4.7.4. – *The operators introduced above satisfy the following identities for every discrete function ϕ :*

$$\hat{\Delta}_N^E \phi = - \left(\theta_N^u \frac{\partial^2}{\partial \vec{u} \partial \vec{u}} + \theta_N^v \frac{\partial^2}{\partial \vec{v} \partial \vec{v}} \right) \phi$$

$$K_N^E \phi = \kappa_N \cdot \phi$$

$$\Gamma_N^E \phi = -\frac{1}{2} \left(\frac{\partial \phi}{\partial \vec{u}} \frac{\partial \theta_N^v}{\partial \vec{u}} + \frac{\partial \phi}{\partial \vec{u}} \frac{\partial \theta_N^v}{\partial \vec{u}} + \frac{\partial \phi}{\partial \vec{v}} \frac{\partial \theta_N^u}{\partial \vec{v}} + \frac{\partial \phi}{\partial \vec{v}} \frac{\partial \theta_N^u}{\partial \vec{v}} \right) = -\text{grad} \phi \cdot \text{grad} \theta_N$$

and

$$K_N^I \phi = -\kappa_N \cdot \bar{\phi}.$$

The operators \mathcal{E}_N^E and \mathcal{E}_N^I become negligible as N goes to infinity, in the sense of the following proposition:

PROPOSITION 4.7.5. – *There exists a sequence $\varepsilon_N = \mathcal{O}(N^{-1})$ with $\varepsilon_N > 0$ such that for all N and all functions $\phi \in C^2(\mathcal{Q}_N(\Sigma))$, we have*

$$\begin{aligned} \|\mathcal{E}_N^I \phi\|_{C^{0,\alpha}} &\leq \varepsilon_N \|\phi\|_{C^{2,\alpha}}, \quad \text{and} \quad \|\mathcal{E}_N^E \phi\|_{C^{0,\alpha}} \leq \varepsilon_N \|\phi\|_{C^{2,\alpha}}, \\ \|\mathcal{E}_N^I \phi\|_{C^0} &\leq \varepsilon_N \|\phi\|_{C^2} \quad \text{and} \quad \|\mathcal{E}_N^E \phi\|_{C^0} \leq \varepsilon_N \|\phi\|_{C^2}. \end{aligned}$$

CHAPTER 5

LIMIT OPERATOR

5.1. Computation of the limit operator

We denote by Δ_σ (resp. Δ_Σ) the Laplace-Beltrami operator associated to the Riemannian metric g_σ (resp. g_Σ) on Σ .

THEOREM 5.1.1. – *Let k be an integer such that $k \geq 2$. For every sequence of discrete functions $\psi_{N_k} \in C^2(\mathcal{Q}_{N_k}(\Sigma))$, converging in the C_w^k -sense toward a pair of functions (ϕ^+, ϕ^-) , we have*

$$\Delta_{N_k} \psi_{N_k} \xrightarrow{C^{k-2}} \Xi(\phi^+, \phi^-),$$

where $\Xi(\phi^+, \phi^-)$ is the pair of functions defined by

$$\begin{aligned} \Xi(\phi^+, \phi^-) = & \left(\theta \Delta_\sigma \phi^+ - g_\sigma(d\phi^+, d\theta) + (K + E)(\phi^+ - \phi^-), \right. \\ & \left. \theta \Delta_\sigma \phi^- - g_\sigma(d\phi^-, d\theta) + (K + E)(\phi^- - \phi^+) \right), \end{aligned}$$

where K is the Gauß curvature of g_Σ and E is a nonnegative function on Σ defined at (5.1).

Proof. – The result is a trivial consequence of Proposition 4.7.4 and the convergence of the coefficients of the operator. The only non trivial fact that must be proved is the following lemma:

LEMMA 5.1.2. – *We have the identity*

$$K + E = -g \left(\frac{\partial^3 \ell}{\partial u^2 \partial v}, \frac{\partial \ell}{\partial v} \right) - g \left(\frac{\partial^3 \ell}{\partial v^2 \partial u}, \frac{\partial \ell}{\partial u} \right),$$

where K is the Gauß curvature of the metric g^Σ and E is the nonnegative function on Σ defined via the second fundamental form \mathbf{II} of $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$

$$(5.1) \quad E = 2g \left(\frac{\partial^2 \ell}{\partial u \partial v}^\perp, \frac{\partial^2 \ell}{\partial u \partial v}^\perp \right) = 2g \left(\mathbf{II} \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right), \mathbf{II} \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) \right).$$

Proof. – Recall the standard formula for the Gauß curvature K of the metric $g^\Sigma = \ell^*g = \theta g_\sigma$, conformal to the flat metric g_σ :

$$K = \frac{1}{2}\theta\Delta_\sigma \log \theta.$$

Using the classical identity

$$\theta\Delta_\sigma \log \theta = \Delta_\sigma \theta + \theta^{-1}g_\sigma(d\theta, d\theta)$$

and using the fact that

$$\theta = g\left(\frac{\partial \ell}{\partial u}, \frac{\partial \ell}{\partial u}\right) = g\left(\frac{\partial \ell}{\partial v}, \frac{\partial \ell}{\partial v}\right),$$

we compute

$$(5.2) \quad \frac{\partial \theta}{\partial v} = 2g\left(\frac{\partial^2 \ell}{\partial u \partial v}, \frac{\partial \ell}{\partial u}\right), \quad \frac{\partial \theta}{\partial u} = 2g\left(\frac{\partial^2 \ell}{\partial u \partial v}, \frac{\partial \ell}{\partial v}\right),$$

hence

$$\frac{\partial^2 \theta}{\partial v^2} = 2g\left(\frac{\partial^3 \ell}{\partial u \partial v^2}, \frac{\partial \ell}{\partial u}\right) + 2g\left(\frac{\partial^2 \ell}{\partial u \partial v}, \frac{\partial^2 \ell}{\partial u \partial v}\right)$$

and

$$\frac{\partial^2 \theta}{\partial u^2} = 2g\left(\frac{\partial^3 \ell}{\partial u^2 \partial v}, \frac{\partial \ell}{\partial u}\right) + 2g\left(\frac{\partial^2 \ell}{\partial u \partial v}, \frac{\partial^2 \ell}{\partial u \partial v}\right).$$

In particular

$$\begin{aligned} g_\sigma(d\theta, d\theta) &= \left|\frac{\partial \theta}{\partial u}\right|^2 + \left|\frac{\partial \theta}{\partial v}\right|^2 \\ &= 4g\left(\frac{\partial^2 \ell}{\partial u \partial v}, \frac{\partial \ell}{\partial u}\right)^2 + 4g\left(\frac{\partial^2 \ell}{\partial u \partial v}, \frac{\partial \ell}{\partial v}\right)^2, \end{aligned}$$

thanks to Formula (5.2). The fact that $\frac{\partial \ell}{\partial u}$ and $\frac{\partial \ell}{\partial v}$ is an orthogonal family of vectors of g -norm $\sqrt{\theta}$ implies that any vector $V \in \mathbb{R}^{2n}$ satisfies the identity

$$g\left(V, \frac{\partial \ell}{\partial u}\right)^2 + g\left(V, \frac{\partial \ell}{\partial v}\right)^2 = \theta g(V^T, V^T),$$

where V^T is the g -orthogonal projection of V onto the plane spanned by $\frac{\partial \ell}{\partial u}$ and $\frac{\partial \ell}{\partial v}$. In other words, V^T is the g -orthogonal projection onto the tangent plane to $\ell(\Sigma)$. Therefore

$$\theta^{-1}g_\sigma(d\theta, d\theta) = 4g\left(\frac{\partial^2 \ell}{\partial u \partial v}{}^T, \frac{\partial^2 \ell}{\partial u \partial v}{}^T\right)$$

and

$$\Delta_\sigma \theta = -2g\left(\frac{\partial^3 \ell}{\partial u \partial v^2}, \frac{\partial \ell}{\partial u}\right) - 2g\left(\frac{\partial^3 \ell}{\partial u^2 \partial v}, \frac{\partial \ell}{\partial v}\right) - 4g\left(\frac{\partial^2 \ell}{\partial u \partial v}, \frac{\partial^2 \ell}{\partial u \partial v}\right).$$

In conclusion

$$\begin{aligned} 2K &= \theta \Delta_\sigma \log \theta \\ &= \Delta_\sigma \theta + \theta^{-1} g_\sigma(d\theta, d\theta) \\ &= -2g\left(\frac{\partial^3 \ell}{\partial u \partial v^2}, \frac{\partial \ell}{\partial u}\right) - 2g\left(\frac{\partial^3 \ell}{\partial u^2 \partial v}, \frac{\partial \ell}{\partial v}\right) - 4g\left(\frac{\partial^2 \ell^\perp}{\partial u \partial v}, \frac{\partial^2 \ell^\perp}{\partial u \partial v}\right), \end{aligned}$$

where \perp denotes the component of a vector orthogonal to the tangent space to ℓ at a point. \square

COROLLARY 5.1.3. – *For all integers $k \geq 0$,*

$$\kappa_N \xrightarrow{\partial^k} K + E.$$

The coefficients of Δ_N are now all understood asymptotically. This completes the proof of the theorem. \square

DEFINITION 5.1.4. – *The operator defined by*

$$\begin{aligned} \Xi(\phi^+, \phi^-) &= \left(\theta \Delta_\sigma \phi^+ - g_\sigma(d\phi^+, d\theta) + (K + E)(\phi^+ - \phi^-), \right. \\ &\quad \left. \theta \Delta_\sigma \phi^- - g_\sigma(d\phi^-, d\theta) + (K + E)(\phi^- - \phi^+) \right) \end{aligned}$$

is called the limit operator of Δ_N .

REMARK 5.1.5. – In particular, the limit operator Ξ is elliptic. This fact will be crucial to derive uniform discrete Schauder estimates for Δ_N .

5.2. Kernel of the limit operator

PROPOSITION 5.2.1. – *A pair of smooth functions (ϕ^+, ϕ^-) is an element of the kernel of the limit operator Ξ if, and only if, there exists some real constants c_0 and c_1 such that*

$$\phi^+ = c_0 + c_1 \theta^{-1}, \quad \phi^- = c_0 - c_1 \theta^{-1}.$$

with $c_1 = 0$, unless the function E vanishes identically on Σ . In particular the kernel of Ξ has dimension 1 or 2 depending on the vanishing of E .

Proof. – The proposition is proved by a straightforward argument using integration by part. A few formulae are needed in order to give a streamlined proof:

LEMMA 5.2.2. – *For every smooth function $f : \Sigma \rightarrow \mathbb{R}$, we have*

$$d^{*\sigma} \theta df = \theta \Delta_\sigma f - g_\sigma(df, d\theta),$$

where d^{σ} is the adjoint of d with respect to the L^2 -inner product induced by g_σ . On the other hand, we have*

$$\theta d^{*\sigma} \theta^{-1} d\theta f = \theta \Delta_\sigma f - g_\sigma(df, d\theta) + 2Kf,$$

where K is the Gauß curvature of g_Σ .

Proof. – For every 1-form β and every function w on Σ , we have $d^{*\sigma}(w\beta) = -*_\sigma d*_\sigma(w\beta) = -*_\sigma d(w*_\sigma\beta) = -*_\sigma(dw \wedge *_\sigma\beta + wd*_\sigma\beta) = wd^{*\sigma}\beta - *_\sigma g_\sigma(dw, \beta)\text{vol}_\sigma = wd^{*\sigma}\beta - g_\sigma(dw, \beta)$. In conclusion

$$d^{*\sigma}(w\beta) = wd^{*\sigma}\beta - g_\sigma(dw, \beta).$$

The first formula of the lemma follows from the above identity. For second identity, we have $\theta^{-1}d(\theta f) = fd \log \theta + df$. Now, $d^{*\sigma}\theta^{-1}d(\theta f) = fd^{*\sigma}d \log \theta - g_\sigma(df, d \log \theta) + d^{*\sigma}df$. We use the fact that the Gauß curvature of g_Σ is given by the formula $2K = \theta\Delta_\sigma \log \theta$ and deduce the second identity of the lemma. \square

We may now complete the proof of Proposition 5.2.1. Let ϕ^\pm be a solution of the system

$$\begin{aligned} \theta\Delta_\sigma\phi^+ - g_\sigma(d\phi^+, d\theta) + (K + E)(\phi^+ - \phi^-) &= 0 \\ \theta\Delta_\sigma\phi^- - g_\sigma(d\phi^-, d\theta) + (K + E)(\phi^- - \phi^+) &= 0. \end{aligned}$$

Adding up the two equations gives the identity

$$\theta\Delta_\sigma(\phi^+ + \phi^-) - \langle d(\phi^+ + \phi^-), d\theta \rangle_\sigma = 0 = d^{*\sigma}\theta d(\phi^+ + \phi^-)$$

by Lemma 5.2.2. Integrating against $\phi^+ + \phi^-$ using the L^2 -inner product induced by g_σ gives $0 = \langle d^{*\sigma}\theta d(\phi^+ + \phi^-), \phi^+ + \phi^- \rangle_{L^2} = \langle \theta d(\phi^+ + \phi^-), d(\phi^+ + \phi^-) \rangle_{L^2}$. Since θ is positive, this forces

$$\phi^+ + \phi^- = 2c_0,$$

for some constant c_0 .

On the other hand the difference of the two equations provides the identity

$$\theta\Delta_\sigma(\phi^+ - \phi^-) - g_\sigma(d(\phi^+ - \phi^-), d\theta) + 2(K + E)(\phi^+ - \phi^-) = 0.$$

by Lemma 5.2.2 we deduce that

$$\theta d^{*\sigma}\theta^{-1}d\theta(\phi^+ - \phi^-) + 2E(\phi^+ - \phi^-) = 0.$$

Integrating the above equation against $\theta(\phi^+ - \phi^-)$ provides the identity

$$\langle \theta d(\phi^+ - \phi^-), d(\phi^+ - \phi^-) \rangle_{L^2} + \langle 2E(\phi^+ - \phi^-), \phi^+ - \phi^- \rangle_{L^2} = 0.$$

Now θ is positive and E is nonnegative, so the two terms of the LHS are non negative: they must vanish both. The vanishing of the first term forces

$$\phi^+ - \phi^- = 2\theta^{-1}c_1,$$

for some real constant c_1 . The vanishing of the second term implies that $c_1 = 0$ unless E vanishes identically on Σ . \square

5.3. Degenerate families of quadrangulations

Proposition 5.2.1 leads us to distinguish two types of constructions.

Recall that the construction of $\mathcal{Q}_N(\Sigma)$ depends on the choice of a Riemannian universal cover $p : \mathbb{R}^2 \rightarrow \Sigma$ for the flat metric g_σ on Σ . Such cover are not unique. They may be, for instance, precomposed with a rotation of \mathbb{R}^2 . Equivalently, we may replace the canonical basis of \mathbb{R}^2 by a rotated basis, which also provides rotated (u, v) -coordinates.

We introduce a definition of degeneracy, bearing on pairs (p, ℓ) , that consists of an isotropic immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ and a choice of Riemannian cover $p : \mathbb{R}^2 \rightarrow \Sigma$ for a flat metric g_σ , in the conformal class of the induced metric g_Σ .

DEFINITION 5.3.1. – *We say that the pair (p, ℓ) is degenerate, if the function $E : \Sigma \rightarrow \mathbb{R}$ defined by (5.1) vanishes identically. Otherwise, we say that the pair (p, ℓ) is nondegenerate.*

EXAMPLE 5.3.2. – An example of degenerate pair is provided by the map

$$\ell : \mathbb{R}^2 \longrightarrow \mathbb{C} \otimes \mathbb{C} \simeq \mathbb{R}^4$$

defined by

$$\ell(x, y) = (\exp(2\pi i u), \exp(2\pi i v)) \in \mathbb{C}^2,$$

where (x, y) are the canonical coordinates of \mathbb{R}^2 and (u, v) are the rotated coordinates defined by (3.12). This map clearly satisfies

$$(5.3) \quad \frac{\partial^2 \ell}{\partial u \partial v} = 0.$$

Moreover, ℓ is invariant under the lattice Γ spanned by $\frac{e_1 + e_2}{\sqrt{2}}$ and $\frac{e_2 - e_1}{\sqrt{2}}$. Hence ℓ descends to a quotient map denoted $\ell : \mathbb{R}^2/\Gamma \rightarrow \mathbb{C}^2$. We obtain a pair (p, ℓ) , where $p : \mathbb{R}^2 \rightarrow \mathbb{R}^2/\Gamma$ is the canonical projection, which is degenerate in the sense of Definition 5.3.1 by (5.3).

Degenerate pairs can create additional technical difficulties. Nevertheless, they may be taken care of with some additional caution (cf. § 5.6). Or they can just be avoided according to the following proposition:

PROPOSITION 5.3.3. – *Given a pair (p, ℓ) , there always exists a rotation r of \mathbb{R}^2 such that $(p \circ r, \ell)$ is non degenerate.*

Proof. – The (u, v) coordinates of \mathbb{R}^2 induce an orthonormal basis of tangent vectors of Σ for the metric g_σ , denoted $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$. If (p, ℓ) is degenerate, \mathbf{II} must vanishes identically on this pair of vector fields. If $(p \circ r, \ell)$ is degenerate for every rotation of \mathbb{R}^2 , the second fundamental form must also vanish for every pair of tangent vectors obtained by rotating the basis $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$. This means that \mathbf{II} vanishes on every pair of orthogonal tangent vectors for g_σ . Since g_Σ is conformal to g_σ , this means that \mathbf{II} must vanish for every pair of orthogonal tangent vectors for g_Σ . This is a contradiction according to the following lemma:

LEMMA 5.3.4. – *For any immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, where Σ is a closed surface diffeomorphic to a torus, there exists a point $x \in \Sigma$ and an orthogonal basis of tangent vectors U, V for the induced metric g_Σ , such that the second fundamental form satisfies $\mathbf{II}(U, V) \neq 0$.*

Proof. – We choose a point $x \in T_x \Sigma$. Assume that $\mathbf{II}(U, V)$ vanishes for every orthonormal basis (U, V) of $T_x \Sigma$. Notice that in this case $(U+V, U-V)$ is an orthogonal basis, hence, by assumption $\mathbf{II}(U+V, U-V) = 0$, and we have

$$\mathbf{II}(U, U) = \mathbf{II}(U+V, U-V) + \mathbf{II}(V, V) = \mathbf{II}(V, V).$$

By the Gauß Theorema Egregium, the curvature K of g_Σ is given by

$$K = -g(\mathbf{II}(U, V), \mathbf{II}(V, U)) + g(\mathbf{II}(U, U), \mathbf{II}(V, V)).$$

According to our discussion, we deduce that

$$K = g(\mathbf{II}(U, U), \mathbf{II}(U, U)) \geq 0.$$

By the Gauß-Bonnet formula, a torus with nonnegative curvature has vanishing curvature. Thus $K = 0$, and as a corollary $\mathbf{II}(U, U) = 0$, which implies that $\mathbf{II} = 0$. In conclusion the image of $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ is totally geodesic. The only totally geodesic surfaces of \mathbb{R}^{2n} are 2-planes. This forces the image of ℓ to be contained in a plane. This is not possible for an immersion of a compact surface. \square

In conclusion there is a choice of rotation r such that $(p \circ r, \ell)$ is nondegenerate, which proves the proposition. \square

5.4. Schauder Estimates

The following result is a consequence of a theorem of Thomée, stated in a broader context [10], for various elliptic finite difference operators, in the case of domains of \mathbb{R}^n covered by square lattices of step $h = N^{-1}$. We provide here a statement adapted to the torus Σ identified to quotients \mathbb{R}^2/Γ_N endowed with its spaces of discrete functions.

THEOREM 5.4.1 (Thomée type theorem). – *There exists a constant $c_1 > 0$ such that for all $N \geq 0$ and for all functions $\psi \in C^2(Q_N(\Sigma))$, we have*

$$\|P_N \psi\|_{C_w^{0,\alpha}} + \|\psi\|_{C^0} \geq c_1 \|\psi\|_{C_w^{2,\alpha}},$$

where

$$P_N = \hat{\Delta}_N^E + \Gamma_N^E.$$

Proof. – Proposition 4.7.4 can be readily used to prove an analog of Theorem 5.1.1 for the operators P_N . In other words, for every $k \geq 2$, for every sequence $\psi_{N_j} \in C_+^2(Q_{N_j}(\Sigma))$ such that (cf. (2.5) and Lemma 2.3.4 as well)

$$(5.4) \quad \psi_{N_j} \xrightarrow{C^k} \phi \quad \Rightarrow \quad P_{N_j} \psi_{N_j} \xrightarrow{C^{k-2}} \Delta_\ell \phi.$$

The operators P_N admit canonical lifts $\tilde{P}_N : C_+^2(Q_N(\mathbb{R}^2)) \rightarrow C_+^2(Q_N(\mathbb{R}^2))$. The elliptic operator Δ_ℓ can also be lifted as an elliptic operator with smooth coefficients $\tilde{\Delta}_\ell$ acting on functions on the plane. By Property (5.4), the discrete operators $\tilde{P}_N : C_+^2(Q_N(\mathbb{R}^2)) \rightarrow C_+^2(Q_N(\mathbb{R}^2))$ converge toward the elliptic operator $\tilde{\Delta}_\ell$. This implies that the sequence of discrete operators \tilde{P}_N is consistent with the elliptic operator $\tilde{\Delta}_\ell$ and that the operators \tilde{P}_N must be elliptic, for N sufficiently large, in the sense of Thomée [10].

We consider a fundamental domain \mathcal{D} of the action of Γ on \mathbb{R}^2 . For $r_0 > 0$ sufficiently large, $\mathcal{D} \subset B(0, r_0)$. We define

$$\Omega_0 = B(0, r_0 + 1), \quad \Omega_1 = B(0, r_0 + 2) \text{ and } \Omega_2 = \mathbb{R}^2,$$

where $B(0, r)$ is an Euclidean ball of \mathbb{R}^2 or radius r , centered at the origin. By definition we have compact embeddings of the domains $\Omega_0 \Subset \Omega_1 \Subset \Omega_2$.

The finite differences (3.17) and (3.18) used to obtain the discrete finite difference operators \tilde{P}_N correspond to the finite differences defined in [10], modulo a translation operator for the retrograde differences. It follows that [10, Theorem 2.1] applies in our setting: there exists a constant $c > 0$ such that for every N sufficiently large and $\phi \in C_+^2(\mathbb{R}^2)$,

$$(5.5) \quad \|\phi\|_{\mathcal{C}_T^{2,\alpha}(\Omega_1)} \leq c \left\{ \|\tilde{P}_N \phi\|_{\mathcal{C}_T^{0,\alpha}(\Omega_2)} + \|\phi\|_{\mathcal{C}_T^0(\Omega_2)} \right\}.$$

REMARK 5.4.2. – In the above notations, the $\mathcal{C}_T^{k,\alpha}(\Omega_i)$ -norm on $C_+^2(Q_N(\mathbb{R}^2))$ are the norms defined in [10], using only forward differences. For $\Omega_2 = \mathbb{R}^2$, these norms coincide with the $\mathcal{C}^{k,\alpha}$ -norms introduced at § 3.7.

If $\Omega = B(0, R)$, the definition of the norms given at (3.10) and (3.21) has to be modified slightly for the $\mathcal{C}_T^{k,\alpha}(\Omega)$ -norm. In order to describe what has to be modified, assume for a moment that ϕ is a discrete function defined only on the set of vertices of $\mathcal{G}_N^+(\mathbb{R}^2)$ contained in $\bar{\Omega}$. Notice that the finite differences $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ are defined on a smaller set, and the second order partial derivative on an even smaller set, etc. The $\mathcal{C}_T^{k,\alpha}(\Omega)$ -norms are defined similarly to the $\mathcal{C}^{k,\alpha}$ -norms, by taking the corresponding sup on a smaller set of vertices. Namely, the vertices of $\mathcal{G}_N^+(\mathbb{R}^2)$ contained in $\bar{\Omega}$ where the relevant partial derivatives are well defined.

For $\psi \in C_+^2(Q_N(\Sigma))$, we define the lift $\phi_N = \psi \circ p_N$. By Remark 5.4.2, since $\Omega_2 = \mathbb{R}^2$, the RHS of (5.5) applied to ϕ_N is equal to

$$c \left\{ \|\tilde{P}_N \phi_N\|_{\mathcal{C}^{0,\alpha}} + \|\phi_N\|_{\mathcal{C}^0} \right\}.$$

By definition of the discrete Hölder norms, $\|\psi\|_{\mathcal{C}^0} = \|\phi_N\|_{\mathcal{C}^0}$ and since $\tilde{P}_N \phi_N = (P_N \psi) \circ p_N$, we have

$$\|\tilde{P}_N \phi_N\|_{\mathcal{C}^{0,\alpha}} = \|P_N \psi\|_{\mathcal{C}^{0,\alpha}}.$$

Thus by (5.5)

$$(5.6) \quad \|\psi \circ p_N\|_{\mathcal{C}_T^{2,\alpha}(\Omega_1)} \leq c \left\{ \|P_N \psi\|_{\mathcal{C}^{0,\alpha}} + \|\psi\|_{\mathcal{C}^0} \right\}.$$

We conclude using the following result

LEMMA 5.4.3. – *There exists a constant $c' > 0$ such that for every N sufficiently large and all $\psi \in C_+^2(Q_N(\Sigma))$*

$$c' \|\psi\|_{\mathcal{C}^{2,\alpha}} \leq \|\psi \circ p_N\|_{\mathcal{C}_T^{2,\alpha}(\Omega_1)}.$$

Proof of the lemma. – By definition, Ω_0 contains a fundamental domain \mathcal{D} of Γ , and furthermore $\mathcal{D} \Subset \Omega_0$. By construction, the lattices Γ_N admit fundamental domains \mathcal{D}_N which converge (say in Hausdorff distance) toward \mathcal{D} . Therefore $\mathcal{D}_N \Subset \Omega_0$ for all N sufficiently large.

In particular every vertex $\mathbf{z} \in \mathcal{G}_N^+(\Sigma)$ admits a lift $\tilde{\mathbf{z}} \in \mathcal{G}_N^+(\mathbb{R}^2)$ via p_N such that $\mathbf{z} \in \Omega_0$. This shows that

$$\|\psi\|_{\mathcal{C}^0} \leq \|\psi \circ p_N\|_{\mathcal{C}_T^0(\Omega_1)}.$$

If N is sufficiently large, the finite differences of order 1 or 2 of $\psi \circ p_N$ are well defined at $\tilde{\mathbf{z}}$ depend only on values taken by the function on the domain $\overline{\Omega}_1$. It follows by Remark 5.4.2 that

$$(5.7) \quad \|\psi \circ p_N\|_{\mathcal{C}^2} \leq \|\psi \circ p_N\|_{\mathcal{C}_T^2(\Omega_1)}.$$

If ξ is any discrete function in $C^0(\mathcal{G}_N^+(\mathbb{R}^2))$, for every pair of vertices $\mathbf{v}_0, \mathbf{v}'$ of $\mathcal{G}_N^+(\mathbb{R}^2)$ with $\mathbf{v}_0 \in \overline{\Omega}_0$ and $\mathbf{v}' \notin \overline{\Omega}_1$, we have

$$\frac{|\xi(\mathbf{v}_0) - \xi(\mathbf{v}')|}{\|\mathbf{v}_0 - \mathbf{v}'\|^\alpha} \leq |\xi(\mathbf{v}_0) - \xi(\mathbf{v}')| \leq 2\|\xi\|_{\mathcal{C}^0}$$

since $\|\mathbf{v}_0 - \mathbf{v}'\| \geq 1$. We apply this inequality to the second order finite differences of $\psi \circ p_N$. This shows that the $\mathcal{C}^{2,\alpha}$ -norm of $\psi \circ p_N$ is controlled by its \mathcal{C}^2 -norm and its $\mathcal{C}_T^{2,\alpha}(\Omega_1)$ -norm. Hence by (5.7) the $\mathcal{C}_T^{2,\alpha}(\Omega_1)$ -norm controls the $\mathcal{C}^{2,\alpha}$ -norm. \square

Using the lemma and (5.6), we deduce that for every N sufficiently large and $\psi \in C_+^2(Q_N(\Sigma))$, we have

$$c' \|\psi\|_{\mathcal{C}^{2,\alpha}} = c' \|\psi \circ p_N\|_{\mathcal{C}^{2,\alpha}} \leq c' \left\{ \|P_N \psi\|_{\mathcal{C}^{0,\alpha}} + \|\psi\|_{\mathcal{C}^0} \right\}.$$

This proves the theorem for N sufficiently large and $\psi \in C_+^2(Q_N(\Sigma))$. The same result holds if $\psi \in C_-^2(Q_N(\Sigma))$. For a general ψ , we use the decomposition in components $\psi = \psi^+ + \psi^-$ and the theorem follows, for N sufficiently large, by definition of the weak Hölder norms. If the theorem holds for N sufficiently large, it holds for every N since $C^2(Q_N(\Sigma))$ is finite dimensional, and all norms are equivalent. \square

COROLLARY 5.4.4. – *There exists a constant $c_2 > 0$ such that for all $N \geq 0$ and for all functions $\phi \in C^2(Q_N(\Sigma))$, we have*

$$\|\Delta_N \phi\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi\|_{\mathcal{C}^0} \geq c_2 \|\phi\|_{\mathcal{C}_w^{2,\alpha}}.$$

Proof. – We use the decomposition $\phi = \phi^+ + \phi^-$ and prove the corollary in the case of the operator

$$\Delta'_N = \hat{\Delta}_N^E + \Gamma_N^E + K_N^E + K_N^I$$

first.

Then $\Delta'_N \phi = \psi = \psi^+ + \psi^-$, where $\psi^+ = P_N \phi^+ + \kappa_N(\phi^+ - \bar{\phi}^-)$ and $\psi^- = P_N \phi^- + \kappa_N(\phi^- - \bar{\phi}^+)$. Since κ_N converges in the sense of Lemma 5.1.3, we deduce that $\|\kappa_N\|_{\mathcal{C}_w^{0,\alpha}}$ is uniformly bounded for all N . Thus a $\mathcal{C}_w^{0,\alpha}$ -bound on ϕ provides a $\mathcal{C}_w^{0,\alpha}$ -bound on $\kappa_N \phi^\pm$. Similarly a $\mathcal{C}_w^{0,\alpha}$ -bound provides $\mathcal{C}_w^{0,\alpha}$ -bound on $\bar{\phi}^\pm$. In other words, there exists a constant $c' > 0$ independent of N and ϕ such that

$$\|\kappa_N(\phi^+ - \bar{\phi}^-)\|_{\mathcal{C}_w^{0,\alpha}} \leq c' \|\phi\|_{\mathcal{C}_w^{0,\alpha}} \text{ and } \|\kappa_N(\phi^- - \bar{\phi}^+)\|_{\mathcal{C}_w^{0,\alpha}} \leq c' \|\phi\|_{\mathcal{C}_w^{0,\alpha}}.$$

It follows that

$$\|\Delta'_N \phi\|_{\mathcal{C}_w^{0,\alpha}} + 2c' \|\phi\|_{\mathcal{C}_w^{0,\alpha}} \geq \|P_N \phi\|_{\mathcal{C}_w^{0,\alpha}},$$

and by Theorem 5.4.1

$$(5.8) \quad \|\Delta'_N \phi\|_{\mathcal{C}_w^{0,\alpha}} + (2c' + 1) \|\phi\|_{\mathcal{C}_w^{0,\alpha}} \geq c \|\phi\|_{\mathcal{C}_w^{2,\alpha}}.$$

We are not quite finished since we have a $\mathcal{C}_w^{0,\alpha}$ -estimate for ϕ in the above inequality rather than a \mathcal{C}^0 -estimate as in the corollary. We prove a weaker version of the corollary first: we show that there exists a constant $c'' > 0$ such that for all N and for all ϕ ,

$$(5.9) \quad \|\Delta'_N \phi\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi\|_{\mathcal{C}^0} \geq c'' \|\phi\|_{\mathcal{C}_w^{0,\alpha}}.$$

If this is true, the corollary trivially follows in the case of Δ'_N from (5.8) and (5.9). Finally, Proposition 4.7.5 completes the proof in the case of $\Delta_N = \Delta'_N + \mathcal{E}_N^E + \mathcal{E}_N^I$.

Assume that (5.9) does not hold. Then there exists a sequence of discrete functions $\phi_{N_k} \in \mathcal{C}^2(\mathcal{Q}_{N_k}(\Sigma))$ with the property that

$$\|\phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} = 1, \quad \|\Delta'_{N_k} \phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} \rightarrow 0 \text{ and } \|\phi_{N_k}\|_{\mathcal{C}^0} \rightarrow 0.$$

Using Inequality (5.8), we obtain a uniform $\mathcal{C}_w^{2,\alpha}$ -bound on ϕ_{N_k} . By the Ascoli-Arzelà Theorem 3.10.6, we may assume up to extraction of a subsequence, that ϕ_{N_k} converges in the \mathcal{C}_w^2 -sense toward a pair of functions (ϕ^+, ϕ^-) on Σ . Since the convergence is \mathcal{C}_w^2 hence \mathcal{C}^0 , the condition $\|\phi_{N_k}\|_{\mathcal{C}^0} \rightarrow 0$ forces $\phi^+ = \phi^- = 0$. This implies that $\phi_{N_k} \xrightarrow{\mathcal{C}_w^2} (0, 0)$, and in particular $\|\phi_{N_k}\|_{\mathcal{C}_w^2} \rightarrow 0$. Since the \mathcal{C}_w^2 -discrete norm controls the $\mathcal{C}_w^{0,\alpha}$ -discrete norm, this contradicts the assumption $\|\phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} = 1$. \square

5.5. Spectral gap

We define the discrete functions

$$\mathbf{1}_N^\pm \in C^0(\mathcal{C}_N^\pm(\Sigma)) \simeq C_\pm^2(\mathcal{Q}_N(\Sigma))$$

by

$$\langle \mathbf{1}_N^\pm, \mathbf{z} \rangle = \begin{cases} 1 & \text{if } \mathbf{z} \in \mathfrak{C}_0(\mathcal{G}_N^\pm(\Sigma)) \\ 0 & \text{if } \mathbf{z} \in \mathfrak{C}_0(\mathcal{G}_N^\mp(\Sigma)). \end{cases}$$

We also define the discrete functions $\mathbf{1}_N, \zeta_N \in C^0(\mathcal{G}_N(\Sigma))$ by

$$\begin{aligned} \mathbf{1}_N &= \mathbf{1}_N^+ + \mathbf{1}_N^- \\ \zeta_N &= \theta_N^{-1} \cdot \mathbf{1}_N^+ - \theta_N^{-1} \cdot \mathbf{1}_N^-, \end{aligned}$$

where θ_N is any discrete function, sufficiently close to θ_N^u or θ_N^v . For instance, we put

$$\theta_N = \frac{1}{2}(\theta_N^u + \theta_N^v).$$

We define the spaces of discrete functions $\mathcal{K}_N \subset C^0(\mathcal{G}_N(\Sigma))$ by

$$(5.10) \quad \mathcal{K}_N = \begin{cases} \mathbb{R} \cdot \mathbf{1}_N & \text{if } (p, \ell) \text{ is nondegenerate,} \\ \mathbb{R} \cdot \mathbf{1}_N \oplus \mathbb{R} \cdot \zeta_N & \text{in the degenerate case.} \end{cases}$$

In addition, we denote by

$$\mathcal{K}_N^\perp \subset C^2(\mathcal{Q}_N(\Sigma))$$

the orthogonal complement of \mathcal{K}_N , with respect to the $\langle\langle \cdot, \cdot \rangle\rangle$ -inner product.

REMARKS 5.5.1. – The function $\mathbf{1}_N$ and more generally, any constant function, is contained in the kernel of the operator Δ_N . Indeed, $\Delta_N = \delta_N \delta_N^*$, but $d^* \mathbf{1}_N = 0$ by Formula (4.11).

— The sequence of discrete functions ζ_N converges toward the pair of functions $(\theta^{-1}, -\theta^{-1})$, at least in the \mathcal{C}_w^2 -sense. Whenever ℓ is degenerate, we must have

$$\Delta_N \zeta_N \xrightarrow{\mathcal{C}_w^0} (0, 0),$$

by Theorem 5.1.1 and Proposition 5.2.1.

— The kernel of Δ_N is at least 1 dimensional. If ℓ is degenerate, our next result at Theorem 5.5.2, implies that for N sufficiently large, $\ker \Delta_N$ has dimension at most 2. Although ζ_N may not belong to $\ker \Delta_N$, the previous remark shows that this function is approximately in the kernel. In this sense, \mathcal{K}_N may be thought of as an approximate kernel of Δ_N .

THEOREM 5.5.2. – *There exists a real constant $c_3 > 0$ such that, for all positive integers N sufficiently large and for all discrete function $\phi \in \mathcal{K}_N^\perp$, we have*

$$\|\Delta_N \phi\|_{\mathcal{C}_w^{0,\alpha}} \geq c_3 \|\phi\|_{\mathcal{C}_w^{2,\alpha}}.$$

Proof. – We are assuming that ℓ is degenerate. Since the proof in the nondegenerate case is completely similar, we leave the details to the reader. We start by proving a weaker version of the theorem:

LEMMA 5.5.3. – *There exists a real constant $c_4 > 0$ such that, for all positive integers N sufficiently large and for all discrete function $\phi \in \mathcal{K}_N^\perp$, we have*

$$\|\Delta_N \phi\|_{\mathcal{C}_w^{0,\alpha}} \geq c_4 \|\phi\|_{\mathcal{C}^0}.$$

Proof of Lemma 5.5.3. – Assume that the result is false. Then there exists a sequence $\phi_{N_k} \in \mathcal{K}_{N_k}^\perp$ such that

$$\forall k \quad \|\phi_{N_k}\|_{\mathcal{C}^0} = 1, \text{ and } \|\Delta_{N_k} \phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} \longrightarrow 0.$$

Using Corollary 5.4.4, we deduce a $\mathcal{C}^{2,\alpha}$ -bound on ϕ_{N_k} . Thanks to the Ascoli-Arzelà Theorem 3.10.6, we may assume that ϕ_{N_k} converges in the weak \mathcal{C}^2 -sense, up to further extraction:

$$(5.11) \quad \phi_{N_k} \xrightarrow{\mathcal{C}_w^2} (\phi^+, \phi^-).$$

By Theorem 5.1.1, we conclude that

$$\Delta_{N_k} \phi_{N_k} \xrightarrow{\mathcal{C}^0} \Xi(\phi^+, \phi^-).$$

The condition $\|\Delta_{N_k} \phi_{N_k}\|_{\mathcal{C}^{0,\alpha}} \rightarrow 0$ implies that $\|\Delta_{N_k} \phi_{N_k}\|_{\mathcal{C}^0} \rightarrow 0$, which shows that the limit is $(0, 0)$. Therefore

$$(5.12) \quad (\phi^+, \phi^-) \in \ker \Xi.$$

We are assuming now that we are in the degenerate case as before. The nondegenerate case is treated similarly. By assumption ϕ_{N_k} is orthogonal to \mathcal{K}_N , hence $\langle\langle \phi_{N_k}, \mathbf{1}_{N_k} \rangle\rangle = \langle\langle \phi_{N_k}, \zeta_{N_k} \rangle\rangle = 0$. Since all these discrete functions converge in the \mathcal{C}^0 -sense, we deduce that the limit also satisfy the orthogonality relation, that is

$$\langle\langle (\phi^+, \phi^-), (\mathbf{1}, \mathbf{1}) \rangle\rangle = \langle\langle (\phi^+, \phi^-), (\theta^{-1}, -\theta^{-1}) \rangle\rangle = 0.$$

In other words (ϕ^+, ϕ^-) is L^2 -orthogonal to $\ker \Xi$. In view of (5.12) we deduce that

$$\phi^+ = \phi^- = 0,$$

and by (5.11), we deduce that

$$\|\phi_{N_k}\|_{\mathcal{C}^0} \longrightarrow 0,$$

which contradicts the assumption $\|\phi_{N_k}\|_{\mathcal{C}^0} = 1$. This completes the proof of the lemma. \square

By Lemma 5.5.3, we have for every $\phi \in \mathcal{K}_N^\perp$

$$(1 + c_4^{-1}) \|\Delta_N \phi\|_{\mathcal{C}_w^{0,\alpha}} \geq \|\Delta_N \phi\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi\|_{\mathcal{C}^0}.$$

By Corollary 5.4.4, the RHS is an upper bound for $c_2 \|\phi\|_{\mathcal{C}_w^{2,\alpha}}$. The constant,

$$c_3 = \frac{c_2}{1 + c_4^{-1}}$$

satisfies the theorem, which completes the proof. \square

COROLLARY 5.5.4. – 1. *If (p, ℓ) is nondegenerate, then for every N sufficiently large, the kernel of Δ_N is given by $\ker \Delta_N = \mathcal{K}_N = \mathbb{R} \cdot \mathbf{1}_N$. Furthermore \mathcal{K}_N^\perp is preserved by Δ_N which induces an isomorphism $\Delta_N : \mathcal{K}_N^\perp \rightarrow \mathcal{K}_N^\perp$.*

2. More generally, including the case where (p, ℓ) is degenerate, there is a direct sum decomposition for every N sufficiently large

$$C^2(\mathcal{Q}_N(\Sigma)) = \mathcal{K}_N \oplus \Delta_N(\mathcal{K}_N^\perp),$$

and a constant c_6 independent of N , such that for all $\phi \in C^2(\mathcal{Q}_N(\Sigma))$ decomposed according to the above splitting as $\phi = \bar{\phi} + \phi^\Delta$, we have

$$(5.13) \quad c_6 \|\phi\|_{\mathcal{C}_w^{0,\alpha}} \geq \|\bar{\phi}\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi^\Delta\|_{\mathcal{C}_w^{0,\alpha}}.$$

Proof. – The first statement is a consequence of the second statement: In the nondegenerate case, \mathcal{K}_N is one dimensional and $\mathcal{K}_N \subset \ker \Delta_N$. Hence $\Delta(\mathcal{K}_N^\perp)$ has codimension at most 1 in $C^2(\mathcal{Q}_N(\Sigma))$. By the second statement the codimension is exactly 1. Therefore $\ker \Delta_N = \mathcal{K}_N$. The rest of the statement follows using the fact that Δ_N is selfadjoint.

We merely have to prove the second statement of the corollary. We start by proving that we have a splitting as claimed. Suppose that the intersection $\mathcal{K}_N \cap \Delta(\mathcal{K}_N^\perp)$ is not reduced to 0 for arbitrarily large N . Then we may find a sequence ϕ_{N_k} contained in the intersections and such that $\|\phi_{N_k}\|_{\mathcal{C}^0} = 1$.

We notice that $\|\mathbf{1}_N\|_{\mathcal{C}^0} = 1$ and that $\|\zeta_N\|_{\mathcal{C}^0}$ converges toward a positive constant, since ζ_N converges toward the pair of functions $(\theta^{-1}, -\theta^{-1})$. Since $\phi_{N_k} \in \mathcal{K}_{N_k}$, we may write

$$\phi_{N_k} = a_k \mathbf{1}_{N_k} \text{ resp. } \phi_{N_k} = a_k \mathbf{1}_{N_k} + b_k \zeta_{N_k} \text{ in the degenerate case.}$$

We deduce that the uniform \mathcal{C}^0 -bound on ϕ_{N_k} provides a uniform bound on the coefficients a_k and b_k . We may after extracting a suitable subsequence assume that the coefficients converge as k goes to infinity. In particular ϕ_{N_k} converges toward an element of $\ker \Xi$, say in the \mathcal{C}^0 -sense. By construction we have a uniform \mathcal{C}_w^1 -bound on Φ_{N_k} , which provides a uniform $\mathcal{C}_w^{0,\alpha}$ -bound.

On the other hand $\phi_{N_k} \in \Delta_{N_k}(\mathcal{K}_{N_k}^\perp)$ so that there exists a sequence $\psi_{N_k} \in \mathcal{K}_{N_k}^\perp$ with $\phi_{N_k} = \Delta_{N_k} \psi_{N_k}$. By Theorem 5.5.2, the uniform $\mathcal{C}_w^{0,\alpha}$ -bound on ϕ_{N_k} provides a uniform $\mathcal{C}_w^{2,\alpha}$ -bound on ψ_{N_k} . By Ascoli-Arzelà Theorem 3.10.6, we may assume that ψ_{N_k} converges in the \mathcal{C}_w^2 -sense toward a limit (ψ^+, ψ^-) after extraction. It follows that $\phi_{N_k} = \Delta_{N_k} \psi_{N_k}$ converges in the \mathcal{C}^0 -sense toward $\Xi(\psi^+, \psi^-)$. In conclusion ϕ_{N_k} converges in the \mathcal{C}^0 -sense to an element of $\text{Im} \Xi \cap \ker \Xi = \{0\}$.

In conclusion, the limit of ϕ_{N_k} must be the pair of functions $(0, 0)$, which contradicts the fact that $\|\phi_{N_k}\|_{\mathcal{C}^0} = 1$. Thus

$$\mathcal{K}_N \cap \Delta_N(\mathcal{K}_N^\perp) = \{0\},$$

for all sufficiently large N . By Theorem 5.5.2, we know that the restriction of Δ_N to \mathcal{K}_N^\perp is injective provided N is large enough. For dimensional reasons, we have a splitting

$$\mathcal{K}_N \oplus \Delta_N(\mathcal{K}_N^\perp) = C^2(\mathcal{Q}_N(\Sigma)).$$

We now proceed to the last part of the second statement. If the control (5.13) does not hold, we find a sequence of discrete functions $\phi_{N_k} \in C^2(\mathcal{Q}_{N_k}(\Sigma))$ with decompositions

$$\phi_{N_k} = \bar{\phi}_{N_k} + \phi_{N_k}^\Delta,$$

and the property that

$$\|\phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} \rightarrow 0 \text{ and } \|\bar{\phi}_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi_{N_k}^\Delta\|_{\mathcal{C}_w^{0,\alpha}} = 1.$$

The $\mathcal{C}_w^{0,\alpha}$ -bound on $\bar{\phi}_{N_k}$ provides a uniform \mathcal{C}^0 -bound. As in the first part of the proof, we may use this bound to show that, up to extraction of a subsequence, $\bar{\phi}_{N_k}$ converges in the \mathcal{C}_w^1 -sense toward a limit $(\bar{\phi}^+, \bar{\phi}^-) \in \ker \Xi$.

Similarly, the $\mathcal{C}_w^{0,\alpha}$ bound on $\phi_{N_k}^\Delta$ can be used to show that, up to extraction of a subsequence, the sequence converges in the \mathcal{C}^0 -sense toward a limit $(\phi_\Delta^+, \phi_\Delta^-)$ in the image of Ξ .

Eventually, we may assume that ϕ_{N_k} converges in the \mathcal{C}^0 -sense toward a limit $(\bar{\phi}^+ + \phi_\Delta^+, \bar{\phi}^- + \phi_\Delta^-) \in \ker \Xi \oplus \text{Im} \Xi$. The fact that $\|\phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} \rightarrow 0$ implies that $\|\phi_{N_k}\|_{\mathcal{C}^0} \rightarrow 0$ and by uniqueness of the limit, we deduce that $\bar{\phi}^\pm = 0$ and $\phi_\Delta = 0$.

However, $\bar{\phi}_{N_k}$ converges in the stronger, says, \mathcal{C}_w^1 -sense, hence $\|\bar{\phi}_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} \rightarrow 0$. We deduce that

$$\|\bar{\phi}_{N_k}\| = \|\phi_{N_k} - \phi_{N_k}^\Delta\|_{\mathcal{C}_w^{0,\alpha}} \leq \|\phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi_{N_k}^\Delta\|_{\mathcal{C}_w^{0,\alpha}} \rightarrow 0,$$

which contradicts the assumption $\|\phi_{N_k}\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi_{N_k}^\Delta\|_{\mathcal{C}_w^{0,\alpha}} = 1$. □

5.6. Modified construction in the degenerate case

The situation for degenerate pairs (p, ℓ) came as a surprise to us. Our first guess was that the operators Δ_N should converge in a reasonable sense toward the operator involved in the smooth setting (2.5). Consequently, we expected the kernel of Δ_N to be one dimensional, at least for N large enough. The first clue that this was not true came from a local model: in this model, we do not choose Σ to be a torus, but a copy of \mathbb{R}^2 embedded in \mathbb{R}^{2n} as an isotropic Euclidean plane identified to \mathbb{R}^2 with its quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$. Then one can check that the function $\mathbf{1}_N^+ - \mathbf{1}_N^-$ belongs to the kernel of δ_N^* directly from the Formula (4.11).

The presence of a 2-dimensional almost kernel \mathcal{K}_N in the degenerate case will create some trouble for solving our problem. We may overcome them by changing slightly our construction.

5.6.1. The setup. – We start with a degenerate pair (p_S, ℓ_S) , where

$$\ell_S : S \longrightarrow \mathbb{R}^4$$

is an isotropic immersion and S is a surface diffeomorphic to an oriented torus. We carry out the constructions of quadrangulations $\mathcal{Q}_N(S)$, graphs $\mathcal{G}_N(S)$ exactly as in the case of Σ (cf. §3), except one crucial detail. The lattice group $\Gamma(S)$ of the covering map $p_S : \mathbb{R}^2 \rightarrow S$ admits oriented basis $(\gamma_1(S), \gamma_2(S))$. This is where comes the difference with §3.2: we choose a best approximation $\gamma_2^N(S) \in \Lambda_N^{\text{ch}}$ of $\gamma_2(S)$ and $\gamma_1^N(S) \in \Lambda_N \setminus \Lambda_N^{\text{ch}}$ for $\gamma_1(S)$. Notice that in the case of Σ , both $\gamma_i^N(S)$ were chosen in Λ_N^{ch} .

This minor change still allows us to construct families of quadrangulations and checkers graph. The only difference is that action of the lattice

$$\Gamma_N(S) = \text{span}\{\gamma_1^N(S), \gamma_2^N(S)\}$$

does not preserve the connected components of the decomposition

$$\mathcal{G}_N(\mathbb{R}^2) = \mathcal{G}_N^+(\mathbb{R}^2) \cup \mathcal{G}_N^-(\mathbb{R}^2),$$

and this splitting does not descend as a splitting of $\mathcal{G}_N(S)$. In particular discrete functions $\phi \in C^2(\mathcal{Q}_N(S))$ do not split into a positive and negative component. However, we may construct the constant function $\mathbf{1}_N \in C^2(\mathcal{Q}_N(S))$, which is the constant 1 on every face of the quadrangulation.

5.6.2. The double cover. – We define

$$\Gamma = \text{span}\{\gamma_1, \gamma_2\},$$

where

$$\gamma_1 = 2\gamma_1(S) \text{ and } \gamma_2 = \gamma_2(S).$$

The quotient $\Sigma = \mathbb{R}^2/\Gamma$ comes with a covering map of index 2

$$(5.14) \quad \Phi^S : \Sigma \rightarrow S,$$

and an action of $G \simeq \Gamma/\Gamma(S) \simeq \mathbb{Z}_2$ on Σ by deck transformations. We define accordingly

$$\Gamma_N = \text{span}\{\gamma_1^N, \gamma_2^N\},$$

where

$$\gamma_1^N = 2\gamma_1^N(S) \text{ and } \gamma_2^N = \gamma_2^N(S).$$

Notice that $2\Lambda_N \subset \Lambda_N^{\text{ch}}$, hence by definition $\gamma_i^N \in \Lambda_N^{\text{ch}}$. We also have double covers

$$\Phi_N^S : \Sigma \rightarrow S$$

with deck transformations $G_N \simeq \Gamma_N/\Gamma_N(S) \simeq \mathbb{Z}_2$ which come from the canonical projections $\mathbb{R}^2/\Gamma_N \rightarrow \mathbb{R}^2/\Gamma_N(S)$. In particular there are canonical embeddings of discrete functions spaces induced by pullback

$$(\Phi_N^S)^* : C^2(\mathcal{Q}_N(S)) \longrightarrow C^2(\mathcal{Q}_N(\Sigma)).$$

The action of G_N induces an action on $C^2(\mathcal{Q}_N(\Sigma))$ and the image of $(\Phi_N^S)^*$ consists of the discrete functions which are G_N -invariant.

5.6.3. Meshes and operators for the modified construction. – Like for Σ , we may define the samples $\tau_N^S \in \mathcal{M}_N(S) = C^0(Q_N(S)) \otimes \mathbb{R}^{2n}$ of the map $\ell_S : S \rightarrow \mathbb{R}^{2n}$, the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ and the operators δ_N, δ_N^* , etc. Using the canonical projections

$$(\Phi_N^S)^* : C^0(Q_N(S)) \otimes \mathbb{R}^{2n} \longrightarrow C^0(Q_N(\Sigma)) \otimes \mathbb{R}^{2n},$$

we see that the pullbacks satisfy $\tau_N = (\Phi_N^S)^* \tau_N(S)$. In other words, they are also the samples of the lifted isotropic immersion $\ell = \ell_S \circ \Phi^S : \Sigma \rightarrow \mathbb{R}^{2n}$. Then τ_N also induces operators denoted δ_N, δ_N^* and Δ_N which commute with the pullback operation, by naturality of the construction.

5.6.4. Spectral gap for the degenerate case. – All the norms defined on $C^2(Q_N(\Sigma))$ induce norms on $C^2(Q_N(S))$ via the pullbacks $(\Phi_N^S)^*$, denoted in the same way. For instance, for $\phi \in C^2(Q_N(S))$, we have

$$\|\phi\|_{\mathcal{C}^{2,\alpha}} = \|\phi \circ \Phi_N^S\|_{\mathcal{C}^{2,\alpha}}.$$

Then we prove the following result:

THEOREM 5.6.5. – *Let $\ell_S : S \rightarrow \mathbb{R}^{2n}$ be an isotropic immersion of an oriented surface diffeomorphic to a torus with a conformal cover $p : \mathbb{R}^2 \rightarrow S$. There exists a constant $c_5 > 0$ such that for every N sufficiently large and every $\phi \in C^2(Q_N(S))$ with $\langle\langle \phi, \mathbf{1}_N \rangle\rangle = 0$, we have*

$$\|\Delta_N \phi\|_{\mathcal{C}^{0,\alpha}} \geq c_5 \|\phi\|_{\mathcal{C}^{2,\alpha}}.$$

Proof. – We choose N sufficiently large, so that the assumptions of Theorem 5.5.2 are satisfied. Let $\phi \in C^2(Q_N(S))$ be a discrete function such that $\langle\langle \phi, \mathbf{1}_N \rangle\rangle = 0$. There may be some ambiguity in our notations, so we should emphasize that $(\Phi_N^S)^* \mathbf{1}_N$ is equal to $\mathbf{1}_N \in C^2(Q_N(\Sigma))$.

We consider the pullback $\tilde{\phi}_N = \phi \circ \Phi_N^S$ of ϕ regarded as an element of $C^2(Q_N(\Sigma))$. By definition of inner products and pullbacks by 2-fold covers, we have

$$\langle\langle \phi, \mathbf{1}_N \rangle\rangle = \frac{1}{2} \langle\langle \tilde{\phi}_N, \mathbf{1}_N \rangle\rangle,$$

hence, by assumption, $\langle\langle \tilde{\phi}_N, \mathbf{1}_N \rangle\rangle = 0$.

Notice that the action of $G_N = \langle \Upsilon_N \rangle \simeq \mathbb{Z}_2$ on $C^2(Q_N(\Sigma))$ respects the inner product $\langle\langle \cdot, \cdot \rangle\rangle$. Since $\gamma_1^N(S) \notin \Lambda_N^{\text{ch}}$, we also have

$$\Upsilon_N \cdot \mathbf{1}_N^+ = \mathbf{1}_N^- \text{ and conversely } \Upsilon_N \cdot \mathbf{1}_N^- = \mathbf{1}_N^+.$$

By construction the discrete function θ_N is G_N invariant. Thus

$$\Upsilon_N \cdot \zeta_N = \Upsilon_N \cdot (\theta_N^{-1}(\mathbf{1}_N^+ - \mathbf{1}_N^-)) = \theta_N^{-1}(-\mathbf{1}_N^+ + \mathbf{1}_N^-) = -\zeta_N.$$

The above property implies that any G_N -invariant discrete function is orthogonal to ζ_N :

$$\langle\langle \tilde{\phi}_N, \zeta_N \rangle\rangle = \langle\langle \Upsilon_N \cdot \tilde{\phi}_N, \Upsilon_N \cdot \zeta_N \rangle\rangle = \langle\langle \tilde{\phi}_N, -\zeta_N \rangle\rangle,$$

therefore

$$\langle\langle \tilde{\phi}_N, \zeta_N \rangle\rangle = 0.$$

In conclusion $\tilde{\phi}_N$ is orthogonal to \mathcal{K}_N and we may apply Theorem 5.5.2 to $\tilde{\phi}_N$, which proves the theorem with $c_5 = c_3$. \square

REMARK 5.6.6. – Notice that Theorem 5.6.5 applies whether the pair (p, ℓ_S) is degenerate or nondegenerate. The applications are different from Theorem 5.5.2, in the sense that we are dealing with different type of quadrangulations and meshes. For instance the spaces of quadrangular meshes \mathcal{M}_N admits a shear action whereas $\mathcal{M}_N(S)$ does not. Indeed the checkers graph associated to $\mathcal{Q}_N(S)$ is connected whereas the checkers graph of $\mathcal{Q}_N(\Sigma)$ is not.

CHAPTER 6

FIXED POINT THEOREM

6.1. Fixed point equation

All the tools have been introduced in order to be able to apply the contraction mapping principle. We consider an isotropic immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$, its sequence of samples $\tau_N \in \mathcal{M}_N$ as before and the map

$$F_N : C^2(\mathcal{Q}_N(\Sigma)) \longrightarrow C^2(\mathcal{Q}_N(\Sigma)),$$

defined by

$$F_N(\phi) = \mu_N^r(\tau_N - J\delta_N^*\phi).$$

Solving the equation $F_N(\phi) = 0$ provides an isotropic perturbation of the sample mesh τ_N .

REMARK 6.1.1. – The perturbative approach introduced here is an analog of Theorem 2.4.1 in the smooth setting. Indeed, let us denote by $f_N : \Sigma \rightarrow \mathbb{R}^{2n}$ a smooth perturbation of $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ and h a smooth function on Σ . The perturbation $\tau_N - J\delta_N^*\phi$ is a discrete analog of the smooth perturbation $K(h, f_N) = \exp_{f_N}(-ih) = f_N - \mathfrak{J}\delta_{f_N}^*h$. Thus the equation $F_N(\phi) = 0$ is the discrete analog of the equation $F(h, f_N) = 0$ (cf. (2.7)) in the smooth setting.

The differential of the map is given by $DF_N|_0 \cdot \phi = -D\mu_N^r|_{\tau_N} \circ J\delta_N^*(\phi) = \delta_N\delta_N^*\phi$ hence

$$DF_N|_0 \cdot \phi = \Delta_N\phi.$$

As pointed out in Lemma 4.1.7, the map μ_N^r is quadratic. According to Definition 4.1.8 and (4.4) one can write

$$\Delta_N\phi = 2\Psi_N^r(\tau_N, \phi)$$

and

$$F_N(\phi) = \eta_N + \Delta_N\phi + \mu_N^r(J\delta_N^*\phi),$$

where $\eta_N = F_N(0)$ is the error term. We introduce the space

$$\mathcal{H}_N = \Delta_N(\mathcal{K}_N^\perp),$$

where \mathcal{K}_N is the almost kernel of Δ_N defined at (5.10). By Corollary 5.5.4, we have a direct sum decomposition

$$C^2(\mathcal{Q}_N(\Sigma)) = \mathcal{K}_N \oplus \mathcal{H}_N,$$

for every N sufficiently large.

We define the Green operator G_N of Δ_N by

$$G_N(\psi) = \begin{cases} 0 & \text{if } \psi \in \mathcal{K}_N, \\ \phi \in \mathcal{K}_N^\perp & \text{with the property that } \Delta_N \phi = \psi \text{ if } \psi \in \mathcal{H}_N. \end{cases}$$

The Green operator is bounded independently of N , which is a crucial property for the application of the fixed point principle:

PROPOSITION 6.1.2. – *There exists a constant $c_8 > 0$ such that for all positive integers N and $\phi \in C^2(\mathcal{Q}_N(\Sigma))$, we have*

$$c_8 \|\phi\|_{\mathcal{C}_w^{0,\alpha}} \geq \|G_N(\phi)\|_{\mathcal{C}_w^{2,\alpha}}.$$

Proof. – For every $N \geq N_0$ sufficiently large, we may use the decomposition $\phi = \bar{\phi} + \phi_\Delta$ and the fact that

$$(6.1) \quad c_6 \|\phi\|_{\mathcal{C}_w^{0,\alpha}} \geq \|\bar{\phi}\|_{\mathcal{C}_w^{0,\alpha}} + \|\phi_\Delta\|_{\mathcal{C}_w^{0,\alpha}},$$

thanks to Corollary 5.5.4. By definition $\phi_\Delta = \Delta_N \psi$ for some $\psi \in \mathcal{K}_N^\perp$, and $G_N(\phi) = \psi$. By Theorem 5.5.2 and (6.1), we deduce that

$$c_6 \|\phi\|_{\mathcal{C}_w^{0,\alpha}} \geq \|\phi_\Delta\|_{\mathcal{C}_w^{0,\alpha}} \geq c_3 \|\psi\|_{\mathcal{C}_w^{2,\alpha}}.$$

This proves the proposition. □

Notice that by definition, G_N takes values in \mathcal{K}_N^\perp and has kernel \mathcal{K}_N . If $\phi \in \mathcal{K}_N^\perp$, we have $G_N \circ \Delta_N \phi = \phi$. Therefore

$$G_N \circ F_N(\phi) = \phi + G_N(\eta_N + \mu_N^r(J\delta_N^* \phi)).$$

For $\phi \in \mathcal{K}_N^\perp$, the equation

$$F_N(\phi) \in \mathcal{K}_N$$

is equivalent to

$$\phi = T_N(\phi),$$

where

$$T_N : \mathcal{K}_N^\perp \longrightarrow \mathcal{K}_N^\perp$$

is the map defined by

$$T_N(\phi) = -G_N(\eta_N + \mu_N^r(J\delta_N^* \phi)).$$

We merely need to apply the fixed point principle to the map T_N .

6.2. Contracting map

Notice that

$$\begin{aligned} T_N(\phi) - T_N(\phi') &= G_N \circ \mu_N^r(J\delta_N^*\phi) - G_N \circ \mu_N^r(J\delta_N^*\phi') \\ &= \frac{1}{2}G_N \circ \Psi_N^r(J\delta_N^*\phi' + J\delta_N^*\phi, J\delta_N^*\phi' - J\delta_N^*\phi), \end{aligned}$$

hence

$$(6.2) \quad T_N(\phi) - T_N(\phi') = \frac{1}{2}G_N \circ \Psi_N^r(J\delta_N^*(\phi' + \phi), J\delta_N^*(\phi' - \phi)).$$

PROPOSITION 6.2.1. – *There exists a constant $c_7 > 0$ such that for every $\phi, \phi' \in C^2(Q_N(\Sigma))$, we have*

$$\|\Psi_N^r(J\delta_N^*\phi, J\delta_N^*\phi')\|_{\mathcal{C}_w^{0,\alpha}} \leq c_7 \|\phi\|_{\mathcal{C}_w^{2,\alpha}} \|\phi'\|_{\mathcal{C}_w^{2,\alpha}}.$$

Proof. – Recall that, for $\tau \in \mathcal{M}_N$, $\mu_N^r(\tau) \in C^2(Q_N(\Sigma))$ is given by the formula

$$\langle \mu_N^r(\tau), \mathbf{f} \rangle = \frac{1}{2} \omega(\mathcal{U}_\tau(\mathbf{f}), \mathcal{V}_\tau(\mathbf{f})).$$

We deduce that

$$\langle \Psi_N^r(\tau, \tau'), \mathbf{f} \rangle = \frac{1}{4} \left[\omega(\mathcal{U}_\tau(\mathbf{f}), \mathcal{V}_{\tau'}(\mathbf{f})) + \omega(\mathcal{U}_{\tau'}(\mathbf{f}), \mathcal{V}_\tau(\mathbf{f})) \right],$$

and it follows that for some universal constant $c'_7 > 0$, we have

$$(6.3) \quad \|\Psi_N^r(\tau, \tau')\|_{\mathcal{C}_w^{0,\alpha}} \leq c'_7 \left[\|\mathcal{U}_\tau\|_{\mathcal{C}_w^{0,\alpha}} \|\mathcal{V}_{\tau'}\|_{\mathcal{C}_w^{0,\alpha}} + \|\mathcal{V}_\tau\|_{\mathcal{C}_w^{0,\alpha}} \|\mathcal{U}_{\tau'}\|_{\mathcal{C}_w^{0,\alpha}} \right].$$

LEMMA 6.2.2. – *There exists a universal constant $c''_7 > 0$ such that for all discrete function ϕ and $\tau = J\delta_N^*\phi$, we have*

$$\|\mathcal{U}_\tau\|_{\mathcal{C}_w^{0,\alpha}} \text{ and } \|\mathcal{V}_\tau\|_{\mathcal{C}_w^{0,\alpha}} \leq c''_7 \|\phi\|_{\mathcal{C}_w^{2,\alpha}}.$$

Proof. – We carry out the proof in the case of \mathcal{U}_τ , as the proof for \mathcal{V}_τ is almost identical. Using the index notations, we have

$$\langle \mathcal{U}_\tau, \mathbf{f}_{kl} \rangle = \frac{1}{N} \left(\langle \delta_N^*\phi, \mathbf{v}_{k+1,l+1} \rangle - \langle \delta_N^*\phi, \mathbf{v}_{k,l} \rangle \right).$$

Using the expression of δ_N^* , we obtain

$$\begin{aligned} \langle \mathcal{U}_\tau, \mathbf{f}_{kl} \rangle &= \frac{1}{N^2} \left((\phi_{k+1,l} D_{k+1,l}^u - \phi_{k,l-1} D_{k,l-1}^u) - (\phi_{k,l+1} D_{k,l+1}^u - \phi_{k-1,l} D_{k-1,l}^u) \right) \\ &\quad + (\phi_{k+1,l+1} D_{k+1,l+1}^v - 2\phi_{kl} D_{kl}^v + \phi_{k-1,l-1} D_{k-1,l-1}^v). \end{aligned}$$

The first line in the above computation is related to the second order finite difference $\frac{\partial^2}{\partial u \partial v}$ of ϕ whereas the second line is related to the finite difference $\frac{\partial^2}{\partial u^2}$ of ϕ . The fact the renormalized diagonals converge smoothly allows to control the $\mathcal{C}_w^{0,\alpha}$ -norms of these terms using the $\mathcal{C}_w^{2,\alpha}$ -norm of ϕ . \square

Inequality (6.3) together with Lemma 6.2.2 completes the proof of the proposition. \square

COROLLARY 6.2.3. – For all $\varepsilon > 0$ there exists $N_0 \geq 1$ and $\delta > 0$ such that for all $N \geq N_0$, $\phi, \phi' \in \mathcal{X}_N^\perp$, such that $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leq \delta$ and $\|\phi'\|_{\mathcal{C}^{2,\alpha}} \leq \delta$, we have

$$\|T_N(\phi) - T_N(\phi')\|_{\mathcal{C}_w^{2,\alpha}} \leq \varepsilon \|\phi - \phi'\|_{\mathcal{C}_w^{2,\alpha}}.$$

Proof. – By (6.2), Proposition 6.1.2 and Proposition 6.2.1

$$\begin{aligned} \|T_N(\phi) - T_N(\phi')\|_{\mathcal{C}_w^{2,\alpha}} &= \frac{1}{2} \|G_N \Psi_N^r (J\delta_N^*(\phi' + \phi), J\delta_N^*(\phi' - \phi))\|_{\mathcal{C}_w^{2,\alpha}} \\ &\leq \frac{c_7 c_8}{2} \|\phi + \phi'\|_{\mathcal{C}_w^{2,\alpha}} \|\phi - \phi'\|_{\mathcal{C}_w^{2,\alpha}}. \end{aligned}$$

In conclusion we may choose $\delta = \frac{\varepsilon}{c_7 c_8}$, which proves the corollary. \square

6.3. Fixed point principle

The idea, as usual is to check whether the sequence $T_N^k(0)$ converges. If so, the limit must be a fixed point of T_N . We have the following classical proposition

PROPOSITION 6.3.1. – Let $(\mathbb{E}, \|\cdot\|)$ be a finite dimensional (or Banach) normed vector space and $T : \mathbb{E} \rightarrow \mathbb{E}$ an application such that

1. There exists $\delta > 0$ such that the restriction of T to the closed ball \bar{B}_δ of \mathbb{E} , centered at 0 with radius δ , is $\frac{1}{2}$ -contractant, i.e.,

$$\forall x, y \in \mathbb{E}, \|x\| \leq \delta \text{ and } \|y\| \leq \delta \Rightarrow \|T(x) - T(y)\| \leq \frac{1}{2} \|x - y\|.$$

2. $\|T(0)\| \leq \frac{\delta}{2}$.

Then the sequence (t_k) defined by $t_0 = 0$ and $t_{k+1} = T(t_k)$ converges to an element $t_\infty \in \mathbb{E}$ with $\|t_\infty\| \leq \delta$. Furthermore, t_∞ is a fixed point for T . Such fixed point are unique in the ball \bar{B}_δ . In addition, we have $\|t_\infty\| \leq 2\|T(0)\|$.

Proof. – The uniqueness of fixed points is a trivial consequence of the contracting property of T in the ball $\bar{B}_\delta = \{x \in \mathbb{E} \mid \|x\| \leq \delta\}$.

For the convergence, we show first by induction that t_k remains in \bar{B}_δ for all k : this is the case for $t_0 = 0$ and t_1 by assumption. Assume now that if $t_0, \dots, t_{k-1} \in \bar{B}_\delta$. Then

$$\|t_k - t_{k-1}\| = \|T(t_{k-1}) - T(t_{k-2})\| \leq \frac{1}{2} \|t_{k-1} - t_{k-2}\|$$

and by induction

$$\|t_k - t_{k-1}\| = \frac{1}{2^p} \|t_{k-p} - t_{k-p-1}\|.$$

In particular

$$\|t_k - t_{k-1}\| = \frac{1}{2^{k-1}} \|t_1 - t_0\| = \frac{1}{2^{k-1}} \|T(0)\|.$$

In turn we have

$$t_k - t_0 = (t_k - t_{k-1}) + (t_{k-1} - t_{k-2}) + \dots + (t_1 - t_0)$$

and by the triangle inequality,

$$\|t_k\| \leq \|T(0)\| \sum_{j=0}^{k-1} \frac{1}{2^j} = \|T(0)\| \frac{1 - \frac{1}{2^k}}{1 - \frac{1}{2}} \leq 2\|T(0)\| \leq \delta,$$

so that $t_k \in \bar{B}_\delta$. This completes the induction and shows that t_k remains in \bar{B}_δ .

Eventually, we just have to prove that t_k converges. But this is clear since

$$t_{k+p} - t_k = (t_{k+p} - t_{k+p-1}) + (t_{k+p-2} - t_{k+p-1}) + \cdots + (t_{k+1} - t_k)$$

and by the triangle inequality

$$\|t_{k+p} - t_k\| \leq \|t_{k+1} - t_k\| \sum_{j=0}^{\infty} \frac{1}{2^j} \leq 2\|t_{k+1} - t_k\| \leq \frac{2}{2^{k-1}} \|T(0)\|,$$

which shows that t_k is Cauchy hence convergent in the closed ball \bar{B}_δ . The fact that the limit of t_k is a fixed point of T is clear from the definition of the sequence, by uniqueness of the limit. \square

We obtain the following result

THEOREM 6.3.2. – *There exists a positive integer N_0 and a real number $\delta > 0$ such that for all $N \geq N_0$ there exists a unique $\phi_N \in \mathcal{X}_N^\perp$ that satisfies*

$$\|\phi_N\|_{C_w^{2,\alpha}} \leq \delta \text{ and } F_N(\phi_N) \in \mathcal{X}_N.$$

Furthermore the sequence satisfies $\|\phi_N\|_{C_w^{2,\alpha}} = \mathcal{O}(N^{-1})$.

PROPOSITION 6.3.3. – *Let $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ be an isotropic immersion and $p : \mathbb{R}^2 \rightarrow \Sigma$ a conformal cover introduced before, such that the pair (p, ℓ) is nondegenerate. Then the meshes*

$$\rho_N = \tau_N - J\delta_N^* \phi_N \in \mathcal{M}_N,$$

where ϕ_N is defined by Theorem 6.3.2 for every $N \geq N_0$ are isotropic.

Proof of Proposition 6.3.3. – By definition $\mu_N^r(\rho_N) \in \mathcal{X}_N$. By nondegeneracy, $\mathcal{X}_N = \mathbb{R}\mathbf{1}_N$, so that $\mu_N^r(\rho_N) = \lambda\mathbf{1}_N$ for some constant λ . We deduce that

$$\langle\langle \mu_N^r(\rho_N), \mathbf{1}_N \rangle\rangle = \lambda \langle\langle \mathbf{1}_N, \mathbf{1}_N \rangle\rangle.$$

This quantity does not vanish unless $\lambda = 0$. But $\langle\langle \mu_N^r(\rho_N), \mathbf{1}_N \rangle\rangle$ is the total symplectic area of the mesh ρ_N , which has to vanish by Stokes theorem, since the symplectic form of \mathbb{R}^4 is exact. In conclusion $\lambda = 0$ so that $\mu_N^r(\rho_N) = 0$. \square

6.4. Proof of Theorem C

We merely need to gather the previous technical results so that the proof and our main result follows as a corollary.

Proof of Theorem C. – Let $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ be a smooth isotropic immersion. By Proposition 5.3.3, we may always assume that the conformal cover $p : \mathbb{R}^2 \rightarrow \Sigma$ is chosen in such a way that the pair (p, ℓ) is non degenerate. By Proposition 6.3.3, the quadrangular meshes ρ_N provided by Theorem 6.3.2, for N sufficiently large, are isotropic. The estimate $\|\phi_N\|_{\mathcal{C}_w^{2,\alpha}} = \mathcal{O}(N^{-1})$ implies that

$$\sup_{\mathbf{v} \in \mathcal{C}_0(\mathcal{Q}_N(\Sigma))} \|\rho_N(\mathbf{v}) - \tau_N(\mathbf{v})\|_{\mathcal{C}_w^{1,\alpha}} = \mathcal{O}(N^{-1}).$$

It follows that

$$\sup_{\mathbf{v} \in \mathcal{C}_0(\mathcal{Q}_N(\Sigma))} \|\rho_N(\mathbf{v}) - \ell(\mathbf{v})\| = \mathcal{O}(N^{-1}),$$

which proves the theorem. \square

6.5. The degenerate case

If (p, ℓ) is a degenerate pair, Theorem 6.3.2 still provides a family of quadrangular meshes ρ_N with the property that $\rho_N \in \mathcal{K}_N$. However ρ_N may not be an isotropic mesh since \mathcal{K}_N may not reduce to constants. This difficulty can be taken care of by working \mathbb{Z}_2 -equivariantly. Given a degenerate pair (p_S, ℓ_S) , we construct modified quadrangulations $\mathcal{Q}_N(S)$ as in § 5.6. Using the notation introduced at § 5.6, we consider the lifted pair (p, ℓ) given by $p = p_S \circ \Phi^S$ and $\ell = \ell_S \circ \Phi^S$, where $\Phi^S : \Sigma \rightarrow S$ is a double cover introduced at (5.14). The pair (p, ℓ) is degenerate as well. Using Theorem 6.3.2, we find a corresponding family of quadrangular meshes ρ_N . All these construction are G_N -equivariant. In particular $\rho_N \in \mathcal{K}_N$ is also G_N -invariant. We have

$$\rho_N = a_N \mathbf{1}_N + b_N \zeta_N,$$

where ρ_N and $\mathbf{1}_N$ are G_N -invariant. However ζ_N is G_N -anti-invariant (cf. proof of Theorem 5.6.5), which implies that $b_N = 0$. We conclude that $a_N = 0$ as in the proof of Proposition 6.3.3.

In conclusion ρ_N descends to the G_N -quotient as an isotropic quadrangular mesh $\rho_N^S \in \mathcal{M}_N(S)$ and we have proved the following result

PROPOSITION 6.5.1. – *Let (p_S, ℓ_S) be any pair, where $\ell_S : S \rightarrow \mathbb{R}^{2n}$ is an isotropic immersion and $p_S : \mathbb{R}^2 \rightarrow S$ an associated conformal cover. Let $\tau_N^S \in \mathcal{M}_N(S)$ be the family of samples of ℓ_S . Then, there exists a family of isotropic quadrangular meshes $\rho_N^S \in \mathcal{M}_N(S)$ such that*

$$\max_{\mathbf{v}} \|\rho_N^S(\mathbf{v}) - \tau_N^S(\mathbf{v})\| = \mathcal{O}(N^{-1}).$$

REMARK 6.5.2. – The approach presented in Proposition 6.5.1 appears as a good solution to treat our perturbation problem in a uniform manner, whether or not the pair (p, ℓ) is degenerate. The main flaw of such technique, relying on \mathbb{Z}_2 -equivariant constructions, is that the moduli spaces $\mathcal{M}_N(S)$ do not admit a shear action as defined in § 4.2 (this is due to the connectedness of the checkers graph of $\mathcal{Q}_N(S)$). Unfortunately, the shear action is used in a crucial way at § 7 to obtain generic quadrangular meshes that will allow to construct piecewise linear immersions as in Theorem A.

CHAPTER 7

FROM QUADRANGULATIONS TO TRIANGULATIONS

The previous section was devoted to the construction of isotropic meshes associated to quadrangulations, sufficiently close to a given smooth isotropic immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$. In this section, we explain how to define a nearby isotropic piecewise linear map as an approximation of ℓ . The idea is to pass from an isotropic quadrangulation to an isotropic triangulation.

7.1. From quadrilaterals to pyramids

The goal of this section is to explain how to pass from an isotropic quadrilateral to an isotropic pyramid, by adding one apex to the quadrilateral. We start by studying a single isotropic quadrilateral (A_0, A_1, A_2, A_3) , where A_i are points in \mathbb{R}^{2n} . We shall use the notations

$$(7.1) \quad D_0 = \overrightarrow{A_0A_2} \text{ and } D_1 = \overrightarrow{A_1A_2},$$

for the two diagonals of the quadrilateral. Recall that the quadrilateral is isotropic if, and only if

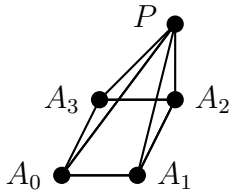
$$\omega(D_0, D_1) = 0.$$

REMARK 7.1.1. – If the diagonals of an isotropic quadrilateral are linearly independent vectors of \mathbb{R}^{2n} , this implies that

$$L = \mathbb{R}D_0 \oplus \mathbb{R}D_1$$

is an isotropic plane of \mathbb{R}^{2n} .

DEFINITION 7.1.2. – A pyramid is given by five points (P, A_0, A_1, A_2, A_3) of \mathbb{R}^{2n} . The four points of quadrilateral (A_0, \dots, A_3) , called the base of the pyramid and the apex $P \in \mathbb{R}^{2n}$. If the four triangles given by (PA_iA_{i+1}) , where i is understood as an index modulo 4, are contained in isotropic planes of \mathbb{R}^{2n} , we say that the pyramid is an isotropic pyramid (cf. Figure 1).

FIGURE 1. Pyramid with apex P and base (A_0, A_1, A_2, A_3)

The following lemma shows a first relation between isotropic quadrilaterals and isotropic pyramids:

LEMMA 7.1.3. – *The base of an isotropic pyramid is an isotropic quadrilateral.*

Proof. – The result is obtained as a trivial consequence of the Stokes theorem, or by elementary algebraic manipulations. \square

Conversely, we have the following result:

LEMMA 7.1.4. – *Let $Q = (A_0, \dots, A_3)$ be an isotropic quadrilateral of \mathbb{R}^{2n} with linearly independent diagonals. We denote by W'_Q be the symplectic orthogonal of the vector space spanned by the sides of the quadrilateral Q . Let W_Q be the set of points $P \in \mathbb{R}^{2n}$ which are the apexes of isotropic pyramids with base given by the quadrilateral Q . Then W_Q is an affine subspace of \mathbb{R}^{2n} with underlying vector space W'_Q . Its dimension is $2n - 2$ if Q is flat and $2n - 3$ otherwise.*

Proof. – We are looking for a solution of the linear system of four equations

$$\omega(\overrightarrow{PA_i}, \overrightarrow{A_iA_{i+1}}) = 0,$$

where $0 \leq i \leq 3$. Put

$$(7.2) \quad X = \overrightarrow{GP},$$

where G is by convention the barycenter of the quadrilateral. The system can be expressed as

$$\omega(X, \overrightarrow{A_iA_{i+1}}) = \omega(\overrightarrow{GA_i}, \overrightarrow{A_iA_{i+1}}).$$

The LHS correspond to a linear map with kernel W'_Q .

If the quadrilateral is flat, it is contained in an isotropic affine plane parallel to $L = \mathbb{R}D_0 \oplus \mathbb{R}D_1$. Any point P in the plane of the quadrilateral is the apex of an isotropic pyramid. Furthermore, the space of solutions is an affine space of codimension 2.

If the quadrilateral is not flat, then $\dim W'_Q = 2n - 3$ and the LHS of the linear system has rank 3. The condition that the quadrilateral is isotropic is precisely the compatibility condition, that insures that the RHS of the equations is in the image of the Linear map. We conclude that the system of equations admits a $2n - 3$ -dimensional affine space of solutions. \square

Lemma 7.1.4 is an excellent tool for passing from isotropic meshes associated to quadrangulations to isotropic meshes associated to triangulations and, in turns, to piecewise linear isotropic maps. One issue, that has to be dealt with, is how C^0 -estimates are preserved and also, whether the piecewise linear map induced by this construction are still immersions. Indeed, Lemma 7.1.4 does not provide any information about the distance from W_Q to the quadrilateral.

7.1.5. Optimal apex. – There exists large families of isotropic pyramids as shown by Lemma 7.1.4. In this section we introduce some particular solutions of the corresponding linear system, called *optimal pyramids* and *optimal apex*.

We use the notations introduced in the proof of Lemma 7.1.4. Again, we consider an isotropic quadrilateral $Q = (A_0, \dots, A_3)$. We are assuming that Q has linearly independent diagonals D_0, D_1 . Hence the diagonals span an isotropic plane $L = \mathbb{R}D_0 \oplus \mathbb{R}D_1$. We may consider its complexification

$$(7.3) \quad L^{\mathbb{C}} = L \oplus JL,$$

and the corresponding orthogonal complex (and symplectic) splitting

$$\mathbb{R}^{2n} = L^{\mathbb{C}} \oplus M.$$

Notice that the real dimension of $L^{\mathbb{C}}$ is 4.

We are looking for a point $P \in \mathbb{R}^{2n}$ solution of the linear system

$$(7.4) \quad \omega(\overrightarrow{GP}, \overrightarrow{A_i A_{i+1}}) = \gamma_i \quad 0 \leq i \leq 3,$$

where

$$(7.5) \quad \gamma_i = \omega(\overrightarrow{GA_i}, \overrightarrow{GA_{i+1}}).$$

According to Lemma 7.1.4, the affine space of solutions W'_Q has dimension $2n - 2$ or $2n - 3$ in \mathbb{R}^{2n} depending on the flatness of the quadrilateral. We may reduce to particular solutions by adding the constraint

$$(7.6) \quad \overrightarrow{PG} \in L^{\mathbb{C}}.$$

We use the notation $X = \overrightarrow{GP}$. A quadrilateral (A_0, \dots, A_3) is determined by specifying its barycenter G , the side vector $V = \overrightarrow{A_0 A_1}$ and the diagonals D_0 and D_1 . We first compute the terms γ_i of the RHS in terms of these quantities. By definition

$$\begin{aligned} 4\overrightarrow{A_0 G} &= \overrightarrow{A_0 A_1} + \overrightarrow{A_0 A_2} + \overrightarrow{A_0 A_3} &= D_0 + D_1 + 2V, \\ 4\overrightarrow{A_1 G} &= \overrightarrow{A_1 A_0} + \overrightarrow{A_1 A_2} + \overrightarrow{A_1 A_3} &= D_0 + D_1 - 2V, \\ 4\overrightarrow{A_2 G} &= \overrightarrow{A_2 A_0} + \overrightarrow{A_2 A_1} + \overrightarrow{A_2 A_3} &= -3D_0 + D_1 + 2V, \\ 4\overrightarrow{A_3 G} &= \overrightarrow{A_3 A_0} + \overrightarrow{A_3 A_1} + \overrightarrow{A_3 A_2} &= D_0 - 3D_1 - 2V. \end{aligned}$$

Hence

$$\begin{aligned}
 16\gamma_0 &= \omega(D_0 + D_1 + 2V, D_0 + D_1 - 2V) \\
 &= 4\omega(V, D_0 + D_1), \\
 16\gamma_1 &= \omega(D_0 + D_1 - 2V, -3D_0 + D_1 + 2V) \\
 &= 4\omega(V, D_0 - D_1), \\
 (7.7) \quad 16\gamma_2 &= \omega(-3D_0 + D_1 + 2V, D_0 - 3D_1 - 2V) \\
 &= -4\omega(V, D_0 + D_1), \\
 16\gamma_3 &= \omega(D_0 - 3D_1 - 2V, D_0 + D_1 + 2V) \\
 &= -4\omega(V, D_0 - D_1).
 \end{aligned}$$

Let D'_0 and D'_1 be the basis of L defined by the orthogonality conditions

$$\forall i, j \in \{0, 1\}, \quad g(D_i, D'_j) = \delta_{ij}$$

and put

$$(7.8) \quad B_i = JD'_i, \quad i \in \{0, 1\},$$

which are a basis for JL . Notice that by definition

$$\omega(D_i, B_j) = g(D_i, D_j) = \delta_{ij}.$$

We may express the vectors X and V using the basis (D_0, D_1, B_0, B_1) of $L^{\mathbb{C}}$ as

$$(7.9) \quad X = a_0D_0 + a_1D_1 + b_0B_0 + b_1B_1,$$

$$(7.10) \quad V = \alpha_0D_0 + \alpha_1D_1 + \beta_0B_0 + \beta_1B_1 + V_M,$$

where $\alpha_i, \beta_i, a_i, b_i \in \mathbb{R}$ and $V_M \in M$. By (7.7) (7.9) and (7.10), we have

$$\begin{aligned}
 4\gamma_0 &= \beta_0 + \beta_1, \\
 4\gamma_1 &= \beta_0 - \beta_1, \\
 4\gamma_2 &= -\beta_0 - \beta_1, \\
 4\gamma_3 &= -\beta_0 + \beta_1.
 \end{aligned}$$

The linear system (7.4) with constraint (7.6) is equivalent to (after adding up lines)

$$(7.11) \quad \begin{cases} \omega(X, V) = \gamma_0, \\ \omega(X, D_0) = \gamma_0 + \gamma_1, \\ \omega(X, D_1) = \gamma_1 + \gamma_2, \end{cases}$$

where we have removed the last redundant equation. Eventually, a solution of (7.11) is given by

$$(7.12) \quad \begin{cases} a_0\beta_0 + a_1\beta_1 - b_0\alpha_0 - b_1\alpha_1 = \frac{1}{4}(\beta_0 + \beta_1), \\ b_0 = -\frac{1}{2}\beta_0, \\ b_1 = \frac{1}{2}\beta_1, \end{cases}$$

i.e., the solutions X are given by

$$(7.13) \quad X = a_0 D_0 + a_1 D_1 - \frac{\beta_0}{2} B_0 + \frac{\beta_1}{2} B_1,$$

where a_0 and a_1 satisfy the affine equation

$$(7.14) \quad \beta_0 a_0 + \beta_1 a_1 = \frac{1}{4}(\beta_0 + \beta_1) + \frac{\beta_1 \alpha_1 - \beta_0 \alpha_0}{2}.$$

In conclusion, any solution (a_0, a_1) of the affine equation

$$(7.15) \quad \beta_0 a_0 + \beta_1 a_1 = \xi(V),$$

where

$$\xi(V) := \frac{\beta_0(1 - 2\alpha_0) + \beta_1(1 + 2\alpha_1)}{4}$$

provides a solution to our linear system. If the orthogonal projection of V onto JL does not vanish, we have $(\beta_0, \beta_1) \neq (0, 0)$ and the above equation defines a line, which, in turn defines a one dimensional space of solutions $X \in L^C$. We summarize our computations in the following lemma:

LEMMA 7.1.6. – *Assume that $Q = (A_0, \dots, A_3)$ is an isotropic quadrilateral with linearly independent diagonals in \mathbb{R}^{2n} . Assume that the orthogonal projection of Q in L^C is not a flat quadrilateral. Then set of points $P \in G + L^C$ which are the apex of an isotropic pyramid over Q , form a 1-dimensional affine space.*

Under the assumptions of the lemma, we may consider a particular solution given by

$$(7.16) \quad X = \xi(V) \frac{\beta_0}{\beta_0^2 + \beta_1^2} D_0 + \xi(V) \frac{\beta_1}{\beta_0^2 + \beta_1^2} D_1 - \frac{\beta_0}{2} B_0 + \frac{\beta_1}{2} B_1.$$

The above solution corresponds to the apex P , which is the closest point to the barycenter G , with the property that the corresponding pyramid is isotropic and $\overline{GP} \in L^C$. This leads us to the following definition:

DEFINITION 7.1.7. – *Let (A_0, \dots, A_3) be an isotropic quadrilateral of \mathbb{R}^{2n} with linearly independent diagonals and G be its barycenter. The closest point P to G in $G + L^C$ such that (P, A_0, \dots, A_3) isotropic is called the optimal apex, and the corresponding pyramid an optimal isotropic pyramid.*

REMARK 7.1.8. – If the orthogonal projection of the quadrilateral in L^C is flat, then the optimal apex is just the barycenter G of the quadrilateral. If it is not flat the optimal apex is given by $X = \overline{GP}$, where X is given by Formula (7.16).

Optimal pyramids enjoy nice properties. We first point out that they are almost always non degenerate in the sense of the following lemma:

LEMMA 7.1.9. – *Let (A_0, \dots, A_3) be an isotropic quadrilateral such that its orthogonal projection on $L^{\mathbb{C}}$ is not flat. Using the above notations, let V' be the orthogonal projection of V on $L^{\mathbb{C}}$ and X be the optimal solution. Then D_0, D_1, V', X is a basis of $L^{\mathbb{C}}$, unless $\beta_0 = 0$ or $\beta_1 = 0$. If $\beta_0\beta_1 \neq 0$ the rays of the optimal isotropic pyramid $\overrightarrow{PA_i}$, for $0 \leq i \leq 3$ are linearly independent.*

Proof. – Easy manipulations on vectors show that the vector space spanned by D_0, D_1, V', X is also spanned by D_0, D_1 and the vectors

$$\beta_0 B_0 + \beta_1 B_1 \text{ and } -\beta_0 B_0 + \beta_1 B_1.$$

The two above vectors belong to JL and they are linearly independent if, and only if

$$\beta_0\beta_1 \neq 0,$$

which proves the lemma as the second statement is an immediate consequence of the first. \square

7.2. \mathcal{C}^0 -estimates for optimal pyramids

DEFINITION 7.2.1. – *A quadrilateral of \mathbb{R}^{2n} with orthonormal diagonals (D_0, D_1) is called an orthonormogonal quadrilateral. If the diagonals satisfy*

$$\forall i, j \in \{0, 1\} \quad \left| g(D_i, D_j) - \delta_{ij} \right| \leq \varepsilon,$$

for some $\varepsilon > 0$, we say that they are ε -orthonormal. Under this assumption, we say that the quadrilateral is ε -orthonormogonal.

By continuity, we have the following result:

LEMMA 7.2.2. – *For every pair $D_0, D_1 \in \mathbb{R}^{2n}$ of linearly independent vectors, we define $D'_0, D'_1 \in \text{span}(D_0, D_1)$ by the orthogonality relations*

$$g(D_i, D'_j) = \delta_{ij}, \forall i, j \in \{0, 1\}.$$

Then D'_0, D'_1 is a basis of $\text{span}(D_0, D_1)$. Furthermore, for every $\varepsilon' > 0$ there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ and every ε -orthonormal family (D_0, D_1) , the family (D'_0, D'_1) is ε' -orthonormal.

REMARK 7.2.3. – We shall assume from now on that the quadrilateral is ε -orthonormogonal, with $\varepsilon > 0$ sufficiently small, so that D_0, D_1 are linearly independent, $\|D_i\| \leq 2$ and $\|D'_i\| \leq 2$.

PROPOSITION 7.2.4. – *There exist $C > 0$ and $\varepsilon > 0$ such that for every ε -orthonormogonal isotropic quadrilateral of diameter d , the diameter d' of the corresponding optimal isotropic pyramid satisfies*

$$d' \leq C(d + 1).$$

Loosely stated, the above proposition says that, for every isotropic quadrilateral which is almost orthonormal, the diameter of the optimal isotropic pyramid is commensurate with the diameter of the quadrilateral.

Proof. – If the projection of the quadrilateral in L^C is flat, then the optimal apex coincide with the barycenter of the quadrilateral and the proposition is obvious. Thus, we will assume that the projection of the quadrilateral is not flat in the rest of the proof.

As $\varepsilon \rightarrow 0$, the basis D_0, D_1, B_0, B_1 becomes almost orthonormal. In particular, there exists $\varepsilon > 0$ sufficiently small such that under the assumptions of the proposition, we have

$$\max(|\alpha_0|, |\alpha_1|, |\beta_0|, |\beta_1|) \leq 2\|V\|.$$

Then Formula (7.16) for the optimal solution $X = a_0D_0 + a_1D_1 + b_0B_0 + b_1B_1$ shows that all the coefficients a_i and b_i are controlled by $\|V\| + 1$ (up to multiplication by a universal constant). Now,

$$\|X\| \leq |a_0|\|D_0\| + |a_1|\|D_1\| + |b_0|\|B_0\| + |b_1|\|B_1\|.$$

According to Remark 7.2.3, if ε is sufficiently small, we have

$$\|D_i\| \text{ and } \|D'_j\| \leq 2.$$

Hence $\|B'_j\| \leq 2$ and it follows from the triangle inequality that

$$\|X\| \leq 2(|a_0| + |a_1| + |b_0| + |b_1|).$$

This shows that the distance between the optimal apex and the center of gravity of the quadrilateral is controlled by $\|V\| + 1$, up to multiplication by a universal constant. The diameters of the quadrilateral controls $\|V\|$, hence the diameter of the quadrilateral controls $\|X\|$ and the lemma follows. \square

7.3. Many quadrilaterals and pyramids

Every faces of an isotropic quadrangular mesh $\tau \in \mathcal{M}_N$ can be seen as a collection of isotropic quadrilaterals of \mathbb{R}^{2n} . In this section we explain how to define particular triangulations $\mathcal{T}_N(\Sigma)$ as a refinement of the quadrangulations $\mathcal{Q}_N(\Sigma)$. Then we explain how to deduce an isotropic quadrangular mesh $\tau' \in \mathcal{M}'_N = C^0(\mathcal{T}_N(\Sigma))$ from τ .

7.3.1. Triangulations obtained by refinement. – We define triangulation $\mathcal{T}_N(\mathbb{R}^2)$ by replacing each face \mathbf{f} of $\mathcal{Q}_N(\mathbb{R}^2)$ with its barycenter $\mathbf{z}_{\mathbf{f}} \in \mathbb{R}^2$. The barycenter $\mathbf{z}_{\mathbf{f}}$ is joined to the vertices of the face \mathbf{f} by straight line segments. We also add four faces given by the four triangles which appear as in the picture below. This operation is better understood by drawing a local picture of the corresponding *CW*-complexes of \mathbb{R}^2 :

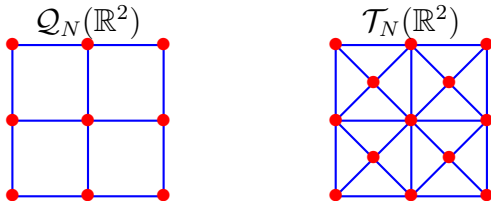


FIGURE 2. Triangular refinement of a quadrangulation

As explained in § 3.3 in the case of quadrangulations, the triangulations $\mathcal{T}_N(\mathbb{R}^2)$ descend to Σ via the covering map $p_N : \mathbb{R}^2 \rightarrow \Sigma$. The resulting triangulation of Σ is denoted $\mathcal{F}_N(\Sigma)$. We define a moduli space of mesh associated to such triangulation

$$\mathcal{M}'_N = C^0(\mathcal{F}_N(\Sigma)) \otimes \mathbb{R}^{2n}.$$

7.3.2. Optimal triangulation of isotropic quadrangular mesh. – Let $\tau \in \mathcal{M}_N$ be an isotropic quadrangular mesh. In addition, we are assuming that the quadrilateral of \mathbb{R}^{2n} associated to each face \mathbf{f} of $\mathcal{Q}_N(\Sigma)$ via τ have linearly independent diagonals. For each face \mathbf{f} of $\mathcal{Q}_N(\Sigma)$, the mesh τ associates an isotropic quadrilateral with linearly independent diagonals. We associate an optimal apex $P_{\mathbf{f}} \in \mathbb{R}^{2n}$ to such a quadrilateral. Then, we define a triangular mesh $\tau' \in \mathcal{M}'_N$ as follows:

- If \mathbf{v} is a vertex of $\mathcal{Q}_N(\Sigma)$, we define $\tau'(\mathbf{v}) = \tau(\mathbf{v})$.
- If \mathbf{z} is a vertex of $\mathcal{F}_N(\Sigma)$ which is not a vertex of $\mathcal{Q}_N(\Sigma)$, it is the barycenter of a face \mathbf{f} of $\mathcal{Q}_N(\Sigma)$ and we put $\tau'(\mathbf{z}) = P_{\mathbf{f}}$, where $P_{\mathbf{f}}$ is the optimal vertex defined via τ .

This leads us to the following definition

DEFINITION 7.3.3. – Let $\tau \in \mathcal{M}_N$ be an isotropic quadrangular mesh with linearly independent diagonals. The triangular mesh $\tau' \in \mathcal{M}'_N$ defined above is called the optimal triangulation of the isotropic mesh τ .

By construction, we have the following obvious property:

PROPOSITION 7.3.4. – Let $\tau \in \mathcal{M}_N$ be an isotropic quadrangular mesh with linearly independent diagonals. The optimal triangulation $\tau' \in \mathcal{M}'_N$ of the quadrangular mesh τ defines a piecewise linear map $\ell' : \Sigma \rightarrow \mathbb{R}^{2n}$, which is isotropic.

7.4. Approximation by isotropic triangular mesh

In Theorem 6.3.2, we construct a sequence of isotropic quadrangular meshes $\rho_N \in \mathcal{M}_N$ out of a smooth isotropic immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$. By construction,

$$\rho_N = \tau_N - J\delta_N^* \phi_N,$$

where $\|\phi_N\|_{C^{2,\alpha}} = \mathcal{O}(N^{-1})$. By Proposition 4.3.1, the renormalized diagonals of τ_N converge towards the partial derivatives of ℓ . Thus, the same holds for ρ_N , i.e.,

$$(7.17) \quad \mathcal{U}_{\rho_N}^\pm \longrightarrow \frac{\partial \ell}{\partial u} \quad \text{and} \quad \mathcal{V}_{\rho_N}^\pm \longrightarrow \frac{\partial \ell}{\partial v}.$$

In particular the diagonals are linearly independent for every N sufficiently large and we may define an optimal isotropic triangulation $\rho'_N \in \mathcal{M}'_N$ associated to ρ_N as in the previous section. It turns out that the triangular meshes ρ'_N are also good \mathcal{C}^0 approximations of the map ℓ in the sense of the following proposition:

PROPOSITION 7.4.1. – *There exists a constant $C > 0$, and $N_0 > 0$ such that for every integer $N \geq N_0$ and every vertex $\mathbf{v} \in \mathcal{F}_N(\Sigma)$*

$$\|\ell(\mathbf{v}) - \rho'_N(\mathbf{v})\| \leq \frac{C}{N}.$$

Proof. – Since $\|\phi_N\|_{C_w^{2,\alpha}} = \mathcal{O}(N^{-1})$, we deduce that $\|\phi_N\|_{C_w^1} = \mathcal{O}(N^{-1})$. It follows that there exists a constant $C_1 > 0$, such that $\|\rho_N(\mathbf{v}) - \tau_N(\mathbf{v})\| = \|\delta_N^* \phi_N(\mathbf{v})\| \leq C_1 N^{-1}$ for every N sufficiently large and every vertex \mathbf{v} of $\mathcal{Q}_N(\Sigma)$. In such case, we have $\tau_N(\mathbf{v}) = \ell(\mathbf{v})$ and $\rho_N(\mathbf{v}) = \rho'_N(\mathbf{v})$ so that

$$(7.18) \quad \|\ell(\mathbf{v}) - \rho'_N(\mathbf{v})\| \leq \frac{C_1}{N}.$$

If \mathbf{v} is a vertex of $\mathcal{F}_N(\Sigma)$ but not a vertex of $\mathcal{Q}_N(\Sigma)$, it is associated to a face \mathbf{f} of the quadrangulation and $\rho'_N(\mathbf{v})$ is the optimal apex associated to \mathbf{f} and ρ_N , by definition of ρ'_N (cf. § 7.3.2). The renormalized diagonals $\mathcal{U}_{\rho_N}^\pm$ and $\mathcal{V}_{\rho_N}^\pm$ converge toward $\frac{\partial \ell}{\partial u}$ and $\frac{\partial \ell}{\partial v}$ by (7.17). The partial derivatives $\frac{\partial \ell}{\partial u}$ and $\frac{\partial \ell}{\partial v}$ are orthogonal, with norm $\sqrt{\theta}$. Therefore

$$(7.19) \quad \frac{1}{\sqrt{\theta_N}} \mathcal{U}_{\rho_N}^\pm, \quad \frac{1}{\sqrt{\theta_N}} \mathcal{V}_{\rho_N}^\pm$$

converge toward a pair of smooth orthonormal vector fields on Σ . In particular, there exists N_0 such that for all $N \geq N_0$, the vectors fields (7.19) are ε -orthonormal, where $\varepsilon > 0$ is chosen according to Proposition 7.2.4. Since θ is a positive smooth function on a compact surface, it is bounded above and below by positive constants. Since $\theta_N^\pm \rightarrow \theta$, it follows that θ_N is also uniformly bounded above and below by positive constants for N sufficiently large. Using Proposition 7.2.4 with the rescaled pyramid, we deduce that the apex \mathbf{v} is close to all the vertices \mathbf{z} of \mathbf{f} in the sense that, for some constant $C_2 > 0$ independent of N_0 , \mathbf{v} and \mathbf{z} , we have

$$(7.20) \quad \|\rho'_N(\mathbf{v}) - \rho_N(\mathbf{z})\| \leq \frac{C_2}{N}.$$

Since ℓ is smooth, there exists a constant $C_3 > 0$ such that for every pair of points $w_1, w_2 \in \Sigma$ contained in the same face of $\mathcal{Q}_N(\Sigma)$, we have

$$(7.21) \quad \|\ell(w_1) - \ell(w_2)\| \leq \frac{C_3}{N}.$$

In particular, for \mathbf{v} and \mathbf{z} as above,

$$\|\ell(\mathbf{v}) - \rho'_N(\mathbf{v})\| \leq \|\ell(\mathbf{v}) - \ell(\mathbf{z})\| + \|\ell(\mathbf{z}) - \rho_N(\mathbf{z})\| + \|\rho_N(\mathbf{z}) - \rho'_N(\mathbf{v})\|.$$

Since \mathbf{z} and \mathbf{v} belong to the same face, $\|\ell(\mathbf{v}) - \ell(\mathbf{z})\| \leq C_3 N^{-1}$ by (7.21). The second term satisfies $\|\ell(\mathbf{z}) - \rho_N(\mathbf{z})\| \leq C_1 N^{-1}$ by (7.18) and the third term $\|\rho_N(\mathbf{z}) - \rho'_N(\mathbf{v})\| \leq C_2 N^{-1}$ by (7.20). The proposition follows, with $C = C_1 + C_2 + C_3$. \square

We deduce the following result, which proves the first part of Theorem A

THEOREM 7.4.2. – *The piecewise linear maps $\ell_N : \Sigma \rightarrow \mathbb{R}^{2n}$ associated to the triangular meshes ρ'_N are isotropic. Furthermore*

$$\|\ell - \ell_N\|_{\mathcal{C}^0} = \mathcal{O}(N^{-1}),$$

where $\|\cdot\|_{\mathcal{C}^0}$ denotes the usual \mathcal{C}^0 -norm for maps $\Sigma \rightarrow \mathbb{R}^{2n}$.

Proof. – The first part of the theorem is obvious. By definition of an isotropic triangular mesh, the piecewise linear map ℓ_N associated to ρ'_N is isotropic.

The following lemma is a trivial consequence of the convergence statement of Proposition 7.4.1. This roughly says that the triangles of the mesh ρ'_N have diameter of order $\mathcal{O}(N^{-1})$.

LEMMA 7.4.3. – *There exists a constant $C_4 > 0$ such that for every N sufficiently large and every pair of vertices $\mathbf{v}_1, \mathbf{v}_2$ of $\mathcal{F}_N(\Sigma)$ which belong to the same face*

$$\|\rho'_N(\mathbf{v}_1) - \rho'_N(\mathbf{v}_2)\| \leq \frac{C_4}{N}.$$

Lemma 7.4.3 applied to the piecewise linear maps ℓ_N shows that there exists a constant $C_5 > 0$ such that for every N sufficiently large and $w_1, w_2 \in \Sigma$ which belong to the same triangular face of $\mathcal{F}_N(\Sigma)$, we have

$$(7.22) \quad \|\ell_N(w_1) - \ell_N(w_2)\| \leq \frac{C_5}{N}.$$

For N sufficiently large, we may assume the control (7.21). For $w \in \Sigma$ and N sufficiently large, we choose a vertex \mathbf{v} of the face of $\mathcal{F}_N(\Sigma)$ which contains w . Then

$$\|\ell_N(w) - \ell(w)\| \leq \|\ell_N(w) - \ell_N(\mathbf{v})\| + \|\ell_N(\mathbf{v}) - \ell(\mathbf{v})\| + \|\ell(\mathbf{v}) - \ell(w)\|.$$

The first term is bounded by (7.22), the second term is bounded by Proposition 7.4.1 and the third is bounded by (7.21). This proves the theorem. \square

7.5. Piecewise linear immersions

Recall that a piecewise linear map is an immersion if, and only if, it is a locally injective map. The piecewise linear isotropic approximations ℓ_N of a smooth isotropic immersion $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ considered at § 7.4 are only close in \mathcal{C}^0 -norm by Theorem 7.4.2. Since this estimate is rather weak, we cannot deduce from this fact that ℓ_N is an immersion for N sufficiently large. However there are many free parameters in our construction:

- The distortion action on \mathcal{M}_N preserves isotropic meshes.
- The apex of each isotropic pyramid with fixed base lies in an affine space of dimension at least $2n - 3$.

These parameters can be used to construct piecewise linear isotropic immersions, at least when the dimension of the target space is sufficiently large, which turns out to be $n \geq 3$.

REMARK 7.5.1. – If ℓ is an immersion, showing that the piecewise linear isotropic surfaces ℓ_N which we construct converge to ℓ in $\mathcal{C}^{0,\alpha}$ norm would be enough to get piecewise linear isotropic immersions. Unfortunately, even though it follows from the proof of Theorem C in Section 6 that the ℓ_N converge to ℓ in $\mathcal{C}_w^{k,\alpha}$ -norms, we cannot get better control than \mathcal{C}^0 away from diagonal directions due to the shear action.

7.5.2. Perturbed meshes without flat faces. – We start by perturbing the isotropic quadrangular meshes $\rho_N \in \mathcal{M}_N$ constructed at Theorem 6.3.2. Our goal is to perturb ρ_N by the shear action, to make sure that the quadrilateral associated to each face of the mesh satisfy the following proposition and, in particular, are not flat in \mathbb{R}^{2n} .

PROPOSITION 7.5.3. – *For every N sufficiently large, there exists $T_N \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ such that for every $s > 0$ small enough, the quadrangular mesh*

$$\rho_N^s = sT_N \cdot \rho_N$$

satisfies the following properties:

1. *For each face of the quadrangular mesh ρ_N^s , the orthogonal projection of the corresponding quadrilateral onto the complex space generated by its diagonals (cf. (7.3)) is not flat.*
2. *For every vertex \mathbf{v} of $Q_N(\Sigma)$, the four vectors of \mathbb{R}^{2n} , associated via ρ_N^s to the four edges with vertex \mathbf{v} , span a 3-dimensional subspace of \mathbb{R}^{2n} . Furthermore any triplet obtained as a subset of the four above vectors is a linearly independent family.*
3. *The associated triangular meshes $(\rho_N^s)' \in \mathcal{M}'_N$ have generic pyramids. In other words, for every vertex \mathbf{v} of $\mathcal{T}_N(\Sigma)$ which is not a vertex of $Q_N(\Sigma)$, the four vectors of \mathbb{R}^{2n} associated to the four edges of the mesh $(\rho_N^s)'$ at \mathbf{v} are linearly independent.*

Proof. – We use the notations introduced at the beginning of § 7: for a quadrilateral $Q = (A_0, \dots, A_3)$ of \mathbb{R}^{2n} , we denote by X the vector defined by (7.2), D_0, D_1 the diagonals defined (7.1) and by B_0, B_1 the vectors (7.8) of JL (cf. (7.3)).

The condition that the projection of the quadrilateral onto L^C is flat is equivalent to $\beta_0 = \beta_1 = 0$, where the β_i has been defined in (7.10). Assume that the projection is flat. Then for every $T_+ \in \mathbb{R}^{2n}$ not contained in the hyperplanes $\beta_0 = 0$ or $\beta_1 = 0$, the projection of the quadrilateral $Q_s = (A_0 + sT_+, A_1, A_2 + sT_+, A_3)$ is not flat for every $s > 0$. Furthermore the optimal pyramid with base Q_s is generic in the sense of Lemma 7.1.9.

Let ρ_N be the isotropic quadrangular mesh considered in hypothesis of the proposition. We choose $T_+ \in \mathbb{R}^{2n}$ which satisfies the above property, for every quadrilateral associated to faces of the mesh ρ_N with flat projection onto the space of complexified directions. This is possible, since we merely need to choose $T_+ \in \mathbb{R}^{2n}$ away from a finite collection of 2-planes. Then $\rho_N^s := (sT_+, 0) \cdot \rho_N$ satisfies the items (1) and (3) of the proposition provided $s > 0$ is sufficiently small.

We just have to show that the condition (2) can be satisfied for a suitable choice of deformations. Given a vertex \mathbf{v} of $Q_N(\Sigma)$ we consider the four diagonals $D_{\mathbf{v}, \mathbf{f}}^{\rho_N}$ for the four quadrangular faces \mathbf{f} with vertex \mathbf{v} . The renormalized diagonals of the mesh converge toward the partial derivatives of ℓ at \mathbf{v} (cf. (7.17)) as $N \rightarrow \infty$. Since ℓ is an immersion, this shows that the four vectors $D_{\mathbf{v}, \mathbf{f}}^{\rho_N}$ span a space of dimension 2 or 3 for every N sufficiently large. If this space is 3-dimensional, (2) is satisfied with $s = 0$ and nothing needs to be done. Assume that the space of diagonals is 2-dimensional. The four vertices connected by an edge to \mathbf{v} define four points of \mathbb{R}^{2n} via ρ_N . By assumption, these points lie in an affine plane of \mathbb{R}^{2n} . If $\rho_N(\mathbf{v})$ does not belong to this plane, then (2) is satisfied. Otherwise, we require the additional condition that T_+ does not belong to the plane spanned by the diagonals. We have to consider every vertex \mathbf{v} as above and this adds a finite number of conditions for choosing T_+ . A finite family of proper subspaces of a vector space never covers the entire space. Thus it is possible to find the desired T_+ . This concludes the proof of the proposition with $T_N = (T_+, 0)$. \square

COROLLARY 7.5.4. – *Given N large enough, for every $s > 0$ sufficiently small, the isotropic triangulation $(\rho_N^s)'$ defines a piecewise linear map $\ell_{N,s} : \Sigma \rightarrow \mathbb{R}^{2n}$ which is an immersion at every point $w \in \Sigma$ which does not belong to the 1-skeleton of $Q_N(\Sigma)$. In particular $\ell_{N,s}$ is an immersion at almost every point of Σ .*

Proof. – By linearity, it is sufficient to check that $\ell_{N,s}$ is an immersion at every vertex of $\mathcal{F}_N(\Sigma)$ which is not a vertex of $Q_N(\Sigma)$. But this is clear for N large enough and $s > 0$ sufficiently small, by Proposition 7.5.3, item (3). \square

REMARK 7.5.5. – The above corollary proves the second part of Theorem A concerning piecewise linear isotropic immersions when $n = 2$. Indeed, the the 1-skeleton of $Q_N(\Sigma)$ is a finite union of meridians of the torus Σ .

7.5.6. Further perturbations by moving apexes. – We are going to apply further isotropic perturbations to the triangular meshes $(\rho_N^s)' \in \mathcal{M}'_N$, so that that the corresponding piecewise linear map is also an immersion along the 1-skeleton of $\mathcal{Q}_N(\Sigma)$.

By definition, $(\rho_N^s)'$ is defined from the quadrangular mesh ρ_N^s , by adding the apex of an optimal isotropic pyramid for each face of $\mathcal{Q}_N(\Sigma)$. The definition of an optimal pyramid is somewhat arbitrary: for N large enough and $s > 0$ sufficiently small, every face of ρ_N^s satisfy Proposition 7.5.3, item (1). Hence, for each face of ρ_N^s , the affine space of apexes of isotropic pyramids is $2n - 3$ -dimensional. We deduce the following lemma:

LEMMA 7.5.7. – *For N large enough and $s > 0$ sufficiently small, there exists a family of isotropic deformations of the triangular isotropic mesh $(\rho_N^s)'$. This family is obtained by moving each vertex of $\mathcal{F}_N(\Sigma)$ which does not belong to $\mathcal{Q}_N(\Sigma)$ within a $2n - 3$ -dimensional affine space.*

The key observation, that will make Lemma 7.5.7 useful for our purpose, is that $2n - 3 \geq 3$ for $n \geq 3$. In particular, we deduce the following proposition:

PROPOSITION 7.5.8. – *Assume that $n \geq 3$, and N is sufficiently large. Then, for every $s > 0$ sufficiently small, there exist isotropic triangular meshes arbitrarily close to $(\rho_N^s)'$, which define piecewise linear immersions $\Sigma \hookrightarrow \mathbb{R}^{2n}$.*

Proof. – As in Corollary 7.5.4, showing that a map is an immersion is a purely local matter. We draw a local picture of the triangular mesh $(\rho_N^s)'$, near the image O of vertex \mathbf{v} of $\mathcal{Q}_N(\Sigma)$. In Figure 3, the bullet labeled O actually represents $\rho_N^s(\mathbf{v}) \in \mathbb{R}^{2n}$. Similarly, all the points P_i , and $A_{ij} \in \mathbb{R}^{2n}$ of the picture are images of corresponding vertices of $\mathcal{F}_N(\Sigma)$ by the triangular isotropic mesh $(\rho_N^s)'$. Notice that the black and blue bullets are prescribed by the quadrangular mesh ρ_N^s , whereas the red bullets are defined by its triangular refinement $(\rho_N^s)'$. More specifically, the red bullets are the optimal apexes of the corresponding optimal isotropic pyramids.

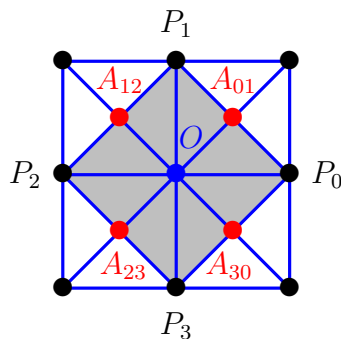


FIGURE 3. Local perturbations of a triangular mesh

We are now looking for a perturbation $(\rho_N^s)''$ of $(\rho_N^s)'$ by moving the points A_{ij} . We denote $\ell'_{N,s}$ (resp. $\ell''_{N,s}$) the piecewise linear maps associated the triangular mesh $(\rho_N^s)''$ (resp. $(\rho_N^s)''$).

1. The property of being an immersion is stable under small deformations. Thus, for sufficiently small perturbation, Corollary 7.5.4 holds for $\ell''_{N,s}$ as well. In particular, $\ell''_{N,s}$ is an immersion at every point $w \in \Sigma$ contained in the interior of one of the four faces of $\mathcal{Q}_N(\Sigma)$, with vertex \mathbf{v} (the four smaller square in the figure).
2. Suppose that we can choose a perturbation so that that $\ell''_{N,s}$ is an immersion at the vertex \mathbf{v} (corresponding to the point O). By linearity, this implies that $\ell''_{N,s}$ is an immersion at every interior point $w \in \Sigma$ of the union of shaded faces of the triangulation $\mathcal{T}_N(\Sigma)$ (with gray color on the picture).

If we are able to show that there exists a perturbation, which satisfies the condition (2) as above, we deduce, together with the above property (1), that the piecewise linear map $\ell''_{N,s}$ is an immersion at every interior point w of the union of the four faces of $\mathcal{Q}_N(\Sigma)$ with vertex \mathbf{v} (the big square in Figure 3). In conclusion, if we have proved the following lemma:

LEMMA 7.5.9. – *If $(\rho_N^s)''$ is a triangular mesh sufficiently close to $(\rho_N^s)'$, such that the corresponding piecewise linear map $\ell''_{N,s} : \Sigma \rightarrow \mathbb{R}^{2n}$ is an immersion at every vertex of $\mathcal{Q}_N(\Sigma)$, then $\ell''_{N,s}$ is an immersion at every point of Σ .*

We merely need to show that there exists an isotropic perturbation $(\rho_{N,s})''$ which satisfies the hypothesis of Lemma 7.5.9 and the proof of the proposition will be complete.

Consider the mesh $(\rho_N^s)'$ represented locally by Figure 3. There are $2n - 3$ degrees of freedom for perturbing each red vertex A_{ij} , in such a way that the triangular mesh remains isotropic. We would like to put them in general position, so that the piecewise linear map is an immersion at O . First, notice that the local injectivity is partially satisfied by $(\rho_N^s)'$ for every $s > 0$ sufficiently small. Indeed, by Corollary 7.5.4, two contiguous triangles of the mesh $(\rho_N^s)'$ in a common pyramid, for instance (OP_0A_{01}) and $(OA_{01}P_1)$, are contained in distinct planes intersecting along a line of \mathbb{R}^{2n} , which in this particular case is (OA_{01}) .

Consider now the two triangles of (OP_0A_{01}) and $(OA_{30}P_0)$ of \mathbb{R}^{2n} . There are two possibilities:

1. The line (OA_{30}) is not contained in the plane of the triangle (OP_0A_{01}) , the two triangles lie in distinct plane intersecting along the line (OP_0) .
2. The line (OA_{30}) is contained in the plane of the triangle (OP_0A_{01}) . In this case, the associated piecewise linear map is not locally injective at O .

In the second situation, we can always find an arbitrarily small perturbation of the point A_{30} which brings us back to the first situation, such that the associated piecewise linear map is still isotropic. Indeed, as pointed out there is a $2n - 3 \geq 3$ dimensional family of points A_{30} such that provide isotropic perturbation. There is a least. Such

space cannot be contained in the plane of (OP_0A_{01}) for obvious dimensional reasons. Thus, we may find the wanted arbitrarily small isotropic perturbations of A_{30} such that we are in the first situation.

We consider now the case where we have two non contiguous triangles, for instance (OP_0A_{01}) and (OP_1A_{12}) . We know that the three lines OP_0 , OA_{01} and OP_1 span a 3-dimensional space by Corollary 7.5.4. By moving slightly A_{12} within its $(2n - 3)$ -dimensional family of isotropic perturbation, we can make sure that the intersection of the planes containing the triangles (OP_0A_{01}) and (OP_1A_{12}) reduces to the point O .

There are other situations that we should handle as well. For instance, we consider the triangles (OP_0A_{01}) and $(OA_{12}P_2)$. By Proposition 7.5.3, item (2), the lines (OP_0) and (OP_2) are distinct. Up to a small isotropic perturbation by moving A_{01} within its $2n - 3$ -dimensional family, we may assume that A_{01} does not belong to the plane (OP_0P_2) . By moving A_{12} similarly, we may assume that A_{12} does not belong to the plane that contains the triangle (OP_0A_{01}) . Eventually, the two planes that contain (OP_0A_{01}) and $(OA_{12}P_2)$, after perturbation, intersect at a single point O .

Other cases are dealt with similarly. Eventually we have proved that there are arbitrarily small isotropic deformations of $(\rho_N^s)'$, obtained by moving the points A_{ij} , such that the eight triangles of the mesh around O lie in distinct planes. In particular, the corresponding piecewise linear map is an immersion at the vertex \mathbf{v} .

By induction, we can apply further similar perturbation, so that the isotropic piecewise linear map is an immersion at every vertex of $\mathcal{Q}_N(\Sigma)$. This proves the proposition. \square

7.6. Proof of Theorem A

Gathering our results provides a complete proof one of our main results:

Proof of Theorem A. – The existence of \mathcal{C}^0 -approximations of smooth isotropic immersions $\ell : \Sigma \rightarrow \mathbb{R}^{2n}$ by piecewise linear isotropic maps is proved in Theorem 7.4.2. The statement for existence of approximations by piecewise linear isotropic immersions is proved at Proposition 7.5.8. The remaining case, for $n = 2$, is a consequence of Corollary 7.5.4. \square

CHAPTER 8

DISCRETE MOMENT MAP FLOW

The moduli space $\mathcal{M} = \{f : \Sigma \rightarrow M, f^*[\omega] = 0\}$, where Σ is a closed surface endowed with an area form σ was introduced at §2. If M is a Kähler manifold, then \mathcal{M} has an induced formal Kähler structure $(\mathcal{M}, \mathfrak{J}, \mathfrak{g}, \Omega)$. The group $\mathcal{G} = \text{Ham}(\Sigma, \sigma)$ acts isometrically on \mathcal{M} . The action is Hamiltonian, with moment map $\mu : \mathcal{M} \rightarrow C_0^\infty(\Sigma)$, given by $\mu(f) = \frac{f^*\omega}{\sigma}$. In this setting, a natural moment map flow is defined (cf. §2.2) by

$$\frac{df}{dt} = -\frac{1}{2} \text{grad} \|\mu\|^2.$$

The properties of the above flow shall be studied in a sequel to this work [6]. For the time being, we merely provide a numerical simulation of the above flow, implemented in the program *Discrete Moment Map Flow (DMMF)*, hosted on the webpage:

<http://www.math.sciences.univ-nantes.fr/~rollin/index.php?page=flow>.

The idea is to approximate the flow, which is an evolution equation on an infinite dimensional space of maps, by an analog finite dimensional approximation. The finite dimensional flow is expected to converge in some sense to the infinite dimensional flow as $N \rightarrow \infty$, at least in favorable situations, but this is part of a broader project to be expanded later in [6].

8.1. Definition of the finite dimensional flow

In the rest of this section, we focus on the case where $M = \mathbb{R}^4$, with its standard Kähler structure and Σ is a surface diffeomorphic to a torus, endowed covering map $p : \mathbb{R}^2 \rightarrow \Sigma$ with Γ , its group of deck transformation, which is a lattice of \mathbb{R}^2 . This data allows to define the quadrangulations $\mathcal{Q}_N(\Sigma)$. The space of quadrangular meshes \mathcal{M}_N is seen as a discrete analog of the moduli space \mathcal{M} . The moment map μ has a discrete version as well, given by

$$\mu_N^r : \mathcal{M}_N \rightarrow C^2(\mathcal{Q}_N(\Sigma)).$$

The space of discrete functions $C^2(\mathcal{Q}_N(\Sigma))$ is also understood as a discrete analog of $C^\infty(\Sigma)$. Recall that this space of discrete functions is endowed with an inner product

$\langle\langle \cdot, \cdot \rangle\rangle$, which is an analog of the L^2 -inner product induced by σ (cf. § 4.5) and denoted $\langle\langle \cdot, \cdot \rangle\rangle$ as well. We denote by $\| \cdot \|$ the norm induced by the inner product $\langle\langle \cdot, \cdot \rangle\rangle$. Then

$$\begin{aligned} D\|\mu_N^r\|_{|\tau}^2 \cdot V &= 2\langle\langle D\mu_N^r|_{|\tau} \cdot V, \mu_N^r(\tau) \rangle\rangle \\ &= -2\langle\langle D\mu_N^r|_{|\tau} \circ J \cdot JV, \mu_N^r(\tau) \rangle\rangle \\ &= 2\langle\langle \delta_\tau(JV), \mu_N^r(\tau) \rangle\rangle \\ &= 2\langle\langle JV, \delta_\tau^* \mu_N^r(\tau) \rangle\rangle, \end{aligned}$$

hence

$$(8.1) \quad -\frac{1}{2}D\|\mu_N^r\|_{|\tau}^2 \cdot V = \langle\langle V, J\delta_\tau^* \mu_N^r(\tau) \rangle\rangle,$$

where

$$\delta_\tau = -D\mu_N^r|_{|\tau} \circ J.$$

Its adjoint δ_τ^* is defined by $\langle\langle \delta_\tau V, \phi \rangle\rangle = \langle\langle V, \delta_\tau^* \phi \rangle\rangle$. For each map $u : \mathcal{M}_N \rightarrow C^2(Q_N(\Sigma))$, we may define a formal gradient vector field on the moduli space

$$\text{grad } u : \mathcal{M}_N \longrightarrow C^0(Q_N(\Sigma)) \otimes \mathbb{R}^4$$

by $Du|_{|\tau} \cdot V = \langle\langle \text{grad } u|_{|\tau}, V \rangle\rangle$. Thus, by (8.1),

$$-\frac{1}{2}\text{grad}\|\mu_N^r(\tau)\|^2 = J\delta_\tau^* \mu_N^r(\tau)$$

and we can define a downward gradient flow by

$$\frac{d\tau}{dt} = -\frac{1}{2}\text{grad}\|\mu_N^r\|^2,$$

which is equivalent to

$$(8.2) \quad \boxed{\frac{d\tau}{dt} = J\delta_\tau^* \mu_N^r(\tau).}$$

DEFINITION 8.1.1. – A solution $\tau_t : I \rightarrow \mathcal{M}_N$ of the ordinary differential Equation (8.2), where I is an open interval of \mathbb{R} , is called a solution of the discrete moment map flow.

REMARK 8.1.2. – The discrete moment map flow is an ordinary differential equation with smooth coefficients on the affine space \mathcal{M}_N . The solution exists for short time but might blowup in finite time. The general behavior of the flow will be addressed in a sequel to this work [6].

The flow has typical properties of ODE with smooth coefficients:

PROPOSITION 8.1.3. – Assume that $\tau_t : [0, C) \rightarrow \mathcal{M}_N$ is a maximal solution of the discrete moment map flow. If τ_t is bounded for $t \in [0, C)$, then $C = +\infty$. If in addition τ_t converges to some τ_∞ , then $\mu_N^r(\tau_\infty) \in \ker \delta_{\tau_\infty}^*$.

REMARK 8.1.4. – If the function $\|\mu_N^r\|^2$ on \mathcal{M}_N were Morse, any bounded flow τ_t would automatically converge toward a critical point of the function. Although we are not trying to prove this fact, all our experiments with the DMMF program seem to indicate that the flow is generically bounded and convergent. If $\ker \delta_{\tau_\infty} = 0$, the conclusion of Proposition 8.1.3 implies that $\mu_N^r(\tau_\infty)$ is a constant discrete function and, by Stokes theorem, τ_∞ must be an isotropic quadrangular mesh. Notice that the fact that the kernel of the operator δ_τ^* is 1-dimensional holds for generic τ according to Proposition 4.6.7. This is also confirmed by all the experiments using the DMMF program.

Proof. – If C is finite, and $\frac{d\tau}{dt}$ is bounded, then τ_t must converge to some τ_C as $t \rightarrow C$. This contradicts the fact that C is maximal. Hence, if C is finite, $\frac{d\tau}{dt}$ must be unbounded. In particular τ_t cannot remain in a bounded set, as the RHS of the evolution equation would be bounded. In conclusion, if τ_t is bounded, we have $C = +\infty$. If τ_t converges towards τ_∞ , the limit must be a fixed point of the flow and $\delta_{\tau_\infty}^* \mu_N^r(\tau_\infty) = 0$. \square

REMARKS 8.1.5. – The kernel of the operator δ_τ^* contains the constants. In the general setting the kernel may not be reduced to constant and Proposition 8.1.3 does not allow to conclude that limits of the discrete flow are isotropic quadrangular meshes. It seems reasonable, especially in view of the experimental results of §8.2, to expect that the flow is always trapped in a compact set of \mathcal{M}_N . The study of these questions shall be carried out in a sequel to this work [6].

8.2. Implementation of the discrete flow

8.2.1. Particular lattices. – Recall that the quadrangulations $\mathcal{Q}_N(\Sigma)$ are defined by identifying the torus Σ with a quotient, via the diffeomorphism $\Phi : \mathbb{R}^2/\Gamma \rightarrow \Sigma$ induced by the covering map $p : \mathbb{R}^2 \rightarrow \Sigma$. We merely have to make a choice for the lattice Γ , in order to define $\mathcal{Q}_N(\Sigma)$ and a corresponding discrete moment map flow. This choice is arbitrary and a sufficiently sophisticated program could deal with any choice. This is not the case of the DMMF program, however, which is based on the choice of lattice

$$\Gamma'' = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2,$$

and surface $\Sigma = \mathbb{R}^2/\Gamma''$ already introduced at §3.5. Then $\Gamma'' \subset \Lambda_N$ for every positive integer N . The quadrangulation $\mathcal{Q}_N(\mathbb{R}^2)$ descends as a quadrangulation of the quotient $\mathcal{Q}_N(\Sigma)$. The quadrangulation has N^2 vertices and a mesh in \mathcal{M}_N can be stored as an $N \times N$ array with entries in \mathbb{R}^4 .

8.2.2. The Euler method. – It is easy to provide numerical approximations of an ODE such as the discrete moment map flow by the Euler method. We consider discrete time values $t_i = i\Delta t$, where i is an integer and $\Delta t > 0$ is a small time step increment. Starting at time $t_0 = 0$ with a mesh $\tau_0 \in \mathcal{M}_N$, we compute τ_1, τ_2 , etc., as follows: given $\tau_i \in \mathcal{M}_N$ at time $t = i\Delta t$ we compute

$$V_i = \delta_{\tau_i}^* \mu_N^r(\tau_i)$$

and define

$$\tau_{i+1} = \tau_i + \Delta t \cdot JV_i.$$

The above computations are easy to carry out and the operator δ_τ^* is explicitly given by Lemma 4.6.5. Starting from any quadrangular mesh, we can compute the above flow very quickly in real time on an ordinary machine, whenever N is not too big (for instance $N \leq 100$ on our laptop).

8.2.3. Visualization. – A choice has to be made for the visualization of each mesh $\tau \in \mathcal{M}_N$ on a computer screen. The basic idea is to choose a projection of \mathbb{R}^4 on a 3-dimensional manifold and represent a mesh as a surface in a 3-dimensional space. We explain now the choice made in the DMMF program, which may not be the best for certain situations: we perform a radial projection of the vertices of τ onto the unit sphere \mathbb{S}^3 of \mathbb{R}^4 , centered at the origin. This projection is followed by a stereographic projection of the sphere minus a point onto one of its tangent spaces identified to \mathbb{R}^3 . Once the positions of the projections of the vertices of τ in \mathbb{R}^3 are computed, we can draw the quadrilateral associated to the faces in \mathbb{R}^3 . A library like OpenGL allows to represent a quadrangular mesh of \mathbb{R}^3 in perspective. We fill the faces with a range of colors which depends on the symplectic density of each face (i.e., the value of $\mu_N^r(\tau)$ on this face). In addition, motions of the mouse are used to precompose these projections with Euclidean rotations of \mathbb{R}^4 . This technique allows the user to look at surfaces from various angles using the mouse.

8.2.4. The DMMF code. – We found out that the *processing* language, which is a Java dialect, was extremely efficient to code the DMMF program. The source code and more information on the technical aspects of the DMMF program are available on the homepage:

<http://www.math.sciences.univ-nantes.fr/~rollin/index.php?page=flow>.

The program starts the flow by sampling various examples of parametrized tori in \mathbb{R}^4 . From an experimental point of view, our numerous observations seem to indicate that the flow should always converges, that the convergence is fast, and that the limits are isotropic. Figure 1 shows an example of (static) output of the DMMF program. This quadrangular mesh has diameter of order 1 and symplectic density of order 10^{-8} . The reader is encouraged to experiment directly the more interactive and dynamic aspects of the program.

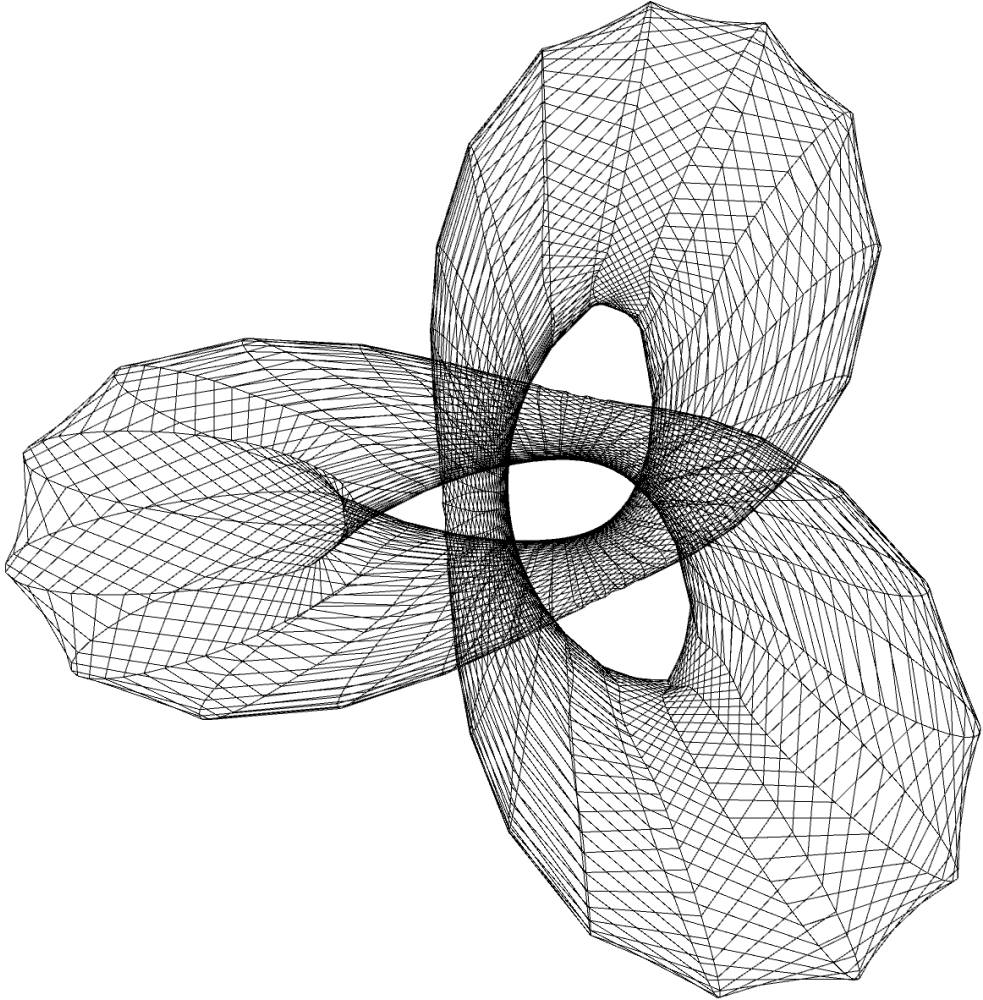


FIGURE 1. Isotropic quadrangular mesh

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We consider smooth isotropic immersions from the 2-dimensional torus into \mathbb{R}^{2n} , for $n \geq 2$. When $n = 2$ the image of such map is an immersed Lagrangian torus of \mathbb{R}^4 . We prove that such isotropic immersions can be approximated by arbitrarily C^0 -close piecewise linear isotropic maps. If $n \geq 3$ the piecewise linear isotropic maps can be chosen so that they are piecewise linear isotropic immersions as well.

The proofs are obtained using analogies with an infinite dimensional moment map geometry due to Donaldson. As a byproduct of these considerations, we introduce a numerical flow in finite dimension, whose limits provide, from an experimental perspective, many examples of piecewise linear Lagrangian tori in \mathbb{R}^4 . The DMMF program, which is freely available, is based on the Euler method and shows the evolution equation of discrete surfaces in real time, as a movie.

Nous considérons des immersions lisses et isotropes du tore de dimension 2 vers \mathbb{R}^{2n} , pour $n \geq 2$. Quand $n = 2$ l'image d'une telle application est un tore lagrangien immergé de \mathbb{R}^4 . Nous démontrons que de telles immersions isotropes peuvent être approximées au sens C^0 , par des applications linéaires par morceaux et isotropes arbitrairement proches. Si $n \geq 3$, il est possible de choisir des applications linéaires par morceaux qui sont de plus des immersions.

Les démonstrations reposent sur des analogies avec une géométrie et une application moment en dimension infinie introduites par Donaldson. Nous en déduisons un flot en dimension finie, dont les limites, du point de vue expérimental, produisent de nombreux exemples de tores lagrangiens linéaires par morceaux de \mathbb{R}^4 . Le programme libre DMMF, basé sur la méthode d'Euler, montre l'équation d'évolution sous forme de film.