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Nouvelle série LOCAL INDEX THEOREM
AND TRACE INTEGRALS
FOR CAUCHY-RIEMANN MANIFOLDS
WITH S^1 ACTION

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**HEAT KERNEL ASYMPTOTICS,
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Jih-Hsin Cheng, Chin-Yu Hsiao, I-Hsun Tsai

Abstract. – Among the transversally elliptic operators initiated by Atiyah and Singer, Kohn’s \square_b operator on CR manifolds with S^1 action is a natural one of geometric significance for complex analysts. Our first main result establishes an asymptotic expansion for the heat kernel of such an operator with values in its Fourier components, which involves a contribution in terms of a distance function from lower dimensional strata of the S^1 -action. Our second main result computes a local index density, in terms of *tangential* characteristic forms, on such manifolds including *Sasakian manifolds* of interest in String Theory, by showing that certain non-trivial contributions from strata in the heat kernel expansion will eventually cancel out by applying Getzler’s rescaling technique to off-diagonal estimates. This leads to a local result which can be thought of as a type of local index theorem on these CR manifolds. We give examples of these CR manifolds, some of which arise from Brieskorn manifolds. Moreover in some cases, we can reinterpret Kawasaki’s Hirzebruch-Riemann-Roch formula for a complex orbifold equipped with an orbifold holomorphic line bundle, as an index theorem obtained by a single integral over a smooth CR manifold. We achieve this without use of equivariant cohomology method and our method can naturally drop the contributions arising from lower dimensional strata as done in previous works.

Résumé (Étude asymptotique du noyau de la chaleur, indice local et traces sur les variétés de Cauchy-Riemann avec action d’un cercle)

Le laplacien de Kohn sur une variété de Cauchy-Riemann (CR) avec action transverse d’un cercle est un exemple important pour l’analyse complexe d’un opérateur transversalement elliptique. Nous établissons ici un développement asymptotique du noyau de la chaleur de ses coefficients de Fourier, qui inclut une contribution des strates singulières de l’action du cercle. Nous calculons ensuite une densité d’indice locale pour ces opérateurs en montrant, à l’aide de techniques dues à Getzler, que

certaines contributions des strates singulières non-triviales dans le développement du noyau de la chaleur s'annulent ici. Ce résultat, que l'on peut interpréter comme un théorème d'indice local sur ces variétés CR, s'applique notamment aux variétés de Sasaki qui sont importantes en théorie des cordes. Nous donnons également des exemples concrets de telles variétés CR, issues notamment des variétés de Brieskorn. De plus, nous pouvons réinterpréter dans certains cas la version du théorème de Hirzebruch-Riemann-Roch pour un orbifold complexe muni d'un fibré orbifold en droites complexes due à Kawasaki comme une formule d'indice. Notre méthode évite le recours à la cohomologie équivariante et les annulations des termes issus des strates singulières surviennent naturellement.

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CHAPTER 1

INTRODUCTION AND STATEMENT OF THE RESULTS

1.1. Introduction and Motivation

Let $(X, T^{1,0}X)$ be a compact (with no boundary) CR manifold of dimension $2n + 1$ and let $\bar{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. For a smooth function u , we say u is CR if $\bar{\partial}_b u = 0$. In CR geometry, it is crucial to be able to produce many CR functions. When X is strongly pseudoconvex and the dimension of X is greater than or equal to five, it is well-known that the space of global smooth CR functions $H_b^0(X)$ is infinite dimensional. When X is weakly pseudoconvex or the Levi form of X has negative eigenvalues, the space of global CR functions could be trivial. In general, it is very difficult to determine when the space $H_b^0(X)$ is large.

A clue to the above phenomenon arises from the following. By $\bar{\partial}_b^2 = 0$, one has the $\bar{\partial}_b$ -complex: $\cdots \rightarrow \Omega^{0,q-1}(X) \rightarrow \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X) \rightarrow \cdots$ and the Kohn-Rossi cohomology group: $H_b^q(X) := \frac{\text{Ker } \bar{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X)}{\text{Im } \bar{\partial}_b : \Omega^{0,q-1}(X) \rightarrow \Omega^{0,q}(X)}$. As in complex geometry, to understand the space $H_b^0(X)$, one could try to establish, in the CR case, a Hirzebruch-Riemann-Roch theorem for $\sum_{j=0}^n (-1)^j \dim H_b^j(X)$, an analog of the Euler characteristic, and to prove vanishing theorems for $H_b^j(X)$, $j \geq 1$. The first difficulty with such an approach lies in the fact that $\dim H_b^j(X)$ could be infinite for some j . Let's say more about this in the following.

If X is strongly pseudoconvex and of dimension ≥ 5 , it is well-known that $\bar{\partial}_b : \Omega^{0,0}(X) \rightarrow \Omega^{0,1}(X)$ is not hypoelliptic and for $q \geq 1$, $q \neq n$, $\square_b^{(q)} : \Omega^{0,q}(X) \rightarrow \Omega^{0,q}(X)$ is hypoelliptic so that $\dim H_b^0(X) = \infty$ and $\dim H_b^q(X) < \infty$ for $q \geq 1$, $q \neq n$. In other cases if the Levi form of X has exactly one negative, $n - 1$ positive eigenvalues on X and $n > 3$, it is well-known that $\dim H_b^1(X) = \infty$, $\dim H_b^{n-1}(X) = \infty$ and $\dim H_b^q(X) < \infty$, for $q \notin \{1, n - 1\}$. With these possibly infinite dimensional spaces, we have the trouble with defining the Euler characteristic $\sum_{j=0}^n (-1)^j \dim H_b^j(X)$ properly.

Another line of thought lies in the fact that the space $H_b^q(X)$ is related to the Kohn Laplacian $\square_b^{(q)} = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^* : \Omega^{0,q}(X) \rightarrow \Omega^{0,q}(X)$. One can try to define the associated heat operator $e^{-t\square_b^{(q)}}$ by using spectral theory and then the small t behavior of $e^{-t\square_b^{(q)}}$ is closely related to the dimension of $H_b^q(X)$. Unfortunately without any Levi curvature assumption, $\square_b^{(q)}$ is not hypoelliptic in general and it is unclear how to determine the small t behavior of $e^{-t\square_b^{(q)}}$. Even if $\square_b^{(q)}$ is hypoelliptic, it is still difficult to calculate the local density.

We are led to ask the following question.

QUESTION 1.1. – *Can we establish some kind of heat kernel asymptotic expansions for Kohn Laplacian and obtain a CR Hirzebruch-Riemann-Roch theorem (not necessarily the usual ones) on some class of CR manifolds?*

It turns out that the class of CR manifolds with S^1 action is a natural choice for the above question. On this class of CR manifolds, the geometrical significance of Kohn's \square_b operator is connected with transversally elliptic operators initiated by Atiyah and Singer [2] (see Folland and Kohn [35, p.113]). Three dimensional (strongly pseudoconvex) CR manifolds with S^1 action have been intensively studied back to 1990s in relation to the CR embeddability problem. We refer the reader to the works [26] and [49] of Charles Epstein and Laszlo Lempert respectively. (see more comments on this in Section 1.3). Another related paper is about CR structure on Seifert manifolds by Kamishima and Tsuboi [45] (cf. Remark 1.16). In our present paper the CR manifold with S^1 action is not restricted to the three dimensional case.

To motivate our approach, let's first look at the case which can be reduced to complex geometry. Consider a compact complex manifold M of dimension n and let $(L, h^L) \rightarrow M$ be a holomorphic line bundle over M , where h^L denotes a Hermitian fiber metric of L . Let $(L^*, h^{L^*}) \rightarrow M$ be the dual bundle of (L, h^L) and put $X = \{v \in L^*; |v|_{h^{L^*}}^2 = 1\}$. We call X the circle bundle of (L^*, h^{L^*}) . It is clear that X is a compact CR manifold of dimension $2n + 1$. Clearly X is equipped with a natural (globally free) S^1 action (by acting on the circular fiber). Let $T \in C^\infty(X, TX)$ be the real vector field induced by the S^1 action, that is, $Tu = \frac{\partial}{\partial \theta}(u(e^{-i\theta} \circ x))|_{\theta=0}$, $u \in C^\infty(X)$. This S^1 action is *CR and transversal*, i.e., $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and $CT(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_xX$ respectively. For each $m \in \mathbb{Z}$ and $q = 0, 1, 2, \dots, n$, put

$$\begin{aligned} \Omega_m^{0,q}(X) &:= \{u \in \Omega^{0,q}(X); Tu = -imu\} \\ &= \{u \in \Omega^{0,q}(X); u(e^{-i\theta} \circ x) = e^{-im\theta}u(x), \forall \theta \in [0, 2\pi[\}. \end{aligned}$$

Since $\bar{\partial}_b T = T\bar{\partial}_b$, we have $\bar{\partial}_{b,m} = \bar{\partial}_b : \Omega_m^{0,q}(X) \rightarrow \Omega_m^{0,q+1}(X)$. We consider the cohomology group: $H_{b,m}^q(X) := \frac{\text{Ker } \bar{\partial}_{b,m} : \Omega_m^{0,q}(X) \rightarrow \Omega_m^{0,q+1}(X)}{\text{Im } \bar{\partial}_{b,m} : \Omega_m^{0,q-1}(X) \rightarrow \Omega_m^{0,q}(X)}$, and call it the m -th S^1 Fourier component of the Kohn-Rossi cohomology group.

The following result can be viewed as the starting point of this paper. Note $\Omega^{0,q}(M, L^m)$ denotes the space of smooth sections of $(0, q)$ forms on M with values in L^m (m -th power of L) and $H^q(M, L^m)$ the q -th $\bar{\partial}$ -Dolbeault cohomology group with values in L^m .

THEOREM 1.2. – *For every $q = 0, 1, 2, \dots, n$, and every $m \in \mathbb{Z}$, there is a bijective map $A_m^{(q)} : \Omega_m^{0,q}(X) \rightarrow \Omega^{0,q}(M, L^m)$ such that $A_m^{(q+1)}\bar{\partial}_{b,m} = \bar{\partial}A_m^{(q)}$ on $\Omega_m^{0,q}(X)$. Hence, $\Omega_m^{0,q}(X) \cong \Omega^{0,q}(M, L^m)$ and $H_{b,m}^q(X) \cong H^q(M, L^m)$. In particular $\dim H_{b,m}^q(X) < \infty$ and $\sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X) = \sum_{j=0}^n (-1)^j \dim H^j(M, L^m)$.*

Theorem 1.2 is probably known to the experts. As a precise reference is not easily available (see, however, Folland and Kohn [35] p.113), we will give a proof of Theorem 1.2 in Section 1.5 for the convenience of the reader.

In this paper by *Kodaira Laplacian* we mean the Laplacian $\square_m^{(q)}$ on L^m -valued $(0, q)$ forms (on M) associated with the $\bar{\partial}$ operator, a term we borrow from the work of Ma and Marinescu [50]. Let $e^{-t\square_m^{(q)}}$ be the associated heat operator. It is well-known that $e^{-t\square_m^{(q)}}$ admits an asymptotic expansion as $t \rightarrow 0^+$. Consider $B_m(t) := (A_m^{(q)})^{-1} \circ e^{-t\square_m^{(q)}} \circ A_m^{(q)}$ ($A_m^{(q)}$ as in the theorem above). Let $\square_{b,m}^{(q)}$ be the Kohn Laplacian (on X) acting on (the m -th S^1 Fourier component of) $(0, q)$ forms, with $e^{-t\square_{b,m}^{(q)}}$ the associated heat operator.

A word of caution is in order. We made no use of metrics for stating Theorem 1.2. However, to define those Laplacians above an appropriate choice of metrics is needed (for *adjoint* of an operator) so that $A_m^{(q)}$ of Theorem 1.2 also preserves the chosen metrics. With this set up it is fundamental that (cf. Proposition 5.1)

$$(1.1) \quad e^{-t\square_{b,m}^{(q)}} = ((A_m^{(q)})^{-1} \circ e^{-t\square_m^{(q)}} \circ A_m^{(q)}) \circ Q_m = B_m(t) \circ Q_m = Q_m \circ B_m(t) \circ Q_m,$$

where $Q_m : \Omega^{0,q}(X) \rightarrow \Omega_m^{0,q}(X)$ is the orthogonal projection. Hence the asymptotic expansion of $e^{-t\square_m^{(q)}}$ and (1.1) lead to an asymptotic expansion

$$(1.2) \quad e^{-t\square_{b,m}^{(q)}}(x, x) \sim t^{-n}a_n^{(q)}(x) + t^{-n+1}a_{n-1}^{(q)}(x) + \dots$$

One goal of this work is to establish a formula similar to (1.2) (which is however not exactly of this form) on any CR manifold with S^1 action. More precisely, due to the assumption that the S^1 action is only locally free, it turns out that $e^{-t\square_{b,m}^{(q)}}(x, x)$ cannot have the standard asymptotic expansion as (1.2). Rather, our asymptotic expansion involves a contribution in terms of a *distance function* from lower dimensional strata of the S^1 action. (See (1.18) in Theorem 1.3 for details and for our first main result.) It should be emphasized that no pseudoconvexity condition is assumed.

Roughly speaking, on the regular part of X we have

$$(1.3) \quad e^{-t\square_{b,m}^{(q)}}(x, x) \sim t^{-n}a_n^{(q)}(x) + t^{-n+1}a_{n-1}^{(q)}(x) + \dots \quad \text{mod } O\left(t^{-n}e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}\right).$$

On the whole X we have, however,

$$(1.4) \quad e^{-t\Box_{b,m}^{(q)}}(x, x) \sim t^{-n} A_n^{(q)}(t, x) + t^{-n+1} A_{n-1}^{(q)}(t, x) + \dots$$

The difference between (1.4) and (1.3) lies in that $A_s^{(q)}(t, x)$ in (1.4) cannot be t -independent for all s and are not canonically determined (by our method) while $a_s^{(q)}(x)$ in (1.3) are t -independent for all s and are canonically determined. This t -dependence presents a great distinction between our asymptotic expansion and those in the previous literature. It appears to have a big influence on the formulation and proof of the relevant index theorems and trace integrals. See Section 7 for more comments.

In addition to the introduction of a distance function \hat{d} in (1.3) our generalization has another feature, which is pertinent to the third topic of this paper, as follows. A heat kernel result for orbifolds obtained in 2008 by Dryden, Gordon, Greenwald and Webb for the case of Laplacian on functions (see (1.30) and [23]) and independently by Richardson ([59]) seems to suggest that integrating (1.3) over X is basically a power series in $t^{\frac{1}{2}}$. See (1.30) for more. To see such a possible connection, one considers X as a fiber space over X/S^1 which is then an orbifold, and presumes boldly an analogy with “(1.2) for the orbifold case”. Then by the above result [23], integrating (1.3) over X might give an asymptotic expansion which is a power series at most in the *fractional* power $t^{\frac{1}{2}}$ of t (cf. Theorem 1.14) (while for the case where the S^1 action is globally free, such as in the circle bundle above, the asymptotic expansion is expressed in the *integral* power of t). However, our further study shows that the coefficients of t^j for j being half-integral necessarily vanish in our present case (irrespective of the local or global freeness of the S^1 action). Despite that there is no nontrivial fractional power in the t -expansion, the *corrections*/contributions associated with the stratification of the locally free S^1 action do arise nontrivially in a proper sense. Some explicit computations about these extra terms are worked out in the main result of the final section (Section 7) regarded as the third topic of this paper.

As far as the asymptotic expansion is concerned, we remark that the approach of using Kodaira Laplacian on M (downstairs) as done above is no longer applicable to the general CR case, as the contribution of a distance function on X involved in our expansion cannot be easily foreseen by use of objects in the space downstairs (an orbifold in general). (However, for *trace integrals* on invariant functions, cf. Section 7, like $\sum_m e^{-t\lambda_m}$ denoted by $I(t)$ in certain Riemannian cases, $I(t)$ has been studied asymptotically with the help of the underlying/quotient manifold/orbifold, cf. [59, p. 2316-2317]. See also Proposition 5.1, Remark 5.4.) We must work on the entire X from scratch with the operator being only transversally elliptic (on X). (See HRR theorem below for another instance of this idea.) Furthermore, as we make no assumption on (strong) pseudoconvexity of X , this renders the techniques usually useful in this direction by previous work (e.g., [4]) hardly adequate in our case. Our current approach is essentially independent of the previous methods. This technicality

partly accounts for the length of the present paper (see Section 1.6 for an outline of proof and Section 7 for a comparison with the previous work).

We expect that the coefficients $a_s^{(q)}(x)$ in (1.3) are related to some geometric quantities. For $q = 0$, function case with strong pseudoconvexity, we refer the reader to the paper of Beals, Greiner, and Stanton [4]. In this regard, Chern-Moser invariants (see [18]) or Tanaka-Webster invariants (see [62] or [64]) should be used to express these coefficients. In our present situation (without assumptions on pseudoconvexity) it is however more natural to use geometric quantities adapted to the S^1 invariance property, so that a notion of *tangential curvature* arises (with the associated *tangential characteristic forms*, cf. Section 2.3) and enters into the coefficients of our asymptotic expansion. It essentially comes back to the Tanaka-Webster curvature in the strongly pseudoconvex case (cf. Remark 1.9).

The mathematics (existence, asymptotics etc.) of equivariant/transversal heat kernels in the Riemannian situation (including that of Riemannian foliations) have been studied in recent years and last decades. For a comparison between these developments and our results, we postpone the survey, together with that of trace integrals, until Section 7.

Back to the special case of the circle bundle X over a compact complex manifold M , the Hirzebruch-Riemann-Roch Theorem or Atiyah-Singer index Theorem, together with Theorem 1.2, tells us that

$$(1.5) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X) = \sum_{j=0}^n (-1)^j \dim H^j(M, L^m) = \int_M \mathrm{Td}(T^{1,0}M) \mathrm{ch}(L^m),$$

in terms of standard characteristic classes on M . Let's reformulate (1.5) in geometric terms on X rather than on M :

$$(1.6) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X) = \frac{1}{2\pi} \int_X \mathrm{Td}_b(T^{1,0}X) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0,$$

where $\mathrm{Td}_b(T^{1,0}X)$ denotes the *tangential Todd class* of $T^{1,0}X$ and $e^{-m \frac{d\omega_0}{2\pi}}$ denotes the Chern polynomial of the Levi curvature and ω_0 is the uniquely determined global real 1-form (see Section 2.2 and Section 2.3 for the precise definitions).

Our second main result turns to any (abstract) CR manifold X with (locally free) S^1 action (but with no assumption on pseudoconvexity); we see that the above Euler characteristic has an index interpretation related to $\bar{\partial}_b + \bar{\partial}_b^*$ on X (see (3.12) and (3.13)). We are able to establish (1.6) (cf. Corollary 1.13) on such X based on our asymptotic expansion for the heat kernel $e^{-t \square_{b,m}^{(q)}}(x, x)$ and a type of *McKean-Singer* formula on X (see Corollaries 4.8 and 5.16).

As an application to complex (orbifold) geometry, it is worth noting a comparison between the present result and a result of Kawasaki on Hirzebruch-Riemann-Roch theorem (HRR theorem for short) on a complex orbifold N ([47]) (which plays the

role as our M above). Our formula simplifies Kawazaki's original one for a certain class of orbifolds. Indeed, the contribution of the lower dimensional strata of the orbifold appearing in Kawazaki's formula vanishes in ours, see Subsection 1.4.2, Theorem 1.28 the remarks following it. In our view this simplification does not appear obvious at all within the original approach of Kawasaki because in his approach the contributions from the (lower dimensional) strata of the orbifold cannot be avoided (unless it is proved to be vanishing) even if the total space of the (orbifold) circle bundle is smooth. Conceptually speaking one may attribute such a simplification to one's working on the entire (smooth) X rather than on the downstairs M (as Kawasaki), a strategy already employed for the asymptotic expansion above and proving useful again in this context of (CR) index theorem. We remark that the vanishing of the contribution of strata also occurs in a related context studied by these works [55], [34] (see also discussions after Theorem 1.28).

In short our second main result (Theorem 1.10) computes a local index density in terms of *tangential* characteristic forms, which is to show that certain non-trivial contributions (cf. $t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}$ of (1.3)) in the heat kernel expansion will eventually cancel out in the index density computation. We can do this by applying Getzler's rescaling technique to the *off-diagonal estimate* (not needed in the classical index theorems). As, to the best of our knowledge, an appropriate term for such a result about the local density hasn't appeared in the literature yet, we shall follow the classical cases and call it a *local index theorem* on these CR manifolds (Corollary 1.13), including *Sasakian manifolds* of interest in String Theory.

With reference to the questions in the beginning of this Introduction, for further application of our results to CR geometry it is important to produce many CR functions or CR sections. Namely we hope to know when $H_b^0(X, E)$ or $H_{b,m}^0(X, E)$ is large (see Questions 1.18, 1.19 and 1.23 in Section 1.3). Progress towards this circle of questions seems limited (Section 1.3). We can now develop a tool for tackling some of these questions. The idea here is to combine our version of CR index theorem with a sort of vanishing theorem for higher cohomology groups, which is intimately related to a version of Grauert-Riemenschneider criterion adapted to the CR case. This methodology turns out to be effective for those CR manifolds studied in this paper.

In Section 1.3 we apply our CR index theorem to prove a CR version of Grauert-Riemenschneider criterion, and produce many CR functions on a weakly pseudoconvex CR manifold with transversal S^1 action and many CR sections on some class of CR manifolds, which give answers to some long-standing questions in several complex variables and CR geometry. In Section 1.4 we provide an abundance of examples of those CR manifolds studied in the present paper, some of which arise from Brieskorn manifolds (generalized Hopf manifolds).

There is another index theory of geometric significance, developed by Charles Epstein. He studied the so called relative index of a pair of embeddable CR structures

through their Szegő projectors in a series of papers (see [27], [28], [29], [30] and [31]). On the other hand, Erik van Erp derived an index formula for subelliptic operators on a contact manifold (see [32], [33]). Moreover, recent work of Paradan and Vergne [55] gave an expression for the index of transversally elliptic operators which is an integral of compactly supported equivariant form on the cotangent bundle; see also Fitzpatrick [34] for related directions. Brüning, Kamber and Richardson [13], [12] computed the index of an equivariant transversally elliptic operator as a sum of integrals over blow ups of the strata of the group action.

Finally it is natural to ask for a generalization from the action of S^1 to that of other Lie groups or even to foliations (cf. Subsection 7.1 for references). As we will discuss in Section 7, the asymptotic expansion in the form (1.4) is thereby indicated as a sort of remedy for (1.3). In this way the “distance function” \hat{d} shall be involved. All of this seems to be best illustrated in the present S^1 case. It appears also conceivable that these features shall be preserved for generalization in a certain (as yet unknown) way. This paper may be presented or read in token of a prototype for further study into much more complicated, diversified situations.

1.2. Main theorems

We shall now formulate the main results. We refer to Section 2.2 and Section 2.3 for some notations and terminologies used here. After the background material, we will discuss in the sequel i) asymptotic expansions, ii) a local index theorem and iii) trace integrals.

1.2.1. Background. – Let $(X, T^{1,0}X)$ be a compact connected CR manifold with a transversal CR locally free S^1 action $e^{-i\theta}$, where $T^{1,0}X$ is a CR structure of X . X is of dimension $2n + 1$ throughout this paper.

Let $T \in C^\infty(X, TX)$ be the real vector field induced by the S^1 action and let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle \omega_0, T \rangle = 1$, $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$.

Associated with the S^1 action of X it is natural to consider various geometric objects admitting an S^1 action. In the following, to streamline the exposition we shall freely use the notion of *rigid* objects: “rigid bundles,” “rigid metrics” etc., and refer to Definitions 2.3, 2.4 and 2.5 for the precise meanings. (See also the work of Baouendi-Rothschild-Treves [3, Definition II.2] for a similar use of this term.) It suffices to say here that this notion of rigid objects is nothing but an equivalent way (by using metric) to consider objects (originally defined without assumption on metric) which admit (compatible) S^1 actions (or S^1 invariance, subject to the proper context) provided one starts with a CR manifold with an S^1 action (cf. Theorem 2.11).

Henceforth let E be a rigid CR vector bundle over X , equipped with a rigid Hermitian metric $\langle \cdot | \cdot \rangle_E$. We note that $T^{1,0}X$ is known to be a rigid complex vector bundle (see the work of Baouendi-Rothschild-Treves [3]) with a rigid Hermitian metric $\langle \cdot | \cdot \rangle$ satisfying extra properties (not specified here, cf. (2.5)). Let $\langle \cdot | \cdot \rangle_E$ be the Hermitian metric on $T^{*0,\bullet}X \otimes E$ induced by those on E and $\mathbb{C}TX$. Denoting by $dv_X = dv_X(x)$ the volume form on X induced by the Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ we get the natural global L^2 inner product $(\cdot | \cdot)_E$ on $\Omega^{0,\bullet}(X, E)$.

As remarked in Introduction, for $u \in \Omega^{0,\bullet}(X, E)$, $Tu \in \Omega^{0,\bullet}(X, E)$ is defined and $T\bar{\partial}_b = \bar{\partial}_b T$. For $m \in \mathbb{Z}$, put

$$\begin{aligned} \Omega_m^{0,\bullet}(X, E) &:= \{u \in \Omega^{0,\bullet}(X, E); Tu = -imu\} \\ &= \{u \in \Omega^{0,\bullet}(X, E); (e^{-i\theta})^*u = e^{-im\theta}u, \quad \forall \theta \in [0, 2\pi[\}, \end{aligned}$$

where $(e^{-i\theta})^*$ denotes the pull-back by the map $e^{-i\theta} : X \rightarrow X$ of S^1 action.

Write $L^2(X, T^{*0,\bullet}X \otimes E)$ (resp. $L_m^2(X, T^{*0,\bullet}X \otimes E)$) for the L_2 -completion of $\Omega^{0,\bullet}(X, E)$ (resp. $\Omega_m^{0,\bullet}(X, E)$) with respect to $(\cdot | \cdot)_E$.

By $T\bar{\partial}_b = \bar{\partial}_b T$ one defines $\bar{\partial}_{b,m} : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E)$ as the restriction of $\bar{\partial}_b$ on $\Omega_m^{0,\bullet}$. Write

$$\bar{\partial}_b^* : \Omega^{0,\bullet}(X, E) \rightarrow \Omega^{0,\bullet}(X, E), \quad \text{resp.} \quad \bar{\partial}_{b,m}^* : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E)$$

for the formal adjoint of $\bar{\partial}_b$ (under $(\cdot | \cdot)_E$), resp. $\bar{\partial}_{b,m}$. Since $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$ are rigid, one sees

$$(1.7) \quad \begin{aligned} T\bar{\partial}_b^* &= \bar{\partial}_b^* T \quad \text{on } \Omega^{0,\bullet}(X, E), \\ \bar{\partial}_{b,m}^* &= \bar{\partial}_b^*|_{\Omega_m^{0,\bullet}} : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E), \quad \forall m \in \mathbb{Z}. \end{aligned}$$

Let $A_m : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E)$ be a certain smooth zeroth order operator with $TA_m = A_m T$ and $A_m : \Omega_m^{0,\pm}(X, E) \rightarrow \Omega_m^{0,\mp}(X, E)$ (arising from a CR version of Spin^c Dirac operator, cf. Definition 4.3). Put

$$(1.8) \quad \tilde{D}_{b,m} := \bar{\partial}_{b,m} + \bar{\partial}_{b,m}^* + A_m : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E)$$

and let

$$(1.9) \quad \tilde{D}_{b,m}^* : \Omega^{0,\bullet}(X, E) \rightarrow \Omega^{0,\bullet}(X, E)$$

be the formal adjoint of $\tilde{D}_{b,m}$ (with respect to $(\cdot | \cdot)_E$).

We have $\tilde{\square}_{b,m}$, given by

$$(1.10) \quad \tilde{\square}_{b,m} := \tilde{D}_{b,m}^* \tilde{D}_{b,m} : \Omega^{0,\bullet}(X, E) \rightarrow \Omega^{0,\bullet}(X, E),$$

which denotes the m -th *modified* Kohn Laplacian, thought of as Spin^c Kohn Laplacian (cf. Definition 4.3 and the paragraph below it). We extend $\tilde{\square}_{b,m}$ by

$$(1.11) \quad \tilde{\square}_{b,m} : \text{Dom } \tilde{\square}_{b,m} (\subset L_m^2(X, T^{*0,\bullet}X \otimes E)) \rightarrow L_m^2(X, T^{*0,\bullet}X \otimes E),$$

with $\text{Dom } \tilde{\square}_{b,m} := \{u \in L_m^2(X, T^{*0,\bullet}X \otimes E); \tilde{\square}_{b,m}u \in L_m^2(X, T^{*0,\bullet}X \otimes E)\}$ in which $\tilde{\square}_{b,m}u$ is defined in the sense of distribution.

We will show in Section 3 that $\tilde{\square}_{b,m}$ is self-adjoint, $\text{Spec } \tilde{\square}_{b,m}$ is a discrete subset of $[0, \infty[$ and for $\nu \in \text{Spec } \tilde{\square}_{b,m}$, ν is an eigenvalue of $\tilde{\square}_{b,m}$ with finite multiplicities $d_\nu < \infty$. Let $\{f_1^\nu, \dots, f_{d_\nu}^\nu\}$ be an orthonormal frame for the eigenspace of $\tilde{\square}_{b,m}$ with eigenvalue ν . The (smooth) heat kernel $e^{-t\tilde{\square}_{b,m}}(x, y)$ can be given by

$$(1.12) \quad e^{-t\tilde{\square}_{b,m}}(x, y) = \sum_{\nu \in \text{Spec } \tilde{\square}_{b,m}} \sum_{j=1}^{d_\nu} e^{-\nu t} f_j^\nu(x) \wedge (f_j^\nu(y))^\dagger,$$

where $(f_j^\nu(y))^\dagger$ denotes the linear map $(f_j^\nu(y))^\dagger : T_y^{*0,\bullet}X \otimes E_y \rightarrow \mathbb{C}$, $u(y) \in T_y^{*0,\bullet}X \otimes E_y \rightarrow \langle u(y) | f_j^\nu(y) \rangle_E \in \mathbb{C}$. Note that $f_j^\nu(x) \wedge (f_j^\nu(y))^\dagger$ is the linear map:

$$\begin{aligned} f_j^\nu(x) \wedge (f_j^\nu(y))^\dagger : T_y^{*0,\bullet}X \otimes E_y &\rightarrow T_x^{*0,\bullet}X \otimes E_x, \\ u(y) \in T_y^{*0,\bullet}X \otimes E_y &\rightarrow f_j^\nu(x) \langle u(y) | f_j^\nu(y) \rangle_E \in T_x^{*0,\bullet}X \otimes E_x. \end{aligned}$$

Let $e^{-t\tilde{\square}_{b,m}} : L^2(X, T^{*0,\bullet}X \otimes E) \rightarrow L_m^2(X, T^{*0,\bullet}X \otimes E)$ be the (continuous) operator associated with the distribution kernel $e^{-t\tilde{\square}_{b,m}}(x, y)$.

Let $e_1(x), \dots, e_d(x)$ be an orthonormal frame of $T_x^{*0,q}X \otimes E_x$ ($q = 0, 1, \dots, n$), and $A \in \text{End}(T_x^{*0,\bullet}X \otimes E_x)$. Put $\text{Tr}^{(q)} A := \sum_{j=1}^d \langle Ae_j | e_j \rangle_E$ and set

$$(1.13) \quad \begin{aligned} \text{Tr } A &:= \sum_{j=0}^n \text{Tr}^{(j)} A, \\ \text{STr } A &:= \sum_{j=0}^n (-1)^j \text{Tr}^{(j)} A. \end{aligned}$$

Let ∇^{TX} be the Levi-Civita connection on TX (with respect to $\langle \cdot | \cdot \rangle$). Then $T^{1,0}X$ is equipped with a connection $\nabla^{T^{1,0}X} := P_{T^{1,0}X} \nabla^{TX}$ where $P_{T^{1,0}X}$ be the projection from $\mathbb{C}TX$ onto $T^{1,0}X$.

Let ∇^E be the connection on E induced by $\langle \cdot | \cdot \rangle_E$ (see Theorem 2.12). Let $\text{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X)$ denote the representative of the tangential Todd class of $T^{1,0}X$ and $\text{ch}_b(\nabla^E, E)$ the representative of the tangential Chern character of E (see Section 2.3 for *tangential* classes).

In what follows we aim to define a *distance function* \hat{d} which plays an important role (for the asymptotic expansion) in this paper. For $x \in X$, we say that the period of x is $\frac{2\pi}{\ell}$, $\ell \in \mathbb{N}$ provided that $e^{-i\theta} \circ x \neq x$ for every $0 < \theta < \frac{2\pi}{\ell}$ and $e^{-i\frac{2\pi}{\ell}} \circ x = x$. Put, for each $\ell \in \mathbb{N}$,

$$(1.14) \quad X_\ell = \{x \in X; \text{the period of } x \text{ is } \frac{2\pi}{\ell}\}$$

and let $p = \min\{\ell \in \mathbb{N}; X_\ell \neq \emptyset\}$. We call X_p the *principal stratum*. It is well-known that if X is connected, then X_p is an open and dense subset of X (see Proposition 1.24 in Meinrenken [52] and Duistermaat-Heckman [25]). Assume

$X = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_k}$, $p =: p_1 < p_2 < \cdots < p_k$. Put $X_{\text{sing}} = X_{\text{sing}}^1 := \bigcup_{j=2}^k X_{p_j}$, $X_{\text{sing}}^r := \bigcup_{j=r+1}^k X_{p_j}$, $k-1 \geq r \geq 1$. Set $X_{\text{sing}}^k := \emptyset$. Note $p_1 | p_j$ for $1 \leq j \leq k$ (cf. Remark 1.17).

Let $d(\cdot, \cdot)$ denotes the standard Riemannian distance with respect to the given Hermitian metric. Take ζ

$$(1.15) \quad 0 < \zeta < \inf \left\{ \frac{2\pi}{p_k}, \left| \frac{2\pi}{p_r} - \frac{2\pi}{p_{r+1}} \right|, r = 1, \dots, k-1 \right\}.$$

Set, for $x \in X$ and $r = 1, 2, \dots, k$,

$$(1.16) \quad \hat{d}_\zeta(x, X_{\text{sing}}^r) := \inf \left\{ d(x, e^{-i\theta}x); \zeta \leq \theta \leq \frac{2\pi}{p_r} - \zeta \right\}.$$

This notation reflects the fact that $\hat{d}_\zeta(x, X_{\text{sing}}^r)$ is equivalent to the ordinary distance $d(x, X_{\text{sing}}^r)$ (see below). Note by definition $\hat{d}_\zeta(x, X_{\text{sing}}^k) (= \hat{d}_\zeta(x, \emptyset)) > 0$ for all $x \in X$. We remark that for any $0 < \zeta, \zeta_1$ satisfying (1.15), $\hat{d}_\zeta(x, X_{\text{sing}}^r)$ and $\hat{d}_{\zeta_1}(x, X_{\text{sing}}^r)$ are equivalent (as far as the estimate in Theorem 1.3 below is concerned). We shall denote $\hat{d}(x, X_{\text{sing}}^r) := \hat{d}_\zeta(x, X_{\text{sing}}^r)$.

Remark that, by examining the definition $\hat{d}(x, X_{\text{sing}}^r) = 0$ if and only if $x \in X_{\text{sing}}^r$. Further, for $\varepsilon > 0$ there is a $\delta > 0$ such that $\hat{d}(x, X_{\text{sing}}^r) \geq \delta$ provided $x \in X$ satisfies (the ordinary distance) $d(x, X_{\text{sing}}^r) \geq \varepsilon$. It is thus convenient to think of $\hat{d}(x, X_{\text{sing}}^r)$ as a distance function from x to X_{sing}^r .

Indeed in Theorem 6.7 for a strongly pseudoconvex X there is a constant $C \geq 1$ such that

$$\frac{1}{C} d(x, X_{\text{sing}}^r) \leq \hat{d}(x, X_{\text{sing}}^r) \leq C d(x, X_{\text{sing}}^r), \quad \forall x \in X.$$

1.2.2. Asymptotic expansion of the heat kernel $e^{-t\tilde{\square}_{b,m}}(x, x)$. – For $p_r, m \in \mathbb{N}$, put $\delta_{p_r|m} := \delta_{\mathbb{N}}(\frac{m}{p_r})$, that is, $\delta_{p_r|m} = 1$ if $p_r | m$ and $\delta_{p_r|m} = 0$ if $p_r \nmid m$. With the distance function \hat{d} , we state the first main result of this paper (see Section 6 for a proof).

THEOREM 1.3. – *Suppose $(X, T^{1,0}X)$ is a compact, connected CR manifold (of dimension $2n+1$) with a transversal CR locally free S^1 action and let E be a rigid CR vector bundle over X . With the notations above, there exist $a_s(t, x) (= a_{s,m}(t, x)) \in C^\infty(\mathbb{R}_+ \times X, \text{End}(T^{*0,\bullet}X \otimes E))$ with $|a_s(t, x)| \leq C$ on $\mathbb{R}_+ \times X$ where $C > 0$ is independent of $t, s = n, n-1, \dots$, such that*

$$(1.17) \quad e^{-t\tilde{\square}_{b,m}}(x, x) \sim t^{-n} a_n(t, x) + t^{-n+1} a_{n-1}(t, x) + \cdots \quad \text{as } t \rightarrow 0^+.$$

(See Definition 5.5 for “ \sim ”.)

Moreover, there exist $\alpha_s(x) (= \alpha_{s,m}(x)) \in C^\infty(X, \text{End}(T^{*0,\bullet}X \otimes E))$, $s = n, n-1, \dots$, satisfying the following property. Given any differential operator

$P_\ell : C^\infty(X, T^{*0, \bullet} X \otimes E) \rightarrow C^\infty(X, T^{*0, \bullet} X \otimes E)$ of order $\ell \in \mathbb{N}_0$, there exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that

$$(1.18) \quad \left| P_\ell \left(a_s(t, x) - p_r \delta_{p_r | m} \alpha_s(x) \right) \right| \leq C_0 t^{-\frac{\ell}{2}} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}}, \quad \forall t \in \mathbb{R}_+, \quad \forall x \in X_{p_r}$$

$r = 1, \dots, k$.

The following is immediate from the proof of Theorem 1.3.

COROLLARY 1.4. – Suppose $(X, T^{1,0} X)$ is a compact, connected CR manifold with a transversal CR locally free S^1 action and let E be a rigid CR vector bundle over X . With the notations above, for any $r = 1, \dots, k$, any differential operator $P_\ell : C^\infty(X, T^{*0, \bullet} X \otimes E) \rightarrow C^\infty(X, T^{*0, \bullet} X \otimes E)$ of order $\ell \in \mathbb{N}_0$, every $N_0 \in \mathbb{N}$ with $N_0 \geq N_0(n)$ for some $N_0(n)$, there are $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ such that

$$(1.19) \quad \left| P_\ell \left(e^{-t \tilde{\square}_{b,m}}(x, x) - p_r \delta_{p_r | m} \sum_{j=0}^{N_0} t^{-n+j} \alpha_{n-j}(x) \right) \right| \leq C_{N_0} \left(t^{-n+N_0+1-\frac{\ell}{2}} + t^{-n-\frac{\ell}{2}} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}} \right), \quad \forall x \in X_{p_r}, \quad \forall 0 < t < \delta.$$

The expansions in (1.18) and (1.19) involve a contribution in terms of a *distance function* from the lower dimensional strata of the S^1 action. The analogous phenomena are encountered in other settings, for instance for the asymptotic of the Bergman kernel on orbifolds appearing in [19] (see also [50, Thm 5.4.10]) and for the heat kernel of the Kodaira Laplacian on complex orbifolds in the recent paper [57].

In the following we supplement these results with a number of remarks before going further.

REMARK 1.5. – It should be noticed that $p_r \delta_{p_r | m} = \sum_{s=1}^{p_r} e^{\frac{2\pi(s-1)}{p_r} m i}$.

REMARK 1.6. – We shall now see that if one wants an asymptotic expansion of $e^{-t \tilde{\square}_{b,m}}(x, x)$ to be valid around each $x \in X$ (cf. Definition 5.5), then (1.17) is basically optimal (i.e., in general, $a_s(t, x, y)$ cannot be t -independent for all s). For U open with $\bar{U} \subset X_p$, a tradition-like formula (assuming $p = 1$ for simplicity)

$$(1.20) \quad e^{-t \tilde{\square}_{b,m}}(x, x) \sim C(t^{-n} \alpha_n(x) + t^{-n+1} \alpha_{n-1}(x) + \dots)$$

is valid for $x \in U$ and $C = 1$ (as follows from (1.18) for $l = 0$) whereas for $x \in X_{p_r}$, $r \geq 2$, an asymptotic expansion (for $p_r \mid m$) with $C = p_r$ is valid around an open subset ($\ni x$) of the stratum X_{p_r} . Since $e^{-t \tilde{\square}_{b,m}}(x, y)$ is going to be a well defined smooth kernel, it is easily seen that those functions $\alpha_s(x)$ ($s = n, n-1, \dots$) satisfying Theorem 1.3 are unique (if they exist). (We notice that $a_s(t, x)$ in (1.17) are not canonically defined by our method which is subject to choice of BRT trivializations, cf. (5.42) and Subsection 2.4.) In short, the above suggests that an asymptotic expansion

of the form as (1.20) can only be true *in the piecewise sense* with respect to strata. See also Subsection 7.1.

To confirm this, one uses Theorem 1.14 (see Theorems 7.20 and 7.24 for a more precise version) by noting $\int_X \text{Tr} \alpha_s^+(x) dv_X(x) = S_{1,s}^+$ in Theorem 7.20 (which is taking the “even” part of the Laplacian). Hence one can interpret the trace integral result (obtained by integrating $\text{Tr} e^{-t \hat{\square}_{b,m}}(x, x)$ over X) as one that gives extra nonzero *correction terms*, cf. the second line in (1.32) or the third line in (7.57).

It follows that if there exists a global asymptotic expansion (not just in the piecewise sense) such as (1.17), then not all of $a_s(t, x)$ can be independent of t . Otherwise, if all $a_s(t, x)$ are independent of t , it would be of the form (1.20) globally by assumption ($C = 1$ if $p = 1$), so by integrating the trace over X , there would be no correction terms as discussed above. To say more, $e^{-t \hat{\square}_{b,m}}(x, x)$ cannot have any asymptotic expansion of the form $t^{m_1} \beta_{m_1}(x) + t^{m_2} \beta_{m_2}(x) + \dots$ (globally) $m_1 < m_2 < \dots \in \mathbb{R}, \beta_{m_1}(x), \beta_{m_2}(x), \dots$ continuous functions on X . Otherwise by equating it to (1.17), each $a_s(t, x)$ would be rendered independent of t , absurd as just remarked (see the next remark for argument independent of Theorems 1.14, 7.20).

The next remark shows that $a_s(t, x)$ for the particular $s = n$ must be dependent on t (nontrivially). This part will not use Theorem 1.14.

REMARK 1.7. – In the above remark a certain discontinuity in the form (1.20) for, say $x \in X_p$ and $x \in X_{p_2}$ seems to appear. We shall now explore it. If the (Gaussian-like) term to the right of (1.18) is examined, it arises from a precise integral (see (6.8)). To show that this integral is generally nontrivial, regardless of whether our estimate given by (1.18) is a fine or crude one, we are actually going to show that the term for $s = n$ in (1.18)

$$a_n(t, x) - p_r \delta_{p_r|m} \alpha_n(x)$$

is nontrivial. For the sake of illustration we assume that $X = X_1 \cup X_2$, that is $p = 1$ and $p_2 = 2$, and take m to be an even number. For $x \in X_1$, by (1.18) (for $l=0$) and $p = 1$, we see that $a_n(t, x) = \alpha_n(x) + r_n(t, x)$

$$(1.21) \quad |r_n(t, x)| \lesssim e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}.$$

As our $\alpha_n(x)$ essentially arises from a local Kodaira Laplacian (see (6.1), similar to discussion after Theorem 1.2), it is well known that $\alpha_n(x)$, as the coefficient of the leading term (in the t -expansion of the heat kernel for Kodaira Laplacian), is constant in x with $\text{Tr} \alpha_n > 0$ (cf. [36, Lemma 4.1.4 and Section 4.4]). By continuity (a_s and α_s being globally continuous functions)

$$(1.22) \quad a_n(t, x) = \alpha_n(x) + r_n(t, x)$$

remains true on X_2 . For $x_0 \in X_2$, the estimate of (1.18) is given by (with $p_2 = 2$ and discussion after (1.16) for $\hat{d}(x_0, \emptyset) > 0$)

$$(1.23) \quad a_n(t, x_0) = 2\alpha_n(x_0) + O(t^\infty).$$

By (1.22) and (1.23) it follows $r_n(t, x_0) = \alpha_n(x_0) + O(t^\infty)$ so $r_n(t, x) \approx \alpha_n(x)$ around x_0 as $t \rightarrow 0$, giving $|r_n(t, x)| \geq \epsilon > 0$ nearby x_0 for some constant ϵ independent of x and t . But this would be absurd by (1.21) if r_n were independent of t (taking $x \in X_1, \neq x_0$ near x_0 so that $|r_n(t, x)| \geq \epsilon$ and letting $t \rightarrow 0$ in (1.21)). Hence $a_n(t, x)$ cannot be independent of t either, as desired.

REMARK 1.8. – To discuss the estimate (1.19), let's take \hat{d} in (1.19) to be d for convenience (as remarked previously \hat{d} is equivalent to the ordinary distance function d at least in the strongly pseudoconvex case, cf. Theorem 6.7). Take $P_l = \text{id}$ (so $l = 0$). The term to the rightmost of (1.19) appears as a Gaussianlike term. As $t \rightarrow 0$, this term tends to a sort of Dirac delta function supported along the strata X_{sing}^r (with an extra singular factor $t^{-\frac{a-1}{2}}$, $a = \dim X_{\text{sing}}^r$). This may conceptually explain the piecewise continuity nature just discussed in Remarks 1.6 and 1.7 if the asymptotic expansion is to be expressed in something, without t -dependence, such as $\alpha_s(x)$. Conversely, the estimate as (1.19) involving a type of Dirac delta function is conceptually reasonable under the piecewise continuity phenomenon in terms of $\alpha_s(x)$. For more about this, some quantitative information may be available by Theorems 1.14, 7.20 and 7.24.

REMARK 1.9. – We make a short comment on the coefficients $a_s(t, x)$ or $\alpha_j(x)$ in (1.18) (the difference between $a_s(t, x)$ and $p\alpha_s(x)$ (at a given $x \in X_p$) is $O(t^\infty)$ by (1.18); this is partly explained conceptually right below). For the standard (elliptic) case (of Dirac type) it is well-known that the coefficients of a heat kernel along the diagonal (by taking trace) are expressible in terms of the curvature and its covariant derivatives (e.g., [36]). In our transversally elliptic case (without bundle E for simplicity) if S^1 action is globally free, it follows from the standard case above (cf. (1.1)-(1.2)) that these coefficients of the (transversal) heat kernel are expressible in terms of the *tangential* curvature (and its derivatives) (cf. Section 2.3). In the locally free case the same results can be achieved in view of the proof of Theorem 1.3, which basically arises from a procedure of patching and successive approximations based on the local (transversal) heat kernels that give the asymptotic approximations of the final (transversal) heat kernel (see Section 1.6 for details of an outline). Since the local kernels can be so expressed as just said (at least on the principal stratum), it follows from the asymptotic approximation (e.g., Theorems 2.23 and 2.30 of [5] or Theorem 5.15 in our case) that the same (expression in tangential curvature and its derivatives) can be said for the global kernel (on the principal stratum then followed by continuous extension of this global kernel on X). It is also of interest to consider the integral version of these coefficients, which is the topic of Section 7 of this paper.

1.2.3. A local index theorem for CR manifolds with S^1 action. – Here we discuss issues related to the index theorem we will prove. We recall that the term to the left of the inequality in (1.18) is basically nontrivial by Remark 1.7. In our formulation of index theorems, the contribution arising from such a term is expected to be removed.

This can be done when $\tilde{\square}_{b,m}$ is the Spin^c Kohn Laplacian (cf. (4.12)). In this case, we show that taking supertrace in (1.19) ($P_l = \text{id}$) and applying Getzler’s rescaling technique to the off-diagonal estimate (see Subsection 1.6.3 for more) yield that the singular part t^{-n} to the rightmost of (1.19) can be removed (see Subsection 1.6.4 and Section 6 for a proof). More precisely

THEOREM 1.10. – *Suppose $(X, T^{1,0}X)$ is a compact, connected CR manifold with a transversal CR locally free S^1 action and let E be a rigid CR vector bundle over X . With the notations above, if $\tilde{\square}_{b,m}$ is the Spin^c Kohn Laplacian (see (4.12)), then for $r = 1, \dots, k$ and every $N_0 \in \mathbb{N}$ with $N_0 \geq N_0(n)$ for some $N_0(n)$, there are $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ such that (STr denoting supertrace, cf. (1.13))*

$$(1.24) \quad \left| \text{STr} e^{-t\tilde{\square}_{b,m}}(x, x) - p_r \delta_{p_r|m} \sum_{j=0}^{N_0} t^{-n+j} \text{STr} \alpha_{n-j}(x) \right| \leq C_{N_0} \left(t^{-n+N_0+1} + e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}} \right), \quad \forall 0 < t < \delta, \quad \forall x \in X_{p_r},$$

and

$$(1.25) \quad \begin{aligned} \text{STr} \alpha_\ell(x) &= 0 \quad \text{for } 1 \leq \ell \leq n, \\ \text{STr} \alpha_0(x) dv_X(x) &= \frac{1}{2\pi} \left[\text{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge \text{ch}_b(\nabla^E, E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0 \right]_{2n+1}(x) \end{aligned}$$

where $[\dots]_{2n+1}$ denotes the part of $(2n + 1)$ -form.

As Spin^c objects can be simplified in the Kähler case, so can the Spin^c Kohn Laplacian in the CR Kähler case, to which we turn now.

DEFINITION 1.11. – We say that X is CR Kähler if there is a closed form $\Theta \in C^\infty(X, T^{*1,1}X)$ such that $\Theta(Z, \bar{Z}) > 0$, for all $Z \in C^\infty(X, T^{1,0}X)$. We call Θ a CR Kähler form on X .

When X is a strongly pseudoconvex CR manifold with a transversal CR locally free S^1 action, the closed form $d\omega_0$ satisfies $d\omega_0(Z, \bar{Z}) > 0$, for all $Z \in C^\infty(X, T^{1,0}X)$. Hence X is CR Kähler.

A quasi-regular Sasakian manifold is also a CR Kähler manifold. We recall that for a compact smooth manifold X of $\dim X = 2n + 1, n \geq 1$, the triple (X, g, α) where g is a Riemannian metric and α is a real 1-form is called a Sasakian manifold if the cone $\mathcal{C}(X) = \{(x, t) \in X \times \mathbb{R}_{>0}\}$ is a Kähler manifold with complex structure J and Kähler form $t^2 d\alpha + 2t dt \wedge \alpha$ compatible with the metric $t^2 g + dt \otimes dt$ (see [6], [9], [54]). As a consequence, X is a compact strongly pseudoconvex CR manifold and the Reeb vector field ξ , defined by $\alpha(\cdot) = g(\xi, \cdot)$, induces a transversal CR \mathbb{R} action on X . If the orbits of this \mathbb{R} action are compact, the Sasakian structure is called quasi-regular. In this case, the Reeb vector field generates a locally free transversal CR S^1 action on X . We

can thus identify a compact quasi-regular Sasakian manifold with a compact strongly pseudoconvex CR manifold $(X, T^{1,0}X)$ equipped with a transversal CR locally free S^1 action such that the induced vector field of the S^1 action coincides with the Reeb vector field on X (see [53], [54]).

Let X be a CR Kähler manifold with a transversal CR locally free S^1 action. If $\langle \cdot | \cdot \rangle$ is induced by a CR Kähler form on X , then $\square_{b,m}$ is equal to the Spin^c Kohn Laplacian. By Theorem 1.10, we immediately obtain a version of *local index theorem* on CR Kähler manifolds with transversal CR locally free S^1 action (which include the compact quasi-regular Sasakian manifolds as a special case by above). These results are discussed below.

For a proof of the following, see the beginning of Subsection 1.6.4 and the discussion leading to Proposition 5.9):

COROLLARY 1.12 (CR Kähler case of Theorem 1.10). – *Suppose $(X, T^{1,0}X)$ is a compact, connected CR Kähler manifold with a transversal CR locally free S^1 action and assume that $\langle \cdot | \cdot \rangle$ is induced by a CR Kähler form on X . Let E be a rigid CR vector bundle over X . With the notations above, for $r = 1, \dots, k$ and every $N_0 \in \mathbb{N}$ with $N_0 \geq N_0(n)$ for some $N_0(n)$, there are $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ such that*

$$(1.26) \quad \left| \text{STr} e^{-t \square_{b,m}}(x, x) - p_r \delta_{p_r|m} \sum_{j=0}^{N_0} t^{-n+j} \text{STr} \alpha_{n-j}(x) \right| \leq C_{N_0} \left(t^{-n+N_0+1} + e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}} \right), \quad \forall 0 < t < \delta, \quad \forall x \in X_{p_r},$$

and

$$(1.27) \quad \begin{aligned} & \text{STr} \alpha_\ell(x) = 0 \quad \text{for } 1 \leq \ell \leq n, \\ & \text{STr} \alpha_0(x) dv_X(x) \\ &= \frac{1}{2\pi} \left[\text{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge \text{ch}_b(\nabla^E, E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0 \right]_{2n+1}(x). \end{aligned}$$

We are in a position to state an index theorem (including a local index theorem in the CR Kähler case). Recall that $\bar{\partial}_{b,m} := \bar{\partial}_b : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E)$, $m \in \mathbb{Z}$, and a $\bar{\partial}_{b,m}$ -complex:

$$\bar{\partial}_{b,m} : \dots \rightarrow \Omega_m^{0,q-1}(X, E) \rightarrow \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E) \rightarrow \dots$$

The q -th $\bar{\partial}_{b,m}$ Kohn-Rossi cohomology group (regarded as the m -th Fourier component of the ordinary q -th Kohn-Rossi cohomology group) is

$$H_{b,m}^q(X, E) := \frac{\text{Ker } \bar{\partial}_{b,m} : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E)}{\text{Im } \bar{\partial}_{b,m} : \Omega_m^{0,q-1}(X, E) \rightarrow \Omega_m^{0,q}(X, E)}.$$

We will prove in Theorem 3.7 that $\dim H_{b,m}^q(X, E) < \infty$ (for each $m \in \mathbb{Z}$ and $q = 0, 1, 2, \dots, n$) without any Levi curvature assumption.

In Corollary 4.8 (see also Remark 4.9) we have a *McKean-Singer type formula* in our CR case: for every $t > 0$,

$$(1.28) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = \int_X \text{STr} e^{-t \widehat{\square}_{b,m}}(x, x) dv_X.$$

Combining (1.28), (1.24) and (1.25) and noting $e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})}{t}}$ is bounded by 1 and rapidly decays to 0 for x in the principal stratum as $t \rightarrow 0$, we conclude the following form of an index theorem on our CR manifolds (see Section 2.3 for the precise meanings of $\text{Td}_b(T^{1,0}X)$ and $\text{ch}_b(E)$ below):

COROLLARY 1.13 (CR Index Theorem, cf. Corollary 6.5). – *Suppose $(X, T^{1,0}X)$ is a compact, connected CR manifold with a transversal CR locally free S^1 action and let E be a rigid CR vector bundle over X . Then*

$$(1.29) \quad \begin{aligned} & \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) \\ &= p\delta_{p|m} \frac{1}{2\pi} \int_X \text{Td}_b(T^{1,0}X) \wedge \text{ch}_b(E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0, \end{aligned}$$

where $\text{Td}_b(T^{1,0}X)$ denotes the tangential Todd class of $T^{1,0}X$ and $\text{ch}_b(E)$ denotes the tangential Chern character of E .

For a connection with other works on index theorems by different formulations and methods, we refer to comments that come after Theorem 1.28 and to the sixth paragraph in Subsection 7.1.

1.2.4. Trace integrals in terms of geometry of the S^1 stratification. – This is the third and the last topic of this paper. For some applications (e.g., see a natural connection with Remark 1.6), one studies the asymptotic behavior of $\int_X \text{Tr} e^{-t \widehat{\square}_{b,m}}(x, x) dv_X(x)$. More comments on the historical respect come in the beginning of Section 7. Suppose M is an orbifold of (real) dimension k and $H(t, x, x)$ is the associated heat kernel on the diagonal for the standard Laplacian on M . It is known in 2008 [23] (for Laplacian on functions; see also Richardson [59, Theorem 3.5]) that

$$(1.30) \quad \int_M H(t, x, x) dv_X(x) \sim t^{-\frac{k}{2}} a_k + t^{-\frac{k}{2} + \frac{1}{2}} a_{k-\frac{1}{2}} + t^{-\frac{k}{2} + 1} a_{k-1} + t^{-\frac{k}{2} + \frac{3}{2}} a_{k-\frac{3}{2}} + \dots,$$

where $a_s \in \mathbb{R}$ is independent of t , $s = k, k - \frac{1}{2}, k - 1, \dots$. A novelty is that apart from the overall $t^{-\frac{k}{2}}$ the expansion is a power series in $t^{\frac{1}{2}}$.

By a strategy partly in connection with the proof of Theorem 1.3, we obtain an expansion of the trace integral in the same spirit as (1.30). We find that in our case, the expansion is a power series still in *integral* power of t . However, there appear various *corrections* (depending on m) supported on each stratum (cf. (7.56) and (7.61)) in contrast to the expansion in the globally free case (of S^1 action).

More precisely, we have (see Theorems 7.20 and 7.24 for more information and proof):

THEOREM 1.14 (Cf. Theorems 7.20, 7.24). – *With notations in Theorem 1.3 and assumption that the S^1 action is locally free but not globally free, let e be the number (which is even) defined to be the minimum of the (real) codimensions of connected components M of X_{p_ℓ} for all $\ell \geq 2$. For $s = n, n - 1, \dots$, we have*

$$(1.31) \quad \int_X \text{Tr } a_{s,m}(t, x) dv_X(x) \sim q_{s,0} + tq_{s,1} + t^2q_{s,2} \cdots \quad \text{as } t \rightarrow 0^+,$$

where $a_{s,m}(t, x) (= a_s(t, x))$ is as in (1.17) and $q_{s,j} \in \mathbb{R}$ is independent of t (dependent on m though), $j = 0, 1, 2, \dots$. Similarly, as $t \rightarrow 0^+$,

$$(1.32) \quad \int_X \text{Tr } e^{-t\tilde{\square}_{b,m}}(x, x) dv_X(x) \sim p\delta_{p|m}(t^{-n}c_n + t^{-n+1}c_{n-1} + t^{-n+2}c_{n-2} + \cdots) \\ + t^{-n+\frac{e}{2}}\tilde{c}_{n-\frac{e}{2}} + O(t^{-n+\frac{e}{2}+1}).$$

These coefficients satisfy the following. For an $\ell \geq 2$, write $\{M_{\ell,\gamma_\ell}\}_{\gamma_\ell}$ (possibly empty for some ℓ) for those connected components M_{ℓ,γ_ℓ} of X_{p_ℓ} with the codimension $\text{codim } M_{\ell,\gamma_\ell} = e$. Set $S_{\ell,\gamma_\ell,s,m} = \int_{M_{\ell,\gamma_\ell}} \text{Tr } \alpha_{s,m} dv_{M_{\ell,\gamma_\ell}}$ where $\alpha_{s,m} (= \alpha_s)$ is as in (1.18) and the numerical factor

$$(1.33) \quad D_{\ell,m} = (\sqrt{\pi})^e \sum_{\substack{c,h \in \mathbb{N}, (h,c)=1 \\ c > 1, c|p_\ell, c|p}} \frac{e^{-i\frac{2\pi h}{c}m}}{\left| e^{i\frac{2\pi h}{c}p} - 1 \right|^e} \quad (> 0 \text{ if } p_\ell | m).$$

i) $q_{s,1} = q_{s,2} = \cdots = q_{s,\frac{e}{2}-1} = 0$, $q_{s,0} = p\delta_{p|m} \int_X \text{Tr } \alpha_{s,m} dv_X$ ($s = n, n - 1, n - 2, \dots$).

ii) $q_{s,\frac{e}{2}}$ is (a finite sum) of the form $\sum_{\ell,\gamma_\ell} D_{\ell,m} S_{\ell,\gamma_\ell,s,m}$ ($s = n, n - 1, n - 2, \dots$).

iii) $c_s = \int_X \text{Tr } \alpha_{s,m} dv_X$ ($s = n, n - 1, n - 2, \dots$).

iv) $\tilde{c}_{n-\frac{e}{2}} = (2\pi)^{-(n+1)} \sum_{\ell,\gamma_\ell} D_{\ell,m} \text{vol}(M_{\ell,\gamma_\ell})$, (vol = volume), which is > 0 if $p_\ell | m$ for each ℓ here.

The Laplacian in the work [23] is limited to the Laplacian acting on functions while ours above is not. We remark that in [59, Theorem 3.5] the nontrivial fractional power in $t^{\frac{1}{2}}$ does occur. This is however due partly to a fixed point set of codimension 1 under a reflection isometry (*loc. cit.*, p. 2315). In our CR case, all of the various fixed point submanifolds are of even (real) codimension, cf. i) of Remark 7.22 or [59, p. 23–24]. See Section 7 for a comparison of these methods and results.

It will be of interest to study the geometrical significance of the various coefficients in (1.31) and (1.32) as usually studied in the standard heat kernel case. Some explicit expressions are available by our treatment, e.g., (7.56), (7.61) and Theorems 7.20, 7.24.

Remark that the above results essentially deal with the Gaussian part of the heat kernel, which behaves as a Dirac type delta function supported on (each) stratum. By

contrast, the CR local index theorem as Corollary 1.13 is derived by exploring the non-Gaussian parts of the heat kernel such as the off-diagonal estimate in Theorem 5.10. In spirit, the two approaches are complementary to each other in the present paper, and jointly enhance the understanding of heat kernels for this special class of CR manifolds.

We indicate a connection with deformation of singularities. In algebraic geometry it may or may not be possible to arrive at a smoothing of a singularity. In the case of quotient singularities there are results on their *rigidity* property (i.e., every deformation is trivial) giving the nonexistence of smoothing, under certain conditions (cf. [1], [61]). Here, as an application of our trace integral, there is the following result on the “rigidity” aspect of S^1 action and of the orbifold geometry.

COROLLARY 1.15. – *Let $\{X_t\}_{-\varepsilon < t < \varepsilon}$ be a smooth family of compact CR manifolds with transversal S^1 action (dependent on t smoothly) as before. Suppose that the S^1 action on X_0 is locally, but not globally, free. Then the same holds true for all t small. In the (special) case where X_0 is realized as an (orbifold) S^1 bundle (associated with a metrized holomorphic line bundle L_0) over a compact complex singular orbifold M_0 (see Subsection 1.4.2 for concrete examples), one hence obtains a nonexistence result on the “smoothing” of the holomorphic object (M_0, L_0) as a pair.*

It is possible to assert (and prove) a refined version of “rigidity” at the level of S^1 stratifications via the use of the notion of “types” introduced in Subsection 7.5; the precise formulation is left to the interested reader.

Proof. – Write $X = \bigcup_t X_t$ which is smooth by assumption. Suppose h_0 is a rigid metric on X_0 (cf. Subsection 1.2.1). It is not difficult to obtain a smooth extension h_t on X , giving a rigid metric on X_t for each t . Because of the nontrivial correction terms by Theorem 1.14, the nearby S^1 actions give nontrivial corrections terms too (on each X_t) by the continuity of heat kernels; this can occur only when the actions are locally, but not globally, free by the same theorem. \square

For those interested in algebraic geometry and singularities, the above “rigidity” result appears new, to the best of our knowledge. Note, however, that an orbifold could still be a nonsingular manifold (cf. Subsection 1.4.2 and Theorem 1.30). The proper meaning of “smoothing” in Corollary 1.15 should be carefully understood in the sense of group action within our context above.

Two more remarks go as follows.

REMARK 1.16. – We note that the topological obstruction exists for a CR manifold to admit a transversal CR S^1 action. For instance, a compact strictly pseudoconvex CR 3-manifold must have even first Betti number if admitting a transversal CR S^1 action. The reason is that such a manifold must be pseudohermitian torsion free (see [48]), and this vanishing pseudohermitian torsion implies even first Betti number as shown by Alan Weinstein (see the appendix in [17]). In this paper, we only consider

the S^1 action that is transversal and locally free. Here are two examples:

Example I: $X = \left\{ (z_1, z_2, z_3) \in \mathbb{C}^3; |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1^2 + z_2|^4 + |z_2^3 + z_3|^6 = 1 \right\}$. Then X admits a transversal CR locally free S^1 action: $e^{-i\theta} \circ (z_1, z_2, z_3) = (e^{-i\theta} z_1, e^{-2i\theta} z_2, e^{-6i\theta} z_3)$. It is clear that this S^1 action is not globally free.

Example II: Let X be a compact orientable Seifert 3-manifold. Kamishima and Tsuboi [45] proved that X is a compact CR manifold with a transversal CR locally free S^1 action. X is S^1 -fibered over a possibly singular base (an orbifold).

In Section 1.4, we collect more examples.

REMARK 1.17. – The S^1 action might admit a reduction to a simpler one as $\text{Hom}(S^1, S^1) \neq \text{id}$. Recall that $p_1 = p < p_2 < p_3 < \dots < p_k$, associated with periods of X under the given S^1 action $(e^{-i\theta}, x) \rightarrow e^{-i\theta} \circ x$. Then p divides each p_j , $j > 1$. For, the isotropy subgroup $\mathbb{Z}_p (= \mathbb{Z}/p\mathbb{Z}) \subset S^1$ acts trivially on the principal stratum, which is dense and open, hence on the whole X by continuity. The isotropy subgroups \mathbb{Z}_{p_j} , $j = 2, \dots, k$, on any other stratum must contain \mathbb{Z}_p , giving $\frac{p_j}{p} \in \mathbb{N}$.

One renormalizes the given S^1 action by the new S^1 action satisfying $p = 1$. More precisely, define

$$\begin{aligned} S^1 \times X &\rightarrow X, \\ (e^{-i\theta}, x) &\rightarrow e^{-i\theta} \diamond x := e^{-i\frac{\theta}{p}} \circ x. \end{aligned}$$

The new S^1 -action $(e^{-i\theta}, \diamond)$ has $p = 1$. Let $\tilde{\omega}_0$ be the global real one form with respect to $(e^{-i\theta}, \diamond)$ and let $\tilde{H}_{b,m}^q(X, E)$ be the corresponding cohomology group with respect to $(e^{-i\theta}, \diamond)$. One sees

$$(1.34) \quad \begin{aligned} \tilde{\omega}_0 &= p\omega_0, \\ \tilde{H}_{b,m}^q(X, E) &= H_{b,pm}^q(X, E), \quad \forall m \in \mathbb{Z}, \quad \forall q = 0, 1, 2, \dots, n. \end{aligned}$$

Examining (1.34) and Corollary 1.13 yields that the index formulas in both cases can be transformed to each other.

1.3. Applications

1.3.1. Applications in CR geometry. – In CR geometry, it has been an important issue to produce many CR functions or CR sections. Put

$$H_b^0(X, E) = \{u \in C^\infty(X, E); \bar{\partial}_b u = 0\}.$$

The following belongs to one of the standard questions in this respect.

QUESTION 1.18. – Let X be a compact weakly pseudoconvex CR manifold. When is the space $H_b^0(X, E)$ large? (Pseudoconvex CR manifolds will be briefly reviewed following Definition 2.2.)

In [49] Lempert proved that a three dimensional compact strongly pseudoconvex CR manifold X with a transversal CR locally free S^1 action can be CR embedded into \mathbb{C}^N . In [26] Epstein proved that a three dimensional compact strongly pseudoconvex CR manifold X with a transversal CR globally free S^1 action can be embedded into \mathbb{C}^N by the positive Fourier components.

The embeddability of X by positive Fourier coefficients is related to the behavior of the S^1 action on X . For example, suppose for $f_1, \dots, f_{d_m} \in H_{b,m}^0(X)$ and $g_1, \dots, g_{h_l} \in H_{b,l}^0(X)$ the map

$$\Phi_{m,l} : x \in X \rightarrow (f_1(x), \dots, f_{d_m}(x), g_1(x), \dots, g_{h_l}(x)) \in \mathbb{C}^{d_m+h_l}$$

is a CR embedding. Then, the S^1 action on X naturally induces an S^1 action on $\Phi_{m,l}(X)$, given by the following:

$$(1.35) \quad e^{-i\theta} \circ (z_1, \dots, z_{d_m}, z_{d_m+1}, \dots, z_{d_m+h_l}) \\ = (e^{-im\theta} z_1, \dots, e^{-im\theta} z_{d_m}, e^{-il\theta} z_{d_m+1}, \dots, e^{-il\theta} z_{d_m+h_l}).$$

In short, under a CR embedding by positive Fourier components, one can describe the S^1 action explicitly. Conversely, to study the embedding theorem of those CR manifolds by positive Fourier components, it becomes important to know

QUESTION 1.19. – When is $\dim H_{b,m}^0(X, E) \approx m^n$ for m large?

We shall answer, combining our index theorems with some vanishing theorems (see below), Question 1.18 and Question 1.19 for CR manifolds with transversal CR locally free S^1 action.

Firstly it follows from Corollary 1.13 (by extracting the leading coefficient of the term m^n)

COROLLARY 1.20. – *Under the same assumption as in Corollary 1.13, one has*

$$(1.36) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = rp\delta_{p|m} \frac{m^n}{n!(2\pi)^{n+1}} \int_X (-d\omega_0)^n \wedge \omega_0 + O(m^{n-1}),$$

where r denotes the complex rank of the vector bundle E .

For a vanishing theorem we can repeat the proof of Theorem 2.1 in [43] with minor changes and get

PROPOSITION 1.21. – *Under the same assumption as in Corollary 1.13, suppose further that X is weakly pseudoconvex. Then, for $m \gg 1$ $\dim H_{b,m}^j(X, E) = o(m^n)$, for every $j = 1, 2, \dots, n$.*

Combining Corollary 1.20 and Proposition 1.21 one has

COROLLARY 1.22. – *Under the same assumption as in Proposition 1.21 (with X being weakly pseudoconvex). One has, for $m \gg 1$,*

$$\dim H_{b,m}^0(X, E) = rp\delta_{p|m} \frac{m^n}{n!(2\pi)^{n+1}} \int_X (-d\omega_0)^n \wedge \omega_0 + o(m^n),$$

where r denotes the complex rank of the vector bundle E . In particular, if the Levi form is strongly pseudoconvex at some point of X , then $\dim H_{b,pm}^0(X) \approx m^n$ for $m \gg 1$, and hence $\dim H_b^0(X, E) = \infty$.

These results have provided answers to Question 1.18 and Question 1.19 (for our class of CR manifolds).

For another application, it is of great interest in CR geometry to study whether and when a CR manifold X can be CR embedded into a complex space. It is a classical theorem of L. Boutet de Monvel [8] which asserts that X can be globally CR embedded into \mathbb{C}^N for some $N \in \mathbb{N}$ provided that X is compact (with no boundary), strongly pseudoconvex, and of dimension greater than or equal to five.

When X is not strongly pseudoconvex, the space of global CR functions could even be trivial. As many interesting examples live in the projective space (e.g. the quadric $\{[z] \in \mathbb{C}\mathbb{P}^{N-1}; |z_1|^2 + \dots + |z_q|^2 - |z_{q+1}|^2 - \dots - |z_N|^2 = 0\}$), it is natural to consider a setting analogous to the Kodaira embedding theorem and ask if X can be embedded into the projective space by means of CR sections of a CR line bundle $L \rightarrow X$ or its k -th power L^k .

For a study into the above question it is natural to seek the case where the dimension of the space $H_b^0(X, L^k)$ of CR sections of L^k is large as $k \rightarrow \infty$ (so one may hopefully find many CR sections to carry out the embedding). In this regard the following question is asked by Henkin and Marinescu [51, p.47-48].

QUESTION 1.23. – When is $\dim H_b^0(X, L^k) \approx k^{n+1}$ for k large?

Assume that L is a *rigid* CR line bundle with a *rigid* Hermitian fiber metric h^L (i.e., L a CR line bundle admitting a compatible S^1 action, cf. the beginning of Section 2.2). Let $\mathcal{R}^L \in \Omega_0^2(X)$ be the curvature of L associated to h^L . For a local trivializing (S^1 -invariant) section s of L , $|s(x)|_{h^L}^2 = e^{-2\phi(x)}$ with $T\phi = 0$. Then $\mathcal{R}^L = 2\partial_b\bar{\partial}_b\phi \in \Omega_0^2(X)$. (L^k, h^{L^k}) denotes the k -th power of (L, h^L) .

With Corollary 1.13 one can show

PROPOSITION 1.24. – *With the notations above, for all $\delta > 0$, we have*

$$(1.37) \quad \sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, L^k \otimes E) \\ = r(2\pi)^{-n-1} \frac{1}{n!} k^{n+1} \int_X \int_{[-\delta, \delta]} (i\mathcal{R}_x^L - sd\omega_0(x))^n \wedge \omega_0(x) ds + o(k^{n+1}),$$

as $k \rightarrow +\infty$, where r denotes the complex rank of the vector bundle E .

Proof. – By Corollary 1.13 one can check

$$\begin{aligned}
 (1.38) \quad & \sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, L^k \otimes E) \\
 &= r(2\pi)^{-n-1} \sum_{m \in \mathbb{Z}, |m| \leq k\delta} p\delta_{p|m} \frac{1}{n!} \int_X (ik \mathcal{R}_x^L - md\omega_0(x))^n \wedge \omega_0(x) + o(k^n) \\
 &= r(2\pi)^{-n-1} \sum_{m \in \mathbb{Z}, |m| \leq k\delta} p\delta_{p|m} \frac{k^n}{n!} \int_X (i \mathcal{R}_x^L - \frac{m}{k} d\omega_0(x))^n \wedge \omega_0(x) + o(k^n).
 \end{aligned}$$

Note $p\delta_{p|m} = p$ if $p \mid m$, and 0 otherwise. By this and (1.38) we get

$$\begin{aligned}
 (1.39) \quad & \sum_{m \in \mathbb{Z}, |m| \leq k\delta} \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, L^k \otimes E) \\
 &= r(2\pi)^{-n-1} \sum_{\ell \in \mathbb{Z}, |p\ell| \leq k\delta} \frac{p}{n!} k^n \int_X (i \mathcal{R}_x^L - \frac{p\ell}{k} d\omega_0(x))^n \wedge \omega_0(x) + o(k^n).
 \end{aligned}$$

It is clear that the (Riemann) sum $\sum_{\ell \in \mathbb{Z}, |p\ell| \leq k\delta} \frac{p}{k} \int_X (i \mathcal{R}_x^L - \frac{p\ell}{k} d\omega_0(x))^n \wedge \omega_0(x)$ converges to

$$\int_X \int_{[-\delta, \delta]} (i \mathcal{R}_x^L - sd\omega_0(x))^n \wedge \omega_0(x) ds$$

as $k \rightarrow \infty$. Hence

$$\begin{aligned}
 (1.40) \quad & \sum_{\ell \in \mathbb{Z}, |p\ell| \leq k\delta} \left\{ \frac{p}{n!} k^n \int_X (i \mathcal{R}_x^L - \frac{p\ell}{k} d\omega_0(x))^n \wedge \omega_0(x) \right\} + o(k^n) \\
 &= \left\{ \frac{1}{n!} k^{n+1} \int_X \int_{[-\delta, \delta]} (i \mathcal{R}_x^L - sd\omega_0(x))^n \wedge \omega_0(x) ds \right\} + o(k^{n+1}).
 \end{aligned}$$

Combining (1.40) with (1.39) we have (1.37). \square

The following two results may be viewed as a companion of the Grauert-Riemenschneider criterion in the CR case (with S^1 action). To start with

DEFINITION 1.25. – We say that (L, h^L) is positive at $p \in X$ if the curvature \mathcal{R}_p^L is a positive Hermitian quadratic form over $T_p^{1,0}X$. We say that (L, h^L) is semipositive if for any $x \in X$ there exists a constant $\delta > 0$ such that $\mathcal{R}_x^L - sid\omega_0(x)$ is a semipositive Hermitian quadratic form over $T_x^{1,0}X$ for any $|s| < \delta$.

We can repeat the proof of Theorem 1.24 in [44] with minor changes and get

PROPOSITION 1.26 (Asymptotical vanishing). – *Assume that (L, h^L) is a semipositive CR line bundle over X . Then, for $\delta > 0$, δ small, we have*

$$\sum_{m \in \mathbb{Z}, |m| \leq k\delta} \dim H_{b,m}^j(X, L^k \otimes E) = o(k^{n+1}), \quad j = 1, 2, \dots, n.$$

Combining Proposition 1.24 and Proposition 1.26, we get

COROLLARY 1.27 (Bigness). – *Assume that (L, h^L) is semi-positive. Then, for $\delta > 0$, δ small, we have*

$$(1.41) \quad \sum_{m \in \mathbb{Z}, |m| \leq k\delta} \dim H_{b,m}^0(X, L^k \otimes E) \\ = r(2\pi)^{-n-1} \frac{1}{n!} k^{n+1} \int_X \int_{[-\delta, \delta]} (i \mathcal{R}_x^L - sd\omega_0(x))^n \wedge \omega_0(x) ds + o(k^{n+1}),$$

where r denotes the complex rank of the vector bundle E . In particular, if (L, h^L) is positive at some point of X , then

$$\dim H_{b,m}^0(X, L^k \otimes E) \approx k^{n+1}.$$

The above result yields an answer to Question 1.23 in the case pertinent to our class of CR manifolds.

1.3.2. Kawasaki’s Hirzebruch-Riemann-Roch and Grauert-Riemenschneider criterion for orbifold line bundles. – There is a link between our CR result and a complex geometry result of Kawasaki on Hirzebruch-Riemann-Roch formula over complex orbifolds [47]. Compared to Kawasaki’s, we get a simpler Hirzebruch-Riemann-Roch formula for some class of orbifold line bundles using our second main Result Corollary 1.13. Moreover, from Corollary 1.22 it follows a Grauert-Riemenschneider criterion for orbifold line bundles.

To the aim we shortly review the orbifold geometry and also set up notations. Let M be a manifold and G a compact Lie group. Assume that M admits a G -action:

$$G \times M \rightarrow M, \\ (g, x) \rightarrow g \circ x.$$

We suppose that the action G on M is locally free, that is, for every point $x \in M$, the stabilizer group $G_x = \{g \in G; g \circ x = x\}$ of x is a finite subgroup of G . In this case the quotient space

$$(1.42) \quad M/G$$

is known to be an orbifold. A remark of Kawasaki [46, p. 76] discusses the validity of a converse statement about when a space has a presentation of the form (1.42). (As is well-known, these spaces are actually called V-manifolds by Satake [60], a slightly restrictive class of orbifolds.)

We assume now that M is a compact connected complex manifold with complex structure $T^{1,0}M$. G induces an action on $\mathbb{C}TM$:

$$(1.43) \quad G \times \mathbb{C}TM \rightarrow \mathbb{C}TM, \\ (g, u) \rightarrow g^*u,$$

where $g^* = (g^{-1})_*$ the push-forward by g^{-1} on $\mathbb{C}TM$. Suppose G acts holomorphically, that is $g^*(T^{1,0}M) \subset T^{1,0}M$ for $g \in G$. Put $\mathbb{C}T(M/G) := \mathbb{C}TM/G$, $T^{0,1}(M/G) := T^{0,1}M/G$ and $T^{1,0}(M/G) := T^{1,0}M/G$.

Assume that $T^{0,1}(M/G) \cap T^{1,0}(M/G) = \{0\}$. Then, $T^{1,0}(M/G)$ gives a complex structure on M/G and M/G becomes a complex orbifold. Suppose $\dim_{\mathbb{C}} T^{1,0}(M/G) = n$.

Let L be a G -invariant holomorphic line bundle over M , that is, there exists a choice of transition functions h (defined on open charts U) of L such that $h(g \circ x) = h(x)$ for every $g \in G$, $x \in U$ with $g \circ x \in U$. Suppose that L admits a locally free G -action compatible with that on M , i.e., an action $(g, v) \in G \times L \rightarrow g \circ v \in L$ with the property $\pi(g \circ v) = g \circ (\pi(v))$ (g linearly acts on fibers of L), $\pi : L \rightarrow M$ the projection. Then, L/G is an orbifold holomorphic line bundle over M/G (the fiber is not necessarily a vector space).

The above construction induced by (locally free) G -action on L naturally extends to L^m , the m -th power of L , and L^* , the dual line bundle of L . Thus L^m/G and L^*/G are also orbifold holomorphic line bundles over M/G . Put $(q = 0, 1, 2, \dots, n)$

$$(1.44) \quad \Omega^{0,q}(M/G, L^m/G) := \{u \in \Omega^{0,q}(M, L^m); g^*u = u, \quad \forall g \in G\}.$$

The Cauchy-Riemann operator $\bar{\partial} : \Omega^{0,q}(M, L^m) \rightarrow \Omega^{0,q+1}(M, L^m)$ is G -invariant, hence gives a $\bar{\partial}$ -complex $(\bar{\partial}, \Omega^{0,\bullet}(M/G, L^m/G))$ and the q -th Dolbeault cohomology group:

$$H^q(M/G, L^m/G) := \frac{\text{Ker } \bar{\partial} : \Omega^{0,q}(M/G, L^m/G) \rightarrow \Omega^{0,q+1}(M/G, L^m/G)}{\text{Im } \bar{\partial} : \Omega^{0,q-1}(M/G, L^m/G) \rightarrow \Omega^{0,q}(M/G, L^m/G)}.$$

Let $\text{Tot}(L^*)$ be the space of all non-zero vectors of L^* . Assume that $\text{Tot}(L^*)/G$ is a smooth manifold. Take any G -invariant Hermitian fiber metric h^{L^*} on L^* , set $\tilde{X} = \{v \in L^*; |v|_{h^{L^*}} = 1\}$ and put $X = \tilde{X}/G$. Since $\text{Tot}(L^*)/G$ is a smooth manifold by the foregoing assumption, $X = \tilde{X}/G$ is a smooth manifold. The natural S^1 action on \tilde{X} induces a locally free S^1 action $e^{-i\theta}$ on X . One can check that X is a CR manifold and the S^1 action on X is CR and transversal.

In a similar vein as the proof of Theorem 1.2 one can show (for $q = 0, 1, 2, \dots, n$ and $m \in \mathbb{Z}$)

$$(1.45) \quad H^q(M/G, L^m/G) \cong H_{b,m}^q(X).$$

We pause and introduce some notations. For every $x \in \text{Tot}(L^*)$ and $g \in G$, put $N(g, x) = 1$ if $g \notin G_x$ and $N(g, x) = \inf \{\ell \in \mathbb{N}^*; g^\ell = \text{id}\}$ if $g \in G_x$. Set

$$(1.46) \quad p = \inf \{N(g, x); x \in \text{Tot}(L^*), \quad g \in G_x, \quad g \neq \text{id}\}.$$

It should be noticed that the p in (1.46) corresponds to the p of the S^1 action on X .

Putting together Corollary 1.13, (1.45) and (1.46) gives

THEOREM 1.28. – *With the notations above, recall that we work with assumptions that M is connected and $\text{Tot}(L^*)/G$ is smooth. Then (for every $m \in \mathbb{Z}$)*

$$(1.47) \quad \sum_{j=0}^n (-1)^j \dim H^j(M/G, L^m/G) = p \delta_{p|m} \frac{1}{2\pi} \int_X \text{Td}_b(T^{1,0}X) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0.$$

To compare this result with that of Kawasaki ([47]) we assume $p = 1$ for simplicity. Note X is smooth (yet M/G could be singular). The above integral (1.47) reduces to an integral over the principal stratum of M/G (by integrating ω_0 along the fiber S^1 , which gives 1). It is thus the same as to say that the contributions from the lower dimensional strata *sum up to zero*. As remarked in Introduction a vanishing result as such is not readily available in the formula of [47]. Note that the notion of “orbifold” in [47] is slightly more general than that of Satake (on which the present section is based). As this generality does no real harm to the reasoning above, we omit the details in this regard.

The above result on the vanishing of the contributions from strata may also be reflected in the index formula of the works [55], [34] which study the index of transversally elliptic operators on a smooth compact manifold with the action of a compact Lie group G , by using the framework of equivariant cohomology theory. A remarkable point is that they define the index as a *generalized function* (on G) (also discussed in Atiyah [2], p.9-17). In fact it is not difficult to verify that for the case of the S^1 action, our present m -th index is basically the m -th Fourier component of the corresponding index (in the sense of generalized functions) of theirs (for the case $g = \text{id} \in G$ in [34], [55]).

The consistency of our result with those works above helps to shape our own view towards the asymptotic expansion of a (transversal) heat kernel conceived in this subject.

For examples that satisfy Theorem 1.28 we refer to Section 1.4. There, we construct, among others, an orbifold holomorphic line bundle over a *singular* complex orbifold such that the assumptions of Theorem 1.28 and Corollary 1.29 are fulfilled (see Corollary 1.31 and Subsection 1.4.2 below). Indeed there are ample examples in this respect.

Take any G -invariant Hermitian fiber metric h^L on L and let R^L be the associated curvature R^L . We say that R^L is semipositive if the Hermitian quadratic form $-\partial\bar{\partial} \log h^L$ is semipositive on $T^{1,0}M$. We say that R^L is positive at $p \in M$ if the Hermitian quadratic form $-\partial\bar{\partial} \log h^L$ is positive on $T_p^{1,0}M$.

As promised in the beginning of this section, we obtain now a Grauert-Riemenschneider criterion for orbifold line bundles, upon combining Corollary 1.22 and (1.45).

COROLLARY 1.29. – *With the notations and assumptions above, suppose that there is a G -invariant Hermitian fiber metric h^L on L such that the associated curvature R^L is semipositive and positive at some point of M . Then $\dim H^0(M/G, L^{pm}/G) \approx m^n$ for $m \gg 1$ and p as in (1.46).*

1.4. Examples

In this subsection, some examples of CR manifolds with locally free S^1 action (including those fitting Theorem 1.28 above) are collected.

We first review the construction of generalized Hopf manifolds introduced by Brieskorn and Van de Ven [10].

1.4.1. Generalized Hopf manifolds. – Let $a = (a_1, \dots, a_{n+2}) \in \mathbb{N}^{n+2}$, let $z = (z_1, \dots, z_{n+2})$ be the standard coordinates of \mathbb{C}^{n+2} and let $M(a)$ be the affine algebraic variety given by the equation

$$\sum_{j=1}^{n+2} z_j^{a_j} = 0.$$

If some $a_j = 1$, the variety $M(a)$ is non-singular. Otherwise $M(a)$ has exactly one singular point, namely $0 = (0, \dots, 0)$. Put $\widetilde{M(a)} := M(a) - \{0\}$. Now we define a holomorphic \mathbb{C} -action on $\widetilde{M(a)}$ by

$$t \circ (z_1, \dots, z_{n+2}) = (e^{\frac{t}{a_1}} z_1, \dots, e^{\frac{t}{a_{n+2}}} z_{n+2}), \quad t \in \mathbb{C}, \quad (z_1, \dots, z_{n+2}) \in \widetilde{M(a)}.$$

It is easy to see that the \mathbb{Z} -action on $\widetilde{M(a)}$ is globally free. The equivalence class of $(z_1, \dots, z_{n+2}) \in \mathbb{C}^{n+2}$ with respect to the \mathbb{Z} -action is denoted by $(z_1, \dots, z_{n+2}) + \mathbb{Z}$ and hence

$$H(a) := \widetilde{M(a)}/\mathbb{Z} = \left\{ (z_1, \dots, z_{n+2}) + \mathbb{Z}; (z_1, \dots, z_{n+2}) \in \widetilde{M(a)} \right\}$$

is a compact complex manifold of complex dimension $n + 1$. We call $H(a)$ a (*generalized*) *Hopf manifold*.

Let Γ_a be the discrete subgroup of \mathbb{C} , generated by 1 and $2\pi\alpha i$, where α is the least common multiple of a_1, a_2, \dots, a_{n+2} . Consider the complex 1-torus $T_a = \mathbb{C}/\Gamma_a$. $H(a)$ admits a natural T_a -action. Put $V(a) := H(a)/T_a$. By Holmann [41], $V(a)$ is a complex orbifold. Let $\pi_a : H(a) \rightarrow V(a)$ be the natural projection.

The following is well-known (see the discussion before Proposition 4 in [10]).

THEOREM 1.30. – *Let $p = (z_1, \dots, z_{n+2}) + \mathbb{Z} \in H(a)$. Assume that there are exactly k coordinates z_{j_1}, \dots, z_{j_k} all different from zero, $k \geq 2$. Then, $V(a)$ is non-singular at $\pi_a(p)$ if and only if*

$$\frac{[a_1, \dots, a_{n+2}]}{[a_{j_1}, \dots, a_{j_k}]} = \prod_{\ell \notin \{j_1, \dots, j_k\}} \frac{[a_1, \dots, a_{n+2}]}{[a_1, \dots, a_{\ell-1}, a_{\ell+1}, \dots, a_{n+2}]},$$

where $[m_1, \dots, m_d]$ denotes the least common multiple of $m_1, \dots, m_d \in \mathbb{N}$.

It follows readily

COROLLARY 1.31. – Assume $n \geq 2$ and $(a_1, a_2, \dots, a_{n+2}) = (4b_1, 4b_2, 2b_3, 2b_4, \dots, 2b_{n+2})$, where $b_j \in \mathbb{Z}$, b_j is odd, $j = 1, \dots, n+2$. Let $p = (0, 0, 1, i, 0, 0, \dots, 0) + \mathbb{Z} \in H(a)$. Then, $V(a)$ is singular at $\pi_a(p)$.

The ideas in the next two (sub)subsections are heavily based on Theorem 1.30 and Corollary 1.31.

1.4.2. Smooth orbifold circle bundle over a singular orbifold.– Put

$$(1.48) \quad X := \{(z_1, \dots, z_{n+2}) \in \mathbb{C}^{n+2}; z_1^{a_1} + z_2^{a_2} + \dots + z_{n+2}^{a_{n+2}} = 0, \\ |z_1|^{2a_1} + |z_2|^{2a_2} + |z_3|^{2a_3} + \dots + |z_{n+2}|^{2a_{n+2}} = 1\}.$$

It can be checked that X is a compact *weakly pseudoconvex* CR manifold of dimension $2n+1$ with CR structure $T^{1,0}X := T^{1,0}\mathbb{C}^{n+2} \cap CTX$, where $T^{1,0}\mathbb{C}^{n+2}$ denotes the standard complex structure on \mathbb{C}^{n+2} .

Let α be the least common multiple of a_1, \dots, a_{n+2} . Consider the following S^1 action on X :

$$(1.49) \quad S^1 \times X \rightarrow X, \\ e^{-i\theta} \circ (z_1, \dots, z_{n+2}) \rightarrow (e^{-i\frac{\alpha}{a_1}\theta} z_1, \dots, e^{-i\frac{\alpha}{a_{n+2}}\theta} z_{n+2}).$$

One sees that the S^1 action is well-defined, locally free, CR and transversal. Moreover one has that the quotient X/S^1 is equal to $V(a)$, $a = (a_1, a_2, \dots, a_{n+2})$. Hence, X/S^1 is a complex orbifold.

One sees, by using Corollary 1.31, that the above X/S^1 is *singular* if $n \geq 2$ and $(a_1, \dots, a_{n+2}) = (4b_1, 4b_2, 2b_3, 2b_4, \dots, 2b_{n+2})$, where $b_j \in \mathbb{Z}$, b_j is odd, $j = 1, 2, \dots, n+2$.

We now show that $(X, T^{1,0}X)$ is CR-isomorphic to the (orbifold) circle bundle associated with an orbifold line bundle over $X/S^1 = V(a)$.

To see this and to construct the circle bundle in the first place, let $L = (\widetilde{M}(a) \times \mathbb{C})/\equiv$, where $(z_1, \dots, z_{n+2}, \lambda) \equiv (\tilde{z}_1, \dots, \tilde{z}_{n+2}, \tilde{\lambda})$ if

$$\tilde{z}_j = e^{\frac{m}{a_j}} z_j, \quad j = 1, \dots, n+2, \\ \tilde{\lambda} = e^m \lambda,$$

where $m \in \mathbb{Z}$. We can check that \equiv is an equivalence relation and L is a holomorphic line bundle over $H(a)$. The equivalence class of $(z_1, \dots, z_{n+2}, \lambda) \in \widetilde{M}(a) \times \mathbb{C}$ is denoted by $[(z_1, \dots, z_{n+2}, \lambda)]$. The complex 1-torus T_a action on L is given by the following:

$$(1.50) \quad T_a \times L \rightarrow L, \\ (t + i\theta) \circ [(z_1, \dots, z_{n+2}, \lambda)] \rightarrow [(e^{\frac{t+i\theta}{a_1}} z_1, \dots, e^{\frac{t+i\theta}{a_{n+2}}} z_{n+2}, e^{t-i\frac{\theta}{\alpha}} \lambda)],$$

where α is the least common multiple of a_1, \dots, a_{n+2} . One has that the torus action (1.50) is well-defined and L/T_a is an orbifold line bundle over $H(a)/T_a = V(a)$.

Let $\tau : L \rightarrow L/T_a$ be the natural projection and for $[(z_1, \dots, z_{n+2}, \lambda)] \in L$, we write $\tau([(z_1, \dots, z_{n+2}, \lambda)]) = [z_1, \dots, z_{n+2}, \lambda] + T_a$. One sees that the pointwise norm

$$|[(z_1, \dots, z_{n+2}, \lambda)] + T_a|_{h^{L/T_a}}^2 := |\lambda|^2 \left(|z_1|^{2a_1} + |z_2|^{2a_2} + |z_3|^{2a_3} + \dots + |z_{2n+2}|^{2a_{n+2}} \right)^{-1}$$

is well-defined as a Hermitian fiber metric on L/T_a . The (orbifold) circle bundle $C(L/T_a)$ with respect to $(L/T_a, h^{L/T_a})$ is given by

$$(1.51) \quad \begin{aligned} C(L/T_a) &:= \left\{ v \in L/T_a; |v|_{h^{L/T_a}}^2 = 1 \right\} \\ &= \left\{ [(z_1, \dots, z_{n+2}, \lambda)] + T_a; \right. \\ &\quad \left. |\lambda|^2 = |z_1|^{2a_1} + |z_2|^{2a_2} + |z_3|^{2a_3} + \dots + |z_{2n+2}|^{2a_{n+2}} \right\}. \end{aligned}$$

One sees that $C(L/T_a)$ is a smooth CR manifold with the CR structure

$$T^{1,0}C(L/T_a) := T^{1,0}L/T_a \cap CTC(L/T_a),$$

where $T^{1,0}C(L/T_a)$ denotes the complex structure on L/T_a . Moreover, the orbifold line bundle $L/T_a \rightarrow V(a)$ satisfies a similar situation as in Theorem 1.28 (i.e., the space $X/S^1 = V(a)$ here as the M/G there, is singular and $C(L/T_a)$ as a (orbifold) circle bundle over M/G is smooth). We are ready to give an CR isomorphism of X and the (orbifold) circle bundle $C(L/T_a)$. Note $C(L/T_a)$ admits a nature S^1 action:

$$e^{-i\theta} \circ ([z_1, \dots, z_{n+2}, \lambda] + T_a) = [(z_1, \dots, z_{n+2}, e^{-i\theta}\lambda)] + T_a.$$

Let $\Phi : C(L/T_a) \rightarrow X$ be the smooth map defined as follows. For every $[(z_1, \dots, z_{n+2}, \lambda)] + T_a \in C(L/T_a)$, there is a unique $(\hat{z}_1, \dots, \hat{z}_{n+2}) \in X$ such that

$$[(z_1, \dots, z_{n+2}, \lambda)] + T_a = [(\hat{z}_1, \dots, \hat{z}_{n+2}, 1)] + T_a.$$

Then, $\Phi([(z_1, \dots, z_{n+2}, \lambda)] + T_a) := (\hat{z}_1, \dots, \hat{z}_{n+2}) \in X$. It can be checked that Φ is a CR embedding, globally one to one, onto and the inverse $\Phi^{-1} : X \rightarrow C(L/T_a)$ is also a CR embedding. Moreover $e^{-i\theta} \circ \Phi(x) = \Phi(e^{-i\theta} \circ x)$, $\forall x \in C(L/T_a)$. We conclude Φ is a CR isomorphism.

1.4.3. Family, non-pseudoconvex cases and deformations. – With the notation of Subsection 1.4.1 we assume $a_1 = 1$, so

$$M(a) = \{(z_1, \dots, z_{n+2}) \in \mathbb{C}^{n+2}; z_1 = -z_2^{a_2} - \dots - z_{n+2}^{a_{n+2}}\}.$$

Fix a $q = 2, 3, \dots, n + 1$. Put, for $t \in \mathbb{C}$,

(1.52)

$$X_{q,t} := \{(z_1, \dots, z_{n+2}) + \mathbb{Z} \in H(a); -|z_2^{a_2} + tz_3^{a_3}|^2 - |z_3|^{2a_3} - \dots - |z_q|^{2a_q} + |z_{q+1}|^{2a_{q+1}} + \dots + |z_{n+2}|^{2a_{n+2}} = 0\}.$$

One can check that for each t , $X_{q,t}$ is a compact CR manifold of dimension $2n + 1$ with CR structure $T^{1,0}X_{q,t} := T^{1,0}H(a) \cap CTX_{q,t}$, where $T^{1,0}H(a)$ denotes the natural complex structure inherited by $M(a)$. Note X_{q,t_1} is diffeomorphic to X_{q,t_2} for $t_1, t_2 \in \mathbb{C}$ since they can be connected through a (smooth) family of compact manifolds.

Let \tilde{a} be the least common multiple of a_1, \dots, a_q . Consider the following S^1 action on $X_{q,t}$:

$$(1.53) \quad S^1 \times X_{q,t} \rightarrow X_{q,t}, \\ e^{-i\theta} \circ ((z_1, \dots, z_{n+2}) + \mathbb{Z}) \rightarrow (e^{-i\tilde{a}\theta}(-z_2^{a_2} - \dots - z_q^{a_q}) - z_{q+1}^{a_{q+1}} - \dots - z_{n+2}^{a_{n+2}}, \\ e^{-i\frac{\tilde{a}}{a_2}\theta} z_2, \dots, e^{-i\frac{\tilde{a}}{a_q}\theta} z_q, z_{q+1}, \dots, z_{n+2}) + \mathbb{Z}.$$

One sees that the S^1 action is well-defined, locally free, CR and transversal. This is an example for a family of CR manifolds admitting a transversal CR locally free S^1 action.

Moreover these CR manifolds $X_{q,t}$ are not pseudoconvex.

Now we consider certain CR deformations of a compact CR manifold X with a transversal CR locally free S^1 action. Let $F(x) \in C^\infty(X)$ with $TF = 0$ (T the global real vector field induced by the S^1 action). Let $Z_1, \dots, Z_n \in C^\infty(X, T^{1,0}X)$ be a basis for $T^{1,0}X$. Put

$$(1.54) \quad H^{1,0}X := \{Z_j + Z_j(F)T; j = 1, 2, \dots, n\}.$$

One can check that $H^{1,0}X$ is a CR structure and the S^1 action is locally free, CR and transversal with respect to this new CR structure $H^{1,0}X$ (see (2.10) via the BRT construction).

To see how “new” this CR structure $H^{1,0}X$ is, let’s take X to be a circle bundle associated with a holomorphic line bundle $(L, \|\cdot\|)$ over a compact complex manifold M . Consider a change of metric $\|\cdot\| \rightarrow e^{-2f}\|\cdot\|$ on L and the circle bundle \tilde{X} thus induced by this new metric. By using the Formula (1.56) below one sees that $H^{1,0}X$ of (1.54) for $F = -if$ is equivalent to $T^{1,0}\tilde{X}$. But is $(X, T^{1,0}X)$ CR equivalent to $(\tilde{X}, T^{1,0}\tilde{X})$? The answer is in general no. For instance, spherical CR structures on a certain topological type of X can be obtained by using special metrics on L (cf. [16]). Hence an arbitrary perturbation of the bundle metric, say by the multiplier e^{-2f} , would bring X out of the spherical category. Note that the moduli space of spherical CR structures in [16] is finite dimensional. It follows that for F a purely imaginary function on X , the CR structure $H^{1,0}X$ is in general not CR equivalent to $T^{1,0}X$.

If, however, F is a real function, it is easily seen that the change $(z, \theta) \rightarrow (z, \theta + F)$ is globally defined, hence it gives a diffeomorphism ϕ of X . One sees $\phi_*(T^{1,0}X) = H^{1,0}X$, cf. (1.56) below. So in this case the CR structure $H^{1,0}X$ is equivalent to the original one.

1.5. Proof of Theorem 1.2

We will use the same notations as in Theorem 1.2. Let s be a local trivializing section of L defined on some open set U of M , $|s|_h^2 = e^{-2\phi}$. Let $z = (z_1, \dots, z_n)$ be holomorphic coordinates on U . We identify U with an open set of \mathbb{C}^n and have the local diffeomorphism:

$$(1.55) \quad \tau : U \times]-\varepsilon_0, \varepsilon_0[\rightarrow X, (z, \theta) \mapsto e^{-\phi(z)} s^*(z) e^{-i\theta}, \quad 0 < \varepsilon_0 \leq \pi.$$

Put $D = U \times]-\varepsilon_0, \varepsilon_0[$ as a canonical coordinate patch with (z, θ) canonical coordinates (with respect to the trivialization s) such that $T = \frac{\partial}{\partial \theta}$ (recall that T is the global real vector field induced by the S^1 action). Moreover one has

$$(1.56) \quad \begin{aligned} T^{1,0}X &= \left\{ \frac{\partial}{\partial z_j} - i \frac{\partial \phi}{\partial z_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \dots, n \right\}, \\ T^{0,1}X &= \left\{ \frac{\partial}{\partial \bar{z}_j} + i \frac{\partial \phi}{\partial \bar{z}_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \dots, n \right\}, \end{aligned}$$

and

$$(1.57) \quad T^{*1,0}X = \{dz_j; j = 1, 2, \dots, n\}, \quad T^{*0,1}X = \{d\bar{z}_j; j = 1, 2, \dots, n\}.$$

See also Theorem 2.9 and proof of Proposition 4.2 for similar formulas in the general case of S^1 action.

Let $f(z) \in \Omega^{0,q}(D)$. By (1.57) we may identify f with an element in $\Omega^{0,q}(U)$.

The key object in our proof is the map $A_m^{(q)} : \Omega_m^{0,q}(X) \rightarrow \Omega^{0,q}(M, L^m)$, to be defined as follows. Let $u \in \Omega_m^{0,q}(X)$. We can write $u(z, \theta) = e^{-im\theta} \hat{u}(z)$ (on D) for some $\hat{u}(z) \in \Omega^{0,q}(U)$. Then, on $U \subset M$, we define

$$(1.58) \quad A_m^{(q)} u := s^m(z) e^{m\phi(z)} \hat{u}(z) \in \Omega^{0,q}(U, L^m).$$

We need to check the following.

i) $A_m^{(q)}$ in (1.58) is well-defined, hence gives rise to a global element $A_m^{(q)} u \in \Omega^{0,q}(M, L^m)$.

ii) It satisfies the commutativity $\bar{\partial} A_m^{(q)} = A_m^{(q+1)} \bar{\partial}_b$ (thus induces a map on respective cohomologies).

To check i) let s and s_1 be local trivializing sections of L on an open set U . Let $(z, \theta) \in \mathbb{C}^n \times \mathbb{R}$ and $(z, \eta) \in \mathbb{C}^n \times \mathbb{R}$ be canonical coordinates of D with respect

to s and s_1 respectively (D as the above). Set $|s|_{h^L}^2 = e^{-2\phi}$ and $|s_1|_{h^L}^2 = e^{-2\phi_1}$. We write (on D)

$$(1.59) \quad \begin{aligned} u &= e^{-im\theta} \hat{u}(z) \\ u &= e^{-im\eta} \hat{u}_1(z). \end{aligned}$$

To check i) amounts to the following

$$(1.60) \quad s^m(z) e^{m\phi(z)} \hat{u}(z) = s_1^m(z) e^{m\phi_1(z)} \hat{u}_1(z), \quad \forall z \in U.$$

Let $s_1 = gs$ for g a unit on U . To find relations between ϕ and ϕ_1 , \hat{u} and \hat{u}_1 in terms of g ,

$$|s_1|_{h^L}^2 = e^{-2\phi_1} = |g|^2 |s|_{h^L}^2 = e^{2\log|g| - 2\phi},$$

giving

$$(1.61) \quad \phi_1 = \phi - \log|g|.$$

For \hat{u} and \hat{u}_1 , we first claim the following (τ in (1.55) for (z, θ) and τ_1 the similar one for (z, η))

$$(1.62)$$

If $\tau(z, \theta) = \tau_1(z, \eta)$, then $e^{-i\theta} \left(\frac{g(z)}{\bar{g}(z)}\right)^{\frac{1}{2}} = e^{-i\eta}$ (with a certain branch of the square root).

Proof of the claim (1.62). – Combining (1.55) and (1.61) one sees

$$(1.63) \quad \begin{aligned} \tau(z, \theta) &= s^*(z) e^{-i\theta - \phi(z)} = s_1^*(z) g(z) e^{-i\theta - \phi(z)} \\ &= s_1^*(z) g(z) e^{-i\theta - \phi_1(z) - \log|g(z)|} \\ &= s_1^*(z) \left(\frac{g(z)}{\bar{g}(z)}\right)^{\frac{1}{2}} e^{-i\theta - \phi_1(z)}. \end{aligned}$$

The condition $\tau(z, \theta) = \tau_1(z, \eta)$ is the same as to say, by (1.55),

$$(1.64) \quad s^*(z) e^{-i\theta - \phi(z)} = s_1^*(z) e^{-i\eta - \phi_1(z)}.$$

By (1.63) and (1.64) we deduce that $\left(\frac{g(z)}{\bar{g}(z)}\right)^{\frac{1}{2}} e^{-i\theta} = e^{-i\eta}$, as claimed. \square

Now that the relations (1.61) and (1.62) have been found, the (1.60) follows by using (1.59). Hence $A_m^{(q)} : \Omega_m^{0,q}(X) \rightarrow \Omega^{0,q}(M, L^m)$ is well-defined, proving i) above.

Moreover it is easily checked that $A_m^{(q)}$ is bijective. We omit the detail.

To prove ii) that $\bar{\partial} A_m^{(q)} = A_m^{(q+1)} \bar{\partial}_b$, by (1.56) and (1.57) one sees (on D)

$$(1.65) \quad \bar{\partial}_b u = \bar{\partial}_b (e^{-im\theta} \hat{u}) = \sum_{j=1}^n e^{-im\theta} d\bar{z}_j \wedge \left(\frac{\partial \hat{u}}{\partial \bar{z}_j}(z) + m \frac{\partial \phi}{\partial \bar{z}_j}(z) \hat{u}(z) \right).$$

Hence (1.65) and (1.58) yield

$$(1.66) \quad \begin{aligned} A_m^{(q+1)}(\bar{\partial}_b u) &= s^m(z) e^{m\phi(z)} \sum_{j=1}^n d\bar{z}_j \wedge \left(\frac{\partial \hat{u}}{\partial \bar{z}_j}(z) + m \frac{\partial \phi}{\partial \bar{z}_j}(z) \hat{u}(z) \right) \\ &= s^m(z) \bar{\partial}(e^{m\phi(z)} \hat{u}(z)) \quad \text{on } U, \end{aligned}$$

giving $\bar{\partial}A_m^{(q)} = A_m^{(q+1)}\bar{\partial}_b$. Theorem 1.2 follows.

REMARK 1.32. – The map $A_m^{(q)}$ does not depend on the metrics of the manifolds X and M . In later sections we study the Kohn Laplacian and Kodaira Laplacian on X and M respectively, and try to establish a link between the two Laplacians (with the aim at the Kohn's). In this regard we need equip X and M with appropriate metrics so that $A_m^{(q)}$ thus defined is also compatible with these metrics. Note a localization of this (metrical) construction (cf. Proposition 5.1) paves the way for our subsequent plan in this work.

Some difficulties (and ways out) for a straightforward generalization of the proof for this special case (globally free S^1 action) will be discussed in the subsection below.

1.6. The idea of the proofs of Theorem 1.3, Theorem 1.10 and Corollary 1.13

We will give an outline of main ideas of some proofs. For the proof of Theorem 1.14, some ideas are outlined in the beginning of Section 7. We refer to Section 2.2 and Section 2.3 for notations and terminologies used here. The main technical tool of our method lies in a construction of a heat kernel for the Kohn Laplacian associated to the m -th S^1 Fourier component.

1.6.1. Global difficulties. – For simplicity we assume that X is CR Kähler (cf. Definition 1.11), $E = \mathbb{C}$ and $\langle \cdot | \cdot \rangle$ is induced by a CR Kähler form Θ on X . Write $\bar{\partial}_b^*$ for the adjoint of $\bar{\partial}_b$ with respect to $(\cdot | \cdot)$ and $\bar{\partial}_{b,m}^* = \bar{\partial}_b^* : \Omega_m^{0,q+1}(X) \rightarrow \Omega_m^{0,q}(X)$ with $\Omega_m^{0,+}(X)$ and $\Omega_m^{0,-}(X)$ denoting forms of even and odd degree. Consider

$$D_{b,m}^\pm := \bar{\partial}_{b,m} + \bar{\partial}_{b,m}^* : \Omega_m^{0,\pm}(X) \rightarrow \Omega_m^{0,\mp}(X), \quad m \in \mathbb{Z}$$

and let $\square_{b,m}^+ := D_{b,m}^- D_{b,m}^+ : \Omega_m^{0,+}(X) \rightarrow \Omega_m^{0,+}(X)$ ($\square_{b,m}^- := D_{b,m}^+ D_{b,m}^-$ similarly).

Extending $\square_{b,m}^+$ and $\square_{b,m}^-$ to $L_m^{2,+}(X)$ and $L_m^{2,-}(X)$ (L^2 -completion), respectively, in the standard way, we will show in Theorem 3.5 that $\text{Spec } \square_{b,m}^\pm$ are discrete subsets of $[0, \infty[$ and $\text{Spec } \square_{b,m}^\pm$ consist of eigenvalues of $\square_{b,m}^\pm$.

For $\nu \in \text{Spec } \square_{b,m}^+$, let $\{f_1^\nu, \dots, f_{d_\nu}^\nu\}$ be an orthonormal frame for the eigenspace of $\square_{b,m}^+$ with eigenvalue ν . Write $T^{*0,\bullet}X = \bigoplus_{0 \leq q \leq n} T^{*0,q}X$. $e^{-t\square_{b,m}^+}(x, y) : T_y^{*0,\bullet}X \rightarrow T_x^{*0,+}X$, said to be a heat kernel, is given by (cf. (1.12))

$$(1.67) \quad e^{-t\square_{b,m}^+}(x, y) = \sum_{\nu \in \text{Spec } \square_{b,m}^+} \sum_{j=1}^{d_\nu} e^{-\nu t} f_j^\nu(x) \wedge (f_j^\nu(y))^\dagger.$$

(Similarly we can define $e^{-t\square_{b,m}^-}(x, y)$.)

We will show in Corollary 4.8 (see also Remark 4.9) that we have a *CR McKean-Singer type formula*: for $t > 0$,

$$(1.68) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X) = \int_X \left(\text{Tr } e^{-t\Box_{b,m}^+}(x, x) - \text{Tr } e^{-t\Box_{b,m}^-}(x, x) \right) dv_X.$$

By this formula the proof of our index theorem (cf. Corollary 1.13) is reduced to determining the small t behavior of the function $\left(\text{Tr } e^{-t\Box_{b,m}^+}(x, x) - \text{Tr } e^{-t\Box_{b,m}^-}(x, x) \right)$.

With the kernel $e^{-t\Box_{b,m}^+}(x, y)$ there is associated an operator denoted by $e^{-t\Box_{b,m}^+} : \Omega^{0,+}(X) \rightarrow \Omega_m^{0,+}(X)$. Note the domain is set to be the full space $\Omega^{0,+}(X)$. From (1.67) it follows that the kernel satisfies a heat equation which is expressed in the following operator form

$$(1.69) \quad \frac{\partial e^{-t\Box_{b,m}^+}}{\partial t} + \Box_{b,m}^+ e^{-t\Box_{b,m}^+} = 0$$

and

$$(1.70) \quad e^{-t\Box_{b,m}^+}|_{t=0} = Q_m^+,$$

where $Q_m^+ : L^{2,+}(X) \rightarrow L_m^{2,+}(X)$ is the orthogonal projection.

The main difficulty lies in that the initial condition (1.70) is a *projection operator* rather than an *identity operator* because we are dealing with part of the L^2 space (i.e., the m -th eigenspaces) rather than the whole L^2 space (as in the usual case). In a similar vein, let us quote in a paper of Richardson [58, p. 358]: “A point of difficulty that often arises in this area of research is that the space [...] is not the set of all sections of any vector bundle, and therefore the usual theory of elliptic operators and heat kernels does not apply directly...”. The condition (1.70) eventually leads to the result that the heat kernels $e^{-t\Box_{b,m}^\pm}(x, y)$ do not have the *standard expansions* (as usually seen).

For a better understanding let’s assume that X is a (orbifold) circle bundle of an orbifold line bundle L over a Kähler orbifold M (see Section 1.4 for specific examples). As in Theorem 1.2 (see Section 1.5), one sees bijective maps

$$A_m^\pm : \Omega_m^{0,\pm}(X) \rightarrow \Omega^{0,\pm}(M, L^m),$$

such that $A_m^- \bar{\partial}_b = \bar{\partial} A_m^+$. Let \Box_m^+ be the Kodaira Laplacian with values in $T^{*0,+}M \otimes L^m$ and let $e^{-t\Box_m^+}$ be the associated heat operator. Consider $B_m(t) := (A_m^+)^{-1} \circ e^{-t\Box_m^+} \circ A_m^+$. A_m^\pm are metric-independent (on a given X). To get a link between $\Box_{b,m}^+$ and \Box_m^+ it requires, however, a compatible choice of metrics on X and M . With this done, one checks that $B'_m(t) + \Box_{b,m}^+ B_m(t) = 0$ and $B_m(0) = I$ on $\Omega_m^{0,+}(X)$.

But $B_m(t)$ is *not* the heat operator $e^{-t\Box_{b,m}^+}$. A trivial reason is that $B_m(t)$ is defined on $\Omega_m^{0,+}(X)$ while $e^{-\Box_{b,m}^+}$ is on the whole $\Omega^{0,+}(X)$. In fact one has

$$(1.71) \quad e^{-t\Box_{b,m}^+} = \left((A_m^+)^{-1} \circ e^{-t\Box_m^+} \circ A_m^+ \right) \circ Q_m^+ = B_m(t) \circ Q_m^+ (= Q_m^+ \circ B_m(t) \circ Q_m^+).$$

Let $B_m(t, x, y)$ be the distribution kernel of $B_m(t)$. To emphasize the role played by Q_m^+ in our construction, it is illuminating to note the following (cf. (1.71), (2.2) and (4.17))

$$(1.72) \quad e^{-t\Box_{b,m}^+}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} B_m(t, x, e^{-iu} \circ y) e^{-imu} du.$$

For $x \in X_p$ (the principal stratum), from (1.72) and the much better known kernel $e^{-t\Box_m^+}$ (on the principal stratum of M) it follows

$$(1.73) \quad e^{-t\Box_{b,m}^+}(x, x) \sim t^{-n} a_n^+(x) + t^{-(n-1)} a_{n-1}^+(x) + \dots$$

However for $x \notin X_p$, by lack of the asymptotic expansion of $B_m(t)$ (or $e^{-t\Box_m^+}$ on low dimensional strata of M) it is unclear how one can understand the asymptotic behavior of $e^{-t\Box_{b,m}^+}(x, x)$ by means of (1.72). This presents a major deviation from proof of the globally free case (as Theorem 1.2, cf. (1.1), (1.2)).

To see more clearly the discrepancy between the two cases (locally free and globally free) we note that the expansion (1.73) converges only *locally uniformly* on X_p , due to a nontrivial contribution involving a “distance function” (see Subsection 1.6.3 for more). In fact the expansion of the form (1.73) which is usually seen, cannot hold here (*globally* on X) (cf. Remark 1.6).

It is thus not immediate for one to arrive at a detailed understanding of the (transversal) heat kernel by only using the global argument. Even in the (smooth) orbifold circle bundle case, to understand the asymptotic behavior of the heat operator $e^{-t\Box_{b,m}^+}$ we will still need to work directly on the CR manifold X instead of M .

In this paper we give a construction which is independent of the use of orbifold geometry and is more adapted to CR geometry as our CR manifold X is not assumed to be an orbifold circle bundle of a complex orbifold. Because of the failure of the global argument as just said, we are now led to work on it *locally*. The framework for this is *BRT trivialization* (Section 2.4) which is first treated by Baouendi, Rothschild and Treves [3] in a more general context.

1.6.2. Transition to local situation. – Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialization (see Theorem 2.9). We write $D = U \times]-\varepsilon, \varepsilon[$, where $\varepsilon > 0$ and U is an open set of \mathbb{C}^n . Let $L \rightarrow U$ be a trivial line bundle with a non-trivial Hermitian fiber metric $|1|_{h^L}^2 = e^{-2\varphi}$ (where $\varphi \in C^\infty(D, \mathbb{R})$ is as in Theorem 2.9) and $(L^m, h^{L^m}) \rightarrow U$ be the m -th power of (L, h^L) . Θ (cf. Definition 1.11, recalling that X is CR Kähler as assumed for the moment) induces a Kähler form Θ_U on the complex manifold U . Let $\langle \cdot, \cdot \rangle$ be the Hermitian metric on $\mathcal{C}TU$ (associated with Θ_U), inducing together with h^{L^m} the L^2 inner product $(\cdot, \cdot)_m$ on $\Omega^{0,*}(U, L^m)$.

Let $\bar{\partial}^{*,m} : \Omega^{0,q+1}(U, L^m) \rightarrow \Omega^{0,q}(U, L^m)$ be the formal adjoint of $\bar{\partial}$ with respect to $(\cdot, \cdot)_m$. Put, as the case of $D_{b,m}$ and $\Box_{b,m}$, $D_{B,m}^\pm := \bar{\partial} + \bar{\partial}^{*,m} : \Omega^{0,\pm}(U, L^m) \rightarrow \Omega^{0,\mp}(U, L^m)$ and $\Box_{B,m}^\pm := D_{B,m}^- D_{B,m}^+ : \Omega^{0,+}(U, L^m) \rightarrow \Omega^{0,+}(U, L^m)$.

In Proposition 5.1, by the above choice of metrics in forming Laplacians on two different spaces (D and U), we can provide a link between these Laplacians, asserting that

$$(1.74) \quad e^{-m\varphi} \square_{B,m}^{\pm} (e^{m\varphi} \tilde{u}) = e^{im\theta} \square_{b,m}^{\pm} (u),$$

where as before, $u \in \Omega_m^{0,\pm}(X)$ can be written (on D) as $u(z, \theta) = e^{-im\theta} \tilde{u}(z)$ for some $\tilde{u}(z) \in \Omega^{0,\pm}(U, L^m) \subset \Omega^{0,\pm}(D, L^m)$.

Write $x = (z, \theta)$, $y = (w, \eta)$ (on D). With (1.74) one expects that the heat kernel $e^{-t \square_{b,m}^+}(x, y)$ locally (on D) should be

$$(1.75) \quad e^{-m\varphi(z) - im\theta} e^{-t \square_{B,m}^+}(x, y) e^{m\varphi(w) + im\eta}.$$

Thus one obtains *local* heat kernels on these BRT charts.

We would like to patch them up. Assume that $X = D_1 \cup D_2 \cup \dots \cup D_N$ (where D_j in a BRT trivialization $B_j := (D_j, (z, \theta), \varphi_j)$) with $D_j = U_j \times]-\delta_j, \tilde{\delta}_j[\subset \mathbb{C}^n \times \mathbb{R}$, $\delta_j > 0$, $\tilde{\delta}_j > 0$, U_j is an open set in \mathbb{C}^n).

Let $\chi_j, \tilde{\chi}_j \in C_0^\infty(D_j)$ ($j = 1, 2, \dots, N$). Put

$$(1.76) \quad \begin{aligned} \mathcal{A}_m(t) &= \sum_{j=1}^N \chi_j(x) \left(e^{-m\varphi_j(z) - im\theta} e^{-t \square_{B_j,m}^+}(z, w) e^{m\varphi_j(w) + im\eta} \right) \tilde{\chi}_j(y), \\ \mathcal{P}_m(t) &= \mathcal{A}_m(t) \circ Q_m^+. \end{aligned}$$

It is hoped that $\mathcal{P}_m(0) = Q_m^+$ and $\mathcal{P}'_m(t) + \square_{b,m}^+ \mathcal{P}_m(t)$ is small as $t \rightarrow 0^+$ for certain $\chi_j, \tilde{\chi}_j$. This is related to *asymptotic heat kernel*. But as we will see, this standard patch-up construction does not quite work out in our case.

In short, we will see that in the *locally free* case the nice (pointwise) relation (1.74) between Kodaira and Kohn Laplacians does not quite carry over to the global objects: heat kernels, whose mutual relation is to be seen below by more delicate analysis relevant to the presence of strata beyond the principal stratum.

1.6.3. Local difficulties. – A necessary condition for $\mathcal{P}_m(0) = Q_m^+$ is (cf. Lemma 5.11)

$$(1.77) \quad \sum_{j=1}^N \chi_j(x) \int_{-\pi}^{\pi} \tilde{\chi}_j(w, \eta) |_{w=z} d\eta = 1.$$

For the cut-off functions $\chi_j, \tilde{\chi}_j$ above, a reasonable choice (adapted to BRT trivializations) is the following (for $j = 1, 2, \dots, N$):

- i) $\chi_j(z, \theta) \in C_0^\infty(D_j)$ with $\sum_{j=1}^N \chi_j = 1$ on X ,
- ii) $\tau_j(z) \in C_0^\infty(U_j)$ with $\tau_j(z) = 1$ if $(z, \theta) \in \text{Supp } \chi_j$,
- iii) $\sigma_j \in C_0^\infty(]-\delta_j, \tilde{\delta}_j[)$ with $\int_{-\delta_j}^{\tilde{\delta}_j} \sigma_j(\eta) d\eta = 1$.

Set $\tilde{\chi}_j(y) \equiv \tau_j(w) \sigma_j(\eta)$. Then $\chi_j(x), \tilde{\chi}_j(y)$ satisfy (1.77).

One can check $\mathcal{P}_m(0) = Q_m^+$ and a little more work shows

$$(1.78) \quad \mathcal{P}'_m(t) + \square_{b,m}^+ \mathcal{P}_m(t) = \mathcal{R}_m(t) \circ Q_m^+,$$

where for some k ,

(1.79)

$$\mathcal{R}_m(t) = \sum_{j=1}^N \sum_{\ell=1}^k L_{\ell,j} \left(\chi_j(x) e^{-m\varphi_j(z) - im\theta} \right) P_{\ell,j} \left(e^{-t\Box_{B_j,m}^+}(z,w) \right) e^{m\varphi_j(w) + im\eta} \tilde{\chi}_j(y),$$

$L_{\ell,j}$ is a partial differential operator of order ≥ 1 and ≤ 2 (for all ℓ, j) and $P_{\ell,j}$ is a partial differential operator of order 1 acting on x (for all ℓ, j). Since $\left| e^{-t\Box_{B_j,m}^+}(z,w) \right| \sim \frac{1}{t^n} e^{-\frac{|z-w|^2}{t}}$, there could be terms of the form, say

$$(1.80) \quad P_{\ell,j} \left(e^{-t\Box_{B_j,m}^+}(z,w) \right) \sim \frac{1}{t^n} e^{-\frac{|z-w|^2}{t}} \frac{|z-w|}{t}.$$

To require $\mathcal{P}'_m(t) + \Box_{b,m}^+ \mathcal{P}_m(t)$ to be small (as $t \rightarrow 0^+$) we need (by substituting (1.80) into (1.79) to get singular terms in powers of $\frac{1}{t}$ smooth out):

(1.81)

$$L_{\ell,j} \left(\chi_j(x) e^{-m\varphi_j(z) - im\theta} \right) e^{m\varphi_j(w) + im\eta} \tilde{\chi}_j(y) = 0 \quad \text{if } z \text{ is close to } w \text{ } (|z-w| \lesssim \sqrt{t}).$$

Since χ_j may not be constant on $\text{Supp } \tilde{\chi}_j$ (for some j), it is hard for (1.81) to hold. Despite that in the usual (elliptic) case a construction of the heat kernel using cut-off functions as above is available, in view that a distance function will appear in our asymptotic expression (cf. (1.85) below) it is unclear whether this type of standard construction can be immediately carried out in our case.

It turns out that upon transferring to an *adjoint* version of the original equation one may bypass the aforementioned difficulty (cf. Lemmas 5.11, 5.12), to which we turn now.

For $j = 1, 2, \dots, N$ there exists $A_{B_j,+}(t, z, w)$ ($\in C^\infty(\mathbb{R}_+ \times U_j \times U_j, T^{*0,+}U \boxtimes (T^{*0,+}U)^*)$, cf. Proposition 5.7), regarded as an *adjoint heat kernel*, such that

$$(1.82) \quad \begin{aligned} \lim_{t \rightarrow 0^+} A_{B_j,+}(t) &= I \text{ in } \mathcal{D}'(U, T^{*0,+}U), \\ A'_{B_j,+}(t)u + A_{B_j,+}(t)(\Box_{B_j,m}^+ u) &= 0, \quad \forall u \in \Omega_c^{0,+}(U), \quad \forall t > 0, \end{aligned}$$

and $A_{B_j,+}(t, z, w)$ admits an asymptotic expansion as $t \rightarrow 0^+$ (see (5.19)). Put

$$(1.83) \quad H_j(t, x, y) = \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tilde{\chi}_j(y).$$

Also set

$$(1.84) \quad \Gamma(t) := \sum_{j=1}^N H_j(t) \circ Q_m^+ : \Omega^{0,+}(X) \rightarrow \Omega^{0,+}(X).$$

By using the adjoint equation, we can avoid the difficulty mentioned in (1.81) so that $\Gamma(t)$ gives an asymptotic (adjoint) heat kernel (see that below (1.76)). To get back to the kernel of the original equation, we can now start with the adjoint of $\Gamma(t)$. By carrying out the (standard) method of successive approximation, we can reach the global kernel of the adjoint of (the adjoint of) $e^{-t\Box_{b,m}^+}$ (Section 5.2). This yields the

kernel of $e^{-t\Box_{b,m}^+}$ since $e^{-t\Box_{b,m}^+}$ is self-adjoint. More precisely we can prove that (see Theorem 5.15 and Theorem 6.1)

(1.85)

$$\left\| e^{-t\Box_{b,m}^+}(x, y) - \Gamma(t, x, y) \right\|_{C^0(X \times X)} \leq e^{-\frac{\varepsilon_1}{t}}, \quad \forall t \in (0, \varepsilon_2),$$

$$\Gamma(t, x, x) \sim \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \right) \sum_{j=0}^{\infty} t^{-n+j} \alpha_{n-j}^+(x) \quad \text{mod } O\left(t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}\right), \quad \forall x \in X_p,$$

where $\alpha_s^+(x) \in C^\infty(X, \text{End}(T^{*0,+}X \otimes E))$, $s = n, n-1, \dots$, $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$ some constants and \hat{d} a sort of “distance function” (discussed above Theorem 1.3).

The appearance of this distance function \hat{d} may be attributed to the use of projection Q_m^+ in (1.84) (which picks up the m -th Fourier component; see (5.42) and (6.8)). See below for more about this point. By the first inequality in (1.85) one obtains the (same) asymptotic expansion

(1.86)

$$e^{-t\Box_{b,m}^+}(x, x) \sim \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \right) \sum_{j=0}^{\infty} t^{-n+j} \alpha_{n-j}^+(x) \quad \text{mod } O\left(t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}\right)$$

on X_p . Similar results hold for $e^{-t\Box_{b,m}^-}(x, x)$.

The terms involved in $O\left(t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}\right)$ of (1.86) are singular (due to t^{-n} as $x \rightarrow X_{\text{sing}}$). Only upon taking the *supertrace* can these terms be (partially) canceled (t^{-n} dropping out). That is, for $x \in X_p$

(1.87)

$$\text{Tr } e^{-t\Box_{b,m}^+}(x, x) - \text{Tr } e^{-t\Box_{b,m}^-}(x, x)$$

$$\sim \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \right) \sum_{j=0}^{\infty} t^{-n+j} \left(\text{Tr } \alpha_{n-j}^+(x) - \text{Tr } \alpha_{n-j}^-(x) \right) \quad \text{mod } O\left(e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}\right).$$

To see this conceptually, let’s take, for instance, (1.71) and (1.72) in which along the diagonal (i.e., setting $x = y$ to the left of (1.72)), the off-diagonal contribution (in the term to the right of the same equation) still enters nontrivially (unseen in the usual elliptic case) due to the projection Q_m^+ .

To get estimates on these off-diagonal terms our argument (cf. Theorem 5.10) is based on the rescaling technique of Getzler and on a supertrace identity in Berenzin integral (cf. Prop. 3.21 of [5]), which combine to give the needed (partial) cancelation.

From (1.68), (1.86) and (1.87) it follows

$$(1.88) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) \\ = \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p} mi} \right) \lim_{t \rightarrow 0^+} \int_X \sum_{\ell=0}^n t^{-\ell} \left(\text{Tr } \alpha_{\ell}^+(x) - \text{Tr } \alpha_{\ell}^-(x) \right) dv_X(x).$$

Remark that we have had a (transversal) heat kernel which is put in the disguise of the spectral geometry (1.67), (4.15). To our knowledge no argument in the literature claims that (in the transversally elliptic case) the spectral heat kernel shall have the asymptotic estimates as (1.86). The somewhat lengthy part of our reconstruction of the (transversal) heat kernel (beyond its spectral realization) becomes indispensable as far as our purpose is concerned.

1.6.4. Completion by evaluating local density and by using Spin^c structure. – As above we first treat the case that X is CR Kähler (Definition 1.11). In view of (1.88), to complete the proof of our index theorem (cf. Corollary 1.13) amounts to understanding the small t behavior of the local density

$$\sum_{\ell=0}^n t^{-\ell} \left(\text{Tr } \alpha_{\ell}^+(x) - \text{Tr } \alpha_{\ell}^-(x) \right).$$

Let's be back to the local situation. Fix $x_0 \in X_p$. Let $B_j = (D_j, (z, \theta), \varphi_j)$ ($j = 1, 2, \dots, N$) be BRT trivializations as before. Assume that $x_0 \in D_j$ and $x_0 = (z_j, 0) \in U_j \subset D_j$.

As our heat kernel (on X) is related to the local heat kernel (on U_j), one sees (for some $N_0(n) \geq n$)

$$(1.89) \quad \sum_{\ell=0}^{N_0(n)} t^{-\ell} \left(\text{Tr } \alpha_{\ell}^+(x_0) - \text{Tr } \alpha_{\ell}^-(x_0) \right) \\ = \frac{1}{2\pi} \sum_{j=1}^N \chi_j(x_0) \left(\text{Tr } A_{B_j,+}(t, z_j, z_j) - \text{Tr } A_{B_j,-}(t, z_j, z_j) \right) + O(t),$$

where $A_{B_j,+}(t, z, w)$ is as in (1.82).

By borrowing the *rescaling technique* in [5] and [24] we can show (in a fairly standard manner, cf. Theorem 5.9 or the second half of this section) that for each $j = 1, 2, \dots, N$,

$$(1.90) \quad \left(\text{Tr } A_{B_j,+}(t, z, z) - \text{Tr } A_{B_j,-}(t, z, z) \right) dv_{U_j}(z) \\ = [\text{Tr}(\nabla^{T^{1,0}U_j}, T^{1,0}U_j) \wedge \text{ch}(\nabla^{L^m}, L^m)]_{2n}(z) + O(t), \quad \forall z \in U_j,$$

(dv_{U_j} the induced volume form on U_j) where $\text{Td}(\nabla^{T^{1,0}U_j}, T^{1,0}U_j)$ and $\text{ch}(\nabla^{L^m}, L^m)$ denote the representatives of the Todd class of $T^{1,0}U_j$ and the Chern character of L^m , respectively.

A novelty here is Section 2.3 in which we will introduce *tangential characteristic classes*, *tangential Chern character* and *tangential Todd class* on CR manifolds with S^1 action, so that

$$(1.91) \quad \frac{[\text{Td}(\nabla^{T^{1,0}U_j}, T^{1,0}U_j) \wedge \text{ch}(\nabla^{L^m}, L^m)]_{2n}(z_j)}{dv_{U_j}(z_j)} = \frac{[\text{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge e^{-m\frac{d\omega_0}{2\pi}} \wedge \omega_0]_{2n+1}(x_0)}{dv_X(x_0)},$$

where $\text{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X)$ denotes the representative of the tangential Todd class of $T^{1,0}X$ (associated with the given Hermitian metric (2.9)). From (1.89), (1.90) and (1.91) it follows

$$(1.92) \quad \sum_{\ell=0}^n t^{-\ell} \left(\text{Tr} \alpha_{\ell}^+(x) - \text{Tr} \alpha_{\ell}^-(x) \right) dv_X(x) = \frac{1}{2\pi} \left[\text{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge e^{-m\frac{d\omega_0}{2\pi}} \wedge \omega_0 \right]_{2n+1}(x) + O(t), \quad \forall x \in X_p.$$

(The $O(t)$ term to the rightmost of (1.92) actually vanishes by using (5.22).)

Combining (1.92) and (1.88) we get our index theorem (cf. Corollary 1.13) when X is CR Kähler.

When X is *not CR Kähler*, we still have (1.85), (1.86) and (1.88). The ensuing *obstacles* are more or less known:

- i) the rescaling technique does not quite work well as the local operator $\square_{B_j, m}^+$ in (1.82) is not going to be of Dirac type (in a strict sense);
- ii) it is obscure to understand the small t behavior of $A_{B_j, +}(t, z, z)$ in this case;
- iii) (1.90) is not even true in general.

To overcome this difficulty in the CR case, we follow the classical (yet nonKähler) case and introduce some kind of CR Spin^c Dirac operator on CR manifolds with S^1 action:

$$\tilde{D}_{b, m} = \bar{\partial}_b + \bar{\partial}_b^* + \text{zeroth order term}$$

with modified/ Spin^c Kohn Laplacians $\tilde{\square}_{b, m}^+ = \tilde{D}_{b, m}^* \tilde{D}_{b, m}$, $\tilde{\square}_{b, m}^- = \tilde{D}_{b, m} \tilde{D}_{b, m}^*$.

A word of caution is in order. The above adaptation of the idea of Spin^c structure to our CR case is not altogether straightforward. Locally X is realized as a (portion of a) circle bundle over a small piece of complex manifold (via BRT charts), so presumably there could arise a problem of patching up when this global Spin^c operator is to be formed. See Proposition 4.2 for more.

We will show in Theorem 4.7 the *homotopy invariance* for the index of $\bar{\partial}_b + \bar{\partial}_b^*$, and in Corollary 4.8 a *McKean-Singer formula* for the modified Kohn Laplacians: for $t > 0$,

$$(1.93) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = \int_X \left(\operatorname{Tr} e^{-t\tilde{\square}_{b,m}^+}(x, x) - \operatorname{Tr} e^{-t\tilde{\square}_{b,m}^-}(x, x) \right) dv_X.$$

For $u \in \Omega_m^{0,\pm}(X)$ we can write (on D) $u(z, \theta) = e^{-im\theta}\tilde{u}(z)$ for some $\tilde{u}(z) \in \Omega^{0,\pm}(U, L^m)$ with D in a BRT trivialization $B := (D, (z, \theta), \varphi)$.

A fundamental relation that we will show in Proposition 5.1, based on Proposition 4.2, is that

$$(1.94) \quad e^{-m\varphi}\tilde{\square}_{B,m}^\pm(e^{m\varphi}\tilde{u}) = e^{im\theta}\tilde{\square}_{b,m}^\pm(u),$$

where $\tilde{\square}_{B,m}^\pm = D_{B,m}^* D_{B,m} : \Omega^{0,\pm}(U, L^m) \rightarrow \Omega^{0,\pm}(U, L^m)$ and $D_{B,m} : \Omega^{0,\pm}(U, L^m) \rightarrow \Omega^{0,\mp}(U, L^m)$ the (ordinary) Spin^c Dirac operator (cf. Definition 4.1) with respect to the Chern connection on L^m (induced by h^{L^m}) and the *Clifford connection* on $\Lambda(T^{*0,1}U)$ (induced by the given Hermitian metric on $\Lambda(T^{*0,1}U)$).

It is conceivable that X with the CR structure and $X/S^1 = M$ with the complex structure (if defined) are linked in some way (as Theorem 1.2). To say more, the Result (1.94) asserts a fundamental fact that not only complex/CR geometrically can the two spaces be linked, but *metrically* in the sense of Laplacians they also can. This link is important for our Spin^c approach to the CR case to be possible.

In the remaining let's give an outline with the CR Spin^c Dirac operator when X is not CR Kähler. Although the following ingredients mostly parallel those in the preceding Subsection 1.6.3, the success of this method relies on, among others, the Spin^c structure and the associated Clifford connection. For that reason and for the sake of clarity, we prefer to put down the precise formulas despite the great similarity in expressions as above.

As (1.82), there exists (modified) $\tilde{A}_{B_j,+}(t, z, w)$ such that

$$(1.95) \quad \lim_{t \rightarrow 0^+} \tilde{A}_{B_j,+}(t) = I \text{ in } \mathcal{D}'(U_j, T^{*0,+}U_j),$$

$$\tilde{A}'_{B_j,+}(t)u + \tilde{A}_{B_j,+}(t)(\tilde{\square}_{B_j,m}^+ u) = 0, \quad \forall u \in \Omega_0^{0,+}(U_j), \quad \forall t > 0,$$

and $\tilde{A}_{B_j,+}(t, z, w)$ admits an asymptotic expansion as $t \rightarrow 0^+$ (see (5.19)). Put

$$(1.96) \quad \tilde{H}_j(t, x, y) = \chi_j(x) e^{-m\varphi_j(z) - im\theta} \tilde{A}_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tilde{\chi}_j(y),$$

$$\tilde{\Gamma}(t) = \sum_{j=1}^N \tilde{H}_j(t) \circ Q_m.$$

Similar to (1.85) and (1.86) in Subsection 1.6.3, one has

$$(1.97) \quad \left\| e^{-t\tilde{\square}_{b,m}^+}(x, y) - \tilde{\Gamma}(t, x, y) \right\|_{C^0(X \times X)} \leq e^{-\frac{\epsilon_1}{t}}, \quad \forall t \in (0, \epsilon_2)$$

and

(1.98)

$$\tilde{\Gamma}(t, x, x) \sim \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \right) \sum_{j=0}^{\infty} t^{-n+j} \tilde{\alpha}_{n-j}^+(x) \pmod{O\left(t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}\right)}, \quad \forall x \in X_p,$$

with some constants $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$, giving

(1.99)

$$e^{-t\tilde{\square}_{b,m}^+}(x, x) \sim \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \right) \sum_{j=0}^{\infty} t^{-n+j} \tilde{\alpha}_{n-j}^+(x) \pmod{O\left(t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}\right)}$$

on X_p . Similar results hold for $e^{-t\tilde{\square}_{b,m}^-}(x, x)$.

The novelty here is analogous to (1.87). By taking supertrace we can improve the estimates in (1.99) (see Theorem 6.4) so that t^{-n} is removed:

(1.100)

$$\begin{aligned} & \text{Tr} e^{-t\tilde{\square}_{b,m}^+}(x, x) - \text{Tr} e^{-t\tilde{\square}_{b,m}^-}(x, x) \\ & \sim \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \right) \sum_{j=0}^{\infty} t^{-n+j} \left(\text{Tr} \tilde{\alpha}_{n-j}^+(x) - \text{Tr} \tilde{\alpha}_{n-j}^-(x) \right) \pmod{O\left(e^{-\frac{\hat{d}(x, X_{\text{sing}})^2}{t}}\right)}, \end{aligned}$$

for $x \in X_p$. Hence (1.93) and (1.100) give

(1.101)

$$\sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = \left(\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p}mi} \right) \lim_{t \rightarrow 0^+} \int_X \sum_{\ell=0}^n t^{-\ell} \left(\text{Tr} \tilde{\alpha}_{\ell}^+(x) - \text{Tr} \tilde{\alpha}_{\ell}^-(x) \right) dv_X(x).$$

A key advantage of introducing our CR Spin^c Dirac operator is basically that *Lichnerowicz formulas* hold for $\tilde{\square}_{B,m}^+$ (and $\tilde{\square}_{B,m}^-$). This enables us to apply the *rescaling technique* (this part of rescaling is essentially the same as in classical cases, cf. [5] and [24]) and to obtain that for each $j = 1, 2, \dots, N$,

$$\begin{aligned} (1.102) \quad & \left(\text{Tr} \tilde{A}_{B_j,+}(z, z) - \text{Tr} \tilde{A}_{B_j,-}(z, z) \right) dv_{U_j}(z) \\ & = [\text{Td}(\nabla^{T^{1,0}U_j}, T^{1,0}U_j) \wedge \text{ch}(\nabla^{L^m}, L^m)]_{2n}(z) + O(t), \quad \forall z \in U_j. \end{aligned}$$

Rewriting (1.102) in tangential forms, one has

$$\begin{aligned} (1.103) \quad & \sum_{\ell=0}^n t^{-\ell} \left(\text{Tr} \tilde{\alpha}_{\ell}^+(x) - \text{Tr} \tilde{\alpha}_{\ell}^-(x) \right) dv_X(x) \\ & = \frac{1}{2\pi} \left[\text{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge e^{-m\frac{d\omega_0}{2\pi}} \wedge \omega_0 \right]_{2n+1}(x) + O(t), \end{aligned}$$

for $t > 0$ and $x \in X_p$.

Theorem 1.3, Theorem 1.10 and Corollary 1.13 follows from (1.99), (1.100), (1.101) and (1.103).

The layout of this paper is as follows. In Section 2.1 and Section 2.2, we collect some notations, definitions, terminologies and statements we use throughout. In Section 2.3, we introduce the tangential de Rham cohomology group, tangential Chern character and tangential Todd class on CR manifolds with S^1 action. In Section 2.4, we recall a classical result of Baouendi-Rothschild-Treves [3] which plays an important role in our construction of the heat kernel. We also prove that for a rigid vector bundle F over X there exist rigid Hermitian metric and rigid connection on F . In Section 3, we establish a Hodge theory for Kohn Laplacian in the L^2 space of the m -th S^1 Fourier component. In Section 4, we introduce our CR Spin^c Dirac operator $\tilde{D}_{b,m}$, modified/ Spin^c Kohn Laplacians $\tilde{\square}_{b,m}^\pm$ and prove (1.93). In Section 5, we construct approximate heat kernels for the operators $e^{-t\tilde{\square}_{b,m}^\pm}$ and prove that $e^{-t\tilde{\square}_{b,m}^\pm}(x, y)$ admit asymptotic expansions in the sense as (1.97). In Section 6, we prove (1.98), (1.100), (1.103) and finish the proofs of Theorem 1.3, Theorem 1.10 and Corollary 1.13. In Section 7 we prove Theorem 1.14.

PART I

PREPARATORY FOUNDATIONS

CHAPTER 2

PRELIMINARIES

2.1. Some standard notations

We use the following notations: $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, \mathbb{R} is the set of real numbers, $\overline{\mathbb{R}}_+ := \{x \in \mathbb{R}; x \geq 0\}$. For a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ we set $|\alpha| = \alpha_1 + \dots + \alpha_n$. For $x = (x_1, \dots, x_n)$ we write

$$\begin{aligned} x^\alpha &= x_1^{\alpha_1} \cdots x_n^{\alpha_n}, & \partial_{x_j} &= \frac{\partial}{\partial x_j}, & \partial_x^\alpha &= \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x^\alpha}, \\ D_{x_j} &= \frac{1}{i} \partial_{x_j}, & D_x^\alpha &= D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, & D_x &= \frac{1}{i} \partial_x. \end{aligned}$$

Let $z = (z_1, \dots, z_n)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, be coordinates of \mathbb{C}^n . We write

$$\begin{aligned} z^\alpha &= z_1^{\alpha_1} \cdots z_n^{\alpha_n}, & \bar{z}^\alpha &= \bar{z}_1^{\alpha_1} \cdots \bar{z}_n^{\alpha_n}, \\ \partial_{z_j} &= \frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right), & \partial_{\bar{z}_j} &= \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}} \right), \\ \partial_z^\alpha &= \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z^\alpha}, & \partial_{\bar{z}}^\alpha &= \partial_{\bar{z}_1}^{\alpha_1} \cdots \partial_{\bar{z}_n}^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial \bar{z}^\alpha}. \end{aligned}$$

Let X be a C^∞ orientable paracompact manifold. We denote the tangent and cotangent bundle of X by TX and T^*X respectively, and the complexified tangent and cotangent bundle by $\mathbb{C}TX$ and $\mathbb{C}T^*X$. We write $\langle \cdot, \cdot \rangle$ to denote the pointwise pairing between T^*X and TX and extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}T^*X \times \mathbb{C}TX$.

Let E, F be C^∞ vector bundles over X . We write $F \boxtimes E^*$ for the vector bundle over $X \times X$ with fiber over $(x, y) \in X \times X$ consisting of linear maps from E_y to F_x .

Let $Y \subset X$ be an open subset. The spaces of smooth sections and *distribution sections* of E over Y will be denoted by $C^\infty(Y, E)$ and $\mathcal{D}'(Y, E)$ respectively. Let $\mathcal{E}'(Y, E)$ be the subspace of $\mathcal{D}'(Y, E)$ whose elements are of *compact support* in Y .

For $m \in \mathbb{R}$, we let $H^m(Y, E)$ denote the Sobolev space of order m for sections of E over Y . Put

$$\begin{aligned} H_{\text{loc}}^m(Y, E) &= \{u \in \mathcal{D}'(Y, E); \varphi u \in H^m(Y, E), \varphi \in C_c^\infty(Y)\}, \\ H_{\text{comp}}^m(Y, E) &= H_{\text{loc}}^m(Y, E) \cap \mathcal{E}'(Y, E). \end{aligned}$$

2.2. Set up and terminology

Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X . That is $T^{1,0}X$ is a subbundle of rank n of the complexified tangent bundle $\mathbb{C}TX$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, $\mathcal{V} = C^\infty(X, T^{1,0}X)$.

We assume that X admits an S^1 action: $S^1 \times X \rightarrow X$. We write $e^{-i\theta}$ to denote the S^1 action. Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the S^1 action given by $(Tu)(x) = \frac{\partial}{\partial \theta} (((e^{-i\theta})^*u)(x)) \Big|_{\theta=0}$ for $u \in C^\infty(X)$.

DEFINITION 2.1. – We say that the S^1 action $e^{-i\theta}$ is CR if $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and the S^1 action is *transversal* if for each $x \in X$, $\mathbb{C}T(x) \oplus T_x^{1,0}X \oplus T_x^{0,1}X = \mathbb{C}T_x X$. Moreover, we say that the S^1 action is *locally free* if $T \neq 0$ everywhere.

We assume throughout that $(X, T^{1,0}X)$ is a compact CR manifold with a transversal CR locally free S^1 action $e^{-i\theta}$ with T the global vector field induced by the S^1 action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle \omega_0, u \rangle = 0$ for all $u \in T^{1,0}X \oplus T^{0,1}X$, and $\langle \omega_0, T \rangle = 1$.

DEFINITION 2.2. – For $p \in X$, the *Levi form* \mathcal{L}_p is the Hermitian quadratic form on $T_p^{1,0}X$ given by $\mathcal{L}_p(U, \overline{V}) = -\frac{1}{2i} \langle d\omega_0(p), U \wedge \overline{V} \rangle$, $U, V \in T_p^{1,0}X$.

If the Levi form \mathcal{L}_p is semi-positive definite (resp. positive definite), we say that X is weakly pseudoconvex (resp. strongly pseudoconvex) at p . If the Levi form is semi-positive definite (resp. positive definite) at every point of X , we say that X is weakly pseudoconvex (resp. strongly pseudoconvex).

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$ respectively. Define the vector bundle of (p, q) forms by $T^{*p,q}X = \Lambda^p(T^{*1,0}X) \wedge \Lambda^q(T^{*0,1}X)$.

Let $D \subset X$ be an open subset and E be a complex vector bundle over D . Denote by $\Omega^{p,q}(D, E)$ (resp. $\Omega^{p,q}(D)$) the space of smooth sections of $T^{*p,q}X \otimes E$ (resp. $T^{*p,q}X$) over D and by $\Omega_c^{p,q}(D, E)$ (resp. $\Omega_c^{p,q}(D)$) those elements of *compact support* in D .

Put

$$\begin{aligned} T^{*0,\bullet} X &:= \bigoplus_{j \in \{0,1,\dots,n\}} T^{*0,j} X, \\ T^{*0,+} X &:= \bigoplus_{j \in \{0,1,\dots,n\}, j \text{ is even}} T^{*0,j} X, \\ T^{*0,-} X &:= \bigoplus_{j \in \{0,1,\dots,n\}, j \text{ is odd}} T^{*0,j} X. \end{aligned}$$

Put $\Omega^{0,\bullet}(X, E)$, $\Omega^{0,+}(X, E)$ and $\Omega^{0,-}(X, E)$ in a similar way as above.

Fix $\theta_0 \in]-\pi, \pi[$. Let $(e^{-i\theta_0})^* : \Lambda^r(\mathbb{C}T^*X) \rightarrow \Lambda^r(\mathbb{C}T^*X)$ be the pull-back map, $(e^{-i\theta_0})^* : T_{e^{-i\theta_0} \circ x}^{*p,q} X \rightarrow T_x^{*p,q} X$. Define for $u \in \Omega^{p,q}(X)$

$$(2.1) \quad Tu := \frac{\partial}{\partial \theta} ((e^{-i\theta})^* u)|_{\theta=0} \in \Omega^{p,q}(X).$$

(See also (2.13).)

Clearly

$$(2.2) \quad u(x) = \sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \int_{-\pi}^{\pi} ((e^{-i\theta})^* u)(x) e^{im\theta} d\theta.$$

Let $\bar{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. From the CR property of the S^1 action it follows that (see also (2.14))

$$T\bar{\partial}_b = \bar{\partial}_b T \quad \text{on } \Omega^{0,q}(X).$$

Naturally associated with the S^1 action are the so-called *rigid* objects. See also [3] for a similar use of this term (cf. Definition II.2 of *loc.cit.*).

DEFINITION 2.3. – Let $D \subset X$ be an open set and $u \in C^\infty(D)$. We say that u is *rigid* if $Tu = 0$, u is Cauchy-Riemann (CR for short) if $\bar{\partial}_b u = 0$ and u is a rigid CR function if $\bar{\partial}_b u = 0$ and $Tu = 0$.

DEFINITION 2.4. – Let F be a complex vector bundle of rank r over X . We say that F is *rigid* (resp. CR) if X can be covered by open subsets U_j with trivializing frames $\{f_j^1, f_j^2, \dots, f_j^r\}$ such that the corresponding transition functions are rigid (resp. CR) (in the sense of the preceding definition).

Let F be a rigid (CR) vector bundle over X . In this work, we fix open cover $(U_j)_{j=1}^N$ of X and a family $\{f_j^1, f_j^2, \dots, f_j^r\}_{j=1}^N$ of trivializing frames $\{f_j^1, f_j^2, \dots, f_j^r\}$ on each U_j such that the entries of the transition matrices between different frames $\{f_j^1, f_j^2, \dots, f_j^r\}$ are rigid (CR). We say that a frame f is rigid (CR) if f is rigid (CR) with respect to $\{f_j^1, f_j^2, \dots, f_j^r\}_{j=1}^N$. We can define the operator T on $\Omega^{0,q}(X, F)$. Indeed, every $u \in \Omega^{0,q}(X, F)$ can be written on U_j as $u = \sum u_\ell \otimes f_j^\ell$ and we set $Tu = \sum Tu_\ell \otimes f_j^\ell$. Then Tu is well defined as element of $\Omega^{0,q}(X, F)$, since the entries

of the transition matrices between different frames $\{f_j^1, f_j^2, \dots, f_j^r\}$ are annihilated by T .

Let X be a compact CR manifold with a locally free transversal CR S^1 action. In this work, we say that a trivializing frame f of $T^{1,0}X$ is rigid if f is rigid with respect to BRT frames (see Theorem 2.9 for the meaning of BRT frames). By using duality, we can also define BRT frames for the bundle $\bigoplus_{r=1}^{2n+1} \Lambda^r(\mathbb{C}T^*X)$ and we say that a trivializing frame f of $\bigoplus_{r=1}^{2n+1} \Lambda^r(\mathbb{C}T^*X)$ is rigid if f is rigid with respect to BRT frames.

Let F be a rigid complex vector bundle over X in the sense of Definition 2.4.

DEFINITION 2.5. – Let $\langle \cdot | \cdot \rangle_F$ be a Hermitian metric on F . We say that $\langle \cdot | \cdot \rangle_F$ is a *rigid Hermitian metric* if for every rigid local frames $\{f_1, \dots, f_r\}$ of F , we have $T\langle f_j | f_k \rangle_F = 0$, for $j, k = 1, 2, \dots, r$.

The condition of being rigid is not a severe restriction as far as the S^1 action is concerned. See Theorems 2.11 and 2.12 which we shall prove within the framework of BRT trivializations in the next section.

Henceforth let E be a rigid CR vector bundle over X . Write $\bar{\partial}_b : \Omega^{0,q}(X, E) \rightarrow \Omega^{0,q+1}(X, E)$ for the tangential Cauchy-Riemann operator. Since E is rigid, we can define Tu for $u \in \Omega^{0,q}(X, E)$ (cf. Theorem 2.11) and have

$$(2.3) \quad T\bar{\partial}_b = \bar{\partial}_b T \quad \text{on } \Omega^{0,q}(X, E).$$

For $m \in \mathbb{Z}$, let

$$(2.4) \quad \Omega_m^{0,q}(X, E) := \{u \in \Omega^{0,q}(X, E); Tu = -imu\}$$

and put $\Omega_m^{0,\bullet}(X, E)$, $\Omega_m^{0,+}(X, E)$ and $\Omega_m^{0,-}(X, E)$ in a similar way as above.

Put $\bar{\partial}_{b,m} := \bar{\partial}_b : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E)$ with a $\bar{\partial}_{b,m}$ -complex:

$$\bar{\partial}_{b,m}: \dots \rightarrow \Omega_m^{0,q-1}(X, E) \rightarrow \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E) \rightarrow \dots$$

Define

$$H_{b,m}^q(X, E) := \frac{\text{Ker } \bar{\partial}_{b,m} : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E)}{\text{Im } \bar{\partial}_{b,m} : \Omega_m^{0,q-1}(X, E) \rightarrow \Omega_m^{0,q}(X, E)}.$$

It is instructive to think of $H_{b,m}^q(X, E)$ as the m -th S^1 Fourier component of the q -th $\bar{\partial}_b$ Kohn-Rossi cohomology group.

We will prove in Theorem 3.7 that $\dim H_{b,m}^q(X, E) < \infty$, for $m \in \mathbb{Z}$ and $q = 0, 1, 2, \dots, n$.

We take a rigid Hermitian metric $\langle \cdot | \cdot \rangle_E$ on E (in the sense of Definition 2.5), and a rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ such that

$$(2.5) \quad T \perp (T^{1,0}X \oplus T^{0,1}X), \quad \langle T | T \rangle = 1$$

(and $T^{1,0}X \perp T^{0,1}X$). (This is always possible; see Theorem 2.11 and Theorem 9.2 in [42].)

The Hermitian metric $\langle \cdot | \cdot \rangle$ on CTX induces by duality a Hermitian metric on CT^*X and on the bundles of $(0, q)$ forms $T^{*0, q}X$ ($q = 0, 1, \dots, n$), to be denoted by $\langle \cdot | \cdot \rangle$ too. A Hermitian metric denoted by $\langle \cdot | \cdot \rangle_E$ on $T^{*0, \bullet}X \otimes E$ is induced by those on $T^{*0, \bullet}X$ and E . Let the linear map $A(x, y) \in (T^{*0, \bullet}X \otimes E) \boxtimes (T^{*0, \bullet}X \otimes E)^*|_{(x, y)}$. We write $|A(x, y)|$ to denote the natural matrix norm of $A(x, y)$ induced by $\langle \cdot | \cdot \rangle_E$.

We denote by $dv_X = dv_X(x)$ the induced volume form, and form the global L^2 inner products $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$ on $\Omega^{0, \bullet}(X, E)$ and $\Omega^{0, \bullet}(X)$ respectively, with L^2 -completion $L^2(X, T^{*0, q}X \otimes E)$ and $L^2(X, T^{*0, q}X)$. Similar notation applies to $L_m^2(X, T^{*0, q}X \otimes E)$ and $L_m^2(X, T^{*0, q}X)$ (the completions of $\Omega_m^{0, q}(X, E)$ and $\Omega_m^{0, q}(X)$ with respect to $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$).

Put $L^2(X, T^{*0, \bullet}X \otimes E)$, $L^{2, +}(X, E)$ and $L^{2, -}(X, E)$ in a similar way as above, and $L_m^2(X, T^{*0, \bullet}X \otimes E)$, $L_m^{2, +}(X, E)$ and $L_m^{2, -}(X, E)$ too.

2.3. Tangential de Rham cohomology group, Tangential Chern character and Tangential Todd class

In this section it is convenient to put $\Omega_0^r(X) = \left\{ u \in \bigoplus_{p+q=r} \Omega^{p, q}(X); Tu = 0 \right\}$ for $r = 0, 1, 2, \dots, 2n$ (without any danger of confusion with $\Omega_0^{p, q}$ in the preceding section) and set $\Omega_0^\bullet(X) = \bigoplus_{r=0}^{2n} \Omega_0^r(X)$. Since $Td = dT$ (see (2.3)), we have d -complex:

$$d : \dots \rightarrow \Omega_0^{r-1}(X) \rightarrow \Omega_0^r(X) \rightarrow \Omega_0^{r+1}(X) \rightarrow \dots$$

Define the r -th tangential de Rham cohomology group:

$$\mathcal{H}_{b, 0}^r(X) := \frac{\text{Ker } d : \Omega_0^r(X) \rightarrow \Omega_0^{r+1}(X)}{\text{Im } d : \Omega_0^{r-1}(X) \rightarrow \Omega_0^r(X)}.$$

Put $\mathcal{H}_{b, 0}^\bullet(X) = \bigoplus_{r=0}^{2n} \mathcal{H}_{b, 0}^r(X)$.

Let a complex vector bundle F over X of rank r be *rigid* as in Definition 2.4. We will show in Theorem 2.12 that there exists a connection ∇ on F such that for any rigid local frame $f = (f_1, f_2, \dots, f_r)$ of F on an open set $D \subset X$, the connection matrix $\theta(\nabla, f) = (\theta_{j, k})_{j, k=1}^r$ satisfies

$$\theta_{j, k} \in \Omega_0^1(D),$$

for $j, k = 1, \dots, r$. We call ∇ as such a *rigid connection* on F . Let $\Theta(\nabla, F) \in C^\infty(X, \Lambda^2(\mathbb{C}T^*X) \otimes \text{End}(F))$ be the associated *tangential curvature*.

Let $h(z) = \sum_{j=0}^{\infty} a_j z^j$ be a real power series on $z \in \mathbb{C}$. Set

$$H(\Theta(\nabla, F)) = \text{Tr} \left(h \left(\frac{i}{2\pi} \Theta(\nabla, F) \right) \right).$$

It is clear that $H(\Theta(\nabla, F)) \in \Omega_0^\bullet(X)$.

The following is well-known (see Theorem B.5.1 in Ma-Marinescu [50]).

THEOREM 2.6. – $H(\Theta(\nabla, F))$ is a closed differential form.

That the tangential de Rham cohomology class

$$[H(\Theta(\nabla, F))] \in \mathcal{H}_{b,0}^\bullet(X)$$

does not depend on the choice of rigid connections ∇ is given by

THEOREM 2.7. – *Let ∇' be another rigid connection on F . Then, $H(\Theta(\nabla, F)) - H(\Theta(\nabla', F)) = dA$, for some $A \in \Omega_0^\bullet(X)$.*

Proof. – The idea of the proof is standard. For each $t \in [0, 1]$, put $\nabla_t = (1-t)\nabla + t\nabla'$ which is a rigid connection on F . Set

$$Q_t = \frac{i}{2\pi} \text{Tr} \left(\frac{\partial \nabla_t}{\partial t} h' \left(\frac{i}{2\pi} \Theta(\nabla_t, F) \right) \right).$$

Since ∇_t is rigid, it is easily seen that

$$(2.6) \quad Q_t \in \Omega_0^\bullet(X).$$

It is well-known that (see Remark B.5.2 in Ma-Marinescu [50])

$$(2.7) \quad H(\Theta(\nabla, F)) - H(\Theta(\nabla_0^1 Q_t dt)).$$

From (2.6) and (2.7), the theorem follows. □

For $h(z) = e^z$ put

$$(2.8) \quad \text{ch}_b(\nabla, F) := H(\Theta(\nabla, F)) \in \Omega_0^\bullet(X),$$

and for $h(z) = \log\left(\frac{z}{1-e^{-z}}\right)$ set

$$(2.9) \quad \text{Td}_b(\nabla, F) := e^{H(\Theta(\nabla, F))} \in \Omega_0^\bullet(X).$$

We can now introduce tangential Todd class and tangential Chern character.

DEFINITION 2.8. – The *tangential Chern character* of F is given by

$$\text{ch}_b(F) := [\text{ch}_b(\nabla, F)] \in \mathcal{H}_{b,0}^\bullet(X)$$

and the *tangential Todd class* of F is given by

$$\text{Td}_b(F) = [\text{Td}_b(\nabla, F)] \in \mathcal{H}_{b,0}^\bullet(X).$$

Baouendi-Rothschild-Treves [3] proved that $T^{1,0}X$ is a rigid complex vector bundle over X (cf. the first part of Theorem 2.11 below). The tangential Todd class of $T^{1,0}X$ and tangential Chern character of $T^{1,0}X$ are thus well defined.

The tangential Chern classes can be defined similarly. Put $\det\left(\frac{i\Theta(\nabla, F)}{2\pi}t + I\right) = \sum_{j=0}^r \hat{c}_j(\nabla, F)t^j$. Thus $\hat{c}_j(\nabla, F) \in \Omega_0^{2j}(D)$. By the matrix identity $\det A = e^{\text{Tr}(\log A)}$ and taking $h(z) = \log(1+z)$, one sees $\hat{c}_j(\nabla, F)$ ($j = 0, 1, \dots, r$) is a closed differential form on X and its tangential de Rham cohomology class $[\hat{c}_j(\nabla, F)] \in \mathcal{H}_{b,0}^{2j}(X)$ is independent of the choice of rigid connections ∇ . Put $\hat{c}_j(F) = [\hat{c}_j(\nabla, F)] \in \mathcal{H}_{b,0}^{2j}(X)$.

We call $\hat{c}_j(F)$ the j -th *tangential Chern class* of F , and $\hat{c}(F) = 1 + \sum_{j=1}^r \hat{c}_j(F) \in \mathcal{H}_{b,0}^\bullet(X)$ the *tangential total Chern class* of F .

2.4. BRT trivializations and rigid geometric objects

In this paper, much of our strategy is heavily based on the following result thanks to Baouendi-Rothschild-Treves [3, Proposition I.2]. Note in the following, Z_j corresponds to \bar{L}_j in their proposition. Some geometrical significance related to a certain circle bundle structure will be discussed in the proof of Proposition 4.2.

THEOREM 2.9. – *For every point $x_0 \in X$ there exist local coordinates $x = (x_1, \dots, x_{2n+1}) = (z, \theta) = (z_1, \dots, z_n, \theta)$, $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$, $x_{2n+1} = \theta$, defined in some small neighborhood $D = \{(z, \theta) : |z| < \delta, -\varepsilon_0 < \theta < \varepsilon_0\}$ of x_0 , $\delta > 0$, $0 < \varepsilon_0 < \pi$, such that $(z(x_0), \theta(x_0)) = (0, 0)$ and*

$$(2.10) \quad \begin{aligned} T &= \frac{\partial}{\partial \theta} \\ Z_j &= \frac{\partial}{\partial z_j} - i \frac{\partial \varphi}{\partial z_j}(z) \frac{\partial}{\partial \theta}, \quad j = 1, \dots, n, \end{aligned}$$

where $Z_j(x)$, $j = 1, \dots, n$, form a basis of $T_x^{1,0}X$ for each $x \in D$ and $\varphi(z) \in C^\infty(D, \mathbb{R})$ is independent of θ . We summarize these data by the notation $(D, (z, \theta), \varphi)$. We call $(D, x = (z, \theta), \varphi)$ *BRT trivialization*, $\{Z_j\}_{j=1}^n$ *BRT frame* and we call $x = (z, \theta)$ *canonical coordinates*.

Furthermore, let $(D, (z, \theta), \varphi)$ and $(\tilde{D}, (w, \eta), \tilde{\varphi})$ be two such data on D . Then the coordinate transformation between them on $D \cap \tilde{D}$ satisfies the following. There exist holomorphic functions H_j , $j = 1, \dots, n$, and g on $\{z \in \mathbb{C}^n; |z| < \delta\}$ with g nowhere vanishing, such that

$$(2.11) \quad \begin{aligned} w &= (w_1, \dots, w_n) = H(z) = (H_1(z), \dots, H_n(z)), \\ \eta &= \theta + \arg g(z) \pmod{2\pi} \quad \text{where } \arg g(z) = \text{Im} \log g(z), \\ \tilde{\varphi}(H(z), \overline{H(z)}) &= \varphi(z, \bar{z}) + \log |g(z)|. \end{aligned}$$

REMARK 2.10. – The formulas in (2.11) generalize those in [3, the line below (I.31)] which correspond to the case $g(z) \equiv \text{constant}$. See the proof of Proposition 4.2 for a derivation. It should be noticed that our notation $-\frac{\partial \varphi}{\partial \bar{z}_j}$ is equal to ϕ_{k, \bar{z}_j} ($k = 1$) in [3, the line below (I.31)].

There exist examples that H is not necessarily one to one. Nevertheless, it can be shown that after shrinking D and \tilde{D} properly, it is one to one, hence a biholomorphism.

We call the above triple $(D, (z, \theta), \varphi)$ a *BRT trivialization*. Note for $(z, \theta) \in D$ and $-\pi < \alpha < \pi$, $e^{-i\alpha} \circ (z, \theta) = (z, \theta + \alpha)$ if $\{e^{-it\alpha} \circ (z, \theta)\}_{0 \leq t \leq 1} \subset D$.

By using BRT trivializations some operations simplify, as follows. Under the BRT triple $(D, (z, \theta), \varphi)$ it is clear that

$$\{d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}, 1 \leq j_1 < \cdots < j_q \leq n\}$$

is a basis for $T_x^{*0,q}X$ for every $x \in D$. For $u \in \Omega^{0,q}(X)$, on D we write

$$(2.12) \quad u = \sum_{j_1 < \cdots < j_q} u_{j_1 \dots j_q} d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$

Recall that T is the vector field associated with the S^1 action. We have

$$(2.13) \quad Tu = \sum_{j_1 < \cdots < j_q} (Tu_{j_1 \dots j_q}) d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$

and Tu is independent of the choice of BRT trivializations.

For $\bar{\partial}_b$ on the BRT triple $(D, (z, \theta), \varphi)$ we have

$$(2.14) \quad \bar{\partial}_b = \sum_{j=1}^n d\bar{z}_j \wedge \left(\frac{\partial}{\partial \bar{z}_j} + i \frac{\partial \varphi}{\partial \bar{z}_j}(z) \frac{\partial}{\partial \theta} \right).$$

The rigid objects (discussed in the preceding section) are natural geometric objects pertinent to the S^1 action. In the following X is again a compact connected CR manifold with a transversally CR locally free S^1 action.

THEOREM 2.11. – *Suppose F is a complex vector bundle over X (not necessarily a CR bundle) and admits an S^1 action compatible with that on X . Then F is actually a rigid vector bundle (in the sense of Definition 2.4). Moreover there is a rigid Hermitian metric $\langle \cdot | \cdot \rangle_F$ on F . Conversely if F is a rigid vector bundle, then F admits a compatible S^1 action.*

Proof. – We first work on the existence of a rigid Hermitian metric (assuming F is rigid). Fix $p \in X$ and let $(D, (z, \theta), \varphi)$ be a BRT trivialization around p such that $(z(p), \theta(p)) = (0, 0)$, $(z, \theta) \in \{z \in \mathbb{C}^{n-1} : |z| < \delta\} \times \{\theta \in \mathbb{R} : |\theta| < \delta\}$ for some $\delta > 0$. Put

$A := \{\lambda \in [-\pi, \pi] : \text{there is a local rigid trivializing frame (l.r.t. frame for short)}$

defined on $\{e^{-i\theta} \circ (z, 0); |z| < \varepsilon, \theta \in [-\pi, \lambda + \varepsilon]\}$ for some $0 < \varepsilon < \delta\}$.

Clearly A is a non-empty open set in $[-\pi, \pi]$. We claim $A = [-\pi, \pi]$. (Remark that the l.r.t. frame above is closely related to the *canonical basis* in [3, Definition I.3 without (I.29a)] when E is $T^{1,0}X$.)

It suffices to prove A is closed. Let λ_0 be a limit point of A . For some small $\varepsilon_1 > 0$, there is a l.r.t. frame $\hat{f} = (\hat{f}_1, \dots, \hat{f}_r)$ defined on

$$\{e^{-i\theta} \circ (z, 0); |z| < \varepsilon_1, \lambda_0 - \varepsilon_1 < \theta_1 < \lambda_0 + \varepsilon_1\}.$$

By assumption $\lambda_0 \in \bar{A}$ there exists a l.r.t. frame $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_r)$ defined around $\{e^{-i\theta} \circ (z, 0)\}$ in which $|z| < \varepsilon_2, \theta \in [-\pi, \lambda_0 - \frac{\varepsilon_1}{2}]$ for some $\varepsilon_2 > 0$. Now $\tilde{f} = g\hat{f}$ on

$$\left\{ e^{-i\theta} \circ (z, 0); |z| < \varepsilon_0, \theta \in (\lambda_0 - \varepsilon_1, \lambda_0 - \frac{\varepsilon_1}{2}) \right\}, \quad \varepsilon_0 = \min \{ \varepsilon_1, \varepsilon_2 \},$$

for some rigid $r \times r$ matrix g .

We now patch up the frames. Put , for $\theta \in [-\pi, \lambda_0 - \frac{\varepsilon_1}{2}]$, $f = \tilde{f}$ (on $\{e^{-i\theta} \circ (z, 0)\}$) and for $\theta \in [\lambda_0 - \frac{\varepsilon_1}{2}, \lambda_0 + \varepsilon_1]$, $f = g\hat{f}$ because g is independent of θ . By $\tilde{f} = g\hat{f}$ on the overlapping, f is well-defined as a l.r.t. frame on

$$\left\{ e^{-i\theta} \circ (z, 0); |z| < \varepsilon_0, \theta \in [-\pi, \lambda_0 + \varepsilon_1] \right\}.$$

extending $\theta = \lambda_0$. Thus A is closed as desired.

By the discussion above we can actually find local rigid trivializations W_1, \dots, W_N such that $X = \bigcup_{j=1}^N W_j$ and each $W_j \supset \bigcup_{-\pi \leq \theta \leq \pi} e^{-i\theta} W_j$ (i.e., W_j is S^1 invariant). Take any Hermitian metric $\langle \cdot, \cdot \rangle_F$ on F . Let $\langle \cdot | \cdot \rangle_F$ be the Hermitian metric on F defined as follows. For each $j = 1, 2, \dots, N$, let h_j^1, \dots, h_j^r be local rigid trivializing frames on W_j . Put $\langle h_j^s(x) | h_j^t(x) \rangle_F = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle h_j^s(e^{-iu} \circ x), h_j^t(e^{-iu} \circ x) \rangle_F du$, $s, t = 1, 2, \dots, r$. One sees that $\langle \cdot | \cdot \rangle_F$ is well-defined as a rigid Hermitian metric on F .

By examining the above reasoning we have also proved that if F is rigid, then it admits a natural S^1 action (by declaring the l.r.t. frames as S^1 invariant frames) compatible with that on X .

For the reverse direction if F admits a compatible S^1 action, by using BRT trivializations one can construct S^1 invariant local frames. These invariant local frames can be easily verified to be local rigid frames, or equivalently the transition functions between them are annihilated by T due to the S^1 invariant property. Hence F is rigid by definition. \square

We shall now prove

THEOREM 2.12. – *Assume the complex vector bundle F is rigid (on X). There exists a rigid connection on F . And if F is equipped with a rigid Hermitian metric, there exists a rigid connection compatible with this Hermitian metric. Suppose F is furthermore CR (and rigid) equipped with a rigid Hermitian metric h . Then there exists a unique rigid connection (see the second paragraph of Subsection 2.3) ∇^F compatible with h such that ∇^F induces a Chern connection on U_j (of any given BRT chart), and that the S^1 invariant sections are its parallel sections along S^1 orbits of X .*

Proof. – Let ∇ be a connection on F . For any $g \in S^1$ considering $g^*\nabla$ on g^*F which is F by using Theorem 2.11 and summing over g (in analogy with the construction of a rigid Hermitian metric above), one obtains a rigid connection on F . Suppose F has a rigid Hermitian metric $\langle \cdot | \cdot \rangle_F$ and a connection ∇ compatible with $\langle \cdot | \cdot \rangle_F$. One readily sees that the rigid connection resulting from the preceding procedure of summation, is still compatible with $\langle \cdot | \cdot \rangle_F$. For the last statement, note that i)

given a rigid CR bundle F with a rigid Hermitian metric h and any BRT chart $D_j = U_j \times]-\varepsilon, \varepsilon[$, (F, h) can descend to U_j as a holomorphic vector bundle with the inherited metric, and ii) for a holomorphic vector bundle with a Hermitian metric, the Chern connection is canonically defined. Combining i) with ii) and using the S^1 invariant local frames (cf. proof of Theorem 2.11), one can construct a canonical connection ∇_j^F on U_j . Then by using (2.10), (2.11) and the canonical property of the Chern connection, one sees that these ∇_j^F patch up to form a global connection ∇^F on X , satisfying the property as stated in the proposition. \square

CHAPTER 3

A HODGE THEORY FOR $\square_{b,m}^{(q)}$

A Hodge decomposition theorem for $\bar{\partial}_b$ on pseudoconvex CR manifolds has been well developed. See [15, Section 9.4] for a nice presentation in some respects; see also [62]. Our goal of this section is to develop an analogous theory for $\square_{b,m}^{(q)}$ on CR manifolds with transversal CR locally free S^1 action (irrespective of pseudoconvexity). Much of what follows appears to parallel the corresponding part of Hodge theory in complex geometry.

Besides the relevance to the index theorem on CR manifolds, the present theory has an application to our proof of homotopy invariance of index (Theorem 4.7).

As before, X is a compact CR manifold with a transversal CR locally free S^1 action. Let $\bar{\partial}_b^* : \Omega^{0,q+1}(X, E) \rightarrow \Omega^{0,q}(X, E)$ ($q = 0, 1, 2, \dots, n$) be the formal adjoint of $\bar{\partial}_b$ with respect to $(\cdot | \cdot)_E$. Put $\square_b^{(q)} := \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b : \Omega^{0,q}(X, E) \rightarrow \Omega^{0,q}(X, E)$. T is the vector field on X induced by the S^1 action, $T\bar{\partial}_b = \bar{\partial}_b T$ and $\bar{\partial}_{b,m} := \bar{\partial}_b|_{\Omega_m^{0,q}} : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q+1}(X, E)$ on eigenspaces of the S^1 action ($\forall m \in \mathbb{Z}$).

Recall that $\langle \cdot | \cdot \rangle_E$ is rigid. One sees $T\bar{\partial}_b^* = \bar{\partial}_b^* T$ so that $\bar{\partial}_b^*|_{\Omega_m^{0,q+1}} : \Omega_m^{0,q+1}(X, E) \rightarrow \Omega_m^{0,q}(X, E)$ is the same as the formal adjoint $\bar{\partial}_{b,m}^*$ of $\bar{\partial}_{b,m}$. Form $\square_{b,m}^{(q)} = \bar{\partial}_{b,m} \bar{\partial}_{b,m}^* + \bar{\partial}_{b,m}^* \bar{\partial}_{b,m} : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q}(X, E)$. We have $\square_{b,m}^{(q)} = \square_b^{(q)}|_{\Omega_m^{0,q}(X, E)}$.

On a general compact CR manifold, there is a fundamental result that follows from Kohn's L^2 estimates. (See [15, Theorem 8.4.2]). Adapting it to our present situation, we can state the result as follows.

THEOREM 3.1. – *For every $s \in \mathbb{N}_0$, there is a constant $C_s > 0$ such that*

$$\|u\|_{s+1} \leq C_s \left(\left\| \square_b^{(q)} u \right\|_s + \|Tu\|_s + \|u\|_s \right), \quad \forall u \in \Omega^{0,q}(X, E),$$

where $\|\cdot\|_s$ denotes the usual Sobolev norm of order s on X .

Theorem 3.1 restricted to $\Omega_m^{0,q}(X, E)$ yields

COROLLARY 3.2. – Fix $m \in \mathbb{Z}$. For every $s \in \mathbb{N}_0$, there is a constant $C_s > 0$ such that

$$\|u\|_{s+1} \leq C_s \left(\left\| \square_{b,m}^{(q)} u \right\|_s + \|u\|_s \right), \quad \forall u \in \Omega_m^{0,q}(X, E).$$

This suggests that a good regularity theory might exist on our X . Observe that, although $\square_b^{(q)}$ is not elliptic on X , $\square_b^{(q)} - T^2$ is and restricts to $\square_{b,m}^{(q)} + m^2$ on $\Omega_m^{0,q}(X, E)$. In fact, without using the above theorems all of the following results are essentially proven by standard results in elliptic theory.

Write $\text{Dom } \square_{b,m}^{(q)} := \{u \in L_m^2(X, T^{*0,q}X \otimes E); \square_{b,m}^{(q)} u \in L_m^2(X, T^{*0,q}X \otimes E)\}$ where $\square_{b,m}^{(q)} u$ is defined in the sense of distribution. $\square_{b,m}^{(q)}$ is extended by

$$(3.1) \quad \square_{b,m}^{(q)} : \text{Dom } \square_{b,m}^{(q)} (\subset L_m^2(X, T^{*0,q}X \otimes E)) \rightarrow L_m^2(X, T^{*0,q}X \otimes E).$$

LEMMA 3.3. – We have $\text{Dom } \square_{b,m}^{(q)} = L_m^2(X, T^{*0,q}X \otimes E) \cap H^2(X, T^{*0,q}X \otimes E)$.

Proof. – For the inclusion put $v = \square_{b,m}^{(q)} u \in L_m^2(X, T^{*0,q}X \otimes E)$.

Then $(\square_{b,m}^{(q)} - T^2)u = v + m^2u \in L_m^2(X, T^{*0,q}X \otimes E)$. Since $(\square_b^{(q)} - T^2)$ is elliptic, we conclude $u \in H^2(X, T^{*0,q}X \otimes E)$. The reverse inclusion is clear. \square

LEMMA 3.4. – We have that $\square_{b,m}^{(q)} : \text{Dom } \square_{b,m}^{(q)} (\subset L_m^2(X, T^{*0,q}X \otimes E)) \rightarrow L_m^2(X, T^{*0,q}X \otimes E)$ is self-adjoint.

Proof. – Since the similar extension of $\square_b^{(q)}$ on $L^2(X, T^{*0,q}X \otimes E)$ is self-adjoint and its restriction to (an invariant subspace) $L_m^2(X, T^{*0,q}X \otimes E)$ gives $\square_{b,m}^{(q)}$, $\square_{b,m}^{(q)}$ is also self-adjoint. \square

Let $\text{Spec } \square_{b,m}^{(q)} \subset [0, \infty[$ denote the spectrum of $\square_{b,m}^{(q)}$ (Davies [20]).

PROPOSITION 3.5. – $\text{Spec } \square_{b,m}^{(q)}$ is a discrete subset of $[0, \infty[$. For any $\nu \in \text{Spec } \square_{b,m}^{(q)}$, ν is an eigenvalue of $\square_{b,m}^{(q)}$ and the eigenspace

$$\mathcal{E}_{m,\nu}^q(X, E) := \{u \in \text{Dom } \square_{b,m}^{(q)}; \square_{b,m}^{(q)} u = \nu u\}$$

is finite dimensional with $\mathcal{E}_{m,\nu}^q(X, E) \subset \Omega_m^{0,q}(X, E)$.

Proof. – $\square_b^{(q)} - T^2 \equiv \Delta$ is a second order elliptic operator. By standard elliptic theory, Δ and hence $\Delta + m^2$, satisfy the statement of the proposition on the (invariant) subspace $L_m^2(X, T^{*0,q}X \otimes E) \supset \Omega_m^{0,q}(X, E)$. On it T^2 acts as $-m^2$, the proposition follows for $\square_{b,m}^{(q)}$ which is $\Delta + m^2$ on $L_m^2(X, T^{*0,q}X \otimes E)$. \square

A role analogous to the Green's operator in the ordinary Hodge theory is given as follows. Let

$$N_m^{(q)} : L_m^2(X, T^{*0,q}X \otimes E) \rightarrow \text{Dom } \square_{b,m}^{(q)}$$

be the partial inverse of $\square_{b,m}^{(q)}$ and let

$$\Pi_m^{(q)} : L_m^2(X, T^{*0,q}X \otimes E) \rightarrow \text{Ker } \square_{b,m}^{(q)}$$

be the orthogonal projection. We have

$$(3.2) \quad \begin{aligned} \square_{b,m}^{(q)} N_m^{(q)} + \Pi_m^{(q)} &= I \quad \text{on } L_m^2(X, T^{*0,q}X \otimes E), \\ N_m^{(q)} \square_{b,m}^{(q)} + \Pi_m^{(q)} &= I \quad \text{on } \text{Dom } \square_{b,m}^{(q)}. \end{aligned}$$

LEMMA 3.6. – We have $N_m^{(q)} : \Omega_m^{0,q}(X, E) \rightarrow \Omega_m^{0,q}(X, E)$.

Proof. – A slight variant of the standard argument applies as $\square_b^{(q)}$ is almost elliptic. Let $u \in \Omega_m^{0,q}(X, E)$ and put $N_m^{(q)}u = v \in L_m^2(X, T^{*0,q}X \otimes E)$. By (3.2), $(I - \Pi_m^{(q)})u = \square_{b,m}^{(q)}v$, giving

$$(3.3) \quad (\square_{b,m}^{(q)} - T^2)v = (I - \Pi_m^{(q)})u + m^2v.$$

By Proposition 3.5, $\text{Ker } \square_{b,m}^{(q)}$ consists of smooth sections, so $\Pi_m^{(q)}u$ is smooth and

$$(3.4) \quad (I - \Pi_m^{(q)})u \in \Omega_m^{0,q}(X, E).$$

By combining (3.3) and (3.4) and noting $\square_b^{(q)} - T^2$ is elliptic, the standard technique in elliptic regularity applies to give $v \in \Omega_m^{0,q}(X, E)$. \square

The following is a version of “harmonic realization” of cohomology.

THEOREM 3.7. – For every $q \in \{0, 1, 2, \dots, n\}$ and every $m \in \mathbb{Z}$, we have

$$(3.5) \quad \text{Ker } \square_{b,m}^{(q)} = \mathcal{E}_{m,0}^q(X, E) \cong H_{b,m}^q(X, E).$$

As a consequence $\dim H_{b,m}^q(X, E) < \infty$ by Proposition 3.5.

Proof. – The argument is mostly standard (although $\square_b^{(q)}$ is not elliptic). Consider the map

$$(3.6) \quad \begin{aligned} \tau_m^q : \text{Ker } \bar{\partial}_{b,m} \cap \Omega_m^{0,q}(X, E) &\rightarrow \text{Ker } \square_{b,m}^{(q)}, \\ u &\mapsto \Pi_m^{(q)}u. \end{aligned}$$

Clearly τ_m^q is surjective. Put $M_m^q := \{\bar{\partial}_{b,m}u; u \in \Omega_m^{0,q-1}(X, E)\}$. The theorem follows if one shows

$$(3.7) \quad \text{Ker } \tau_m^q = M_m^q.$$

It is easily seen $M_m^q \subset \text{Ker } \tau_m^q$ since $M_m^q \perp \text{Ker } \square_{b,m}^{(q)}$. For the converse let $u \in \text{Ker } \tau_m^q$ so $\Pi_m^{(q)} u = 0$. From (3.2) we have

$$(3.8) \quad u = \square_{b,m}^{(q)} N_m^{(q)} u + \Pi_m^{(q)} u = (\bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b) N_m^{(q)} u.$$

We claim that

$$(3.9) \quad \bar{\partial}_b^* \bar{\partial}_b N_m^{(q)} u = 0.$$

One sees, by using $\bar{\partial}_b^2 = 0$ for the first equality below,

$$(3.10) \quad (\bar{\partial}_b^* \bar{\partial}_b N_m^{(q)} u \mid \bar{\partial}_b^* \bar{\partial}_b N_m^{(q)} u)_E = (\bar{\partial}_b^* \bar{\partial}_b \square_{b,m}^{(q)} N_m^{(q)} u \mid N_m^{(q)} u)_E = (\bar{\partial}_b^* \bar{\partial}_b (I - \Pi_m^{(q)}) u \mid N_m^{(q)} u)_E \\ = (\bar{\partial}_b^* \bar{\partial}_b u \mid N_m^{(q)} u)_E,$$

which is zero because $u \in \text{Ker } \bar{\partial}_{b,m}$ by (3.6), giving the claim (3.9). By (3.8) and (3.9),

$$(3.11) \quad u = \bar{\partial}_b \bar{\partial}_b^* N_m^{(q)} u,$$

with $\bar{\partial}_b^* N_m^{(q)} u \in \Omega_m^{0,q-1}(X, E)$ by Lemma 3.6. By (3.11), $u \in M_m^q$, yielding the desired inclusion $\text{Ker } \tau_m^q \subset M_m^q$. Hence (3.7). \square

Let $D_{b,m} := \bar{\partial}_b + \bar{\partial}_b^* : \Omega_m^{0,+}(X, E) \rightarrow \Omega_m^{0,-}(X, E)$ with extension $D_{b,m}$

$$D_{b,m} : \text{Dom } D_{b,m} (\subset L_m^{2,+}(X, E)) \rightarrow L_m^{2,-}(X, E),$$

$\text{Dom } D_{b,m} = \{u \in L_m^{2,+}(X, E); \text{distribution } D_{b,m} u \in L_m^{2,-}(X, E)\}$. The Hilbert space adjoint of $D_{b,m}$ with respect to $(\cdot \mid \cdot)_E$ is given by $D_{b,m}^* : \text{Dom } D_{b,m}^* (\subset L_m^{2,-}(X, E)) \rightarrow L_m^{2,+}(X, E)$.

Combining Proposition 3.5 and Theorem 3.7, one can verify (as in standard Hodge theory)

THEOREM 3.8. – *With the notation above*

$$(3.12) \quad \text{Ker } D_{b,m} = \bigoplus_{\substack{q \in \{0,1,\dots,n\} \\ q \text{ even}}} \text{Ker } \square_{b,m}^{(q)} (\subset \Omega_m^{0,+}(X, E)), \\ \text{Ker } D_{b,m}^* = \bigoplus_{\substack{q \in \{0,1,\dots,n\} \\ q \text{ odd}}} \text{Ker } \square_{b,m}^{(q)} (\subset \Omega_m^{0,-}(X, E)).$$

Put $\text{ind } D_{b,m} := \dim \text{Ker } D_{b,m} - \dim \text{Ker } D_{b,m}^*$. Hence, together with Theorem 3.7,

$$(3.13) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = \text{ind } D_{b,m}.$$

CHAPTER 4

MODIFIED KOHN LAPLACIAN (Spin^c KOHN LAPLACIAN)

We are prepared by Theorem 3.8 to see that to calculate $\sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E)$ is the same as to calculate the index $\text{ind } D_{b,m}$. To do so effectively we need to modify the Dirac type operator $D_{b,m}$ hence the standard Kohn Laplacian because the modified versions $\tilde{D}_{b,m}$, $\tilde{\square}_{b,m}$ will have a manageable heat kernel that suits our purpose better for the CR non-Kähler case (cf. Remark 4.9). Lastly we shall give an argument for the homotopy invariance, and obtain $\text{ind } D_{b,m} = \text{ind } \tilde{D}_{b,m}$.

The main idea here is borrowed from that of classical cases. But as the CR manifold X is not assumed to be a (orbifold) circle bundle globally, there could arise the problem of patching (from local constructions to the global one). Part of the technicality in the beginning of this section lies in a careful treatment in this regard.

We recall some basics of Clifford connection and Spin^c Dirac operator. For more details we refer to Chapter 1 in [50] and [24].

Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialization with $D = U \times]-\varepsilon, \varepsilon[$ where $\varepsilon > 0$ and U is an open set of \mathbb{C}^n . Using φ in B , we let $\langle \cdot, \cdot \rangle$ be the Hermitian metric on CTU induced by that on D

$$(4.1) \quad \left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right\rangle = \left\langle \frac{\partial}{\partial z_j} - i \frac{\partial \varphi}{\partial z_j}(z) \frac{\partial}{\partial \theta} \mid \frac{\partial}{\partial z_k} - i \frac{\partial \varphi}{\partial z_k}(z) \frac{\partial}{\partial \theta} \right\rangle, \quad j, k = 1, 2, \dots, n$$

(cf. Theorem 2.9). By (2.10) and Theorem 2.9, the above metric is actually intrinsically defined.

The $\langle \cdot, \cdot \rangle$ induces Hermitian metrics on $T^{*0,q}U$ still denoted by $\langle \cdot, \cdot \rangle$ and a Riemannian metric g^{TU} on TU .

For any $v \in TU$ with decomposition $v = v^{(1,0)} + v^{(0,1)} \in T^{1,0}U \oplus T^{0,1}U$, let $\bar{v}^{(1,0),*} \in T^{*0,1}U$ be the metric dual of $v^{(1,0)}$ with respect to $\langle \cdot, \cdot \rangle$. That is, $\bar{v}^{(1,0),*}(u) = \langle v^{(1,0)}, \bar{u} \rangle$ for all $u \in T^{0,1}U$.

The Clifford action v on $\Lambda(T^{*0,1}U) := \bigoplus_{q=0}^n T^{*0,q}U$ is defined by

$$(4.2) \quad c(v)(\cdot) = \sqrt{2}(\bar{v}^{(1,0),*} \wedge (\cdot) - i_{v^{(0,1)}}(\cdot))$$

(where \wedge and i denote the exterior and interior product respectively).

Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{1,0}U$ with respect to $\langle \cdot, \cdot \rangle$ with dual frame $\{w^j\}_{j=1}^n$. Write

$$(4.3) \quad e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \bar{w}_j) \text{ and } e_{2j} = \frac{i}{\sqrt{2}}(w_j - \bar{w}_j), \quad j = 1, 2, \dots, n,$$

for an orthonormal frame of TU . Let ∇^{TU} be the Levi-Civita connection on TU (with respect to g^{TU}), and ∇^{\det} be the Chern connection on the determinant line bundle $\det(T^{1,0}U)$ (with $\langle \cdot, \cdot \rangle$), with connection forms Γ^{TU} and Γ^{\det} associated to the frames $\{e_j\}_{j=1}^{2n}$ and $w_1 \wedge \dots \wedge w_n$. That is,

$$(4.4) \quad \begin{aligned} \nabla_{e_j}^{TU} e_\ell &= \Gamma^{TU}(e_j)e_\ell, \quad j, \ell = 1, 2, \dots, 2n, \\ \nabla^{\det}(w_1 \wedge \dots \wedge w_n) &= \Gamma^{\det} w_1 \wedge \dots \wedge w_n. \end{aligned}$$

The *Clifford connection* ∇^{Cl} on $\Lambda(T^{*0,1}U)$ is defined for the frame

$$\{\bar{w}^{j_1} \wedge \dots \wedge \bar{w}^{j_q}; 1 \leq j_1 < \dots < j_q \leq n\}$$

by the local formula

$$(4.5) \quad \nabla^{\text{Cl}} = d + \frac{1}{4} \sum_{j,\ell=1}^{2n} \langle \Gamma^{TU} e_j, e_\ell \rangle c(e_j)c(e_\ell) + \frac{1}{2} \Gamma^{\det}.$$

In general a Levi-Civita connection ∇ cannot be compatible with the complex structure unless a certain extra condition is imposed such as Kähler condition on the metric. Alternatively one takes the orthogonal projection $P_{T^{1,0}X} \nabla$ to produce a connection on $T^{1,0}X$. One key point above is that the Clifford connection ∇^{Cl} (regardless of Kähler condition nor orthogonal projection) defines a *Hermitian connection* (connection compatible with the underlying Hermitian metric) on $\Lambda(T^{*0,1}U)$ (see Proposition 1.3.1 in [50]).

Let's be back to the CR case. In the same notation as before, $\Omega^{0,q}(U, E)$ denotes the space of $(0, q)$ forms on U with values in E , $\Omega^{0,+}(U, E)$ the even part and $\Omega^{0,-}(U, E)$ the odd part of $\Omega^{0,\bullet}(U, E)$ etc.

Assume X is equipped with a CR bundle E which is rigid. Being rigid E can descend as a holomorphic vector bundle over U . We may assume that E is (holomorphically) trivial on U (possibly after shrinking U). A rigid Hermitian (fiber) metric $\langle \cdot | \cdot \rangle_E$ descends to a Hermitian (fiber) metric $\langle \cdot | \cdot \rangle_E$ on E over U . Let ∇^E be the Chern connection on E associated with $\langle \cdot | \cdot \rangle_E$ (over U).

We still denote by ∇^{Cl} the connection on $\Lambda(T^{*0,1}U) \otimes E$ induced by ∇^{Cl} and ∇^E .

DEFINITION 4.1. – The Spin^c Dirac operator D_B is defined by

$$(4.6) \quad D_B = \frac{1}{\sqrt{2}} \sum_{j=1}^{2n} c(e_j) \nabla_{e_j}^{\text{Cl}} : \Omega^{0,\bullet}(U, E) \rightarrow \Omega^{0,\bullet}(U, E).$$

It is well-known that D_B is formally self-adjoint (see Proposition 1.3.1 and equation (1.3.1) in [50]) and $D_B : \Omega^{0,\pm}(U, E) \rightarrow \Omega^{0,\mp}(U, E)$.

Write $\bar{\partial}^* : \Omega^{0,q+1}(U, E) \rightarrow \Omega^{0,q}(U, E)$ for the adjoint of $\bar{\partial} : \Omega^{0,q}(U, E) \rightarrow \Omega^{0,q+1}(U, E)$ with respect to the L^2 inner product on $\Omega^{0,q}(U, E)$ induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot | \cdot \rangle_E$ ($q = 0, 1, 2, \dots, n-1$). Then, by Theorem 1.4.5 in [50]

$$(4.7) \quad D_B = \bar{\partial} + \bar{\partial}^* + A_B : \Omega^{0,\pm}(U, E) \rightarrow \Omega^{0,\mp}(U, E),$$

where $A_B : \Omega^{0,\pm}(U, E) \rightarrow \Omega^{0,\mp}(U, E)$ is a smooth zeroth order operator and $A_B = A_B(z)$, independent of θ . Moreover, A_B does not depend on E and acts only on the factor $\Omega^{0,\bullet}(X)$. Note that A_B as an operator $\Omega^{0,\bullet}(U, E) \rightarrow \Omega^{0,\bullet}(U, E)$ is self-adjoint because both D_B and $\bar{\partial} + \bar{\partial}^*$ are so.

The following is instrumental in forming a global operator from local ones, whose proof is based on canonical coordinates of BRT trivializations. Note for $u \in \Omega^{0,q}(D, E)$ ($q = 0, 1, 2, \dots, n$) with $u = u(z)$, i.e., u is independent of θ , we may identify such u with an element in $\Omega^{0,q}(U, E)$ by using (2.12) (and vice versa).

PROPOSITION 4.2. – *Let $B = (D, (z, \theta), \varphi)$ and $\tilde{B} = (D, (w, \eta), \tilde{\varphi})$ be two BRT trivializations with $D = U \times]-\varepsilon, \varepsilon[$ for $\varepsilon > 0$ and an open set U of \mathbb{C}^n . Let $A_B, A_{\tilde{B}} : \Omega^{0,\pm}(U, E) \rightarrow \Omega^{0,\mp}(U, E)$ be the operators given by (4.7). Fix an $m \in \mathbb{Z}$. For $u \in \Omega_m^{0,\pm}(X, E)$ we can write $u(= u|_D) = e^{-im\theta}v(z) = e^{-im\eta}\tilde{v}(w)$ for some $v(z), \tilde{v}(w) \in \Omega^{0,\pm}(U, E)$. Then*

$$(4.8) \quad e^{-im\theta}A_B(v(z)) = e^{-im\eta}A_{\tilde{B}}(\tilde{v}(w)) \quad \text{on } D.$$

Proof. – Although we shall only use part of the coordinate transformation of BRT trivializations (2.11)

$$(4.9) \quad w = H(z), \quad \bar{\partial}H(z) = 0; \quad \eta = \theta + \arg g(z); \quad \tilde{\varphi}(H(z), \overline{H(z)}) = \varphi(z, \bar{z}) + \log |g(z)|,$$

let's give a geometrical interpretation of how the above can be obtained for an independent interest. This complements the treatment of [3]. See Subsection 5.1 (cf. Remark 5.3) for its use in the construction of a modified Kodaira Laplacian.

To see (4.9), we are going to realize D (possibly after shrinking it) as (part of) the total space of a circle bundle associated with a trivial holomorphic line bundle L over a complex manifold $U \subset \mathbb{C}^n$. More precisely suppose L is equipped with a Hermitian metric such that a local basis 1 has $\|1\| = e^{-2\phi(z)}$ and $Y = \{(z, \lambda) \in \mathbb{C}^{n+1}; |\lambda|^2 e^{2\phi} = 1\}$ is the circle bundle inside the L^* . Write $\rho = |\lambda|^2 e^{2\phi} - 1$ and $\lambda|_Y = e^{-\phi - i\theta}$. One has $T^{0,1}Y = (\text{Ker } \bar{\partial}\rho) \cap T^{0,1}\mathbb{C}^{n+1}$. In terms of (z, θ) coordinates on Y , one has $T^{0,1}Y = \text{Span} \left(\left\{ \frac{\partial}{\partial \bar{z}_j} + i \frac{\partial \phi}{\partial \bar{z}_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \dots, n \right\} \right)$ (and $T^{1,0}Y = \overline{T^{0,1}Y}$) because the RHS is checked to be contained in $\text{Ker } \bar{\partial}\rho$ and it has the correct dimension. In view of Theorem 2.9 by taking the above ϕ to be φ of Z_j in (2.10) and mapping (z, θ) of D to $(z, e^{-\phi(z) - i\theta})$ of Y , it will be seen that D is realized in this way as a portion of Y .

For this realization, we compare the above with [3, (0.1) or Theorem I.2] and set up the (holomorphic) transformation in coordinates: setting our φ , λ , θ and z (on U) above to be ϕ , e^{iw} , $-s_1$ and z respectively in [3, (0.1) and Theorem I.2]). Then our Z_j in (2.10) corresponds to \bar{L}_j in [3, Proposition I.2]. One sees that the ambient complex space of [3] is (locally) biholomorphic to part of the above L^* . For the next purpose, let us give an intrinsic formulation of this complex space from another point of view. Let $\mathbb{R}_{>0}$ be the set of positive real numbers. Consider $D \times \mathbb{R}_{>0}$ and equip it with the complex structure J defined as follows. $J|_{TU}$ is set to be the complex structure of U ; it suffices to define J on $]-\varepsilon, \varepsilon[\times \mathbb{R}_{>0}$ with coordinates s, r : $J(\partial/\partial r) = -\frac{1}{r}\partial/\partial s$, $J(\frac{1}{r}\partial/\partial s) = \partial/\partial r$. Note this definition of J is independent of choice of BRT trivializations (since $Z_j \subset T^{1,0}Y$ identified with $T^{1,0}U$, gives a local basis of $T^{1,0}D$; see Theorem 2.9, and $\{z\} \times]-\varepsilon, \varepsilon[$ for each $z \in U$ is (part of) an S^1 orbit in X). J is seen to be (equivalent to) the complex structure on L^* (with $(z, s, r) \in D \times \mathbb{R}_{>0}$ and $(z, re^{is}) \in L^*$ in correspondence), hence an integrable complex structure.

Let now $\tilde{B} = (D, (w, \eta), \tilde{\varphi})$ be any other BRT chart. Correspondingly we will denote the associated objects by the same notation as in B but topped with a tilde. Note that in \tilde{B} the set defined by $w = w_0$ for a fixed w_0 is part of an S^1 orbit in X ; the same can be said with B . Conversely, any S^1 orbit of X is described (locally) by the θ parameter in any BRT charts with z -coordinates being fixed. By using $(D \times \mathbb{R}_{>0}, J)$ above, one has $L^* \cong \tilde{L}^*$ (locally) by a biholomorphism F that preserves respective fibers (since these are S^1 orbits, described by θ parameters in each chart and hence must be in correspondence via $D \times \mathbb{R}_{>0}$ by the property of BRT charts as just remarked). Further, one sees that F restricted to fibers has to be linear, hence F is a bundle isomorphism. Geometrically this picture is essentially the same as a local change of holomorphic coordinates on the base manifold U and a change of a local basis of L^* by $\tilde{e}^*(z) = g(z)e^*(z)$ with $\|e^*\|^2 = e^{2\varphi}$, $\|\tilde{e}^*\|^2 = e^{2\tilde{\varphi}}$ for some nowhere vanishing holomorphic function $g(z)$ on U . The above transformation Formula (4.9) easily follows from this concrete realization.

Using the above transformation (4.9), one claims

$$(4.10) \quad \begin{aligned} D_B &= D_{\tilde{B}} \quad \text{on } \Omega^{0,\pm}(U, E), \\ A_B &= A_{\tilde{B}} \quad \text{on } \Omega^{0,\pm}(U, E). \end{aligned}$$

To see this for the case without E , note D is just realized as (part of) the total space of a circle bundle of a holomorphic line bundle. Clearly $\bar{\partial}_U + \bar{\partial}_U^*$ does not depend on choice of holomorphic coordinates on U ; that is, $\bar{\partial}_U + \bar{\partial}_U^*$ is an intrinsic object (because the Hermitian metric used for $\bar{\partial}_U^*$ is intrinsic, see (4.1)). The same idea can be applied to D_B which is defined above as an intrinsic object too. Therefore (4.10) holds with the change of coordinates in (4.9) (using only $w = H(z)$). In the general case where

E is not necessarily the trivial bundle \mathbb{C} , the reasoning is basically unchanged. Hence the claim (4.10).

By (4.9) $v(z) = e^{-imG(z)}\tilde{v}(w)$ ($G(z) = \arg g(z)$), hence by (4.10)

$$\begin{aligned} e^{-im\theta}A_B(v(z)) &= e^{-im\theta}A_B(e^{-imG(z)}\tilde{v}(w)) \\ &= e^{-im\theta}A_{\tilde{B}}(e^{-imG(z)}\tilde{v}(w)) = e^{-im\eta}A_{\tilde{B}}(\tilde{v}(w)), \end{aligned}$$

proving the proposition. \square

We are now ready to introduce a global operator:

DEFINITION 4.3. – For every $m \in \mathbb{Z}$, let $A_m : \Omega_m^{0,\pm}(X, E) \rightarrow \Omega_m^{0,\mp}(X, E)$ be the linear operator defined as follows. Let $u \in \Omega_m^{0,\pm}(X, E)$. Then, $v := A_m u$ is an element in $\Omega_m^{0,\mp}(X, E)$ such that for every BRT trivialization $B := (D, (z, \theta), \varphi)$ ($D = U \times]-\varepsilon, \varepsilon[$, $\varepsilon > 0$, U an open set in \mathbb{C}^n) we have $v|_D = e^{-im\theta}A_B(\tilde{u})(z)$ where $u = e^{-im\theta}\tilde{u}(z)$ on D for some $\tilde{u} \in \Omega^{0,\pm}(U, E)$ and A_B is given in (4.7).

In view of Proposition 4.2, Definition 4.3 is well-defined.

We are now in a position to define the modified Kohn Laplacian (Spin^c Kohn Laplacian) including a type of CR Spin^c Dirac operator $\tilde{D}_{b,m}$. One goal of this part is to express the index of $\tilde{D}_{b,m}$ in an integral form of the heat kernel density (cf. Proposition 4.6).

The treatment below mostly follows traditional cases except the use of the projection operator Q_m^\pm together with its explicit expression in integral (see (4.16) and (4.17)).

By using A_m in Definition 4.3 we consider

$$(4.11) \quad \begin{aligned} \tilde{D}_{b,m} &= \bar{\partial}_b + \bar{\partial}_b^* + A_m : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E), \\ \tilde{D}_{b,m}^\pm &= \bar{\partial}_b + \bar{\partial}_b^* + A_m : \Omega_m^{0,\pm}(X, E) \rightarrow \Omega_m^{0,\mp}(X, E) \end{aligned}$$

with the formal adjoint $\tilde{D}_{b,m}^*$ on $\Omega_m^{0,\bullet}(X, E)$. We remark that $\tilde{D}_{b,m}^* = \tilde{D}_{b,m}$ on $\Omega_m^{0,\bullet}(X, E)$. Now, as A_B is self-adjoint on $\Omega^{0,\bullet}(U, E)$ as aforementioned, it follows that A_m is self-adjoint on $\Omega_m^{0,\bullet}(X, E)$. That $\tilde{D}_{b,m}$ is self-adjoint follows as $\bar{\partial}_b + \bar{\partial}_b^*$ is also self-adjoint.

The *modified/Spin^c Kohn Laplacian* is given by

$$(4.12) \quad \begin{aligned} \tilde{\square}_{b,m} &:= \tilde{D}_{b,m}^* \tilde{D}_{b,m} : \Omega_m^{0,\bullet}(X, E) \rightarrow \Omega_m^{0,\bullet}(X, E) \\ \tilde{\square}_{b,m}^+ &:= \tilde{D}_{b,m}^* \tilde{D}_{b,m}|_{\Omega_m^{0,+}(X, E)} = \tilde{D}_{b,m}^- \tilde{D}_{b,m}^+ : \Omega_m^{0,+}(X, E) \rightarrow \Omega_m^{0,+}(X, E), \end{aligned}$$

where $\tilde{\square}_{b,m}^- := \tilde{D}_{b,m}^+ \tilde{D}_{b,m}^-$. We extend $\tilde{\square}_{b,m}^+$ and $\tilde{\square}_{b,m}^-$ by

$$(4.13) \quad \tilde{\square}_{b,m}^\pm : \text{Dom } \tilde{\square}_{b,m}^\pm (\subset L_m^{2,\pm}(X, E)) \rightarrow L_m^{2,\pm}(X, E),$$

where $\text{Dom } \tilde{\square}_{b,m}^\pm := \{u \in L_m^{2,\pm}(X, E); \text{distribution } \tilde{\square}_{b,m}^\pm u \in L_m^{2,\pm}(X, E)\}$.

Clearly $\tilde{\square}_{b,m} - T^2$ (where T is the real vector field induced by the S^1 action) is (the restriction of) an elliptic operator on X since $(\bar{\partial}_b + \bar{\partial}_b^*)^2 - T^2$ is.

Put as usual, the Sobolov spaces (cf. Subsection 2.1) $H^{s,+}(X, E)$, $H^{s,-}(X, E)$ the even and odd part of $H^s(X, T^{*0,\bullet} \otimes E)$. In the same vein as Lemma 3.3 and Lemma 3.4 one has

$$(4.14) \quad \text{Dom } \tilde{\square}_{b,m}^{\pm} = L_m^{2,\pm}(X, E) \cap H^{2,\pm}(X, E), \quad \tilde{\square}_{b,m}^+ \text{ and } \tilde{\square}_{b,m}^- \text{ are self-adjoint.}$$

Further, for the spectrum $\text{Spec } \tilde{\square}_{b,m}^+ \subset [0, \infty[$ (resp. $\text{Spec } \tilde{\square}_{b,m}^- \subset [0, \infty[$) one has (similar to Proposition 3.5)

PROPOSITION 4.4. – $\text{Spec } \tilde{\square}_{b,m}^+$ is a discrete subset of $[0, \infty[$. For any $\mu \in \text{Spec } \tilde{\square}_{b,m}^+$, μ is an eigenvalue of $\tilde{\square}_{b,m}^+$ and the eigenspace

$$\tilde{\mathcal{E}}_{m,\nu}^+(X, E) := \{u \in \text{Dom } \tilde{\square}_{b,m}^+; \tilde{\square}_{b,m}^+ u = \nu u\}$$

is finite dimensional with $\tilde{\mathcal{E}}_{m,\nu}^+(X, E) \subset \Omega_m^{0,+}(X, E)$. Similar results hold for the case of $\tilde{\square}_{b,m}^-$.

The following can be proved by standard argument.

LEMMA 4.5. – We have $\text{Spec } \tilde{\square}_{b,m}^+ \cap]0, \infty[= \text{Spec } \tilde{\square}_{b,m}^- \cap]0, \infty[$, and for every $0 \neq \mu \in \text{Spec } \tilde{\square}_{b,m}^+$, $\dim \tilde{\mathcal{E}}_{m,\mu}^+(X, E) = \dim \tilde{\mathcal{E}}_{m,\mu}^-(X, E)$.

We are going to introduce a *McKean-Singer* type formula (Corollary 4.8). Let F be a complex vector bundle over X of rank r with a Hermitian metric $\langle \cdot | \cdot \rangle_F$. Let $A(x, y) \in C^\infty(X \times X, F \boxtimes F^*)$. For every $u \in C^\infty(X, F)$, $\int_X A(x, y)u(y)dv_X(y) \in C^\infty(X, F)$ is defined in a fairly standard manner.

Much of what follows parallels the classical cases except that Q_m^+ is introduced in our case. For $\nu \in \text{Spec } \tilde{\square}_{b,m}^{\pm}$ let $P_{m,\nu}^{\pm} : L^{2,\pm}(X, E) \rightarrow \tilde{\mathcal{E}}_{m,\nu}^{\pm}(X, E)$ be the orthogonal projections (with respect to $(\cdot | \cdot)_E$), and $P_{m,\nu}^{\pm}(x, y) (\in C^\infty(X \times X, (T^{*0,\pm} X \otimes E) \boxtimes (T^{*0,\pm} X \otimes E)^*))$ the distribution kernels of $P_{m,\nu}^{\pm}$.

The heat kernels of $\tilde{\square}_{b,m}^+$ and $\tilde{\square}_{b,m}^-$ are given by

$$(4.15) \quad e^{-t\tilde{\square}_{b,m}^{\pm}}(x, y) = P_{m,0}^{\pm}(x, y) + \sum_{\nu \in \text{Spec } \tilde{\square}_{b,m}^{\pm}, \nu > 0} e^{-\nu t} P_{m,\nu}^{\pm}(x, y)$$

with the associated continuous operators $e^{-t\tilde{\square}_{b,m}^{\pm}} : \Omega^{0,\pm}(X, E) \rightarrow \Omega_m^{0,\pm}(X, E) \subset \Omega^{0,\pm}(X, E)$. $e^{-t\tilde{\square}_{b,m}^{\pm}}$ is self-adjoint on $\Omega^{0,\pm}(X, E)$.

Remark that the heat kernels (4.15) are smooth. For, the eigenfunctions involved (in the equivalent form as (1.67)) are still eigenfunctions of $(\tilde{\square}_{b,m} - T^2)$ hence eigenfunctions of an elliptic operator. In the elliptic case, one has the Gårding type inequality which estimates the various Sobolev norms of the eigenfunctions, and hence mainly

by Sobolev embeddings, gives eventually the smoothness of the heat kernels (cf. [36, Lemmas 1.6.3 and 1.6.5]).

An important operator is given by the orthogonal projection

$$(4.16) \quad Q_m^\pm : L^{2,\pm}(X, E) \rightarrow L_m^{2,\pm}(X, E)$$

(for the m -th Fourier component). Fourier analysis with (2.2) gives

$$(4.17) \quad Q_m^\pm u = \frac{1}{2\pi} \int_{-\pi}^{\pi} ((e^{-i\theta})^* u)(x) e^{im\theta} d\theta, \quad \forall u \in \Omega^{0,\pm}(X, E).$$

The explicit expression (4.17) turns out to be crucial to many (unconventional) estimates later.

It is fairly standard (note Q_m^+ in the second line below) to obtain (by (4.15))

$$(4.18) \quad \begin{aligned} \left(\frac{\partial}{\partial t} + \tilde{\square}_{b,m}^\pm\right)(e^{-t\tilde{\square}_{b,m}^\pm} u) &= 0, \quad \forall u \in \Omega^{0,\pm}(X, E), \quad \forall t > 0, \\ \lim_{t \rightarrow 0^+} (e^{-t\tilde{\square}_{b,m}^\pm} u) &= Q_m^\pm u, \quad \forall u \in \Omega^{0,\pm}(X, E). \end{aligned}$$

For $\nu \in \text{Spec } \tilde{\square}_{b,m}^+$, let $\{f_1^\nu, \dots, f_{d_\nu}^\nu\}$ be an orthonormal basis for $\tilde{\mathcal{E}}_{m,\nu}^+(X, E)$. Define

$$(4.19) \quad \text{Tr } P_{m,\nu}^+(x, x) := \sum_{j=1}^{d_\nu} |f_j^\nu(x)|_E^2 \in C^\infty(X),$$

which is equal to $\text{Tr } P_{m,\nu}^+(x, x) = \sum_{j=1}^{d_\nu} \langle P_{m,\nu}^+(x, x) e_j(x) | e_j(x) \rangle_E$ where $\{e_j(x)\}_j$ is any orthonormal basis of $T_x^{*0,+} X \otimes E_x$. Define $\text{Tr } P_{m,\mu}^-(x, x)$ similarly.

Clearly $d_\nu^\pm = \int_X \text{Tr } P_{m,\nu}^\pm(x, x) dv_X(x)$.

Put $\text{Tr } e^{-t\tilde{\square}_{b,m}^\pm}(x, x) := \text{Tr } P_{m,0}^\pm(x, x) + \sum_{\nu \in \text{Spec } \tilde{\square}_{b,m}^\pm, \nu > 0} e^{-\nu t} \text{Tr } P_{m,\nu}^\pm(x, x)$, so

$$(4.20) \quad \int_X \text{Tr } e^{-t\tilde{\square}_{b,m}^\pm}(x, x) dv_X(x) = \dim \tilde{\mathcal{E}}_{m,0}^\pm(X, E) + \sum_{\nu \in \text{Spec } \tilde{\square}_{b,m}^\pm, \nu > 0} e^{-\nu t} \dim \tilde{\mathcal{E}}_{m,\nu}^\pm(X, E).$$

Combining Lemma 4.5 and (4.20) gives

$$(4.21) \quad \int_X \left(\text{Tr } e^{-t\tilde{\square}_{b,m}^+}(x, x) - \text{Tr } e^{-t\tilde{\square}_{b,m}^-}(x, x) \right) dv_X(x) = \dim \text{Ker } \tilde{\square}_{b,m}^+ - \dim \text{Ker } \tilde{\square}_{b,m}^-.$$

As in Theorem 3.8 one has

$$(4.22) \quad \text{Ker } \tilde{D}_{b,m} = \text{Ker } \tilde{\square}_{b,m}^+ \subset \Omega_m^{0,+}(X, E), \quad \text{Ker } \tilde{D}_{b,m}^* = \text{Ker } \tilde{\square}_{b,m}^- \subset \Omega_m^{0,-}(X, E).$$

Put $\text{ind } \tilde{D}_{b,m} := \dim \text{Ker } \tilde{D}_{b,m} - \dim \text{Ker } \tilde{D}_{b,m}^*$. We express the index (by (4.22) and (4.21)) as

PROPOSITION 4.6. – For every $t > 0$, we have

$$(4.23) \quad \text{ind } \tilde{D}_{b,m} = \int_X \left(\text{Tr } e^{-t\tilde{\square}_{b,m}^+}(x, x) - \text{Tr } e^{-t\tilde{\square}_{b,m}^-}(x, x) \right) dv_X(x).$$

The invariance of the index is expressed by the following (some aspects on $\text{ind } D_{b,m}$ refer to Theorem 3.8).

THEOREM 4.7 (Homotopy invariance). – We have $\text{ind } D_{b,m} = \text{ind } \tilde{D}_{b,m}$.

To summarize (with Theorem 4.7, Proposition 4.6 and (3.13)) we have a McKean-Singer formula (cf. Corollary 5.16 for McKean-Singer (II)).

COROLLARY 4.8 (McKean-Singer (I)). – Fix $m \in \mathbb{Z}$. For $t > 0$, we have

$$(4.24) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = \int_X \left(\text{Tr } e^{-t\tilde{\square}_{b,m}^+}(x, x) - \text{Tr } e^{-t\tilde{\square}_{b,m}^-}(x, x) \right) dv_X(x).$$

REMARK 4.9. – To compare with the original Kohn Laplacian, a similar formula (as Corollary 4.8)

$$\sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = \int_X \left(\text{Tr } e^{-t\Box_{b,m}^+}(x, x) - \text{Tr } e^{-t\Box_{b,m}^-}(x, x) \right) dv_X(x)$$

holds. When X is not CR Kähler, it is obscure, by the experience from classical cases, to calculate the density $\text{Tr } e^{-t\Box_{b,m}^+}(x, x) - \text{Tr } e^{-t\Box_{b,m}^-}(x, x)$ with the original Kohn Laplacian. The introduction of the modified Kohn Laplacians replacing $\Box_{b,m}^\pm$ by $\tilde{\square}_{b,m}^\pm$ is expected to facilitate this calculation. But because of the unconventional asymptotic expansion of $e^{-t\tilde{\square}_{b,m}^\pm}(x, x)$ some novelty beyond the classical cases shows up (as mentioned in Introduction). It should be noted that when X is CR Kähler, $\tilde{\square}_{b,m}^\pm = \Box_{b,m}^\pm$.

To prove Theorem 4.7 in the remaining of this section, observe that it is nothing but a statement of *homotopy invariance* of index. For, with A_m a global operator (see Definition 4.3), putting $L_t = \bar{\partial}_{b,m} + \bar{\partial}_{b,m}^* + tA_m : \Omega_m^{0,+}(X, E) \rightarrow \Omega_m^{0,-}(X, E)$ for $t \in [0, 1]$, gives the homotopy between $L_0 = D_{b,m}$ and $L_1 = \tilde{D}_{b,m}$.

Remark that there have been proofs for results of this type; for instance, see [5] using heat kernel method and [7] using functional analysis method (both not exactly formulated in the above form though). To make it accessible to a wider readership, we include a (comparatively) self-contained and short proof. It is amusing to note that the Hodge theory in Section 3 is useful at certain points of our proof.

Some preparations are in order. We extend L_t by setting

$$\text{Dom } L_t = \{u \in L_m^{2,+}(X, E); \text{distribution } L_t u \in L_m^{2,-}(X, E)\}$$

so that $L_t : \text{Dom } L_t (\subset L_m^{2,+}(X, E)) \rightarrow L_m^{2,-}(X, E)$. Write L_t^* for the Hilbert space adjoint of L_t .

Let $H_m^{1,+}(X, E)$ be the completion of $\Omega_m^{0,+}(X, E)$ with respect to the Hermitian inner product

$$Q(u, v) = (u | v)_E + (\bar{\partial}_b u | \bar{\partial}_b v)_E + (\bar{\partial}_b^* u | \bar{\partial}_b^* v)_E.$$

Clearly $H_m^{1,+}(X, E) \subset \text{Dom } L_t, \forall t \in \mathbb{R}$. One can show that $H_m^{1,+}(X, E) = \text{Dom } L_t$. Let $t = 0$. Assume $L_0 f = u$ with $f, u \in L_m^{2,+}$ and also assume $f \perp \text{Ker } \square_{b,m}^+$ since for any smooth $g, f + g \in H_m^{1,+}$ iff $f \in H_m^{1,+}$. Using the partial inverse N_m in (3.2) of our Hodge theory in Section 3, we have $L_0 N_m f = N_m L_0 f = N_m u$ since L_0 commutes with N_m as in the ordinary Hodge theory. Now $L_0^* L_0 N_m f = \square_{b,m}^+ N_m f = f$ by (3.2), one has $f = L_0^* N_m u$. But $N_m u$ increases the (Sobolev) order of regularity of u by 2 and then $L_0^* N_m u$ decreases by 1, the regularity of f is of order 1. By localization, with a partition of unity, on an open subset D in place of X and by the Formula (2.14) in BRT charts $D = U \times]-\delta, \delta[$ for $\bar{\partial}_b$, it follows from the standard Gårding's inequality (e.g., [37, p. 93]) that the above $Q(\cdot, \cdot)$ is equivalent to the Sobolev norm of order one (on the m -th component). Hence $f \in H_m^{1,+}$. For $t \neq 0$, since $L_t = L_0 + tA_m$ with A_m a smooth zeroth order operator, it follows $\text{Dom } L_t = \text{Dom } L_0$.

Consider $H_0 := H_m^{1,+}(X, E) \oplus \text{Ker } L_0^*$ and $H_1 = L_m^{2,-}(X, E) \oplus \text{Ker } L_0$. Let $(\cdot | \cdot)_{H_0}$ and $(\cdot | \cdot)_{H_1}$ be inner products on H_0 and H_1 respectively, given by

$$\begin{aligned} ((f_1, g_1) | (f_2, g_2))_{H_0} &= Q(f_1, f_2) + (g_1 | g_2)_E, \\ ((\tilde{f}_1, \tilde{g}_1) | (\tilde{f}_2, \tilde{g}_2))_{H_1} &= (\tilde{f}_1 | \tilde{f}_2)_E + (\tilde{g}_1 | \tilde{g}_2)_E. \end{aligned}$$

Let $P_{\text{Ker } L_0}$ denote the orthogonal projection onto $\text{Ker } L_0$ with respect to $(\cdot | \cdot)_E$.

Let $A_t : H_0 \rightarrow H_1$ be the (continuous) linear map defined as follows. For $(u, v) \in H_0$,

$$A_t(u, v) = (L_t u + v, P_{\text{Ker } L_0} u) \in H_1.$$

LEMMA 4.10. – *There is a $r > 0$ such that $A_t : H_0 \rightarrow H_1$ is invertible, for every $0 \leq t \leq r$.*

Proof. – We first claim that

$$(4.25) \quad A_0 \text{ is invertible.}$$

If $A_0(u, v) = 0$ for some $(u, v) \in H_0$, then

$$(4.26) \quad \text{i) } L_0 u = -v \in \text{Ker } L_0^*, \quad \text{ii) } P_{\text{Ker } L_0} u = 0.$$

By (4.26)

$$(4.27) \quad (L_0 u | L_0 u)_E = -(L_0 u | v)_E = -(u | L_0^* v)_E = 0,$$

giving $u \in \text{Ker } L_0$. Hence by ii) of (4.26) we obtain $u = 0$, giving also $v = 0$ by i) of (4.26). We have proved that A_0 is injective.

We shall now prove that A_0 is surjective. Let $(a, b) \in H_1$. First we note $L_0 : \text{Dom } L_0 \rightarrow L_m^{2,-}(X, E)$ has an L^2 closed range, so

$$(4.28) \quad a = L_0 \alpha + \beta, \quad \alpha \in H_m^{1,+}(X, E), \quad \alpha \perp \text{Ker } L_0, \quad \beta \in (\text{Rang } L_0)^\perp = \text{Ker } L_0^*.$$

Another way to see (4.28) is to use $\square_{b,m}^- N_m^- + \Pi_m^- = I$ (on L_m^-) of (3.2) (for the “-” case) of Hodge theory in Section 3, and obtain $a = L_0 L_0^* N_m^- a + \Pi_m^- a$ where $\Pi_m^- a \in \text{Ker } L_0^*$ (cf. Theorem 3.8) and $L_0^* N_m^- a \in H_m^{1,+}(X, E)$ as mentioned above this lemma. In either way, by (4.28) one sees

$$A_0(\alpha + b, \beta) = (L_0(\alpha + b) + \beta, P_{\text{Ker } L_0}(\alpha + b)) = (L_0\alpha + \beta, b) = (a, b).$$

Thus A_0 is surjective. The claim (4.25) follows.

Let $A_0^{-1} : H_1 \rightarrow H_0$ be the inverse of A_0 . It follows from open mapping theorem that A_0^{-1} is continuous.

To finish the proof the following argument based on geometric series is standard. Write $A_t = A_0 + R_t$, where $R_t : H_0 \rightarrow H_1$ is continuous and there is a constant $c > 0$ such that $\|R_t u\|_{H_1} \leq ct \|u\|_{H_0}$, for $u \in H_0$. Put

$$\begin{aligned} H_t &= I - A_0^{-1} R_t + (A_0^{-1} R_t)^2 - (A_0^{-1} R_t)^3 + \cdots, \\ \tilde{H}_t &= I - R_t A_0^{-1} + (R_t A_0^{-1})^2 - (R_t A_0^{-1})^3 + \cdots. \end{aligned}$$

Since A_0^{-1} is continuous, $H_t : H_0 \rightarrow H_0$ and $\tilde{H}_t : H_1 \rightarrow H_1$ are well-defined as continuous maps for small $t \geq 0$. Moreover $A_t \circ (H_t \circ A_0^{-1}) = I$ (on H_1) and $(A_0^{-1} \circ \tilde{H}_t) \circ A_t = I$ (on H_0), giving right and left inverses of A_t for small $t \geq 0$. Hence the lemma follows. \square

For $t \in [0, 1]$ write $L_t^* : \text{Dom } L_t^* (\subset L_m^{2,-}(X, E)) \rightarrow L_m^{2,+}(X, E)$ for the adjoint of L_t with respect to $(\cdot | \cdot)_E$. Similar to L_0 and L_1 , one has $\dim \text{Ker } L_t < \infty$ and $\dim \text{Ker } L_t^* < \infty$ (with $\text{Ker } L_t \subset \Omega_m^{0,+}(X, E)$, $\text{Ker } L_t^* \subset \Omega_m^{0,-}(X, E)$).

Put $\text{ind } L_t := \dim \text{Ker } L_t - \dim \text{Ker } L_t^*$.

LEMMA 4.11. – *There is a $r_0 > 0$ such that $\text{ind } L_t = \text{ind } L_0$, for every $0 \leq t \leq r_0$.*

Proof. – Let $r > 0$ be as in Lemma 4.10. We first show that

$$(4.29) \quad \text{ind } L_0 \leq \text{ind } L_t, \quad \forall 0 \leq t \leq r.$$

Fix $0 \leq t \leq r$. We define

$$B : \text{Ker } L_t^* \oplus \text{Ker } L_0 \rightarrow \text{Ker } L_t \oplus \text{Ker } L_0^*$$

as follows. Let $(a, b) \in \text{Ker } L_t^* \oplus \text{Ker } L_0$. By Lemma 4.10,

$$A_t : H_m^{1,+}(X, E) \oplus \text{Ker } L_0^* \rightarrow L_m^{2,-}(X, E) \oplus \text{Ker } L_0$$

is invertible. There is a unique $(u, v) \in H_m^{1,+}(X, E) \oplus \text{Ker } L_0^* = \text{Dom } L_t \oplus \text{Ker } L_0^*$ such that $A_t(u, v) = (a, b)$. Let $P_{\text{Ker } L_t} : L_m^{2,+}(X, E) \rightarrow \text{Ker } L_t$ be the orthogonal projection with respect to $(\cdot | \cdot)_E$. Then the above map B is defined by

$$B(a, b) := (P_{\text{Ker } L_t} u, v) \in \text{Ker } L_t \oplus \text{Ker } L_0^*.$$

We claim that B is injective. If so, then

$$(4.30) \quad \dim \text{Ker } L_t^* + \dim \text{Ker } L_0 \leq \dim \text{Ker } L_t + \dim \text{Ker } L_0^*,$$

i.e., $\dim \text{Ker } L_0 - \dim \text{Ker } L_0^* \leq \dim \text{Ker } L_t - \dim \text{Ker } L_t^*$, yielding the desired (4.29).

For the claim that B is injective, if $B(a, b) = (0, 0)$ for some $(a, b) \in \text{Ker } L_t^* \oplus \text{Ker } L_0$, write $(u, v) \in (H_m^{1,+}(X, E) \oplus \text{Ker } L_0^*)$ such that $A_t(u, v) = (a, b)$. As $(0, 0) = B(a, b) = (P_{\text{Ker } L_t} u, v)$, $P_{\text{Ker } L_t} u = 0$ and $v = 0$. Using the definition of A_t , one has

$$(4.31) \quad A_t(u, v) = A_t(u, 0) = (L_t u, P_{\text{Ker } L_0} u) = (a, b) \in \text{Ker } L_t^* \oplus \text{Ker } L_0$$

to give $a = L_t u \in \text{Ker } L_t^*$, hence $(a|a)_E = (a|L_t u)_E = (L_t^* a|u)_E = 0$ gives $L_t u = a = 0$ so that $u \in \text{Ker } L_t$, i.e., $u = P_{\text{Ker } L_t} u$ by definition. It follows that $u = 0$ since $P_{\text{Ker } L_t} u = 0$ as just seen. With $u = 0$ and (4.31) one sees $(a, b) = (L_t u, P_{\text{Ker } L_0} u) = (0, 0)$, giving the injectivity of B .

By the same argument, $\text{ind } L_0^* \leq \text{ind } L_t^*$ for small t . By $\text{ind } L_t^* = -\text{ind } L_t$, $\text{ind } L_0 \geq \text{ind } L_t$ holds. This and (4.29) prove the lemma. \square

Proof of Theorem 4.7. – Let

$I_0 := \{r \in [0, 1]; \text{ there is an } \varepsilon > 0 \text{ such that } \text{ind } L_t = \text{ind } L_0, \forall t \in (r - \varepsilon, r + \varepsilon) \cap [0, 1]\}$.

$I_0 \neq \emptyset$ is open by Lemma 4.11. Around a limit point r_∞ of I_0 , by the same type of argument in the proof of Lemma 4.11 and Lemma 4.10 (replacing $t = 0$ by $t = r_\infty$ in H_0 , H_1 and A_0), one finds $\text{ind } L_t = \text{ind } L_{r_\infty}$ for $t \in (r_\infty - \varepsilon_0, r_\infty + \varepsilon_0)$ with some $\varepsilon_0 > 0$. This implies I_0 is closed in $[0, 1]$, so $I_0 = [0, 1]$. \square

CHAPTER 5

ASYMPTOTIC EXPANSIONS FOR THE HEAT KERNELS OF THE MODIFIED KOHN LAPLACIANS

In view of the McKean-Singer formula (Corollary 4.8), one of the goals is to calculate the local density (i.e., the term to the right of (4.24)). It consists in obtaining an asymptotic expansion for the heat kernel of the modified Kohn Laplacian (Spin^c Kohn Laplacian), to which we base our approach on two main steps. While the first step is motivated by the globally free case (see Theorem 1.2), it will be replaced by a local version within the framework of BRT trivialisations (Section 2.4). A crucial off-diagonal estimate is going to be done in this subsection (cf. Theorem 5.10). In the second step we use the *adjoint* version of the heat equation to construct a global heat kernel with an asymptotic expansion related to *local kernels*.

5.1. Heat kernels of the modified Kodaira Laplacians on BRT trivialisations

This subsection is motivated by the globally free case (cf. Theorem 1.2). Here the emphasis is made on the localization of the argument including the Spin^c structure (which is needed for explicit local formulas of the heat kernel density). An important heat kernel estimate, termed as *off-diagonal estimate*, will be established in Theorem 5.10.

It is worth remarking that in the statement and proof of Theorem 1.2, we make no use of harmonic theory. In the locally free case, by contrast, it will be an important step to relate the (modified) Kohn Laplacian to (modified) Kodaira Laplacian (see discussion after that theorem). Since these Laplacians are defined via certain *adjoints*, suitable matching of metrics involved in both Laplacians must be done as an essential step.

We will use the same notations as in Section 4. Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialisaton (with $D = U \times]-\varepsilon, \varepsilon[$, $\varepsilon > 0$ and U an open subset of \mathbb{C}^n , cf. Subsection 2.4). For $x \in D$ wrtie $z = z(x)$ and $\theta = \theta(x)$. Since E is rigid and CR, equipped

with a rigid Hermitian (fiber) metric $\langle \cdot | \cdot \rangle_E$, (as in Section 4) E descends as a (holomorphically trivial) vector bundle over U (possibly after shrinking U) and $\langle \cdot | \cdot \rangle_E$ as a Hermitian (fiber) metric on E over U .

Let $L \rightarrow U$ be a trivial (complex) line bundle with a non-trivial Hermitian fiber metric $|1|_{h^L}^2 = e^{-2\varphi}$ (φ as in the above BRT triple B). Write $(L^m, h^{L^m}) \rightarrow U$ for the m -th power of (L, h^L) . Let $\Omega^{0,q}(U, E \otimes L^m)$ be the space of $(0, q)$ forms on U with values in $E \otimes L^m$ ($q = 0, 1, 2, \dots, n$). As usual, $\Omega^{0,+}(U, E \otimes L^m)$ and $\Omega^{0,-}(U, E \otimes L^m)$ denote forms of even and odd degree.

To start with the matching of the metrics we let $\langle \cdot, \cdot \rangle$ be the Hermitian metric on CTU given by (cf. (4.1))

$$(5.1) \quad \left\langle \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k} \right\rangle = \left\langle \frac{\partial}{\partial z_j} - i \frac{\partial \varphi}{\partial z_j}(z) \frac{\partial}{\partial \theta} \mid \frac{\partial}{\partial z_k} - i \frac{\partial \varphi}{\partial z_k}(z) \frac{\partial}{\partial \theta} \right\rangle, \quad j, k = 1, 2, \dots, n.$$

$\langle \cdot, \cdot \rangle$ induces Hermitian metrics on $T^{*0,q}U$ (bundle of $(0, q)$ forms on U), denoted also by $\langle \cdot, \cdot \rangle$. These metrics induce Hermitian metrics on $T^{*0,q}U \otimes E$, still denoted by $\langle \cdot | \cdot \rangle_E$.

Let (\cdot, \cdot) be the L^2 inner product on $\Omega^{0,q}(U, E)$ induced by $\langle \cdot, \cdot \rangle, \langle \cdot | \cdot \rangle_E$, and similarly $(\cdot, \cdot)_m$ the L^2 inner product on $\Omega^{0,q}(U, E \otimes L^m)$ induced by $\langle \cdot, \cdot \rangle, \langle \cdot | \cdot \rangle_E$ and h^{L^m} .

Let $\bar{\partial}_{L^m} : \Omega^{0,q}(U, E \otimes L^m) \rightarrow \Omega^{0,q+1}(U, E \otimes L^m)$, ($q = 0, 1, 2, \dots, n-1$), be the Cauchy-Riemann operator. Let

$$\bar{\partial}_{L^m}^* : \Omega^{0,q+1}(U, E \otimes L^m) \rightarrow \Omega^{0,q}(U, E \otimes L^m)$$

be the formal adjoint of $\bar{\partial}_{L^m}$ with respect to $(\cdot, \cdot)_m$.

An essential operator that enters our picture is the following one (of Dirac type).

$$(5.2) \quad D_{B,m} (= D_{B,m}^{+-}) := \bar{\partial}_{L^m} + \bar{\partial}_{L^m}^* + A_B : \Omega^{0,+}(U, E \otimes L^m) \rightarrow \Omega^{0,-}(U, E \otimes L^m),$$

where $A_B : \Omega^{0,+}(U, E \otimes L^m) \rightarrow \Omega^{0,-}(U, E \otimes L^m)$ is as in (4.7) (replacing E there by $E \otimes L^m$ here) and

$$(5.3) \quad D_{B,m}^* : \Omega^{0,-}(U, E \otimes L^m) \rightarrow \Omega^{0,+}(U, E \otimes L^m)$$

the formal adjoint of $D_{B,m}$ with respect to $(\cdot, \cdot)_m$. (Note $D_{B,m}$ on the full $\Omega^{0,\bullet} = \Omega^{0,+} \oplus \Omega^{0,-}$ is self-adjoint; see the line below (4.6). But we prefer to use the above $D_{B,m}^*$ in the present context.) Note also L with the metric h^L depends on the choice of a BRT trivialization. However, A_B is indeed an intrinsic object; we refer to Remark 5.3 in this regard.

One has the *modified/Spin^c Kodaira Laplacian*:

$$(5.4) \quad \tilde{\square}_{B,m}^+ := D_{B,m}^* D_{B,m} : \Omega^{0,+}(U, E \otimes L^m) \rightarrow \Omega^{0,+}(U, E \otimes L^m).$$

One may define $\tilde{\square}_{B,m}^- : \Omega^{0,-}(U, E \otimes L^m) \rightarrow \Omega^{0,-}(U, E \otimes L^m)$ analogously (by starting with $D_{B,m}^{-+}$ or $D_{B,m}^*$).

The following fact appears fundamental in itself. It is instrumental to our construction of a heat kernel (cf. (5.46)) (See, however, remarks after its proof).

PROPOSITION 5.1. – *Using the above notations, let $u \in \Omega_m^{\pm}(X, E)$. On D we can write $u(z, \theta) = e^{-im\theta} \tilde{u}(z)$ for some $\tilde{u}(z) \in \Omega^{0,\pm}(U, E)$. Recall that the modified Kohn Laplacian $\square_{b,m}^{\pm}$ is given in (4.12). We write s for the local basis 1^m of L^m . Then*

$$(5.5) \quad e^{-m\varphi} \tilde{\square}_{B,m}^{\pm} (e^{m\varphi} \tilde{u} \otimes s) = (e^{im\theta} \tilde{\square}_{b,m}^{\pm} (u)) \otimes s.$$

Without any danger of confusion we may write

$$(5.6) \quad e^{-m\varphi} \tilde{\square}_{B,m}^{\pm} (e^{m\varphi} \tilde{u}) = e^{im\theta} \tilde{\square}_{b,m}^{\pm} (u).$$

Proof. – One may work out this result by explicit computations. The following gives a somewhat conceptual proof. The idea is that one continues to match the objects on U and on $D (\subset X)$. (In this way it turns out that no explicit computations of these Laplacians in local coordinates are needed.)

We define $\chi : \Omega^{0,q}(U, E) \rightarrow \Omega^{0,q}(U, E \otimes L^m)$ ($q = 0, 1, 2, \dots, n$) by $\tilde{v}(z) \rightarrow \tilde{v}(z) e^{m\varphi(z)} \otimes s(z)$ for $\tilde{v} \in \Omega^{0,q}(U, E)$. Note χ preserves the (pointwise) norms. Equivalently $\chi(e^{-m\varphi} \tilde{v}) = \tilde{v} \otimes s$.

We define $\delta \tilde{v} = \bar{\partial} \tilde{v} + m(\bar{\partial} \varphi) \wedge \tilde{v}$ for $\tilde{v} \in \Omega^{0,q}(U, E)$ where $\bar{\partial} : \Omega^{0,q}(U, E) \rightarrow \Omega^{0,q+1}(U, E)$. One may verify

$$(5.7) \quad \bar{\partial}_{L^m} \circ \chi = \chi \circ \delta \quad \text{on } \Omega^{0,q}(U, E).$$

Indeed, by $\chi(e^{-m\varphi} \bar{\partial} \tilde{u}) = \bar{\partial} \tilde{u} \otimes s = \bar{\partial}_{L^m} (\tilde{u} \otimes s)$, one sees the term to the left of (5.7): $\bar{\partial}_{L^m} \circ \chi(e^{-m\varphi} \tilde{u}) (= \bar{\partial}_{L^m} (\tilde{u} \otimes s)) = \chi(e^{-m\varphi} \bar{\partial} \tilde{u})$. Further, by using definition of δ one computes $e^{m\varphi} \delta(e^{-m\varphi} \tilde{u}) = \bar{\partial} \tilde{u}$. Then $\chi(e^{-m\varphi} \bar{\partial} \tilde{u}) = \chi(e^{-m\varphi} (e^{m\varphi} \delta(e^{-m\varphi} \tilde{u})))$, which is $\chi(\delta(e^{-m\varphi} \tilde{u}))$, giving the term to the right of (5.7) and proving (5.7).

Since χ is norm-preserving, we have also

$$(5.8) \quad \bar{\partial}_{L^m}^* \circ \chi = \chi \circ \delta^*$$

between respective adjoints. Combining (5.7) and (5.8) gives for $\square_{B,m} \equiv (\bar{\partial}_{L^m}^* + \bar{\partial}_{L^m})^2$ and $\Delta \equiv (\delta^* + \delta)^2$

$$(5.9) \quad \square_{B,m} \circ \chi = \chi \circ \Delta.$$

By (2.14) for $\bar{\partial}_b$, one computes, for $g = e^{-im\theta} \tilde{g} \in \Omega_m^{0,q}(U \times]-\varepsilon, \varepsilon[, E) = \Omega_m^{0,q}(D, E)$

$$(5.10) \quad e^{im\theta} \bar{\partial}_b (e^{-im\theta} \tilde{g}(z)) = \delta(\tilde{g}(z)).$$

Write the map $\chi_1 : \Omega_m^{0,q}(D, E) \rightarrow \Omega^{0,q}(U, E)$ for $\chi_1(g) = \chi_1(e^{-im\theta} \tilde{g}) = \tilde{g}$, equivalently, $\chi_1(g) = e^{im\theta} g$. Note χ_1 preserves the respective (pointwise) norms (cf. (5.1)).

By (5.10) one sees (with $\bar{\partial}_{b,m} = \bar{\partial}_b|_{\Omega_m^{0,q}}$)

$$(5.11) \quad \chi_1 \circ \bar{\partial}_{b,m} = \delta \circ \chi_1 \quad \text{on } \Omega_m^{0,q}(D, E).$$

By (2.5) the L^2 inner product on $\Omega_m^{0,q}(D, E)$ is clearly $2\varepsilon(\cdot, \cdot)$ with the L^2 inner product (\cdot, \cdot) on $\Omega^{0,q}(U, E)$. Thus, in the same way as (5.9) by using (5.11) we have for $\square_{b,m} \equiv (\bar{\partial}_{b,m}^* + \bar{\partial}_{b,m})^2$ (and $\Delta \equiv (\delta^* + \delta)^2$ as above)

$$(5.12) \quad \chi_1 \circ \square_{b,m} = \Delta \circ \chi_1 \quad \text{on } \Omega_m^{0,q}(D, E).$$

Combining (5.12) and (5.9) yields

$$(5.13) \quad \square_{b,m} = (\chi\chi_1)^{-1} \circ \square_{B,m} \circ (\chi\chi_1)$$

By $\chi\chi_1(e^{-im\theta}\tilde{u}) = e^{m\varphi}\tilde{u} \otimes s$ and $(\chi\chi_1)^{-1}(\tilde{v} \otimes s) = e^{-im\theta}e^{-m\varphi}\tilde{v}$, one obtains

$$(5.14) \quad (\square_{b,m}u) \otimes s = e^{-im\theta}e^{-m\varphi}(\square_{B,m}e^{m\varphi}(\tilde{u} \otimes s)) \quad \text{for } u = e^{-im\theta}\tilde{u} \in \Omega_m^{0,q}(D, E),$$

giving $e^{-m\varphi}\square_{B,m}(e^{m\varphi}\tilde{u}) = e^{im\theta}\square_{b,m}(u)$ in notation similar to (5.6).

For modified Laplacians, from the definition of the zeroth order operator $A_m : \Omega_m^{0,+}(X, E) \rightarrow \Omega_m^{0,-}(X, E)$ (see Definition 4.3), it is clear that (in notation similar to (5.6))

$$(5.15) \quad e^{-m\varphi}A_B(e^{m\varphi}\tilde{u}) = e^{im\theta}A_m(u).$$

In a way similar to (5.14) it follows by using (5.15) that

$$e^{-m\varphi}D_{B,m}(e^{m\varphi}\tilde{u}) = e^{im\theta}\tilde{D}_{b,m}(u)$$

hence easily that

$$e^{-m\varphi}\tilde{\square}_{B,m}^+(e^{m\varphi}\tilde{u}) = e^{im\theta}\tilde{\square}_{b,m}^+(u)$$

proving the proposition. \square

REMARK 5.2. – Remark that one might be led by Proposition 5.1 to reduce the study of Kohn $\tilde{\square}_{b,m}^\pm$ to that of Kodaira $\tilde{\square}_{B,m}$. Indeed such a reduction works quite well in the globally free case (see discussion following Theorem 1.2 in Introduction). In the locally free case (of S^1 action), however, a naive thought of using the Kodaira Laplacian and its associated (local) heat kernels for a better understanding of the heat kernel in Kohn's case is not directly accessible (see remarks following proof of Theorem 5.15). Namely the associated heat kernels of the two Laplacians cannot be easily linked as (5.5) seems to suggest. This reflects the fact that the associated heat kernels, rather than Laplacians themselves, are objects which are more global in nature. More in this regard will be pursued in the coming Subsection 5.2 and Section 6.

REMARK 5.3. – The definition of A_B in (5.2) depends on a BRT triple, and the same can be said with Proposition 5.1. To see that A_B has an intrinsic meaning, one uses the transformation of BRT coordinates as shown in the proof of Proposition 4.2. The geometrical construction given there shows that locally X is part of a circle bundle inside the L^* (with metric induced by that of L) over U , and the quantities such as φ , z and θ in a BRT triple are associated with geometric ones as metric for a local basis (of L), coordinates on the base U and (part of) a holomorphic coordinate on

fibers (of L^*) respectively. The transformation in these quantities with another choice of a BRT chart is nothing more than a change of holomorphic coordinates of the same line bundle. It follows that A_B is intrinsic in a proper sense. A similar explanation can be given to Proposition 5.1 too (although we do not strictly need this intrinsic property in what follows).

REMARK 5.4. – In the case of certain Riemannian foliations, it is known that the Laplacian downstairs and Laplacian upstairs (in a suitable generalized sense) can be related in spirit similar to that in our proposition above. See [59, p. 2310-2311].

As remarked in Subsection 1.6.3, to suit our purpose we will actually be considering *adjoint* heat equation and *adjoint* heat kernel first.

To proceed further, some notations are in order. Let M be a C^∞ orientable paracompact manifold of dimension m with a vector bundle F over it.

DEFINITION 5.5. – Let $A(t, x) \in C^\infty(\mathbb{R}_+ \times M, F)$. We write

$$A(t, x) \sim t^k b_{-k}(x, t) + t^{k+1} b_{-k+1}(x, t) + t^{k+2} b_{-k+2}(x, t) + \dots \text{ as } t \rightarrow 0^+,$$

$$b_s(x, t) \in C^\infty(M, F) \text{ a possibly } t\text{-dependent smooth function,}$$

$s = -k, -k+1, -k+2, \dots$, for $k \in \mathbb{Z}$, provided that for every $\ell \in \mathbb{N}$, there is a constant $M_1(m, \ell)$ such that for every compact set $K \subseteq M$, every $M_0 \in \mathbb{N}$ with $M_0 \geq M_0(m)$ for some $M_0(m)$ ($m = \dim M$), there are constants $C_{\ell, K, M_0} > 0$, $\varepsilon_0 > 0$ independent of t such that

$$\left| A(t, x) - \sum_{j=0}^{M_0} t^{k+j} b_{-k+j}(x, t) \right|_{C^\ell(K)} \leq C_{\ell, K, M_0} t^{M_0 - M_1(m, \ell)}, \quad \forall 0 < t < \varepsilon_0.$$

REMARK 5.6. – In the important case of the heat kernel $p_t(x, y)$ of a generalized Laplacian on a compact Riemannian manifold B of dimension β , $M = B \times B$ is of dimension $m = 2\beta$ and $k = -\frac{\beta}{2}$. One can take $M_0(m) = \lceil \frac{\beta}{2} \rceil + 1$ and $M_1(m, \ell) = \frac{\beta + \ell}{2}$. See [5, Theorem 2.30]. In this case $b_s(t, x)$ for all s can be taken to be independent of t .

The novelty above is that b_s could be nontrivially dependent on t (in contrast to the conventional case of an asymptotic expansion for heat kernels).

Let $T^{*0,+}U$ and $T^{*0,-}U$ denote forms of even degree and odd degree in $T^{*0,\bullet}U$, respectively as before. If $T(z, w) \in (T^{*0,+}U \otimes E) \boxtimes (T^{*0,+}U \otimes E)^*|_{(z,w)}$, write $|T(z, w)|$ for the standard pointwise matrix norm of $T(z, w)$ induced by $\langle \cdot, \cdot \rangle$ and $\langle \cdot | \cdot \rangle_E$.

Suppose $G(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{*0,+}U \otimes E) \boxtimes (T^{*0,+}U \otimes E)^*)$. As usual, we denote $G(t) : \Omega_c^{0,+}(U, E) \rightarrow \Omega_c^{0,+}(U, E)$ (resp. $G'(t)$) the continuous operator associated with the kernel $G(t, z, w)$ (resp. $\frac{\partial G(t, z, w)}{\partial t}$). ($\Omega_c^{0,+}(U, E)$ denotes elements of compact support in U).

We are now ready to consider the heat operators associated with $\tilde{\square}_{B,m}^+$ and $\tilde{\square}_{B,m}^-$ in an *adjoint* version. By using the Dirichlet heat kernel construction (see [38] or [14]) we can obtain the theorem stated in the following form.

PROPOSITION 5.7. – *There exists an*

$$A_{B,+m}(t, z, w) =: A_{B,+}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{*0,+}U \otimes E) \boxtimes (T^{*0,+}U \otimes E)^*)$$

such that

$$(5.16) \quad \begin{aligned} \lim_{t \rightarrow 0^+} A_{B,+}(t) &= I \text{ in } \mathcal{D}'(U, T^{*0,+}U \otimes E), \\ A'_{B,+}(t)u + A_{B,+}(t)(\tilde{\square}_{B,m}^+ u) &= 0, \quad \forall u \in \Omega_c^{0,+}(U, E), \quad \forall t > 0, \end{aligned}$$

and $A_{B,+}(t, z, w)$ satisfies the following: (I) For every compact set $K \Subset U$ and every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0^n$, every $\gamma \in \mathbb{N}_0$, there are constants $C_{\gamma, \alpha_1, \alpha_2, \beta_1, \beta_2, K} > 0$, $\varepsilon_0 > 0$ and $P \in \mathbb{N}$ independent of t such that

$$(5.17) \quad \left| \partial_t^\gamma \partial_z^{\alpha_1} \partial_{\bar{z}}^{\alpha_2} \partial_w^{\beta_1} \partial_{\bar{w}}^{\beta_2} A_{B,+}(t, z, w) \right| \leq C_{\gamma, \alpha_1, \alpha_2, \beta_1, \beta_2, K} t^{-P} e^{-\varepsilon_0 \frac{|z-w|^2}{t}}, \quad \forall (t, z, w) \in \mathbb{R}_+ \times K \times K.$$

(II) Let $g \in \Omega_c^{0,+}(U, E)$. For every $\alpha_1, \alpha_2 \in \mathbb{N}_0^n$ and every compact set $K \Subset U$, there is a $C_{\alpha_1, \alpha_2, K} > 0$ independent of t such that

$$(5.18) \quad \begin{aligned} \sup \{ |\partial_z^{\alpha_1} \partial_{\bar{z}}^{\alpha_2} (A_{B,+}(t)g)(z)|; z \in K \} \\ \leq C_{\alpha_1, \alpha_2, K} \sum_{\beta_1, \beta_2 \in \mathbb{N}_0^n, |\beta_1| + |\beta_2| \leq |\alpha_1| + |\alpha_2|} \sup \left\{ \left| \partial_z^{\beta_1} \partial_{\bar{z}}^{\beta_2} g(z) \right|; z \in U \right\}. \end{aligned}$$

(III) $A_{B,+}(t, z, w)$ admits an asymptotic expansion in the following sense (see Definition 5.5 for \sim). For some $K_{B,+}(t, z, w)$

$$(5.19) \quad \begin{aligned} A_{B,+}(t, z, w) &= e^{-\frac{h_+(z,w)}{t}} K_{B,+}(t, z, w), \\ K_{B,+}(t, z, w) &\sim t^{-n} b_n^+(z, w) + t^{-n+1} b_{n-1}^+(z, w) + \dots \\ &\quad + b_0^+(z, w) + t b_{-1}^+(z, w) + \dots \quad \text{as } t \rightarrow 0^+, \\ b_s^+(z, w) (= b_{s,m}^+(z, w)) &\in C^\infty(U \times U, (T^{*0,+}U \otimes E) \boxtimes (T^{*0,+}U \otimes E)^*), \end{aligned}$$

$s = n, n-1, n-2, \dots$, where $h_+(z, w) \in C^\infty(U \times U, \mathbb{R}_+)$ with $h_+(z, z) = 0$ for every $z \in U$ and for every compact set $K \Subset U$, there is a constant $C_K > 1$ such that $\frac{1}{C_K} |z-w|^2 \leq h_+(z, w) \leq C_K |z-w|^2$.

In (5.16) with $\tilde{\square}_{B,m}^-$ in place of $\tilde{\square}_{B,m}^+$, corresponding statements (with $A_{B,-}, K_{B,-}$ etc.) for $\tilde{\square}_{B,m}^-$ hold true as well.

REMARK 5.8. – One may use a (weaker) version in the sense of *asymptotic heat kernel*. More precisely, it was shown in [24, p. 96] that there is a

$$A(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{*0,+}U \otimes E) \boxtimes (T^{*0,+}U \otimes E)^*)$$

with the property that $A(t, z, w)$ satisfies (5.17)-(5.19) and

$$(5.20) \quad \lim_{t \rightarrow 0^+} A_B(t) = I \text{ in } \mathcal{D}'(U, T^{*0,+}U \otimes E),$$

$$A'_B(t)u + A_B(t)(\tilde{\square}_{B,m}^+ u) = Q(t)u, \quad \forall u \in \Omega_c^{0,+}(U, E), \quad \forall t > 0,$$

where $Q(t)$ is a smoothing operator on U with distribution kernel $Q(t, x, y) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{*0,+}U \otimes E) \boxtimes (T^{*0,+}U \otimes E)^*)$ and $Q_t = O(t^{+\infty})$. The notation $Q_t = O(t^{+\infty})$ means that for every $\ell \in \mathbb{N}$, every compact set $K \Subset U$ and every $N > 0$ there is a $\varepsilon > 0$ and a constant $C_{\ell,K,N} > 0$ such that

$$(5.21) \quad |Q(t, x, y)|_{C^\ell(K \times K)} \leq C_{\ell,K,N} t^N, \quad \text{for every } 0 < t < \varepsilon.$$

The asymptotic heat kernels are not unique. Let $\hat{A}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^{*0,+}U \otimes E) \boxtimes (T^{*0,+}U \otimes E)^*)$ and assume that $\hat{A}(t, z, w)$ satisfies (5.17)-(5.19) and (5.20). Then, $\hat{A}(t, z, w) - A(t, z, w) = O(t^{+\infty})$ in the sense of (5.21) and the symbols $b_j^+(z, w)$, $j = n, n-1, \dots$, in the expansion (5.19) are unique (cf. [24, p. 96]).

We are interested in calculating $\text{Tr } b_s^+(z, z) - \text{Tr } b_s^-(z, z)$ ($s = n, n-1, \dots, 0$) (where $\text{Tr } b_s^\pm(z, z) = \sum_j \langle b_s^\pm(z, z) e_j \mid e_j \rangle_E$ for any orthonormal frame e_j of $T_z^{*0,+}U \otimes E_z$). The idea relies on *Lichnerowicz formulas* for (modified/Spin^c Kodaira Laplacians) $\tilde{\square}_{B,m}^+$ and $\tilde{\square}_{B,m}^-$ (cf. Theorem 1.3.5 and Theorem 1.4.5 in [50]) so that the (by now standard) rescaling technique in [5] and [24] can apply well.

To state the result precisely, we introduce some notations. Let ∇^{TU} be the Levi-Civita connection on $\mathbb{C}TU$ with respect to $\langle \cdot, \cdot \rangle$. Let $P_{T^{1,0}U}$ be the natural projection from $\mathbb{C}TU$ onto $T^{1,0}U$. $\nabla^{T^{1,0}U} := P_{T^{1,0}U} \nabla^{TU}$ is a connection on $T^{1,0}U$. Let $\nabla^{E \otimes L^m}$ be the (Chern) connection on $E \otimes L^m \rightarrow U$ (induced by $\langle \cdot, \cdot \rangle_E$ and h^{L^m} , see Theorem 2.12). Let $\Theta(\nabla^{T^{1,0}U}, T^{1,0}U) (\in C^\infty(U, \Lambda^2(\mathbb{C}T^*U) \otimes \text{End}(T^{1,0}U)))$ and $\Theta(\nabla^{E \otimes L^m}, E \otimes L^m) (\in C^\infty(U, \Lambda^2(\mathbb{C}T^*U) \otimes \text{End}(E \otimes L^m)))$ be the associated curvatures. As in complex geometry, put

$$\text{Td}(\nabla^{T^{1,0}U}, T^{1,0}U) = e^{\text{Tr}(h(\frac{i}{2\pi} \Theta(\nabla^{T^{1,0}U}, T^{1,0}U)))}, \quad h(z) = \log\left(\frac{z}{1-e^{-z}}\right),$$

$$\text{ch}(\nabla^{E \otimes L^m}, E \otimes L^m) = \text{Tr}\left(\tilde{h}\left(\frac{i}{2\pi} \Theta(\nabla^{E \otimes L^m}, E \otimes L^m)\right)\right), \quad \tilde{h}(z) = e^z.$$

Then the above calculation leads to the following.

$$(5.22) \quad \left(\text{Tr } b_s^+(z, z) - \text{Tr } b_s^-(z, z)\right) = 0, \quad s = n, n-1, \dots, 1,$$

$$\left(\text{Tr } b_0^+(z, z) - \text{Tr } b_0^-(z, z)\right) dv_U(z) = \left[\text{Td}(\nabla^{T^{1,0}U}, T^{1,0}U) \wedge \text{ch}(\nabla^{E \otimes L^m}, E \otimes L^m)\right]_{2n}(z),$$

$\forall z \in U$, where $[\dots]_{2n}$ denotes the $2n$ -form part.

As the calculation to be performed here is almost entirely the same as in the standard case, we omit the detail.

Let ∇^{TX} be the Levi-Civita connection on TX with respect to $\langle \cdot | \cdot \rangle$ and ∇^E the connection on E associated with $\langle \cdot | \cdot \rangle_E$ (cf. Theorem 2.12). In similar notation as above $\nabla^{T^{1,0}X} := P_{T^{1,0}X} \nabla^{TX}$ is a connection on $T^{1,0}X$.

Since $\nabla^{T^{1,0}X}$ and ∇^E are rigid, in view of compatibility of metrics (and connections) in (5.1) and Theorem 2.12, one sees that $\forall (z, \theta) \in D$ (for ω_0 see lines below Definition 2.1):

$$(5.23) \quad \begin{aligned} \mathrm{Td}(\nabla^{T^{1,0}U}, T^{1,0}U)(z) &= \mathrm{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X)(z, \theta) \\ \mathrm{ch}(\nabla^{E \otimes L^m}, E \otimes L^m)(z) &= \left(\mathrm{ch}_b(\nabla^E, E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \right)(z, \theta) \end{aligned}$$

and

$$(5.24) \quad \begin{aligned} \left[\mathrm{Td}(\nabla^{T^{1,0}U}, T^{1,0}U) \wedge \mathrm{ch}(\nabla^{E \otimes L^m}, E \otimes L^m) \right]_{2n}(z) \wedge d\theta \\ = \left[\mathrm{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge \mathrm{ch}_b(\nabla^E, E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0 \right]_{2n+1}(z, \theta). \end{aligned}$$

To sum up we arrive at the following (by (5.22), (5.24) and $dv_U \wedge d\theta = dv_X$ on D cf. (2.5))

PROPOSITION 5.9. – *With the notations above, we have*

$$(5.25) \quad \begin{aligned} \left(\mathrm{Tr} b_0^+(z, z) - \mathrm{Tr} b_0^-(z, z) \right) dv_X(z, \theta) \\ = \left[\mathrm{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge \mathrm{ch}_b(\nabla^E, E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0 \right]_{2n+1}(z, \theta), \quad \forall (z, \theta) \in D. \end{aligned}$$

To state the final technical result of this subsection, we first identify $T_{z_2}^{*0, \bullet} U \otimes E_{z_2}$ with $T_{z_1}^{*0, \bullet} U \otimes E_{z_1}$ (by parallel transport along geodesics joining $z_1, z_2 \in U$), so we can identify $T \in (T^{*0, \bullet} U \otimes E) \boxtimes (T^{*0, \bullet} U \otimes E)^* \big|_{(z_1, z_2)}$ with an element in $\mathrm{End}(T_{z_1}^{*0, \bullet} U \otimes E_{z_1})$. With this identification, write

$$(5.26) \quad \mathrm{Tr}_{z_1} T := \sum_{j=1}^d \langle T e_j | e_j \rangle_E,$$

where e_1, \dots, e_d is an orthonormal frame of $T_{z_1}^{*0, \bullet} U \otimes E_{z_1}$.

In the proof of Theorem 1.10 (see Theorem 6.4), somewhat surprisingly, as deviated from the classical case, we need to estimate the off-diagonal terms $\mathrm{Tr}_z b_s^+(z, w) - \mathrm{Tr}_z b_s^-(z, w)$ for each s . For this, the following can be considered as another application of the rescaling technique (and an identity in Berenzin integral as usual).

THEOREM 5.10 (Off-diagonal estimate). – *With the notations above, we have*

$$(5.27) \quad \mathrm{Tr}_z b_s^+(z, w) - \mathrm{Tr}_z b_s^-(z, w) = O(|z - w|^{2s})$$

locally uniformly on $U \times U$, $s = n, n-1, \dots, 1$.

Proof. – Recall that E is (holomorphically) trivial on U . Let e_1, \dots, e_{2n} be an orthonormal basis for T_0^*U . For $f \in T_0^*U$, let $c(f) \in \text{End}(T_0^{*0, \bullet}U)$ be the natural Clifford action of f (see (4.2) or [5]). As usual, for every strictly increasing multi-index $J = (j_1, \dots, j_q)$ we set $|J| := q$, $e_J := e_{j_1} \wedge \dots \wedge e_{j_q}$ and $c(e_J) = c(e_{j_1}) \cdots c(e_{j_q})$. For $T \in \text{End}(T_0^{*0, \bullet}U)$, we can always write $T = \sum'_{|J| \leq 2n} c(e_J)T_J$ ($T_J \in \mathbb{C}$), where \sum' denotes the summation over strictly increasing multiindices. For $k \leq 2n$, we put

$$(5.28) \quad [T]_k := \sum'_{|J|=k} T_J e_J \quad (\in \mathbb{C}T_0^{*k}U).$$

and a similar expression for $[T]$ (without the subscript k). We identify T with $[T]$ without any danger of confusion. We say that $\text{ord } T \leq k$ if $T_J = 0$, for all $|J| > k$, and $\text{ord } T = k$ if $\text{ord } T \leq k$ and $[T]_k \neq 0$.

A crucial result for our need here is an identity in Berenzin integral (see [5, Proposition 3.21, Definition 3.4 and (1.28)]) which asserts that if $\text{ord } T < 2n$ then $\text{STr } T = 0$ (see (1.13) for the definition of supertrace there) and

$$(5.29) \quad \text{STr } T = (-2i)^{2n} \text{STr } T_{J_0} c(e_{J_0}), \quad J_0 = (1, 2, \dots, 2n).$$

Recall the identification $T_x^{*0, \bullet}U \cong T_0^{*0, \bullet}U$ just mentioned above the theorem, so that a smooth function $F(x) \in (T^{*0, \bullet}U) \boxtimes (T^{*0, \bullet}U)^*|_{(0,x)}$ is identified with a function $x \rightarrow F(x) \in \text{End}(T_0^{*0, \bullet}U)$, giving a Taylor expansion

$$F(x) = \sum_{\alpha \in \mathbb{N}_0^{2n}, |\alpha| \leq P} x^\alpha F_\alpha + O(|x|^{P+1}), \quad F_\alpha \in \text{End}(T_0^{*0, \bullet}U).$$

We are ready to apply Getzler’s rescaling technique to off-diagonal estimates. Consider $A_B(t, x, y) \equiv A_B(t, z, w) := A_{B,+}(t, z, w) \oplus A_{B,-}(t, z, w)$ (cf. Proposition 5.7) and let $\chi \in C_c^\infty(U)$ with $\chi = 1$ near $z = 0$. Let

$$(5.30) \quad r(u, t, x) := \sum_{k=1}^{2n} u^{-\frac{k}{2}+n} [\chi(\sqrt{u}x) A_B(ut, 0, \sqrt{u}x)]_k.$$

Note that A_B is actually identified with $[A_B]$ similarly to the case of T above, so that the k -form part ($k > n$) of (5.30) makes sense.

It is well-known that (see [5]) $\lim_{u \rightarrow 0} r(u, t, x) = g(t, x) \in C^\infty(\mathbb{R}_+ \times \mathbb{C}^n, \mathbb{C}T^{*\bullet}\mathbb{C}^n)$ in C^∞ -topology locally uniformly ($\mathbb{C}T^{*\bullet}\mathbb{C}^n = \bigoplus_{k=0}^{k=2n} \Lambda^k \mathbb{C}T^{*\bullet}\mathbb{C}^n$). In particular, $\lim_{u \rightarrow 0} r(u, 1, x)_{2n} = g(1, x)_{2n}$ in C^∞ -topology locally uniformly, for their $2n$ -form parts.

Let $b_s(z, w) := b_s^+(z, w) \oplus b_s^-(z, w)$, $s = n, n-1, \dots$ (cf. (5.19)). One sees

$$(5.31) \quad r(u, 1, x)_{2n} = e^{-\frac{h_+(0, \sqrt{u}x)}{u}} \chi(\sqrt{u}x) \left(u^{-n} [b_n(0, \sqrt{u}x)]_{2n} + u^{-n+1} [b_{n-1}(0, \sqrt{u}x)]_{2n} + \dots \right).$$

Since $\lim_{u \rightarrow 0} e^{-\frac{h_+(0, \sqrt{ux})}{u}}$ converges to a smooth function in C^∞ -topology locally uniformly on \mathbb{C}^n (see (5.19) in Proposition 5.7), we deduce that

$$(5.32) \quad \lim_{u \rightarrow 0} \left(u^{-n} [b_n(0, \sqrt{ux})]_{2n} + u^{-n+1} [b_{n-1}(0, \sqrt{ux})]_{2n} + \dots \right) = \hat{g}(x) \in C^\infty(\mathbb{C}^n, \mathcal{CT}^{*\bullet} \mathbb{C}^n)$$

in C^∞ -topology locally uniformly. Fix $P \gg 1$. Write

$$(5.33) \quad \hat{g}(x) = \sum_{\alpha \in \mathbb{N}_0^{2n}, |\alpha| \leq P} \hat{g}_\alpha x^\alpha + O(|x|^{P+1}).$$

and for each $s = n, n-1, \dots$,

$$(5.34) \quad b_s(0, x) = \sum_{\alpha \in \mathbb{N}_0^{2n}, |\alpha| \leq P} b_{s,\alpha} x^\alpha + O(|x|^{P+1}).$$

Hence

$$(5.35) \quad [b_s(0, \sqrt{ux})]_{2n} = \sum_{\alpha \in \mathbb{N}_0^{2n}, |\alpha| \leq P} u^{\frac{|\alpha|}{2}} [b_{s,\alpha}]_{2n} x^\alpha + u^{\frac{P+1}{2}} O(|x|^{P+1}), \quad s = n, n-1, \dots,$$

and from (5.32), (5.33) and (5.35) it follows that for every $\alpha \in \mathbb{N}_0^{2n}$

$$(5.36) \quad \lim_{u \rightarrow 0} \left(u^{-n+\frac{|\alpha|}{2}} [b_{n,\alpha}]_{2n} + u^{-n+1+\frac{|\alpha|}{2}} [b_{n-1,\alpha}]_{2n} + \dots \right) = \hat{g}_\alpha.$$

With (5.36) we conclude

$$(5.37) \quad [b_{n,\alpha}]_{2n} = 0, \quad \forall |\alpha| < 2n.$$

Combining (5.37) and (5.34), we see

$$\mathrm{Tr}_0 b_n^+(0, w) - \mathrm{Tr}_0 b_n^-(0, w) = O(|w|^{2n}).$$

We can repeat the method above for the second leading term, and deduce similarly

$$\mathrm{Tr}_0 b_s^+(0, w) - \mathrm{Tr}_0 b_s^-(0, w) = O(|w|^{2s}), \quad s = n-1, n-2, \dots, 1.$$

The theorem follows. □

5.2. Heat kernels of the modified Kohn Laplacians (Spin^c Kohn Laplacians)

Based on Proposition 5.1, one is tempted to patch up the local heat kernels of the modified Kodaira Laplacian constructed in Proposition 5.7 to form a global heat kernel for the modified Kohn Laplacian. This is no problem in the globally free case (of the S^1 action). In the locally free case, however, some delicate points arise as the relation of the two Laplacians given in the above proposition is, by nature, a local property, whereas the heat kernels are global objects. See discussions after proof of Proposition 5.1 and of Theorem 5.15 with (5.56) for more.

As remarked in Subsection 1.6.3, if we use the *adjoint* version of the original heat equation, it becomes more effective to go over the desired process of patching up. It

is worth noting that an important role, mostly unseen traditionally, is played by the projection Q_m^\pm in our situation.

Assume $X = D_1 \cup D_2 \cup \dots \cup D_N$ with each $B_j := (D_j, (z, \theta), \varphi_j)$ a BRT trivialization. A slightly more complicated set up is as follows. Write for each j , $D_j = U_j \times]-\delta_j, 2\tilde{\delta}_j[\subset \mathbb{C}^n \times \mathbb{R}$, $\delta_j > 0$, $\tilde{\delta}_j > 0$, $U_j = \{z \in \mathbb{C}^n; |z| < \gamma_j\}$. Put $\hat{D}_j = \hat{U}_j \times]-\frac{\delta_j}{2}, \frac{\tilde{\delta}_j}{2}[$, $\hat{U}_j = \{z \in \mathbb{C}^n; |z| < \frac{\gamma_j}{2}\}$. We suppose $X = \hat{D}_1 \cup \hat{D}_2 \cup \dots \cup \hat{D}_N$.

Here are some cut-off functions, all with values in $[0, 1]$; the choice is adapted to BRT trivializations.

i) $\chi_j(x) \in C_c^\infty(\hat{D}_j)$ with $\sum_{j=1}^N \chi_j = 1$ on X . Put

$$A_j = \left\{ z \in \hat{U}_j; \text{there is a } \theta \in]-\frac{\delta_j}{2}, \frac{\tilde{\delta}_j}{2}[\text{ such that } \chi_j(z, \theta) \neq 0 \right\}.$$

ii) $\tau_j(z) \in C_c^\infty(\hat{U}_j)$ with $\tau_j \equiv 1$ on some neighborhood W_j of A_j .

iii) $\sigma_j \in C_c^\infty(]-\frac{\delta_j}{2}, \frac{\tilde{\delta}_j}{2}[)$ with $\int_{-\delta_j/2}^{\tilde{\delta}_j/2} \sigma_j(\theta) d\theta = 1$.

iv) $\hat{\sigma}_j \in C_c^\infty(]-\delta_j, \tilde{\delta}_j[)$ such that $\hat{\sigma}_j = 1$ on some neighborhood of $\text{Supp } \sigma_j$ and $\hat{\sigma}_j(\theta) = 1$ if there exists z such that $(z, \theta) \in \text{Supp } \chi_j$.

Write $x = (z, \theta)$, $y = (w, \eta) \in \mathbb{C}^n \times \mathbb{R}$. We are going to lift many objects in the preceding subsection defined on U_j to the ones defined on \hat{D}_j via the above cut-off functions.

Let $A_{B_j,+}(t, z, w)$, $K_{B_j,+}$, $h_{j,+}(z, w)$ and $b_{j,s}^+$ ($s = n, n-1, \dots$) be as in Proposition 5.7 and (5.19). Slightly tediously, we put

(5.38)

$$\begin{aligned} H_j(t, x, y) &= H_{j,+}(t, x, y) = \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w) \sigma_j(\eta) \\ G_j(t, x, y) &= G_{j,+}(t, x, y) = \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w). \end{aligned}$$

and (the last two equations are from (5.19))

(5.39)

$$\begin{aligned} \hat{K}_{j,+}(t, x, y) &= \hat{K}_j(t, x, y) = \chi_j(x) e^{-m\varphi_j(z) - im\theta} K_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w) \sigma_j(\eta) \\ \hat{h}_{j,+}(x, y) &= \hat{\sigma}_j(\theta) h_{j,+}(z, w) \hat{\sigma}_j(\eta) \in C_c^\infty(D_j), \quad x = (z, \theta), \quad y = (w, \eta) \\ \hat{b}_{j,s}^+(x, y) &= \chi_j(x) e^{-m\varphi_j(z) - im\theta} b_{j,s}^+(z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w) \sigma_j(\eta), \quad s = n, n-1, \dots \\ \hat{\beta}_{j,s}^+(x, y) &= \chi_j(x) e^{-m\varphi_j(z) - im\theta} b_{j,s}^+(z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w), \quad s = n, n-1, \dots \\ A_{B_j,+}(t, z, w) &= e^{-\frac{h_+(z,w)}{t}} K_{B_j,+}(t, z, w) \\ K_{B_j,+}(t, z, w) &\sim t^{-n} b_{j,n}^+(z, w) + t^{-n+1} b_{j,n-1}^+(z, w) + \dots \\ &\quad + b_{j,0}^+(z, w) + t b_{j,-1}^+(z, w) + \dots \quad \text{as } t \rightarrow 0^+. \end{aligned}$$

Remark that these expressions, apart from cut-off functions, are mainly motivated by the Formulas (5.5), (5.6) of Proposition 5.1. Let $H_j(t)$ be the continuous operator

associated with $H_j(t, x, y)$, for which we put down the expression for later use (cf. (5.44) and (5.46))

(5.40)

$$H_j(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E),$$

$$u \rightarrow \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w) \sigma_j(\eta) u(y) dv_X(y).$$

Consider the patched up kernel (recall that Q_m is the projection on the m -th Fourier component, cf. (4.16))

$$(5.41) \quad \Gamma(t) := \sum_{j=1}^N H_j(t) \circ Q_m : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E)$$

and let $\Gamma(t, x, y) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*)$ be the distribution kernel of $\Gamma(t)$.

For an explicit expression, one sees (using Q_m of (4.17)) that

$$(5.42) \quad \begin{aligned} \Gamma(t, x, y) &= \frac{1}{2\pi} \sum_{j=1}^N \int_{-\pi}^{\pi} ((e^{-iu})_y^* H_j)(t, x, y) e^{-imu} du \\ &= \frac{1}{2\pi} \sum_{j=1}^N \int_{-\pi}^{\pi} e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ y)}{t}} ((e^{-iu})_y^* \hat{K}_j)(t, x, y) e^{-imu} du \\ &\sim t^{-n} a_n^+(t, x, y) + t^{-n+1} a_{n-1}^+(t, x, y) + \dots \quad \text{as } t \rightarrow 0^+, \end{aligned}$$

where we have written

$$(5.43) \quad a_s^+(t, x, y) = \frac{1}{2\pi} \sum_{j=1}^N \int_{-\pi}^{\pi} e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ y)}{t}} ((e^{-iu})_y^* \hat{b}_{j,s}^+)(x, y) e^{-imu} du,$$

$s = n, n-1, n-2, \dots$ Note that $a_s^+(t, x, y)$ and $\hat{b}_{j,s}^+(x, y)$ depend on m . For the initial condition of $\Gamma(t, x, y)$, one has the following.

LEMMA 5.11. – *We have*

$$\lim_{t \rightarrow 0^+} \Gamma(t)u = Q_m u \quad (\text{on } \mathcal{D}'(X, T^{*0,+} X \otimes E)),$$

for $u \in \Omega^{0,+}(X, E)$.

Proof. – For $u \in \Omega^{0,+}(X, E)$, $Q_m u \in \Omega_m^{0,+}(X, E)$, $Q_m u|_{D_j}$ can be expressed as $e^{-im\eta} v_j(w)$ for some $v_j(w) \in \Omega^{0,+}(U_j, E)$. With (5.40) we find (note $A_{B_j,+}(t) = I$

as $t \rightarrow 0$ and $dv_{D_j} = dv_{U_j}d\eta$ by (2.5))

(5.44)

$$\begin{aligned}
\lim_{t \rightarrow 0^+} H_j(t)Q_m u &= \lim_{t \rightarrow 0^+} \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \\
&\quad \cdot \tau_j(w) \sigma_j(\eta) e^{-im\eta} v_j(w) dv_{U_j}(w) d\eta \\
&= \lim_{t \rightarrow 0^+} \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w)} \tau_j(w) v_j(w) dv_{U_j}(w) \\
&= \chi_j(x) e^{-m\varphi_j(z) - im\theta} e^{m\varphi_j(z)} \tau_j(z) v_j(z) \\
&= \chi_j e^{-im\theta} v_j = \chi_j Q_m u.
\end{aligned}$$

With the above, the lemma follows from (5.41) and $\sum_j \chi_j = 1$. \square

$\Gamma(t)$ satisfies an *adjoint type* heat equation *asymptotically* in the following sense (cf. [24, p. 96]).

LEMMA 5.12. – We consider $\tilde{\square}_{b,m}^+ \circ Q_m$ still denoted by $\tilde{\square}_{b,m}^+$. $\Gamma(t, x, y)$ satisfies

$$\Gamma'(t)u + \Gamma(t)\tilde{\square}_{b,m}^+ u = R(t)u, \quad \forall u \in \Omega^{0,+}(X, E),$$

where $R(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E)$ is the continuous operator with distribution kernel $R(t, x, y) (\in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*))$, which satisfies the following. For every $\ell \in \mathbb{N}_0$, there exists an $\varepsilon_0 > 0$, $C_\ell > 0$ independent of t such that

$$(5.45) \quad \|R(t, x, y)\|_{C^\ell(X \times X)} \leq C_\ell e^{-\frac{\varepsilon_0}{t}}, \quad \forall t \in \mathbb{R}_+.$$

Proof. – As in the preceding lemma let $u \in \Omega_m^{0,+}(X, E)$ and write $u|_{D_j} = e^{-im\eta} v_j(w)$ for some $v_j(w) \in \Omega^{0,+}(U_j, E)$ on D_j . By this, (5.5), (5.16) and (5.40) it is a bit tedious but straightforward to compute

(5.46)

$$\begin{aligned}
&H_j'(t)u + H_j(t)\tilde{\square}_{b,m}^+ u \\
&= \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A'_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w) \sigma_j(\eta) u(y) dv_X(y) \\
&\quad + \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w) \sigma_j(\eta) (\tilde{\square}_{b,m}^+ u)(y) dv_X(y) \\
&= \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A'_{B_j,+}(t, z, w) \tau_j(w) e^{m\varphi_j(w)} v_j(w) dv_{U_j}(w) \\
&\quad + \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) \tau_j(w) (\tilde{\square}_{B_j,m}^+ (e^{m\varphi_j} v_j))(w) dv_{U_j}(w)
\end{aligned}$$

$$\begin{aligned}
&= \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A'_{B_j,+}(t, z, w) \tau_j(w) e^{m\varphi_j(w)} v_j(w) dv_{U_j}(w) \\
&\quad + \int \chi_j(x) e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) (\tilde{\square}_{B_j,m}^+ (\tau_j e^{m\varphi_j} v_j))(w) dv_{U_j}(w) \\
&\quad + \int \chi_j(x) S_j(t, x, w) v_j(w) dv_{U_j}(w) \\
&= \int \chi_j(x) S_j(t, x, w) v_j(w) dv_{U_j}(w) = \int \chi_j(x) S_j(t, x, w) e^{im\eta} \sigma_j(\eta) u(y) dv_X(y),
\end{aligned}$$

where

$$\begin{aligned}
S_j(t, x, w) &= e^{-m\varphi_j(z) - im\theta} A_{B_j,+}(t, z, w) [\tau_j(w), \tilde{\square}_{B_j,m}^+] e^{m\varphi_j(w)} \\
&\in C_c^\infty(\mathbb{R}_+ \times D_j \times U_j, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*).
\end{aligned}$$

Note $\tau_j(z) = 1$ for (z, θ) in some small neighborhood of $\text{Supp } \chi_j$. One sees that $\chi_j(x) S_j(t, x, w) = 0$ if (x, w) is in some small neighborhood of (z, z) . Hence by using (5.17) for (5.46) (on $|z - w|$ away from zero), we conclude that for every $\ell \in \mathbb{N}_0$, there is an $\varepsilon > 0$ independent of t such that

$$(5.47) \quad \|\chi_j(x) S_j(t, x, w)\|_{C^\ell(X \times X)} \leq C_\ell e^{-\frac{\varepsilon}{t}}, \quad \forall t \in \mathbb{R}_+.$$

Put $\tilde{R}(t, x, y) := \sum_{j=1}^N \chi_j S_j(t, x, w) e^{im\eta} \sigma_j(\eta) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*)$ and set

$$R(t, x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{((e^{-iu})_y^* R)}(t, x, y) e^{-imu} du.$$

Let $R(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E)$ be the continuous operator with distribution kernel $R(t, x, y)$. Note that $R(t) = R(t) \circ Q_m$ (cf. (5.42) and (5.41)). By (5.47), $R(t, x, y)$ satisfies (5.45) and by (5.46) and (5.41), one sees $\Gamma'(t)u + \Gamma(t)\tilde{\square}_{b,m}^+ u = R(t)u$, $\forall u \in \Omega^{0,+}(X, E)$. The lemma follows. \square

To get back to the original heat equation from its adjoint version, it suffices to take the adjoints $\Gamma^*(t)$ of $\Gamma(t)$ and $R^*(t)$ of $R(t)$ (with respect to $(\cdot | \cdot)_E$) because $\tilde{\square}_{b,m}^+$ is self-adjoint. Hence combining Lemma 5.11 and Lemma 5.12 one obtains the following (asymptotic) heat kernel.

THEOREM 5.13. – *With the notations above, we have*

$$\lim_{t \rightarrow 0^+} \Gamma^*(t)u = Q_m u \quad \text{on } \mathcal{D}'(X, T^{*0,+} X \otimes E)$$

for $u \in \Omega^{0,+}(X, T^{*0,+} X \otimes E)$ and $(\tilde{\square}_{b,m}^+ \circ Q_m)$ still denoted by $\tilde{\square}_{b,m}^+$ below, which is self-adjoint)

$$\frac{\partial \Gamma^*(t)}{\partial t} u + \tilde{\square}_{b,m}^+ \Gamma^*(t)u = R^*(t)u, \quad \forall u \in \Omega^{0,+}(X, E),$$

where $R^*(t)$ is the continuous operator with the distribution kernel $R^*(t, x, y)$ satisfying a similar estimate as $R(t)$ in Lemma 5.12.

Based on the above theorem, one way to solving our heat equation resorts to the method of *successive approximation*. This part of reasoning is basically standard. But because of the important role played by Q_m^+ in the final result (cf. *McKean-Singer (II)* in Corollary 5.16), for the convenience of the reader we sketch some details and refer the full details to, e.g., [5, Section 2.4].

To start with, suppose $A(t)$, $B(t)$ and $C(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E)$ are continuous operators with distribution kernels $A(t, x, y)$, $B(t, x, y)$ and $C(t, x, y) \in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+}X \otimes E) \boxtimes (T^{*0,+}X \otimes E)^*)$. Define the (continuous) operator $(A \# B)(t) : \Omega^{0,+}(X, E) \rightarrow \Omega^{0,+}(X, E)$ with distribution kernel

$$(5.48) \quad \begin{aligned} (A \# B)(t, x, y) &:= \int_0^t \int_X A(t-s, x, z) B(s, z, y) dv_X(z) ds \\ &(\in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+}X \otimes E) \boxtimes (T^{*0,+}X \otimes E)^*)). \end{aligned}$$

It is standard that $((A \# B) \# C)(t) = (A \# (B \# C))(t)$, denoted in common by

$$(A \# B \# C)(t).$$

(The generalization to more operators is similar.)

The method of successive approximation results in a solution (which is actually unique by Theorem 5.15 below) to our heat equation, as follows.

PROPOSITION 5.14. – i) (*Existence*) Fix $\ell \in \mathbb{N}$, $\ell \geq 2$. There is an $\epsilon > 0$ such that the sequence

$$(5.49) \quad \Lambda(t) := \Gamma^*(t) - (\Gamma^* \# R^*)(t) + (\Gamma^* \# R^* \# R^*)(t) - \dots$$

converges in $C^\ell((0, \epsilon) \times X \times X, (T^{*0,+}X \otimes E) \boxtimes (T^{*0,+}X \otimes E)^*)$ and

$$(5.50) \quad \begin{aligned} \Lambda(t) &: \Omega^{0,+}(X, E) \rightarrow C^\ell(X, T^{*0,+}X \otimes E) \cap L_m^{2,+}(X, E), \\ \lim_{t \rightarrow 0^+} \Lambda(t)u &= Q_m u \quad \text{on } \mathcal{D}'(X, T^{*0,+}X \otimes E), \quad \forall u \in \Omega^{0,+}(X, E), \\ \Lambda'(t)u + \tilde{\square}_{b,m}^+ \Lambda(t)u &= 0, \quad \forall u \in \Omega^{0,+}(X, E). \end{aligned}$$

ii) (*Approximation*) Write $\Lambda(t, x, y)$, ($\in C^\ell((0, \epsilon) \times X \times X, (T^{*0,+}X \otimes E) \boxtimes (T^{*0,+}X \otimes E)^*)$) for the distribution kernel of $\Lambda(t)$. Then there exists an $\epsilon_0 > 0$ independent of t such that

$$(5.51) \quad \|\Lambda(t, x, y) - \Gamma^*(t, x, y)\|_{C^\ell(X \times X)} \leq e^{-\frac{\epsilon_0}{t}}, \quad \forall t \in (0, \epsilon_0).$$

With $\tilde{\square}_{b,m}^-$ in place of $\tilde{\square}_{b,m}^+$ in (5.50), the corresponding statements for $\tilde{\square}_{b,m}^-$ (with Λ_-, Γ_- etc.) hold true as well.

Proof. – We sketch a proof of ii) and comment on i). For notational convenience we set $Z = R^*$, $Z^2 = R^* \# R^*$, $Z^3 = R^* \# R^* \# R^*$ etc. as defined in (5.48) with $\|\cdot\|_\ell$ as the C^ℓ -norm on $X \times X$. By using (5.45) (for R^*) one sees that there are $1 > \delta_0, \delta_1 > 0$ such that for all $t \in (0, \delta_0)$,

$$(5.52) \quad \|Z\|_\ell \leq \frac{1}{2}e^{-\frac{\delta_1}{t}}, \quad \|Z^2\|_\ell \leq \frac{1}{2^2}e^{-\frac{\delta_1}{t}}, \dots$$

Similarly from the estimate of $\Gamma^*(t)$ (see (5.17)) with the above (5.52) we conclude that for all $t \in (0, \delta_0)$,

$$(5.53) \quad \|\Gamma^* \# Z\|_\ell \leq \frac{C_1}{2}e^{-\frac{\delta_1}{t}}, \quad \|\Gamma^* \# Z^2\|_\ell \leq \frac{C_1}{2^2}e^{-\frac{\delta_1}{t}}, \dots,$$

where $C_1 > 0$ is some constant. Hence the sequence (5.49) converges (in $C^\ell((0, \epsilon) \times X \times X, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*)$) and (5.51) holds.

It takes slightly more work to verify (5.50) in i). Let $q^k(t, x, y)$ be the $(k+1)$ -th term in (5.49). One verifies directly by computation of the convolution that

$$(5.54) \quad \partial_t q^k(t, x, y) + \tilde{\square}_{b,m}^+ q^k(t, x, y) = Z^k(t, x, y) + Z^{k+1}(t, x, y)$$

(cf. [5, (2) of Lemma 2.22]). Since $\Lambda(t, x, y)$ is the alternating sum of these q^k , by the good estimates (5.52) and (5.53), one interchanges the order of the action of $(\partial_t + \tilde{\square}_{b,m}^+)$ on $\Lambda(t, x, y)$ with the summation. By telescoping with (5.54), one finds that the heat Equation (5.50) of i) is satisfied (cf. [5, Theorem 2.23]). \square

The uniqueness part of the above theorem is included in the following. (Note $e^{-t\tilde{\square}_{b,m}^+}(x, y)$ is as in (4.15).)

THEOREM 5.15. – i) (*Uniqueness*) We have $e^{-t\tilde{\square}_{b,m}^+}(x, y) = \Lambda(t, x, y) (\in C^\ell((0, \epsilon) \times X \times X, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*))$. Hence by (5.51), for every $\ell \in \mathbb{N}_0$ there exist $\epsilon_0 > 0$ and $\epsilon > 0$ (independent of t) such that

$$(5.55) \quad \left\| e^{-t\tilde{\square}_{b,m}^+}(x, y) - \Gamma(t, x, y) \right\|_{C^\ell(X \times X)} \leq e^{-\frac{\epsilon_0}{t}}, \quad \forall t \in (0, \epsilon).$$

As a consequence $e^{-t\tilde{\square}_{b,m}^+}(x, y)$ and $\Gamma(t, x, y)$ are the same in the sense of asymptotic expansion (as defined in Definition 5.5).

ii) (*Asymptotic expansion*) More explicitly one has (cf. (5.42))

(5.56)

$$e^{-t\tilde{\square}_{b,m}^+}(x, y) \sim t^{-n} a_n^+(t, x, y) + t^{-n+1} a_{n-1}^+(t, x, y) + \dots + a_0^+(t, x, y) + t a_{-1}^+(t, x, y) \\ + \dots \quad \text{as } t \rightarrow 0^+,$$

$$a_s^+(t, x, y) (= a_{s,m}^+(t, x, y)) = \frac{1}{2\pi} \sum_{j=1}^N \int_{-\pi}^{\pi} e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ y)}{t}} ((e^{-iu})_y^* \hat{b}_{j,s}^+)(x, y) e^{-imu} du$$

$$(\in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*)),$$

$s = n, n-1, n-2, \dots$

Similar statements hold for the case of $e^{-t\widehat{\square}_{b,m}^-}(x, y)$ as well.

Proof. – The argument for the uniqueness part is standard. To sketch it, there is the following trick (cf. [5, Lemma 2.16]) in which we shall use heat equations for both kernels (cf. (4.18) and (5.50)). For $0 < t < \epsilon$ (ϵ as in Proposition 5.14) and $f, g \in \Omega^{0,+}(X, E)$

$$\begin{aligned} 0 &= \int_0^t \frac{\partial}{\partial s} \left((\Lambda(t-s)f | e^{-s\widehat{\square}_{b,m}^+} g)_E \right) ds \\ &= (Q_m f | e^{-t\widehat{\square}_{b,m}^+} g)_E - (\Lambda(t)f | Q_m g)_E \\ &= (f | e^{-t\widehat{\square}_{b,m}^+} g)_E - (\Lambda(t)f | g)_E \\ &= (e^{-t\widehat{\square}_{b,m}^+} f | g)_E - (\Lambda(t)f | g)_E, \end{aligned}$$

proving that $e^{-t\widehat{\square}_{b,m}^+}(x, y) = \Lambda(t, x, y)$.

The estimates (5.55) and (5.56) follow from (5.51) and (5.42) ($e^{-t\widehat{\square}_{b,m}^+}$ is self-adjoint). \square

Remark that by Proposition 5.1 it was tempting to speculate that the heat kernel for (modified/Spin^c) Kohn Laplacian might be (at least asymptotically) the (local) heat kernel for (modified/Spin^c) Kodaira Laplacian. This is however too much to be true as suggested by the above Theorem 5.15 because the asymptotic expansion of (modified) Kohn Laplacian involves a nontrivial t -dependence in $a_s(t, x, y)$ (cf. Remark 1.6 and Remark 1.7).

We are ready to establish a link between our index and the heat kernel density of (modified) Kodaira Laplacian. For $a_\ell^+(t, x, x)$ in (5.56), define $\text{Tr } a_\ell^+(t, x, x) = \sum_{s=1}^d \langle a_\ell^+(t, x, x) e_s(x) | e_s(x) \rangle_E$ as usual, where $\{e_s(x)\}_s$ an orthonormal frame (of $T_x^{*0,+} X \otimes E_x$). (Similar notation and definition apply to the case of $a_\ell^-(t, x, x)$.)

To sum up from Corollary 4.8 and (5.56), there is a second form of *McKean-Singer* type formula for the index in our case (cf. Corollary 4.8 for the first form).

COROLLARY 5.16 (McKean-Singer (II)). – *We have*

$$(5.57) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) = \lim_{t \rightarrow 0^+} \int_X \sum_{\ell=0}^n t^{-\ell} \left(\text{Tr } a_\ell^+(t, x, x) - \text{Tr } a_\ell^-(t, x, x) \right) dv_X(x).$$

By this result we are now reduced to computing $a_s(t, x, x)$ in the following section.

PART II

PROOFS OF MAIN THEOREMS

CHAPTER 6

PROOFS OF THEOREMS 1.3 AND 1.10

We are in a position to prove the main results of this paper. A new ingredient is the notion of “distance function” \hat{d} (see (1.16) for the definition and Theorem 6.7 for its property). This function naturally appears when we compute $a_s(t, x, x)$ in the form of an integral (5.56). In the remaining part of this section we prove that this “distance function” is equivalent to the ordinary distance function at least in the strongly pseudoconvex case (Theorem 6.7).

Theorem 1.3 is proved in Theorem 6.1, Remark 6.2, Corollary 6.3 together with Theorem 5.15; Theorem 1.10 proved in Theorem 6.4 and in (6.15).

In the same notations as before recall that $X_{p_\ell} = \left\{ x \in X; \text{the period of } x \text{ is } \frac{2\pi}{p_\ell} \right\}$, $1 \leq \ell \leq k$ with $p|p_\ell$ (all ℓ) and $p = p_1$. X_p is open and dense in X . See the discussion preceding Theorem 1.3 for more detail.

Let $G_j(t, x, y)$ be as in (5.38). (Notations set up in (5.38)–(5.43) will be useful in what follows.) By the construction of $G_j(t, x, y)$, it is clear that

(6.1)

$$\frac{1}{2\pi} \sum_{j=1}^N G_j(t, x, x) \sim t^{-n} \alpha_n^+(x) + t^{-n+1} \alpha_{n-1}^+(x) + \dots \quad \text{as } t \rightarrow 0^+,$$

$$\alpha_s^+(x) = \frac{1}{2\pi} \sum_{j=1}^N \hat{\beta}_{j,s}^+(x, y)|_{y=x} \in C^\infty(X, \text{End}(T^{*0,+} X \otimes E)), \quad s = n, n-1, \dots$$

$\alpha_s^+(x)$ are independent of choice of BRT trivialization charts (in view of Remark 5.8). It is perhaps instructive to think of these as the data of the asymptotic expansion associated with the “underlying Kodaira Laplacian” (cf. *loc. cit.* and Proposition 5.1) regardless of the existence of a genuine “underlying space”.

Recall the asymptotic expansions of $\Gamma(t, x, y)$ and $e^{-t} \hat{\square}_{b,m}^+(x, y)$ (they coincide by Theorem 5.15), in which we have $a_s^+(t, x, y) (\in C^\infty(\mathbb{R}_+ \times X \times X, (T^{*0,+} X \otimes E) \boxtimes (T^{*0,+} X \otimes E)^*))$, $s = n, n-1, \dots$, cf. (5.56) or (5.42). By the construction, $\Gamma(t, x, y)$

and $a_s(t, x, y)$ of (5.56) depend on the choice of BRT charts. (The authors do not know whether there exists a canonical choice of $a_s(t, x, y)$ in this respect.)

We are now ready to give a proof of the following.

THEOREM 6.1 (Cf. Theorem 1.3). – *For every $N_0 \in \mathbb{N}$ with $N_0 \geq N_0(n)$ for some $N_0(n)$, there exist $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ such that*

$$(6.2) \quad \left| \sum_{j=0}^{N_0} t^{-n+j} a_{n-j}^+(t, x, x) - p_r \delta_{p_r | m} \sum_{j=0}^{N_0} t^{-n+j} \alpha_{n-j}^+(x) \right| \leq C_{N_0} \left(t^{-n+N_0+1} + t^{-n} e^{\frac{-\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}} \right), \quad \forall x \in X_{p_r} \ (r = 1, \dots, k), \quad \forall 0 < t < \delta.$$

Proof. – For simplicity, we only prove Theorem 6.1 for $r = 1$. The proof for $r > 1$ is similar.

As in the beginning of Section 5.2, there are BRT trivializations $B_j := (D_j, (z, \theta), \varphi_j)$, $j = 1, \dots, N$. We write

$$D_j = U_j \times]-2\delta_j, 2\tilde{\delta}_j[, \quad \hat{D}_j = \hat{U}_j \times]-\frac{\delta_j}{2}, \frac{\tilde{\delta}_j}{2}[\quad (\subset \mathbb{C}^n \times \mathbb{R}),$$

with $U_j = \{z \in \mathbb{C}^n; |z| < \gamma_j\}$, $\hat{U}_j = \{z \in \mathbb{C}^n; |z| < \gamma_j/2\}$ for some $\delta_j > 0$, $\tilde{\delta}_j > 0$, $\gamma_j > 0$.

Assume $X = \hat{D}_1 \cup \dots \cup \hat{D}_N$. In the following we let $\delta_j = \tilde{\delta}_j = \zeta$ (all j), ζ satisfy (1.15) with $4|\zeta| < \frac{2\pi}{p}$.

It is easily verified that there is an $\hat{\varepsilon}_0 > 0$ such that $(d(\cdot, \cdot))$ the ordinary distance function on X

$$(6.3) \quad \hat{\varepsilon}_0 d((z_1, \theta_1), (z_2, \theta_1)) \leq |z_1 - z_2| \leq \frac{1}{\hat{\varepsilon}_0} d((z_1, \theta_1), (z_2, \theta_1)), \quad \forall (z_1, \theta_1), (z_2, \theta_1) \in D_j, \\ \hat{\varepsilon}_0 d((z_1, \theta_1), (z_2, \theta_1))^2 \leq h_{j,+}(z_1, z_2) \leq \frac{1}{\hat{\varepsilon}_0} d((z_1, \theta_1), (z_2, \theta_1))^2, \quad \forall (z_1, \theta_1), (z_2, \theta_1) \in D_j,$$

where $h_{j,+}(z, w)$ is as in (5.19).

Recall that the modified distance \hat{d} which is defined in (1.16). We are going to compare \hat{d} with (6.3).

Fix $x_0 \in X_p$. Suppose $x_0 \in \hat{D}_j$ for some $j = 1, 2, \dots, N$ and also suppose $x_0 = (z, 0)$ on D_j .

Some crucial remarks are in order.

i) For $0 \leq |u| \leq 2\zeta$ the action of e^{-iu} on x_0 is only moving along the “angle” direction (due to the assumption that a BRT trivialization D_j is valid here), i.e., the z -coordinates of x_0 and $e^{-iu} \circ x_0$ are the same.

ii) Let a $u_0 \in [2\zeta, \frac{2\pi}{p} - 2\zeta]$ be given. Assume that the action by e^{-iu_0} on x_0 still belongs to \hat{D}_j with a coordinate $e^{-iu_0} \circ x_0 = (\tilde{z}, \tilde{\eta})$. Then it could happen that $\tilde{z} \neq z$

because the orbit $\{e^{-iv} \circ x_0\}$ for $2\zeta \leq v \leq \frac{2\pi}{p} - 2\zeta$ may partly lie outside of D_j . We will show in (6.4) below that indeed $\tilde{z} \neq z$ in this case.

Remark that the above ii) is basically the reason responsible for why the contribution of our distance function \hat{d} enters, as seen shortly. The question about whether the condition $e^{-iu_0} \circ x_0 \in \hat{D}_j$ of ii) is vacuous or not will be discussed below (equivalent to whether J below is an empty set or not).

We shall now formulate the above ii) more precisely. If $x_0 \in \hat{D}_j$, we claim the following.

(6.4) Suppose $e^{-i\theta} \circ x_0 = (\tilde{z}, \tilde{\eta})$ also belongs to \hat{D}_j for some $\theta \in [2\zeta, \frac{2\pi}{p} - 2\zeta]$.

Then $|z - \tilde{z}| \geq \hat{\varepsilon}_0 \hat{d}(x_0, X_{\text{sing}}) (> 0)$.

Proof of the claim. – By $(\tilde{z}, \tilde{\eta}) \in \hat{D}_j$ one has $e^{i\tilde{\eta}} \circ (\tilde{z}, \tilde{\eta}) = (\tilde{z}, 0)$ equivalently $e^{-i\tilde{\eta}} \circ (\tilde{z}, 0) = (\tilde{z}, \tilde{\eta})$ (by the above i) as $|\tilde{\eta}| \leq \frac{\zeta}{2}$ here). One sees (by (6.3) and isometry of S^1 action for the first inequality below)

$$(6.5) \quad \begin{aligned} |\tilde{z} - z| &\geq \hat{\varepsilon}_0 d(e^{i\tilde{\eta}} \circ (\tilde{z}, \tilde{\eta}), e^{i\tilde{\eta}} \circ (z, \tilde{\eta})) = \hat{\varepsilon}_0 d(e^{i\tilde{\eta}} \circ (e^{-i\theta} \circ x_0), x_0) \\ &\geq \hat{\varepsilon}_0 \inf \left\{ d(e^{-iu} \circ e^{-i\theta} \circ x_0, x_0); |u| \leq \frac{\zeta}{2} \right\} \\ &\geq \hat{\varepsilon}_0 \inf \left\{ d(e^{-i\hat{\theta}} \circ x_0, x_0); \zeta \leq \hat{\theta} \leq \frac{2\pi}{p} - \zeta \right\} \\ &= \hat{\varepsilon}_0 \hat{d}(x_0, X_{\text{sing}}) \end{aligned}$$

(see (1.16) for the definition of \hat{d}), as claimed. □

Remark that a sharp result in this direction (6.4) is proved in Lemma 7.6.

We continue with the proof of the theorem. We need to estimate $\Gamma(t) = \sum_{j=1}^N H_j(t) \circ Q_m$ for the first summation to the left of (6.2). By definition (see (5.42)) this is in turn to estimate

$$(6.6) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} ((e^{-iu})_y^* H_j)(t, x_0, x_0) e^{-imu} du$$

and sum over $j = 1, \dots, N$.

We first assume that in (6.6), $x_0 = (z, 0)$ in \hat{D}_j and $x_0 \notin \hat{D}_k$ for any other $k \neq j$.

To work on (6.6) we shall divide $[-\pi, \pi]$ in (6.6) into two types.

The first type is to estimate $\frac{1}{2\pi} \int_{-2\zeta}^{2\zeta} ((e^{-iu})_y^* H_j)(t, x_0, x_0) e^{-imu} du$. Note if $u \in [-2\zeta, 2\zeta]$ and $e^{-iu} \circ x_0 = (z_u, \theta_u)$, then $(z_u, \theta_u) = (z, u)$ (by i) above (6.4)), i.e., $e^{-iu} \circ (z, 0) = (z, u)$. Hence, by (5.38) the factor $e^{im\eta}$ in H_j is going to be e^{imu} and this is canceling off the term e^{-imu} in the integral (6.6). By (5.19), $h_+(z, z_u) = h_+(z, z) = 0$ (also by (6.3)) and the factor $e^{-\frac{h_+}{t}}$ of $A_{B_j, +}$ in H_j of (5.38) becomes 1. Finally we note that for σ_j in H_j of (5.38), $\int_I \sigma_j(\theta) d\theta = 1$, $I = [-\frac{\zeta}{2}, \frac{\zeta}{2}]$.

To sum up, by (5.38) and (5.19) one obtains the following ($x_0 \notin \hat{D}_k$ for $k \neq j$)

$$(6.7) \quad \frac{1}{2\pi} \int_{-2\zeta}^{2\zeta} ((e^{-iu})_y^* H_j)(t, x_0, x_0) e^{-imu} du \\ \sim \left(t^{-n} \alpha_n^+(x_0) + t^{-(n-1)} \alpha_{n-1}^+(x_0) + \dots \right) \text{ as } t \rightarrow 0^+,$$

where $\alpha_s^+(x)$, $s = n, n-1, \dots$, are as in (6.1).

For the second type suppose $u \in [2\zeta, \frac{2\pi}{p} - 2\zeta]$. Note the action by e^{-iu} on x_0 may change the z coordinate of x_0 by ii) above (6.4). We let J be the subset of those $u \in [2\zeta, \frac{2\pi}{p} - 2\zeta] \equiv E$ that $e^{-iu} \circ x_0 = (z_u, \theta_u)$ belongs to \hat{D}_j (then $z_u \neq z$ by (6.4)), and J' be the complement of J in E . One finds, for some $\varepsilon_0 > 0, \delta > 0$ and $C_0 > 0$ (independent of j, x_0), that

$$(6.8) \quad \left| \frac{1}{2\pi} \int_{u \in [2\zeta, \frac{2\pi}{p} - 2\zeta]} ((e^{-iu})_y^* H_j)(t, x_0, x_0) e^{-imu} du \right| \\ \leq \frac{1}{2\pi} \int_{u \in J} |((e^{-iu})_y^* H_j)(t, x_0, x_0) e^{-imu}| du \\ + \frac{1}{2\pi} \int_{u \in J'} |((e^{-iu})_y^* H_j)(t, x_0, x_0) e^{-imu}| du \\ \leq C_0 t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x_0, X_{\text{sing}})^2}{t}}, \quad \forall 0 < t < \delta,$$

where the integral over J' vanishes because the cut-off function σ_j in H_j of (5.38) gives $\sigma_j(\eta(y)) = 0$ for $y = e^{-iu} \circ x_0 \notin \hat{D}_j$ as $\sigma_j = 0$ outside \hat{D}_j (see lines above (5.38)), and the second inequality arises from applying (6.3) and (6.4) to $h_+(z, z_u)$ in H_j (see (5.38) and (5.19)).

Is J an empty set? We remark that the top term in (6.8) is in general nonzero (by combining (6.7) and Remark 1.7 for $p = 1$). Hence $J \neq \emptyset$ in general. There is a geometrical way to see the claim that for some open subset V of X , if $x_0 \in V$, then $J \neq \emptyset$. For simplicity assume $X = X_1 \cup X_2$, i.e., $p = 1$ and $p_2 = 2$. Choose $y \in X_2$. Let $g = e^{-i\frac{2\pi}{p_2}} \in S^1$. Fix a neighborhood $U \subset \hat{D}_j$ of y in X . Since $g \circ y = y$, by continuity argument there are neighborhoods N_1, N_2 of y, g in X, S^1 respectively such that the action $h \circ x \in U$ if $(h, x) \in N_2 \times N_1$. Choose $N_1 \subset \hat{D}_j$, N_2 small and set $V \equiv N_1 \setminus X_2$. It follows that for these $x_0 \in V$, $J \neq \emptyset$ since $N_2 \subset J$. This result also accounts for the necessity of the remark ii) above (6.4) and hence that of a certain extra contribution (e.g., \hat{d}) in estimates (6.8).

Suppose $p = 1$. Then (6.7) and (6.8), Definition 5.5 for \sim and Remark 5.6 (by noting that $\dim X = 2n + 1$, $\dim U_j = 2n$, $M_0(m) = n + 1$, $m = 4n = 2\beta$, $M_1(m, \ell) = n$

for $\ell = 0$) immediately lead to

$$(6.9) \quad \left| \Gamma(t, x_0, x_0) - \sum_{j=0}^{N_0} t^{-n+j} \alpha_{n-j}^+(x_0) \right| \leq C_{N_0} \left(t^{-n+N_0} + C_0 t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x_0, X_{\text{sing}})^2}{t}} \right), \quad N_0 \geq N_0(n), \quad \forall 0 < t < \delta.$$

Now adding $t^{-n+N_0+1} \alpha_{n-N_0-1}^+$ to (6.9) and subtracting it we improve t^{-n+N_0} by t^{-n+N_0+1} . Hence the estimate (6.2) of the theorem for $p = 1$.

Suppose $p > 1$. Then one has an extra $p - 1$ sectors in $[-\pi, \pi]$ (obtained by shifting the above first sector $s = 1$ by a common $(s - 1)\frac{2\pi}{p}$):

$$(6.10) \quad [(s - 1)\frac{2\pi}{p} - 2\zeta, (s - 1)\frac{2\pi}{p} + 2\zeta], [(s - 1)\frac{2\pi}{p} + 2\zeta, s\frac{2\pi}{p} - 2\zeta], \quad s = 1, \dots, p$$

($s = p + 1$ identified with $s = 1$) over which the integrals correspond to types I (6.7) and II (6.8) respectively. One may check without difficulty that the version of the claim (6.4) adapted to these sectors holds true as well. On each of these sectors, a simple (linear) change of variable for u , which is to bring the intervals of the integration on these sectors back to those in (6.7) and (6.8), produces the extra numerical factor in sum (by $e^{-imu} du$ in (6.6)) : $\sum_{s=1}^p e^{\frac{2\pi(s-1)}{p} mi} = p\delta_{p|m}$ as expressed in (6.2).

Finally, note that we have assumed $x_0 = (z, 0) \in \hat{D}_j$. In this case the above argument appears *symmetrical* in writing. This (assumption) is however not essential. Since we shall also adopt a similar assumption in Section 7, we give an outline about the *asymmetrical* situation (i.e., $x_0 = (z, \theta)$, $\theta \neq 0$) of the argument. By going over the same process, one sees the following. i) If $x = (z, v)$, with $0 < v \leq \frac{\zeta}{2}$, the intervals in (6.7), (6.8) shall be replaced by $[-2\zeta - v, 2\zeta - v]$, $[2\zeta - v, \frac{2\pi}{p} - 2\zeta - v]$ (thought of as translated by a common $-v$) with the new integrals denoted by (6.7)', (6.8)', respectively; ii) $[-2\zeta - v, 2\zeta - v] \supseteq [-\zeta, \zeta]$ hence $\int \sigma(u) du$ is still 1 in (6.7)' ; iii) In the proof of claim (6.5), $e^{i\tilde{\eta}}$ should be replaced by $e^{i\gamma}$ with $\gamma = \tilde{\eta} - v$, $(\tilde{z}, 0)$ by (\tilde{z}, v) and $\theta \in [2\zeta, \frac{2\pi}{p} - 2\zeta]$ by $\theta \in [2\zeta - v, \frac{2\pi}{p} - 2\zeta - v]$ throughout (6.4) and (6.5). One can check that the reasoning in (6.5) remains basically unchanged, and the conclusion of (6.5) holds true as well in this modified case; iv) By the preceding ii) and iii), the results corresponding to (6.7)' and (6.8)' hold true. Hence the asymmetrical situation follows.

We have also assumed $x_0 \notin \hat{D}_k$ for $k \neq j$. This condition is unimportant if we take the preceding asymmetrical situation into account (for $x_0 \in \hat{D}_k$, $k \neq j$ in the general case), and note that there is a hidden partition of unity in $\{H_j\}_{j=1, \dots, N}$.

An alternative of the above is to use the kernel $e^{-t\hat{\square}_{b,m}^+}(x, y)$ in place of $\Gamma(t, x, y)$ and $a_s^+(t, x, y)$. An advantage is that $e^{-t\hat{\square}_{b,m}^+}(x, y)$ is independent of BRT charts, so that for a given point x_0 we can take a covering of X by convenient BRT charts for the previous special conditions to be satisfied (e.g., $x_0 = (z, 0)$, $x_0 \in \hat{D}_j$ for exactly

one j etc.). By the asymptotic property between $e^{-t\widehat{\square}_{b,m}^+}(x, y)$ and $\Gamma(t, x, y)$ (5.55), this also leads to Theorem 6.1. \square

REMARK 6.2. – For the relation between $a_s^+(t, x, y)|_{y=x}$ and $\alpha_s(x)$ as stated in (1.18) of Theorem 1.3, the method of the above proof works. By using (setting $y = x$ below)

$$(6.11) \quad a_s^+(t, x, y) = \frac{1}{2\pi} \sum_{j=1}^N \int_{-\pi}^{\pi} e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ y)}{t}} ((e^{-iu})_y^* \hat{b}_{j,s}^+)(x, y) e^{-imu} du$$

(see (5.56)) with the same reasoning as (6.7), (6.8) and (6.10), one obtains (1.18) for $P_\ell = \text{id}$. For $\ell > 0$, (1.18) follows using

$$(6.12) \quad \partial_x e^{-\frac{x^2}{t}} = -2t^{-\frac{1}{2}} \left(\frac{x^2}{t}\right)^{1/2} e^{-\frac{x^2}{t}} = O(t^{-\frac{1}{2}})$$

in (6.11) and by noting Remark 5.6 as in (6.9). Remark that if one extracts the corresponding coefficients of t^{-s} in (6.2) of Theorem 6.1 and uses the Result (6.2), the estimate appears to be $e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})}{t}} + O(t^\infty)$ which is slightly weaker than above (due to $O(t^\infty)$).

For similar estimates with regard to C^l topology we have the following (cf. (6.12) and Remark 5.6).

COROLLARY 6.3. – *In the same notation as above, for any differential operator $P_\ell : C^\infty(X, T^{*0,+}X \otimes E) \rightarrow C^\infty(X, T^{*0,+}X \otimes E)$ of order $\ell \in \mathbb{N}$ and every $N_0 \geq N_0(n)$ for some $N_0(n)$,*

$$(6.13) \quad \left| P_\ell \left(\sum_{j=0}^{N_0} t^{-n+j} a_{n-j}^+(t, x, x) - p_r \delta_{p_r|m} \sum_{j=0}^{N_0} t^{-n+j} \alpha_{n-j}^+(x) \right) \right| \leq C_{N_0} \left(t^{-n+N_0+1-\frac{\ell}{2}} + t^{-n-\frac{\ell}{2}} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}} \right), \quad \forall 0 < t < \delta, \quad \forall x \in X_{p_r},$$

for some $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ independent of x .

Note the singular behavior t^{-n} (in the term to the rightmost of (6.2)). So the estimate (6.2) is not directly applicable to the proof of our local index theorem. That is, computation involving a_s cannot be automatically reduced to computation involving α_s as soon as $x (\in X_p)$ approaches X_{sing} . Intuitively $t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}})^2}{t}}$ goes to a kind of Dirac delta function (along X_{sing}) as $t \rightarrow 0$ (apart from a factor of the form $\frac{1}{t^\beta}$, some $\beta > 0$). So after integrating (6.2) over X , a nonzero contribution due to this term could appear or even blow up as $t \rightarrow 0$. A more precise analysis along this line will be taken up in the study of *trace integrals* in Section 7.

Fortunately, the abovementioned singular behavior can be removed (t^{-n} dropping out completely) after taking the *supertrace*, so that the index density for our need

does exist. (However, as far as the full kernel is concerned, a certain estimate such as that in Theorem 6.1 is unavoidable as discussed in Remark 1.8).

We shall now take up this improvement on (6.2) under supertrace. We formulate it as follows, whose proof is heavily based on the off-diagonal estimate obtained in Theorem 5.10.

THEOREM 6.4 (Cf. Theorem 1.10). – *With the notations above, for every $N_0 \in \mathbb{N}$, $N_0 \geq N_0(n)$ for some $N_0(n)$, there exist $\varepsilon_0 > 0$, $\delta > 0$ and $C_{N_0} > 0$ such that*

$$(6.14) \quad \left| \operatorname{Tr} e^{-t\tilde{\square}_{b,m}^+}(x, x) - \operatorname{Tr} e^{-t\tilde{\square}_{b,m}^-}(x, x) - p_r \delta_{p_r|m} \sum_{j=0}^{N_0} t^{-n+j} \left(\operatorname{Tr} \alpha_{n-j}^+(x) - \operatorname{Tr} \alpha_{n-j}^-(x) \right) \right| \leq C_{N_0} \left(t^{-n+N_0+1} + e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}} \right), \quad \forall 0 < t < \delta, \quad \forall x \in X_{p_r},$$

The implication of Theorem 6.4 yields a link between the two identities arising from Corollary 5.16 and Proposition 5.9 together with (5.22):

$$(6.15) \quad \lim_{t \rightarrow 0^+} \int_X \sum_{\ell=0}^n t^{-\ell} \left(\operatorname{Tr} a_{\ell}^+(t, x, x) - \operatorname{Tr} a_{\ell}^-(t, x, x) \right) dv_X(x) = \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) \\ \lim_{t \rightarrow 0^+} \int_X \sum_{\ell=0}^n t^{-\ell} \left(\operatorname{Tr} \alpha_{\ell}^+(x) - \operatorname{Tr} \alpha_{\ell}^-(x) \right) dv_X(x) \\ = \frac{1}{2\pi} \int_X \operatorname{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge \operatorname{ch}_b(\nabla^E, E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0(x).$$

It follows that the two in (6.15) are equal because in (6.14), $e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\text{sing}}^r)^2}{t}} (\leq 1) \rightarrow 0$ in L^1 by Lebesgue's dominated convergence theorem as $t \rightarrow 0^+$ on X_p . We arrive now at an index theorem for our class of CR manifolds.

COROLLARY 6.5 (Cf. Corollary 1.13). – *We have*

$$(6.16) \quad \sum_{j=0}^n (-1)^j \dim H_{b,m}^j(X, E) \\ = p \delta_{p|m} \frac{1}{2\pi} \int_X \left[\operatorname{Td}_b(\nabla^{T^{1,0}X}, T^{1,0}X) \wedge \operatorname{ch}_b(\nabla^E, E) \wedge e^{-m \frac{d\omega_0}{2\pi}} \wedge \omega_0 \right]_{2n+1}(x),$$

where $[\dots]_{2n+1}$ denotes the $(2n+1)$ -form part.

We turn now to the proof of Theorem 6.4.

Proof of Theorem 6.4. – For simplicity, we only prove Theorem 6.4 for $r = 1$. The proof for $r > 1$ is similar. Adopting the same notations as in the proof of Theorem 6.1 (e.g., $B_j, D_j, \hat{D}_j \dots$), we shall follow a similar line of thought as in Theorem 6.1.

Fix $x_0 \in X_p$. As $e^{-t\hat{\square}_{b,m}^+}(x, y)$ and $\Gamma(t, x, y)$ are asymptotically the same (Theorem 5.15), we also break the desired estimate at $x = x_0$ into two types of integrals corresponding to (6.7) and (6.8).

One integral is over $I = [-2\zeta, 2\zeta]$ and the other over I' , the complement of I in $[-\pi, \pi]$. The first type gives rise to the first term to the right of (6.14) almost the same way as (6.7).

The key of this proof lies in the second type which corresponds to (6.8). It is estimated over I' , as in (6.17) below. (Here we rewrite H_j in a convenient form, in terms of $\hat{h}_{j,+}, \hat{K}_{j,+}$ of (5.39), reminiscent of an analogous relation $A_{B,+} = e^{-\frac{h_+}{t}} K_{B,+}$ in (5.19).)

$$(6.17) \quad \frac{1}{2\pi} \sum_{j=1}^N \left| \int_{u \in I'} e^{-\frac{\hat{h}_{j,+}(x_0, e^{-iu} \circ x_0)}{t}} \left(((e^{-iu})_y^* \text{Tr } \hat{K}_{j,+})(t, x_0, x_0) - ((e^{-iu})_y^* \text{Tr } \hat{K}_{j,-})(t, x_0, x_0) \right) du \right|.$$

We shall now show that there exist $\varepsilon_0 > 0$ and $C > 0$ (independent of x_0) such that (6.17) is bounded above by

$$(6.18) \quad C e^{-\frac{\varepsilon_0 \hat{d}(x_0, X_{\text{sing}})^2}{t}}$$

for small $t \in \mathbb{R}_+$. To see this we first note that for $k \geq 0$,

$$(6.19) \quad e^{-\varepsilon \frac{x^2}{t}} \left(\frac{x^2}{t} \right)^k \leq C_{k,\varepsilon} e^{-\varepsilon \frac{x^2}{2t}}$$

for some constant $C_{k,\varepsilon}$ independent of x and $t > 0$. Write $x_0 = (z, \theta)$ and $e^{-iu} \circ x_0 = (z_u, \theta_u)$ in BRT coordinates. Since $\hat{h}_{j,+}(x_0, e^{iu} \circ x_0)$ is essentially $h_+(z, z_u) \approx |z - z_u|^2$, we have

$$(6.20) \quad e^{-\frac{\hat{h}_{j,+}(x_0, e^{iu} \circ x_0)}{t}} \leq e^{-2c_1 \frac{|z - z_u|^2}{t}} \leq e^{-c_1 \frac{\varepsilon_0 \hat{d}(x_0, X_{\text{sing}})^2}{t}} e^{-c_1 \frac{|z - z_u|^2}{t}}$$

for some constant $c_1 > 0$ by using (6.4) for \hat{d} . By using the off-diagonal estimate of Theorem 5.10 and by (5.19), (5.39) for linking b_\bullet with K_\bullet , one obtains the following estimate from (6.20)

$$(6.21) \quad \left| e^{-\frac{\hat{h}_{j,+}(x_0, e^{-iu} \circ x_0)}{t}} \left(((e^{-iu})_y^* \text{Tr } \hat{K}_{j,+})(t, x_0, x_0) - ((e^{-iu})_y^* \text{Tr } \hat{K}_{j,-})(t, x_0, x_0) \right) \right| \\ \leq e^{-c_1 \frac{\varepsilon_0 \hat{d}(x_0, X_{\text{sing}})^2}{t}} \sum_{k=0}^n \text{constants} \cdot e^{-c_1 \frac{|z - z_u|^2}{t}} \left(\frac{|z - z_u|^{2k}}{t^k} + O(t) \right)$$

for (6.17). Now one readily obtains the bound (6.18) from (6.21) and (6.19).

Combining the above estimates for integrals of the first type and second type (6.17), we obtain (6.14) in the way similar to (6.9) (with t^{-n} dropping out of $t^{-n}e^{-\frac{\varepsilon_0 d^2}{t}}$). \square

In the remaining part of this section we give a geometric meaning for $\hat{d}(x, X_{\text{sing}}^r)$ (when X is strongly pseudoconvex). To this aim it is useful to use another equivalent form of the function \hat{d} , as follows (without any pseudoconvexity condition on X).

LEMMA 6.6. – *There exists a small constant $\varepsilon_0 > 0$ (satisfying (1.15) at least) with the following property. Fix an ε with $0 < \varepsilon \leq \varepsilon_0$. For $x \in X$ define another “distance function” \hat{d}_2 by (for a fixed ℓ)*

$$\hat{d}_2(x, X_{\text{sing}}^{\ell-1}) = \inf \left\{ d(x, e^{-i\theta} \circ x); \frac{2\pi}{p_\ell} - \varepsilon \leq \theta \leq \frac{2\pi}{p_\ell} + \varepsilon \right\}$$

($X_{\text{sing}}^{\ell-1} = X_{p_\ell} \cup X_{p_{\ell+1}} \dots$). Then $\hat{d}_2(x, X_{\text{sing}}^{\ell-1})$ is equivalent to $\hat{d}(x, X_{\text{sing}}^{\ell-1})$. (Namely $\frac{1}{C_{\ell, \varepsilon}} \hat{d}_2 \leq \hat{d} \leq C_{\ell, \varepsilon} \hat{d}_2$ for some constant $C_{\ell, \varepsilon}$ independent of x).

We postpone the proof of the lemma until after Theorem 6.7.

For technical reasons we impose a pseudoconvex condition on X in the following although the same result is expected to hold without this condition.

THEOREM 6.7. – *With the notations above, assume that X is strongly pseudoconvex. Then there is a constant $C \geq 1$ such that*

$$\frac{1}{C} d(x, X_{\text{sing}}^r) \leq \hat{d}(x, X_{\text{sing}}^r) \leq C d(x, X_{\text{sing}}^r), \quad \forall x \in X.$$

Proof. – For simplicity, we assume that $X = X_1 \cup X_2$, i.e., $p = 1, p_2 = 2$, so that $X_{\text{sing}} \equiv X_{\text{sing}}^1 = X_2$ ($r = 1$) by definition. For the general case, the proof is essentially the same. Let $f, g : X \rightarrow [0, +\infty[$ be positive functions. We write $f(x) \approx g(x)$ if there is a constant $C > 1$ such that

$$\frac{1}{C} g(x) \leq f(x) \leq C g(x),$$

for every $x \in X$. By Lemma 6.6, for every (small and fixed) $\varepsilon > 0$ we have

$$(6.22) \quad \hat{d}(x, X_{\text{sing}}) \approx \inf \{ d(x, e^{-i\theta} \circ x); \pi - \varepsilon \leq \theta \leq \pi + \varepsilon \}.$$

Since X is strongly pseudoconvex, it is well-known that (see [40]) there exists a CR embedding:

$$(6.23) \quad \begin{aligned} \Phi : X &\rightarrow \mathbb{C}^N, \\ x &\mapsto (f_1(x), \dots, f_N(x)) \end{aligned}$$

with $f_j \in H_{b, m_j}^0(X)$ for some $m_j \in \mathbb{N}$ ($j = 1, \dots, N$).

We assume that m_1, \dots, m_s are odd numbers and m_{s+1}, \dots, m_N are even numbers. By $p = 1$ and $p_2 = 2$ one sees that (cf. (1.35))

$$(6.24) \quad x \in X_{\text{sing}} \text{ if and only if } f_1(x) = \dots = f_s(x) = 0$$

so that

$$(6.25) \quad d(x, X_{\text{sing}})^2 \approx \sum_{j=1}^s |f_j(x)|^2, \quad \forall x \in X.$$

Now, by using the embedding Theorem (6.23) (together with (1.35)) we have

$$(6.26) \quad d(x, e^{-i\pi} \circ x)^2 \approx \sum_{j=1}^N |f_j(x) - f_j(e^{-i\pi} \circ x)|^2 = 4 \sum_{j=1}^s |f_j(x)|^2 \approx d(x, X_{\text{sing}})^2$$

and hence for every $\pi - \varepsilon \leq \theta \leq \pi + \varepsilon$ ($\varepsilon > 0$ small)

$$(6.27) \quad d(x, e^{-i\theta} \circ x)^2 \approx \sum_{j=1}^N |f_j(x) - f_j(e^{-i\theta} \circ x)|^2 \\ \geq \sum_{j=1}^s |(1 - e^{-im_j\theta})f_j(x)|^2 \approx \sum_{j=1}^s |f_j(x)|^2 \approx d(x, X_{\text{sing}})^2.$$

By (6.27) we conclude that

$$(6.28) \quad \inf \{d(x, e^{-i\theta} \circ x); \pi - \varepsilon \leq \theta \leq \pi + \varepsilon\} \approx d(x, X_{\text{sing}}).$$

Combining (6.22) and (6.28) we have proved the theorem. \square

We give now:

proof of Lemma 6.6. – In the following we write $\hat{d}(x) = \hat{d}(x, X_{\text{sing}}^{\ell-1})$ and $\hat{d}_2(x) = \hat{d}_2(x, X_{\text{sing}}^{\ell-1})$ for a fixed ℓ . For an illustration we assume $X = X_1 \cup X_2$, i.e., $p_1 = 1$, $p_2 = 2$, and $x \in X_1$ ($\hat{d}_2 = \hat{d} = 0$ for $x \in X_2$.) Write I for the complement of $I' \equiv]\pi - \varepsilon, \pi + \varepsilon[$ in $[\zeta, 2\pi - \zeta] \equiv K$ (where ζ satisfies (1.15) and $\varepsilon > 0$ a small constant to be specified, cf. the line above (6.35)). By definition $\hat{d}_2 \geq \hat{d} (= \hat{d}_\zeta)$ (\hat{d}_2 is to take inf over I' while \hat{d} over K , and $I' \subset K$).

It remains to see $\hat{d}_2 \leq C\hat{d}$ for some C . Put

$$f_S(x) = \inf_{\theta \in S} \{d(x, e^{-i\theta} \circ x)\}$$

for a set S . We claim that there exists a c , $1 > c > 0$,

$$(6.29) \quad f_I(x) \geq c$$

for all $x \in X_1$. Indeed for each $x \in X$ and for any $\theta \in I = [\zeta, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \zeta]$ one sees $x \neq e^{-i\theta} \circ x$. So (6.29) follows by a compactness argument. Let $M \geq 1$ be an upper bound of \hat{d}_2 . We claim

$$(6.30) \quad \hat{d}_2(x) \leq \frac{M}{c} \hat{d}(x), \quad x \in X.$$

Note $\hat{d}(x) = f_K(x) = \min\{f_I(x), f_{I'}(x)\}$ and $\hat{d}_2 = f_{I'}$. Suppose $f_K(y) < f_I(y)$. Then $f_K(y) = f_{I'}(y)$, i.e., $\hat{d}(y) = \hat{d}_2(y)$ and (6.30) holds for these y (as $\frac{M}{c} > 1$). If $f_K(y) \geq f_I(y)$ (for some $y \in X_1$), then $f_K(y) = f_I(y)$, giving $\hat{d}(y) \geq c$ by (6.29).

For these y , (6.30) still holds. In any case we have proved (6.30) for $x \in X_1$, hence for $x \in X$ ($\hat{d} = \hat{d}_2 = 0$ at $x \in X_2$).

For another illustration, in the same notation as above except that say, $X = X_1 \cup X_2 \cup X_4$ (i.e., $p_3 = 4$). We are going to prove the lemma for the case $\ell = 2$ (with $x \in X_1$, as $\hat{d} = \hat{d}_2 = 0$ at $x \notin X_1$ for $\ell = 2$).

With the above I, I' and K , let J be the complement of $J' \equiv]\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon[\cup]\frac{3\pi}{2} - \varepsilon, \frac{3\pi}{2} + \varepsilon[$ in I .

It follows, similarly as (6.29), that there exists a $c_2, 1 > c_2 > 0$ such that

$$(6.31) \quad f_J(x) \geq c_2, \quad \forall x \in X.$$

Let $\{W_\alpha\}_\alpha$ be the set of connected components of X_4 . Each $y \in X_4$ is a fixed point of the subgroup $\mathbb{Z}_4 = \{1, e^{i\frac{\pi}{2}}, e^{i\pi}, e^{i\frac{3\pi}{2}}\}$ of S^1 ; write $\lambda_{i,\alpha}(g)$ for all the eigenvalues of the isotropy (and isometric) action of $g \in \mathbb{Z}_4$ on $T_y X$ for $y \in W_\alpha$. All of them are independent of the choice of $y \in W_\alpha$. Let

$$C_M = \max_{1 \neq g \in \mathbb{Z}_4, \lambda_{i,\alpha}(g)} \{|\lambda_{i,\alpha}(g) - 1|\} > 0; \quad c_m = \min_{1 \neq g \in \mathbb{Z}_4, \lambda_{i,\alpha}(g) \neq 1} \{|\lambda_{i,\alpha}(g) - 1|\} > 0.$$

Let $B = \{x \in X; \hat{d}_2(x) \geq (\frac{C_M}{c_m} + 1)\frac{M}{c_2}\hat{d}(x) > 0\}$ ($M \geq 1$ as above). Clearly $B \subset X_1$ (zero distance for $x \in X_2 \cup X_4$). We claim that

$$(6.32) \quad \overline{B} \cap X_4 = \emptyset.$$

To see (6.32) suppose otherwise. Let $y_n \in B$ and $y_n \rightarrow y \in X_4$ as $n \rightarrow \infty$. Observe that $f_K(y_n) \neq f_{I'}(y_n)$ for all n because the equality $\hat{d}(y_n) = \hat{d}_2(y_n)$ (note $f_K = \hat{d}$ and $f_{I'} = \hat{d}_2$) clearly contradicts the definition of B with $y_n \in B$. By $K = I' \cup J' \cup J$, we are left with two possibilities for a y_n

$$(6.33) \quad \begin{aligned} \text{i) } & f_K(y_n) = f_{J'}(y_n) \\ \text{ii) } & f_K(y_n) = f_J(y_n). \end{aligned}$$

Suppose i). By examining the isotropy (and isometric) action of \mathbb{Z}_4 at $y \in X_4$, one sees that both ratios below

$$(6.34) \quad \frac{d(y_n, e^{\beta\pi} \circ y_n)}{d(y_n, e^{\beta\frac{\pi}{2}} \circ y_n)}, \quad \frac{d(y_n, e^{\beta\pi} \circ y_n)}{d(y_n, e^{\beta\frac{3\pi}{2}} \circ y_n)}$$

are bounded above by $\frac{C_M}{c_m} + \frac{1}{4}$ as $n \gg 1$. Since I' and J' are ε -neighborhoods around π and $\{\frac{\pi}{2}, \frac{3\pi}{2}\}$ respectively, by choosing a sufficiently small ε (say $\varepsilon \leq \varepsilon_0$) one sees from (6.34)

$$(6.35) \quad \frac{f_{I'}(y_n)}{f_{J'}(y_n)} \leq \frac{C_M}{c_m} + \frac{1}{2}, \quad n \gg 1.$$

We claim that this contradicts $y_n \in B$. Note $f_K = \hat{d}$ and $f_{I'} = \hat{d}_2$ so that the assumption i) $f_K(y_n) = f_{J'}(y_n)$ amounts to $\hat{d}(y_n) = f_{J'}(y_n)$ and (6.35) gives

$$(6.36) \quad \frac{\hat{d}_2(y_n)}{\hat{d}(y_n)} \leq \frac{C_M}{c_m} + \frac{1}{2}, \quad n \gg 1.$$

By $y_n \in B$, (6.36) contradicts the definition of B .

Suppose ii) of (6.33). By (6.31), $f_J(x) \geq c_2$ for all $x \in X$ hence by $f_K = \hat{d}$ and ii) of (6.33), one obtains $\hat{d}(y_n) \geq c_2$, giving $\hat{d}_2(y_n) \geq (\frac{C_M}{c_m} + 1)M$ by using $y_n \in B$, which is absurd since $\hat{d}_2 \leq M$ by assumption. The claim (6.32) is proved by contradictions in i) and ii) of (6.33).

Granting the claim (6.32) we have $\overline{B} \subset X_1 \cup X_2$ (which is open in X). Since for $\theta \in I$ and $x \in X_1 \cup X_2$ (in particular for $x \in \overline{B}$) $x \neq e^{-i\theta} \circ x$, by compactness there exists a c_3 , $1 > c_3 > 0$ satisfying (as in (6.29))

$$(6.37) \quad f_I(x) \geq c_3$$

for all $x \in \overline{B}$. One asserts that

$$(6.38) \quad \hat{d}_2(x) \leq \left(\frac{C_M}{c_m} + 1\right) \frac{M}{c_2 c_3} \hat{d}(x), \quad \forall x \in \overline{B}.$$

The argument is similar. By $f_K = \min\{f_I, f_{I'}\}$, a) $f_K(x) = f_{I'}(x)$ or b) $f_K(x) = f_I(x)$. a) If $f_K(x) = f_{I'}(x)$, then by $f_K = \hat{d}$ and $f_{I'} = \hat{d}_2$, $\hat{d}(x) = \hat{d}_2(x)$; b) if $f_K(x) = f_I(x)$, then by (6.37) and $f_K = \hat{d}$, $\frac{\hat{d}(x)}{c_3} \geq 1$ (for $x \in \overline{B}$). In both cases a) and b), (6.38) holds (by $M \geq 1$ an upper bound of \hat{d}_2 and $\frac{1}{c_2}, \frac{1}{c_3} > 1$).

Finally, Since the same inequality of (6.38) holds for all x outside B by definition of B (with $\hat{d}_2 = \hat{d} = 0$ for $x \in X_2 \cup X_4$), the equivalence between \hat{d} and \hat{d}_2 (for all $x \in X$) is proved.

The proof for the general case clearly flows from the similar pattern as above (although tedious). We shall omit the detail. \square

CHAPTER 7

TRACE INTEGRALS AND PROOF OF THEOREM 1.14

7.1. A setup, including a comparison with recent developments

There is a vast literature about heat kernels on manifolds. A comparison between the previous results and those of ours in the present paper shall now be discussed before we proceed further. A concise account of the (ordinary) heat kernel in diversified aspects is given in Richardson [59] and references therein. A generalization of the heat kernel to orbit spaces of a group Γ (of isometries) acting on a manifold M dates back to the seminal work of H. Donnelly in late '70s [21], [22]. Among others, Donnelly calculated the asymptotic expansion of the trace of the ordinary heat kernel on M restricted to Γ -invariant functions (here Γ -action is assumed to be properly discontinuous on M). Brüning and Heintze in '84 [11] studied the equivariant trace with Γ replaced by a compact group G of isometries (including the trace restricted to G -invariant eigenfunctions). A similar study (of trace) into the orbifold case has been made recently in [59] and [23]. In all of these works the asymptotic expansion of the (ordinary) heat kernel is more or less regarded as known. The questions or techniques come down partly to that used in Donnelly [21] where the contributions to the trace integral are shown to be essentially supported on the fixed point set of the group action.

In a closely related direction some authors consider the case of Riemannian foliations. In this regard, if the orbits of a group acting by isometries are of the same dimension, this forms an example of a Riemannian foliation. For a Riemannian foliation, one is usually restricted to the space of *basic functions* which are constant on leaves of the foliation. Similar ideas apply to give *basic forms*. The *basic Laplacian* and *basic heat kernel* $K_B(t, x, y)$ can then be defined. Over decades there has been much study into the existence part of the basic heat kernel $K_B(t, x, y)$, which is finally proven in great generality by E. Park and K. Richardson in '96 [56]. Another proof on the existence is found in '98 [58], which gives a specific formula for $K_B(t, x, y)$ and allows them to obtain an asymptotic formula for $K_B(t, x, x)$. We denote the trace integral (on basic functions) by $\text{Tr } e^{-t\Delta_B}$ (which is $\sum_m e^{-t\lambda_m}$ for certain eigenvalues

with multiplicities). In [59] and [58] the trace integral is also denoted by $K_B(t)$ which will be avoided here due to a possible confusion. We shall dwell upon this important point after the next paragraph.

Let's first pause for a moment for comparison. *For the part of the trace integral*, the basic technique based on Donnelly is also employed here so that the extra contributions, if exist, are expected to be supported on the (lower dimensional) strata. One of our features, however, is Lemma 7.6 which leads to a precise information about the Gaussianlike term of the heat kernel and facilitates our ensuing asymptotic expansion (of the trace integral) in explicit expressions essentially based only on the data given by the ordinary (Kodaira) heat kernel (hence computable in a sense, cf. Remarks 7.25, 1.9). In the process we also need to sum over the group elements (Subsections 7.2, 7.3) and patch up these local sums over X (Subsections 7.4, 7.5). *For the part of the asymptotic expansion*, our present heat kernel by its very definition is similar to the K_B above. Yet objects beyond the basic forms, allowing a generalization in the equivariant sense, indexed by $m(\in \mathbb{Z})$ in our notation (with $m = 0$ corresponding to the case for K_B), with bundle-values, are considered here. Since we allow CR nonKähler case, suitable Spin^c structure in our CR version need be devised and equipped here in order for the rescaling technique of Getzler and our discovery of the off-diagonal estimate (Theorem 5.10) to go through. In this regard, it is not obvious at all (to us) whether the existence theory in the Riemannian case as above can be directly applied to our case. Indeed, besides the need of the Spin^c structure, our proof of the heat kernel is heavily based on the feature of the group action on CR manifolds, encoded by the BRT trivialization (Subsection 2.4), through the use of the *adjoint* version of the original equation (Subsection 1.6). Above all, it lies in the following how our approach distinguishes itself from those of others.

Notably, a seeming inconsistency could occur. That is, a discovery in the works [59] and [58] reveals that the so obtained asymptotic expansion for $K_B(t, x, x)$ cannot be integrated (over x) to give the asymptotics of the trace (integral). This perhaps takes one by surprise. See p. 2304 of [59] and remark in p. 379 of [58]. Despite this, the work [59] manages to prove an asymptotic expansion for the trace integral (on basic functions) by using the work [11] (rather than by integrating the asymptotics of $K_B(t, x, x)$ obtained therein). In this way, some nontrivial logarithmic terms are to appear unless they are proved to be vanishing. A conjecture has thus been introduced by K. Richardson in '10 [59, Conjecture 2.5] to the effect that in the Riemannian setting as above, for the (special) case of the isometric group action on a compact manifold, the logarithmic terms in the asymptotic expansion of the trace integral $\text{Tr} e^{-t\Delta_B}$ must vanish and under a mild assumption (on orientation), there shall be no fractional powers in t (except possibly an overall fractional power in t). It is worth mentioning that the works [58] and [59] discuss a number of interesting examples pertinent to the aforementioned peculiar phenomenon. Despite that the seeming inconsistency is

consistent with examples by explicit computations, it remains conceptually unclear how this phenomenon comes about.

Our present work affirms the above conjecture of Richardson (with extension to the S^1 -equivariant case) in the special case of CR manifolds studied here (see Theorems 7.20, 1.14). One key point for all of this lies in (1.4) with t -dependent coefficients in t powers, which is regarded as the asymptotic expansion one shall be dealing with in this paper, rather than a classical looking one (1.3) (which is similar in nature to those proposed and studied in [58], [59]). See also our Remarks 1.6, 1.7 and 1.8, which are closely related to the above singular behavior of a classical formulation of asymptotic expansion. Put simply, the formulation (1.3) of an asymptotic expansion leads to certain discontinuities of the t -coefficients along the strata (cf. [58, (4.7)] for a concrete example). A remedy for (1.3) by (1.4) is mainly made via the introduction of a “distance function” (see Theorem 1.3). Eventually, in this work we can restore the trace integral as the integration of our (unconventional) asymptotic expansion of the relevant heat kernel (see Definition 5.5 for the meaning of our asymptotic expansion). Thus, our trace integral and our asymptotic expansion of the heat kernel jointly clarify (with our class of manifolds) the somewhat undesirable phenomenon which is as mentioned above.

To go from the trace integral to the index theorem (thought of as a supertrace integral) is usually not immediate. To the knowledge of the authors, the argument for the proof of index theorems by using trace integrals remains unclarified (cf. Remark 7.26). Completely new ideas might be required; see [13], [12] for very interesting ideas. In the present paper, we couldn’t make our understanding of the (transversal) heat kernel (for our class of CR manifolds) complete without employing the rescaling technique of Getzler and the off-diagonal estimate (Theorem 5.10) adapted to our setting. These results explore in depth the non-Gaussian terms of our (transversal) heat kernel, in contrast to the Gaussianlike term explored in the trace integral here. With these two parts together, our approach studies the meaningful separate aspects of the heat kernel in a unified manner, hence results in an (local) index theorem and the trace integral. These point to the differences between our approaches/results and those of the recent development.

We turn now to our proof of the trace integral. The line of thought in the proof involves four stages.

In the first stage while the proof in the beginning echoes that in last section, we shall make use of Lemma 6.6 and Theorem 6.7 to handle the distance function \hat{d} . (Here we assume the strongly pseudoconvex condition on X .) After this initial step, we shall take a different approach that supersedes the previous one, which is more quantitative in nature without the strongly pseudoconvex condition on X (hence without using Lemma 6.6 and Theorem 6.7). This approach is partly based on the differential geometric information of the various isotropy actions associated with the

fixed point sets (strata) of the S^1 action. This allows us to learn more precise details about the heat kernel of Kohn Laplacian, hence to refine the computation in (7.13) which is basically qualitative. (See (7.13) for a kind of Dirac delta functions associated with the strata.) Remark that one key point here is the notion of *type* which is initially designed for the need of computation. In the fourth stage it is attached to the S^1 stratification closely.

In the second, third and fourth stages, the treatment goes in line with that in the first stage and is mostly technical so as to integrate the results obtained in the first stage in a well organized manner. The non unique way (subject to choice of BRT trivializations) of giving the asymptotic expansion of $e^{-t\widehat{\square}_{b,m}^+}(t, x, y)$ (cf. Theorem 6.1) leaves us the freedom of choosing convenient BRT charts to work out some computations. The salient fact that $e^{-t\widehat{\square}_{b,m}^+}(t, x, y)$ is an intrinsic object (yet not directly computable), thus is independent of choice of BRT trivializations, is essential to giving intrinsic meanings to some BRT-dependent computations (cf. the contrast between Propositions 7.16 and 7.18 on η_s -terms). This conceptual understanding turns out to be crucial to our final result. The extension of the previous notion “type” to the S^1 stratification is the last conceptual step for the completion of the proof.

As before, X ($\dim X = 2n + 1$) is a compact connected CR manifold with a transversal CR locally free S^1 action. To proceed with the proof of Theorem 1.14, assume

$$(7.1) \quad X = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_k},$$

where

$$(7.2) \quad X_{p_\ell} = \bigcup_{\gamma=1}^{s_\ell} X_{p_\ell(\gamma)}, \quad (s_{\ell=1} = 1),$$

as a disjoint union of (connected) submanifolds $X_{p_\ell(\gamma)}$ (\overline{X}_{p_ℓ} , being the fixed point set of an isometry $e^{-i\frac{2\pi}{p_\ell}}$, is a submanifold (possibly disconnected)).

Write $e_{\ell(\gamma)}$ for the (real) codimension of $X_{p_\ell(\gamma)}$ in X . When there is no danger of confusion, we may drop γ and write e_ℓ for $e_{\ell(\gamma)}$. Recall that $X_{\text{sing}}^{\ell-1} = X_{p_\ell} \cup X_{p_{\ell+1}} \cup \cdots$.

We follow the notations in Subsection 5.2 and the beginning of the last section. Thus $B_j := (D_j, (z, \theta), \varphi_j)$ ($j = 1, 2, \dots, N$) with $D_j = U_j \times]-2\delta_j, 2\tilde{\delta}_j[$, $U_j = \{z \in \mathbb{C}^n; |z| < \gamma_j\}$ and similarly $\hat{D}_j = \hat{U}_j \times]-\frac{\delta_j}{2}, \frac{\tilde{\delta}_j}{2}[$, $\hat{U}_j = \{z \in \mathbb{C}^n; |z| < \frac{\gamma_j}{2}\}$. We let $\delta_j = \tilde{\delta}_j = \zeta$, $j = 1, 2, \dots, N$ and assume $X = \hat{D}_1 \cup \cdots \cup \hat{D}_N$. As before, we assume that $\zeta > 0$ satisfies (1.15).

7.2. Local angular integral

Recall that $\hat{h}_{j,+}(x, y)$, $\hat{b}_{j,s}^+(x, y)$ of (5.39) (to be given below); $a_s^+(t, x, y)$ involves a certain integral over $[0, 2\pi]$ (cf. (5.43)), $s = n, n - 1, \dots$. One key step is the following

local version. That is, the (trace) integral of the form

$$(7.3) \quad I_t = I_t^{(j)}(p_\ell, g) \equiv \frac{1}{2\pi} \int_{\frac{2\pi}{p_\ell} - \varepsilon}^{\frac{2\pi}{p_\ell} + \varepsilon} \int_X g(x) e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ x)}{t}} ((e^{-iu})^*_y \text{Tr} \hat{b}_{j,s}^+)(x, x) e^{-imu} dv_X(x) du.$$

The trace “Tr” here is actually well defined despite a slight abuse of notation about $((e^{-iu})^*_y \hat{b}_{j,s}^+)(x, x)$ at the second variable (see the line above (2.2)).

Recall that the expressions in (5.39) (to be used in what follows):

$$(7.4) \quad \begin{aligned} \hat{h}_{j,+}(x, y) &= \hat{\sigma}_j(\theta) h_{j,+}(z, w) \hat{\sigma}_j(\eta) \in C_c^\infty(D_j), \quad x = (z, \theta), \quad y = (w, \eta) \\ \hat{b}_{j,s}^+(x, y) &= \chi_j(x) e^{-m\varphi_j(z) - im\theta} b_{j,s}^+(z, w) e^{m\varphi_j(w) + im\eta} \tau_j(w) \sigma_j(\eta), \quad s = n, n-1, \dots \end{aligned}$$

with suitable cut-off functions χ_j, τ_j, σ_j and $\hat{\sigma}_j$ defined there.

There will be cases for the Result (7.3). We need some preparations and notations.

For $I_t = I_t(p_\ell, g)$ of (7.3), take a point $x_0 \in \text{Supp } g \cap \overline{X}_{p_\ell}$, then $x_0 \in \overline{X}_{p_\ell(\gamma_\ell)}$ for a $\gamma_\ell \in \{1, \dots, s_\ell\}$. Locally at x_0 there are higher dimensional strata

$$(7.5) \quad \overline{X}_{p_{i_1(\gamma_{i_1})}} = X \supseteq \overline{X}_{p_{i_2(\gamma_{i_2})}} \supseteq \dots \supseteq \overline{X}_{p_{i_f(\gamma_{i_f})}} \supseteq \overline{X}_{p_{i_{f+1}(\gamma_{i_{f+1}})}} = \overline{X}_{p_\ell(\gamma_\ell)}$$

passing through x_0 where $i_1 = 1 < i_2 < \dots < i_f < i_{f+1} = \ell, \in \{1, 2, \dots, \ell - 1, \ell\}$. Here (to be useful later) $p_{i_1} | p_{i_2} \dots | p_{i_f} | p_\ell$ (by Remark 1.17 similarly). We claim that the strata in (7.5) are uniquely determined by x_0 . Fix $\ell \in \{1, \dots, k\}$, where $k \in \mathbb{N}$ is introduced in (7.1). We first show that the decomposition

$$(7.6) \quad \overline{X}_{p_\ell} = \bigcup_{\gamma=1}^{s_\ell} \overline{X}_{p_\ell(\gamma)}, \quad (s_{\ell=1} = 1),$$

is a disjoint union of connected components. We write $\overline{X}_{p_\ell} = \bigcup_{\gamma=1}^{s'_\ell} A_\gamma$ as a disjoint union of connected components. Below the set difference $A \setminus B$ is also denoted as $A - B$. Then we have

$$(7.7) \quad X_{p_\ell} = \left(A_1 \setminus (\overline{X}_{p_\ell} - X_{p_\ell}) \right) \cup \left(A_2 \setminus (\overline{X}_{p_\ell} - X_{p_\ell}) \right) \cup \dots \cup \left(A_{s'_\ell} \setminus (\overline{X}_{p_\ell} - X_{p_\ell}) \right).$$

Since $\overline{X}_{p_\ell} - X_{p_\ell} \subset \bigcup_{j>\ell} (X_{p_j} \cap \overline{X}_{p_\ell})$, $\overline{X}_{p_\ell} - X_{p_\ell}$ is a submanifold of \overline{X}_{p_ℓ} of real codimension ≥ 2 (see Remark 7.22). We conclude that $A_\gamma \setminus (\overline{X}_{p_\ell} - X_{p_\ell})$ is connected, for every $\gamma = 1, \dots, s'_\ell$ and thus the decomposition (7.7) is the same as the decomposition (7.2) and hence $s'_\ell = s_\ell$. We may assume that $A_\gamma \setminus (\overline{X}_{p_\ell} - X_{p_\ell}) = X_{p_\ell(\gamma)}$, for every $\gamma = 1, 2, \dots, s_\ell$. Then $A_\gamma = X_{p_\ell(\gamma)} \cup (\overline{X}_{p_\ell} - X_{p_\ell}) = \overline{X}_{p_\ell(\gamma)}$, for every $\gamma = 1, \dots, s_\ell$. We have proved that the decomposition (7.6) is a disjoint union of connected components. Hence, for every $p_u < p_\ell$, there is at most one connected component of \overline{X}_{p_u} containing $\overline{X}_{p_\ell(\gamma_\ell)}$. Thus, the strata in (7.5) are uniquely determined as claimed.

From now on, our smooth function $g(x)$ will be chosen to be of good support (see Definition 7.7 below for the meaning of good support), Thus Definition 7.1 below is independent of choice of point x_0 . Fix a $i_\ell, i_1 \leq i_\ell \leq i_f$. We say

DEFINITION 7.1. – i) The *type* $\tau(I_t)$ of $I_t(p_\ell, g)$ at x_0 is

$$\tau(I_t) := (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \dots, i_f(\gamma_{i_f}), i_{f+1}(\gamma_{i_{f+1}}))$$

where $i_1 = \gamma_{i_1} = 1$ and $i_{f+1} = \ell$ always. The length $l(\tau(I_t, x_0))$ of the type is $f + 1$. $I_t(p_\ell, g)$ is said to be of *simple type* if in $\tau(I_t, x_0)$, $(i_1, i_2, \dots, i_{f+1}) = (1, 2, \dots, \ell - 1, \ell)$.

ii) Two given types

$$\begin{aligned} \tau(I_t(p_{\ell_1}, g_1)) &= (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \dots, i_{f_1+1}(\gamma_{i_{f_1+1}})) \\ \tau(I_t(p_{\ell_2}, g_2)) &= (j_1(\gamma'_{j_1}), j_2(\gamma'_{j_2}), \dots, j_{f_2+1}(\gamma'_{j_{f_2+1}})) \end{aligned}$$

are said to be in the same *class* provided a) $f_1 = f_2 := f$, $\ell_1 = \ell_2$, $i_1 = j_1, i_2 = j_2, \dots, i_f = j_f$ and b) the codimensions of the corresponding strata coincide: $e_{\ell_1(\gamma_{\ell_1})} = e_{\ell_2(\gamma'_{\ell_2})}$, $e_{i_1(\gamma_{i_1})} = e_{j_1(\gamma'_{j_1})}$, $e_{i_2(\gamma_{i_2})} = e_{j_2(\gamma'_{j_2})}$, \dots , $e_{i_f(\gamma_{i_f})} = e_{j_f(\gamma'_{j_f})}$.

iii) As above $I_t = I_t(p_\ell, g)$, suppose $\text{Supp } g \cap \overline{X}_{p_\ell} = \emptyset$, equivalently $\text{Supp } g \subset \bigcup_{q=1}^{\ell-1} X_{p_{\ell-q}}$. We say $\tau(I_t)$ is of *trivial type*.

In the final subsection, the notion of “type” will be naturally extended to each connected submanifold $X_{p_{\ell(\gamma)}}$ in the strata. By this, the influence of the geometry of the S^1 stratification on the heat kernel trace integral will become more evident.

Most numerical results in what follows will only depend on the equivalence classes of types. But for the sake of notational convenience, we assume I_t to be of simple type or trivial type in the proposition below. In the following proposition, ii) and Case a) of iii) are basically of trivial type; i) and Case b) of iii) are of simple type. The modification to the general type is basically only complicated in notation and will be treated later.

PROPOSITION 7.2. – Suppose $x_0 \in \hat{D}_j$. Then there exist a neighborhood $\tilde{\Omega}$ ($\Subset \hat{D}_j$) of x_0 and an $\tilde{\varepsilon} > 0$ (depending on x_0) such that for every $\Omega \subset \tilde{\Omega}$, every $g(x) \in C_c^\infty(\Omega)$ we have the following for I_t of (7.3) with any $\varepsilon \leq \tilde{\varepsilon}$.

i) $\ell = 1$. For $x = (z, v) \in \hat{D}_j$ write $z(x) = z$ and $\theta(x) = v$.

$$I_t = e^{-\frac{2\pi i}{p} m} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \int_X g(x) \chi_j(x) \text{Tr } b_{j,s}^+(z, z) \tau_j(z) \sigma_j(v + \psi) dv_X(x) d\psi.$$

In particular, I_t is a constant independent of t . (Note it is $b_{j,s}^+$ instead of $\hat{b}_{j,s}^+$ here; the same can be said with (7.8) below.)

ii) Suppose $e^{-i\frac{2\pi}{p_\ell}} \circ x_0 \notin \hat{D}_j$ (here $\ell = 2, 3, \dots, k$). Then $I_t = 0$.

iii) Suppose $e^{-i\frac{2\pi}{p_\ell}} \circ x_0 \in \hat{D}_j$ (here $\ell = 2, 3, \dots, k$).

Case a) $x_0 \in \bigcup_{q=1}^{q=\ell-1} X_{p_{\ell-q}}$. Then $I_t \sim O(t^\infty)$ as $t \rightarrow 0^+$.

Case b) $x_0 \notin \bigcup_{q=1}^{q=\ell-1} X_{p_{\ell-q}}$ and $x_0 \in \overline{X}_{p_{\ell(\gamma_\ell)}} \subset \overline{X}_{p_\ell}$. Take local coordinates ($e_\ell = e_{\ell(\gamma_\ell)}$) for some $\gamma_\ell = 1, \dots, s_\ell$)

$$y = (y_1, \dots, y_{2n+1}) = (\hat{y}, Y) \text{ with } \hat{y} = (y_1 \cdots, y_{e_\ell}) \text{ and } Y = (y_{e_\ell+1}, \dots, y_{2n+1})$$

defined on Ω such that

$$\overline{X}_{p_\ell} \cap \Omega = \{y \in \Omega; y_1 = \cdots = y_{e_\ell} = 0\}.$$

Assume (possibly after shrinking Ω about x_0) $\Omega = \bigcup_{j \in \{1, \dots, k\}} (X_{p_j(\gamma_j)} \cap \Omega)$ (for some $\gamma_j = 1, \dots, s_j$) which is seen to be (by assumption of simple type)

$$(\overline{X}_{p_\ell(\gamma_\ell)} \cap \Omega) \bigcup_{q=1}^{\ell-1} (X_{p_{\ell-q}(\gamma_{\ell-q})} \cap \Omega).$$

Write $e_{\ell-q+1} - e_{\ell-q}$ for the codimension of $X_{p_{\ell-q+1}}$ in $\overline{X}_{p_{\ell-q}}$ where $p_\mu = p_{\mu(\gamma_\mu)}$ for $\mu = \ell - q + 1$ and $\mu = \ell - q$ respectively. If $y = (z, \theta)$ (in BRT coordinates), write $z(y)$ for z and if $y = (0, Y)$, write Y for $(0, Y)$ and $z(Y)$ for $z(y)$. Similar notation for $\theta(Y)$ etc.

Then ($e_\ell = e_{\ell(\gamma_\ell)}$)

$$I_t = b_{s, \frac{e_\ell}{2}}^{(j)} t^{\frac{e_\ell}{2}} + b_{s, \frac{e_{\ell+1}}{2}}^{(j)} t^{\frac{e_{\ell+1}}{2}} + \cdots,$$

where the first coefficient $b_{s, \frac{e_\ell}{2}}^{(j)}$ is given by

$$(7.8) \quad b_{s, \frac{e_\ell}{2}}^{(j)} = \pi^{\frac{e_\ell}{2}} e^{-\frac{2\pi i}{p_\ell} m} \prod_{q=1}^{\ell-1} \left| e^{\frac{i2\pi}{p_\ell} p_{\ell-q}} - 1 \right|^{-(e_{\ell-q+1} - e_{\ell-q})} \times \\ \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \int_{\overline{X}_{p_\ell(\gamma_\ell)}} g(Y) \chi_j(Y) \text{Tr} b_{j,s}^+(z(Y), z(Y)) \tau_j(z(Y)) \sigma_j(\theta(Y) + u) dv_{\overline{X}_{p_\ell(\gamma_\ell)}}(Y) du.$$

In particular, for $s = n$ (cf. $\dim X = 2n + 1$), (7.8) for $b_{n, \frac{e_\ell}{2}}^{(j)}$ simplifies by using $\text{Tr} b_{j,n}^+(z, z) \equiv (2\pi)^{-n}$.

Proof. – Write $x_0 = (z_0, \theta_0)$. For simplicity, assume $\theta_0 = 0$ without loss of generality (cf. the last three paragraphs of the proof of Theorem 6.1 for a similar situation). Note that the existence of $\tilde{\Omega}$ and $\tilde{\varepsilon}$ in the statement above will be obvious from the proof below and we shall not refer to them explicitly.

To see i), we note that $e^{-i\frac{2\pi}{p}} = \text{id}$ on X (because it is so on X_p by definition which is dense (and open) in X). For $x = (z, v)$ lying in the BRT neighborhood \hat{D}_j and for $u = \frac{2\pi}{p} \pm \varepsilon$ such that $e^{-iu} \circ x = e^{\pm i\varepsilon} \circ x$ lies in D_j ($\supset \hat{D}_j$), one has $e^{\pm i\varepsilon} \circ x = (z, v \mp \varepsilon)$ by construction of BRT charts D_j . In this case $e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ x)}{t}} \equiv 1$ since $\hat{h}_{j,+}(x, e^{-iu} \circ x) = 0$ by $h_{j,+}(z, z) = 0$ of (7.4). The same reasoning applies to $((e^{-iu})_y^* \text{Tr} \hat{b}_{j,s}^+)(x, x)$ to reach $\text{Tr} b_{j,s}^+(z, z)$. Now choose a neighborhood $\Omega \Subset \hat{D}_j$ of x_0 then a small $\varepsilon > 0$ (depending on x_0) such that $e^{\pm i\varepsilon} \circ x$ lies in D_j for $x \in \Omega$. As $g \in C_c^\infty(\Omega)$, we can apply the above argument for these x by making $\psi = u - \frac{2\pi}{p}$ ($|\psi| \leq \varepsilon$) so that $e^{-iu} \circ x = e^{-i\psi} \circ x = (z, v + \psi)$. In (7.4) one thus has $\theta = v$, $w = z$ and $\eta = v + \psi$. By these remarks, i) of the proposition follows from (7.3) and (7.4).

For ii) of the proposition, with $x_0 \in \hat{D}_j$ yet $e^{-i\frac{2\pi}{p\ell}} \circ x_0 \notin \hat{D}_j$, by continuity of S^1 action there exist a neighborhood Ω ($\subseteq \hat{D}_j$) of x_0 and an $\varepsilon > 0$ such that $e^{-iu} \circ x \notin \hat{D}_j$ for $x \in \Omega$ and $u \in]\frac{2\pi}{p\ell} - \varepsilon, \frac{2\pi}{p\ell} + \varepsilon[$. Hence for these $x \in \Omega$, $\hat{h}_{j,+}(x, e^{-i\frac{2\pi}{p\ell}} \circ x) = 0$ because of the cut-off function τ_j (of compact support in $\hat{U}_j \subset \hat{D}_j$) involved in $\hat{h}_{j,+}$ (see (7.4)), giving $I = 0$ in (7.3).

For case a) of iii), the assumption gives $x_0 \in X_{p\ell-q}$, $q \in \{1, 2, \dots, \ell - 1\}$. Further, by assumption $e^{-i\frac{2\pi}{p\ell}} \circ x_0 \in \hat{D}_j$ we write $e^{-i\frac{2\pi}{p\ell}} \circ x_0 = (\tilde{z}_0, \tilde{\theta}_0)$ with $|\tilde{\theta}_0| < \frac{\zeta}{2}$. We claim $\tilde{z}_0 \neq z_0$. The line of argument is slightly different from that in (6.4). Suppose $\tilde{z}_0 = z_0$. Then by $e^{i\tilde{\theta}_0} \circ (e^{-i\frac{2\pi}{p\ell}} \circ x_0) = e^{i\tilde{\theta}_0} \circ (\tilde{z}_0, \tilde{\theta}_0) = (\tilde{z}_0, 0) = (z_0, 0) = x_0$ (recall that $\theta_0 = 0$ in the beginning of the proof). Hence,

$$(7.9) \quad \frac{2\pi}{p\ell} - \tilde{\theta}_0 = m \frac{2\pi}{p\ell-q}, \quad m \in \mathbb{Z}$$

by assumption $x_0 \in X_{p\ell-q}$. But $|\tilde{\theta}_0| < \frac{\zeta}{2}$ and ζ is assumed to satisfy (1.15), so the above equality is absurd, proving the claim $\tilde{z}_0 \neq z_0$ by contradiction.

Now that $\tilde{z}_0 \neq z_0$, there exists a neighborhood Ω of x_0 and an $\varepsilon > 0$ (dependent on x_0) such that for $x \in \Omega$ and $\theta \in]\frac{2\pi}{p\ell} - \varepsilon, \frac{2\pi}{p\ell} + \varepsilon[$, writing $e^{-i\theta} \circ x = (\tilde{z}, \tilde{\theta})$ and $x = (z, \theta)$ one has $|\tilde{z} - z| \geq \frac{1}{2} |\tilde{z}_0 - z_0| \equiv \delta$ by using continuity of S^1 action at $x = x_0$ and $\theta = \frac{2\pi}{p\ell}$. From the property of $\hat{h}_{j,+}(x, y)$ (which is essentially $|z - w|^2$, cf. (5.19) and (5.39)) one sees that I of (7.3) gives

$$(7.10) \quad \frac{1}{2\pi} \int_{\frac{2\pi}{p\ell} - \varepsilon}^{\frac{2\pi}{p\ell} + \varepsilon} \int_X g(x) e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ x)}{t}} ((e^{-iu})_y^* \text{Tr } b_{j,s}^+(x, x) e^{-imu} dv_X(x) du = O(t^\infty),$$

as $t \rightarrow 0^+$ (for $g \in C_c^\infty(\Omega)$) simply because the exponential term in (7.10) decays rapidly if $|\tilde{z} - z| \geq \delta$ here, proving case a) of iii) of the proposition.

To prove case b) of iii), we first give an estimate under an additional assumption that X is strongly pseudoconvex, then we will drop this assumption and carry out some refined computations to complete the proof.

Since x_0 is a fixed point of $e^{-\frac{2\pi i}{p\ell}}$ by assumption, by continuity of S^1 action there exist an open subset Ω of x_0 and a small constant $0 < \varepsilon < \frac{\zeta}{2}$ such that $e^{-i\theta} \circ x \in \hat{D}_j$ for $x \in \Omega$ and $\theta \in]\frac{2\pi}{p\ell} - \varepsilon, \frac{2\pi}{p\ell} + \varepsilon[$. We assume that Ω is small, say contained in the BRT chart \hat{D}_j , and satisfies the local coordinates of case b) of iii). For $x = (z, v) \in \Omega$ and $\theta \in]\frac{2\pi}{p\ell} - \varepsilon, \frac{2\pi}{p\ell} + \varepsilon[$, write $e^{-i\theta} \circ x = (\tilde{z}, \tilde{v}) \in \hat{D}_j$.

We claim that there exists a positive continuous function $f_1(x)$ such that

$$(7.11) \quad h_{j,+}(z, \tilde{z}) \geq f_1(x) d(x, X_{p\ell})^2, \quad \forall x \in \Omega,$$

where $h_{j,+}$ is as in (7.4) (cf. (5.19)). (Here is the only place where we use the assumption that X is strongly pseudoconvex.)

Granting the claim (7.11), with the local coordinates in iii), suppose $y(x_0) = Y(x_0) = 0$. Rewrite the quotient $\frac{d(y, X_{p_\ell})^2}{(|y_1|^2 + \dots + |y_{e_\ell}|^2)}$ as

$$(7.12) \quad d(y, X_{p_\ell})^2 = f_2(y)(|y_1|^2 + \dots + |y_{e_\ell}|^2), \quad \forall y \in \hat{\Omega},$$

where $f_2(y)$ is a positive continuous function.

With (7.11) and (7.12), we estimate $\hat{h}_{j,+}$ below and have the following (see (7.4) or (5.39) and note $g(x) \in C_0^\infty(\Omega)$, Ω small).

$$(7.13) \quad \begin{aligned} I_t &= \frac{1}{2\pi} \int_{\frac{2\pi}{p_\ell} - \varepsilon}^{\frac{2\pi}{p_\ell} + \varepsilon} \int_X g(x) e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ x)}{t}} ((e^{-iu})_y^* \text{Tr } b_{j,s}^+) (x, x) e^{-imu} dv_X(x) du \\ &\leq \frac{1}{2\pi} \int_{\frac{2\pi}{p_\ell} - \varepsilon}^{\frac{2\pi}{p_\ell} + \varepsilon} \int_X g(x) e^{-\frac{f_1(x) d(x, X_{p_\ell})^2}{t}} ((e^{-iu})_y^* \text{Tr } b_{j,s}^+) (x, x) e^{-imu} dv_X(x) du \\ &= \frac{1}{2\pi} \int_{\frac{2\pi}{p_\ell} - \varepsilon}^{\frac{2\pi}{p_\ell} + \varepsilon} \int_X g(y) e^{-\frac{f_1(y) f_2(y) (|y_1|^2 + \dots + |y_{e_\ell}|^2)}{t}} ((e^{-iu})_y^* \text{Tr } b_{j,s}^+) (y, y) e^{-imu} dv_X(y) du \\ &\sim c_{s, \frac{e_\ell}{2}}^{(j)} t^{\frac{e_\ell}{2}} + c_{s, \frac{e_\ell+1}{2}}^{(j)} t^{\frac{e_\ell+1}{2}} + \dots \text{ as } t \rightarrow 0^+, \end{aligned}$$

where the last step is obtained by a change-of-variable (rescaling y_i by $\sqrt{t}y_i$, $i = 1, \dots, e_\ell$, $e_\ell \geq 1$ as $\ell \geq 2$) and $c_{s,k}^{(j)} \in \mathbb{R}$ is independent of t ($k = \frac{e_\ell}{2}, \frac{e_\ell+1}{2}, \dots$).

We are left with the proof of the claim (7.11). Part of the argument echos that for (6.4). We first estimate $|z - \tilde{z}|^2$. Without any danger of confusion we omit “ \circ ” in what follows. By $(z, 0) = e^{iv}x$ and $(\tilde{z}, 0) = e^{i\tilde{v}}(e^{-i\theta}x)$, $|z - \tilde{z}|$ is equivalent to $d(e^{iv}x, e^{i\tilde{v}}(e^{-i\theta}x))$ (cf. (6.3)) which is the same as $d(x, e^{-iv}(e^{i\tilde{v}}(e^{-i\theta}x)))$. As $(\tilde{z}, \tilde{v}), (z, v) \in \hat{D}_j$, one has $\tilde{v}, v \leq \frac{\zeta}{2}$. By choosing ζ, ε to be (much) less than the ε_0 of Lemma 6.6, one sees $d(x, e^{-iv}(e^{i\tilde{v}}(e^{-i\theta}x))) \geq \hat{d}_2(x, X_{\text{sing}}^{\ell-1})$ of Lemma 6.6. By the same lemma

$$(7.14) \quad \hat{d}_2(x, X_{\text{sing}}^{\ell-1}) \text{ is equivalent to } \hat{d}(x, X_{\text{sing}}^{\ell-1}).$$

As $|z - \tilde{z}|^2$ is also equivalent to $h_{j,+}(z, \tilde{z})$ of (7.11) (cf. (6.3)) and (7.14) is equivalent to $d(x, X_{\text{sing}}^{\ell-1})$ by Theorem 6.7 with our assumption X is strongly pseudoconvex, we have now shown

$$(7.15) \quad h_{j,+}(z, \tilde{z}) \geq c \cdot d(x, X_{\text{sing}}^{\ell-1})^2$$

for some constant $c > 0$. In view that $X_{\text{sing}}^{\ell-1} = X_{p_\ell} \cup X_{p_{\ell+1}} \dots$ and the assumption that $x_0 \in \bar{X}_{p_\ell}$, one sees, possibly after shrinking Ω , $d(x, X_{\text{sing}}^{\ell-1}) = d(x, \bar{X}_{p_\ell}) = d(x, X_{p_\ell})$. Hence we have reached (7.11) from (7.15), as desired.

Following (7.13) we shall now make some accurate computations for case b) of iii) of this proposition.

Henceforward we do not assume that X is strongly pseudoconvex; we will not use Lemma 6.6 and Theorem 6.7 as used above.

Write $\frac{2\pi}{p_\ell} = \omega$ and $u = \psi + \omega$. In coordinates of case b) of iii), in view of (7.13) one seeks to identify, among others,

$$(7.16) \quad \lim_{t \rightarrow 0^+} \frac{\hat{h}_{j,+}((\sqrt{t}\hat{y}, Y), e^{-iu}(\sqrt{t}\hat{y}, Y))}{t},$$

where we have rescaled $\hat{y} \rightarrow \sqrt{t}\hat{y}$, and we omit “o” for the $e^{-iu} \in S^1$ action.

Since the fixed point set of an isometry is totally geodesic, we assume Y to be a system of geodesic coordinates at $Y = 0$ of \overline{X}_{p_ℓ} , as (\hat{y}, Y) the geodesic coordinates at $(0, 0)$ of X . We choose $Y = 0$ in (7.16) to simplify the notation. Expressed in BRT coordinates, $(\sqrt{t}\hat{y}, 0) = (z_0, v_0)$ and $e^{-i\omega}(\sqrt{t}\hat{y}, 0) = (z_1, v_1)$ (by continuity of S^1 action, for t small $e^{-i\omega}(\sqrt{t}\hat{y}, 0) \in \hat{D}_j$ since $e^{-i\omega}(0, 0) = (0, 0)$).

One sees $e^{-iu}(\sqrt{t}\hat{y}, 0) = e^{-i\psi}(z_1, v_1) = (z_1, v_1 + \psi)$ for $|\psi| \leq \varepsilon$. By (7.4) one sees

$$(7.17) \quad \hat{h}_{j,+}((\sqrt{t}\hat{y}, 0), e^{-iu}(\sqrt{t}\hat{y}, 0)) \text{ is now reduced to } h_{j,+}(z_0, z_1),$$

which is independent of ψ for u in the ε -neighborhood of ω .

Namely $\hat{h}_{j,+}((\sqrt{t}\hat{y}, 0), e^{-iu}(\sqrt{t}\hat{y}, 0)) \equiv \hat{h}_{j,+}((\sqrt{t}\hat{y}, 0), e^{-i\omega}(\sqrt{t}\hat{y}, 0))$ for u in the ε -neighborhood of ω .

Now $x_0 = (0, 0)$ is a fixed point of $e^{-i\omega}$ ($\omega = \frac{2\pi}{p_\ell}$). One can see that $T_{x_0}X$ under the isotropy action induced by $e^{-i\omega}$ decomposes as an orthogonal direct sum of eigenspaces (where $N(S/M)$ denotes the normal bundle of a submanifold S in an ambient manifold M with $N_p(S/M)$ the fiber of $N(S/M)$ at p , and $\overline{X}_{p_\mu} = \overline{X}_{p_\mu(\gamma_\mu)}$ for some $\gamma_\mu = 1, 2, \dots, s_\mu; \mu = 1, 2, \dots, \ell$)

$$T_{x_0}\overline{X}_{p_\ell}, N_{x_0}(\overline{X}_{p_\ell}/\overline{X}_{p_{\ell-1}}), N_{x_0}(\overline{X}_{p_{\ell-1}}/\overline{X}_{p_{\ell-2}}) \cdots, N_{x_0}(\overline{X}_{p_2}/\overline{X}_p)$$

associated with eigenvalues

$$(7.18) \quad 1, e^{i\omega p_{\ell-1}}, e^{i\omega p_{\ell-2}}, \dots, e^{i\omega p} \text{ respectively.}$$

For instance, assume $\ell = 2$ and take $g = e^{\frac{2\pi i}{p_2}}$. Set $q = \frac{p_2}{p} \in \mathbb{N}$. On $N_{x_0}(\overline{X}_{p_2}/\overline{X}_p)$, $g \neq \text{id}$ and $g^q = \text{id}$. Hence $v \in N_{x_0}(\overline{X}_{p_2}/\overline{X}_p)$ is rotated by the angle $\frac{2\pi}{q}$ which is ωp .

The goal in what follows is to prove the claim that for $q = 1, \dots, \ell - 1$,

$$(7.19) \quad \lim_{t \rightarrow 0^+} \frac{\hat{h}_{j,+}((\sqrt{t}\hat{y}, 0), e^{-i\omega}(\sqrt{t}\hat{y}, 0))}{t} = |e^{i\omega p_{\ell-q}} - 1|^2 \|\hat{y}\|^2 \text{ for } \hat{y} \in N_{x_0}(\overline{X}_{\ell-q+1}/\overline{X}_{\ell-q})$$

or equivalently, in the notation above (see (7.17))

$$(7.20) \quad \lim_{t \rightarrow 0^+} \frac{h_{j,+}(z_0, z_1)}{t} = |e^{i\omega p_{\ell-q}} - 1|^2 \|\hat{y}\|^2 \text{ (for } \hat{y} \in N_{x_0}(\overline{X}_{\ell-q+1}/\overline{X}_{\ell-q}))$$

where $\|\cdot\|$ denotes the norm with respect to the metric tensor of X at x_0 .

Our proof of claim (7.19) is based on the following sequence of lemmas.

LEMMA 7.3. – *With the notation above, for $\hat{y} \in N_{x_0}(\overline{X}_{\ell-q+1}/\overline{X}_{\ell-q})$ we have*

$$\lim_{t \rightarrow 0^+} \frac{d_X^2((\sqrt{t}\hat{y}, 0), e^{-i\omega}(\sqrt{t}\hat{y}, 0))}{t} = |e^{i\omega p_{\ell-q}} - 1|^2 \|\hat{y}\|^2,$$

where d_X denotes the distance on X .

Proof. – On $T_{(0,0)}X$ the action induced by $e^{-i\omega}$ rotates the tangent vector \hat{y} by the angle $\omega p_{\ell-q}$. Hence the lemma follows from the well known fact that in a Riemannian manifold (M, g) , if a, b in M are the images of A, B in T_pM by the exponential map at $p \in M$, then as $(a, b) \rightarrow (p, p)$

$$(7.21) \quad \lim \frac{d_M(a, b)}{\|A - B\|} \rightarrow 1,$$

where $\|\cdot\|$ is g at p (cf. [39, Proposition 9.10]). □

SUBLEMMA 7.4. – *Suppose N is a Riemannian submanifold of a Riemannian manifold M . Then the respective distance functions on M and on N are infinitesimally the same. More precisely, suppose in N , $p_n \neq q_n$ for all $n \in \mathbb{N}$, and $p_n, q_n \rightarrow p$ as $n \rightarrow \infty$ for a given point $p \in N$. Then $\lim_{n \rightarrow \infty} \frac{d_M(p_n, q_n)}{d_N(p_n, q_n)} = 1$. Moreover, suppose $t_{n,M}$ (resp. $t_{n,N}$) in $T_{p_n}M$ are the unit tangent vectors along which the minimal geodesics in M (resp. N) join p_n and q_n . Then $\lim_n(t_{n,M} - t_{n,N}) = 0$.*

Proof. – Suppose the special case $p_n = p$ for all n . Let γ_n be a geodesic (with unit speed) of N joining p and q_n , and $\beta_n = \exp_p^{-1}(\gamma_n) \subset T_pM$. Write $l_n(t)$ for the length of (part of) β_n (with the parameter going from 0 to t) measured with the metric $g_{ij} = 1 + O(|x|^2)$ in geodesic coordinates (at p). Write $\|v\|$ for the Euclidean norm of a vector $v \in T_pM$ expressed in geodesic coordinates. Given a curve $\beta(t) \subset T_pM$, $\beta(0) = p$, $\dot{\beta}(0) \neq 0$, one sees the length function $l(t) = \int_0^t \sqrt{\langle \dot{\beta}(t), \dot{\beta}(t) \rangle_{g_{ij}}} dt$ satisfies $\left| \frac{\|\beta(t)\| - l(t)}{\|\beta(t)\|} \right| = O(t) \leq Ct$ for a (locally bounded) quantity C which depends only, apart from β , on the local geometry at p (uniformly). Clearly this implies the lemma if q_n is assumed to approach p along a given geodesic γ of N . If q_n approaches p along different geodesics γ_n , since these geodesics can be uniformly controlled by the local geometry around p , the same results hold as well. For the general case where p_n are different, the similar argument using the control by local geometry implies $\left| \frac{d_M(p_n, q_n) - d_N(p_n, q_n)}{d_M(p_n, q_n)} \right| \leq C(d_M(p_n, q_n))$. The assertion about the unit tangent vectors can be proved similarly. □

SUBLEMMA 7.5. – *Let N be a differentiable manifold equipped with two Riemannian metrics g and h , and $p \in N$. Assume that $g(p) = h(p)$. Suppose that in N , $p_n \neq q_n$ for all $n \in \mathbb{N}$ such that $\lim_n p_n = \lim_n q_n = p$. Then $\lim_{n \rightarrow \infty} \frac{d_g(p_n, q_n)}{d_h(p_n, q_n)} = 1$ where d_\bullet denotes the (metric-dependent) distance function on N .*

Proof. – The result is local; assume $N \subset \mathbb{R}^n$ as an open subset. By comparison to a fixed Euclidean metric, we assume g is Euclidean inherited from \mathbb{R}^n . By reasoning similar to the previous sublemma, it is seen that for $n \gg 1$, $d_n(p_n, q_n)$ is basically $(1 \pm C \max\{\|p_n - p\|, \|q_n - p\|\})d_g(p_n, q_n)$ (where $\|\cdot\|$ denotes the Euclidean norm) with a uniform bound C . The assertion follows. \square

The last lemma (as our main lemma) is as follows. (This lemma can be viewed as a sharp version of the important claim (6.4) in the proof of Theorem 6.1, which bears upon the reason why our distance function \hat{d} arises.)

LEMMA 7.6. – *In the previous notation, write $p = (z(p), \theta(p))$ and $q = (z(q), \theta(q))$ in (z, θ) coordinates on the BRT chart $\hat{D}_j = \hat{U}_j \times]-\frac{\zeta}{2}, \frac{\zeta}{2}[$. We omit the subscript j in what follows. Let $S = S^1 \circ p$ be the S^1 -orbit of p and $N(S/X)$ be the normal bundle of S in X identified with the orthogonal complement of TS in $TX|_S$. Suppose $p_n \neq q_n$ for all n and $p_n, q_n \rightarrow p \in X$ as $n \rightarrow \infty$ such that $D_n = \exp_{X,p}^{-1}(p_n)$, $A_n = \exp_{X,p}^{-1}(q_n) \in N_p(S/X)$. In the case where $p_n \neq p$ and $q_n \neq p$ (for all n), suppose that the angle at p given by the vectors D_n and A_n are bounded away from 0 as $n \rightarrow \infty$. Then (see (4.1) for the metric on U giving d_U below)*

$$(7.22) \quad \lim_{n \rightarrow \infty} \frac{d_X(p_n, q_n)}{d_U(z(p_n), z(q_n))} = 1.$$

In particular $z(p_n) \neq z(q_n)$ for n large.

Proof. – In this proof we take the same notation $U = \hat{U}_j$ seated as an embedded submanifold of X with $\theta = 0$. As in (the proof of) Sublemma 7.4, we first assume $p_n = p$ for all n . By applying the S^1 isometries we assume $p = (z(p), 0) \in U$. By using the construction of our rigid metric on X (cf. (2.5) and (4.1)) it may be assumed that at p , $N_p(S/X) = T_p U$. To see this, by the geometrical interpretation of BRT transformations in (the proof of) Proposition 4.2 one can adjust the BRT coordinates such that $d\phi(p) = 0$ (similar to the well known fact that for a hermitian metric h of a holomorphic line bundle L on a complex manifold, at any given point p one has $dh(p) = 0$ up to a change of local frames of L). This gives $T_p^{0,1}D = \{\frac{\partial}{\partial \bar{z}_j} + i \frac{\partial \phi}{\partial \bar{z}_j}(z) \frac{\partial}{\partial \theta}; j = 1, 2, \dots, n\} = \{\frac{\partial}{\partial \bar{z}_j}; j = 1, 2, \dots, n\}$ (and $T^{1,0}D = \overline{T^{0,1}D}$) (cf. *loc. cit.*). It easily follows $N_p(S/X) = T_p U$ as claimed. A word of caution is in order. The (intrinsic) geometrical interpretation for BRT charts (cf. *loc. cit.* and remarks after (4.1)) shows that the asserted (7.22) is independent of choice of BRT coordinates (on that particular BRT neighborhood). On the other hand, U considered as an embedded submanifold of X as done above does depend on the choice of BRT coordinates.

The first reduction step is as follows. U is endowed with another (Riemannian) metric inherited, as a submanifold, from that of X . This is different from the metric originally defined (on U , cf. (4.1)), yet the two metrics coincide at p as can be seen

above. By Sublemma 7.5 it is enough to prove (7.22) with this inherited metric. Without the danger of confusion we shall adopt the same notation $d_U(\cdot, \cdot)$ for the new distance function in what follows.

Fix an n and set $q = q_n$, $q^{i\theta(q)} \circ q = (z(q), 0) \in U$. Put $A = \exp_{X,p}^{-1}(q)$, $B = \exp_{X,p}^{-1}(q') \in T_p X$, so $\|A\| = d_X(p, q)$, $\|B\| = d_X(p, q')$ (where $T_p X$ is equipped with the Euclidean metric $\|\cdot\|$ in geodesic coordinates). We are going to prove, as $n \rightarrow \infty$,

$$(7.23) \quad \lim \frac{\|A\|}{\|B\|} = 1.$$

One has $d_X(p, q')/d_U(p, q') \rightarrow 1$ by Sublemma 7.4. Hence, to prove (7.22) for this special case $p_n = p$ is the same as to prove $d_X(p, q)/d_X(p, q') \rightarrow 1$ which is (7.23) above.

To see (7.23) (hence (7.22)), we first argue (7.24) below. Let $L \subset T_p X$ be the line determined by A, B , i.e., $L = \{A + t(B - A); t \in \mathbb{R}\}$. Then

$$(7.24) \quad L \text{ is approximately orthogonal to } A \text{ and to } B \text{ (when } n \text{ is large)}.$$

Note that $A \in N_p(S/X) = T_p U$ from the condition of the lemma, and that B is nearly lying on $T_p U$ (with a small angle between B and $T_p U$) by using Sublemma 7.4 on tangents. Let $\Gamma_1 = \{e^{i\theta} \circ q\}_{\theta \in [0, \theta(q)]} \subset X$ ($\theta(q) \geq 0$, say) joining q and q' and $\Gamma = \exp_{X,p}^{-1} \Gamma_1 \subset T_p X$ joining A and B . Recall that the vector field T induced by the S^1 action is orthogonal to U at p as mentioned above, hence $T_{q'} \Gamma_1 \perp T_{q'} U$ approximately (as $n \gg 1$). On the other hand, by $T_p S \perp T_p U$ and $\Gamma_1 \approx S$ (as $n \gg 1$), one sees by $A \in T_p U$ that $T_A \Gamma \perp T_p U$ approximately (as vector subspaces in $T_p X$). In sum, if q, q' are close to p (so $T_{q'} U$ close to $T_p U$), then $T_A \Gamma \perp T_p U$, $T_B \Gamma \perp T_p U$ approximately (cf. the foliation argument below); for this we write $\Gamma \perp A, B$ approximately. We are ready to prove (7.24). Pulling back the S^1 foliation locally around p via $\exp_{X,p}$ in the same way as Γ was obtained from Γ_1 , there is a foliation \mathcal{F} around p in $T_p X$ in which (part of) Γ lies as a leaf. Write $p \in \Gamma_0 (\subset \exp_{X,p}^{-1} S) \in \mathcal{F}$. As $n \gg 1$, the line L determined by $A, B \in \Gamma$ tends to the tangent line ($= T_p S$) to Γ_0 at p (since the leaf Γ of \mathcal{F} tends to the leaf Γ_0). Hence by using the uniform continuity for \mathcal{F} around p , L is close to lines \tilde{L} tangent to leaves $\tilde{\Gamma}$ of \mathcal{F} if $\tilde{\Gamma}$ are nearby Γ_0 and \tilde{L} nearby $T_p S$. In particular, L is close to the tangent lines $T_A \Gamma, T_B \Gamma$ (as $n \gg 1$). Now that $\Gamma \perp A, B$ approximately as just shown, giving readily $T_A \Gamma \perp A, T_B \Gamma \perp B$ approximately, this in turn yields $L \perp A, B$ approximately ($n \gg 1$), proving (7.24).

For q' close to p , by simple Euclidean geometry (on $T_p X$), $\|A - B\|$ is rather small in comparison to $\|A\|$ and $\|B\|$ by using $L \perp A, B$ approximately (7.24), i.e., $\|A - B\| = o(\|A\|), o(\|B\|)$. By using law of cosines, one can obtain (7.23), yielding the special case $p_n = p$ of the lemma. As this step appears crucial and will be instrumental to the general case, we prefer to supply some details as follows.

Take a triangle with vertices T_i ($i = 1, 2, 3$), angles α_i at T_i and δ_i the length of the side facing T_i . Suppose $\alpha_2 \leq \alpha_3$ and both $\approx \frac{\pi}{2}$. Set $\alpha_2 = \frac{\pi}{2} - \alpha$, $0 < \alpha \ll 1$. Let D sit on the line L_1 determined by T_2 and T_3 such that the line L_2 determined by T_1 and D is perpendicular to L_1 . Assume first that D sits between T_2 and T_3 . Then $\delta_1 = \delta_3 \sin \theta_1 + \delta_2 \sin \theta_2$ where θ_1, θ_2 (with $\theta_1 + \theta_2 = \alpha_1$) are angles given by L_2 and the two sides at T_1 . Thus $\frac{\delta_1}{\delta_3} \leq 2 \sin \alpha_1$ ($\frac{\delta_2}{\delta_3} \leq 1$ by $\alpha_2 \leq \alpha_3$). If D sits outside the triangle, then $\delta_1 = \delta_3 \sin \alpha - \delta_2 \sin \theta_3$, $\theta_3 = \alpha - \alpha_1$, so $\frac{\delta_1}{\delta_3} \leq \sin \alpha$. One obtains $\frac{\delta_1}{\delta_3} \rightarrow 0$ if both $\alpha_1 \rightarrow 0$ and $\alpha \rightarrow 0$. By $\delta_1^2 = \delta_2^2 + \delta_3^2 - 2\delta_2\delta_3 \cos \alpha_1$, one has

$$\left(1 - \frac{\delta_2}{\delta_3}\right)^2 = \left(\frac{\delta_1}{\delta_3}\right)^2 - \frac{2\delta_2}{\delta_3}(1 - \cos \alpha_1) \leq \left(\frac{\delta_1}{\delta_3}\right)^2 \leq \sin \alpha + 2 \sin \alpha_1$$

giving $\frac{\delta_2}{\delta_3} \rightarrow 1$ if both $\alpha_2, \alpha_3 \approx \frac{\pi}{2}$ (hence $\alpha, \alpha_1 \approx 0$). As said, this yields (7.23).

We draw some consequences in order for the general case. If $\alpha_1 \rightarrow 0$ (by $\alpha_2, \alpha_3 \rightarrow \pi/2$), then the two sides at α_1 are close to each other, i.e., $\lim(A/\|A\| - B/\|B\|) = 0$ (as $q \rightarrow p$). One sees that if $C = \exp_{U,p}^{-1}(q') \in T_p U$, then by using (7.23) and Sublemma 7.4 on tangents via B ,

$$(7.25) \quad a) \lim \|A/\|C\| = 1, \quad b) \lim(A/\|A\| - C/\|C\|) = 0.$$

We are ready to prove the general case $p_n \neq p$. Write $D = \exp_{X,p}^{-1}(p_n)$, $F = \exp_{U,p}^{-1}(p'_n)$ in the same way as $A = \exp_{X,p}^{-1}(q_n)$, $C = \exp_{U,p}^{-1}(q'_n)$ above. With D, F in place of A, C in (7.25) one has the same results for D, F :

$$(7.26) \quad a) \lim \|D/\|F\| = 1, \quad b) \lim(D/\|D\| - F/\|F\|) = 0.$$

In view of (7.21) one has $\|D - A\|/d_X(p_n, q_n) \rightarrow 1, \|F - C\|/d_U(p'_n, q'_n) \rightarrow 1$. Hence to prove (7.22), i.e., $d_X(p_n, q_n)/d_U(p'_n, q'_n) \rightarrow 1$, is the same as to show $\lim \|D - A\|/\|F - C\| = 1$. This is intuitively clear by (7.25), (7.26) (which alludes to $A \approx C$ and $D \approx F$) provided that the angle given by the two vectors D and A at p (hence by F and C at p , cf. b) of (7.25) and (7.26)) is not approaching zero. This is precisely the condition given in the lemma. For the rigor of this argument one may use law of cosines without difficulty. Hence the lemma follows. \square

Proof of claim (7.20). – By combining Lemma 7.6 and Lemma 7.3 we can finish the proof of the claim (7.20) provided that $h_j(z_1, z_2) = d_{U_j}^2(z_1, z_2)$. But this is a standard fact for the heat kernels of Dirac and Laplacian type (see [5, Theorem 2.29]); see also the famous result of S. R. S. Varadhan[63] for a generalization in this regard. \square

Proof of Proposition 7.2 resumed. – We are now ready to prove case b) of iii) of Proposition 7.2. To work on the integral I_t of (7.3) we are going to refine the computation contained in (7.13). Indeed, case b) of iii) can be obtained if one notes the following $\alpha) - \epsilon)$ (part of them similar to the proof of i) of this proposition):

$$\alpha) \int_{-\infty}^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{|a|};$$

$$\beta) \text{ using (7.19) (with } q = 1, 2, \dots, \ell - 1) \text{ for } e^{-\frac{\hat{h}_j^+}{t}} \text{ in (7.3);}$$

γ) change of variable $u = \psi + \frac{2\pi}{p_\ell}$ in (7.3);

δ) in (7.3), by rescaling $\hat{y} \rightarrow \sqrt{t}\hat{y}$, $\hat{b}_{j,s}^+((\sqrt{t}\hat{y}, Y), e^{-iu} \circ (\sqrt{t}\hat{y}, Y))$ replacing $\hat{b}_{j,s}^+(x, e^{-iu} \circ x)$, tends to $\hat{b}_{j,s}^+((0, Y), e^{-iu} \circ (0, Y)) = \hat{b}_{j,s}^+((0, Y), e^{-i\psi} \circ (0, Y))$ because $(0, Y) \in \overline{X}_{p_\ell}$. But $|\psi|$ being small ($\leq \varepsilon$), $e^{-i\psi} \circ (0, Y)$ does not change the $z((0, Y)) (= z(Y))$ coordinate in \hat{D}_j , giving $\hat{b}_{j,s}^+((0, Y), e^{-i\psi} \circ (0, Y)) = b_{j,s}^+(z(Y), z(Y))$ (up to cut-off functions);

ϵ) as $t \rightarrow 0$, $\sigma_j(\eta) = \sigma_j(\theta(e^{-iu} \circ (0, Y))) = \sigma_j(\theta(e^{-i\psi} \circ (0, Y))) = \sigma_j(\theta(Y) + \psi)$. With $\eta = \theta + \psi$ and $u = \psi + 2\pi/p_\ell$ in (7.3) and (7.4), a cancellation occurs for the three exponentials there; eventually a numerical factor $e^{-i2\pi m/p_\ell}$ is pulled out. And instead of $\text{Tr } \hat{b}_{j,s}^+$ in I_t of (7.3), we are reduced to $\chi_j \text{Tr } b_{j,s}^+$ (no “hat” on $b_{j,s}^+$ here) as put down in this proposition.

The formula for the coefficient $b_{s, \frac{\varepsilon_\ell}{2}}^{(j)}$ of (7.8) follows from $\alpha) - \epsilon)$ above.

Finally, for $s = n$, it is well-known that (dropping j here) $\text{Tr } b_n^+(z, z)$ in the integral (7.8) being the leading coefficient term in the asymptotic expansion of the (Spin^c) Kodaira heat kernel, is constant in z and equals $(4\pi)^{-n} \cdot (\text{rk}(\wedge T^{*0,1}(U))) = (2\pi)^{-n}$ ([36, (a) of Theorem 4.4.1], cf. [5, Theorem 2.41]). \square

7.3. Global angular integral

To work out the global version (i.e., the integration on $[0, 2\pi]$) it is natural to consider not only (an ε -neighborhood of) $2\pi/p_\ell$ but also all their multiples $s2\pi/p_\ell$, $s \in \mathbb{N}$, $s \leq p_\ell$. The analysis will thus partly depend on whether $s2\pi/p_\ell = s'2\pi/p_{\ell'}$ for some $s, s', p_\ell, p_{\ell'}$ or not. One needs a systematic control of the behavior in this regard. Further, since the result will appear as a sum over these ε -neighborhoods, to organize this sum in a manageable way is also desirable. We shall now mainly deal with these issues in this subsection.

There are minor duplication and perhaps discrepancy in notation between this subsection and the preceding one. But, it would have appeared cumbersome if we had set up this generality right in the preceding subsection (as the resulting proof would have become much less illuminating).

Recall that the S^1 -period of X denoted by $\frac{2\pi}{p} > \frac{2\pi}{p_2} > \dots > \frac{2\pi}{p_k}$ with $p|p_\ell$, $1 \leq \ell \leq k$ (cf. Remark 1.17) and the stratum X_{p_ℓ} (the set of points of period $\frac{2\pi}{p_\ell}$) is a disjoint union of connected submanifolds $\bigcup_{1 \leq \gamma_\ell \leq s_\ell} X_{p_\ell(\gamma_\ell)}$.

DEFINITION 7.7. – Fix a smooth function $g \neq 0$ on X . We say that g is of *good support* if the following conditions are satisfied.

$$\begin{aligned}
 & \text{i) } \text{Supp } g \subset X_{p_{i_1}(\gamma_{i_1})} \cup X_{p_{i_2}(\gamma_{i_2})} \cup \cdots \cup X_{p_{i_t}(\gamma_{i_t})} \cup X_{p_{i_{t+1}}(\gamma_{i_{t+1}})}, \\
 & \qquad 1 = i_1 < i_2 < \cdots < i_{t+1} \leq k, \\
 (7.27) \quad & \text{ii) } \text{Supp } g \cap X_{p_{i_s}(\gamma_{i_s})} \neq \emptyset, \quad \forall s, 1 \leq s \leq t+1, \\
 & \text{iii) } \overline{X}_{p_{i_1}(\gamma_{i_1})} \supseteq \overline{X}_{p_{i_2}(\gamma_{i_2})} \supseteq \cdots \supseteq \overline{X}_{p_{i_t}(\gamma_{i_t})} \supseteq \overline{X}_{p_{i_{t+1}}(\gamma_{i_{t+1}})},
 \end{aligned}$$

where the number k appearing in i) is introduced in (7.1),

Since the decomposition (7.6) is a disjoint union of connected components, it is clear that given any $x_0 \in X$ there exists a neighborhood $\Omega \ni x_0$ such that every nontrivial $g(x) \in C_0^\infty(\Omega)$ is of good support in the sense above.

DEFINITION 7.8. – Let $g(x)$ be a smooth function on X of good support in the sense above, (7.27). Let $c \in \mathbb{N}$. We define a number $i(c, g) = i(c)$ associated with c and g as follows:

- i) $i(c, g) := \ell \geq 2$ if the following is satisfied a) $c \mid p_\ell$, $\ell = i_s$ for some s , $2 \leq s \leq t+1$ and b) $c \nmid p_{i_{s'}}$ for all $s' < s$.
- ii) $i(c, g) := 1$ if $c \mid p$ ($p = p_{i_1}$). This is independent of g .
- iii) $i(c, g) := \infty$ if $c \nmid p_{i_s}$ for each s with $1 \leq s \leq t+1$.

It is easily seen that $p_{i(c)} \mid p_{i_s}$ for each i_s with $i(c) \leq i_s \leq i_{t+1}$ if $i(c) \neq \infty$. Indeed, $p_{i_1} \mid p_{i_2} \mid \cdots \mid p_{i_{t+1}}$ (cf. Remark 1.17 for a similar case).

By the above definition, one sees

LEMMA 7.9. – Let $x \in \text{Supp } g$, $i(c) \neq \infty$ and $h \in \mathbb{N}$ with $(h, c) = 1$. Then $e^{-i\frac{2\pi h}{c}} \circ x = x$ if and only if $x \in \overline{X}_{p_{i(c)}}$.

Let $h \in \mathbb{N}$ with $(h, c) = 1$ and $h < c$. We consider the integral similar to (7.3) for $i(c) \neq \infty$:

$$\begin{aligned}
 (7.28) \quad I_t &= I_t^{(j)}(p_{i(c)}, g, \frac{h}{c}) \\
 &\equiv \frac{1}{2\pi} \int_{\frac{2\pi h}{c} - \varepsilon}^{\frac{2\pi h}{c} + \varepsilon} \int_X g(x) e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ x)}{t}} ((e^{-iu})_y^* \text{Tr } \hat{b}_{j,s}^+)(x, x) e^{-imu} dv_X(x) du.
 \end{aligned}$$

The above extends to the case $i(c) = \infty$ simply by formally setting $I_t^{(j)}(p_{i(c)=\infty}, g, \frac{h}{c})$ to be the integral to the right of (7.28).

DEFINITION 7.10. – i) Set $i(c)$ of Definition 7.8 to be ℓ . Assume $\ell \neq \infty$. Define the type $\tau(I_t(p_{i(c)}, g, \frac{h}{c}))$ to be $\tau(I_t(p_\ell, g))$ where $\tau(I_t(p_\ell, g))$ is given in i) of Definition 7.1.

ii) The notions of *simple type* and *class* are defined similarly. (Note there is no trivial type for $\ell \neq \infty$.)

- iii) If $i(c) = \infty$, then $\text{Supp } g \cap X_a = \emptyset$ where X_a is the fixed point set of $a = e^{-i2\pi/c} \in S^1$ (whether X_a is empty or not depends on c). In this case we define it to be the *trivial type* (cf. iii) of Definition 7.1).

REMARK 7.11. – The notion of type concerns only the local stratification (of the S^1 action) at x_0 around which $\Omega \supset \text{Supp } g$ is a small neighborhood. Thus, with a small open subset $\Omega \subset X$ one may associate the type $\tau(\Omega)$ without referring to any kind of integral. In the fourth subsection, we will basically adopt this viewpoint for our purpose.

First, one examines the case $i(c, g) = \infty$, which turns out to be inessential.

LEMMA 7.12. – *Let $c \in \mathbb{N}$. There exists a (finite) covering of BRT trivializations on X with the following property. For any smooth function g of small compact support (cf. Definition 7.7), suppose $i(c, g) = \infty$ and x_0 with $g(x_0) \neq 0$, given. Then there exist a small open set $\Omega \ni x_0$ and a small $\tilde{\varepsilon} > 0$ such that for any $\chi \in C_0^\infty(\Omega)$, if one replaces g by χg in the integral I_t of (7.28), this I_t with any $0 < \varepsilon \leq \tilde{\varepsilon}$, equals 0, or $O(t^\infty)$ as $t \rightarrow 0^+$.*

Proof. – By the definition of $i(c, g) = \infty$, if the fixed point set X_a of $a = e^{-i2\pi/c}$ is empty, one chooses a covering of BRT trivializations $\{D_j\}_j$ (cf. lines above Subsection 7.2) such that $\hat{D}_j \cap a^h(\hat{D}_j) = \emptyset$ for all j and h with $(h, c) = 1$. In this case the remaining argument by using the continuity of the S^1 action, is almost the same as ii) of Proposition 7.2, yielding $I_t = 0$. If X_a is not empty, one chooses any (finite) covering of BRT trivializations (such as the one given prior to Subsection 7.2). Then it may occur the extra case $a^h \circ x_0 \in \hat{D}_j$ for some $\hat{D}_j \ni x_0$ if the choice of g is such that x_0 is very near X_a . In this case the remaining argument is essentially similar to Case a) of iii) of Proposition 7.2, giving rise to $I_t = O(t^\infty)$ as $t \rightarrow 0^+$. □

To compute I_t of (7.28), we assume the *simple type* condition for I (when it is not of trivial type) as given in Definition 7.10. Combining Lemma 7.12 we have the following corollary, as a generalization of Proposition 7.2.

COROLLARY 7.13. – *Notations and the simple type condition as above. Assume that the covering by BRT trivializations satisfies Lemma 7.12. Let $c \in \mathbb{N}$, $x_0 \in X$, $\Omega \ni x_0$ an open subset and $g \in C_0^\infty(\Omega)$ (of good support as above). Then the $\varepsilon > 0$ (in I_t) and Ω can be chosen to satisfy the following.*

- a) *The same results (for computing $I_t = I_t^{(j)}(p_{i(c)}, g, \frac{1}{c})$ of (7.28)) hold true as in Proposition 7.2 provided that one adopts the replacement of $e^{-i2\pi/p}$, $e^{-i2\pi/p_\ell}$ by $e^{-i2\pi/c}$ in i), ii) and iii) of the statement, and ℓ (not the one in $\frac{2\pi}{p_\ell}$) by $i(c)$ in Cases a), b) of iii) throughout (so $p_{\ell-q} \rightarrow p_{i(c)-q}$, $e_\ell \rightarrow e_{i(c)}$, $e_{\ell-q+1} \rightarrow e_{i(c)-q+1}$, $e_{\ell-q} \rightarrow e_{i(c)-q}$ and $\gamma_\ell \rightarrow \gamma_{i(c)}$, $X_\ell \rightarrow X_{i(c)}$, $X_{\ell-q} \rightarrow X_{i(c)-q}$ etc. in*

Case b)). Note that after the replacement, $i(c) = \infty$ in Case a) and $i(c) \neq \infty$ in Case b), of iii).

- b) More generally, for $h \in \mathbb{N}$, $(h, c) = 1$ and $h < c$, with the replacement $\frac{2\pi}{c} \rightarrow \frac{2\pi h}{c}$ (and ℓ not the one in $\frac{2\pi}{p_\ell}$ by $i(c)$ in Cases a), b) of iii)), the same results (for computing $I_t = I_t^{(j)}(p_{i(c)}, g, \frac{h}{c})$ of (7.28)) hold true as well.

Proof of Corollary 7.13. – One sees that with the replacement of $\frac{2\pi}{p_\ell}$ by $\frac{2\pi}{c}$ or $\frac{2\pi h}{c}$, the condition on c (in Definition 7.8) renders the argument in proof of Proposition 7.2 essentially unchanged. For instance, with (7.9) replaced by $\frac{2\pi h}{c} - \tilde{\theta}_0 = m \frac{2\pi}{p_{\ell-q}}$, taking ζ smaller does the job. Further, with substitution of ℓ (not in $\frac{2\pi}{p_\ell}$) by $i(c)$, the distinct eigenvalues $\{1, e^{i\frac{2\pi}{p_\ell} p_{\ell-1}}, e^{i\frac{2\pi}{p_\ell} p_{\ell-2}}, \dots, e^{i\frac{2\pi}{p_\ell} p}\}$ of the isotropy action (of $e^{-i\frac{2\pi}{p_\ell}}$ at x_0) (cf. (7.18)) are changed to $\{1, e^{i\frac{2\pi h}{c} p_{i(c)-1}}, e^{i\frac{2\pi h}{c} p_{i(c)-2}}, \dots, e^{i\frac{2\pi h}{c} p}\}$ (of $e^{-i\frac{2\pi h}{c}}$ at x_0) (which remain distinct). □

Let $c \in \mathbb{N}$ and g a smooth function on X of good support as above, with $i(c) (= i(c, g))$ in Definition 7.8. We are going to associate certain numerical factors. For a contrast, we will give them for cases of the simple type and the general type separately (cf. Definitions 7.1 and 7.10).

For the simple type, the numerical factor $d_c = d_{c,g,m}$ is set to be

(7.29)

i) $i(c) = 1$

if $c > 1$, $d_c (= d_{c,g,m}) := \sum_{h \in \mathbb{N}, h < c, (h,c)=1} e^{-\frac{2\pi i}{c} hm}$; if $c = 1$, $d_c := 1$.

ii) $\infty > i(c) \geq 2$

$$d_c (= d_{c,g,m}) := (\sqrt{\pi})^{e_{i(c)}} \sum_{h \in \mathbb{N}, h < c, (h,c)=1} \frac{e^{-i\frac{2\pi h}{c} m}}{\prod_{q=1}^{i(c)-1} \left| e^{i\frac{2\pi h}{c} p_{i(c)-q}} - 1 \right|^{e_{i(c)-q+1} - e_{i(c)-q}}}$$

iii) $i(c) = \infty$

$d_c (= d_{c,g,m}) := 1$.

REMARK 7.14. – For the general type, the $d_{c,g,m}$ for $i(c) \geq 2$ should be modified as follows.

In notation of Definition 7.10, let it be given

$$\tau(I_t(p_{i(c)}, g, \frac{h}{c})) = (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \dots, i_f(\gamma_{i_f}), i_{f+1}(\gamma_{i_{f+1}}))$$

where $i_1 = \gamma_{i_1} = 1, i_{f+1} = i(c) \neq \infty$, say. (In the previous Definition 7.1, $i_{f+1} = \ell$. Here we have $i_{f+1} = i(c)$.)

One sees that the eigenvalues of the isotropy action of $e^{-\frac{2\pi i}{c} h}$ (at $x_0 \in \overline{X}_{p_{i(c)}(\gamma_{i(c)})}$) are:

(7.30) $e^{i\frac{2\pi h}{c} p_{i_1}}, e^{i\frac{2\pi h}{c} p_{i_2}}, \dots, e^{i\frac{2\pi h}{c} p_{i_f}}, 1$

(because by writing $e^{-\frac{2\pi i}{c}h} = (e^{-\frac{2\pi i}{p_\ell}})^j$ if $h/c = j/p_\ell$ with $\ell = i(c)$, the eigenvalues are λ^j where $\lambda = e^{i\omega p_{i_1}}, e^{i\omega p_{i_2}}, \dots, \omega = \frac{2\pi}{p_\ell}$, cf. (7.18)) with multiplicities (where $e_{i_1(\gamma_{i_1})} = 0$)

$$(7.31) \quad e_{i_2(\gamma_{i_2})} - e_{i_1(\gamma_{i_1})}, \dots, e_{i_f(\gamma_{i_f})} - e_{i_{f-1}(\gamma_{i_{f-1}})}, e_{i(c)(\gamma_{i(c)})} - e_{i_f(\gamma_{i_f})}, \dim X - e_{i(c)(\gamma_{i(c)})}.$$

Write, for $1 \leq r \leq f$,

$$(7.32) \quad \lambda_r := e^{i\frac{2\pi h}{c}p_{i_r}}; \quad m_r := e_{i_{r+1}(\gamma_{i_{r+1}})} - e_{i_r(\gamma_{i_r})} \quad (\text{with } i_{f+1} := i(c)).$$

Numerical factors d_c for the general type. – Given $\tau(I_t) = (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \dots, i_f(\gamma_{i_f}), i_{f+1}(\gamma_{i_{f+1}}))$ of general type, $I_t = I_t(p_\ell, g, \frac{h}{c})$ with $i(c) = \ell$, the numerical factors d_c similar to (7.29) are defined as follows.

If c is of $i(c) = 1$ or ∞ , then d_c is the same as d_c in i), iii) of (7.29). For c with $\infty > i(c) \geq 2$,

$$(7.33) \quad \infty > i(c) (= i(c, g)) \geq 2, \quad d_c := (\sqrt{\pi})^{e_{i(c)(\gamma_{i(c)})}} \sum_{h \in \mathbb{N}, h < c, (h, c) = 1} \frac{e^{-i\frac{2\pi h}{c}m}}{\prod_{r=1}^{i_f} |\lambda_r - 1|^{m_r}},$$

where we recall that $e_{i(c)(\gamma_{i(c)})} = \text{codim } X_{p_{i(c)(\gamma_{i(c)})}}$ and that $\tau(\Omega)$ is introduced in Remark 7.11.

Note that the $\Omega (\ni \text{Supp } g)$ is chosen to be small enough so that there is no mixing of types. The factor d_c of (7.33) is well defined.

We turn now to the global version (over the entire $[0, 2\pi]$) of the integral I_t (of (7.3) or (7.28)).

Let

$$(7.34) \quad J_{s,t} = J_{s,t}^{(j)} = J_{s,t,m}^{(j)}(g(x)) \equiv \frac{1}{2\pi} \int_X g(x) e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ x)}{t}} ((e^{-iu})_y^* \text{Tr } \hat{b}_{j,s}^+) (x, x) e^{-imu} dv_X(x),$$

where $g(x) \in C_0^\infty(\Omega)$ with Ω satisfying Proposition 7.2. We assume $g(x)$ is of good support in the sense of Definition 7.7.

We are going to compute $\int_0^{2\pi} J_{s,t} du$, a $[0, 2\pi]$ -integral version of the integral I in (7.3).

To this aim, the main tool is the corollary below which is a reformulation of Corollary 7.13. But prior to this, let's write, for $x_0 \in X$, $g(x) \in C_0^\infty(\Omega)$ with Ω a small

neighborhood at x_0 ,

$$(7.35)$$

$$\ell \geq 1,$$

$$I_{\ell(\gamma_\ell),s}^{+,(j)} (= I_{\overline{X}_{p\ell(\gamma_\ell)},s,m}^{+,(j)}(g)) \quad (z(Y) = z(x) \text{ below; } b_{j,s}^+ = b_{j,s,m}^+ \text{ without "hat" over it})$$

$$\equiv \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} \int_{\overline{X}_{p\ell(\gamma_\ell)}} g(Y) \chi_j(Y) \text{Tr} b_{j,s}^+(z(Y), z(Y)) \tau_j(z(Y)) \sigma_j(\theta(Y) + u) dv_{\overline{X}_{p\ell(\gamma_\ell)}}(Y) du$$

$$\ell = 1, \quad I_{1,s}^{+,(j)} (= I_{X,s,m}^{+,(j)} = I_{\overline{X},s}^{+,(j)}(g) = I_{X,s}^{+,(j)}(g))$$

$$d_c = d_{c,g,m,\tau(\Omega)} \text{ cf. (7.33), also written as } d_{c,g,m,I} \text{ with } I = I_{i(c)(\gamma_{i(c)}),s}^{+,(j)}.$$

The proof of the previous results in the case of simple type remains basically unchanged for the case of general type. With the numerical factor d_c introduced in (7.29) and (7.33), one sees the following.

COROLLARY 7.15. – Notations as above with the general type $\tau(I_t)$ allowed (cf. Definitions 7.1 and 7.10). Assume that the covering by BRT trivializations $\{\hat{D}_j\}_j$ satisfies Lemma 7.12. Let $c \in \mathbb{N}$ and $x_0 \in X$. For an ε , write $\mu_c := \bigcup_{h \in \mathbb{N}, h < c, (h,c)=1} \left] \frac{2\pi h}{c} - \varepsilon, \frac{2\pi h}{c} + \varepsilon \right[$ for $c > 1$ and $\lambda_1 :=]-\varepsilon, \varepsilon[$ for $c = 1$. Write $\Omega \subset X$ for an open subset with $x_0 \in \Omega$. Then the $\varepsilon > 0$ and Ω can be chosen to satisfy the following. (Recall that with respect to $\{\hat{D}_j\}_j$ we write $J_{s,t}^{(j)} = J_{s,t}^{(j)}(g)$ for any given $g \in C_0^\infty(\Omega)$ of good support in the sense of Definition 7.7.)

i) Suppose $x_0 \in X_p$. Then

$$\text{if } i(c) = 1, \quad \int_{\mu_c} J_{s,t}^{(j)} du = d_c I_{X,s}^{+,(j)};$$

$$\text{if } i(c) \geq 2 \text{ (giving } i(c) = \infty \text{ here), } \int_{\mu_c} J_{s,t}^{(j)} du = 0 \text{ or } \sim O(t^\infty) \text{ (as } t \rightarrow 0^+).$$

The $\ell \geq 2$ in the following ii) and iii) is such that $x_0 \in X_{p_\ell}$ (so $x_0 \in X_{p\ell(\gamma_\ell)}$, for some γ_ℓ).

ii) Assume $e^{-\frac{i2\pi h}{c}} \circ x_0 \notin \hat{D}_j$ (giving $i(c) = \infty$). Then $\int_{\mu_c} J_{s,t}^{(j)} du = 0$.

iii) Assume $e^{-\frac{i2\pi h}{c}} \circ x_0 \in \hat{D}_j$. Then (as $t \rightarrow 0^+$)

$$\text{if } i(c) \geq \ell + 1 \text{ (giving } i(c) = \infty \text{ here), } \int_{\mu_c} J_{s,t}^{(j)} du = 0 \text{ or } \sim O(t^\infty);$$

$$\text{if } 2 \leq i(c) \leq \ell, \quad \int_{\mu_c} J_{s,t}^{(j)} du \sim d_c I_{i(c)(\gamma_{i(c)}),s}^{+,(j)} \sqrt{t}^{e_{i(c)}(\gamma_{i(c)})} + O(\sqrt{t}^{e_{i(c)}(\gamma_{i(c)})+1});$$

$$\text{if } i(c) = 1, \quad \int_{\mu_c} J_{s,t}^{(j)} du = d_c I_{X,s}^{+,(j)},$$

where we note $d_c = d_{c,g,m,I}$ in (7.35) with $I = I_{i(c)(\gamma_{i(c)}),s}^{+,(j)}$.

We are now ready to compute $\int_0^{2\pi} J_{s,t} du$.

Let μ_c be as in Corollary 7.15 (with a given $g(x)$ of good support).

$$(7.36) \quad \begin{aligned} S_1 &= \{c \in \mathbb{N}; i(c, g) = 1\}, \\ S_2 &= \{c \in \mathbb{N}; \infty \geq i(c, g) \geq 2 \text{ and } c \mid p_\ell \text{ for some } \ell, 2 \leq \ell \leq k\} \\ \Lambda &= \Lambda_1 \cup \Lambda_2 \quad \text{with} \quad \Lambda_1 = \bigcup_{c \in S_1} \mu_c, \quad \Lambda_2 = \bigcup_{c \in S_2} \mu_c; \quad N = [0, 2\pi] \setminus \Lambda. \end{aligned}$$

By using (7.36) and corollaries above one has the following.

PROPOSITION 7.16. – *Let $x_0 \in X$. Then we can find $\{\hat{D}_j\}_{j=1}^K$ a finite covering of X by BRT charts that satisfy Lemma 7.12 and $\varepsilon > 0$ (for Λ_1, Λ_2 in i), ii), resp. below) and an open set Ω of x_0 , such that for every $g(x) \in C_c^\infty(\Omega)$ of good support in the sense of Definition 7.7), and every $s = n, n - 1, \dots$, one has the following.*

i) $\int_{\Lambda_1} J_{s,t}^{(j)} du = \left(\sum_{q=1}^p e^{-\frac{i2\pi qm}{p}}\right) I_{X,s}^{+, (j)}$, for every $j \in \{1, 2, \dots, K\}$.

ii)

Case a) If $x_0 \in X_p$, $\int_{\Lambda_2} J_{s,t}^{(j)} du \sim 0$ or $O(t^\infty)$ (as $t \rightarrow 0^+$), for every $j \in \{1, 2, \dots, K\}$.

Case b) If $x_0 \in X_{p_\ell}$ with $\ell \geq 2$,

$$\int_{\Lambda_2} J_{s,t}^{(j)} du \sim \left(\sum_{2 \leq i(c) \leq \ell} (d_c I_{i(c)(\gamma_{i(c)}, s)}^{+, (j)} \sqrt{t}^{e_{i(c)}(\gamma_{i(c)})} + O(\sqrt{t}^{e_{i(c)}(\gamma_{i(c)})+1})) \right) + O(t^\infty) \text{ (as } t \rightarrow 0^+),$$

for every $j \in \{1, 2, \dots, K\}$, where $d_c = d_{c,g,m,I}, I = I_{i(c)(\gamma_{i(c)}, s)}^{+, (j)}$, cf. (7.33) and (7.35).

iii) For every $s = n, n - 1, \dots$ and every $j \in \{1, 2, \dots, K\}$, write $\int_N J_{s,t}^{(j)}(g) du := \eta_{s,t}^{(j)}(g)$. Then, for every $P \in \mathbb{N}$, there are constants $C > 0$ and $c_1 > 0$ which are dependent on $\{\hat{D}_j\}_{j=1}^K, \Omega, P$ and independent of g and t , such that

$$(7.37) \quad \left| \sum_{j=1}^K \sum_{s=-n}^P t^s \eta_{s,t}^{(j)}(g) \right| \leq C t^{P+1} \int_X |g(x)| dv_X(x), \text{ for every } 0 < t < c_1.$$

Proof. – The action by $e^{-i\theta}$ with $\theta \in N$ is fixed point free on X . Each point in X has a distinguished neighborhood $\hat{\Omega}$ such that if $x \in \hat{\Omega}$ and $\theta \in N$, $e^{-i\theta} \circ x \notin \hat{\Omega}$ by using continuity of the S^1 action. By compactness, we can take a finite covering $\{\hat{D}_j\}_{j=1}^K$ of BRT charts satisfying Lemma 7.12 (as there are only finitely many c here), such that there exist an $\varepsilon > 0$ (for Λ_1, Λ_2 in i), ii), resp.) and an open set Ω of x_0 (small enough) with $\Omega \subseteq \hat{D}_1$ and $\Omega \cap \hat{D}_j = \emptyset$ if $j \geq 2$, and if $x \in \Omega$ and $\theta \in N$, then $e^{-i\theta} \circ x \notin \hat{D}_1$.

To see i), by i) and iii) of Corollary 7.15 it suffices to note $\sum_{c, i(c)=1} d_c = \sum_{q=1}^p e^{-\frac{i2\pi qm}{p}}$ by (7.29). Case a) of ii) follows directly from i) of *loc. cit.* whereas Case b) from iii) of *loc. cit.* by writing $\int_{\Lambda_2} J_{s,t} du = \sum_{2 \leq i(c) \leq \ell} \int_{\mu_c} J_{s,t} du + \sum_{i(c) \geq \ell+1} \int_{\mu_c} J_{s,t} du$.

To see iii), For every $j \in \{1, 2, \dots, \tilde{K}\}$, let $H_j(t)$ be the operator constructed in (5.40) with respect to the BRT charts $\{\hat{D}_j\}_{j=1}^{\tilde{K}}$. Note that $x \in \Omega$ and $\theta \in N$, we have $e^{-i\theta} \circ x \notin \hat{D}_1$. From this observation, it is easy to see that

$$(7.38) \quad H(t, x, e^{-iu} \circ x) = O(t^{+\infty}), \quad \text{for every } u \in N \text{ and every } y \in \Omega.$$

From (7.38), we get (7.37). □

7.4. Patching up angular integrals over X ; proof for the simple type

We are going to study the main issue

$$(7.39) \quad \int_X \text{Tr } a_s^+(t, x, x) dv_X(x),$$

where we recall that (by (5.43))

$$(7.40) \quad \begin{aligned} a_s^+(t, x, y) &= a_{s,m}^+(t, x, y) \quad (\hat{b}_{j,s}^+ = \hat{b}_{j,s,m}^+) \\ &= \frac{1}{2\pi} \sum_{j=1}^N \int_{-\pi}^{\pi} e^{-\frac{\hat{h}_{j,+}(x, e^{-iu} \circ y)}{t}} ((e^{-iu})_y^* \hat{b}_{j,s}^+)(x, y) e^{-imu} du, \quad s = n, n-1, n-2, \dots \end{aligned}$$

For this, one would like to patch up those integrals $\int_0^{2\pi} J_{s,t}^{(j)} du$ of the last subsection over j . However, $a_s^+(t, x, y)$ is not canonically defined by our method and is in fact dependent on the choice of BRT charts. A direct study of it appears inefficient (unless one sticks to a fixed covering of BRT charts).

It turns out to be more effective if instead, one studies its equivalence (cf. (5.56) in the asymptotic sense):

$$(7.41) \quad \int_X \text{Tr } e^{-t \hat{\square}_{b,m}^+}(x, x) dv_X(x)$$

in which $e^{-t \hat{\square}_{b,m}^+}(x, y)$ is of course independent of choice of BRT charts.

Suppose an $s = n, n-1, \dots$ are given. Assume that the BRT covering $\{\hat{D}_j\}_j$ satisfies Proposition 7.16 in which by using compactness, one can find a (finite) covering $\{\Omega_\alpha\}_\alpha$ of X , $\Omega_\alpha \Subset \hat{D}_j$ if $\hat{D}_j \cap \Omega_\alpha \neq \emptyset$, and an $\varepsilon > 0$ such that the conclusion i), ii) and iii) of that proposition hold with each of these Ω_α and this ε . As indicated in Proposition 7.2, whenever necessary, one can shrink the size of Ω_α without changing ε . For $\rho = \frac{\varepsilon}{2}$, we assume (possibly after shrinking Ω_α and using compactness) for each α, j , and for some (possibly big) $m > 1$,

$$(7.42) \quad \theta\text{-coordinates of } \Omega_\alpha, \hat{D}_j \text{ lie inside of } [-\rho, \rho], [-m\rho, m\rho] \text{ respectively.}$$

Let $\{g_\alpha(x)\}_\alpha$ be a partition of unity *subordinate* to this covering (i.e., $\text{Supp } g_\alpha \Subset \Omega_\alpha$). We further assume each g_α is of good support in the sense of Definition 7.7. One sees that as $t \rightarrow 0^+$

$$(7.43) \quad \int_X g_\alpha(x) \text{Tr } e^{-t\hat{\square}_{b,m}^+}(x, x) dv_X(x) \sim \sum_j \sum_{s=n, n-1, \dots} t^{-s} \int_0^{2\pi} J_{s,t}^{(j)}(g_\alpha(x)) du,$$

where the term to the right is computed with respect to any given BRT covering of X , including but not restricted to, the previous $\{\hat{D}_j\}_j$. Hence at each stage of the computation we may choose convenient BRT charts for the need (as far as the asymptotic expansion as $t \rightarrow 0^+$ is concerned).

By Proposition 7.16, (7.43) is reduced to computing $I_{\ell,s}^{(j)}(g_\alpha)$ (see (7.35)) (for a fixed g_α).

Henceforth, in the following we fix an (arbitrarily given) α . As aforementioned, we are free to reset the BRT charts $\{\hat{D}_j\}_j$ (with certain cut-off functions). To do so, we make the following definition for convenience.

DEFINITION 7.17. – Fix an $x_0 \in X$. $\{\hat{D}_j\}_j$ ($\hat{D}_j \subset D_j$ etc. notations as in the beginning of this section) a (finite) covering of X , is said to be a covering by *distinguished BRT charts at x_0* provided that $x_0 \in \hat{D}_j$ for some j and $x_0 \notin \hat{D}_k$ for all $k \neq j$.

Now, we can further assume that for the above fixed α and for an $x \in \Omega_\alpha$, $\{\hat{D}_j\}_j$ is distinguished at x in the sense of Definition 7.17. In fact we can assume a little more that $\Omega_\alpha \Subset \hat{D}_{j_0}$ and $\Omega_\alpha \cap \hat{D}_k = \emptyset$ for $k \neq j_0$; namely $\{\hat{D}_j\}_j$ is distinguished at x for each $x \in \Omega_\alpha$. Also, we assume that (7.42) is satisfied.

We shall now choose the cut-off function σ_{j_0} , in notation of (7.35), that satisfies (see lines above (5.38))

$$(7.44) \quad \int_{-m\rho}^{m\rho} \sigma_{j_0}(u) du = \int_{-\rho}^{\rho} \sigma_{j_0}(u) du = 1, \quad \text{Supp } \sigma_{j_0} \subset]-\rho, \rho[$$

and choose $\chi_{j_0} \equiv 1$, so $\tau_{j_0} \equiv 1$, on $\bar{\Omega}_\alpha$ (see *loc. cit.*).

With the above setup, some simplifications for (7.43) occur. Firstly,

$$(7.45) \quad \int_X g_\alpha(x) \text{Tr } e^{-t\hat{\square}_{b,m}^+}(x, x) dv_X(x) \sim \sum_{s=n, n-1, \dots} t^{-s} \int_0^{2\pi} J_{s,t}^{(j_0)}(g_\alpha(x)) du.$$

We are reduced, by Proposition 7.16, to computing the integrals in (7.35).

Secondly, in notation of (7.35) there is an angular integral

$$(7.46) \quad \int_{-\varepsilon}^{\varepsilon} \sigma_{j_0}(\theta(Y) + u) du.$$

For a fixed $Y \in \Omega_\alpha$, $\theta(Y) \in [-\rho, \rho]$ by (7.42), and for u going through $[-\varepsilon, \varepsilon] = [-2\rho, 2\rho]$ of (7.46), one sees that $\theta(Y) + u$ covers $[-\rho, \rho]$, it follows from (7.44) that the angular integral (7.46) is 1.

Thirdly, by the above condition on χ_{j_0} and τ_{j_0} one obtains, with (7.46) $\equiv 1$, the following for (7.35).

$$(7.47) \quad I_{\ell(\gamma_\ell),s}^{+,(j_0)}(g_\alpha(x)) (= I_{\overline{X}_{p\ell(\gamma_\ell)},s}^{+,(j_0)}(g_\alpha(x))) = \frac{1}{2\pi} \int_{\overline{X}_{p\ell(\gamma_\ell)}} g_\alpha(Y) \text{Tr } b_{j_0,s}^+(z(Y), z(Y)) dv_{\overline{X}_{p\ell(\gamma_\ell)}}(Y)$$

($\ell \geq 1$; $s = n, n - 1, \dots$).

Finally, recall that $\alpha_s^+(x)$ is as in our main Result Theorem 1.3 (cf. Theorem 6.1) defined in (6.1) which is independent of choice of BRT charts and cut-off functions (cf. Remarks 1.6 and 5.8). Indeed one sees, for $x \in \Omega_\alpha$,

$$(7.48) \quad \frac{1}{2\pi} b_{j_0,s}^+(z(x), z(x)) = \alpha_s^+(x) (= \alpha_{s,m}^+(x))$$

by (6.1) and the choice of distinguished BRT charts here.

In sum, since the above applies to each Ω_α in the covering $\{\Omega_\alpha\}_\alpha$, by (7.47) and (7.48) we have reached the following *invariant expressions* (independent of choice of BRT coverings)

$$(7.49) \quad \begin{aligned} (k \geq \ell \geq 1) \quad S_{\ell(\gamma_\ell),s}^+(g_\alpha) & (= S_{\overline{X}_{p\ell(\gamma_\ell)},s,m}^+(g_\alpha)) \\ & \equiv S_{\ell(\gamma_\ell),s}^{+,(j_0)}(g_\alpha) = \int_{\overline{X}_{p\ell(\gamma_\ell)}} g_\alpha(Y) \text{Tr } \alpha_{s,m}^+(Y, Y) dv_{\overline{X}_{p\ell(\gamma_\ell)}}(Y) \\ S_{\ell(\gamma_\ell),s}^+ & (= S_{\overline{X}_{p\ell(\gamma_\ell)},s,m}^+) \\ & \equiv \sum_\alpha S_{\ell(\gamma_\ell),s}^+(g_\alpha) = \int_{\overline{X}_{p\ell(\gamma_\ell)}} \text{Tr } \alpha_{s,m}^+(Y, Y) dv_{\overline{X}_{p\ell(\gamma_\ell)}}(Y). \end{aligned}$$

Now (7.43) and (7.45) can be given, by using (7.49) and Proposition 7.16, as follows.

First, we classify the set $\{\Omega_\alpha\}_\alpha$ by writing

$$(7.50) \quad \chi(\alpha) (= \chi(\Omega_\alpha)) = \ell \text{ if } \Omega_\alpha \cap X_{p\ell} \neq \emptyset \text{ and for any } \ell' > \ell, \Omega_\alpha \cap X_{p\ell'} = \emptyset, \quad 1 \leq \ell, \ell' \leq k.$$

Note d_c below (with a specific c) is given without ambiguity (cf. (7.33)) by the local nature of Ω_α and g_α .

PROPOSITION 7.18. – *With the notation above and in terms of the functions of (7.49), let α and any $m \in \mathbb{N}$ be given. Then we have, as $t \rightarrow 0^+$, for g_α with*

Supp $g_\alpha \subseteq \Omega_\alpha$ such that $\chi(\alpha) = \ell$,

(7.51)

$$\int_X g_\alpha(x) \text{Tre}^{-t\widehat{\square}_{b,m}^+}(x, x) dv_X(x) \sim \sum_{s=n, n-1, \dots} t^{-s} A_s(g_\alpha) \quad \text{where } A_s(g_\alpha) \text{ is given by}$$

$$\begin{aligned} A_s(g_\alpha) &= \left(\sum_{q=1}^p e^{-\frac{i2\pi qm}{p}} \right) S_{X,s}^+(g_\alpha) \\ &+ \sum_{2 \leq i(c) \leq \ell} (d_c S_{i(c)(\gamma_{i(c)}),s}^+(g_\alpha) \sqrt{t}^{e_{i(c)}(\gamma_{i(c)})} + O(\sqrt{t}^{e_{i(c)}(\gamma_{i(c)})+1})) \\ &+ \eta_s(g_\alpha), \end{aligned}$$

where $i(c) = i(c, g_\alpha)$, $d_c = d_{c, g_\alpha, m, I}$, $I = S_{i(c)(\gamma_{i(c)}),s}^+(g_\alpha)$ by (7.35) and (7.33), and $\eta_s(g_\alpha)$ (which equals $\sum_j \eta_{s,t}^{(j)}(g_\alpha)$ in notation of Proposition 7.16 and distinguished BRT charts at x_0) satisfies the following estimate: For every $P \in \mathbb{N}$, there are constants $C > 0$, $c_1 > 0$, where C, c_1 depend on P and are independent of g_α and t , such that

$$\left| \sum_{s=-n}^P t^s \eta_{s,t}(g_\alpha) \right| \leq C t^{P+1} \int_X |g_\alpha(x)| dv_X(x), \quad \text{for every } 0 < t < c_1.$$

In the remaining of this subsection, to streamline the argument we assume the simplest case that

- i) each X_{p_ℓ} , $1 \leq \ell \leq k$, is connected
- ii) $X = \overline{X}_p \supseteq \overline{X}_{p_2} \supseteq \overline{X}_{p_3} \cdots \supseteq \overline{X}_{p_k}$.

(We postpone the general case to the next subsection.) One sees $p|p_2| \cdots |p_k$.

Hence all types reduce to simple types (cf. Definition 7.10).

In this case, the argument γ_ℓ in $\ell(\gamma_\ell)$ will henceforth be dropped throughout the remaining of this subsection.

It will take a bit more work to sum (7.51) over α . The numerical factor d_c (cf. (7.29)) in this simplified case satisfies the following. For smooth functions g, g' of good support (Definition 7.7), if $i(c, g) = i(c, g') (\leq \infty)$, then $d_{c,g,m} = d_{c,g',m}$. It is useful to set, for $g = g_\alpha$ with $\chi(\alpha) \geq \ell$ in ii) below ($\chi(\alpha)$ as in (7.50)),

$$\begin{aligned} \text{i) } D_{1,g} (= D_{1,g,m}) &\equiv \sum_{c, i(c,g)=1} d_{c,g,m} = \sum_{q=1}^p e^{-\frac{i2\pi qm}{p}}; \\ \text{ii) } (k \geq \ell \geq 2) \quad D_{\ell,g} (= D_{\ell,g,m}) &\equiv \sum_{c, i(c,g)=\ell} d_{c,g,m}. \end{aligned} \tag{7.52}$$

Suppose $\alpha, \beta \in \bigcup_{\ell' \geq \ell} \chi^{-1}(\ell')$. As said above, one sees $D_{\ell, g_\alpha} = D_{\ell, g_\beta}$ for $1 \leq \ell \leq k$ (because $i(c, g_\alpha) = \ell$ if and only if $i(c, g_\beta) = \ell$ here). We write

DEFINITION 7.19. – $D_\ell (= D_{\ell,m}) := D_{\ell,g_\alpha} (= D_{\ell,g_\alpha,m})$ for any α with $\chi(\alpha) \geq \ell$, $1 \leq \ell \leq k$.

By using (7.51) of Proposition 7.18 and Definition 7.19, one sees, for $\alpha \in \chi^{-1}(\ell)$,

$$\begin{aligned}
 \alpha \in \chi^{-1}(\ell), \quad & \int_X g_\alpha(x) \operatorname{Tr} e^{-t \widehat{\square}_{b,m}^+}(x, x) dv_X(x) \sim \sum_{s=n, n-1, \dots} t^{-s} \times \\
 & \eta_{-s,t}(g_\alpha) + D_1 S_{1,s}^+(g_\alpha) + \sum_{c, i(c)=2} (d_c S_{2,s}^+(g_\alpha) \sqrt{t}^{e_2} + O(\sqrt{t}^{e_2+1})) + \dots \\
 (7.53) \quad & + \sum_{c, i(c)=\ell} (d_c S_{\ell,s}^+(g_\alpha) \sqrt{t}^{e_\ell} + O(\sqrt{t}^{e_\ell+1})) \\
 & = \sum_s t^{-s} \left(\eta_{-s,t}(g_\alpha) + D_1 S_{1,s}^+(g_\alpha) + (D_2 S_{2,s}^+(g_\alpha) \sqrt{t}^{e_2} + O(\sqrt{t}^{e_2+1})) + \dots \right. \\
 & \quad \left. + (D_\ell S_{\ell,s}^+(g_\alpha) \sqrt{t}^{e_\ell} + O(\sqrt{t}^{e_\ell+1})) \right).
 \end{aligned}$$

Combining (7.50) and (7.53) yields the following as $t \rightarrow 0^+$ (where $\{\alpha\}_\alpha = \chi^{-1}(1) \cup \chi^{-1}(2) \cup \chi^{-1}(3) \dots$):

$$\begin{aligned}
 (7.54) \quad & \int_X \operatorname{Tr} e^{-t \widehat{\square}_{b,m}^+}(x, x) dv_X(x) \left(= \sum_\alpha \int_X g_\alpha(x) \operatorname{Tr} e^{-t \widehat{\square}_{b,m}^+}(x, x) dv_X(x) \right) \\
 & \sim \sum_{s=n, n-1, \dots} t^{-s} \left(\left(\sum_{\alpha \in \chi^{-1}(1)} D_1 S_{1,s}^+(g_\alpha) \right) \right. \\
 & \quad + \left(\left(\sum_{\alpha \in \chi^{-1}(2)} D_1 S_{1,s}^+(g_\alpha) \right) + \left(\sum_{\alpha \in \chi^{-1}(2)} D_2 S_{2,s}^+(g_\alpha) \sqrt{t}^{e_2} + O(\sqrt{t}^{e_2+1}) \right) \right) \\
 & \quad + \left(\left(\sum_{\alpha \in \chi^{-1}(3)} D_1 S_{1,s}^+(g_\alpha) \right) + \left(\sum_{\alpha \in \chi^{-1}(3)} D_2 S_{2,s}^+(g_\alpha) \sqrt{t}^{e_2} + O(\sqrt{t}^{e_2+1}) \right) \right) \\
 & \quad \left. + \left(\sum_{\alpha \in \chi^{-1}(3)} D_3 S_{3,s}^+(g_\alpha) \sqrt{t}^{e_3} + O(\sqrt{t}^{e_3+1}) \right) \right) + \dots + \sum_\alpha \eta_{-s,t}(g_\alpha).
 \end{aligned}$$

We rearrange (7.54) as (only keeping terms in leading order)

$$\begin{aligned}
 (7.55) \quad & \sum_s t^{-s} \times \left(\sum_\alpha \eta_{-s,t}(g_\alpha) + \left(\sum_{\alpha \in \chi^{-1}(1)} D_1 S_{1,s}^+(g_\alpha) + \sum_{\alpha \in \chi^{-1}(2)} D_1 S_{1,s}^+(g_\alpha) \right) \right. \\
 & \quad + \sum_{\alpha \in \chi^{-1}(3)} D_1 S_{1,s}^+(g_\alpha) + \dots \\
 & \quad + \left(\left(\sum_{\alpha \in \chi^{-1}(2)} D_2 S_{2,s}^+(g_\alpha) \sqrt{t}^{e_2} + \dots \right) \right. \\
 & \quad \left. + \left(\sum_{\alpha \in \chi^{-1}(3)} D_2 S_{2,s}^+(g_\alpha) \sqrt{t}^{e_2} + \dots \right) + \left(\sum_{\alpha \in \chi^{-1}(4)} \dots \right) + \dots \right)
 \end{aligned}$$

$$\begin{aligned}
 &+ \left(\left(\sum_{\alpha \in \chi^{-1}(3)} D_3 S_{3,s}^+(g_\alpha) \sqrt{t}^{e_3} + \dots \right) \right. \\
 &\quad \left. + \left(\sum_{\alpha \in \chi^{-1}(4)} D_3 S_{3,s}^+(g_\alpha) \sqrt{t}^{e_3} + \dots \right) + \left(\sum_{\alpha \in \chi^{-1}(5)} \dots \right) + \dots \right)
 \end{aligned}$$

which equals (by (7.49) and $S_{\ell,s}^+(g_\alpha) = 0$ for $\alpha \in \chi^{-1}(\ell')$ with $\ell' < \ell$),

$$\begin{aligned}
 (7.56) \quad \sum_{s=n, n-1, \dots} t^{-s} \times &\left(\sum_{\alpha} \eta_{-s,t}(g_\alpha) + D_1 S_{1,s}^+ + (D_2 S_{2,s}^+ \sqrt{t}^{e_2} + O(\sqrt{t}^{e_2+1})) \right) \\
 &+ (D_3 S_{3,s}^+ \sqrt{t}^{e_3} + O(\sqrt{t}^{e_3+1})) + \dots + \dots.
 \end{aligned}$$

For $s = n$, we have $S_{\ell,n}^+ = \frac{1}{2\pi} (2\pi)^{-n} \text{vol}(X_{p_\ell})$ (see iii) of Proposition 7.2).

For the term given by the sum $\sum_{\alpha} \eta_{s,t}(g_\alpha)$ in (7.56), by Proposition 7.18 we obtain that for every $P \in \mathbb{N}$, there are constants $C > 0, c_1 > 0$, where C, c_1 depend on P and are independent of g_α and t , such that

$$\left| \sum_{\alpha} \sum_{s'=-n}^P t^{s'} \eta_{s',t}(g_\alpha) \right| \leq C t^{P+1} \sum_{\alpha} \int_X |g_\alpha(x)| dv_X(x), \quad \text{for every } 0 < t < c_1.$$

By the definition of asymptotic expansion (cf. Definition 5.5), the term $\sum_{\alpha} \eta_{s,t}(g_\alpha)$ becomes immaterial to the exact form of the asymptotic expansion.

Further, the asymptotic expansion of (7.39) basically follows from that of $\int_X \text{Tr} e^{-t \hat{\square}_{b,m}^+}(x, x) dv_X(x)$.

We have now proved (part of) the main result of this section.

THEOREM 7.20 (Cf. Theorem 1.14). – *Suppose $X = \overline{X}_p \supseteq \overline{X}_{p_2} \supseteq \dots \supseteq X_{p_k} = \overline{X}_{p_k}$ with each stratum X_{p_ℓ} a connected submanifold. Let $a_s^+(t, x, y)$ ($= a_{s,m}^+(t, x, y)$), $s = n, n-1, \dots$, be as in (5.56). Write e_2 for the (real) codimension of X_{p_2} (which is an even number, cf. Remark 7.22 below). (Recall that the numerical factors $D_{\ell,m}$ are as given in Definition 7.19 and the integrals $S_{\ell,s}^+$ ($= S_{\ell,s,m}^+$) in (7.49) with subscripts simplified in the present case.) Then the following holds.*

i) As $t \rightarrow 0^+$,

$$\begin{aligned}
 (7.57) \quad \int_X \text{Tr} e^{-t \hat{\square}_{b,m}^+}(x, x) dv_X(x) &\sim D_{1,m} ((2\pi)^{-1} (2\pi t)^{-n} \text{vol}(X) + t^{-n+1} S_{1,n-1}^+ + t^{-n+2} S_{1,n-2}^+ + \dots) \\
 &+ (2\pi)^{-(n+1)} D_{2,m} \text{vol}(X_{p_2}) t^{-n+\frac{e_2}{2}} + O(t^{-n+\frac{e_2}{2}+1}).
 \end{aligned}$$

In particular, by $p \delta_{p|m} \sum_{q=1}^p e^{-\frac{2\pi i}{p} qm} = p$ for $p \mid m$ and 0 otherwise, one has $D_{1,m} = p$ if $p \mid m$. If $p_2 \mid m$ (thus $p \mid m$ too), then $D_{1,m}, D_{2,m} > 0$.

ii) In the asymptotic expansion (7.57), all the coefficients of t^j for j being half-integral, vanish. Hence, the remaining term in (7.57) is $O(t^{-n+\frac{e_2}{2}+1})$.

iii) As a consequence of (7.56) and ii)

$$(7.58) \quad \int_X \text{Tr} a_{s,m}^+(t, x, x) dv_X(x) \sim D_{1,m} S_{1,s}^+ + D_{2,m} S_{2,s}^+ t^{\frac{\epsilon_2}{2}} + O(t^{\frac{\epsilon_2}{2}+1}), \quad \text{as } t \rightarrow 0^+.$$

The similar results hold true for the case $\int_X \text{Tr} e^{-t\hat{\square}_{b,m}^-}(x, x) dv_X(x)$ and $\int_X \text{Tr} a_{s,m}^-(t, x, x) dv_X(x)$ as well.

Proof. – It remains to prove ii) of the theorem. Recall that the last two paragraphs of the proof of Proposition 7.2, especially the item δ) there. In the present case, by scaling ($\hat{y} \rightarrow \sqrt{t}\hat{y}$) and using (7.19), it reduces to computing the expansion (in \sqrt{t}) of

$$(7.59) \quad \begin{aligned} \text{a)} \quad & \hat{b}_{j,s}^+((\sqrt{t}\hat{y}, Y), e^{-iu} \circ (\sqrt{t}\hat{y}, Y)) \\ \text{b)} \quad & dv_X(x) \end{aligned}$$

for a fixed u .

Write $g_u(x) = e^{-iu} \circ x$ and

$$\hat{b}_{j,s}^+((\sqrt{t}\hat{y}, Y), e^{-iu} \circ (\sqrt{t}\hat{y}, Y)) = (\hat{b}_{j,s} \circ (\text{id}, g_u))((\sqrt{t}\hat{y}, Y), (\sqrt{t}\hat{y}, Y)).$$

By δ) mentioned above, $g_u(0, Y)$ is only away from $(0, Y)$ by a small difference ($\leq \epsilon$) in their θ -coordinates, hence by continuity, $g_u((\sqrt{t}\hat{y}, Y))$ lies in an $O(2\epsilon)$ -small neighborhood of $(0, Y)$ (as $t \rightarrow 0^+$), giving that the Taylor expansion of $\hat{b}_{j,s}(x, y)$, $x = (\sqrt{t}\hat{y}, Y)$, $y = e^{-iu} \circ x$, around $x = y = (0, Y) \equiv Y \equiv 0$ can be done in terms of integral powers of $\sqrt{t}\hat{y}_i$ where \hat{y}_i is in \hat{y} . Hence the coefficients of the t^j for j being half-integral must involve an odd power of some variable \hat{y}_i in \hat{y} . Since \hat{y} sits in an even dimensional space (cf. i) of Remark 7.22 below), $dv_X(x)$ is of integral power in t . With a), b) of (7.59), by using i) the claim (7.19), ii) $\int_{-\infty}^{\infty} e^{-\hat{y}_i^2} \hat{y}_i^n d\hat{y}_i = 0$ for an odd number n and iii) for a polynomial $P(x)$, $\int_{1/\sqrt{t}}^{\infty} e^{-x^2} P(x) dx \sim O(t^\infty)$ (as $t \rightarrow 0^+$), our assertion about the asymptotic expansion in ii) of the theorem follows. \square

REMARK 7.21. – One may think of the second line in (7.57) as the *main terms* which remind one of the close relation between the Kodaira Laplacian and Kohn Laplacian (cf. Proposition 5.1). However, one key point in this paper is the idea that if the S^1 action is locally free (but not globally free) on X , then this relation cannot be altogether extended to their heat kernels. In this regard the *correction terms* exist, and consist in the third line of (7.57) linked up with the higher strata of the (locally free) S^1 action beyond the principal stratum.

REMARK 7.22. – i) Let us prove that e_ℓ , the codimension of X_{p_ℓ} , is even. X_{p_ℓ} is S^1 invariant; $TX_{p_\ell} = (\mathbb{R}T^\perp \cap TX_{p_\ell}) \oplus \mathbb{R}T|_{X_{p_\ell}}$ where $\mathbb{R}T$ is the line subbundle of TX generated by $\partial/\partial\theta$ such that $\mathbb{R}T \oplus \mathbb{R}T^\perp = TX$. In a BRT chart $U \times]\epsilon, \epsilon[$ we denote $U \times \{0\} (\subset X)$ by \tilde{U} . Write $(\mathbb{R}T^\perp \cap TX_{p_\ell})|_{\tilde{U} \cap X_{p_\ell}} \equiv E$. For any given $p \in U$, with $\tilde{p} = p \times \{0\}$, one may choose a BRT coordinate such that $E_{\tilde{p}} \subset T_{\tilde{p}}\tilde{U}$ (see the proof of Lemma 7.6 where $T_{\tilde{p}}\tilde{U} = \mathbb{R}T^\perp|_{\tilde{p}}$). Without loss of generality we may assume $\tilde{p} \in X_{p_\ell}$.

Write $g = e^{i2\pi/p_\ell} (\in S^1)$ which is CR and an isometry with the fixed point set \overline{X}_{p_ℓ} . By $dg \circ J = J \circ dg = J$ on $E_{\tilde{p}}$ with J the complex structure of $T_{\tilde{p}}\tilde{U}$, $E_{\tilde{p}}$ is invariant under J . It follows E is of even dimension, so X_{p_ℓ} is of odd dimension, i.e., e_ℓ is even. ii) For $1 \leq \ell \leq k$ we can write $X_{p_\ell} \rightarrow M_{p_\ell}$ for a complex manifold M_{p_ℓ} , as an S^1 fiber bundle. The quantities $\alpha_s^\pm(x)$ are S^1 invariant by construction (see (7.47)-(7.49)), hence descend to M_{p_ℓ} . One sees $S_{\overline{X}_{\ell,s}}^\pm (\equiv S_{\ell,s}^\pm) = \frac{2\pi}{p_\ell} S_{M_{p_\ell},s}^\pm (S_{M_{p_\ell},s}^\pm = \int_{M_{p_\ell}} \text{Tr} \alpha_s^\pm dv_{M_{p_\ell}})$. Here, the metric on M_{p_ℓ} (cf. $dv_{M_{p_\ell}}$) is defined in a way similar to that given in (4.1). This suggests a question of how the heat kernels (for the locally free S^1 action) of the present paper may be connected with (certain suitably defined) heat kernels in the orbifold base X/S^1 . In a certain Riemannian setting, some work in a similar direction has been done (cf. [59, Theorems 3.5, 3.6]).

7.5. Types for S^1 stratifications; proof for the general type

Lastly, to modify the above reasoning to the case beyond the simple type is essentially not difficult. Suppose, say, X_{p_2} has several connected components Y_i such that the simple type condition is assumed along each component Y_i . Then, clearly the above argument applies to the individual Y_i and the result is just to sum up over i . Without assuming the simple type condition on Y_i , say, inside some Y_i the next stratum X_{p_3} has several components Z_j or some components Z_j are seated even outside of each Y_i . Then by localization argument along each Z_j just as done above, one repeats the pattern similarly. The process continues.

We are now motivated to transplant the notion of “type,” “class” in Definition 7.1 for the integral I of (7.3) into the geometry of the stratification of the S^1 action.

For a connected component $X_{p_{\ell(\gamma_\ell)}} \subset X_{p_\ell}$, $\gamma_\ell \in \{1, \dots, s_\ell\}$, contained in the higher dimensional connected components of the strata

$$(X =) \overline{X}_{p_{i_1(\gamma_{i_1})}} \supseteq \overline{X}_{p_{i_2(\gamma_{i_2})}} \cdots \supseteq \overline{X}_{p_{i_f(\gamma_{i_f})}} \supseteq \overline{X}_{p_{i_{f+1}(\gamma_{i_{f+1}})}} = \overline{X}_{p_{\ell(\gamma_\ell)}},$$

where $i_1 = 1 < i_2 < \dots < i_f < i_{f+1} = \ell \in \{1, 2, \dots, \ell - 1, \ell\}$, we define its type $\tau(X_{p_{\ell(\gamma_\ell)}})$ by

$$(7.60) \quad \tau(X_{p_{\ell(\gamma_\ell)}}) \equiv \tau(\ell(\gamma_\ell)) := (i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \dots, i_f(\gamma_{i_f}), i_{f+1}(\gamma_{i_{f+1}})),$$

$i_1 = \gamma_{i_1} = s_1 = 1; i_{f+1} = \ell$. One has $p_{i_1} | p_{i_2} | \dots | p_{i_{f+1}}$ (cf. Remark 1.17 for a similar case).

The notions such as *simple type*, *class* and *length* $l(\tau)$ are defined similarly, cf. Definition 7.1. (No definition of *trivial type* is given here. See iii) of Definition 7.23 below in which $i(c, [\tau]) = \infty$ corresponds to the trivial type, cf. iii) of Definition 7.10.)

Recall that if $M \subset N$ is a finite disjoint union of submanifolds M_j , then the dimension of M is $\max_j \{\dim_{\mathbb{R}} M_j\}$ and the codimension of M is $\dim_{\mathbb{R}} N - \dim_{\mathbb{R}} M$.

The following definition, which is bit tedious yet bears a great similarity as previously, is set up for the immediate use in the general situation.

DEFINITION 7.23. – i) Write $\nu_\ell = \{[\tau]; \tau = \tau(X_{p_\ell(\gamma_\ell)}), \gamma_\ell = 1, 2, \dots, s_\ell\}$ for the set of equivalence classes of types $\tau = \tau(X_{p_\ell(\gamma_\ell)})$ of connected components $X_{p_\ell(\gamma_\ell)}$ in X_{p_ℓ} .

ii) Write (similar to (7.49), (7.48))

$$S_{\ell(\gamma_\ell),s}^+ (= S_{X_{p_\ell(\gamma_\ell)},s,m}^+) = \int_{\overline{X}_{p_\ell(\gamma_\ell)}} \text{Tr } \alpha_s^+(Y, Y) dv_{\overline{X}_{p_\ell(\gamma_\ell)}}(Y) \quad (\alpha_s^+ = \alpha_{s,m}^+)$$

associated with $X_{p_\ell(\gamma_\ell)}$.

iii) Let $[\tau] = [(i_1(\gamma_{i_1}), i_2(\gamma_{i_2}), \dots, i_f(\gamma_{i_f}), i_{f+1}(\gamma_{i_{f+1}}))]$ be given. If $c \mid p_1$, define $i(c, [\tau]) = 1$. (Hence it is independent of $[\tau]$.) If $c \nmid p_1$ and $c \mid p_\ell$, $\ell = i_s$ for some s , $2 \leq s \leq f + 1$, such that $c \nmid i_{s'}$ for all $s' < s$. Then $i(c, [\tau]) := \ell \geq 2$. If $c \nmid p_{i_s}$ for $1 \leq s \leq t + 1$, $i(c, [\tau]) := \infty$. We may write $i(c)$ for $i(c, [\tau])$.

iv) For $i(c, [\tau]) \geq 2$, define the numerical factors $d_{c,m, [\tau]}$ correspondingly as in (7.33). For $i(c) = 1$, define $d_{c,m, [\tau]} = d_c$ (which is independent of τ) as in (7.29).

v) For a given $[\tau]$, if $i(c) = 1$, then define the weight factors D_1 as in (7.52) (which is independent of τ) and if $i(c) \neq 1, \infty$, define $D_{\ell, [\tau]} (= D_{\ell, m, [\tau]}) \equiv \sum_{c, i(c)=\ell} d_{c,m, [\tau]}$.

vi) Write $e_{i_q, [\tau]} \equiv e_{i_q(\gamma_{i_q})}$ with $\tau(\ell(\gamma_\ell)) = (i_1(\gamma_{i_1}), \dots, i_q(\gamma_{i_q}), \dots, i_{f+1}(\gamma_{i_{f+1}}))$ for the codimension of $X_{p_{i_q}(\gamma_{i_q})}$. Obviously, $e_{i_1, [\tau]} < e_{i_2, [\tau]} < \dots$. For $[\tau] \in \nu_\ell$, write $e_{[\tau]} \equiv e_{i_{f+1}(\gamma_{i_{f+1}})}$, i.e., $e_{\ell(\gamma_\ell)}$.

vii) Write $e = \min_{\substack{[\tau] \in \nu_\ell \\ 2 \leq \ell \leq k}} e_{[\tau]} (= \min_{\tau, \ell(\tau)=2} e_{[\tau]}$ by vi) above) and for $\ell \geq 2$,

$$\hat{\nu}_\ell = \{[\hat{\tau}_\ell] \in \nu_\ell; \hat{\tau}_\ell = (1, \ell(\gamma_\ell)) \text{ of length two such that } e_{\ell(\gamma_\ell)} = e, \text{ i.e., } e_{[\hat{\tau}_\ell]} = e\} \subset \nu_\ell.$$

Of course, it is not ruled out that for some values of ℓ , $\hat{\nu}_\ell$ could be an empty set. Intuitively, one thinks of e as the minimal codimension among those connected components $X_{p_\ell(\gamma_\ell)}$ such that if $\overline{X}_{p_{\ell'}} \supsetneq X_{p_\ell(\gamma_\ell)}$, then $X_{p_{\ell'}} = X_{p_1}$ the principal stratum.

viii) For a fixed $[\kappa] \in \nu_\ell$ in i), write (see (7.60) for $\tau(\ell(\gamma_\ell)) \equiv \tau(X_{p_\ell(\gamma_\ell)})$)

$$Z_{[\kappa],s} = \sum_{\substack{\gamma_\ell, [\tau(\ell(\gamma_\ell))] = [\kappa] \\ 1 \leq \gamma_\ell \leq s_\ell}} S_{\ell(\gamma_\ell),s}^+.$$

For the case of general type, we can obtain results corresponding to (7.54), (7.55) and (7.56) (yet complicated in expressions here). We are content with summarizing the final result as follows.

$$(7.61) \quad \int_X \text{Tre}^{-t} \widehat{\square}_{b,m}^+(x, x) dv_X(x) \sim \sum_{s=n, n-1, \dots} t^{-s} \times \left(D_1 S_{1,s}^+ + \sum_{[\kappa_2] \in \nu_2} (D_{2, [\kappa_2]} Z_{[\kappa_2],s} \sqrt{t}^{e_{[\kappa_2]}} + O(\sqrt{t}^{e_{[\kappa_2]}+1})) \right. \\ \left. + \sum_{[\kappa_3] \in \nu_3} (D_{3, [\kappa_3]} Z_{[\kappa_3],s} \sqrt{t}^{e_{[\kappa_3]}} + O(\sqrt{t}^{e_{[\kappa_3]}+1})) + \dots \right).$$

The following main result of this subsection parallels Theorem 7.20 in the last subsection. By comparison, to collect the coefficients for the next leading order in t or \sqrt{t} in (7.61) here, we have a slightly more complicated summation (regarded as

part of *corrections* as indicated in Remark 7.21) in the formula below. Note that the conversion to the stated form of Theorem 1.14 is nothing but a direct consequence of an examination (slightly tedious) of the various definitions here.

THEOREM 7.24 (Cf. Theorem 1.14). – *Notations as in Theorem 7.20 without assuming the conditions of connectedness and simple-type there. The weight factors $D_{\ell,m, [\tau]}$, the integrals $S_{\ell(\gamma_\ell),s}^+$ ($= S_{\ell(\gamma_\ell),s,m}^+$), e , $\hat{\tau}_\ell$ etc. are just given above. One has the following.*

i) As $t \rightarrow 0^+$,

$$(7.62) \quad \int_X \text{Tr} e^{-t \hat{\square}_{b,m}^+}(x, x) dv_X(x) \sim D_{1,m} ((2\pi)^{-1} (2\pi t)^{-n} \text{vol}(X) + t^{-n+1} S_{1,n-1}^+ + t^{-n+2} S_{1,n-2}^+ + \dots) + t^{-n+\frac{e}{2}} \left(\sum_{\substack{[\hat{\tau}_\ell] \in \hat{\nu}_\ell \\ 2 \leq \ell \leq k}} D_{\ell,m, [\hat{\tau}_\ell]} Z_{[\hat{\tau}_\ell],n} \right) + O(t^{-n+\frac{e+1}{2}})$$

(where recall that $Z_{[\hat{\tau}_\ell],n} = (2\pi)^{-(n+1)} \sum_{\substack{\gamma_\ell, 1 \leq \gamma_\ell \leq s_\ell \\ \text{codim } X_{p_\ell(\gamma_\ell)} = e}} \text{vol}(X_{p_\ell(\gamma_\ell)}) > 0$ in the locally free case of the S^1 action). If $p_\ell \mid m$ (thus $p_1 \mid m$ too), then $D_{1,m}, D_{\ell,m, [\hat{\tau}_\ell]} > 0$.

ii) In the asymptotic expansion (7.62), all the coefficients of t^j for j being half-integral, vanish.

iii) As a consequence of (7.61) and ii),

$$(7.63) \quad \int_X \text{Tr} a_{s,m}^+(t, x, x) dv_X(x) \sim D_{1,m} S_{1,s}^+ + t^{\frac{e}{2}} \left(\sum_{\substack{[\hat{\tau}_\ell] \in \hat{\nu}_\ell \\ 2 \leq \ell \leq k}} D_{\ell,m, [\hat{\tau}_\ell]} Z_{[\hat{\tau}_\ell],s} \right) + O(t^{\frac{e}{2}+1})$$

(where $Z_{[\hat{\tau}_\ell],s} = \sum_{\substack{\gamma_\ell, 1 \leq \gamma_\ell \leq s_\ell \\ \text{codim } X_{p_\ell(\gamma_\ell)} = e}} S_{\ell(\gamma_\ell),s}^+$).

The similar results hold for $\int_X \text{Tr} e^{-t \hat{\square}_{b,m}^-}(x, x) dv_X(x)$ and $\int_X \text{Tr} a_{s,m}^-(t, x, x) dv_X(x)$.

REMARK 7.25. – The quantities involved above are computable in the sense that they are basically reduced to those involved in the (ordinary) Kodaira heat kernel by ii) of Definition 7.23, cf. Remark 1.9.

REMARK 7.26. – It is not obvious how one can compute the *supertrace* integral, hence our index Theorem 6.4, Corollary 6.5 solely by techniques similar to those derived in (7.62), partly because here we are not using the *off-diagonal estimate* of Theorem 5.10 which is partly based on a cancelation result in Berenzin integral (see the proof of that theorem). These results (of estimate and cancelation) appear to lie beyond what the geometry of the stratifications can reveal as done in this Section 7.

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Among the transversally elliptic operators initiated by Atiyah and Singer, Kohn's \square_b operator on CR manifolds with S^1 action is a natural one of geometric significance for complex analysts. Our first main result establishes an asymptotic expansion for the heat kernel of such an operator with values in its Fourier components, which involves a contribution in terms of a distance function from lower dimensional strata of the S^1 -action. Our second main result computes a local index density, in terms of *tangential* characteristic forms, on such manifolds including *Sasakian manifolds* of interest in String Theory, by showing that certain non-trivial contributions from strata in the heat kernel expansion will eventually cancel out by applying Getzler's rescaling technique to off-diagonal estimates. This leads to a local result which can be thought of as a type of local index theorem on these CR manifolds. We give examples of these CR manifolds, some of which arise from Brieskorn manifolds. Moreover in some cases, we can reinterpret Kawasaki's Hirzebruch-Riemann-Roch formula for a complex orbifold equipped with an orbifold holomorphic line bundle, as an index theorem obtained by a single integral over a smooth CR manifold. We achieve this without use of equivariant cohomology method and our method can naturally drop the contributions arising from lower dimensional strata as done in previous works.

Le laplacien de Kohn sur une variété de Cauchy-Riemann (CR) avec action transverse d'un cercle est un exemple important pour l'analyse complexe d'un opérateur transversalement elliptique. Nous établissons ici un développement asymptotique du noyau de la chaleur de ses coefficients de Fourier, qui inclut une contribution des strates singulières de l'action du cercle. Nous calculons ensuite une densité d'indice locale pour ces opérateurs en montrant, à l'aide de techniques dues à Getzler, que certaines contributions des strates singulières non-triviales dans le développement du noyau de la chaleur s'annulent ici. Ce résultat, que l'on peut interpréter comme un théorème d'indice local sur ces variétés CR, s'applique notamment aux variétés de Sasaki qui sont importantes en théorie des cordes. Nous donnons également des exemples concrets de telles variétés CR, issues notamment des variétés de Brieskorn. De plus, nous pouvons réinterpréter dans certains cas la version du théorème de Hirzebruch-Riemann-Roch pour un orbifold complexe muni d'un fibré orbifold en droites complexes due à Kawasaki comme une formule d'indice. Notre méthode évite le recours à la cohomologie équivariante et les annulations des termes issus des strates singulières surviennent naturellement.