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STABLE FORMALITY QUASI-ISOMORPHISMS FOR HOCHSCHILD COCHAINS

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STABLE FORMALITY QUASI-ISOMORPHISMS FOR HOCHSCHILD COCHAINS

V. A. Dolgushev

Abstract. – We consider L_∞-quasi-isomorphisms for Hochschild cochains whose structure maps admit "graphical expansion". We introduce the notion of stable formality quasi-isomorphism which formalizes such an L_∞-quasi-isomorphism. We define a homotopy equivalence on the set of stable formality quasi-isomorphisms and prove that the set of homotopy classes of stable formality quasi-isomorphisms form a torsor for the group corresponding to the zeroth cohomology of the full (directed) graph complex. This result may be interpreted as a complete description of homotopy classes of formality quasi-isomorphisms for Hochschild cochains in the "stable setting".

Résumé **(Quasi-isomorphismes stables de formalité pour les cochaînes de Hochschild)**

Nous considérons des L_∞-quasi-isomorphismes pour les cochaînes de Hochschild dont les applications structurelles admettent une « expansion graphique ». Nous introduisons la notion de quasi-isomorphisme stable de formalité qui formalise les L_∞-quasi-isomorphismes de ce genre. Nous définissons une équivalence homotopique sur l'ensemble des quasi-isomorphismes stables de formalité. Nous prouvons que l'ensemble des classes homotopiques de quasi-isomorphismes stables de formalité est un torseur pour le groupe correspondant à la cohomologie de degré zéro du graphe-complexe complet (direct). Ce résultat peut-être interprété comme une description complète des classes homotopiques de quasi-isomorphismes de formalité pour les cochaînes de Hochschild dans le « cadre stable ».

CONTENTS

CHAPTER 1

INTRODUCTION

When a difficult problem is solved, it becomes even more challenging to describe all possible solutions to that problem. In this paper we propose a framework in which this interesting question can be answered completely for Kontsevich's formality conjecture [**29**] on Hochschild cochain complex.

Kontsevich's formality conjecture [**29**] states that there exists an L_∞ quasi-isomorphism from the graded Lie algebra V_A of polyvector fields on an affine space to the dg Lie algebra of Hochschild cochains $C^\bullet(A)$ of the algebra of functions A on this affine space.

In plain English the question was to find an infinite collection of maps

$$U_n : \big(V_A\big)^{\otimes n} \to C^\bullet(A), \qquad n \geq 1 \tag{1.1}$$

compatible with the action of symmetric groups and satisfying an intricate sequence of relations. The first relation says that U_1 is a map of complexes, the second relation says that U_1 is compatible with the Lie brackets up to homotopy with U_2 serving as a chain homotopy and so on.

In his groundbreaking paper [**31**] M. Kontsevich proposed a construction of such an L_∞ quasi-isomorphism over reals. His construction is "natural" in the following sense. Given polyvector fields $v_1, v_2, \dots, v_n \in V_A$, the n-th component U_n produces a Hochschild cochain via contracting indices of derivatives of various orders of polyvector fields and of functions which enter as arguments for this cochain.

Thus each term in U_n can encoded by a directed graph with two types of vertices: vertices of one type are reserved for polyvector fields and vertices of another type are reserved for functions.

In this paper we formalize the notion of L_∞ quasi-isomorphism for Hochschild cochains which are "natural" in the above sense. In other words, each term in U_n is encoded by a graph with two types of vertices and all the desired identities hold universally, i.e., on the level of linear combinations of graphs.

Such formality quasi-isomorphisms are defined for affine spaces of all [(1)] (finite) dimensions simultaneously. This is why we refer to them as *stable formality quasi-isomorphisms* (SFQs). We show that the notion of homotopy equivalence of formality quasi-isomorphisms can also be formulated in this "stable setting". Thus we can talk about homotopy classes of stable formality quasi-isomorphisms.

In this paper we show (see Theorem 6.8) that the set of homotopy classes of SFQs form a torsor for a pro-unipotent group which is obtained by exponentiating the Lie algebra $H^0(\mathsf{dfGC})$, where dfGC denotes the full (directed) version of Kontsevich's graph complex [**29**, Section 5].

Following [(2)] T. Willwacher [**39**] the group $\exp(H^0(\mathsf{dfGC}))$ is isomorphic to the Grothendieck-Teichmüller group GRT_1 introduced by V. Drinfeld in [**15**]. Thus combining Theorem 6.8 with the result of T. Willwacher [**39**], we conclude that the set of homotopy classes of SFQs is a GRT_1-torsor.

Since a formality quasi-isomorphism for Hochschild cochains provides us with a bijection between equivalence classes of star products and equivalence classes of formal Poisson structures, the result may be interpreted as a complete description of all (deformation) quantization procedures.

To give a precise definition of an SFQ, we recall [**25, 26**] that an open-closed homotopy algebra (OC-algebra) is a pair $(\mathcal{V}, \mathcal{A})$ of cochain complexes with the following data:

— an L_∞-structure on $\mathcal{V}$,
— an A_∞-structure on $\mathcal{A}$ and
— an L_∞-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of $\mathcal{A}$.

We denote by OC the 2-colored dg operad which governs open-closed homotopy algebras. It is known that OC is free as an operad in the category of graded vector spaces. Furthermore, OC is the cobar construction of a (2-colored) cooperad closely connected with the homology of Voronov's Swiss Cheese operad [**36**].

Let us also denote by KGra the 2-colored operad which is "assembled" from graphs used in Kontsevich's paper [**31**]. This 2-colored operad extends the operad dGra of directed labeled graphs and acts naturally on the pair "polyvector fields V_A and polynomials A". (See Section 3 for more details.)

Let us observe that any map of (dg) operads from OC to KGra induces an open-closed homotopy algebra on the pair (V_A, A). So we define an SFQ as a map of (dg) operads from OC to KGra subject to a few "boundary conditions". These conditions guarantee that

1. In fact they are also defined for any $\mathbb{Z}$-graded affine space.
2. See [**13**, Corollary 3.6] for more precise statement.

— the L_∞-structure on polyvector fields coincides with the Lie algebra structure given by the Schouten-Nijenhuis bracket,
— the A_∞-structure on A coincides with the usual associative (and commutative) algebra structure on polynomials, and
— the L_∞-morphism from polyvector fields V_A to Hochschild cochains $C^\bullet(A)$ starts with the Hochschild-Kostant-Rosenberg [**24**] embedding

$$V_A \hookrightarrow C^\bullet(A).$$

This operadic definition allows us to introduce a natural notion of homotopy equivalence on the set of SFQs. We give this definition using an interpretation of SFQs as Maurer-Cartan (MC) elements of an auxiliary dg Lie algebra.

Let us denote by $\mathscr{Z}^0(\mathsf{dfGC})$ the Lie algebra of degree zero cocycles of the full directed version dfGC of Kontsevich's graph complex [**29**, Section 5]. It is not hard to see that $\mathscr{Z}^0(\mathsf{dfGC})$ is a pro-nilpotent Lie algebra. Hence it can be exponentiated to the group $\exp\big(\mathscr{Z}^0(\mathsf{dfGC})\big)$.

We show that the group $\exp\big(\mathscr{Z}^0(\mathsf{dfGC})\big)$ acts on SFQs and this action descends to an action of the group $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes of SFQs.

Finally, we prove that this action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes is simply transitive.

Specialists can probably start reading this paper with Section 3. The goal of Section 2 is mostly to fix conventions and remind a few constructions for colored (co)operads. In Section 3, we define the operad of graded vector spaces dGra and its 2-colored extension KGra. In this section we also introduce a natural action of KGra on the pair "polyvector fields V_A and polynomials A". In Section 4, we remind the (dg) operad OC which governs open-closed homotopy algebras [**26**]. In Section 5, we introduce SFQs and define a notion of homotopy equivalence between them. Section 6 is devoted to the full graph complex dfGC and its "action" on SFQs. The main result of this paper (Theorem 6.8) is stated at the end of Section 6. Its proof occupies Section 7 and 8 and it depends on a few technical statements which are proved in appendices at the end of the paper.

Acknowledgment. – In many respects this work was inspired by papers [**31, 29, 28, 30**] and [**39**] and I would like to thank Thomas Willwacher for numerous discussions of his work [**39**] and for his comments on the original draft of this paper. The ideas of paper [**38**] by Thomas Willwacher were used to streamline the proof of the fact that the action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on the set of homotopy classes of SFQs is free. In this respect, the current version of this manuscript benefited from paper [**38**]. I would like to thank Chris Rogers for collaboration on [**11, 13**], and for numerous discussions. The results of this paper were presented at multiple seminars, at XXX Workshop on Geometric Methods in Physics in June of 2011 (Bialowieza, Poland) and at Geometry and Physics (GAP) XI in August of 2013 (Pittsburgh, PA). I would like to thank Murray Gerstenhaber and Jim Stasheff for the comments they made during my talk on this paper at their Deformation Theory Seminar. I would like to thank an anonymous referee for carefully reading my manuscript and for her/his remarks. I would like to thank Elena Roubtsov and Volodya Roubtsov for their help with the French version of the abstract for this paper. I would like to thank my brother-in-law Igal Vainer for his help and encouragement. I would also like to acknowledge the following NSF grants DMS 0856196, DMS-1124929, DMS-1161867, and DMS-1501001.

Many years ago, my mother abandoned her unfinished PhD thesis in mathematics to be able to devote more time to my brother and me when we were kids. I would like to thank my mother for her devotion to us and *humbly devote this paper to her.*

CHAPTER 2

PRELIMINARIES

We denote by $\mathbb{K}$ a field of characteristic zero. Our underlying symmetric monoidal category $\mathfrak{C}$ is either the category $\mathsf{grVect}_{\mathbb{K}}$ of $\mathbb{Z}$-graded $\mathbb{K}$-vector spaces or the category $\mathsf{Ch}_{\mathbb{K}}$ of unbounded *cochain* complexes of $\mathbb{K}$-vector spaces. In this paper, we use exclusively cohomological conventions. The notation ad_ξ is reserved for the adjoint action $[\xi,\ ]$ of a vector ξ in a Lie algebra and the expression $\mathrm{CH}(x,y)$ denotes the Campbell-Hausdorff series in variables x and y.

The notation S_n is reserved for the group of permutations of the set $\{1,2,\ldots,n\}$ and $\mathrm{Sh}_{p_1,p_2,\ldots,p_k}$, with $p_i \geq 0$ and $p_1+p_2+\cdots+p_k = n$, denotes the subset of $(p_1,p_2,\ldots,p_k)$-shuffles in S_n, i.e.,

$$\begin{aligned}\mathrm{Sh}_{p_1,p_2,\ldots,p_k} = \{\sigma \in S_n \mid \sigma(1) < \cdots < \sigma(p_1), \\ \sigma(p_1+1) < \cdots < \sigma(p_1+p_2), \ldots, \sigma(n-p_k+1) < \cdots < \sigma(n)\Big\}.\end{aligned} \tag{2.1}$$

We often denote by id the identity element of S_n without specifying the number n.

We denote by Com (resp. As) the operad which governs commutative (and associative) algebras without unit (resp. associative algebras without unit). The notation Lie is reserved for the operad which governs Lie algebras. Dually, we denote by coCom (resp. coAs) the cooperad which governs cocommutative (and coassociative) coalgebras without counit (resp. coassociative coalgebras without counit).

The notation Λ is reserved for the following collection in $\mathsf{grVect}_{\mathbb{K}}$

$$\Lambda(n) = \begin{cases} \mathbf{s}^{1-n}\mathrm{sgn}_n & \text{if } n \geq 1, \\ \mathbf{0} & \text{if } n = 0, \end{cases} \tag{2.2}$$

where sgn_n is the sign representation of S_n.

The collection (2.2) is equipped with a natural structure of an operad and a natural structure of a cooperad. Namely, the i-th elementary insertion and the i-th elementary co-insertion are given by the formula

$$1_n \circ_i 1_k = (-1)^{(1-k)(n-i)} 1_{n+k-1} \tag{2.3}$$

and the formula

$$\Delta_i(1_{n+k-1}) = (-1)^{(1-k)(n-i)} 1_n \otimes 1_k, \tag{2.4}$$

respectively. Here 1_m denotes the generator $\mathbf{s}^{1-m}1 \in \mathbf{s}^{1-m}\mathrm{sgn}_m$.

For an operad $\mathcal{O}$ (resp. a cooperad $\mathcal{C}$) we denote by $\Lambda\mathcal{O}$ (resp. $\Lambda\mathcal{C}$) the operad (resp. the cooperad) which is obtained from $\mathcal{O}$ (resp. $\mathcal{C}$) by tensoring with Λ. For example, a $\Lambda\mathsf{Lie}$-algebra in $\mathsf{grVect}_{\mathbb{K}}$ is a graded vector space $\mathcal{V}$ equipped with the binary operation:

$$\{,\} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$$

of degree -1 satisfying the identities:

$$\{v_1, v_2\} = (-1)^{|v_1||v_2|}\{v_2, v_1\},$$

$$\{\{v_1, v_2\}, v_3\} + (-1)^{|v_1|(|v_2|+|v_3|)}\{\{v_2, v_3\}, v_1\} + (-1)^{|v_3|(|v_1|+|v_2|)}\{\{v_3, v_1\}, v_2\} = 0,$$

where v_1, v_2, v_3 are homogeneous vectors in $\mathcal{V}$.

The operad $\Lambda\mathsf{Lie}$ has the following free resolution

$$\Lambda\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\mathsf{coCom}), \tag{2.5}$$

which we use to define an ∞-version of $\Lambda\mathsf{Lie}$-algebra structure. Thus a $\Lambda\mathsf{Lie}_\infty$-*structure* on a cochain complex $\mathcal{V}$ is a MC element Q in the Lie algebra

$$\mathrm{Coder}\big(\Lambda^2\mathsf{coCom}(\mathcal{V})\big)$$

of coderivations of the cofree coalgebra $\Lambda^2\mathsf{coCom}(\mathcal{V})$ subject to the auxiliary technical condition

$$Q\big|_{\mathcal{V}} = 0.$$

A $\Lambda\mathsf{Lie}_\infty$-*morphism* between $\Lambda\mathsf{Lie}$-algebras $(\mathcal{V}, Q)$ and $(\mathcal{W}, \widetilde{Q})$ is a homomorphism of the cofree coalgebras

$$\Lambda^2\mathsf{coCom}(\mathcal{V}) \qquad \text{and} \qquad \Lambda^2\mathsf{coCom}(\mathcal{W})$$

compatible with the differentials $\partial_{\mathcal{V}} + \mathrm{ad}_Q$ and $\partial_{\mathcal{W}} + \mathrm{ad}_{\widetilde{Q}}$ on $\Lambda^2\mathsf{coCom}(\mathcal{V})$ and $\Lambda^2\mathsf{coCom}(\mathcal{W})$, respectively.

It is not hard to see that $\Lambda\mathsf{Lie}_\infty$-algebra structures on a cochain complex $\mathcal{V}$ are in a natural bijection with L_∞-algebra structures on $\mathbf{s}^{-1}\mathcal{V}$. Moreover, it is very easy to switch back and forth between these algebra structures. However, for our purposes, it is much more convenient to work with the operad (2.5) versus the operad $\mathrm{Cobar}(\Lambda\mathsf{coCom})$ which governs L_∞-algebras. So, in the bulk of the paper, we adhere to the former choice.

A directed graph Γ consists of two finite sets $V(\Gamma)$, $E(\Gamma)$ and a map $\mathfrak{e} : E(\Gamma) \to V(\Gamma) \times V(\Gamma)$. Elements of $V(\Gamma)$ are called vertices and elements of $E(\Gamma)$ are called edges. In this paper, we consider exclusively graphs without loops (i.e., cycles of length one). In other words, the image of the map $\mathfrak{e}$ has the empty intersection with the diagonal in $V(\Gamma) \times V(\Gamma)$. Although we do consider graphs with the empty set of edges, we will tacitly assume that the set of vertices is always non-empty.

For example, the graph Γ shown in Figure 2.1 has $V(\Gamma) = \{1, 2, 3, 4, 5\}$ and $E(\Gamma) = \{a, b, c, d\}$ with $\mathfrak{e}(a) = (3, 1)$, $\mathfrak{e}(b) = (3, 2)$, and $\mathfrak{e}(c) = \mathfrak{e}(d) = (2, 3)$.

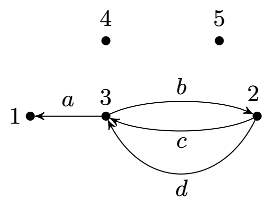


FIGURE 2.1. An example of a directed graph

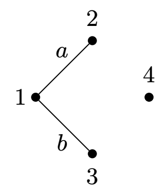


FIGURE 2.2. An undirected graph Γ'

An undirected graph (or simply a graph) Γ consists of two finite sets $V(\Gamma)$, $E(\Gamma)$ and a map $\mathfrak{e} : E(\Gamma) \to V(\Gamma)^{[2]}$, where $V(\Gamma)^{[2]}$ is the set of all unordered pairs of (distinct) elements of $V(\Gamma)$. For example, the graph Γ' shown in Figure 2.2 has $V(\Gamma') = \{1, 2, 3, 4\}$, $E(\Gamma') = \{a, b\}$, $\mathfrak{e}(a) = \{1, 2\} = \{2, 1\}$, and $\mathfrak{e}(b) = \{1, 3\} = \{3, 1\}$.

A valency of a vertex v in a (directed) graph Γ is the number of edges incident to v. For example, the valency of vertex 2 in the graph in Figure 2.1 is 3 and the valency of vertex 1 in the graph in Figure 2.2 is 2.

In this paper, we mostly deal with directed graphs which do not have multiple edges with the same direction. For such graphs Γ, we will identify $E(\Gamma)$ with the corresponding subset of ordered pairs of vertices. Furthermore, if a graph Γ' is undirected and has no multiple edges then we will identify $E(\Gamma')$ with the corresponding subset of unordered pairs of vertices. For example, for the graph Γ' in Figure 2.2, $E(\Gamma')$ can be identified with the set of unordered pairs $\{\{1, 2\}, \{1, 3\}\}$.

2.1. Trees

A connected graph without cycles is called a tree. In this paper, we tacitly assume that all trees are rooted and the root vertex has always valency 1. (Such trees are sometimes called *planted*). The remaining vertices of valency 1 are called *leaves*. A vertex is called *internal* if it is neither a root nor a leaf. We always orient trees in the direction towards the root. Thus every internal vertex has at least 1 incoming edge and exactly 1 outgoing edge. An edge adjacent to a leaf is called *external*.

Let us recall that for every planar tree $\mathbf{t}$ the set of its vertices is equipped with a natural total order. To define this total order on the set $V(\mathbf{t})$ of all vertices of $\mathbf{t}$ we introduce the function

$$\mathscr{N} : V(\mathbf{t}) \to V(\mathbf{t}). \tag{2.6}$$

To a non-root vertex v the function $\mathscr{N}$ assigns the next vertex along the (unique) path connecting v to the root vertex. Furthermore $\mathscr{N}$ sends the root vertex to the root vertex.

Let v_1, v_2 be two distinct vertices of $\mathbf{t}$. If v_1 lies on the path which connects v_2 to the root vertex then we declare that $v_1 < v_2$. Similarly, if v_2 lies on the path which connects v_1 to the root vertex then we declare that $v_2 < v_1$. If neither of the above options realize then there exist numbers k_1 and k_2 such that

$$\mathscr{N}^{k_1}(v_1) = \mathscr{N}^{k_2}(v_2) \tag{2.7}$$

but

$$\mathscr{N}^{k_1-1}(v_1) \neq \mathscr{N}^{k_2-1}(v_2).$$

Since the tree $\mathbf{t}$ is planar the set of $\mathscr{N}^{-1}(\mathscr{N}^{k_1}(v_1))$ is equipped with a total order. Furthermore, since both vertices $\mathscr{N}^{k_1-1}(v_1)$ and $\mathscr{N}^{k_2-1}(v_2)$ belong to the set $\mathscr{N}^{-1}(\mathscr{N}^{k_1}(v_1))$, we may compare them with respect to this order. We declare that, if $\mathscr{N}^{k_1-1}(v_1) < \mathscr{N}^{k_2-1}(v_2)$, then $v_1 < v_2$. Otherwise we set $v_2 < v_1$.

It is not hard to see that the resulting relation $<$ on $V(\mathbf{t})$ is indeed a total order.

We have an obvious bijection between the set of edges $E(\mathbf{t})$ of a tree $\mathbf{t}$ and the subset of vertices:

$$V(\mathbf{t}) \setminus \{\text{root vertex}\}. \tag{2.8}$$

This bijection assigns to a vertex v in (2.8) its outgoing edge.

Thus the canonical total order on the set (2.8) gives us a natural total order on the set of edges $E(\mathbf{t})$.

For our purposes we also extend the total orders on the sets $V(\mathbf{t}) \setminus \{\text{root vertex}\}$ and $E(\mathbf{t})$ to the disjoint union

$$\Big(V(\mathbf{t}) \setminus \{\text{root vertex}\}\Big) \sqcup E(\mathbf{t}) \tag{2.9}$$

by declaring that a vertex is bigger than its outgoing edge. For example, the root edge is the minimal element in the set (2.9).

2.1.1. Colored trees, labeled colored trees. – Let Ξ be a non-empty finite totally ordered set. We will call elements of Ξ colors.

Let $\mathbf{t}$ be a tree and v be an internal vertex of $\mathbf{t}$. Let us denote by $E_v(\mathbf{t})$ the set of edges terminating at v. Recall that a planar structure on a tree $\mathbf{t}$ is nothing but a choice of total orders on the sets $E_v(\mathbf{t})$ for all internal vertices v.

A Ξ-*colored planar tree* is a planar tree $\mathbf{t}$ equipped with a map

$$c_{\mathbf{t}} : E(\mathbf{t}) \to \Xi,$$

which satisfies the following condition

CONDITION 2.1. – *The restriction of the map $c_{\mathbf{t}}$ to the subset $E_v(\mathbf{t}) \subset E(\mathbf{t})$*

$$c_{\mathbf{t}}\big|_{E_v(\mathbf{t})} \ : \ E_v(\mathbf{t}) \to \Xi$$

is a monotonous function for every internal vertex v.

We refer to the value $c_{\mathbf{t}}(e)$ of $c_{\mathbf{t}}$ at e as *the color of the edge e.*

Using the obvious bijection between the leaves and the external edges we assign to each leaf the color of its adjacent edge. We denote the resulting color function by $c_{\mathbf{t},l}$

$$c_{\mathbf{t},l} : L(\mathbf{t}) \to \Xi, \tag{2.10}$$

where $L(\mathbf{t})$ is the set of leaves of $\mathbf{t}$.

Using the function (2.10) we split the set $L(\mathbf{t})$ into the disjoint union

$$L(\mathbf{t}) = \bigsqcup_{\chi \in \Xi} c_{\mathbf{t},l}^{-1}(\chi). \tag{2.11}$$

We now define a *labeled* Ξ*-colored planar tree* as a Ξ-colored planar tree $\mathbf{t}$ equipped with (not necessarily monotonous) injective maps

$$\mathfrak{l}_\chi : \{1, 2, \dots, n_\chi\} \to c_{\mathbf{t},l}^{-1}(\chi), \tag{2.12}$$

where n_χ are non-negative integers satisfying the obvious condition $n_\chi \le |c_{\mathbf{t},l}^{-1}(\chi)|$. The collection of numbers $\{n_\chi\}_\chi$ is considered as a part of the data incorporated in a labeling of a tree.

Leaves belonging to the union

$$\bigsqcup_{\chi \in \Xi} \mathfrak{l}_\chi(\{1, 2, \dots, n_\chi\})$$

are called *labeled.* Furthermore, a vertex x of a labeled colored planar tree $\mathbf{t}$ is called *nodal* if it is neither a root vertex, nor a labeled leaf. We denote by $V_{\mathrm{nod}}(\mathbf{t})$ the set of all nodal vertices of $\mathbf{t}$. Keeping in mind the canonical total order on the set of all vertices of $\mathbf{t}$ we say things like "the first nodal vertex," "the second nodal vertex," and "the i-th nodal vertex".

EXAMPLE 2.2. – *In this paper, the set Ξ is often the two-element set*[1] $\{\mathfrak{c}, \mathfrak{o}\}$ *with* $\mathfrak{c} < \mathfrak{o}$*. Figure 2.3 gives us an example of a labeled* $\{\mathfrak{c}, \mathfrak{o}\}$*-colored (or simply* 2*-colored) planar tree. Throughout this paper edges of color* $\mathfrak{c}$ *are drawn solid and edges of color* $\mathfrak{o}$ *are drawn dashed. In addition, we use small white circles for nodal vertices and small black circles for labeled leaves and the root vertex. Figure 2.4 shows an example of a*

1. The notation for colors comes from string theory **[42]**. $\mathfrak{o}$ refers to open strings and $\mathfrak{c}$ refers to closed strings.

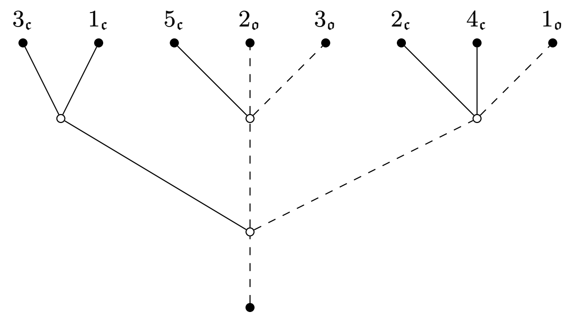


FIGURE 2.3. Solid edges carry the color $\mathfrak{c}$ and dashed edges carry the color $\mathfrak{o}$

labeled 2-colored planar tree which has two unlabeled leaves (a.k.a. two univalent nodal vertices).

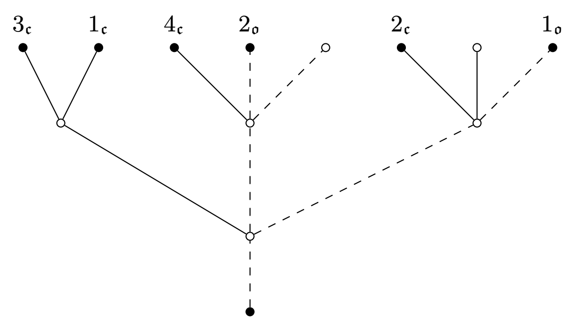


FIGURE 2.4. The 4-th and the 6-th nodal vertices are univalent

Ξ-colored planar corollas will play an important role. In particular, we will need a map which assigns a Ξ-colored planar corolla $\kappa(\mathbf{t})$ to a labeled Ξ-colored planar tree $\mathbf{t}$. To define this map we observe that Ξ-colored planar corollas are in bijection with the arrays $\{n_\chi; \chi_{\rm root}\}_{\chi \in \Xi}$ where n_χ are non-negative integers and $\chi_{\rm root}$ is an element in Ξ. More precisely, the array $\{n_\chi; \chi_{\rm root}\}_{\chi \in \Xi}$ corresponding to a Ξ-colored planar corolla $\mathbf{q}$ has $\chi_{\rm root}$ equal to the color of the root edge of $\mathbf{q}$ and

$$n_\chi = |c^{-1}_{\mathbf{q},l}(\chi)|. \tag{2.13}$$

For example, the 2-colored corolla depicted on Figure 2.5 corresponds to the array $\{2, 1; \mathfrak{o}\}$.

The degenerate array $\{n_\chi = 0; \chi_{\rm root}\}_{\chi \in \Xi}$ *is* allowed and it corresponds to the corolla depicted in Figure 2.6.

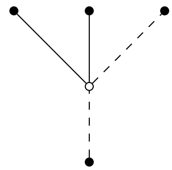


FIGURE 2.5. The corolla corresponding to the array $\{2, 1; \mathfrak{o}\}$

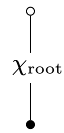


FIGURE 2.6. The corolla corresponding to the degenerate array $\{n_\chi = 0; \chi_{\text{root}}\}_{\chi \in \Xi}$

We now notice that every labeled Ξ-colored planar tree $\mathbf{t}$ gives us the array $\{n_\chi; \chi_{\text{root}}\}_{\chi\in\Xi}$ with χ_{root} being the color of the root edge of $\mathbf{t}$ and n_χ being the numbers which enter the labeling (2.12) of the tree $\mathbf{t}$. We denote by $\kappa(\mathbf{t})$ the Ξ-colored planar corolla corresponding to this array.

For example, the corolla $\kappa(\mathbf{t})$ corresponding to the labeled 2-colored planar tree $\mathbf{t}$ in Figure 2.3 is shown in Figure 2.7. Similarly, the corolla $\kappa(\mathbf{t}')$ corresponding to the labeled 2-colored planar tree $\mathbf{t}'$ in Figure 2.4 is shown in Figure 2.8.

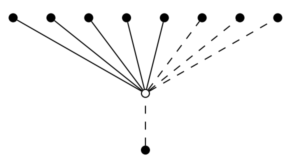


FIGURE 2.7. The corolla $\kappa(\mathbf{t})$.

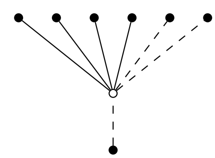


FIGURE 2.8. The corolla $\kappa(\mathbf{t}')$.

REMARK 2.3. – *It is clear that, if Ξ is a one-point set, then Ξ-colored planar trees are exactly non-colored planar trees and Ξ-colored corollas are in bijection with non-negative integers.*

2.1.2. Groupoid of labeled (colored) planar trees. – For our purposes we need to upgrade the set of labeled Ξ-colored planar trees to a groupoid Tree^{Ξ}. Objects of Tree^{Ξ} are labeled Ξ-colored planar trees and morphisms are *non-planar* isomorphisms of the corresponding trees compatible with labeling and coloring in the following sense: an isomorphism ϕ from $\mathbf{t}$ to $\mathbf{t}'$ sends the leaf of $\mathbf{t}$ with label i to the leaf [2] of $\mathbf{t}'$ with

2. In particular, a nodal vertex can only be sent a nodal vertex.

label i; furthermore, if the edge originating at $v \in V(\mathbf{t})$ carries the color χ then the edge originating at $\phi(v) \in V(\mathbf{t}')$ carries the same color χ.

EXAMPLE 2.4. – *Let us denote by* $\mathbf{t}$ *the labeled* 2*-colored planar tree depicted in Figure 2.3. The tree* $\mathbf{t}_1$ *in Figure 2.9 is isomorphic to* $\mathbf{t}$ *while the tree* $\mathbf{t}_2$ *in Figure 2.10 is not isomorphic to* $\mathbf{t}$*.*

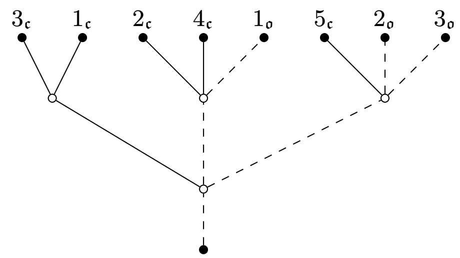


FIGURE 2.9. The labeled 2-colored tree $\mathbf{t}_1$

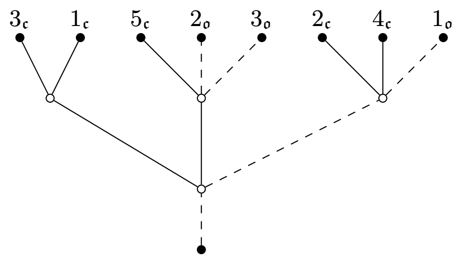


FIGURE 2.10. The labeled 2-colored tree $\mathbf{t}_2$

An object of Tree^{Ξ} *may have a non-trivial automorphism. For example, the tree shown in Figure 2.11 has a non-identity automorphism which switches the two univalent nodal vertices.*

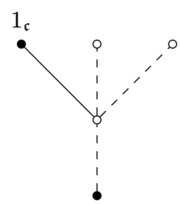


FIGURE 2.11. This labeled tree has a non-trivial automorphism which switches the two univalent nodal vertices

We tacitly assume that each labeled colored planar tree has at least one nodal vertex. In other words, the degenerate labeled tree shown on Figure 2.12 is not considered as an object of Tree^{Ξ}.

It is easy to see that, if the corollas $\kappa(\mathbf{t})$ and $\kappa(\mathbf{t}')$ corresponding to labeled Ξ-colored planar trees $\mathbf{t}$ and $\mathbf{t}'$ are different, then there are no morphisms between $\mathbf{t}$ and $\mathbf{t}'$. Therefore the groupoid Tree^{Ξ} splits into the disjoint union

$$\mathsf{Tree}^{\Xi} = \bigsqcup_{\mathbf{q}} \mathsf{Tree}^{\Xi}(\mathbf{q}), \tag{2.14}$$

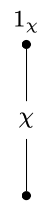


FIGURE 2.12. This tree is not considered as an object of Tree^{Ξ}

where $\mathsf{Tree}^{\Xi}(\mathbf{q})$ is the full subcategory of labeled Ξ-colored planar trees $\mathbf{t}$ satisfying the condition

$$\kappa(\mathbf{t}) = \mathbf{q} \tag{2.15}$$

and the union (2.14) is taken over all Ξ-colored planar corollas.

For every Ξ-colored planar corolla $\mathbf{q}$, we introduce the group

$$S_{\mathbf{q}} = \prod_{\chi \in \Xi} S_{n_\chi}, \tag{2.16}$$

where $n_\chi = c^{-1}_{\mathbf{q},l}(\chi)$. This group acts in the obvious way on the groupoid $\mathsf{Tree}^{\Xi}(\mathbf{q})$ by permuting labels of leaves with the same colors.

We reserve the notation $\mathsf{Tree}^{\Xi}_2(\mathbf{q})$ for the full subcategory of $\mathsf{Tree}^{\Xi}(\mathbf{q})$ whose objects are labeled Ξ-colored planar trees with exactly two nodal vertices. For example, if $\Xi = \{\mathfrak{c} < \mathfrak{o}\}$ and $\mathbf{q}$ is the corolla corresponding the array $(n, k; \chi)$ then the set of isomorphism classes of objects in $\mathsf{Tree}^{\Xi}_2(\mathbf{q})$ is in bijection with the set

$$\bigsqcup_{0 \le p \le n} \bigsqcup_{0 \le q \le k} \big\{ (\sigma, \tau, \chi_1) \mid \sigma \in \mathrm{Sh}_{p,n-p},\ \tau \in \mathrm{Sh}_{q,k-q},\ \chi_1 \in \{\mathfrak{c}, \mathfrak{o}\} \big\}, \tag{2.17}$$

where $\mathrm{Sh}_{p,q}$ is the set of (p,q)-shuffles (see the beginning of Section 2).

Namely, if the root edge of the corolla $\mathbf{q}$ carries the color $\mathfrak{c}$ then the bijection assigns to an element $(\sigma, \tau, \mathfrak{c})$ (resp. $(\sigma, \tau, \mathfrak{o})$) the isomorphism class of the labeled 2-colored planar tree depicted in Figure 2.13 (resp. 2.14). If the root edge of the corolla $\mathbf{q}$

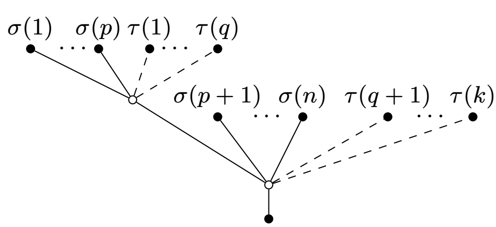


FIGURE 2.13. Here $\sigma \in \mathrm{Sh}_{p,n-p}$ and $\tau \in \mathrm{Sh}_{q,k-q}$

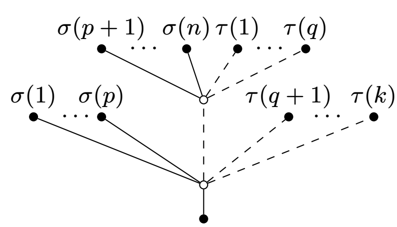


FIGURE 2.14. Here $\sigma \in \mathrm{Sh}_{p,n-p}$ and $\tau \in \mathrm{Sh}_{q,k-q}$

carries the color $\mathfrak{o}$ then we need to replace the solid root edges of the trees depicted in Figures 2.13 and 2.14 by dashed edges.

As we mentioned above, the case when Ξ is the one-point set corresponds to non-colored labeled planar trees. In this case, corollas can be identified with non-negative integers and the groupoid Tree of labeled planar trees splits into the disjoint union

$$\mathsf{Tree} = \bigsqcup_{n \geq 0} \mathsf{Tree}(n), \tag{2.18}$$

where $\mathsf{Tree}(n)$ is the groupoid of labeled planar trees with exactly n labeled leaves. We refer to objects of $\mathsf{Tree}(n)$ as *n-labeled planar trees.*

By analogy with $\mathsf{Tree}_2^{\Xi}(\mathbf{q})$, we reserve the notation $\mathsf{Tree}_2(n)$ for the full sub-groupoid of $\mathsf{Tree}(n)$ whose objects are n-labeled planar trees with exactly 2 nodal vertices. It is not hard to see that isomorphism classes of $\mathsf{Tree}_2(n)$ are in bijection with the union

$$\bigsqcup_{0 \leq p \leq n} \mathrm{Sh}_{p,n-p},$$

where $\mathrm{Sh}_{p,n-p}$ denotes the set of $(p, n-p)$-shuffles in S_n. The bijection assigns to a $(p, n-p)$-shuffles τ the isomorphism class of the planar tree depicted in Figure 2.15. Note that the "degenerate case" $p = 0$ we get a labeled planar tree with one nodal vertex of valency 1.

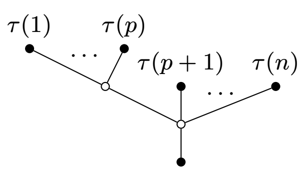


FIGURE 2.15. Here τ is a $(p, n-p)$-shuffle

2.1.3. Insertion of (colored) trees. – Let $\mathbf{t}$ be a Ξ-colored tree and x be a nodal vertex of $\mathbf{t}$. We denote by $\kappa(x)$ the Ξ-colored corolla formed by the edges adjacent to x.

If $\widetilde{\mathbf{t}}$ be another Ξ-colored labeled planar tree and its i-th nodal vertex x_i satisfies the condition

$$\kappa(\mathbf{t}) = \kappa(x_i), \tag{2.19}$$

then we can define the insertion $\bullet_i$ of the tree $\mathbf{t}$ into the i-th nodal vertex of $\widetilde{\mathbf{t}}$. For the resulting planar tree $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ we have

$$\kappa(\widetilde{\mathbf{t}} \bullet_i \mathbf{t}) = \kappa(\widetilde{\mathbf{t}}). \tag{2.20}$$

To build the tree $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$, we follow these steps:

— first, we denote by $E_{i,\chi}(\widetilde{\mathbf{t}})$ the set of edges of color χ terminating at the i-th nodal vertex of $\widetilde{\mathbf{t}}$. Since $\widetilde{\mathbf{t}}$ is planar, the set $E_{i,\chi}(\widetilde{\mathbf{t}})$ comes with a total order;
— second, we erase the i-th nodal vertex of $\widetilde{\mathbf{t}}$;
— third, we identify the root edge of $\mathbf{t}$ with the edge of $\widetilde{\mathbf{t}}$ which originated at the i-th nodal vertex;
— finally, we identify external edges of $\mathbf{t}$ which have *labeled leaves* with edges in the union

$$\bigsqcup_{\chi\in\Xi} E_{i,\chi}(\widetilde{\mathbf{t}})$$

following this rule: the external edge with color χ and label j gets identified with the j-th edge in the set $E_{i,\chi}(\widetilde{\mathbf{t}})$. In doing this, we keep the same planar structure on $\mathbf{t}$, so, in general, branches of $\widetilde{\mathbf{t}}$ move around.

EXAMPLE 2.5. – *Figure 2.18 shows the result of the insertion* $\widetilde{\mathbf{t}}\bullet_1\mathbf{t}$ *of the labeled* 2*-colored planar tree* $\mathbf{t}$ *(depicted in Figure 2.17) into the first nodal vertex of the labeled* 2*-colored planar tree* $\widetilde{\mathbf{t}}$ *(depicted in Figure 2.16).*

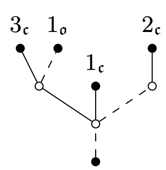


FIGURE 2.16. A labeled 2-colored planar tree $\widetilde{\mathbf{t}}$

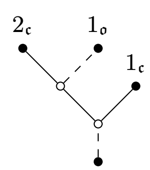


FIGURE 2.17. A labeled 2-colored planar tree $\mathbf{t}$

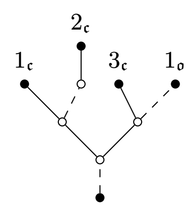


FIGURE 2.18. The result of the insertion $\widetilde{\mathbf{t}}\bullet_1\mathbf{t}$

The algorithm for constructing $\widetilde{\mathbf{t}}\bullet_1\mathbf{t}$ *is illustrated in Figure 2.19*

2.2. Colored operads and their dual versions

2.2.1. Colored collections. – Let us recall that a Ξ-colored collection in a symmetric monoidal category $\mathfrak{C}$ is given by the data:

— For each Ξ-colored planar corolla $\mathbf{q}$ we have an object

$$P(\mathbf{q})\in\mathfrak{C}$$

equipped with a left action of the group $S_{\mathbf{q}}$ (2.16).

Morphisms of Ξ-colored collections are defined in the obvious way.

In the case $\Xi=\{\mathfrak{c}<\mathfrak{o}\}$ we will denote the object corresponding to a corolla $\mathbf{q}$ by

$$P(n,k)^{\chi},$$

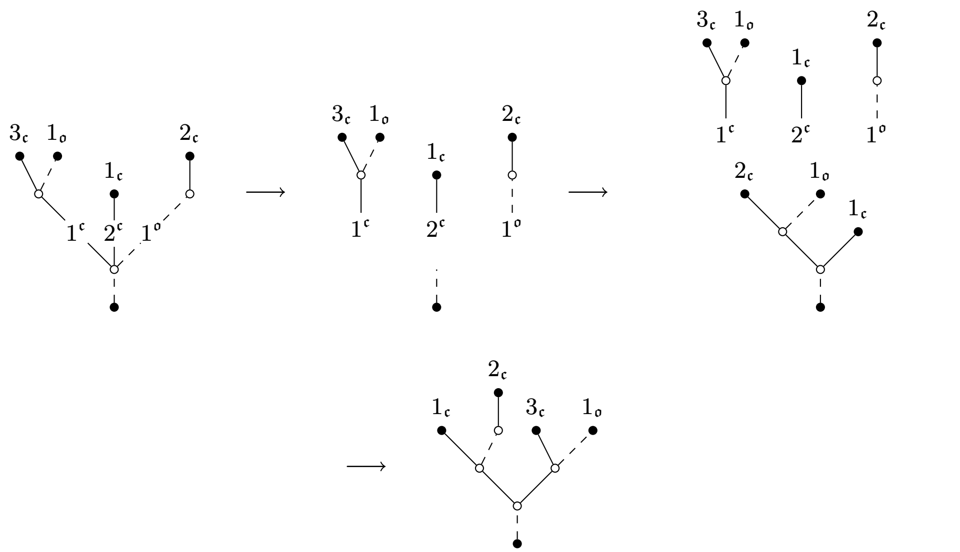


FIGURE 2.19. Algorithm for constructing $\widetilde{\mathbf{t}} \bullet_1 \mathbf{t}$

where $n = |c^{-1}_{\mathbf{q},l}(\mathfrak{c})|$, $k = |c^{-1}_{\mathbf{q},l}(\mathfrak{o})|$, and χ is the color of the root edge.

Given a Ξ-colored collection P in $\mathfrak{C}$ we introduce a covariant functor

$$P : \mathsf{Tree}^{\Xi} \to \mathfrak{C} \tag{2.21}$$

from the groupoid Tree^{Ξ} of labeled Ξ-colored planar trees to $\mathfrak{C}$.

To a labeled Ξ-colored planar tree $\mathbf{t}$, the functor P assigns the object

$$P(\mathbf{t}) = \bigotimes_{x \in V_{\text{nod}}(\mathbf{t})} P(\kappa(x)), \tag{2.22}$$

where $V_{\text{nod}}(\mathbf{t})$ is the set of all nodal vertices of $\mathbf{t}$, $\kappa(x)$ is the Ξ-colored planar corolla formed by all edges of $\mathbf{t}$ adjacent to x, and the order of the factors agrees with the total order on the set $V_{\text{nod}}(\mathbf{t})$.

To define P on the level of morphisms, we use the action of the group (2.16) on $P(\mathbf{q})$ and the braiding of the symmetric monoidal category in the obvious way. For example, let $\mathbf{t}$ and $\mathbf{t}_1$ be 2-colored trees depicted in Figures 2.3 and 2.9, respectively. For these trees we have

$$P(\mathbf{t}) = P(1,2)^{\mathfrak{o}} \otimes P(2,0)^{\mathfrak{c}} \otimes P(1,2)^{\mathfrak{o}} \otimes P(2,1)^{\mathfrak{o}},$$
$$P(\mathbf{t}_1) = P(1,2)^{\mathfrak{o}} \otimes P(2,0)^{\mathfrak{c}} \otimes P(2,1)^{\mathfrak{o}} \otimes P(1,2)^{\mathfrak{o}}.$$

The functor P sends the unique morphism $\phi : \mathbf{t} \to \mathbf{t}_1$ to

$$P(\phi) = (\mathrm{id}, \sigma_{12}) \otimes 1 \otimes \beta,$$

where $(\mathrm{id}, \sigma_{12})$ is the non-identity element of the group $S_1 \times S_2$ and β is the braiding

$$\beta : P(1,2)^{\mathfrak{o}} \otimes P(2,1)^{\mathfrak{o}} \to P(2,1)^{\mathfrak{o}} \otimes P(1,2)^{\mathfrak{o}}.$$

2.2.2. Colored (pseudo)operads. – Let $\mathbf{q}$ be a Ξ-colored planar corolla. We say that the corolla $\mathbf{q}$ is *naturally labeled* if the map

$$\mathfrak{l}_\chi : \{1, 2, \dots, n_\chi\} \to c^{-1}_{\mathbf{t},l}(\chi)$$

is a monotonous bijection for every $\chi \in \Xi$. An example of a naturally labeled corolla is depicted in Figure 2.20. The degenerate corolla shown in Figure 2.6 is considered

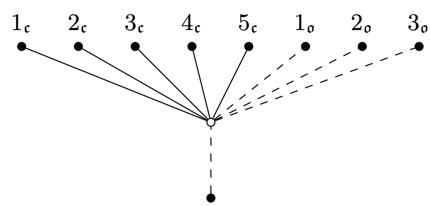


FIGURE 2.20. An example of a naturally labeled 2-colored corolla

as a naturally labeled corolla by convention.

For our purposes it is convenient to use the following definition of a colored pseudo-operad.

DEFINITION 2.6. – *A Ξ-colored pseudo-operad is a Ξ-colored collection P equipped with multiplication maps*

$$\mu_{\mathbf{t}} : \mathrm{P}(\mathbf{t}) \to P(\kappa(\mathbf{t})) \tag{2.23}$$

defined for every labeled Ξ-colored planar trees $\mathbf{t}$ and subject to the following axioms:

— *If $\mathbf{q}$ is a naturally labeled Ξ-colored planar corolla then*

$$\mu_{\mathbf{q}} = \mathrm{id}_{P(\mathbf{q})}. \tag{2.24}$$

— *The operation $\mu_{\mathbf{t}}$ is $S_{\kappa(\mathbf{t})}$-equivariant. Namely, for every labeled Ξ-colored planar tree $\mathbf{t}$ we have*

$$\mu_{\sigma(\mathbf{t})} = \sigma \circ \mu_{\mathbf{t}}, \qquad \forall \sigma \in S_{\kappa(\mathbf{t})}. \tag{2.25}$$

— *For every morphism $\lambda : \mathbf{t} \to \mathbf{t}'$ in Tree^{Ξ} we have*

$$\mu_{\mathbf{t}'} \circ \mathrm{P}(\lambda) = \mu_{\mathbf{t}}. \tag{2.26}$$

— *To formulate the associativity axiom, we consider a triple* $(\widetilde{\mathbf{t}}, x, \mathbf{t})$ *where* $\widetilde{\mathbf{t}}$ *is a labeled* Ξ*-colored planar tree,* x *is the* i*-th nodal vertex of* $\widetilde{\mathbf{t}}$*, and* $\mathbf{t}$ *is a labeled* Ξ*-colored planar tree such that* $\kappa(\mathbf{t}) = \kappa(x)$*. The associativity axiom states that for each such triple we have*

$$\mu_{\widetilde{\mathbf{t}}} \circ (1 \otimes \cdots \otimes 1 \otimes \underbrace{\mu_{\mathbf{t}}}_{i\text{-th spot}} \otimes 1 \otimes \cdots \otimes 1) \circ \beta_{\widetilde{\mathbf{t}},x,\mathbf{t}} = \mu_{\widetilde{\mathbf{t}} \bullet_i \mathbf{t}}, \tag{2.27}$$

where $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ *is the tree obtained by inserting* $\mathbf{t}$ *into the* i*-th vertex of* $\widetilde{\mathbf{t}}$ *and* $\beta_{\widetilde{\mathbf{t}},x,\mathbf{t}}$ *is the isomorphism in* $\mathfrak{C}$ *which is "responsible for putting tensor factors in the correct order".*

Morphisms of pseudo-operads are defined in the obvious way.

Let $\mathbf{q}_1$ and $\mathbf{q}_2$ be two naturally labeled Ξ-colored planar corollas such that the root edge of $\mathbf{q}_2$ carries the color χ. Let $m_{\chi'}$ (resp. $n_{\chi'}$) be the number of external edges (if any) of $\mathbf{q}_1$ (resp. $\mathbf{q}_2$) of color $\chi' \in \Xi$

Given a color $\chi \in \Xi$ for which $m_\chi > 0$ and $1 \geq i \geq m_\chi$, we denote by $\mathbf{t}_{i,\chi}$ the labeled Ξ-colored planar tree which is obtained from $\mathbf{q}_1$ and $\mathbf{q}_2$ in two steps. First, we glue $\mathbf{q}_2$ with $\mathbf{q}_1$ by identifying the root edge of $\mathbf{q}_2$ with the external edge of $\mathbf{q}_1$ which carries the color χ and label i. Second, we change the labels on the leaves of the resulting Ξ-colored planar tree following these steps:

— if $\chi' < \chi$ then we shift the labels on leaves in $c^{-1}_{\mathbf{q}_2,l}(\chi')$ up by $m_{\chi'}$;

— we shift the labels on leaves in $c^{-1}_{\mathbf{q}_2,l}(\chi)$ up by $i-1$ and we shift the labels on leaves in $c^{-1}_{\mathbf{q}_1,l}(\chi)$ which are $> i$ up by $n_\chi - 1$;

— if $\chi' > \chi$ then we shift the labels on leaves in $c^{-1}_{\mathbf{q}_1,l}(\chi')$ up by $n_{\chi'}$.

For example, if $\mathbf{q}_1$ and $\mathbf{q}_2$ is the 2-colored corollas depicted in Figures 2.21 and 2.22, respectively, then $\mathbf{t}_{2,\mathfrak{o}}$ is the tree depicted in Figure 2.23. Although the tree $\mathbf{t}_{i,\chi}$

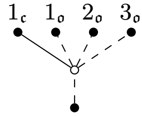


FIGURE 2.21. A 2-colored corola $\mathbf{q}_1$

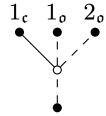


FIGURE 2.22. A 2-colored corola $\mathbf{q}_2$

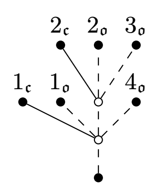


FIGURE 2.23. The labeled 2-colored tree $\mathbf{t}_{2,\mathfrak{o}}$

depends on the corollas $\mathbf{q}_1$ and $\mathbf{q}_2$, we suppress $\mathbf{q}_1$ and $\mathbf{q}_2$ from the notation.

To introduce a structure of a pseudo-operad on a collection P it suffices to specify the multiplications

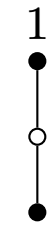
$$\mu_{\mathbf{t}_{i,\chi}} : P(\mathbf{q}_1) \otimes P(\mathbf{q}_2) \to P(\kappa(\mathbf{t}_{i,\chi})) \tag{2.28}$$

for all tuples $(\mathbf{q}_1, \mathbf{q}_2, i, \chi)$. All the remaining multiplications (2.23) can be deduced from (2.28) using axioms of pseudo-operad.

The operations (2.28) are called *elementary insertions* and we will use for them the special notation $\circ_{i,\chi}$. Namely, if $v \in P(\mathbf{q}_1)$ and $w \in P(\mathbf{q}_2)$ then

$$v \circ_{i,\chi} w := \mu_{\mathbf{t}_{i,\chi}}(v, w). \tag{2.29}$$

Let $\chi \in \Xi$ and let $\mathbf{u}_\chi$ be the labeled tree with exactly two edges: the root edge and the external edge, both carrying the color χ:

(2.30) $\mathbf{u}_\chi =$ 

We say that

Definition 2.7. – *P is a Ξ-colored operad if P is a Ξ-colored pseudo-operad with chosen maps (unit maps)*

$$I_\chi : \mathbb{K} \to P(\mathbf{u}_\chi), \tag{2.31}$$

such that the compositions

$$\begin{array}{c} P(\mathbf{q}) \cong P(\mathbf{q}) \otimes \mathbb{K} \stackrel{1 \otimes I_\chi}{\longrightarrow} P(\mathbf{q}) \otimes P(\mathbf{u}_\chi) \stackrel{\mu_{\mathbf{t}_{i,\chi}}}{\longrightarrow} P(\mathbf{q}) \\ P(\mathbf{q}) \cong \mathbb{K} \otimes P(\mathbf{q}) \stackrel{I_\chi \otimes 1}{\longrightarrow} P(\mathbf{u}_\chi) \otimes P(\mathbf{q}) \stackrel{\mu_{\mathbf{t}_{i,\chi}}}{\longrightarrow} P(\mathbf{q}) \end{array} \tag{2.32}$$

coincide with the identity map on $P(\mathbf{q})$ whenever they make sense. Morphisms of Ξ-colored operads are defined in the obvious way.

Remark 2.8. – *For a conventional definition of colored operads we refer the reader to paper* [**2**] *by C. Berger and I. Moerdijk. Due to the observation made in* [**2**, Remark 1.3] *the definition given here is equivalent to the conventional one.*

Example 2.9. – *Let $\Xi = \{\mathfrak{c}, \mathfrak{o}\}$ and $(\mathcal{V}, \mathcal{A})$ be a pair of cochain complexes. The 2-colored collection $\mathsf{End}_{\mathcal{V},\mathcal{A}}$ with*

$$\mathsf{End}_{\mathcal{V},\mathcal{A}}(n,k)^{\mathfrak{c}} = \mathrm{Hom}(\mathcal{V}^{\otimes n} \otimes \mathcal{A}^{\otimes k}, \mathcal{V}),$$

$End_{\mathcal{V},\mathcal{A}}(n,k)^{\mathfrak{o}} = \mathrm{Hom}(\mathcal{V}^{\otimes n} \otimes \mathcal{A}^{\otimes k}, \mathcal{A})$(2.33) is equipped with the obvious structure of a 2-colored operad. $\mathsf{End}_{\mathcal{V},\mathcal{A}}$ is called the endomorphism operad of the pair $(\mathcal{V}, \mathcal{A})$. This example can be obviously generalized to an arbitrary set of colors Ξ.

Example 2.9 plays an important role because an algebra over a Ξ-colored operad P is defined as a family $\{V_\chi\}_{\chi\in\Xi}$ of objects in $\mathfrak{C}$ with an operad morphism from P to $\mathsf{End}_{\{V_\chi\}_{\chi\in\Xi}}$.

2.2.3. Augmentation of colored operads. – The Ξ-colored collection

$$* (\mathbf{q}) = \begin{cases} \mathbb{K} & \text{if } \mathbf{q} = \mathbf{u}_\chi \text{ for some } \chi \in \Xi, \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{2.34}$$

is equipped with a unique structure of a Ξ-colored operad. It is easy to see that $*$ is the initial object in the category of Ξ-colored operads.

A Ξ-colored operad P is called augmented if P comes with an operad morphism

$$\varepsilon : P \to *.$$

For every augmented operad P the kernel of the map $P \to *$ is naturally a pseudo-operad. We denote this pseudo-operad by $P_\circ$.

It is not hard to see that the assignment

$$P \rightsquigarrow P_\circ$$

extends to a functor. According to [3] [**33**, Proposition 21] this functor gives us an equivalence between the category of augmented (colored) operads and the category of (colored) pseudo-operads.

2.2.4. Colored (pseudo)cooperads. – Reversing all arrows in Definition 2.6 we get

Definition 2.10. – *A Ξ-colored pseudo-cooperad is a Ξ-colored collection Q equipped with comultiplication maps*

$$\Delta_{\mathbf{t}} : Q(\kappa(\mathbf{t})) \to \mathrm{Q}(\mathbf{t}) \tag{2.35}$$

defined for every labeled Ξ-colored planar trees $\mathbf{t}$ and subject to the following axioms:

— *If $\mathbf{q}$ is a naturally labeled Ξ-colored planar corolla then*

$$\Delta_{\mathbf{q}} = \mathrm{id}_{Q(\mathbf{q})}. \tag{2.36}$$

— *The operation $\Delta_{\mathbf{t}}$ is $S_{\kappa(\mathbf{t})}$-equivariant. Namely, for every labeled Ξ-colored planar tree $\mathbf{t}$ we have*

$$\Delta_{\sigma(\mathbf{t})} \circ \sigma = \Delta_{\mathbf{t}}, \qquad \forall \sigma \in S_{\kappa(\mathbf{t})}. \tag{2.37}$$

— *For every morphism $\lambda : \mathbf{t} \to \mathbf{t}'$ in Tree^{Ξ} we have*

$$\Delta_{\mathbf{t}'} = \mathrm{Q}(\lambda) \circ \Delta_{\mathbf{t}}. \tag{2.38}$$

3. Although, in paper [**33**] the author considers only non-colored operads, the line of arguments can be easily extended to the colored setting.

— *To formulate the coassociativity axiom we consider a triple* $(\widetilde{\mathbf{t}}, x, \mathbf{t})$ *where* $\widetilde{\mathbf{t}}$ *is a labeled* Ξ*-colored planar tree,* x *is the* i*-th nodal vertex of* $\widetilde{\mathbf{t}}$*, and* $\mathbf{t}$ *is a labeled* Ξ*-colored planar tree such that* $\kappa(\mathbf{t}) = \kappa(x)$*. The coassociativity axiom states that for each such triple we have*

$$(1 \otimes \cdots \otimes 1 \otimes \underbrace{\Delta_{\mathbf{t}}}_{i\text{-th spot}} \otimes 1 \otimes \cdots \otimes 1)\Delta_{\widetilde{\mathbf{t}}} = \Delta_{\widetilde{\mathbf{t}} \bullet_i \mathbf{t}} \circ \beta_{\widetilde{\mathbf{t}}, x, \mathbf{t}}, \tag{2.39}$$

where $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ *is the tree obtained by inserting* $\mathbf{t}$ *into the* i*-th vertex of* $\widetilde{\mathbf{t}}$ *and* $\beta_{\widetilde{\mathbf{t}}, x, \mathbf{t}}$ *is the isomorphism in* $\mathfrak{C}$ *which is "responsible for putting tensor factors in the correct order".*

Morphisms of pseudo-cooperads are defined in the obvious way.

Similarly, reversing arrows in (2.31), (2.32), and Definition 2.7 we get the notion of counit and the definition of a Ξ-colored cooperad.

The Ξ-colored collection (2.34) carries a unique structure of a Ξ-colored cooperad. Furthermore, $*$ is the terminal object in the category of Ξ-colored cooperads.

Dually to augmentation, we define a coaugmentation on a (colored) cooperad Q as a cooperad morphism

$$\varepsilon' : * \to Q.$$

For every coaugmented (colored) cooperad Q the cokernel of coaugmentation naturally forms a (colored) pseudo-cooperad. We denote this pseudo-cooperad by $Q_\circ$.

Just as for (colored) operads, the assignment

$$Q \rightsquigarrow Q_\circ$$

extends to a functor which establishes an equivalence between the category of coaugmented (colored) cooperads and the category of (colored) pseudo-cooperads.

2.3. The convolution Lie algebra

Let $\mathcal{C}$ (resp. $\mathcal{O}$) be a Ξ-colored pseudo-cooperad (resp. Ξ-colored pseudo-operad) in $\mathsf{Ch}_{\mathbb{K}}$.

We consider the following cochain complex

$$\mathrm{Conv}(\mathcal{C}, \mathcal{O}) := \prod_{\mathbf{q}} \mathrm{Hom}_{S_{\mathbf{q}}}(\mathcal{C}(\mathbf{q}), \mathcal{O}(\mathbf{q})), \tag{2.40}$$

where the product is taken over all Ξ-colored planar corollas and the differential comes solely from the ones on $\mathcal{C}$ and $\mathcal{O}$.

Let us denote by $\mathsf{Isom}_2^{\Xi}(\mathbf{q})$ the set of isomorphism classes in [4] $\mathsf{Tree}_2^{\Xi}(\mathbf{q})$. Let us choose for every class $z \in \mathsf{Isom}_2^{\Xi}(\mathbf{q})$ its representative $\mathbf{t}_z$.

Using the trees $\mathbf{t}_z$ we equip the complex (2.40) with the following binary operation

$$f \bullet g(X) = \sum_{z \in \mathsf{Isom}_2^{\Xi}(\mathbf{q})} \mu_{\mathbf{t}_z}\big(f \otimes g(\Delta_{\mathbf{t}_z}(X))\big), \tag{2.41}$$

where $X \in \mathcal{C}(\mathbf{q})$. The axioms of pseudo-(co)operad imply that $\bullet$ is a well-defined operation. Namely, the right hand side of (2.41) does not depend on the choice of representatives $\mathbf{t}_z$ and $f \bullet g$ is $S_{\mathbf{q}}$-equivariant.

We claim that

Proposition 2.11. – *The operation* $\bullet$ (2.41) *equips* $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ *with a pre-Lie algebra structure. In other words,*

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = (-1)^{|g||h|}(f \bullet h) \bullet g - (-1)^{|g||h|} f \bullet (h \bullet g), \tag{2.42}$$

for all homogeneous vectors $f, g, h \in \mathrm{Conv}(\mathcal{C}, \mathcal{O})$.

Proof. – This statement was proved in the more general setting (for PROPs) in [**34**, Section 2.2] by B. Vallette and S. Merkulov. For non-colored (co)operads, a detailed proof can be found in [**11**, Section 4]. □

Proposition 2.11 implies that the operation

$$[f, g] = f \bullet g - (-1)^{|f||g|} g \bullet f \tag{2.43}$$

satisfies the Jacobi identity. Thus, $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ is a Lie algebra in the category $\mathsf{Ch}_{\mathbb{K}}$. Following [**34**], we call $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ the *convolution Lie algebra* of a pair $(\mathcal{C}, \mathcal{O})$.

Using "arity" we can equip the convolution Lie algebra $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ with the natural descending filtration

$$\mathrm{Conv}(\mathcal{C}, \mathcal{O}) = \mathcal{F}_{-1}\mathrm{Conv}(\mathcal{C}, \mathcal{O}) \supset \mathcal{F}_{0}\mathrm{Conv}(\mathcal{C}, \mathcal{O}) \supset \mathcal{F}_{1}\mathrm{Conv}(\mathcal{C}, \mathcal{O}) \supset \cdots,$$

where

$$\mathcal{F}_m \mathrm{Conv}(\mathcal{C}, \mathcal{O}) = \big\{ f \in \mathrm{Conv}(\mathcal{C}, \mathcal{O}) \mid f\big|_{\mathcal{C}(\mathbf{q})} = 0 \forall \text{corollas} \mathbf{q} \text{satisfying } |\mathbf{q}| \le m \big\} \tag{2.44}$$

and $|\mathbf{q}|$ is the total number of incoming edges of the corolla $\mathbf{q}$.

It is easy to see that this filtration is compatible with the Lie bracket and $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ is complete with respect to this filtration. Namely,

$$\mathrm{Conv}(\mathcal{C}, \mathcal{O}) = \lim_m \mathrm{Conv}(\mathcal{C}, \mathcal{O}) \Big/ \mathcal{F}_m \mathrm{Conv}(\mathcal{C}, \mathcal{O}). \tag{2.45}$$

4. Recall that $\mathsf{Tree}_2^{\Xi}(\mathbf{q})$ is the full subcategory of $\mathsf{Tree}^{\Xi}(\mathbf{q})$ whose objects are labeled Ξ-colored planar trees $\mathbf{t}$ with exactly two nodal vertices.

In fact, we may introduce an additional descending filtration $\mathscr{F}^{\chi}_{\bullet}$ on the convolution Lie algebra $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ for each color $\chi \in \Xi$:

$$\mathrm{Conv}(\mathcal{C}, \mathcal{O}) = \mathscr{F}^{\chi}_{-1}\,\mathrm{Conv}(\mathcal{C}, \mathcal{O}) \supset \mathscr{F}^{\chi}_{0}\,\mathrm{Conv}(\mathcal{C}, \mathcal{O}) \supset \mathscr{F}^{\chi}_{1}\,\mathrm{Conv}(\mathcal{C}, \mathcal{O}) \supset \cdots ,$$

where

$$\mathscr{F}^{\chi}_{m}\,\mathrm{Conv}(\mathcal{C}, \mathcal{O}) \tag{2.46}$$

consists of vectors $f \in \mathrm{Conv}(\mathcal{C}, \mathcal{O})$ satisfying these two conditions:

$$f\big|_{\mathcal{C}(\mathbf{q})} = 0 \text{ if the color of the root edge of } \mathbf{q} \text{ is } \chi \textbf{ and } |c^{-1}_{\mathbf{q},l}(\chi)| \le m, \tag{2.47}$$

and

$$f\big|_{\mathcal{C}(\mathbf{q})} = 0 \text{ if the color of the root edge of } \mathbf{q} \text{ is not } \chi \textbf{ and } |c^{-1}_{\mathbf{q},l}(\chi)| \le m-1. \tag{2.48}$$

It is not hard to see that this filtration is compatible with the pre-Lie multiplication $\bullet$ (2.41) on $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ and $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ is complete with respect to this filtration.

2.4. Free Ξ-colored operad

Let Q be a Ξ-colored collection. Following [**2**], the spaces $\Psi\mathbb{OP}(Q)(\mathbf{q})$ of the free Ξ-colored pseudo-operad generated by the collection Q are

$$\Psi\mathbb{OP}(Q)(\mathbf{q}) = \mathrm{colim}\; Q\big|_{\mathsf{Tree}^{\Xi}(\mathbf{q})}, \tag{2.49}$$

where $\mathsf{Tree}^{\Xi}(\mathbf{q})$ is the full subcategory of Tree^{Ξ} whose objects are labeled Ξ-colored planar trees $\mathbf{t}$ satisfying condition (2.15).

The pseudo-operad structure on $\Psi\mathbb{OP}(Q)$ is defined in the obvious way using grafting of trees.

The free Ξ-colored operad $\mathbb{OP}(Q)$ is obtained from $\Psi\mathbb{OP}(Q)$ via adjoining the units.

Unfolding (2.49) we see that $\Psi\mathbb{OP}(Q)(\mathbf{q})$ is the quotient of the direct sum

$$\bigoplus_{\mathbf{t},\kappa(\mathbf{t})=\mathbf{q}} Q(\mathbf{t}) \tag{2.50}$$

by the subspace spanned by vectors of the form

$$(\mathbf{t}, X) - (\mathbf{t}', Q(\lambda)(X)),$$

where $\lambda : \mathbf{t} \to \mathbf{t}'$ is a morphism in $\mathsf{Tree}^{\Xi}(\mathbf{q})$ and $X \in Q(\mathbf{t})$.

Thus it is convenient to represent vectors in $\Psi\mathbb{OP}(Q)$ and in $\mathbb{OP}(Q)$ by labeled Ξ-colored planar trees with nodal vertices decorated by vectors in Q. The decoration is subject to this rule: if $\kappa(x)$ is the corolla formed by all edges adjacent to a nodal vertex x then x is decorated by a vector $v_x \in Q(\kappa(x))$.

If a decorated tree $\mathbf{t}'$ is obtained from a decorated tree $\mathbf{t}$ by applying an element $\sigma \in S_{\kappa(x)}$ to incoming edges of a vertex x and replacing the vector v_x by $\sigma^{-1}(v_x)$ then $\mathbf{t}'$ and $\mathbf{t}$ represent the same vectors in (2.49).

EXAMPLE 2.12. – *Let Q be a 2-colored collection. Figure 2.24 shows a labeled 2-colored tree $\mathbf{t}$ decorated by vectors $v_1 \in Q(1,2)^{\mathfrak{o}}$, $v_2 \in Q(2,0)^{\mathfrak{c}}$ and $v_3 \in Q(1,0)^{\mathfrak{o}}$. Figure 2.25 shows another decorated tree with $v_1' = (\mathrm{id}, \sigma_{12})(v_1)$ and $v_2' = \sigma_{12}(v_2)$, where σ_{12} is the transposition in S_2. According to our discussion, these trees represent the same vector in $\mathbb{OP}(Q)(3,1)^{\mathfrak{o}}$.*

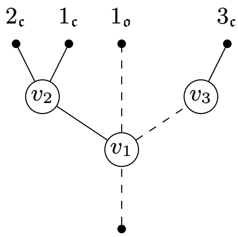


FIGURE 2.24. A 2-colored decorated tree $\mathbf{t}$

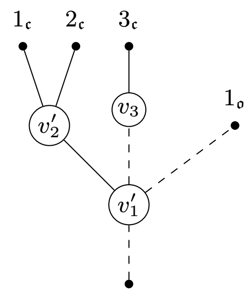


FIGURE 2.25. A 2-colored decorated tree $\widetilde{\mathbf{t}}$. Here $v_1' = (\mathrm{id}, \sigma_{12})(v_1)$ and $v_2' = \sigma_{12}(v_2)$

2.5. The cobar construction in the colored setting

The cobar construction [**16, 19, 20**], [**32**, Section 6.5] is a functor from the category of coaugmented cooperads (in $\mathsf{Ch}_{\mathbb{K}}$) to the category of augmented operads (in $\mathsf{Ch}_{\mathbb{K}}$). It is used to construct free resolutions for operads. In this section, we briefly describe the cobar construction in the colored setting.

Let $\mathcal{C}$ be a coaugmented Ξ-colored cooperad in the category $\mathsf{Ch}_{\mathbb{K}}$ and $\mathcal{C}_\circ$ be the cokernel of coaugmentation. As an operad in the category $\mathsf{grVect}_{\mathbb{K}}$, $\mathrm{Cobar}(\mathcal{C})$ is freely generated by the collection $\mathbf{s}\,\mathcal{C}_\circ$

$$\mathrm{Cobar}(\mathcal{C}) = \mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ). \tag{2.51}$$

Thus, it suffices to define the differential $\partial^{\mathrm{Cobar}}$ on generators $X \in \mathbf{s}\,\mathcal{C}_\circ$.

The differential $\partial^{\mathrm{Cobar}}$ on $\mathrm{Cobar}(\mathcal{C})$ can be written as the sum

$$\partial^{\mathrm{Cobar}} = \partial' + \partial'',$$

with

$$\partial'(X) = -\mathbf{s}\,\partial_{\mathcal{C}}\,\mathbf{s}^{-1} X \tag{2.52}$$

and

$$\partial''(X) = -\bigoplus_{z\in \mathsf{Isom}_2^{\Xi}(\mathbf{q})} (\mathbf{s}\otimes\mathbf{s})\big(\mathbf{t}_z; \Delta_{\mathbf{t}_z}(\mathbf{s}^{-1}X)\big), \tag{2.53}$$

where $X \in \mathbf{s}\,\mathcal{C}_\circ(\mathbf{q})$, $\mathsf{Isom}_2^{\Xi}(\mathbf{q})$ is the set of isomorphism classes in $\mathsf{Tree}_2^{\Xi}(\mathbf{q})$, the tree $\mathbf{t}_z$ is any representative of the class z, and $\partial_{\mathcal{C}}$ is the differential on $\mathcal{C}$.

Properties of comultiplications $\Delta_{\mathbf{t}}$ imply that the right hand side of (2.53) does not depend on the choice of representatives $\mathbf{t}_z$. Furthermore, using the identity $\big(\partial_{\mathcal{C}}\big)^2 = 0$ and the compatibility of $\partial_{\mathcal{C}}$ with comultiplications $\Delta_{\mathbf{t}}$ one easily deduces that

$$\partial'\circ\partial' = 0,$$

and

$$\partial'\circ\partial'' + \partial''\circ\partial' = 0.$$

Finally the coassociativity law (2.39) implies that

$$\partial''\circ\partial'' = 0. \tag{2.54}$$

Let $\mathcal{O}$ be a Ξ-colored operad in $\mathsf{Ch}_{\mathbb{K}}$. We claim that

PROPOSITION 2.13. – *For every coaugmented Ξ-colored cooperad $\mathcal{C}$ (in $\mathsf{Ch}_{\mathbb{K}}$), operad morphisms from* Cobar($\mathcal{C}$) *to $\mathcal{O}$ are in bijection with MC elements of the Lie algebra*

$$\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}), \tag{2.55}$$

where $\mathcal{O}$ is viewed as a Ξ-colored pseudo-operad via the forgetful functor.

Proof. – Since Cobar($\mathcal{C}$) is freely generated by the Ξ-colored collection $\mathbf{s}\,\mathcal{C}_\circ$ any operad morphism

$$F : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$$

is uniquely determined by its restriction to $\mathbf{s}\,\mathcal{C}_\circ$.

Let us denote by α_F the degree 1 element

$$\alpha_F : \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}) \tag{2.56}$$

corresponding to the restriction

$$F\big|_{\mathbf{s}\,\mathcal{C}_\circ} : \mathbf{s}\,\mathcal{C}_\circ \to \mathcal{O}.$$

A direct computation shows that the compatibility of F with the differentials is equivalent to the MC equation on α_F in the Lie algebra (2.55). □

REMARK 2.14. – *It is possible to express the Lie bracket on* $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$ *in terms of the portion* ∂'' (2.53) *of the cobar differential* $\partial^{\mathrm{Cobar}}$. *More precisely, for* $f, g \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$ *and* $X \in \mathcal{C}_\circ$ *we have*

$$[f,g](X) = (-1)^{|g|}\mu\big(f\mathbf{s}^{-1} \otimes g\mathbf{s}^{-1}(\partial''(\mathbf{s}X))\big) - (-1)^{|f||g|}(f \leftrightarrow g), \tag{2.57}$$

where $f\mathbf{s}^{-1}$ *and* $g\mathbf{s}^{-1}$ *act in the obvious way on the tensor factors of* $\partial''(\mathbf{s}X) \in \mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ)$ *and* μ *denotes the multiplication map*

$$\mu : \mathbb{OP}(\mathcal{O}) \to \mathcal{O}.$$

CHAPTER 3

OPERAD dGra AND ITS 2-COLORED EXTENSION KGra

Let us remind from [**39**] the operad (in $\mathsf{grVect}_{\mathbb{K}}$) of directed labeled graphs dGra.

To define the space $\mathsf{dGra}(n)$ we introduce an auxiliary set dgra_n. An element of dgra_n is a directed graph Γ with the set of vertices $\{1, 2, \dots, n\}$ and with a total order on the set of edges. We require that each directed graph Γ in dgra_n has no multiple edges with the same direction [(1)]. An example of an element in dgra_5 is shown in Figure 3.1. We will often use roman numerals to specify a total order on a set of edges.

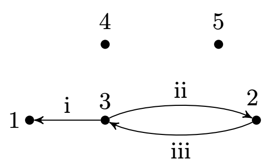


FIGURE 3.1. Roman numerals indicate that $(3,1) < (3,2) < (2,3)$

For example, the roman numerals in Figure 3.1 indicate that $(3,1) < (3,2) < (2,3)$.

The space $\mathsf{dGra}(n)$ is spanned by elements of dgra_n, modulo the relation $\Gamma^\sigma = (-1)^{|\sigma|}\Gamma$, where the graphs Γ^σ and Γ correspond to the same directed labeled graph but differ only by permutation σ of edges. We also declare that the degree of a graph Γ in $\mathsf{dGra}(n)$ equals $-e(\Gamma)$, where $e(\Gamma)$ is the number of edges in Γ. For example, the graph Γ in Figure 3.1 has 3 edges. Thus its degree is -3.

1. This allows us to identify elements of dgra_n with ordered subsets of ordered pairs of numbers $1, 2, \dots, n$. Let us also recall that we do not consider graphs with loops (i.e., cycles of length 1).

3.1. Operad structure on dGra

Let Γ and $\widetilde{\Gamma}$ be graphs representing vectors in $\mathsf{dGra}(n)$ and $\mathsf{dGra}(m)$, respectively.

For $1 \le i \le m$, the vector $\widetilde{\Gamma} \circ_i \Gamma \in \mathsf{dGra}(n+m-1)$ is represented by the sum of graphs $\Gamma_\alpha \in \mathrm{dgra}_{n+m-1}$

$$\widetilde{\Gamma} \circ_i \Gamma = \sum_\alpha \Gamma_\alpha, \tag{3.1}$$

where Γ_α is obtained by "plugging in" the graph Γ into the i-th vertex of the graph $\widetilde{\Gamma}$ and reconnecting the edges incident to the i-th vertex of $\widetilde{\Gamma}$ to vertices of Γ in all possible ways. (The index α refers to a particular way of connecting the edges incident to the i-th vertex of $\widetilde{\Gamma}$ to vertices of Γ.) After reconnecting edges we

— shift all labels on vertices of Γ up by $i-1$, and

— shift the labels on the last $m-i$ vertices of $\widetilde{\Gamma}$ up by $n-1$.

To define the total order on edges of the graph Γ_α we declare that all edges of $\widetilde{\Gamma}$ are smaller than all edges of the graph Γ.

Note that every graph in $\{\Gamma_\alpha\}_\alpha$ is a legitimate element of dgra_{n+m-1} because Γ and $\widetilde{\Gamma}$ have no multiple edges with the same direction and have no loops.

Example 3.1. – *Let $\widetilde{\Gamma}$ (resp. Γ) be the graph depicted in Figure 3.2 (resp. Figure 3.3). The vector $\widetilde{\Gamma} \circ_2 \Gamma$ is shown in Figure 3.4. For the first graph in the sum $\widetilde{\Gamma} \circ_2 \Gamma$ we*

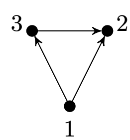


Figure 3.2. A graph $\widetilde{\Gamma} \in \mathrm{dgra}_3$. The order on edges is $(1,2) < (1,3) < (3,2)$

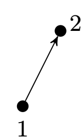


Figure 3.3. A graph $\Gamma \in \mathrm{dgra}_2$

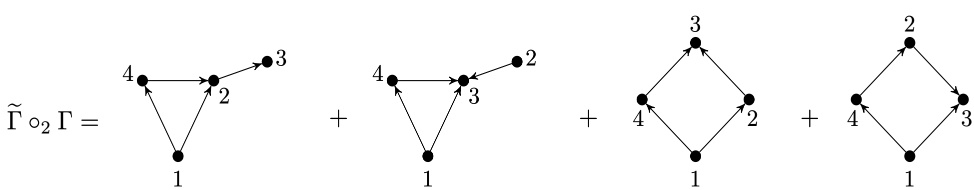


Figure 3.4. The vector $\widetilde{\Gamma} \circ_2 \Gamma \in \mathsf{dGra}(4)$

have $(1,2) < (1,4) < (4,2) < (2,3)$. For the second graph in the sum $\widetilde{\Gamma} \circ_2 \Gamma$ we *have* $(1,3) < (1,4) < (4,3) < (2,3)$. *For the third graph in the sum* $\widetilde{\Gamma} \circ_2 \Gamma$ *we have* $(1,2) < (1,4) < (4,3) < (2,3)$. *Finally, for the last graph in the sum* $\widetilde{\Gamma} \circ_2 \Gamma$ *we have* $(1,3) < (1,4) < (4,2) < (2,3)$.

The symmetric group S_n acts on $\mathsf{dGra}(n)$ in the obvious way by rearranging the labels on vertices. It is not hard to see that insertions (3.1) together with this action of S_n give on dGra an operad structure with the identity element being the unique graph in dgra_1 with no edges.

3.2. 2-colored operad KGra

To define a stable formality quasi-isomorphism (SFQ) we need to upgrade the operad dGra to a 2-colored operad KGra (in $\mathsf{grVect}_{\mathbb{K}}$). The additional spaces of the operad KGra are assembled from the graphs which were used by M. Kontsevich in his groundbreaking paper [**31**]. As far as I understand, T. Willwacher is using this operad in [**40**] under the different name: SGra.

Recall that, following our conventions, $\mathsf{KGra}(n,k)^{\mathfrak{c}}$ denotes the space of operations with n inputs of color $\mathfrak{c}$, k inputs of color $\mathfrak{o}$, and with the color of the output being $\mathfrak{c}$. Similarly, $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ is the space of operations with n inputs of color $\mathfrak{c}$, k inputs of color $\mathfrak{o}$, and with the color of the output being $\mathfrak{o}$.

First, we declare that $\mathsf{KGra}(n,k)^{\mathfrak{c}} = \mathbf{0}$ whenever $k \geq 1$.

Next, for the space $\mathsf{KGra}(n,0)^{\mathfrak{c}}$ $(n \geq 0)$, we have

$$\mathsf{KGra}(n,0)^{\mathfrak{c}} = \mathsf{dGra}(n). \tag{3.2}$$

To define the space $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ we introduce the auxiliary set $\mathrm{dgra}_{n,k}$. An element of the set $\mathrm{dgra}_{n,k}$ is a directed graph Γ with the set of vertices $\{1_{\mathfrak{c}}, \dots, n_{\mathfrak{c}}, 1_{\mathfrak{o}}, \dots, k_{\mathfrak{o}}\}$ and with a total order on the set of its edges. In addition, we require that

— each $\Gamma \in \mathrm{dgra}_{n,k}$ has no multiple edges with the same direction, and

— each $\Gamma \in \mathrm{dgra}_{n,k}$ has no edges originating from any vertex with color $\mathfrak{o}$.

EXAMPLE 3.2. – *Figure 3.5 shows an example of a graph in* $\mathrm{dgra}_{2,3}$. *Black (resp. white) vertices carry the color* $\mathfrak{c}$ *(resp.* $\mathfrak{o}$*). We use separate labels for vertices of color* $\mathfrak{c}$ *and vertices of color* $\mathfrak{o}$. *For example,* $2_{\mathfrak{c}}$ *denotes the second vertex of color* $\mathfrak{c}$ *and* $3_{\mathfrak{o}}$ *denotes the third vertex of color* $\mathfrak{o}$.

The space $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ is spanned by elements of $\mathrm{dgra}_{n,k}$, modulo the relation $\Gamma^{\sigma} = (-1)^{|\sigma|}\Gamma$, where Γ^{σ} and Γ correspond to the same directed labelled graph but differ only by permutation σ of edges. As above, we declare that the degree of a graph Γ in $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ equals $-e(\Gamma)$.

The elementary insertions

$$\mathsf{KGra}(m,0)^{\mathfrak{c}} \otimes \mathsf{KGra}(n,0)^{\mathfrak{c}} \to \mathsf{KGra}(m+n-1,0)^{\mathfrak{c}}$$

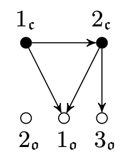


FIGURE 3.5. We equip the edges with the order $(1_\mathfrak{c}, 2_\mathfrak{c}) < (1_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 3_\mathfrak{o})$

are defined in the same way as for dGra. So we proceed to the remaining insertions.

3.3. Elementary insertions $\mathsf{KGra}(m,k)^\mathfrak{o} \otimes \mathsf{KGra}(n,0)^\mathfrak{c} \to \mathsf{KGra}(m+n-1,k)^\mathfrak{o}$

Let Γ and $\widetilde{\Gamma}$ be graphs representing vectors in $\mathsf{KGra}(n,0)^\mathfrak{c}$ and $\mathsf{KGra}(m,k)^\mathfrak{o}$, respectively. Let $1 \le i \le m$.

The vector $\widetilde{\Gamma} \circ_{i,\mathfrak{c}} \Gamma \in \mathsf{KGra}(n+m-1,k)^\mathfrak{o}$ is the sum of graphs $\Gamma_\alpha \in \mathrm{dgra}_{n+m-1,k}$

$$\widetilde{\Gamma} \circ_{i,\mathfrak{c}} \Gamma = \sum_\alpha \Gamma_\alpha, \tag{3.3}$$

where Γ_α is obtained by "plugging in" the graph Γ into the i-th black vertex of the graph $\widetilde{\Gamma}$ and reconnecting the edges incident to this vertex to vertices of Γ in all possible ways. (The index α refers to a particular way of connecting the edges incident to the i-th black vertex of $\widetilde{\Gamma}$ to vertices of Γ.) After reconnecting edges we

— shift all labels on vertices of Γ up by $i-1$, and
— shift labels on the last $m-i$ black vertices of $\widetilde{\Gamma}$ up by $n-1$.

To define the total order on edges of the graph Γ_α we declare that all edges of $\widetilde{\Gamma}$ are smaller than all edges of the graph Γ.

EXAMPLE 3.3. – *The graphs depicted in Figures 3.6 and 3.7 represent vectors* $\widetilde{\Gamma} \in \mathsf{KGra}(2,1)^\mathfrak{o}$ *and* $\Gamma \in \mathsf{KGra}(2,0)^\mathfrak{c}$, *respectively. For the edges of* $\widetilde{\Gamma}$ *we set*

$$(1_\mathfrak{c}, 1_\mathfrak{o}) < (1_\mathfrak{c}, 2_\mathfrak{c}) < (2_\mathfrak{c}, 1_\mathfrak{o}).$$

As above, white vertices carry the color $\mathfrak{o}$ *and black vertices carry the color* $\mathfrak{c}$.

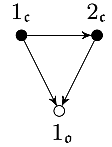


FIGURE 3.6. The graph $\widetilde{\Gamma}$

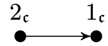


FIGURE 3.7. The graph Γ

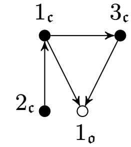


FIGURE 3.8. The graph Γ_1

The vector $\widetilde{\Gamma} \circ_{1,\mathfrak{c}} \Gamma \in \mathsf{KGra}(3,1)^{\mathfrak{o}}$ *is represented by the sum of graphs* Γ_1, Γ_2, Γ_3, Γ_4 *depicted on Figures 3.8, 3.9, 3.10, 3.11, respectively. Following our rule, the edges*

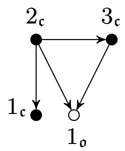


FIGURE 3.9. The graph Γ_2

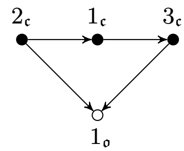


FIGURE 3.10. The graph Γ_3

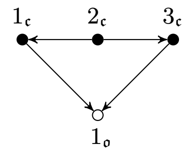


FIGURE 3.11. The graph Γ_4

of Γ_1 *are ordered as follows* $(1_\mathfrak{c}, 1_\mathfrak{o}) < (1_\mathfrak{c}, 3_\mathfrak{c}) < (3_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 1_\mathfrak{c})$. *Similarly, edges of* Γ_2 *carry the order* $(2_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 3_\mathfrak{c}) < (3_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 1_\mathfrak{c})$. *The edges of* Γ_3 *are equipped with the order* $(2_\mathfrak{c}, 1_\mathfrak{o}) < (1_\mathfrak{c}, 3_\mathfrak{c}) < (3_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 1_\mathfrak{c})$. *Finally, for* Γ_4 *we have* $(1_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 3_\mathfrak{c}) < (3_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 1_\mathfrak{c})$.

3.4. Elementary insertions $\mathsf{KGra}(m,p)^{\mathfrak{o}} \otimes \mathsf{KGra}(n,q)^{\mathfrak{o}} \to \mathsf{KGra}(m+n,p+q-1)^{\mathfrak{o}}$

Let Γ and $\widetilde{\Gamma}$ be graphs representing vectors in $\mathsf{KGra}(n,q)^{\mathfrak{o}}$ and $\mathsf{KGra}(m,p)^{\mathfrak{o}}$, respectively. Let $1 \le i \le p$.

The vector $\widetilde{\Gamma} \circ_{i,\mathfrak{o}} \Gamma \in \mathsf{KGra}(m+n,p+q-1)^{\mathfrak{o}}$ is represented by the sum of graphs $\Gamma_\alpha \in \mathrm{dgra}_{m+n,p+q-1}$

$$\widetilde{\Gamma} \circ_{i,\mathfrak{o}} \Gamma = \sum_\alpha \Gamma_\alpha, \tag{3.4}$$

where Γ_α is obtained by "plugging in" the graph Γ into the i-th white vertex of the graph $\widetilde{\Gamma}$ and reconnecting the edges incident to this vertex to vertices of Γ in all possible ways. (The index α refers to a particular way of connecting the edges incident to the i-th white vertex of $\widetilde{\Gamma}$ to vertices of Γ.) After reconnecting edges we

— shift all labels on black vertices of Γ up by m,

— shift all labels on white vertices of Γ up by $i-1$, and finally

— shift all labels on the last $p-i$ white vertices of $\widetilde{\Gamma}$ up by $q-1$.

To define the total order on edges of the graph Γ_α we declare that all edges of $\widetilde{\Gamma}$ are smaller than all edges of the graph Γ.

EXAMPLE 3.4. – *If* $\widetilde{\Gamma}$ *is the graph depicted in Figure 3.5 and* Γ *is the graph depicted in Figure 3.12 then the vector* $\widetilde{\Gamma} \circ_{3,\mathfrak{o}} \Gamma \in \mathsf{KGra}(3,3)$ *is the sum of graphs depicted in Figure 3.13. For the edges of the first graph in this sum we have* $(1_\mathfrak{c}, 2_\mathfrak{c}) < (1_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 3_\mathfrak{o}) < (3_\mathfrak{c}, 3_\mathfrak{o})$. *For the edges of the second graph in this sum we have* $(1_\mathfrak{c}, 2_\mathfrak{c}) < (1_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 1_\mathfrak{o}) < (2_\mathfrak{c}, 3_\mathfrak{c}) < (3_\mathfrak{c}, 3_\mathfrak{o})$.

FIGURE 3.12. A graph $\Gamma \in \mathrm{dgra}_{1,1}$

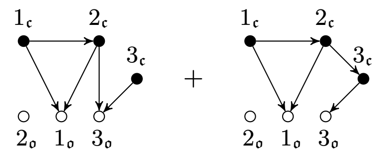
$\widetilde{\Gamma} \circ_{3,\mathfrak{o}} \Gamma =$ 

FIGURE 3.13. The vector $\widetilde{\Gamma} \circ_{3,\mathfrak{o}} \Gamma$

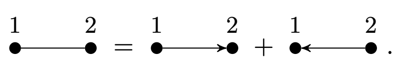
The identity element $\mathbf{u}_{\mathfrak{c}} \in \mathsf{KGra}(1,0)^{\mathfrak{c}}$ (resp. $\mathbf{u}_{\mathfrak{o}} \in \mathsf{KGra}(0,1)^{\mathfrak{o}}$) is represented by the graph in dgra_1 (resp. the graph in $\mathrm{dgra}_{0,1}$) with no edges.

It is straightforward to verify that $\mathbf{u}_{\mathfrak{c}}$, $\mathbf{u}_{\mathfrak{o}}$, and Equations (3.1), (3.3), (3.4) together with the natural action of $S_n \times S_k$ on $\mathsf{KGra}(n,k)^{\mathfrak{o}}$ (resp. S_n on $\mathsf{KGra}(n,0)^{\mathfrak{c}}$) define a structure of a 2-colored operad on KGra in $\mathsf{grVect}_{\mathbb{K}}$.

REMARK 3.5. – *When dealing with elements of* $\mathsf{KGra}(n,0)^{\mathfrak{c}} = \mathsf{dGra}(n)$ *or with elements of* $\mathsf{KGra}(n,0)^{\mathfrak{o}}$*, we will often omit the subscript* $\mathfrak{c}$ *in labels* $1_{\mathfrak{c}}, 2_{\mathfrak{c}}, 3_{\mathfrak{c}}, \cdots$.

REMARK 3.6. – *Let* Γ *be a graph in* dgra_n *(resp.* $\mathrm{dgra}_{n,k}$*) and* e *be an edge of* Γ *which connects two black vertices. We denote by* $f_e(\Gamma)$ *the graph which is obtained from* Γ *by changing the direction of the edge* e*.*

It is convenient to draw the linear combination $\Gamma + f_e(\Gamma)$ *as a graph which is obtained from* Γ *by forgetting the direction of* e*. For example,*

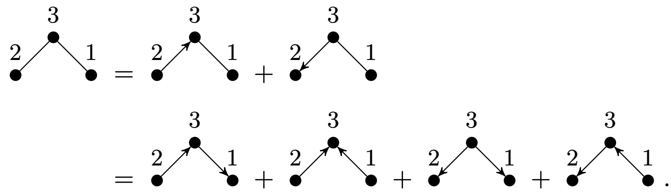
(3.5) 

Similarly, if $e_1, e_2, \ldots, e_p$ *are edges of* Γ *which connect only black vertices and the graph* Γ' *is obtained from* Γ *by forgetting the directions of the edges* $e_1, e_2, \ldots, e_p$*, then* Γ' *denotes the sum*

$$\Gamma' = \sum_{k_i \in \{0,1\}} (f_{e_1})^{k_1} (f_{e_2})^{k_2} \cdots (f_{e_p})^{k_p} (\Gamma).$$

For example,

(3.6)



3.5. The action of the operad KGra on polyvectors and functions

Let A be a free finitely generated commutative algebra (with unit) in $\mathsf{grVect}_{\mathbb{K}}$. We denote by

$$x^1, x^2, \dots, x^d \tag{3.7}$$

generators of A and by $|x^1|, |x^2|, \dots, |x^d|$ their corresponding degrees. We think of A as the algebra of functions on a graded affine space.

Let us denote by V_A the free commutative algebra in $\mathsf{grVect}_{\mathbb{K}}$ generated by

$$x^1, x^2, \dots, x^d, \theta_1, \theta_2, \dots, \theta_d, \tag{3.8}$$

where θ_c carries the degree $1 - |x^c|$. We think of V_A as the algebra of polyvector fields on the corresponding graded affine space.

If all generators x^c have degree 0 then A (resp. V_A) is the algebra of functions (resp. the algebra of polyvector fields) on the affine space $\mathbb{K}^d$. However, for our constructions there is no need to impose any restrictions on degrees of generators (3.7).

We claim that

Proposition 3.7. – *The pair (V_A, A) is naturally an algebra over the 2-colored operad* KGra.

Proof. – For $\Gamma \in \mathrm{dgra}_n$ and $v_1, v_2, \dots, v_n \in V_A$ we set

$$\Gamma(v_1, v_2, \dots, v_n) := \mathrm{mult}_n\Big(\Big[\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}\Big](v_1 \otimes v_2 \otimes \cdots \otimes v_n)\Big), \tag{3.9}$$

where mult_n is the multiplication map

$$\mathrm{mult}_n : \big(V_A\big)^{\otimes n} \to V_A,$$

$$\underline{\Delta}_{(i,j)} = \sum_{c=1}^{d} 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{\theta_c}}_{i\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{x^c}}_{j\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \tag{3.10}$$

if $i < j$,

$$\underline{\Delta}_{(i,j)} = \sum_{c=1}^{d} 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{x^c}}_{j\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{\partial_{\theta_c}}_{i\text{-th slot}} \otimes 1 \otimes \cdots \otimes 1 \tag{3.11}$$

if $j < i$, and the order of factors in the product

$$\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}$$

comes from the order on the set $E(\Gamma)$ of edges of Γ.

To define the action of a graph $\Gamma \in \mathrm{dgra}_{n,k}$ we identify vertices of Γ with the numbers $1, 2, \ldots, n+k$ by using the labels and declaring that all black vertices precede all white vertices. Namely, the black vertex with label i is identified with number i and the white vertex with label j is identified with number $n+j$. Then for $v_1, v_2, \ldots, v_n \in V_A$, and $a_1, a_2, \ldots, a_k \in A$ we set

$$\Gamma(v_1, v_2, \ldots, v_n; a_1, a_2, \ldots, a_k) :=$$

$$\mathrm{mult}_{n,k}\Big(\Big[\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}\Big]\,(v_1 \otimes v_2 \otimes \cdots \otimes v_n \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_k)\Big)\Big|_{\theta_c=0}, \tag{3.12}$$

where $\mathrm{mult}_{n,k}$ is the multiplication map

$$\mathrm{mult}_{n,k} : \big(V_A\big)^{\otimes n} \;\otimes\; A^{\otimes k} \to V_A,$$

$\underline{\Delta}_{(i,j)}$ is defined by Equations (3.10), (3.11), and the order of factors in the product

$$\prod_{(i,j)\in E(\Gamma)} \underline{\Delta}_{(i,j)}$$

comes from the order on the set $E(\Gamma)$ of edges of Γ.

It is not hard to verify that Equations (3.9), (3.12) define an action of KGra on the pair (V_A, A). □

CHAPTER 4

THE 2-COLORED OPERAD OC OF H. KAJIURA AND J. STASHEFF

Inspired by Zwiebach's open-closed string field theory [**42**], H. Kajiura and J. Stasheff introduced in [**26**] open-closed homotopy algebras (OCHA).

An OCHA is a pair of cochain complexes $(\mathscr{V}, \mathscr{A})$ with the following data:

— A $\Lambda\mathsf{Lie}_\infty$-structure on $\mathscr{V}$,

— an A_∞-structure on $\mathscr{A}$, and

— a $\Lambda\mathsf{Lie}_\infty$-morphism from $\mathscr{V}$ to the Hochschild cochain complex $C^\bullet(\mathscr{A})$ of $\mathscr{A}$.

It was shown in [**27**], that OCHAs are governed by a 2-colored operad (in $\mathsf{Ch}_\mathbb{K}$) which we denote by OC. Moreover, as an operad in grVect, OC is freely generated by the 2-colored collection $\mathfrak{oc}$ with the following spaces:

$$\mathfrak{oc}(n,0)^\mathfrak{c} = \mathbf{s}^{3-2n}\mathbb{K}, \qquad n \geq 2, \tag{4.1}$$

$$\mathfrak{oc}(0,k)^\mathfrak{o} = \mathbf{s}^{2-k}\,\mathrm{sgn}_k \otimes \mathbb{K}[S_k], \qquad k \geq 2, \tag{4.2}$$

$$\mathfrak{oc}(n,k)^\mathfrak{o} = \mathbf{s}^{2-2n-k}\,\mathrm{sgn}_k \otimes \mathbb{K}[S_k], \qquad n \geq 1, k \geq 0, \tag{4.3}$$

where sgn_k is the sign representation of S_k. The remaining spaces of the collection $\mathfrak{oc}$ are zero.

Following the description of free colored operads via decorated (and colored) trees (see Section 2.4), we represent generators of OC in $\mathfrak{oc}(n,0)^\mathfrak{c}$ by non-planar labeled corollas with n solid incoming edges (see Figure 4.1). We represent generators of OC in $\mathfrak{oc}(0,k)^\mathfrak{o}$ by planar labeled corollas with k dashed incoming edges (see Figure 4.2). Finally, we use labeled 2-colored corollas with a planar structure given only on the dashed edges to represent generators of OC in $\mathfrak{oc}(n,k)$ (see Figure 4.3).

Using the corolla $\mathsf{t}^\mathfrak{o}_k$ (resp. the corolla $\mathsf{t}^\mathfrak{o}_{n,k}$) depicted in Figure 4.2 (resp. 4.3), we can form a basis of the vector space $\mathfrak{oc}(0,k)^\mathfrak{o}$ (resp. $\mathfrak{oc}(n,k)^\mathfrak{o}$). Namely, the set $\{\sigma(\mathsf{t}^\mathfrak{o}_k) \mid \sigma \in S_k\}$ is a basis of the vector space $\mathfrak{oc}(0,k)^\mathfrak{o}$ and the set $\{(\mathrm{id},\sigma)\big(\mathsf{t}^\mathfrak{o}_{n,k}\big) \mid \sigma \in S_k\}$ is a basis of the vector space $\mathfrak{oc}(n,k)^\mathfrak{o}$.

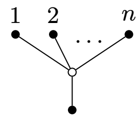


FIGURE 4.1. The non-planar corolla $\mathsf{t}^{\mathfrak{c}}_n$ representing a generator of $\mathfrak{oc}(n,0)^{\mathfrak{c}}$

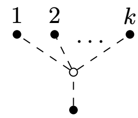


FIGURE 4.2. The 2-colored planar corola $\mathsf{t}^{\mathfrak{o}}_k$ representing a generator of $\mathfrak{oc}(0,k)^{\mathfrak{o}}$

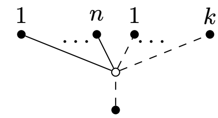


FIGURE 4.3. The 2-colored partially planar corola $\mathsf{t}^{\mathfrak{o}}_{n,k}$ representing a generator of $\mathfrak{oc}(n,k)^{\mathfrak{o}}$

Equations (4.1), (4.2), and (4.3) imply that the corollas $\mathsf{t}^{\mathfrak{c}}_n$, $\mathsf{t}^{\mathfrak{o}}_k$ and $\mathsf{t}^{\mathfrak{o}}_{n,k}$ carry the following degrees:

$$|\mathsf{t}^{\mathfrak{c}}_n| = 3 - 2nn \geq 2, \tag{4.4}$$

$$|\mathsf{t}^{\mathfrak{o}}_k| = 2 - kk \geq 2, \tag{4.5}$$

$$|\mathsf{t}^{\mathfrak{o}}_{n,k}| = 2 - 2n - kn \geq 1, \ k \geq 0. \tag{4.6}$$

We should remark that OC comes from a Koszul operad and this fact was established in beautiful paper [25] by E. Hoefel and M. Livernet.

4.1. The differential on OC

It is convenient to split the differential $\mathscr{D}$ on OC into four summands

$$\mathscr{D} = \mathscr{D}_{\mathsf{Lie}} + \mathscr{D}_{\mathsf{As}} + \mathscr{D}' + \mathscr{D}''. \tag{4.7}$$

Since OC is freely generated by the 2-colored collection $\mathfrak{oc}$, it suffices to define the values of summands $\mathscr{D}_{\mathsf{Lie}}$, $\mathscr{D}_{\mathsf{As}}$, $\mathscr{D}'$, and $\mathscr{D}''$ on corollas $\mathsf{t}^{\mathfrak{c}}_n$, $\mathsf{t}^{\mathfrak{o}}_k$, and $\mathsf{t}^{\mathfrak{o}}_{n,k}$ depicted in Figures 4.1, 4.2, and 4.3, respectively.

For the corolla $\mathsf{t}^{\mathfrak{c}}_n$ we have

$$\mathscr{D}_{\mathsf{As}}(\mathsf{t}^{\mathfrak{c}}_n) = 0, \qquad \mathscr{D}'(\mathsf{t}^{\mathfrak{c}}_n) = 0, \qquad \mathscr{D}''(\mathsf{t}^{\mathfrak{c}}_n) = 0 \tag{4.8}$$

and $\mathscr{D}_{\mathsf{Lie}}(\mathsf{t}^{\mathfrak{c}}_n)$ is the sum shown in Figure 4.4.

$$\mathscr{D}_{\mathsf{Lie}}(\mathsf{t}^{\mathfrak{c}}_n) = -\sum_{p=2}^{n-1} \sum_{\tau \in \mathrm{Sh}_{p,n-p}}$$

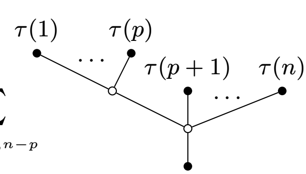


FIGURE 4.4. The value of $\mathscr{D}_{\mathsf{Lie}}$ on $\mathsf{t}^{\mathfrak{c}}_n$

For the corolla $\mathsf{t}^{\mathsf{o}}_k$ we have

$$\mathscr{D}_{\mathsf{Lie}}(\mathsf{t}^{\mathsf{o}}_k) = 0, \qquad \mathscr{D}'(\mathsf{t}^{\mathsf{o}}_k) = 0, \qquad \mathscr{D}''(\mathsf{t}^{\mathsf{o}}_k) = 0, \tag{4.9}$$

and $\mathscr{D}_{\mathsf{As}}(\mathsf{t}^{\mathsf{o}}_k)$ is the sum shown in Figure 4.5.

$$\mathscr{D}_{\mathsf{As}}(\mathsf{t}^{\mathsf{o}}_k) = -\sum_{p=0}^{k-2}\sum_{q=p+2}^{k}(-1)^{p+(k-q)(q-p)}$$

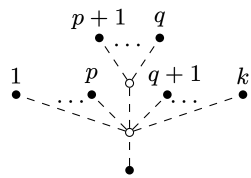


FIGURE 4.5. The value of $\mathscr{D}_{\mathsf{As}}$ on $\mathsf{t}^{\mathsf{o}}_k$

The value of $\mathscr{D}_{\mathsf{Lie}}$ on the corolla $\mathsf{t}^{\mathsf{o}}_{n,k}$ is given by the sum depicted in Figure 4.6 and the value of $\mathscr{D}_{\mathsf{As}}$ on the corolla $\mathsf{t}^{\mathsf{o}}_{n,k}$ is given by the sum depicted in Figure 4.7. The values $\mathscr{D}'(\mathsf{t}^{\mathsf{o}}_{n,k})$ and $\mathscr{D}''(\mathsf{t}^{\mathsf{o}}_{n,k})$ for $n \geq 2$ are defined in Figures 4.8 and 4.9, respectively. Finally, for the corollas $\mathsf{t}^{\mathsf{o}}_{1,k}$ we have

$$\mathscr{D}'(\mathsf{t}^{\mathsf{o}}_{1,k}) = \mathscr{D}''(\mathsf{t}^{\mathsf{o}}_{1,k}) = 0, \qquad \forall\ k \geq 0. \tag{4.10}$$

$$\mathscr{D}_{\mathsf{Lie}}(\mathsf{t}^{\mathsf{o}}_{n,k}) = (-1)^k \sum_{p=2}^{n-1}\sum_{\tau\in \mathrm{Sh}_{p,n-p}}$$

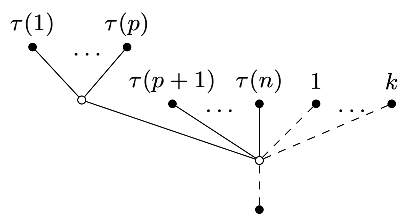


FIGURE 4.6. The value of $\mathscr{D}_{\mathsf{Lie}}$ on $\mathsf{t}^{\mathsf{o}}_{n,k}$

Direct computations show that

$$\big(\mathscr{D}_{\mathsf{As}}\big)^2 = 0, \tag{4.11}$$

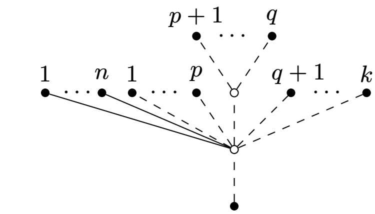
$$\big(\mathscr{D}_{\mathsf{Lie}} + \mathscr{D}'\big)^2 = 0, \tag{4.12}$$

$$\mathscr{D}_{\mathsf{As}} \circ (\mathscr{D}_{\mathsf{Lie}} + \mathscr{D}') + (\mathscr{D}_{\mathsf{Lie}} + \mathscr{D}') \circ \mathscr{D}_{\mathsf{As}} = 0, \tag{4.13}$$

$$(\mathscr{D}_{\mathsf{Lie}} + \mathscr{D}') \circ \mathscr{D}'' + \mathscr{D}'' \circ (\mathscr{D}_{\mathsf{Lie}} + \mathscr{D}') = 0, \tag{4.14}$$

$$\mathscr{D}_{\mathsf{As}} \circ \mathscr{D}'' + \mathscr{D}'' \circ \mathscr{D}_{\mathsf{As}} + \mathscr{D}'' \circ \mathscr{D}'' = 0. \tag{4.15}$$

$$\mathscr{D}_{\mathsf{As}}(\mathsf{t}^{\mathfrak{o}}_{n,k}) = -\sum_{p=0}^{k-2}\sum_{q=p+2}^{k}(-1)^{p+(k-q)(q-p)}$$



$$-\sum_{p=0}^{k-2}\sum_{q=p+2}^{k}(-1)^{p+(k-q)(q-p)}$$

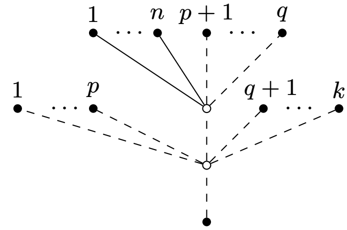


FIGURE 4.7. The value of $\mathscr{D}_{\mathsf{As}}$ on $\mathsf{t}^{\mathfrak{o}}_{n,k}$

$$\mathscr{D}'(\mathsf{t}^{\mathfrak{o}}_{n,k}) = (-1)^k$$

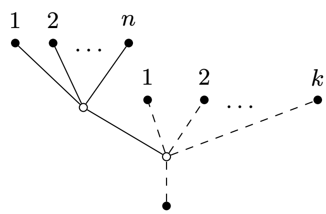


FIGURE 4.8. The value of $\mathscr{D}'$ on $\mathsf{t}^{\mathfrak{o}}_{n,k}$ for $n \geq 2$

$$\mathscr{D}''(\mathsf{t}^{\mathfrak{o}}_{n,k}) = -\sum_{r=1}^{n-1}\sum_{\sigma\in \mathrm{Sh}_{r,n-r}}\sum_{0\leq p\leq q\leq k}(-1)^{p+(k-q)(q-p)}$$

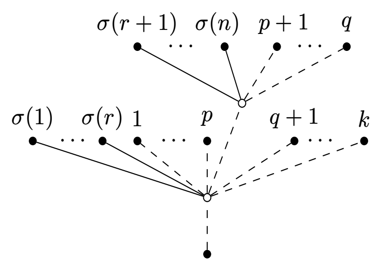


FIGURE 4.9. The value of $\mathscr{D}''$ on $\mathsf{t}^{\mathfrak{o}}_{n,k}$ for $n \geq 2$

REMARK 4.1. – *It is not hard to see that the differential* $\mathscr{D}$ *on* $\mathbb{OP}(\mathfrak{oc})$ *defines on* $\mathbf{s}^{-1}\mathfrak{oc}$ *a structure of* 2*-colored pseudo-cooperad. Thus, if* $\mathfrak{oc}^{\vee}$ *is the* 2*-colored cooperad obtained from* $\mathbf{s}^{-1}\mathfrak{oc}$ *via formally adjoining the counit, then* (1)

$$\mathsf{OC} = \mathrm{Cobar}(\mathfrak{oc}^{\vee}). \tag{4.16}$$

1. This fact was also observed in [7, Section 4.1].

We remark that

$$\mathfrak{oc}^{\vee}(n,0)^{\mathfrak{c}} = \Lambda^2 \mathsf{coCom}(n) \tag{4.17}$$

and

$$\mathfrak{oc}^{\vee}(0,k)^{\mathfrak{o}} = \Lambda \mathsf{coAs}(k). \tag{4.18}$$

4.2. OC-algebras

As we stated above, an OC-algebra is a pair of cochain complexes $(\mathcal{V}, \mathcal{A})$ with the following data:

— A $\Lambda\mathsf{Lie}_\infty$-structure on $\mathcal{V}$,

— an A_∞-structure on $\mathcal{A}$, and

— a $\Lambda\mathsf{Lie}_\infty$-morphism from $\mathcal{V}$ to the Hochschild cochain complex $C^\bullet(\mathcal{A})$ of $\mathcal{A}$.

Let us briefly recall how to get the above data from an operad morphism

$$\mathsf{OC} \to \mathsf{End}_{(\mathcal{V},\mathcal{A})}. \tag{4.19}$$

The desired $\Lambda\mathsf{Lie}_\infty$ structure on $\mathcal{V}$

$$Q : \Lambda^2\mathsf{coCom}_\circ(\mathcal{V}) \to \mathcal{V}$$

comes from the action of corollas $\mathsf{t}^{\mathfrak{c}}_n$ in Figure 4.1 for $n \geq 2$. Namely,

$$Q(v_1,\dots,v_n) = \mathsf{t}^{\mathfrak{c}}_n(v_1,\dots,v_n), \tag{4.20}$$

where $v_1,\dots,v_n \in \mathcal{V}$.

The desired A_∞-structure

$$m : \Lambda\mathsf{coAs}_\circ(\mathcal{A}) \to \mathcal{A}$$

comes from the action of corollas $\mathsf{t}^{\mathfrak{o}}_k$ in Figure 4.2 for $k \geq 2$. Namely,

$$m(a_1,\dots,a_k) = (-1)^{\varepsilon(a_1,\dots,a_k)}\mathsf{t}^{\mathfrak{o}}_k(a_1,\dots,a_k), \tag{4.21}$$

where $a_1,\dots,a_k \in \mathcal{A}$ and

$$\varepsilon(a_1,\dots,a_k) = |a_1|(k-1) + |a_2|(k-2) + \cdots + |a_{k-1}|.$$

Finally the action of corollas $\mathsf{t}^{\mathfrak{o}}_{n,k}$ gives us the desired $\Lambda\mathsf{Lie}_\infty$-morphism from $\mathcal{V}$ to $C^\bullet(\mathcal{A})$

$$U : \Lambda^2\mathsf{coCom}(\mathcal{V}) \otimes T(\mathbf{s}^{-1}\mathcal{A}) \to \mathcal{A}.$$

Namely,

$$U(v_1,\dots,v_n;a_1,\dots,a_k) = (-1)^{\varepsilon'(v_1,\dots,v_n;a_1,\dots,a_k)}\mathsf{t}^{\mathfrak{o}}_{n,k}(v_1,\dots,v_n;a_1,\dots,a_k), \tag{4.22}$$

where

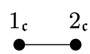
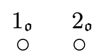
(4.23)

$$\varepsilon'(v_1,\dots,v_n;a_1,\dots,a_k) = k(|v_1|+\cdots+|v_n|) + |a_1|(k-1) + |a_2|(k-2) + \cdots + |a_{k-1}|.$$

CHAPTER 5

STABLE FORMALITY QUASI-ISOMORPHISMS AND THEIR HOMOTOPIES

Several vectors of KGra will play a special role in the definition of a stable formality quasi-isomorphism (SFQ) and in further considerations. These are

(5.1) $\Gamma_{\bullet\!-\!\bullet} =$ , $\qquad \Gamma_{\circ\,\circ} =$ 

and the series of "brooms" for $k \geq 0$ depicted in Figure 5.1.

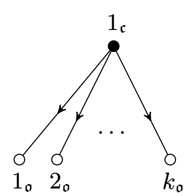
$\Gamma^{\mathrm{br}}_k =$ 

FIGURE 5.1. Edges are ordered in this way $(1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (1_{\mathfrak{c}}, 2_{\mathfrak{o}}) < \cdots < (1_{\mathfrak{c}}, k_{\mathfrak{o}})$

Note that the graph $\Gamma^{\mathrm{br}}_0 \in \mathsf{KGra}(1,0)^{\mathfrak{o}}$ consists of a single black vertex labeled by $1_{\mathfrak{c}}$ and it has no edges.

According to Section 3.5, the 2-colored operad KGra acts on the pair (V_A, A) where A (resp. V_A) is the algebra of functions (resp. the algebra of polyvector fields) on a graded affine space. Hence, every morphism of operads (in $\mathsf{Ch}_{\mathbb{K}}$) $F : \mathsf{OC} \to \mathsf{KGra}$ gives us a $\Lambda\mathsf{Lie}_\infty$-structure on V_A, an A_∞-structure on A and an $\Lambda\mathsf{Lie}_\infty$-morphism from V_A to the Hochschild cochain complex $C^\bullet(A)$ of A. Moreover, this construction works for a graded affine space of *any dimension.* This observation motivates the following definition.

DEFINITION 5.1. – *A stable formality quasi-isomorphism (SFQ) is a morphism of 2-colored operads in the category of cochain complexes*

$$F : \mathsf{OC} \to \mathsf{KGra} \tag{5.2}$$

satisfying the following "boundary conditions":

$$F(\mathsf{t}^{\mathfrak{c}}_n) = \begin{cases} \Gamma_{\bullet\!-\!\bullet} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases} \tag{5.3}$$

$$F(\mathsf{t}^{\mathfrak{o}}_2) = \Gamma_{\circ\,\circ}, \tag{5.4}$$

and

$$F(\mathsf{t}^{\mathfrak{o}}_{1,k}) = \frac{1}{k!}\Gamma^{\mathrm{br}}_k, \tag{5.5}$$

where $\mathsf{t}^{\mathfrak{c}}_n$, $\mathsf{t}^{\mathfrak{o}}_k$, *and* $\mathsf{t}^{\mathfrak{o}}_{n,k}$ *are corollas depicted in Figures 4.1, 4.2, 4.3, respectively, and* $\Gamma_{\bullet\!-\!\bullet}$, $\Gamma_{\circ\,\circ}$ *and* Γ^{br}_k *are the vectors of* KGra *specified in the beginning of this section.*

To interpret the "boundary conditions" we consider the OCHA structure induced by the morphism F (5.2) on the pair (V_A, A).

The first condition (eq. (5.3)) implies that the $\Lambda\mathsf{Lie}_\infty$-structure on polyvector fields induced by the morphism F coincides with the standard Schouten-Nijenhuis algebra structure.

The second condition (eq. (5.4)) implies that the binary operation of the induced A_∞-structure on A coincides with the ordinary (commutative) multiplication. For degree reasons, the image $F(\mathsf{t}^{\mathfrak{o}}_k)$ of the corolla $\mathsf{t}^{\mathfrak{o}}_k$ in $\mathsf{KGra}(0,k)^{\mathfrak{o}}$ is zero for all $k \geq 3$. Thus the induced A_∞-structure on A coincides with the original associative (and commutative) algebra structure.

The third boundary condition (eq. (5.5)) implies that the corresponding $\Lambda\mathsf{Lie}_\infty$-morphism from V_A to $C^\bullet(A)$ starts with the Hochschild-Kostant-Rosenberg embedding. The latter condition guarantees that the induced $\Lambda\mathsf{Lie}_\infty$-morphism is a quasi-isomorphism.

REMARK 5.2. – *It should be mentioned that the map in* (5.2) *is never a quasi-isomorphism of dg operads. Indeed, the restriction of any morphism of dg operads* $F : \mathsf{OC} \to \mathsf{KGra}$ *satisfying the above "boundary conditions" to the spaces* $\mathsf{OC}(n,0)^{\mathfrak{c}}$ *(for* $n \geq 0$*) gives us the morphism of operads* $\Lambda\mathsf{Lie}_\infty \to \mathsf{dGra}$ *which coincides with the composition of the canonical quasi-isomorphism* $\Lambda\mathsf{Lie}_\infty \xrightarrow{\sim} \Lambda\mathsf{Lie}$ *and the standard embedding of operads* $\Lambda\mathsf{Lie} \hookrightarrow \mathsf{dGra}$ [**11**, Section 7.1]. *This composition is not a quasi-isomorphism because the embedding* $\Lambda\mathsf{Lie} \hookrightarrow \mathsf{dGra}$ *is not onto.*

5.1. SFQs as MC elements. Homotopies of SFQs

Due to Proposition 2.13 and Remark 4.1, SFQs are in bijection with MC elements α of the Lie algebra

$$\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra}) \tag{5.6}$$

subject to the three conditions

$$\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_n) = \begin{cases} \Gamma_{\bullet\!-\!\bullet} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases} \tag{5.7}$$

$$\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_2) = \Gamma_{\circ\,\circ}, \tag{5.8}$$

and

$$\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,k}) = \frac{1}{k!}\Gamma^{\mathrm{br}}_k, \tag{5.9}$$

where $\mathsf{t}^{\mathfrak{c}}_n$, $\mathsf{t}^{\mathfrak{o}}_k$, and $\mathsf{t}^{\mathfrak{o}}_{n,k}$ are corollas depicted in Figures 4.1, 4.2, 4.3, respectively, and $\Gamma_{\bullet\!-\!\bullet}$, $\Gamma_{\circ\,\circ}$ and Γ^{br}_k are the vectors of KGra specified in the beginning of this section.

We would like to remark that, since all vectors in $\mathsf{KGra}(0,k)^{\mathfrak{o}}$ have degree zero, we have

$$\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_k) = 0, \tag{5.10}$$

for all $k \geq 3$ and for all degree 1 elements α in (5.6).

In what follows, we denote by α_F the MC element in (5.6) corresponding to an SFQ F.

According to Section 2.3, the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ is equipped with the "arity" filtration $\mathscr{F}_{\bullet}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$, such that $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ is complete with respect to this filtration.

Hence, following general theory from Appendix C, the set of MC elements of the Lie algebra (5.6) is equipped with the action of the pro-unipotent group

$$\exp\Big(\mathscr{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})^0\Big). \tag{5.11}$$

We claim that

Proposition 5.3. – *Degree zero vectors*

$$\xi \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}) \tag{5.12}$$

satisfying the "boundary" condition

$$\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_n) = 0 \qquad \forall n \geq 2 \tag{5.13}$$

form a Lie subalgebra of $\mathscr{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})^0$. *Moreover, if* α *is a MC element of the Lie algebra* $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ *satisfying the boundary conditions* (5.7), (5.8), (5.9) *and* ξ *is a degree zero vector* (5.12) *satisfying* (5.13) *then the MC element*

$$\alpha' = \exp(\xi)\,(\alpha) \tag{5.14}$$

also satisfies conditions (5.7), (5.8), (5.9).

Proof. – First, we observe that every degree zero vector (5.12) satisfies

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}) = 0 \qquad \forall k \geq 0. \tag{5.15}$$

Indeed, since the vector $\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k}$ has degree $-k-1$, $\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,k})$ must be a linear combination of graphs with 1 black vertex, k white vertices, and exactly $k+1$ edges. Since an edge cannot originate at any white vertex, multiple edges with the same direction and loops are not allowed, the set of such graphs is empty.

Similarly, since all vectors in $\mathsf{KGra}(0,k)^{\mathfrak{o}}$ have degree zero, we conclude that

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k}) = 0 \qquad \forall k \geq 2 \tag{5.16}$$

for any degree zero vector (5.12).

The inclusion

$$\xi \in \mathscr{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}). \tag{5.17}$$

follows immediately from Equation (5.15) for $k=0$.

Moreover, the vector $[\xi_1, \xi_2]$ satisfies condition (5.13) if so do both ξ_1 and ξ_2. Thus the first statement of the proposition is proved.

Using (5.13), (5.15), and (5.16), it is easy to see that α' in (5.14) satisfies conditions (5.7), (5.8), (5.9) if so does α. □

We can now give the definition of homotopy between two SFQs:

DEFINITION 5.4. – *We say that an SFQ F (5.2) is homotopy equivalent to $\widetilde{F}$ if the corresponding MC elements*

$$\alpha_F, \alpha_{\widetilde{F}} \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$$

are isomorphic via $\exp(\xi)$*, where ξ is a degree zero element in* $\mathscr{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ *satisfying condition* (5.13)*.*

REMARK 5.5. – *Proposition 5.3 implies that the resulting relation on the set of SFQs is indeed an equivalence relation.*

REMARK 5.6. – *Since* $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}) = \mathscr{F}_0\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$*, every MC element α of* (5.6) *satisfies the condition*

$$\alpha \in \mathscr{F}_0\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}). \tag{5.18}$$

On the other hand, it is not true that a MC element α corresponding to an SFQ belongs to $\mathscr{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$*. Indeed, according to* (5.9)*, we have* $\alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{1,0}) = \Gamma^{\mathrm{br}}_0 \neq 0$.

Using the "boundary conditions" (5.7)*,* (5.8) *and* (5.9) *for MC elements α corresponding to SFQs, it is easy to see that*

$$\alpha \in \mathscr{F}^{\mathfrak{c}}_0\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}), \tag{5.19}$$

where the filtration $\mathscr{F}^{\mathfrak{c}}_{\bullet}$ is defined in (2.46) *(in Section 2.3).*

To explain our motivation behind Definition 5.4 we consider the pair (V_A, A), where A is a finitely generated free commutative algebra in $\mathsf{grVect}_{\mathbb{K}}$ and V_A be the algebra of polyvector fields on the corresponding (graded) affine space.

Recall that an SFQ F gives us a $\Lambda\mathsf{Lie}_\infty$ quasi-isomorphism U_F from V_A to $C^\bullet(A)$ which admits a graphical expansion.

We claim that, if two SFQs F and $\widetilde{F}$ are homotopy equivalent, then the corresponding $\Lambda\mathsf{Lie}_\infty$-morphisms U_F and $U_{\widetilde{F}}$ are also homotopy equivalent. Furthermore, the homotopy between U_F and $U_{\widetilde{F}}$ admits a graphical expansion.

Indeed, according to [**10**, Lemma 2.9] or [**14**, Section 1.3], any $\Lambda\mathsf{Lie}_\infty$-morphism

$$U : V_A \rightsquigarrow C^\bullet(A)$$

is a MC element of the following auxiliary Lie algebra

$$\mathbf{s}\mathrm{Hom}(\mathbf{s}^2 S(\mathbf{s}^{-2} V_A), C^\bullet(A)). \tag{5.20}$$

For the definition of the differential and the Lie bracket on (5.20), see Section 2.1 in [**10**].

Furthermore, following Definition 4.7 in [**10**], two $\Lambda\mathsf{Lie}_\infty$-morphisms

$$U : V_A \rightsquigarrow C^\bullet(A) \qquad \text{and} \qquad \widetilde{U} : V_A \rightsquigarrow C^\bullet(A)$$

are homotopy equivalent if and only if the corresponding MC elements in the Lie algebra (5.20) are isomorphic.

By merely unfolding definitions it is not hard to see that, if MC elements

$$\alpha_F, \alpha_{\widetilde{F}} \in \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$$

are isomorphic via $\exp(\xi)$ for a degree zero vector (5.12) satisfying (5.13), then the MC elements in (5.20) corresponding to U_F and $U_{\widetilde{F}}$ are isomorphic via

$$\exp(\xi'),$$

where ξ' is the degree zero vector in (5.20) given by the formula:

$$\begin{aligned}\xi'(v_1, v_2, \dots, v_n; a_1, a_2, \dots, a_n) \\ = (-1)^{\varepsilon'(v_1,\dots,v_n;a_1,\dots,a_k)} \xi(\mathbf{s}^{-1}\mathfrak{t}^{\mathfrak{o}}_{n,k})(v_1, v_2, \dots, v_n; a_1, a_2, \dots, a_n),\end{aligned} \tag{5.21}$$

where $\varepsilon'(v_1, \dots, v_n; a_1, \dots, a_k)$ is defined in (4.23).

Example 5.7. – *In his famous paper* [**31**] *M. Kontsevich proposed a construction of a* $\Lambda\mathsf{Lie}_\infty$ *quasi-isomorphism from the Lie algebra of polyvector fields* V_A *on* $\mathbb{R}^d$ *to polydifferential operators on* $\mathbb{R}^d$*. The structure maps of this* $\Lambda\mathsf{Lie}_\infty$ *quasi-isomorphism are defined using graphical expansion and the* $\Lambda\mathsf{Lie}_\infty$ *quasi-isomorphism starts with the standard Hochschild-Kostant-Rosenberg embedding. Thus Kontsevich's construction from* [**31**] *gives us an SFQ over any extension of the field* $\mathbb{R}$*. For more details, we refer the reader to* [**8**, Section 2.4]*.*

CHAPTER 6

THE ACTION OF KONTSEVICH'S GRAPH COMPLEX ON STABLE FORMALITY QUASI-ISOMORPHISMS

It is possible to produce new homotopy types of stable formality quasi-isomorphisms using the action of Kontsevich's graph complex on the Lie algebra

$$\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}).$$

We describe this action here.

6.1. Reminder of the full directed graph complex dfGC

Let us consider the Lie algebra (in $\mathsf{grVect}_{\mathbb{K}}$)

$$\mathsf{dfGC} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{dGra}). \tag{6.1}$$

Since

$$\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{dGra}) = \prod_{n=1}^{\infty} \mathsf{s}^{2n-2}\big(\mathsf{dGra}(n)\big)^{S_n}$$

vectors in (6.1) are (possibly infinite) linear combinations

$$\gamma = \sum_{n=1}^{\infty} \gamma_n,$$

where γ_n is an S_n-invariant vector in $\mathsf{dGra}(n)$.

If all graphs in the linear combination $\gamma_n \in (\mathsf{dGra}(n))^{S_n}$ have the same number of edges e then γ_n is a homogeneous vector in dfGC of degree

$$|\gamma_n| = 2n - 2 - e. \tag{6.2}$$

For example, the vector $\Gamma_{\bullet\!-\!\bullet} \in \mathsf{dGra}(2)$ defined in (5.1) is S_2-invariant and hence is a vector in dfGC. According to (6.2), the vector $\Gamma_{\bullet\!-\!\bullet}$ carries degree 1. A direct computation shows that

$$[\Gamma_{\bullet\!-\!\bullet}, \Gamma_{\bullet\!-\!\bullet}] = 0. \tag{6.3}$$

Hence, $\Gamma_{\bullet\!\to\!\bullet}$ is a MC element and it can be used to equip the graded vector space (6.1) with the non-zero differential

$$\partial = \mathrm{ad}_{\Gamma_{\bullet\to\bullet}}. \tag{6.4}$$

DEFINITION 6.1. – *The graded vector space* dfGC (6.1) *with the differential* (6.4) *is called the full directed graph complex.*

For example the graph $\Gamma_{\bullet} \in \mathrm{dgra}_1$ which consists of a single vertex without edges gives us a degree zero vector in dfGC. According to the definition of the Lie bracket on dfGC, we have

$$[\Gamma_{\bullet\to\bullet}, \Gamma_{\bullet}] = \Gamma_{\bullet\to\bullet} \circ_1 \Gamma_{\bullet} + \sigma_{12}(\Gamma_{\bullet\to\bullet} \circ_1 \Gamma_{\bullet}) - \Gamma_{\bullet} \circ_1 \Gamma_{\bullet\to\bullet} = \Gamma_{\bullet\to\bullet} + \Gamma_{\bullet\to\bullet} - \Gamma_{\bullet\to\bullet} = \Gamma_{\bullet\to\bullet}, \tag{6.5}$$

where σ_{12} is the transposition in S_2.

Thus $\Gamma_{\bullet}$ is not a cocycle in dfGC.

According to Section 2.3, the Lie algebra dfGC is equipped with the descending filtration (2.44) such that dfGC is complete with respect to this filtration. Unfolding (2.44), it is easy to see that $\mathcal{F}_m\mathsf{dfGC}$ consists of sums

$$\gamma = \sum_{n=m+1}^{\infty} \gamma_n, \qquad \gamma_n \in \big(\mathsf{dGra}(n)\big)^{S_n}.$$

I.e. $\gamma \in \mathcal{F}_m\mathsf{dfGC}$ if and only if each graph in γ has $\geq m+1$ vertices. For example, $\Gamma_{\bullet\to\bullet} \in \mathcal{F}_1\mathsf{dfGC}$. Therefore the differential (6.4) is compatible with the filtration on dfGC.

Since loops are not allowed, $\Gamma_{\bullet}$ is the only element of dgra_1. Therefore, since $\partial\Gamma_{\bullet} \neq 0$ and the differential ∂ raises the number of vertices up by 1, every cocycle $\gamma \in \mathsf{dfGC}$ has the property [1]

$$\gamma \in \mathcal{F}_1\mathsf{dfGC}. \tag{6.6}$$

Thus, the Lie algebra $H^0(\mathsf{dfGC})$ is pro-nilpotent.

To give an example of a degree zero cocycle in dfGC we consider the tetrahedron in $\mathsf{dGra}(4)$ depicted in Figure 6.1. This graph is invariant with respect to the action of S_4 and hence it can be viewed as a vector in dfGC. According to (6.2), this vector has degree zero. A direct computation shows that it is a cocycle and it is easy to see that this cocycle is non-trivial.

In fact, it was proved in [**39**, Proposition 9.1] that, for every odd number $n \geq 3$, there exists a non-trivial cocycle which has a non-zero coefficient in front of the wheel with n spokes (see Figure 6.2). Note that, labels on vertices do not play an important role because vectors in dfGC are invariant under the action of the symmetric group.

1. Inclusion (6.6) no longer holds if we allow loops. Indeed, a simple computation shows that the single loop $\circlearrowleft \in \mathrm{dgra}_1$ would be a cocycle in dfGC. See [**13**] for more details.

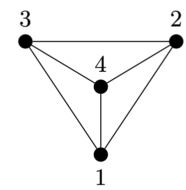


FIGURE 6.1. We may choose this order on the set of edges: $(1,2) < (1,3) < (1,4) < (2,3) < (2,4) < (3,4)$

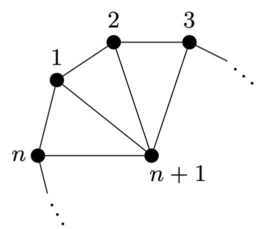


FIGURE 6.2. Here n is an odd integer ≥ 3

Using [**39**, Theorem 1.1] and the ideas sketched in [**39**, Appendix K], one can prove the following statement:

THEOREM 6.2 ([**13**], Corollary 3.6). – *For the full directed graph complex* dfGC*, we have an isomorphism of Lie algebras*

$$H^0(\mathsf{dfGC}) \cong \mathfrak{grt}_1, \tag{6.7}$$

where $\mathfrak{grt}_1$ *is the Grothendieck-Teichmueller Lie algebra* [**1**, Section 4.2]*,* [**15**, Section 6]*.*

REMARK 6.3. – *Let* Γ *be an element in* dgra_n*. We say that a vertex* v *of* Γ *is a pike if* v *has valency* 1 *and the edge adjacent to* v *terminates at* v*. We observe that, due to* [**13**, Proposition 3.5] *any cocycle* γ *in* dfGC *is cohomologous to a cocycle in which all graphs do not have pikes* (2)*.*

2. Note that, in this paper, we work exclusively with the loopless version of the full directed graph complex. So what we denote by dfGC in this paper is denoted by $\mathsf{dfGC}^{\varnothing}$ in [**13**].

6.2. The action of dfGC on SFQs

For our purposes it is convenient to extend the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ (5.6) to the Lie algebra

$$\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{6.8}$$

and view MC elements of $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ as MC elements of its extension $\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$.

Using Equation (4.17), we define a natural embedding of dfGC (6.1) into the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$

$$J : \mathsf{dfGC} \hookrightarrow \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}). \tag{6.9}$$

This embedding is given by the formulas

$$J(\gamma)\big|_{\mathfrak{oc}^{\vee}(n,0)^{\mathfrak{c}}} = \gamma, \qquad J(\gamma)\big|_{\mathfrak{oc}^{\vee}(n,k)^{\mathfrak{o}}} = 0. \tag{6.10}$$

The embedding J is obviously compatible with the Lie brackets and with the filtrations by arity on dfGC and $\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ (see (2.44)). However, we should point out that J is not compatible with the differentials. Indeed, the Lie algebra $\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ carries the zero differential while dfGC carries the non-zero differential (6.4).

Let α_F be a MC element in $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ corresponding to an SFQ. We claim that

PROPOSITION 6.4. – *For every degree zero cocycle $\gamma \in \mathsf{dfGC}$ the equation*

$$\alpha' = \exp(\mathrm{ad}_{J(\gamma)})\alpha_F \tag{6.11}$$

defines a MC element α' in $\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ satisfying conditions (5.7), (5.8), *and* (5.9).

Proof. – It is obvious that α' satisfies the MC equation in $\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$.

Furthermore, since each cocycle in dfGC belongs to $\mathcal{F}_1\mathsf{dfGC}$, the MC element α' belongs to the Lie subalgebra

$$\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}) \subset \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}), .$$

Next, using the cocycle condition for γ

$$[\Gamma_{\bullet\!-\!\bullet}, \gamma] = 0$$

it is not hard to show that α' satisfies condition (5.7).

Finally, it is straightforward to verify that α' also satisfies (5.8) and (5.9). □

Due to Proposition 6.4, the group $\exp\left(\mathcal{Z}^0(\mathsf{dfGC})\right)$ acts on SFQs. In the following proposition we list important properties of this action.

PROPOSITION 6.5. – *Let γ be a degree zero cocycle in* dfGC*. If α and $\widetilde{\alpha}$ are MC elements of* (5.6) *corresponding to homotopy equivalent SFQs F and $\widetilde{F}$, then the MC elements*

$$\exp(\mathrm{ad}_{J(\gamma)})\alpha, \textit{and} \exp(\mathrm{ad}_{J(\gamma)})\widetilde{\alpha}$$

also correspond to homotopy equivalent SFQs.

Furthermore, if

$$\gamma = [\Gamma_{\bullet\!-\!\bullet}, \psi] \tag{6.12}$$

then there exists a degree zero vector

$$\xi \in \mathscr{F}_1 \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$$

for which (5.13) *holds and*

$$\exp(\mathrm{ad}_{J(\gamma)})\alpha = \exp(\mathrm{ad}_\xi)\alpha. \tag{6.13}$$

If, in addition,

$$\psi \in \mathscr{F}_{n-1}\mathsf{dfGC} \tag{6.14}$$

for some $n \geq 2$ then the vector ξ in (6.13) *can be chosen in such a way that*

$$\xi \in \mathscr{F}_n^{\mathfrak{c}} \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}) \tag{6.15}$$

and

$$\xi(\mathbf{s}^{-1}\, \mathfrak{t}^{\mathfrak{o}}_{n,0}) = \psi(1_n), \tag{6.16}$$

where 1_n is the generator $\mathbf{s}^{2-2n}1 \in \mathbf{s}^{2-2n}\mathbb{K} \cong \Lambda^2\mathsf{coCom}(n)$.

Proof. – Since α and $\widetilde{\alpha}$ represent homotopy equivalent SFQs, there exists a degree zero vector

$$\xi \in \mathscr{F}_1 \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$$

for which (5.13) holds and

$$\widetilde{\alpha} = \exp(\mathrm{ad}_\xi)\alpha. \tag{6.17}$$

Applying $\exp(\mathrm{ad}_{J(\gamma)})$ to both sides of Equation (6.17) we get

$$\exp(\mathrm{ad}_{J(\gamma)})\widetilde{\alpha} = \exp(\mathrm{ad}_{J(\gamma)}) \exp(\mathrm{ad}_\xi)\alpha = \exp(\mathrm{ad}_{\widetilde{\xi}}) \left(\exp(\mathrm{ad}_{J(\gamma)})\alpha\right), \tag{6.18}$$

where

$$\widetilde{\xi} = \exp(\mathrm{ad}_{J(\gamma)})\, \xi. \tag{6.19}$$

The vector $\widetilde{\xi}$ obviously belongs to $\mathscr{F}_1 \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$. Furthermore, $\widetilde{\xi}$ satisfies the condition

$$\widetilde{\xi}(\mathbf{s}^{-1}\, \mathfrak{t}^{\mathfrak{c}}_n) = 0, \qquad \forall n \geq 2$$

since so does ξ.

Thus the MC elements

$$\exp(\mathrm{ad}_{J(\gamma)})\alpha, \text{and} \exp(\mathrm{ad}_{J(\gamma)})\widetilde{\alpha}$$

indeed correspond to homotopy equivalent stable formality quasi-isomorphisms.

To prove the second statement, we introduce the Lie algebra (in $\mathsf{grVect}_{\mathbb{K}}$)

$$\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \,\hat{\otimes}\, \mathbb{K}[t], \tag{6.20}$$

where $\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ is considered with the topology coming from the filtration $\mathscr{F}_{\bullet}$ "by arity" (2.44) and $\mathbb{K}[t]$ is considered with the discrete topology.

Let us denote by $\alpha(t)$ the following vector in (6.20)

$$\alpha(t) = \exp(t\, \mathrm{ad}_{J(\gamma)})\alpha. \tag{6.21}$$

It is easy to see that $\alpha(t)$ enjoys the MC equation

$$[\alpha(t), \alpha(t)] = 0 \tag{6.22}$$

and the conditions

$$\alpha(t)\,(\mathbf{s}^{-1}\, \mathsf{t}^{\mathfrak{c}}_{n}) = \begin{cases} \Gamma_{\bullet\!-\!\bullet} & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases} \tag{6.23}$$

$$\alpha(\mathbf{s}^{-1}\, \mathsf{t}^{\mathfrak{o}}_{2}) = \Gamma_{\circ\,\circ}, \tag{6.24}$$

$$\alpha(\mathbf{s}^{-1}\, \mathsf{t}^{\mathfrak{o}}_{1,k}) = \frac{1}{k!}\Gamma^{\mathrm{br}}_{k}, \tag{6.25}$$

since so does α and since γ is cocycle in dfGC. Furthermore, $\alpha(t)$ satisfies the following (formal) differential equation

$$\frac{d}{dt}\,\alpha(t) = [J(\gamma), \alpha(t)] \tag{6.26}$$

with the initial condition

$$\alpha(t)\big|_{t=0} = \alpha. \tag{6.27}$$

Let us now assume that

$$\gamma = [\Gamma_{\bullet\!-\!\bullet}, \psi] \tag{6.28}$$

for a degree -1 vector in dfGC.

Since loops are not allowed and ψ has degree -1, we have

$$\psi \in \mathscr{F}_{1}\mathsf{dfGC}. \tag{6.29}$$

Moreover, since the map J (6.9) is compatible with Lie brackets, we have

$$J(\gamma) = [J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)]$$

and hence the vector $\alpha(t)$ satisfies the equation

$$\frac{d}{dt}\,\alpha(t) = \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], \alpha(t)\big]. \tag{6.30}$$

Let us denote by $\Delta\alpha(t)$ the difference

$$\Delta\alpha(t) = \alpha(t) - J(\Gamma_{\bullet\!-\!\bullet}) \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \hat{\otimes} \mathbb{K}[t]. \tag{6.31}$$

The MC Equation (6.22) for $\alpha(t)$ implies that

$$[J(\Gamma_{\bullet\!-\!\bullet}), \Delta\alpha(t)] + \frac{1}{2}[\Delta\alpha(t), \Delta\alpha(t)] = 0. \tag{6.32}$$

Furthermore, due to Equation (6.23), we have

$$\Delta\alpha(t)\left(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_m\right) = 0 \qquad \forall m \geq 2. \tag{6.33}$$

Using the Jacobi idenity, identity $[J(\Gamma_{\bullet\!-\!\bullet}), J(\Gamma_{\bullet\!-\!\bullet})] = 0$, and Equation (6.32) we rewrite Equation (6.30) as follows

$$\begin{aligned}
\frac{d}{dt}\,\alpha(t) &= \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], J(\Gamma_{\bullet\!-\!\bullet})\big] + \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], \Delta\alpha(t)\big] \\
&= \big[[J(\Gamma_{\bullet\!-\!\bullet}), J(\psi)], \Delta\alpha(t)\big] \\
&= -\big[[J(\psi), \Delta\alpha(t)], J(\Gamma_{\bullet\!-\!\bullet})\big] - \big[[J(\Gamma_{\bullet\!-\!\bullet}), \Delta\alpha(t)], J(\psi)\big] \\
&= -\big[[J(\psi), \Delta\alpha(t)], J(\Gamma_{\bullet\!-\!\bullet})\big] + \frac{1}{2}\big[[\Delta\alpha(t), \Delta\alpha(t)], J(\psi)\big] \\
&= -\big[[J(\psi), \Delta\alpha(t)], J(\Gamma_{\bullet\!-\!\bullet})\big] - \frac{1}{2}\big[[\Delta\alpha(t), J(\psi)], \Delta\alpha(t)\big] - \frac{1}{2}\big[[J(\psi), \Delta\alpha(t)], \Delta\alpha(t)\big] \\
&= -\big[[J(\psi), \Delta\alpha(t)], \alpha(t)\big].
\end{aligned}$$

Thus the vector $\alpha(t)$ (6.21) satisfies the (formal) differential equation

$$\frac{d}{dt}\,\alpha(t) = [\eta(t), \alpha(t)], \tag{6.34}$$

where $\eta(t)$ is the degree zero vector

$$\eta(t) = -[J(\psi), \Delta\alpha(t)] \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \hat{\otimes} \mathbb{K}[t]. \tag{6.35}$$

It is clear that $\eta(t)$ satisfies the conditions

$$\eta(t)\big(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_n\big) = 0 \qquad \forall n \geq 2$$

and

$$\eta(t)\big(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,k}\big) = 0 \qquad \forall k \geq 0.$$

Let us apply Theorem C.6 from Appendix C.1 to the case when $\mathscr{L} = \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ and $\mathfrak{g}$ consists of vectors in $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})^0$ which satisfy (5.13). Due to this theorem, there exists a vector

$$\xi \in \mathscr{F}_1\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra}),$$

which satisfies (5.13) and (6.13).

Thus the second statement of Proposition 6.5 is proved.

To prove the last statement, we observe that (6.14), (6.33), and the identities

$$J(\psi)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,k}) = J(\psi)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{k_1}) = 0 \qquad \forall m \geq 1,\ k \geq 0,\ k_1 \geq 2$$

imply that $\eta(t) \in \mathscr{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \hat{\otimes} \mathbb{K}[t]$ and hence the inclusion in (6.15) holds.

Inclusion (6.14) implies that $\gamma \in \mathscr{F}_n \mathsf{dfGC}$. Combining this fact with (5.18), we conclude that

$$\Delta\alpha(t) - (\alpha - \Gamma_{\bullet\!-\!\bullet}) \in \mathscr{F}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}).$$

Therefore,

$$\eta(t) = -[J(\psi), \alpha - \Gamma_{\bullet\!-\!\bullet}] \mathrm{mod}\, \mathscr{F}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})[[t]].$$

Hence, due to the second part of Theorem C.6, there exists

$$\xi \in \mathscr{F}_1 \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra}),$$

such that (5.13) and (6.13) hold, and we have

$$\begin{aligned}\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) &= -[J(\psi), \alpha - \Gamma_{\bullet\!-\!\bullet}](\mathsf{t}^{\mathfrak{o}}_{n,0}) \\ &= \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} J(\psi)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_{n}) = \Gamma^{\mathrm{br}}_0 \circ_{1,\mathfrak{c}} \psi(1_n) = \psi(1_n).\end{aligned}$$

Thus Equation (6.16) also holds. □

Proposition 6.5 implies that

COROLLARY 6.6. – *The action of* $\exp\left(\mathscr{Z}^0(\mathsf{dfGC})\right)$ *on SFQs descends to an action of* $\exp\left(H^0(\mathsf{dfGC})\right)$ *on homotopy classes of SFQs.*

Proof. – Let γ be a degree zero cocycle of dfGC. Due to the first statement of Proposition 6.5, $\exp(\gamma)$ transforms homotopy equivalent SFQs to homotopy equivalent SFQs.

Thus it remains to prove that, if γ' is cohomologous to γ then MC elements

(6.36) $$\exp(\mathrm{ad}_{J(\gamma')})\alpha \qquad \text{and} \qquad \exp(\mathrm{ad}_{J(\gamma)})\alpha$$

are connected by the action of $\exp(\mathrm{ad}_\xi)$ for a vector

$$\xi \in \mathscr{F}_1 \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

satisfying condition (5.13).

Using the fact that the difference $\gamma' - \gamma$ is exact, it is easy to see that

$$\mathrm{CH}(-\gamma, \gamma')$$

is also exact.

Therefore, due to the second statement of Proposition 6.5, the MC elements

$$\exp(\mathrm{ad}_{J(\gamma)}) \exp(\mathrm{ad}_{J(\gamma')})\, \alpha \qquad \text{and} \qquad \alpha$$

are connected by the action of $\exp(\mathrm{ad}_\xi)$ for a vector $\xi \in \mathscr{F}_1 \mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$ satisfying condition (5.13).

Hence, the MC elements (6.36) represent homotopy equivalent SFQs. □

REMARK 6.7. – *Let A be a finitely generated free commutative algebra in* $\mathsf{grVect}_{\mathbb{K}}$ *and* V_A *be the algebra of polyvector fields on the corresponding (graded) affine space. It is not hard to see that* $\Lambda\mathsf{Lie}_\infty$*-morphisms from* V_A *to* $C^\bullet(A)$ *which correspond to* α_F *and* (6.11) *are connected by the action described in* [**29**, Section 5] *by M. Kontsevich.*

Let us now state the main result of this paper

THEOREM 6.8. – *The pro-unipotent group* $\exp(H^0(\mathsf{dfGC}))$ *acts simply transitively on the set of homotopy classes of stable formality quasi-isomorphisms (SFQs).*

The proof of this theorem occupies the next two sections of the paper and it depends on a few technical statements which are proved in Appendices A and B.

Combining Theorem 6.8 with Theorem 6.2 stated above, we deduce that

COROLLARY 6.9. – *The set of homotopy classes of SFQs form a torsor for the Grothendieck-Teichmueller group* GRT_1*. ,*

REMARK 6.10. – *We should mention that this result agrees very well with Tamarkin's approach* [**23, 35**] *to Kontsevich's formality theorem* [**31, 29**]*. Tamarkin's construction* [**9, 35**, Section 2] *may be viewed as a map from the set of Drinfeld associators*[3] *to the set of homotopy classes of formality quasi-isomorphisms for Hochschild cochains. Due to* [**15**, Proposition 5.5] *and* [**39**, Theorem 1.1]*, both the source and the target of Tamarkin's map are equipped with the actions of the group* GRT_1*. Moreover, according*[4] *to* [**39**, Section 10.1]*, Tamarkin's construction is equivariant with respect to the action of* GRT_1*.*

Section 6 of paper [**37**] *contains a sketch on a version of Tamarkin's construction in "stable" setting. According to this sketch, every SFQ is homotopic to an SFQ which can be extended to a stable* Ger_∞*-morphism from polyvector fields to Hochschild cochains (for some choice of Tamarkin's* Ger_∞*-structure on Hochschild cochains).*

3. Here, we only consider Drinfeld associators whose "braiding" constant is 1. In [**15**, Section 5], this set is denoted by M_1. In [**9**, Section 4.1], this set is denoted by $\mathrm{DrAssoc}_1$.

4. For more details, we refer the reader to [**9**].

CHAPTER 7

THE ACTION OF $\exp\big(H^0(\mathsf{dfGC})\big)$ IS TRANSITIVE

Let α and $\widetilde{\alpha}$ be MC elements of the graded Lie algebra $\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$ corresponding to SFQs.

Since α and $\widetilde{\alpha}$ satisfy conditions (5.7), (5.8), (5.9), and (5.10), the difference

$$\delta\alpha := \widetilde{\alpha} - \alpha \tag{7.1}$$

satisfies the identities

$$\delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_m) = 0, \qquad \delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_k) = 0, \qquad \forall m, k \geq 2, \tag{7.2}$$

and

$$\delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,k}) = 0, \qquad \forall k \geq 0. \tag{7.3}$$

Therefore [1],

$$\delta\alpha \in \mathscr{F}^{\mathfrak{c}}_2\,\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}). \tag{7.4}$$

We will deduce the transitivity of the action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes of SFQs from the following statement

Proposition 7.1. – *If α and $\widetilde{\alpha}$ are MC elements of the graded Lie algebra* (5.6) *corresponding to SFQs and*

$$\widetilde{\alpha} - \alpha \in \mathscr{F}^{\mathfrak{c}}_n\,\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}) \tag{7.5}$$

for some $n \geq 2$ then there exists a degree zero cocycle $\gamma \in \mathscr{F}_{n-1}\mathsf{dfGC}$ and a degree zero vector

$$\xi \in \mathscr{F}^{\mathfrak{c}}_{n-1}\,\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$$

for which (5.13) *holds and*

$$\exp(\mathrm{ad}_\xi)\,\widetilde{\alpha} - \exp(\mathrm{ad}_{J(\gamma)})\alpha \in \mathscr{F}^{\mathfrak{c}}_{n+1}\,\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}). \tag{7.6}$$

1. See conditions (2.47) and (2.48) in Section 2.3.

The proof of this proposition consists of two parts. The first part is given in Section 7.2 and the second part is given in Section 7.3.

7.1. Proposition 7.1 implies that the action is transitive

Using (7.4) and applying Proposition 7.1 recursively, we see that there exist infinite sequences of degree zero vectors

$$\xi_1, \xi_2, \xi_3, \dots \in \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}), \qquad \xi_m \in \mathcal{F}^{\mathfrak{c}}_m \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}) \tag{7.7}$$

$$\gamma_1, \gamma_2, \gamma_3, \dots \in \mathsf{dfGC}, \qquad \gamma_m \in \mathcal{F}_m \mathsf{dfGC} \tag{7.8}$$

such that each ξ_m satisfies (5.13) and

$$\exp(\mathrm{ad}_{\xi_m}) \cdots \exp(\mathrm{ad}_{\xi_1})\, \widetilde{\alpha} - \exp(\mathrm{ad}_{J(\gamma_m)}) \cdots \exp(\mathrm{ad}_{J(\gamma_1)})\alpha \in \mathcal{F}^{\mathfrak{c}}_{m+2} \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra}) \tag{7.9}$$

for every $m \geq 1$.

Since $\mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$ (resp. dfGC) is complete with respect to the filtration $\mathcal{F}^{\mathfrak{c}}_\bullet$ (resp. $\mathcal{F}_\bullet$), the existence of the above sequence implies that the homotopy class of the SFQ corresponding to $\widetilde{\alpha}$ has a representative which lies on the $\exp(\mathcal{Z}^0(\mathsf{dfGC}))$-orbit of the SFQ corresponding to α.

7.2. Taking care of $\delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$

Due to (7.2) and (7.3), the element $\delta\alpha$ is uniquely determined by the vectors

$$\delta\alpha_{m,k} := \delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,k}) \;\in\; \mathsf{KGra}(m,k). \tag{7.10}$$

In addition, since the restriction

$$\delta\alpha\big|_{\mathfrak{oc}_\circ^\vee(m,k)^{\mathfrak{o}}}$$

is $S_m \times S_k$ equivariant and $\delta\alpha$ has degree 1, $\delta\alpha_{m,k}$ may be viewed as a *degree zero* vector in

$$\mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}. \tag{7.11}$$

Condition (7.5) is equivalent to $\delta\alpha \in \mathcal{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}_\circ^\vee, \mathsf{KGra})$. In other words, we know that

$$\delta\alpha_{m,k} = 0 \qquad \forall m \leq n-1,\; k \geq 0. \tag{7.12}$$

So vectors $\delta\alpha_{m,k}$ may be non-zero only for $m \geq n$.

Since both α and $\widetilde{\alpha}$ satisfy the MC equations

$$[\alpha, \alpha] = 0, \qquad [\,\widetilde{\alpha}, \widetilde{\alpha}\,] = 0,$$

the difference $\delta\alpha$ satisfies the equation

$$[\alpha,\delta\alpha]+\frac{1}{2}[\delta\alpha,\delta\alpha]=0. \tag{7.13}$$

Since $\delta\alpha \in \mathscr{F}^{\mathfrak{c}}_n\,\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ},\mathsf{KGra})$,

$$[\delta\alpha,\delta\alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k})=0 \qquad \forall k\geq 0. \tag{7.14}$$

Moreover, it is easy to see that only $\mathscr{D}_{\mathsf{As}}(\mathsf{t}^{\mathfrak{o}}_{n,k})$ may contribute to the expression $[\alpha,\delta\alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k})$.

Unfolding $[\alpha,\delta\alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k})$, we get

$$\begin{aligned}[\alpha,\delta\alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) &= -\big(\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_2)\circ_{2,\mathfrak{o}}\delta\alpha_{n,k-1}+\sum_{i=1}^{k-1}(-1)^i\delta\alpha_{n,k-1}\circ_{i,\mathfrak{o}}\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_2)\\ &\qquad+(-1)^k\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_2)\circ_{1,\mathfrak{o}}\delta\alpha_{n,k-1}\big)\\ &= -\big(\Gamma_{\circ\,\circ}\circ_{2,\mathfrak{o}}\delta\alpha_{n,k-1}+\sum_{i=1}^{k-1}(-1)^i\delta\alpha_{n,k-1}\circ_{i,\mathfrak{o}}\Gamma_{\circ\,\circ}\\ &\qquad+(-1)^k\Gamma_{\circ\,\circ}\circ_{1,\mathfrak{o}}\delta\alpha_{n,k-1}\big).\end{aligned}$$

Thus (7.13), (7.14), and the above calculation imply the following statement:

Claim 7.2. – *For each $k\geq 0$, the vector $\delta\alpha_{n,k}$ is a degree zero cocycle in the cochain complex*

$$\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}}=\mathbf{s}^{2n-2}\bigoplus_{k\geq 0}\mathbf{s}^k\big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n} \tag{7.15}$$

with the differential ∂^{Hoch} given by the formula

(7.16)
$$\begin{aligned}\partial^{\mathrm{Hoch}}(\gamma) &= \Gamma_{\circ\,\circ}\circ_{2,\mathfrak{o}}\gamma-\gamma\circ_{1,\mathfrak{o}}\Gamma_{\circ\,\circ}+\gamma\circ_{2,\mathfrak{o}}\Gamma_{\circ\,\circ}-\cdots\\ &\qquad+(-1)^k\gamma\circ_{k,\mathfrak{o}}\Gamma_{\circ\,\circ}+(-1)^{k+1}\Gamma_{\circ\,\circ}\circ_{1,\mathfrak{o}}\gamma, \quad \gamma\in\mathbf{s}^{2n-2+k}\big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n}.\end{aligned}$$

The cochain complex (7.15) is examined in detail in Appendix A. For now, we use Corollary A.10 and Claim 7.2 to deduce that

Claim 7.3. – *The white vertex of each graph in the linear combination*

$$\delta\alpha_{n,1}\in\mathbf{s}^{2n-1}\big(\mathsf{KGra}(n,1)^{\mathfrak{o}}\big)^{S_n}$$

has valency 1.

7.2.1. Pikes in $\delta\alpha_{n,0}$ can be "killed". – In general linear combinations $\delta\alpha_{m,k}$ (7.10) may contain graphs with a black vertex of valency 1 whose adjacent edge terminates at this vertex. We call such vertices *pikes.*

The following statement says that, if Equation (7.12) holds, then pikes in $\delta\alpha_{n,0}$ can be "killed". More precisely,

CLAIM 7.4. – *If Equation* (7.12) *holds, then there exists a degree zero vector*

$$\xi \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}),$$

such that (5.13) *holds, each graph in the linear combination*

$$\big(\exp([\xi,\ ])\widetilde{\alpha}\big)(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) - \alpha(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) \tag{7.17}$$

does not have pikes, and

$$\big(\exp([\xi,\ ])\widetilde{\alpha}\big) - \alpha \ \in\ \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}). \tag{7.18}$$

Proof. – Let us denote by $\delta\alpha^r_{n,0}$ the linear combination in $\mathsf{KGra}(n,0)^{\mathfrak{o}}$ which is obtained from $\delta\alpha_{n,0}$ by retaining only graphs with exactly r pikes.

Since $\delta\alpha^r_{n,0}$ is a linear combination of graphs without white vertices, it is a cocycle in the complex (B.1) with the differential $\mathfrak{d}$ (B.5) examined in detail in Appendix B. According to Lemma B.3 from this appendix, we have

$$\mathfrak{d}\mathfrak{d}^*(\delta\alpha^r_{n,0}) = r\delta\alpha^r_{n,0}. \tag{7.19}$$

Thus, for the vector

$$\chi_{n-1,1} = -\sum_{r\ge 1}\frac{1}{r}\mathfrak{d}^*(\delta\alpha^r_{n,0}) \in \mathbf{s}^{2(n-1)-1}\big(\mathsf{KGra}(n-1,1)^{\mathfrak{o}}\big)^{S_{n-1}}, \tag{7.20}$$

the linear combination

$$\delta\alpha_{n,0} + \mathfrak{d}(\chi_{n-1,1})$$

does not have pikes.

Next, we define the degree 0 vector $\xi \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ by setting

$$\xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n-1,1}) = \chi_{n-1,1}, \qquad \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{c}}_{m_1}) = \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{k_1}) = \xi(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{m,k}) = 0 \tag{7.21}$$

for all m_1, k_1 and for all pairs $(m,k) \neq (n-1,1)$. By construction, $\xi \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ and satisfies (5.13).

Let us denote by $\widetilde{\alpha}'$ and $\delta\alpha'$ the new MC element

$$\widetilde{\alpha}' := \exp(\mathrm{ad}_{\xi})\widetilde{\alpha} \tag{7.22}$$

and the new difference $\delta\alpha' = \widetilde{\alpha}' - \alpha$, respectively. Clearly,

$$\delta\alpha' = \exp(\mathrm{ad}_{\xi})\delta\alpha + \exp(\mathrm{ad}_{\xi})\alpha - \alpha. \tag{7.23}$$

To prove that each graph in the linear combination $\delta\alpha'(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$ does not have pikes, we observe that, for every $f \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$,

$$[\xi, f](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) = \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\big(\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{1,\mathfrak{o}} f(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0})\big),$$

where $\tau_{n,i}$ is the following family of cycles in S_n

$$\tau_{n,i} := (i, i+1, \ldots, n-1, n). \tag{7.24}$$

Therefore

$$\begin{aligned}\delta\alpha'(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) &= \delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) + [\xi, \alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) \\ &= \delta\alpha_{n,0} + \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\big(\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0})\big) \\ &= \delta\alpha_{n,0} + \sum_{i=1}^{n} (\tau_{n,i}, \mathrm{id})\big(\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_{0}\big) = \delta\alpha_{n,0} + \mathfrak{d}(\chi_{n-1,1}).\end{aligned}$$

As we showed above, all graphs in this linear combination do not have pikes.

It remains to show that $\delta\alpha' \in \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$.

Since $\xi \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ and $\widetilde{\alpha} \in \mathscr{F}^{\mathfrak{c}}_{0}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$,

$$\mathrm{ad}^{k}_{\xi}(\widetilde{\alpha}) \in \mathscr{F}^{\mathfrak{c}}_{k(n-1)}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}).$$

For $k \geq 2$, this observation (together with the inequality $n \geq 2$) implies that $\mathrm{ad}^{k}_{\xi}(\widetilde{\alpha}) \in \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$.

Since $[\xi, \widetilde{\alpha}] \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$ and $[\xi, \widetilde{\alpha}](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_{m}) = 0$ for all m, it is sufficient to show that

$$[\xi, \widetilde{\alpha}](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,k}) = 0, \qquad \forall\, k \geq 0. \tag{7.25}$$

Using the defining equations of ξ (see (7.21)), it is easy to see that $[\xi, \widetilde{\alpha}](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,k}) = 0$ for all $k \neq 2$. As for $k = 2$, we have

$$[\xi, \alpha]\big(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,2}\big) = \chi_{n-1,1} \circ_{1,\mathfrak{o}} \Gamma_{\circ\,\circ} - \Gamma_{\circ\,\circ} \circ_{1,\mathfrak{o}} \chi_{n-1,1} - \Gamma_{\circ\,\circ} \circ_{2,\mathfrak{o}} \chi_{n-1,1} = 0$$

because the white vertex in each graph of the linear combination $\chi_{n-1,1}$ has valency 1.

Claim 7.4 is proved. □

7.2.2. $\delta\alpha_{n,0}$ is a degree zero cocycle in dfGC. – Since $\delta\alpha_{n,0} = \delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$ is a linear combination of graphs with only black vertices and it is invariant with respect to the action of S_n, we may view $\delta\alpha_{n,0}$ as a vector in the full directed graph complex dfGC (see (6.1)).

We will now prove the following statement:

CLAIM 7.5. – *Let α and $\widetilde{\alpha}$ be MC elements in* (5.6) *corresponding to SFQs. If $\delta\alpha$ satisfies* (7.12) *and all graphs in the linear combination $\delta\alpha_{n,0}$ do not have pikes, then $\delta\alpha_{n,0}$ is a degree zero cocycle in* dfGC (6.1).

Proof. – The claim that $\delta\alpha_{n,0}$ is a degree zero vector in dfGC follows immediately from the fact that $\mathsf{t}^{\mathfrak{o}}_{n,0}$ has degree $2-2n$. So we will proceed to the proof of the cocycle condition for $\delta\alpha_{n,0}$.

Since

$$\begin{aligned}\mathscr{D}(\mathsf{t}^{\mathfrak{o}}_{n+1,0}) = & \sum_{p=2}^{n+1}\sum_{\tau\in\mathrm{Sh}_{p,n+1-p}}(\tau,\mathrm{id})\big(\mathsf{t}^{\mathfrak{o}}_{n+2-p,0}\circ_{1,\mathfrak{c}}\mathsf{t}^{\mathfrak{c}}_{p}\big)\\ & -\sum_{r=1}^{n}\sum_{\sigma\in\mathrm{Sh}_{r,n+1-r}}(\sigma,\mathrm{id})\big(\mathsf{t}^{\mathfrak{o}}_{r,1}\circ_{1,\mathfrak{o}}\mathsf{t}^{\mathfrak{o}}_{n+1-r,0}\big),\end{aligned}$$

$\delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_{m}) = 0$ for all m and (7.12) holds, we have

$$\delta\alpha\mathbf{s}^{-1}\otimes\delta\alpha\mathbf{s}^{-1}\,(\mathscr{D}(\mathsf{t}^{\mathfrak{o}}_{n+1,0})) = 0,$$

$$\begin{aligned}\delta\alpha\mathbf{s}^{-1}\otimes\alpha\mathbf{s}^{-1}\,(\mathscr{D}(\mathsf{t}^{\mathfrak{o}}_{n+1,0})) = & \sum_{\tau\in\mathrm{Sh}_{2,n-1}}(\tau,\mathrm{id})\big(\delta\alpha_{n,0}\circ_{1,\mathfrak{c}}\Gamma_{\bullet\!-\!\bullet}\big)\\ & -\sum_{i=1}^{n+1}(\tau_{n+1,i},\mathrm{id})\big(\delta\alpha_{n,1}\circ_{1,\mathfrak{o}}\Gamma^{\mathrm{br}}_{0}\big),\end{aligned}$$

and

$$\alpha\mathbf{s}^{-1}\otimes\delta\alpha\mathbf{s}^{-1}\,(\mathscr{D}(\mathsf{t}^{\mathfrak{o}}_{n+1,0})) = -\sum_{i=1}^{n+1}(\sigma_{n+1,i},\mathrm{id})\big(\Gamma^{\mathrm{br}}_{1}\circ_{1,\mathfrak{o}}\delta\alpha_{n,0}\big), \tag{7.26}$$

where $\sigma_{n+1,i}$ and $\tau_{n+1,i}$ are the cycles $(i, i-1, \ldots, 1)$ and $(i, i+1, \ldots, n+1)$ in S_{n+1}, respectively.

Thus, applying the right hand side of (7.13) to $\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}$, we get the identity for vectors $\delta\alpha_{n,0}$ and $\delta\alpha_{n,1}$:

$$\begin{aligned}\sum_{\tau\in\mathrm{Sh}_{2,n-1}}(\tau,\mathrm{id})\big(\delta\alpha_{n,0}\circ_{1,\mathfrak{c}}\Gamma_{\bullet\!-\!\bullet}\big) & -\sum_{i=1}^{n+1}(\sigma_{n+1,i},\mathrm{id})\big(\Gamma^{\mathrm{br}}_{1}\circ_{1,\mathfrak{o}}\delta\alpha_{n,0}\big)\\ & -\sum_{i=1}^{n+1}(\tau_{n+1,i},\mathrm{id})\big(\delta\alpha_{n,1}\circ_{1,\mathfrak{o}}\Gamma^{\mathrm{br}}_{0}\big) = 0.\end{aligned} \tag{7.27}$$

Notice that Γ^{br}_0 consists of a single black vertex and the insertion $\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0$ is nothing but replacing the single white vertex in each graph of the linear combination $\delta\alpha_{n,1}$ by black vertex with label $n+1$.

On the other hand, Claim 7.3 says that all white vertices in $\delta\alpha_{n,1}$ have valency 1. Thus, for each graph in $\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0$ the black vertex with label $n+1$ is necessarily a pike.

Since $\delta\alpha_{n,0}$ does not have pikes, the sum

$$-\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\big)$$

should necessarily cancel the linear combination L_{pikes} which is obtained from

$$\sum_{\tau \in \mathrm{Sh}_{2,n-1}} (\tau, \mathrm{id})\big(\delta\alpha_{n,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\big) \tag{7.28}$$

by retaining only the graphs with pikes.

It is not hard to see that the linear combination L_{pikes} coincides with the sum [2]

$$\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma^{>}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big), \tag{7.29}$$

where $\tau_{n+1,i}$ is the cycle $(i, i+1, \dots, n+1)$ in S_{n+1} and

$$\Gamma^{>}_{\bullet\!-\!\bullet} := \overset{1}{\bullet}\!\longrightarrow\!\overset{2}{\bullet}\,. \tag{7.30}$$

Thus we conclude that

$$\sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\delta\alpha_{n,1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\big) = \sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma^{>}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big).$$

On the other hand, we have

$$\sum_{i=1}^{n+1} (\sigma_{n+1,i}, \mathrm{id})\big(\Gamma^{\mathrm{br}}_1 \circ_{1,\mathfrak{o}} \delta\alpha_{n,0}\big) = \sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma^{<}_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big),$$

where

$$\Gamma^{<}_{\bullet\!-\!\bullet} = \overset{1}{\bullet}\!\longleftarrow\!\overset{2}{\bullet}\,. \tag{7.31}$$

Therefore identity (7.27) implies that

$$\sum_{\tau \in \mathrm{Sh}_{2,n-1}} (\tau, \mathrm{id})\big(\delta\alpha_{n,0} \circ_{1,\mathfrak{c}} \Gamma_{\bullet\!-\!\bullet}\big) - \sum_{i=1}^{n+1} (\tau_{n+1,i}, \mathrm{id})\big(\Gamma_{\bullet\!-\!\bullet} \circ_{1,\mathfrak{c}} \delta\alpha_{n,0}\big) = 0.$$

In other words, $\delta\alpha_{n,0}$ is indeed a cocycle in dfGC. □

Claim 7.5 has the following corollary.

2. Here we use the fact that $\delta\alpha_{n,0}$ carries an even degree.

Corollary 7.6. – *Let α and $\widetilde{\alpha}$ be MC elements in* (5.6) *corresponding to SFQs. If $\delta\alpha$ satisfies* (7.12), *and all graphs in the linear combination $\delta\alpha_{n,0}$ do not have pikes, then there exists degree zero cocycle $\gamma \in \mathscr{F}_{n-1}\mathsf{dfGC}$ such that*

$$\left(\widetilde{\alpha} - \left(\exp(\mathrm{ad}_{J(\gamma)})\alpha\right)\right)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) \;=\; 0. \tag{7.32}$$

and

$$\widetilde{\alpha} - \left(\exp(\mathrm{ad}_{J(\gamma)})\alpha\right) \;\in\; \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}) \tag{7.33}$$

Proof. – Due to Claim 7.5 the linear combination $\delta\alpha_{n,0}$ is a degree zero cocycle in dfGC. So we set $\gamma := -\delta\alpha_{n,0}$. In other words,

$$\gamma(1_m) := \begin{cases} -\delta\alpha_{n,0} & \text{if } m = n \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where 1_m denotes the generator $\mathbf{s}^{2-2m}1 \in \mathbf{s}^{2-2m}\mathbb{K} \cong \Lambda^2\mathsf{coCom}(m)$. By construction, γ belongs to $\mathscr{F}_{n-1}\mathsf{dfGC}$.

For any degree 1 element $f \in \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ we have

$$[J(\gamma), f](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) = -f(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} \gamma \qquad \text{and} \qquad [J(\gamma), f](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0}) = 0.$$

Therefore,

$$\begin{aligned} \widetilde{\alpha}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) - \left(\exp(\mathrm{ad}_{J(\gamma)})\alpha\right)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) &= \delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) + \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} \gamma \\ &= \delta\alpha_{n,0} + \Gamma^{\mathrm{br}}_0 \circ_{1,\mathfrak{c}} \gamma \\ &= \delta\alpha_{n,0} + \gamma = 0 \end{aligned} \tag{7.34}$$

and Equation (7.32) follows.

To prove (7.33) we observe that

$$J(\gamma) \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}).$$

Combining this observation with the fact that $\alpha \in \mathscr{F}^{\mathfrak{c}}_{0}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ and the inequality $n \geq 2$, it is easy to see that

$$\mathrm{ad}^q_{J(\gamma)}(\alpha) \in \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \qquad \forall\, q \geq 2$$

and

$$[J(\gamma), \alpha] \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}). \tag{7.35}$$

Since γ is a cocycle in dfGC, $[J(\gamma), \alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_{m}) = 0$ for all m. Furthermore,

$$[J(\gamma), \alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,k}) = 0 \qquad \forall k \geq 0 \tag{7.36}$$

since the vector $\mathsf{t}^{\mathfrak{c}}_{n}$ does not show up in $\mathscr{D}(\mathsf{t}^{\mathfrak{o}}_{n-1,k})$.

Thus (7.33) follows from (7.35) and (7.36). □

7.3. Taking care of vectors $\delta\alpha(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,k})$ for $k \geq 1$

Let us prove that following auxiliary statement:

CLAIM 7.7. – *Let q be an integer ≥ 1 and α, $\widetilde{\alpha}$ be MC elements of* (5.6) *corresponding to SFQs. If $\delta\alpha$ satisfies* (7.12) *and*

$$\delta\alpha_{n,k} = 0 \qquad \forall k \leq q-1, \tag{7.37}$$

then there exists a degree zero vector

$$\xi \in \mathcal{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \cap \mathcal{F}_{n+q-2}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}), \tag{7.38}$$

such that (5.13) *holds,*

$$\big(\exp([\xi,\ ])\widetilde{\alpha}\big) - \alpha \ \in\ \mathcal{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}) \tag{7.39}$$

and

$$\big(\exp([\xi,\ ])\widetilde{\alpha}\big)(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,k}) \ =\ \alpha(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,k}) \qquad \forall k \leq q. \tag{7.40}$$

Proof. – The proof of this claim consists of two steps. First, we show that there exists a degree zero vector

$$\xi^{(1)} \in \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}),$$

such that $\xi^{(1)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{c}}_{m}) = 0$ for all m, $\xi^{(1)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{m_1,k}) = 0$ for all pairs $(m_1, k) \neq (n, q-1)$,

$$\big(\exp([\xi^{(1)},\ ])\widetilde{\alpha}\big) - \alpha \ \in\ \mathcal{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}), \tag{7.41}$$

$$\big(\exp([\xi^{(1)},\ ])\widetilde{\alpha}\big)(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,k}) \ =\ \alpha(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,k}) \qquad \forall k \leq q-1 \tag{7.42}$$

and the difference

$$\big(\exp([\xi^{(1)},\ ])\widetilde{\alpha}\big)(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,q}) - \alpha(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,q}) \tag{7.43}$$

satisfies Properties A.2, A.3, i.e., for each graph in (7.43) white vertices have valency 1 and the linear combination (7.43) is anti-symmetric with respect to permutations of labels on white vertices.

Let us denote by $\widetilde{\alpha}^{(1)}$ and $\delta\alpha^{(1)}$ the new MC element

$$\widetilde{\alpha}^{(1)} := \big(\exp([\xi^{(1)},\ ])\widetilde{\alpha}\big) \tag{7.44}$$

and the new difference [3]

$$\delta\alpha^{(1)} := \big(\exp([\xi^{(1)},\ ])\widetilde{\alpha}\big) - \alpha. \tag{7.45}$$

---

3. We also set $\delta\alpha^{(1)}_{m,k} := \delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{m,k})$.

In the second step, we show that there exists a degree zero vector

$$\xi^{(2)} \in \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}),$$

such that $\xi^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_m) = 0$ for all m, $\xi^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m_1,k}) = 0$ for all pairs $(m_1,k) \neq (n-1, q+1)$,

$$\big(\exp([\xi^{(2)},\ ])\widetilde{\alpha}^{(1)}\big) - \alpha \ \in\ \mathscr{F}^{\mathfrak{c}}_n\mathrm{Conv}(\mathfrak{oc}^\vee_\circ, \mathsf{KGra})$$

and

$$\big(\exp([\xi^{(2)},\ ])\widetilde{\alpha}^{(1)}\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) \ =\ \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) \qquad \forall k \le q.$$

Since both vectors $\xi^{(1)}$ and $\xi^{(2)}$ satisfy (5.13), $\xi^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m_1,k}) = 0$ for all pairs $(m_1,k) \neq (n, q-1)$ and $\xi^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m_1,k}) = 0$ for all pairs $(m_1,k) \neq (n-1, q+1)$,

$$\xi^{(1)},\ \xi^{(2)} \ \in\ \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \cap \mathscr{F}_{n+q-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}).$$

Thus the desired vector (7.38) is obtained by setting

$$\xi := \mathrm{CH}(\xi^{(2)}, \xi^{(1)}).$$

Step 1. – If $q = 1$ then the linear combination $\delta\alpha_{n,q}$ already satisfies Properties A.2, A.3 due to Claim 7.3. So, in this case, we proceed to Step 2. In Step 1, it remains to consider the case $q \ge 2$.

Due to Claim 7.2, the vector

$$\delta\alpha_{n,q} \in \mathbf{s}^{2n-2+q}\Big(\mathsf{KGra}(n,q)^{\mathfrak{o}}\Big)^{S_n} \tag{7.46}$$

is a cocycle in the complex (A.1) with the differential (A.2). Thus, Corollary A.9 implies that there exists a vector

$$\psi_{n,q-1} \in \mathbf{s}^{2n+q-3}\Big(\mathsf{KGra}(n,q-1)^{\mathfrak{o}}\Big)^{S_n},$$

such that the difference $\delta\alpha_{n,q} - \partial^{\mathrm{Hoch}}\psi_{n,q-1}$ satisfies Properties A.2, A.3.

So we define $\xi^{(1)}$ by setting

$$\xi^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q-1}) = \psi_{n,q-1}, \qquad \xi^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_{m_1}) = \xi^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{k_1}) = \xi^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,k}) = 0$$

for all $m_1 \ge 2$, $k_1 \ge 2$, and all pairs $(m,k) \neq (n, q-1)$.

It is easy to see that

$$\xi^{(1)} \ \in\ \mathscr{F}^{\mathfrak{c}}_n\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \tag{7.47}$$

and

$$\xi^{(1)} \ \in\ \mathscr{F}_{n+q-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}). \tag{7.48}$$

Since the vector (7.45) can be rewritten as

$$\delta\alpha^{(1)} = \exp(\mathrm{ad}_{\xi^{(1)}})(\delta\alpha) + \exp(\mathrm{ad}_{\xi^{(1)}})(\alpha) - \alpha, \tag{7.49}$$

(7.41) follows from (7.47) and the inclusions

$$\delta\alpha \in \mathscr{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}), \qquad \alpha \in \mathscr{F}^{\mathfrak{c}}_0 \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}).$$

Since $\alpha \in \mathscr{F}_0 \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ (see (5.18)), $n \geq 2$ and $q \geq 2$, inclusion (7.48) implies that

$$\mathrm{ad}^r_{\xi^{(1)}}(\alpha) \in \mathscr{F}_{n+q} \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \qquad \forall r \geq 2 \tag{7.50}$$

and

$$[\xi^{(1)}, \alpha] \in \mathscr{F}_{n+q-2} \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}). \tag{7.51}$$

Moreover, since [4] $\delta\alpha \in \mathscr{F}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ and $n \geq 2$,

$$\mathrm{ad}^r_{\xi^{(1)}}(\delta\alpha) \in \mathscr{F}_{n+q} \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \qquad \forall r \geq 1. \tag{7.52}$$

Combining (7.50), (7.51), (7.52) with

$$\delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall k \leq q-1,$$

we deduce that

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall k \leq q-2, \tag{7.53}$$

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q-1}) = [\xi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q-1}), \tag{7.54}$$

and

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q}) = \delta\alpha_{n,q} + [\xi^{(1)}, \alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q}). \tag{7.55}$$

Thus, to prove (7.42), it remains to show that $\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q-1}) = 0$. The latter follows from (7.54) and the fact that $\mathsf{t}^{\mathfrak{o}}_{n,q-1}$ does not show up in $\mathscr{D}(\mathsf{t}^{\mathfrak{o}}_{n,q-1})$.

Finally computing the right hand side of (7.55), we deduce that

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q}) = \delta\alpha_{n,q} - \partial^{\mathrm{Hoch}}\psi_{n,q-1},$$

which means that $\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q})$ satisfies Properties A.2, A.3.

Step 2. – Since $\widetilde{\alpha}^{(1)}$ and α satisfy MC equations in $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$, the difference $\delta\alpha^{(1)}$ satisfies the equation

$$[\alpha, \delta\alpha^{(1)}] + \frac{1}{2}[\delta\alpha^{(1)}, \delta\alpha^{(1)}] = 0. \tag{7.56}$$

Due to (7.41), $\delta\alpha^{(1)} \in \mathscr{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ and hence $[\delta\alpha^{(1)}, \delta\alpha^{(1)}](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,q-1}) = 0$. So applying the left hand side of (7.56) to $\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,q-1}$ and using (see (7.42))

$$\delta\alpha^{(1)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall k \leq q-1,$$

4. Note that $\delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0})$ can be non-zero.

we get the identity

$$\sum_{i=1}^{n+1}\sum_{p=0}^{q-1}(-1)^p\,(\tau_{n+1,i},\mathrm{id})\big(\delta\alpha^{(1)}_{n,q}\circ_{p+1,\mathfrak{o}}\Gamma^{\mathrm{br}}_0\big)=\Gamma_{\circ\circ}\circ_{2,\mathfrak{o}}\delta\alpha^{(1)}_{n+1,q-2}+\sum_{p=1}^{q-2}(-1)^p\delta\alpha^{(1)}_{n+1,q-2}\circ_{p,\mathfrak{o}}\Gamma_{\circ\circ}+(-1)^{q-1}\Gamma_{\circ\circ}\circ_{1,\mathfrak{o}}\delta\alpha^{(1)}_{n+1,q-2}, \tag{7.57}$$

where $\tau_{n+1,i}$ is the cycle $(i,i+1,\dots,n+1)$ in S_{n+1}.

In other words, the vector

$$\rho_{n+1,q-1}=\sum_{i=1}^{n+1}\sum_{p=1}^{q}(-1)^{p+1}\,(\tau_{n+1,i},\mathrm{id})\big(\delta\alpha^{(1)}_{n,q}\circ_{p,\mathfrak{o}}\Gamma^{\mathrm{br}}_0\big) \tag{7.58}$$

is ∂^{Hoch}-exact in

$$\mathbf{s}^{2(n+1)-2+(q-1)}\big(\mathsf{KGra}(n+1,q-1)\big)^{S_{n+1}}. \tag{7.59}$$

Since the difference (7.43) coincides with $\delta\alpha^{(1)}_{n,q}$, the vector $\delta\alpha^{(1)}_{n,q}$ satisfies Properties A.2, A.3. So, using the antisymmetry of $\delta\alpha^{(1)}_{n,q}$ with respect to the action of S_q on the labels of white vertices, we see that

$$\sum_{p=1}^{q}(-1)^{p+1}\,(\tau_{n+1,i},\mathrm{id})\big(\delta\alpha^{(1)}_{n,q}\circ_{p,\mathfrak{o}}\Gamma^{\mathrm{br}}_0\big)=q\,(\tau_{n+1,i},\mathrm{id})\big(\delta\alpha^{(1)}_{n,q}\circ_{1,\mathfrak{o}}\Gamma^{\mathrm{br}}_0\big).$$

In other words,

$$\rho_{n+1,q-1}=\mathfrak{d}(\delta\alpha^{(1)}_{n,q}),$$

where $\mathfrak{d}$ is the operation defined in (B.5) in Appendix B.

Hence $\rho_{n+1,q-1}$ is a vector in (7.59) satisfying Properties A.2, A.3. Combining this observation with the fact that $\rho_{n+1,q-1}$ is ∂^{Hoch}-exact and using the second claim in Corollary A.9 we conclude that

$$\rho_{n+1,q-1}=0.$$

In other words, $\delta\alpha^{(1)}_{n,q}$ is a cocycle in the cochain complex (B.1) with the differential $\mathfrak{d}$ (B.5).

Since $q\geq 1$, Corollary B.5 from Appendix B implies that there exists a vector (of degree -1)

$$\psi_{n-1,q+1}\in\mathbf{s}^{2(n-1)-2+q+1}\left(\mathsf{KGra}(n-1,q+1)^{\mathfrak{o}}\right)^{S_{n-1}},$$

which satisfies Properties A.2, A.3 and such that

$$\delta\alpha^{(1)}_{n,q}=\mathfrak{d}(\psi_{n-1,q+1}). \tag{7.60}$$

Using $\psi_{n-1,q+1}$, we define the following degree zero vector

$$\xi^{(2)} \in \mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$$

by setting

(7.61)

$$\xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,q+1}) = -\psi_{n-1,q+1}, \qquad \xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{c}}_{m_1}) = \xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{k_1}) = \xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{m,k}) = 0$$

for all m_1, k_1 and for all pairs $(m,k) \neq (n-1, q+1)$.

It is obvious that

$$\xi^{(2)} \in \mathscr{F}_{n+q-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{7.62}$$

and

$$\xi^{(2)} \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}). \tag{7.63}$$

Next we consider the MC element

$$\widetilde{\alpha}^{(2)} = \exp(\mathrm{ad}_{\xi^{(2)}})(\widetilde{\alpha}^{(1)})$$

and rewrite the difference $\delta\alpha^{(2)} := \widetilde{\alpha}^{(2)} - \alpha$ as follows

$$\delta\alpha^{(2)} = \exp(\mathrm{ad}_{\xi^{(2)}})(\delta\alpha^{(1)}) + \exp(\mathrm{ad}_{\xi^{(2)}})(\alpha) - \alpha. \tag{7.64}$$

Since $\delta\alpha^{(1)} \in \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ (see (7.41)) and $\xi^{(2)} \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$,

$$\delta\alpha^{(2)} \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{7.65}$$

or equivalently $\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{m,k}) = 0$ for all $m < n-1$ and $k \geq 0$.

Moreover,

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,k}) = [\xi^{(2)}, \alpha](\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,k}). \qquad \forall\, k \geq 0. \tag{7.66}$$

Since $\xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{m,k}) = 0$ for all $(m,k) \neq (n-1, q+1)$,

$$[\xi^{(2)}, \alpha](\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,k}) = 0 \qquad \forall k \neq q+2 \tag{7.67}$$

and

$$[\xi^{(2)}, \alpha](\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,q+2}) = -\partial^{\mathrm{Hoch}}\xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,q+2}). \tag{7.68}$$

On the other hand $\partial^{\mathrm{Hoch}}\xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,q+2}) = 0$ since the vector $\xi^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,q+2}) = \psi_{n-1,q+2}$ satisfies Property A.2. Thus, combining (7.65) with (7.66), (7.67) and (7.68), we deduce that

$$\delta\alpha^{(2)} \in \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}).$$

To complete Step 2, it remains to show that

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall k \leq q. \tag{7.69}$$

Since $\alpha \in \mathscr{F}_0\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ (see (5.18)), $\delta\alpha^{(1)} \in \mathscr{F}_n\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ and $n+q-1 \geq 2$, inclusion (7.62) implies that

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) = 0 \qquad \forall k < q \tag{7.70}$$

and

$$\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q}) = \delta\alpha^{(1)}_{n,q} + [\xi^{(2)}, \alpha]\big(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q}\big). \tag{7.71}$$

Unfolding the right hand side of (7.71) and using the fact that $\psi_{n-1,q+1}$ is anti-symmetric with respect to permutations of labels on white vertices, we get

$$\begin{aligned}\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q}) &= \delta\alpha^{(1)}_{n,q} + \sum_{p=0}^{k}\sum_{i=1}^{n}(-1)^p(\tau_{n,i},\mathrm{id})\Big(\xi^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,q+1}) \circ_{p+1,\mathfrak{o}} \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0})\Big)\\ &= \delta\alpha^{(1)}_{n,q} - \sum_{p=0}^{q}\sum_{i=1}^{n}(-1)^p(\tau_{n,i},\mathrm{id})\big(\psi_{n-1,q+1} \circ_{p+1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\big)\\ &= (q+1)\sum_{i=1}^{n}(\tau_{n,i},\mathrm{id})\big(\psi_{n-1,q+1} \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0\big),\end{aligned}$$

where $\tau_{n,i}$ is the cycle $(i, i+1, \dots, n-1, n)$ in S_n.

Hence $\delta\alpha^{(2)}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,q}) = 0$ follows from Equation (7.60).

Since (7.69) is proved, Step 2 is complete and so is the proof of Claim 7.7. □

We can now prove the following statement:

Claim 7.8. – *Let α and $\widetilde{\alpha}$ be MC elements corresponding to SFQs. If $\delta\alpha := \widetilde{\alpha} - \alpha$ belongs to* $\mathscr{F}^{\mathfrak{c}}_n\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ *and*

$$\delta\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) = 0,$$

then there exists a degree zero vector $\xi \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ satisfying condition (5.13) *and such that*

$$\exp(\mathrm{ad}_\xi)(\widetilde{\alpha}) - \alpha \;\in\; \mathscr{F}^{\mathfrak{c}}_{n+1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}). \tag{7.72}$$

Proof. – Claim 7.7 implies that there exists an infinite sequence of degree zero vectors

$$\xi_1, \xi_2, \dots, \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}),$$

such that each ξ_r satisfies (5.13),

$$\xi_q \in \mathscr{F}_{n+q-2}\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \qquad \forall q \geq 1,$$

$$\big(\exp(\mathrm{ad}_{\xi_q})\cdots\exp(\mathrm{ad}_{\xi_1})(\widetilde{\alpha}) - \alpha\big) \in \mathscr{F}^{\mathfrak{c}}_n\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}).$$

and

$$\big(\exp(\mathrm{ad}_{\xi_q})\cdots\exp(\mathrm{ad}_{\xi_1})(\widetilde{\alpha})\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k}) = \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,k})$$

for all $k \leq q$.

Since the graded Lie algebra $\mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ is complete with respect to the filtration

$$\mathscr{F}_{\bullet}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \cap \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}),$$

the limit ξ of the sequence

$$\{\, \mathrm{CH}(\xi_q, \mathrm{CH}(\xi_{q-1}, \dots, \mathrm{CH}(\xi_2, \xi_1)..)\,\}_{q\ge 1}$$

exists in $\mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ and it satisfies (7.72). $\square$

7.4. The end of the proof of Proposition 7.1

Let us now put together the results of Sections 7.2 and 7.3 to finish the proof of Proposition 7.1.

Due to Claim 7.4, there exists a degree zero vector

$$\xi^{-\bullet} \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}),$$

such that $\xi^{-\bullet}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_m) = 0$ for all m,

$$\big(\exp(\mathrm{ad}_{\xi^{-\bullet}})\widetilde{\alpha}\,\big) - \alpha \;\in\; \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$$

and each graph in the difference

$$\big(\exp(\mathrm{ad}_{\xi^{-\bullet}})\widetilde{\alpha}\,\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) - \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$$

does not have pikes.

Applying Corollary 7.6 to the MC elements α and setting

$$\widetilde{\alpha}^{\diamond} := \exp(\mathrm{ad}_{\xi^{-\bullet}})\,\widetilde{\alpha}, \tag{7.73}$$

we deduce that there exists a degree zero cocycle $\gamma \in \mathscr{F}_{n-1}\mathsf{dfGC}$ such that

$$\widetilde{\alpha}^{\diamond} - \big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big) \;\in\; \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$$

and

$$\widetilde{\alpha}^{\diamond}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) \;=\; \big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}).$$

Finally, applying Claim 7.8 to the MC elements

$$\alpha^{\diamond} := \exp(\mathrm{ad}_{J(\gamma)})\alpha$$

and $\widetilde{\alpha}^{\diamond}$, we deduce that there exists a degree zero vector

$$\xi^{\sharp} \in \mathscr{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}),$$

such that $\xi^{\sharp}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_m) = 0$ for all m and

$$\exp(\mathrm{ad}_{\xi^{\sharp}})(\widetilde{\alpha}^{\diamond}) - \alpha^{\diamond} \;\in\; \mathscr{F}^{\mathfrak{c}}_{n+1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}).$$

Thus, setting $\xi := \mathrm{CH}(\xi^{\sharp}, \xi^{-\bullet})$, we get that

$$\big(\exp(\mathrm{ad}_{\xi})(\widetilde{\alpha})\big) - \big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big) \;\in\; \mathscr{F}^{\mathfrak{c}}_{n+1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}).$$

In other words, (7.6) is satisfied.

The proof of transitivity of the action of $\exp\left(H^0(\mathsf{dfGC})\right)$ on homotopy classes of SFQs is now complete.

CHAPTER 8

THE ACTION OF $\exp\big(H^0(\mathsf{dfGC})\big)$ IS FREE

Let α be a MC element of $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ representing an SFQ and γ be a degree zero cocycle in dfGC. Let us assume that there exists a degree zero vector

$$\xi \in \mathscr{F}_1\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}), \tag{8.1}$$

which satisfies condition (5.13) and such that

$$\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha = \exp\big(\mathrm{ad}_\xi\big)\alpha. \tag{8.2}$$

Our goal is to show that γ is exact.

Due to Remark 6.3, we assume, without loss of generality, that we deal exclusively with cocycles γ of dfGC which do not involve graphs with pikes.

To prove that γ is exact, we will need the following technical claims which are proved below in Sections 8.2 and 8.3, respectively.

Claim 8.1. – *Let n be an integer ≥ 2, α be a MC element of* $\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$ *corresponding to an SFQ and*

$$\xi \in \mathscr{F}^{\mathfrak{c}}_n\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}) \tag{8.3}$$

be a degree zero vector satisfying condition (5.13). *There exists a degree zero vector*

$$\tilde{\xi} \in \mathscr{F}^{\mathfrak{c}}_n\mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$$

for which (5.13) *holds, all graphs in*

$$\tilde{\xi}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$$

do not have pikes and

$$\exp\big(\mathrm{ad}_{\tilde{\xi}}\big)\alpha = \exp\big(\mathrm{ad}_\xi\big)\alpha. \tag{8.4}$$

CLAIM 8.2. – *Let γ be a degree zero cocycle in* dfGC*, n be an integer ≥ 2 and*

$$\xi \in \mathcal{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}) \tag{8.5}$$

be a degree zero vector satisfying condition (5.13)*. If Equation* (8.2) *holds and all graphs in $\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$ do not have pikes, then*[1]

$$\gamma \in \mathcal{F}_n \mathsf{dfGC}. \tag{8.6}$$

Moreover, there exist a degree -1 vector $\kappa \in \mathcal{F}_{n-1}\mathsf{dfGC}$ and a degree 0 vector

$$\eta \in \mathcal{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$$

satisfying (5.13)*,*

$$\exp\left(\mathrm{ad}_{J(\mathrm{CH}(\partial\kappa,\gamma))}\right)\alpha = \exp\left(\mathrm{ad}_{\mathrm{CH}(\eta,\xi)}\right)\alpha, \tag{8.7}$$

$$\mathrm{CH}(\partial\kappa,\gamma) \in \mathcal{F}_{n+1}\mathsf{dfGC} \qquad \textit{and} \qquad \mathrm{CH}(\eta,\xi) \in \mathcal{F}^{\mathfrak{c}}_{n+1}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra}). \tag{8.8}$$

8.1. Claims 8.1 and 8.2 imply that the action is free

Claims 8.1 and 8.2 imply that, for every degree 0 cocycle γ satisfying (8.2), there exists a sequence

$$\{\kappa_m\}_{m\geq 1}, \qquad \kappa_m \in \mathcal{F}_m \mathsf{dfGC}, \tag{8.9}$$

such that for every $n \geq 1$

$$\mathrm{CH}(\partial\kappa_n, \dots, \mathrm{CH}(\partial\kappa_2, \mathrm{CH}(\partial\kappa_1,\gamma)\cdots) \in \mathcal{F}_{n+2}\mathsf{dfGC}.$$

Since dfGC is complete with respect to the filtration $\mathcal{F}_\bullet\mathsf{dfGC}$, the existence of this sequence implies that γ is indeed a coboundary.

8.2. Proof of Claim 8.1

If $\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$ does not involve graphs with pikes then we set $\tilde{\xi} := \xi$ and Equation (8.4) obviously holds.

Otherwise, we observe that, since $[\alpha,\alpha] = 0$, $\mathrm{ad}_{[\psi,\alpha]}$ acts trivially on α for every degree -1 vector $\psi \in \mathcal{F}_1\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra})$. Hence, we have

$$\exp\left(\mathrm{ad}_{\mathrm{CH}(\xi,[\psi,\alpha])}\right)(\alpha) = \exp(\mathrm{ad}_{\xi})(\alpha). \tag{8.10}$$

We will prove that there exists a degree -1 vector

$$\psi \in \mathcal{F}^{\mathfrak{c}}_{n-1}\mathrm{Conv}(\mathfrak{oc}^{\vee}_{\circ}, \mathsf{KGra}), \tag{8.11}$$

such that the element

$$\tilde{\xi} := \mathrm{CH}(\xi, [\psi,\alpha]) \tag{8.12}$$

1. I.e., all graphs in γ have $\geq (n+1)$ vertices.

— belongs to $\mathscr{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra})$,
— satisfies condition (5.13), and
— all graphs in $\tilde{\xi}(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,0})$ do not have pikes.

Let us set $\xi_{n,0} := \xi(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n,0})$.

Since the graphs in $\xi_{n,0}$ do not have white vertices, the vector $\xi_{n,0}$ is a cocycle in the complex (B.1) with the differential $\mathfrak{d}$ (B.5) (see Appendix B).

Let us denote by $\xi^r_{n,0}$ the linear combination in $\mathsf{KGra}(n,0)^{\mathfrak{o}}$, which is obtained from $\xi_{n,0}$ by retaining only the graphs with exactly r pikes. According to Lemma B.3 from Appendix B, we have

$$\mathfrak{d}\mathfrak{d}^*(\xi^r_{n,0}) = r\xi^r_{n,0}.$$

Thus, if

$$\psi_{n-1,1} := -\sum_{r\geq 1}\frac{1}{r}\mathfrak{d}^*(\xi^r_{n,0}), \tag{8.13}$$

then each graph in the linear combination

$$\xi_{n,0} + \mathfrak{d}(\psi_{n-1,1})$$

does not have pikes.

Next, we define a degree -1 vector (8.11) by setting

$$\psi(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n-1,1}) = \psi_{n-1,1}, \qquad \psi(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{n_2,k_2}) = 0 \quad \forall (n_2,k_2) \neq (n-1,1), \tag{8.14}$$

and

$$\psi(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{c}}_{n_1}) = \psi(\mathbf{s}^{-1}\,\mathfrak{t}^{\mathfrak{o}}_{k_1}) = 0 \qquad \forall n_1, k_1 \geq 2. \tag{8.15}$$

Then we consider the vector

$$\tilde{\xi} := \mathrm{CH}(\xi, [\psi, \alpha]). \tag{8.16}$$

By construction (8.13), all white vertices in graphs in $\psi_{n-1,1}$ have valency one. Hence $\psi_{n-1,1}$ belongs to the kernel of the differential ∂^{Hoch} (7.16). Using this fact, (8.14) and (8.15), it is not hard to show that

$$[\psi, \alpha] \in \mathscr{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}). \tag{8.17}$$

Therefore,

$$\tilde{\xi} \in \mathscr{F}^{\mathfrak{c}}_n \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}).$$

Equation (8.15) implies that $[\psi, \alpha](\mathfrak{t}^{\mathfrak{c}}_{n_1}) = 0$ for all $n_1 \geq 2$. Combining this observation with the fact that ξ satisfies (5.13), we conclude that $\tilde{\xi}$ also satisfies (5.13).

Using (8.14), (8.15), (8.17), and the inequality $n \geq 2$, we get

$$\begin{aligned}\mathrm{CH}(\xi, [\psi, \alpha])\,(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) &= \xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) + [\psi, \alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) \\ &= \xi_{n,0} + \sum_{i=1}^{n} \tau_{n,i}\big(\psi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n-1,1}) \circ_{\mathfrak{o},1} \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0})\big) \\ &= \xi_{n,0} + \sum_{i=1}^{n} \tau_{n,i}\big(\psi_{n-1,1} \circ_{\mathfrak{o},1} \Gamma^{\mathrm{br}}_{0}\big),\end{aligned}$$

where $\tau_{n,i}$ is the cycle $(i, i+1, \dots, n-1, n)$ in S_n.

Thus, by definition of the operator $\mathfrak{d}$ (B.5), we deduce that

$$\tilde{\xi}(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) = \xi_{n,0} + \mathfrak{d}\psi_{n-1,1}.$$

Since each graph in the linear combination $\xi_{n,0} + \mathfrak{d}\psi_{n-1,1}$ does not have pikes, Claim 8.1 is proved.

8.3. Proof of Claim 8.2

Let m be an integer $\leq n$ such that

$$\gamma(1_k) = 0 \qquad \forall k < m, \tag{8.18}$$

i.e., $\gamma \in \mathscr{F}_{m-1}\mathsf{dfGC}$.

Due to (8.18),

$$\begin{aligned}\big(\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}) &= \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}) - \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0}) \circ_{1,\mathfrak{c}} \gamma(1_m) \\ &= \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}) - \Gamma^{\mathrm{br}}_{1,0} \circ_{1,\mathfrak{c}} \gamma(1_m),\end{aligned}$$

i.e.,

$$\big(\exp\big(\mathrm{ad}_{J(\gamma)}\big)\alpha\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}) = \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}) - \gamma(1_m). \tag{8.19}$$

Since $m \leq n$, $\xi \in \mathscr{F}^{\mathfrak{c}}_{n}\mathrm{Conv}(\mathfrak{oc}^{\vee}, \mathsf{KGra})$ and ξ satisfies (5.13), we have

$$[\xi, \mathrm{ad}^{k}_{\xi}(\alpha)](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}) = 0 \qquad \forall k \geq 0$$

and hence

$$\big(\exp\big(\mathrm{ad}_{\xi}\big)\alpha\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}) = \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{m,0}). \tag{8.20}$$

Combining (8.2), (8.19), and (8.20), we conclude that

$$\gamma(1_k) = 0$$

for all $k \leq m$.

Thus inclusion (8.6) indeed holds, i.e.,

$$\gamma(1_k) = 0 \qquad \forall k \leq n. \tag{8.21}$$

Let us deduce from (8.2) that

CLAIM 8.3. – *The white vertex in every graph in*

$$\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,1})$$

has valency 1.

of Claim 8.3. – Evaluating both sides of (8.2) on $\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,2}$, and using (5.13), (8.5) and (8.6), we deduce that

$$\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,2}) = \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,2}) + [\xi,\alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,2}).$$

Hence,

$$[\xi,\alpha](\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,2}) = 0$$

or equivalently

$$\partial^{\mathrm{Hoch}}\big(\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,1})\big) = 0, \tag{8.22}$$

where ∂^{Hoch} is defined in (7.16).

Combining (8.22) with Corollary A.10 from Appendix A, we conclude that the white vertex in each graph in $\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,1})$ must have valency 1.

Thus Claim 8.3 is proved. □

We will now deduce Claim 8.2 by evaluating both sides of (8.2) on $\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}$.

Using (8.21), it is easy to show that

$$\big(\exp(\mathrm{ad}_{J(\gamma)})\alpha\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}) = \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}) - \gamma(1_{n+1}). \tag{8.23}$$

On the other hand, using (8.5) and (5.13), we get that

$$\begin{aligned}\big(\exp(\mathrm{ad}_{\xi})\alpha\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}) = \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}) - &\sum_{\tau\in\mathrm{Sh}_{2,n-1}} \tau\big(\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0}) \circ_{\mathfrak{c},1} \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{c}}_{2})\big)\\ &- \sum_{i=1}^{n+1} \sigma_{n+1,i}\big(\alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,1}) \circ_{\mathfrak{o},1} \xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})\big)\\ &+ \sum_{i=1}^{n+1} \tau_{n+1,i}\big(\xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,1}) \circ_{\mathfrak{o},1} \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{1,0})\big).\end{aligned}$$

Hence,

$$\begin{aligned}\big(\exp(\mathrm{ad}_{\xi})\alpha\big)(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}) = \alpha(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n+1,0}) - &\sum_{\tau\in\mathrm{Sh}_{2,n-1}} \tau\big(\xi_{n,0} \circ_{\mathfrak{c},1} \Gamma_{\bullet\!-\!\bullet}\big)\\ &- \sum_{i=1}^{n+1} \sigma_{n+1,i}\big(\Gamma^{\mathrm{br}}_{1} \circ_{\mathfrak{o},1} \xi_{n,0}\big) + \sum_{i=1}^{n+1} \tau_{n+1,i}\big(\xi_{n,1} \circ_{\mathfrak{o},1} \Gamma^{\mathrm{br}}_{0}\big),\end{aligned} \tag{8.24}$$

where $\xi_{n,0} := \xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,0})$ and $\xi_{n,1} := \xi(\mathbf{s}^{-1}\,\mathsf{t}^{\mathfrak{o}}_{n,1})$.

Combining (8.23) with (8.24), we conclude that
(8.25)

$$\gamma(1_{n+1}) = \sum_{\tau \in \mathrm{Sh}_{2,n-1}} \tau\big(\xi_{n,0} \circ_{\mathfrak{c},1} \Gamma_{\bullet\!-\!\bullet}\big) + \sum_{i=1}^{n+1} \sigma_{n+1,i}\big(\Gamma_1^{\mathrm{br}} \circ_{\mathfrak{o},1} \xi_{n,0}\big) - \sum_{i=1}^{n+1} \tau_{n+1,i}\big(\xi_{n,1} \circ_{\mathfrak{o},1} \Gamma_0^{\mathrm{br}}\big).$$

By Claim 8.3, the white vertex of every graph in $\xi_{n,1}$ has valency 1. Hence every graph in the last sum in the right hand side of (8.25) has a pike. Therefore, since neither $\gamma(1_{n+1})$ nor $\xi_{n,0}$ involve graphs with pikes, the linear combination

$$\sum_{i=1}^{n+1} \tau_{n+1,i}\big(\xi_{n,1} \circ_{\mathfrak{o},1} \Gamma_0^{\mathrm{br}}\big)$$

is obtained from

$$\sum_{\tau \in \mathrm{Sh}_{2,n-1}} \tau\big(\xi_{n,0} \circ_{\mathfrak{c},1} \Gamma_{\bullet\!-\!\bullet}\big)$$

by keeping only graphs with pikes.

Thus the right hand side of (8.25) equals

$$[\Gamma_{\bullet\!-\!\bullet}, \xi_{n,0}],$$

where $\xi_{n,0}$ is viewed as a vector in dfGC.

We set

$$\kappa := -\xi_{n,0}$$

and recall that, due to the second part of Proposition 6.5, there exists a degree 0 vector

$$\eta \in \mathscr{F}_n^{\mathfrak{c}} \mathrm{Conv}(\mathfrak{oc}^\vee, \mathsf{KGra}), \tag{8.26}$$

which satisfies (5.13),

$$\eta(\mathbf{s}^{-1}\,\mathbf{t}^{\mathfrak{o}}_{n,0}) = -\xi_{n,0} \tag{8.27}$$

and such that (8.7) holds.

The desired inclusions in (8.8) follow easily from $\gamma + \partial\xi_{n,0} = 0$, (8.5), (8.6), (8.26), and (8.27).

Claim 8.2 is proved.

We showed that the action of $\exp\big(H^0(\mathsf{dfGC})\big)$ on homotopy classes of SFQs is free. Thus Theorem 6.8 is proved. □

APPENDIX A

A COCHAIN COMPLEX THAT IS CLOSELY CONNECTED WITH THE HOCHSCHILD COMPLEX OF A COFREE COCOMMUTATIVE COALGEBRA

In this appendix we compute the cohomology of the cochain complex

$$\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}} = \mathbf{s}^{2n-2} \bigoplus_{k \geq 0} \mathbf{s}^k \big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n} \tag{A.1}$$

with the differential ∂^{Hoch} given by the formula

(A.2)

$$\partial^{\mathrm{Hoch}}(\gamma) = \Gamma_{\circ\circ} \circ_{2,\mathfrak{o}} \gamma - \gamma \circ_{1,\mathfrak{o}} \Gamma_{\circ\circ} + \gamma \circ_{2,\mathfrak{o}} \Gamma_{\circ\circ} - \cdots$$
$$+ (-1)^k \gamma \circ_{k,\mathfrak{o}} \Gamma_{\circ\circ} + (-1)^{k+1} \Gamma_{\circ\circ} \circ_{1,\mathfrak{o}} \gamma, \quad \gamma \in \mathbf{s}^{2n-2+k}\big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n}.$$

For this purpose we consider a slightly simpler cochain complex

$$\mathsf{KGra}^{\mathrm{Hoch}} = \mathbf{s}^{2n-2} \bigoplus_{k \geq 0} \mathbf{s}^k \mathsf{KGra}(n,k)^{\mathfrak{o}} \tag{A.3}$$

with the differential ∂^{Hoch} defined by the same Formula (A.2).

The cochain complex (A.3) is equipped with the obvious action of the group S_n and (A.1) is nothing but the complex of S_n-invariants.

Example A.1. – *An example of computation of $\partial^{\mathrm{Hoch}}(\Gamma)$ for a graph $\Gamma \in \mathrm{dgra}_{3,1}$ is shown in Figure A.1. Let us say that we chose this order $(1_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}})$ on the set of edges of Γ. The orders on the sets of edges of graphs in the right hand side are inherited from the total order on the edges of Γ in the obvious way. For example, the first graph in the sum on the right hand side has its edges ordered this way: $(1_{\mathfrak{c}}, 3_{\mathfrak{c}}) < (1_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 2_{\mathfrak{o}}) < (3_{\mathfrak{c}}, 1_{\mathfrak{o}})$.*

Before computing the cohomology of (A.3) let us make a couple of remarks about vectors

$$c \in \mathbf{s}^{2n-2+k}\mathsf{KGra}(n,k)^{\mathfrak{o}} \mathrm{or} c \in \mathbf{s}^{2n-2+k}\big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\big)^{S_n} \tag{A.4}$$

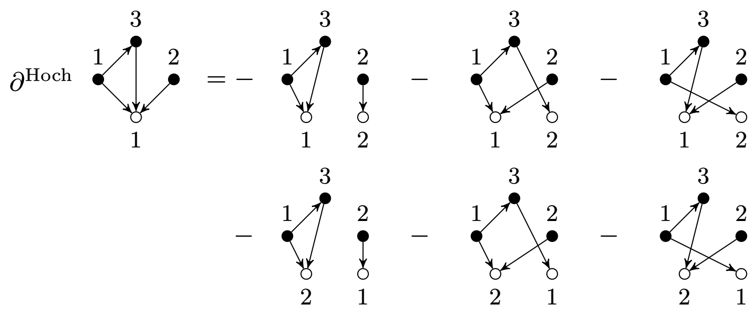


FIGURE A.1. Computing ∂^{Hoch}

satisfying these two properties:

PROPERTY A.2. – *All white vertices in each graph of the linear combination c have valency one.*

PROPERTY A.3. – *For every $\sigma \in S_k$ we have*

$$(\mathrm{id}, \sigma)\,(c) = (-1)^{|\sigma|} c. \tag{A.5}$$

For example, the "brooms" Γ_k^{br} depicted in Figure 5.1 obviously satisfy these properties.

REMARK A.4. – *It is easy to see that every vector* (A.4) *satisfying Properties A.2 and A.3 is closed with respect to ∂^{Hoch}. Furthermore, it is not hard to see that a cocycle c satisfying Properties A.2 and A.3 is trivial if and only if $c = 0$.*

A.1. The Hochschild complex of a cofree cocommutative coalgebra

To compute the cohomology of (A.3) we consider the cofree cocommutative $\mathbb{K}$-coalgebra $\mathscr{C}_r$ with counit co-generated by degree 0 elements $h_1, h_2, \ldots, h_r$.

To the coalgebra $\mathscr{C}_r$ we assign the following cochain complex

$$\mathrm{Hoch}(\mathscr{C}_r) = \bigoplus_{k \geq 0} \mathbf{s}^k \left(\mathscr{C}_r\right)^{\otimes k} \tag{A.6}$$

with the differential

$$\partial^{\mathscr{C}} : \left(\mathscr{C}_r\right)^{\otimes k} \to \left(\mathscr{C}_r\right)^{\otimes (k+1)}$$

given by the formula

$$\partial^{\mathscr{C}}(X) = 1 \otimes X + \sum_{i=1}^{k} (-1)^i (\mathrm{id}, \ldots, \mathrm{id}, \underbrace{\Delta}_{i\text{-th spot}}, \mathrm{id}, \ldots, \mathrm{id})(X) + (-1)^{k+1} X \otimes 1, \tag{A.7}$$

where Δ denotes the comultiplication on $\mathscr{C}_r$.

The complex $\mathrm{Hoch}(\mathscr{C}_r)$ obviously splits into the direct sum of sub-complexes

$$\mathrm{Hoch}(\mathscr{C}_r) = \bigoplus_{m \ge 0} \mathrm{Hoch}(\mathscr{C}_r)_m, \tag{A.8}$$

where $\mathrm{Hoch}(\mathscr{C}_r)_m$ is spanned by tensor monomials with the total degree in co-generators being m.

In [**31**, Section 4.6.1.1] it was proved that

Claim A.5 (Section 4.6.1.1, [**31**]). – *If X is a cocycle in*

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap \ \mathrm{Hoch}(\mathscr{C}_r)_m$$

and $m \neq k$ then X is $\partial^{\mathscr{C}}$-exact. Furthermore, if X is a cocycle in

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap \ \mathrm{Hoch}(\mathscr{C}_r)_m$$

and $m = k$ then there exists

$$\widetilde{X} \in \mathbf{s}^{k-1} \big(\mathscr{C}_r\big)^{\otimes (k-1)} \ \cap \ \mathrm{Hoch}(\mathscr{C}_r)_m,$$

such that

$$X - \partial^{\mathscr{C}}(\widetilde{X}) = \sum_{i_1 i_2 \cdots i_k} \lambda^{i_1 i_2 \cdots i_k} (h_{i_1}, h_{i_2}, \dots, h_{i_k}),$$

where $\lambda^{i_1 i_2 \cdots i_k} \in \mathbb{K}$ and

$$\lambda^{\cdots i_p i_{p+1} \cdots} = -\lambda^{\cdots i_{p+1} i_p \cdots}.$$

Finally a cocycle of the form

$$\sum_{i_1 i_2 \cdots i_k} \lambda^{i_1 i_2 \cdots i_k} (h_{i_1}, h_{i_2}, \dots, h_{i_k}), \qquad \lambda^{\cdots i_p i_{p+1} \cdots} = -\lambda^{\cdots i_{p+1} i_p \cdots} \in \mathbb{K}$$

is exact if and only if all coefficients $\lambda^{i_1 i_2 \cdots i_k} = 0$.

For our purposes we will need the following subcomplex of $\mathrm{Hoch}(\mathscr{C}_r)$:

$$\mathrm{Hoch}'(\mathscr{C}_r) = \{X \in \mathrm{Hoch}(\mathscr{C}_r)_r \mid \text{each co-generator } h_i \text{ appears in the tensor monomial } X \text{ exactly once}\}. \tag{A.9}$$

Using Claim A.5 about cocycles in $\mathrm{Hoch}(\mathscr{C}_r)$ it is easy to deduce an analogous statement for the cochain complex $\mathrm{Hoch}'(\mathscr{C}_r)$:

Claim A.6. – *If X is a cocycle in*

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap \ \mathrm{Hoch}'(\mathscr{C}_r)$$

and $k \neq r$ then X is $\partial^{\mathscr{C}}$-exact. Furthermore, if X is a cocycle in

$$\mathbf{s}^k \big(\mathscr{C}_r\big)^{\otimes k} \ \cap \ \mathrm{Hoch}'(\mathscr{C}_r)$$

and $k = r$ then there exists

$$\widetilde{X} \in \mathbf{s}^{k-1} \big(\mathscr{C}_r\big)^{\otimes (k-1)} \ \cap \ \mathrm{Hoch}'(\mathscr{C}_r),$$

such that

$$X - \partial^{\mathscr{C}}(\widetilde{X}) = \sum_{\sigma \in S_r} (-1)^{|\sigma|} \lambda(h_{\sigma(1)}, h_{\sigma(2)}, \dots, h_{\sigma(r)}),$$

for $\lambda \in \mathbb{K}$. *Finally, the cocycle*

$$\sum_{\sigma \in S_r} (-1)^{|\sigma|} (h_{\sigma(1)}, h_{\sigma(2)}, \dots, h_{\sigma(r)})$$

is non-trivial.

A.2. Computing cohomology of $\mathsf{KGra}^{\mathrm{Hoch}}$ and $\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}}$

Let us now return to the cochain complex $\mathsf{KGra}^{\mathrm{Hoch}}$ (A.3).

It is clear that $\mathsf{KGra}^{\mathrm{Hoch}}$ splits into the direct sum of sub-complexes

$$\mathsf{KGra}^{\mathrm{Hoch}} = \bigoplus_r \mathsf{KGra}^{\mathrm{Hoch}}_r, \tag{A.10}$$

where $\mathsf{KGra}^{\mathrm{Hoch}}_r$ is spanned by graphs with exactly r edges terminating at white vertices.

To compute the cohomology of $\mathsf{KGra}^{\mathrm{Hoch}}_r$ we introduce an auxiliary subspace:

$$\mathsf{KGra}'(n, r) \subset \mathsf{KGra}(n, r), \tag{A.11}$$

which consists of linear combinations of graphs in $\mathrm{dgra}_{n,r}$ with all white vertices (if any) having valency 1.

Let us now suppose that we are given a tensor monomial with k factors

$$X = h_{i_{11}} h_{i_{12}} \cdots h_{i_{1r_1}} \otimes h_{i_{21}} h_{i_{22}} \cdots h_{i_{2r_2}} \otimes \cdots \otimes h_{i_{k1}} h_{i_{k2}} \cdots h_{i_{kr_k}} \in \mathrm{Hoch}'(\mathscr{C}_r) \tag{A.12}$$

and a graph $\Gamma' \in \mathrm{dgra}_{n,r}$ with all white vertices having valency 1. To the pair (X, Γ') we assign a graph $\Gamma \in \mathrm{dgra}_{n,k}$ following these steps:

— First, for each $i \in \{1, 2, \dots, r\}$ we find the number of the tensor factor in (A.12) which contains the co-generator [1] h_i. We denote this number by d_i.

— Second, we erase white vertices of Γ' and attach the resulting free edges to new k white vertices with labels $1, 2, \dots, k$ following this rule: the edged which previously terminated at the white vertex with label i should now terminate at the white vertex with label d_i.

— Finally, in the resulting graph Γ, we keep the same total order on the set of edges as for Γ'.

1. Recall that each co-generator h_i enters the monomial (A.12) exactly once.

Example A.7. – *To a graph* Γ' *depicted in Figure A.2 and the monomial*

$$(h_1 h_2, 1, h_3, 1) \in \mathrm{Hoch}'(\mathscr{C}_3)$$

we should assign the graph Γ *shown on Figure A.3. The total order on the set of edges of* Γ *is inherited from the total order on the set of edges of* Γ'*.*

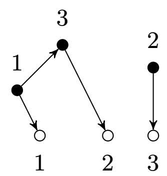


Figure A.2. A graph $\Gamma' \in \mathrm{dgra}_{3,3}$

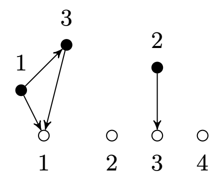


Figure A.3. The graph $\Gamma \in \mathrm{dgra}_{3,4}$

The described procedure gives us an obvious map

$$\Upsilon' : \mathbf{s}^{2n-2}\mathsf{KGra}'(n,r) \otimes \mathrm{Hoch}'(\mathscr{C}_r) \to \mathsf{KGra}^{\mathrm{Hoch}}_r. \tag{A.13}$$

The group S_r acts in the obvious way on the source of the map (A.13) by simultaneously rearranging the labels on white vertices and co-generators of $\mathscr{C}_r$. It is easy to see that Υ' (A.13) descends to an isomorphism

$$\Upsilon : \big(\mathbf{s}^{2n-2}\mathsf{KGra}'(n,r) \otimes \mathrm{Hoch}'(\mathscr{C}_r)\big)_{S_r} \to \mathsf{KGra}^{\mathrm{Hoch}}_r \tag{A.14}$$

from the space

$$\big(\mathbf{s}^{2n-2}\mathsf{KGra}'(n,r) \otimes \mathrm{Hoch}'(\mathscr{C}_r)\big)_{S_r}$$

of S_r-coinvariants to the complex in question $\mathsf{KGra}^{\mathrm{Hoch}}_r$. It is not hard to see that the map (A.14) is compatible with the differential ∂^{Hoch} on $\mathsf{KGra}^{\mathrm{Hoch}}_r$ and the differential on the source coming from $\partial^{\mathscr{C}}$ on $\mathrm{Hoch}'(\mathscr{C}_r)$.

Thus, using Claim A.6, it is not hard to prove the following statement about cohomology of $\mathsf{KGra}^{\mathrm{Hoch}}$ (A.3).

Proposition A.8. – *For every cocycle*

$$\gamma \in \mathbf{s}^{2n-2+k}\mathsf{KGra}(n,k)^{\mathfrak{o}}$$

there exists a vector

$$\gamma_1 \in \mathbf{s}^{2n-2+k-1}\mathsf{KGra}(n,k-1)^{\mathfrak{o}},$$

such that the difference

$$c = \gamma - \partial^{\mathrm{Hoch}}(\gamma_1)$$

satisfies Properties A.2 and A.3. A cocycle c *in* (A.3) *satisfying Properties A.2 and A.3 is trivial if and only if* $c = 0$. , To deduce an analogous statement for the cochain complex $\mathsf{KGra}^{\mathrm{Hoch}}_{\mathrm{inv}}$ (A.1) we need to use the averaging operator

$$\frac{1}{n!}\sum_{\sigma\in S_n}\sigma.$$

More precisely, Proposition A.8 implies that

Corollary A.9. – *For every cocycle*

$$\gamma\in \mathbf{s}^{2n-2+k}\Big(\mathsf{KGra}(n,k)^{\mathfrak{o}}\Big)^{S_n}$$

there exists a vector

$$\gamma_1\in \mathbf{s}^{2n-2+k-1}\Big(\mathsf{KGra}(n,k-1)^{\mathfrak{o}}\Big)^{S_n},$$

such that the difference

$$c=\gamma-\partial^{\mathrm{Hoch}}(\gamma_1)$$

satisfies Properties A.2 and A.3. A cocycle c in the complex (A.1) *satisfying Properties A.2 and A.3 is trivial if and only if $c=0$.* ,

It is clear that for every vector

$$\gamma\in \mathbf{s}^{2n-2}\Big(\mathsf{KGra}(n,0)^{\mathfrak{o}}\Big)^{S_n}$$

$$\partial^{\mathrm{Hoch}}(\gamma)=0. \tag{A.15}$$

Due to this observation Corollary A.9 implies the following statement.

Corollary A.10. – *A vector*

$$\gamma\in \mathbf{s}^{2n-1}\Big(\mathsf{KGra}(n,1)^{\mathfrak{o}}\Big)^{S_n} \tag{A.16}$$

is a cocycle in (A.1) *if and only if the white vertex in each graph in the linear combination γ has valency* 1. *Furthermore, a cocycle γ in $\mathbf{s}^{2n-1}\Big(\mathsf{KGra}(n,1)^{\mathfrak{o}}\Big)^{S_n}$ is trivial if and only if $\gamma=0$.* ,

APPENDIX B

THE COMPLEX OF "HEDGEHOGS"

This appendix is devoted to an auxiliary cochain complex which is assembled from graphs $\Gamma \in \mathrm{dgra}_{m,k}$ satisfying the additional property: *each white vertex of* Γ *has valency* 1. Since such graphs look like hedgehogs we call this cochain complex the complex of "hedgehogs".

This cochain complex and especially Corollary B.5 (proved below) are used in Sections 7 and 8.

We start by introducing the following graded vector space

$$\mathsf{Hg} = \Big\{ \gamma \in \bigoplus_{m,k} \mathbf{s}^{2m-2+k} \big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m} \mid \gamma \text{ obeys Properties A.2, A.3} \Big\} \tag{B.1}$$

and the families of cycles $\tau_{m,i} \in S_m$, and $\sigma_{k,i}, \varsigma_{k,i} \in S_k$

$$\tau_{m,i} := (i, i+1, \dots, m-1, m), \tag{B.2}$$

$$\sigma_{k,i} := (i, i-1, \cdots 2, 1), \tag{B.3}$$

and

$$\varsigma_{k,i} := (1, 2, \dots, i-1, i). \tag{B.4}$$

Next, we denote by $\mathfrak{d}$ the following degree 1 operation on Hg

$$\mathfrak{d}(\gamma) = k \sum_{i=1}^{m+1} (\tau_{m+1,i}, \mathrm{id}) \big(\gamma \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0 \big), \qquad \gamma \in \mathbf{s}^{2m-2+k} \big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}. \tag{B.5}$$

Notice that, since the graph Γ^{br}_0 consists of a single black vertex and has no edges, the insertion $\circ_{1,\mathfrak{o}}$ of Γ^{br}_0 replaces the white vertex with label 1 by a black vertex with label $m+1$ and shifts the labels on the remaining white vertices down by 1.

Using the fact that each linear combination $\gamma \in \mathsf{Hg}$ is anti-symmetric with respect to permutations of labels on white vertices, it is not hard to deduce that

$$\mathfrak{d}^2 = 0. \tag{B.6}$$

Thus $(\mathsf{Hg}, \mathfrak{d})$ is a cochain complex. We call this cochain complex the complex of "hedgehogs".

For our purposes, we need a degree -1 operation

$$\mathfrak{d}^* : \mathsf{Hg} \to \mathsf{Hg}, \tag{B.7}$$

which we will now define. Let γ be a vector in $\mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}$ satisfying Properties A.2, A.3. To compute $\mathfrak{d}^*(\gamma)$ we follow these steps:

— First, we omit in γ all graphs for which the black vertex with label 1 is not a pike. We denote the resulting linear combination in $\mathbf{s}^{2m-2+k}\mathsf{KGra}(m,k)^{\mathfrak{o}}$ by γ'.

— Second, we replace the black vertex with label 1 in each graph of γ' by a white vertex and shift all labels on black vertices down by 1. We assign label 1 to this additional white vertex and shift the labels of the remaining white vertices up by 1. We denote the resulting linear combination in

$$\mathbf{s}^{2(m-1)-2+k+1}\mathsf{KGra}(m-1,k+1)^{\mathfrak{o}}$$

by γ''.

— Finally, we set

$$\mathfrak{d}^*(\gamma) = \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{k+1}(\mathrm{id}, \sigma_{k+1,i})(\gamma''). \tag{B.8}$$

Note that the linear combination γ' is invariant with respect to the action of the group $S_{\{2,3,\ldots,m\}}$. Hence, the linear combination $\mathfrak{d}^*(\gamma)$ is S_{m-1}-invariant. Furthermore, $\mathfrak{d}^*(\gamma)$ obviously satisfies Properties A.2 and A.3.

Remark B.1. – *Notice that*

$$\mathfrak{d}^*(\gamma) = 0 \tag{B.9}$$

if each graph in the linear combination γ does not have pikes.

Example B.2. – *Let us denote by Γ_k the graph depicted in Figure B.1 and let*

$$\gamma = \Gamma_k + (\sigma_{12}, \mathrm{id})\big(\Gamma_k\big), \tag{B.10}$$

where σ_{12} is the transposition in S_2.

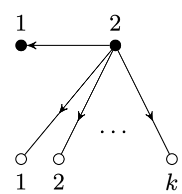


Figure B.1. Edges are ordered in this way $(2_{\mathfrak{c}}, 1_{\mathfrak{c}}) < (2_{\mathfrak{c}}, 1_{\mathfrak{o}}) < (2_{\mathfrak{c}}, 2_{\mathfrak{o}}) < \cdots < (2_{\mathfrak{c}}, k_{\mathfrak{o}})$

It is obvious that γ is a vector in $\mathbf{s}^{k+2}\big(\mathsf{KGra}(2,k)^{\mathfrak{o}}\big)^{S_2}$ satisfying Properties A.2, A.3.

Following the steps outlined above, we get

$$\gamma' = \Gamma_k \qquad \textit{and} \qquad \gamma'' = \Gamma^{\mathrm{br}}_{k+1},$$

where Γ^{br}_k is the family of "brooms" shown on Figure 5.1. Since $\Gamma^{\mathrm{br}}_{k+1}$ is already anti-symmetric with respect to permutations of labels on white vertices,

$$\mathfrak{d}^*(\gamma) = \sum_{i=1}^{k+1} \frac{(-1)^{i-1}}{k+1}(\mathrm{id}, \sigma_{k+1,i})(\Gamma^{\mathrm{br}}_{k+1}) = \Gamma^{\mathrm{br}}_{k+1}.$$

We need the following lemma.

Lemma B.3. – *For every vector*

$$\gamma \in \mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}$$

satisfying Properties A.2, A.3 we have

$$\mathfrak{d}\mathfrak{d}^*(\gamma) + \mathfrak{d}^*\mathfrak{d}(\gamma) = k\gamma + \sum_{r \ge 1} r\gamma_r, \tag{B.11}$$

where γ_r is the linear combination in Hg, which is obtained from γ by retaining the graphs with exactly r pikes.

Proof. – Let us observe that the space

$$\mathbf{s}^{2m-2+k}\big(\mathsf{KGra}(m,k)^{\mathfrak{o}}\big)^{S_m}$$

is spanned by vectors of the form

$$\sum_{\tau \in S_m,\, \sigma \in S_k} (-1)^{|\sigma|}(\tau,\sigma)\big(\Gamma\big), \tag{B.12}$$

where Γ is a graph in $\mathrm{dgra}_{m,k}$ with all white vertices having valency 1.

Thus we may assume, without loss of generality, that

$$\gamma = \sum_{\tau \in S_m,\, \sigma \in S_k} (-1)^{|\sigma|}(\tau,\sigma)\big(\Gamma\big) \tag{B.13}$$

for a graph $\Gamma \in \mathrm{dgra}_{m,k}$ with all white vertices having valency 1.

Using the cycles $\varsigma_{k,i}$ (B.4) we rewrite (B.13) as follows:

$$\gamma = \sum_{\tau \in S_m} \sum_{\sigma' \in S_{\{2,3,\dots,k\}}} (-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{i=1}^{k}(-1)^{i-1}(\mathrm{id},\varsigma_{k,i})(\Gamma)\right), \tag{B.14}$$

where $S_{\{2,3,\dots,k\}}$ denotes the permutation group of the set $\{2,3,\dots,k\}$.

Next, using (B.14) together with the obvious identity

$$\big((\mathrm{id},\varsigma_{k,i})(\Gamma)\big) \circ_{1,\mathfrak{o}} \Gamma^{\mathrm{br}}_0 = \Gamma \circ_{i,\mathfrak{o}} \Gamma^{\mathrm{br}}_0$$

we deduce that

$$\mathfrak{d}(\gamma) = k \sum_{j=1}^{m+1} (\tau_{m+1,j}, \mathrm{id}) \left(\sum_{\tau \in S_m,\, \sigma' \in S_{k-1}} (-1)^{|\sigma'|} (\tau, \sigma') \left(\sum_{i=1}^{k} (-1)^{i-1} \Gamma \circ_{i,\mathfrak{o}} \Gamma_0^{\mathrm{br}} \right) \right)$$

$$= k \sum_{\tau \in S_{m+1},\, \sigma' \in S_{k-1}} (-1)^{|\sigma'|} (\tau, \sigma') \left(\sum_{i=1}^{k} (-1)^{i-1} \Gamma \circ_{i,\mathfrak{o}} \Gamma_0^{\mathrm{br}} \right). \tag{B.15}$$

Let us, first, consider the case when the graph Γ does not have pikes. In this case, due to Remark B.1, we have

$$\mathfrak{d}^*(\gamma) = 0.$$

Furthermore, using (B.15), we get

(B.16)

$$\mathfrak{d}^*\mathfrak{d}(\gamma) = \sum_{j=1}^{k} (-1)^{j-1} (\mathrm{id}, \sigma_{k,j}) \left(\sum_{\tau \in S_m} \sum_{\sigma' \in S_{\{2,3,\dots,k\}}} (-1)^{|\sigma'|} (\tau, \sigma') \left(\sum_{i=1}^{k} (-1)^{i-1} (\mathrm{id}, \varsigma_{k,i})(\Gamma) \right) \right),$$

where $S_{\{2,3,\dots,k\}}$ denotes the permutation group of the set $\{2, 3, \dots, k\}$, and $\varsigma_{k,i}$ is the family of cycles defined in (B.4).

According to (B.14),

$$\sum_{\tau \in S_m} \sum_{\sigma' \in S_{\{2,3,\dots,k\}}} (-1)^{|\sigma'|} (\tau, \sigma') \left(\sum_{i=1}^{k} (-1)^{i-1} (\mathrm{id}, \varsigma_{k,i})(\Gamma) \right) = \gamma. \tag{B.17}$$

Moreover, since γ is antisymmetric with respect to permutations of labels on white vertices,

$$(\mathrm{id}, \sigma_{k,j})(\gamma) = (-1)^{j-1} \gamma.$$

Hence

$$\sum_{j=1}^{k} (-1)^{j-1} (\mathrm{id}, \sigma_{k,j})(\gamma) = k\gamma. \tag{B.18}$$

Therefore, combining (B.16) with (B.17) and (B.18), we get

$$\mathfrak{d}^*\mathfrak{d}(\gamma) = k\gamma. \tag{B.19}$$

Thus, if each graph in a linear combination γ does not have pikes, then Equation (B.11) holds.

Let us now turn to the case when Γ has exactly $r \geq 1$ pikes.

Without loss of generality, we may assume that the pikes of Γ are labeled by $1, 2, \dots, r$.

Let us recall that the vector γ' is obtained from γ by discarding all graphs for which the black vertex with label 1 is not a pike. In our case, the vector γ' can be written as follows:

$$\gamma' = \sum_{\sigma\in S_k}\ \sum_{\tau'\in S_{\{2,3,\dots,m\}}} (-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right), \tag{B.20}$$

where $S_{\{2,3,\dots,m\}}$ denotes the permutation group of the set $\{2,3,\dots,m\}$, and $\varsigma_{m,p}$ is the family of cycles in S_m defined in (B.4).

Using (B.20) we get

$$\mathfrak{d}^*(\gamma) = \sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{k+1}(\mathrm{id},\sigma_{k+1,i})(\gamma'') \tag{B.21}$$

with

$$\gamma'' = \sum_{\sigma\in S_{\{2,3,\dots,k+1\}}}\ \sum_{\tau'\in S_{m-1}} (-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}R_\circ\big((\varsigma_{m,p},\mathrm{id})(\Gamma)\big)\right), \tag{B.22}$$

where $S_{\{2,3,\dots,k+1\}}$ is the group of permutations of the set $\{2,3,\dots,k+1\}$ and $R_\circ$ is the operation which replaces the pike with label 1 by a white vertex with label 1, shifts labels on the remaining white vertices up by 1 and shifts labels on black vertices down by 1.

For the vector $\mathfrak{d}\mathfrak{d}^*(\gamma)$ we get

$$\begin{aligned}\mathfrak{d}\mathfrak{d}^*(\gamma) = {}& \sum_{j=1}^{m}(\tau_{m,j},\,\mathrm{id})\left(\sum_{\sigma\in S_k}\ \sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\right)\\ &+\sum_{j=1}^{m}(\tau_{m,j},\mathrm{id})\left(\sum_{i=2}^{k+1}(-1)^{i-1}\big((\mathrm{id},\sigma_{k+1,i})(\gamma'')\big)\circ_{1,\mathfrak{o}}\Gamma_0^{\mathrm{br}}\right),\end{aligned} \tag{B.23}$$

where the first sum comes from the first term in the sum (B.21) and the second sum comes from the remaining terms in (B.21).

The first sum in (B.23) can be simplified as follows.

$$\begin{aligned}\sum_{j=1}^{m}(\tau_{m,j},\,\mathrm{id})&\left(\sum_{\sigma\in S_k}\ \sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\right)\\ &=\sum_{\sigma\in S_k}\ \sum_{\tau\in S_m}(-1)^{|\sigma|}(\tau,\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\\ &= r\sum_{\sigma\in S_k}\ \sum_{\tau\in S_m}(-1)^{|\sigma|}(\tau,\sigma)(\Gamma) = r\,\gamma.\end{aligned}$$

In other words,

$$\sum_{j=1}^{m}(\tau_{m,j},\,\mathrm{id})\left(\sum_{\sigma\in S_k}\sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}(\varsigma_{m,p},\mathrm{id})(\Gamma)\right)\right)=r\,\gamma. \tag{B.24}$$

To simplify the second sum in (B.23) we notice that the subsets of S_{k+1}

$$\{\sigma_{k+1,i}\circ\sigma \mid \sigma\in S_{\{2,3,\dots,k+1\}},\ 2\le i\le k+1\}$$

and

$$\{\sigma\circ\varsigma_{k+1,i} \mid \sigma\in S_{\{2,3,\dots,k+1\}},\ 2\le i\le k+1\}$$

coincide.

Hence,

(B.25)

$$\sum_{i=2}^{k+1}\frac{(-1)^{i-1}}{k+1}(\mathrm{id},\sigma_{k+1,i})(\gamma'')$$

$$=\frac{1}{k+1}\sum_{\sigma\in S_{\{2,3,\dots,k+1\}}}\sum_{\tau'\in S_{m-1}}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}\sum_{i=2}^{k+1}(-1)^{i-1}(\mathrm{id},\varsigma_{k+1,i})R_\circ\big((\varsigma_{m,p},\mathrm{id})(\Gamma)\big)\right).$$

Next, we introduce operations $\{\mathrm{Cg}^i_p\}_{1\le p\le r,1\le i\le k}$ whose input is our graph Γ and whose outputs are graphs in $\mathrm{dgra}_{m,k}$ with the same properties, i.e., each white vertex of $\mathrm{Cg}^i_p(\Gamma)$ has valency 1 and $\mathrm{Cg}^i_p(\Gamma)$ has exactly r pikes. This operation is illustrated in Figure B.2. More precisely, $\mathrm{Cg}^i_p(\Gamma)$ is obtained from Γ via these steps:

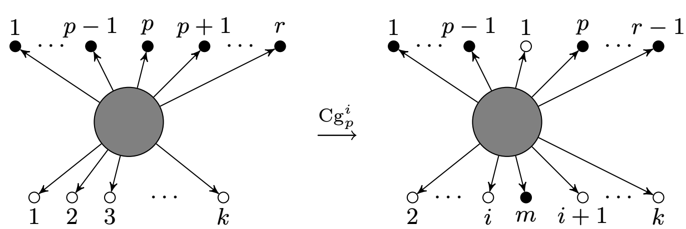


FIGURE B.2. The operation $\Gamma\mapsto\mathrm{Cg}^i_p(\Gamma)$. Gray regions denote subgraphs formed by black vertices which are not pikes

- first, we replace the black vertex with label p by a white vertex and replace the white vertex with label i by a black vertex;
- second, we shift the labels on the black vertices which are $> p$ down by 1;
- third, we shift the labels on the white vertices which are $< i$ up by 1;
- finally, we assign label 1 to the new white vertex and we assign label m to the new black vertex.

Using Equation (B.25) and the graphs $\mathrm{Cg}_p^i(\Gamma)$ we present the second sum in (B.23) in the following way.

$$\begin{aligned}
&\sum_{j=1}^{m}(\tau_{m,j},\mathrm{id})\left(\sum_{i=2}^{k+1}(-1)^{i-1}\big((\mathrm{id},\sigma_{k+1,i})(\gamma'')\big)\circ_{1,\mathfrak{o}}\Gamma_0^{\mathrm{br}}\right)\\
&\quad=\sum_{j=1}^{m}(\tau_{m,j},\mathrm{id})\sum_{\tau'\in S_{m-1}}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau',\sigma)\left(\sum_{p=1}^{r}\sum_{i=2}^{k+1}(-1)^{i-1}\Big(\mathrm{Cg}_p^{i-1}(\Gamma)\Big)\right)\\
&\quad=-\sum_{j=1}^{m}\sum_{\tau'\in S_{m-1}}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau_{m,j}\tau',\sigma)\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\Big(\mathrm{Cg}_p^{i}(\Gamma)\Big)\right)\\
&\quad=-\sum_{\tau\in S_m}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau,\sigma)\Big(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}_p^{i}(\Gamma)\Big).
\end{aligned}\tag{B.26}$$

Combining this observation with Equation (B.24), we conclude that

$$\mathfrak{d}\mathfrak{d}^*(\gamma)=r\,\gamma-\sum_{\tau\in S_m}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\tau,\sigma)\Big(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}_p^{i}(\Gamma)\Big).\tag{B.27}$$

To unfold $\mathfrak{d}^*\mathfrak{d}(\gamma)$, we denote by ω the vector $\mathfrak{d}(\gamma)$ (B.15). By discarding in ω all graphs for which black vertex with label 1 is not a pike we get the expression

(B.28)

$$\begin{aligned}
\omega'=k&\sum_{\tau\in S_{\{2,3,\dots,m+1\}}}\sum_{\sigma'\in S_{k-1}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{i=1}^{k}(-1)^{i-1}(\tau_{m+1,1},\mathrm{id})(\Gamma\circ_{i,\mathfrak{o}}\Gamma_0^{\mathrm{br}})\right)\\
&+k\sum_{\tau\in S_{\{2,3,\dots,m+1\}}}\sum_{\sigma'\in S_{k-1}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}(\varsigma_{m+1,p},\mathrm{id})(\Gamma\circ_{i,\mathfrak{o}}\Gamma_0^{\mathrm{br}})\right).
\end{aligned}$$

Next, replacing the black vertices with label 1 in each graph in ω' by a white vertex with label 1 and shifting the labels of the remaining vertices correspondingly, we get

$$\begin{aligned}
\omega''=k&\sum_{\tau\in S_m}\sum_{\sigma'\in S_{\{2,3,\dots,k\}}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{i=1}^{k}(-1)^{i-1}(\mathrm{id},\varsigma_{k,i})(\Gamma)\right)\\
&+k\sum_{\tau\in S_m}\sum_{\sigma'\in S_{\{2,3,\dots,k\}}}(-1)^{|\sigma'|}(\tau,\sigma')\left(\sum_{p=1}^{r}\sum_{i=1}^{k}(-1)^{i-1}\mathrm{Cg}_p^{i}(\Gamma)\right).
\end{aligned}\tag{B.29}$$

Thus

$$\mathfrak{d}^*\mathfrak{d}(\gamma)=\sum_{j=1}^{k}\frac{(-1)^{j-1}}{k}(\mathrm{id},\sigma_{k,j})(\omega'')\tag{B.30}$$

$$= \sum_{\tau \in S_m} \sum_{\sigma \in S_k} (-1)^{|\sigma|} (\tau, \sigma) \left(\sum_{i=1}^{k} (-1)^{i-1} (\mathrm{id}, \varsigma_{k,i})(\Gamma) \right)$$
$$+ \sum_{\tau \in S_m} \sum_{\sigma \in S_k} (-1)^{|\sigma|} (\tau, \sigma) \left(\sum_{p=1}^{r} \sum_{i=1}^{k} (-1)^{i-1} \mathrm{Cg}^i_p (\Gamma) \right)$$
$$= k\,\gamma + \sum_{\tau \in S_m} \sum_{\sigma \in S_k} (-1)^{|\sigma|} (\tau, \sigma) \left(\sum_{p=1}^{r} \sum_{i=1}^{k} (-1)^{i-1} \mathrm{Cg}^i_p (\Gamma) \right)$$

Combining (B.27) with (B.30) we immediately deduce Equation (B.11).

Lemma B.3 is proved. □

REMARK B.4. – *The cochain complex* Hg (B.1) *with the differential* $\mathfrak{d}$ (B.5) *is very similar to Koszul complex for the exterior algebra. However, the author could not find an elegant way to reduce* Hg *to this well known complex.*

We have the following corollary.

COROLLARY B.5. – *Let* γ *be a vector in*

$$\mathbf{s}^{2m-2+k} \left(\mathsf{KGra}(m,k)^{\mathfrak{o}} \right)^{S_m}$$

satisfying Properties A.2, A.3. If $k \geq 1$ *and* γ *is* $\mathfrak{d}$*-closed then there exists*

$$\widetilde{\gamma} \in \mathbf{s}^{2(m-1)-2+k+1} \left(\mathsf{KGra}(m-1,k+1)^{\mathfrak{o}} \right)^{S_{m-1}},$$

which satisfies Properties A.2, A.3 and such that

$$\gamma = \mathfrak{d}(\widetilde{\gamma}). \tag{B.31}$$

Proof. – Since γ is $\mathfrak{d}$-closed, Equation (B.11) implies that

$$\mathfrak{d}\mathfrak{d}^*(\gamma) = k\gamma + \sum_{r \geq 1} r \gamma_r, \tag{B.32}$$

where γ_r is the linear combination in Hg, which is obtained from γ by retaining the graphs with exactly r pikes.

Since each graph in the image of $\mathfrak{d}$ has at least one pike, Equation (B.32) implies that each graph in the linear combination γ has at least one pike. Hence,

$$\gamma = \sum_{r \geq 1} \gamma_r \tag{B.33}$$

and (B.32) can be rewritten as

$$\mathfrak{d}\mathfrak{d}^*(\gamma) = \sum_{r \geq 1} (k+r) \gamma_r. \tag{B.34}$$

Thus, setting

$$\widetilde{\gamma} = \sum_{r\geq 1} \frac{1}{k+r}\mathfrak{d}^*(\gamma_r) \tag{B.35}$$

we get the desired identity

$$\gamma = \mathfrak{d}(\widetilde{\gamma}).$$

□

APPENDIX C

MAURER-CARTAN (MC) ELEMENTS OF FILTERED LIE ALGEBRAS

Let $\mathcal{L}$ be a Lie algebra in the category $\mathsf{Ch}_{\mathbb{K}}$ of unbounded cochain complexes of $\mathbb{K}$-vector spaces. Let us assume that $\mathcal{L}$ is equipped with a descending filtration

$$\cdots \supset \mathcal{F}_{-1}\mathcal{L} \supset \mathcal{F}_{0}\mathcal{L} \supset \mathcal{F}_{1}\mathcal{L} \supset \mathcal{F}_{2}\mathcal{L} \supset \mathcal{F}_{3}\mathcal{L} \supset \cdots, \tag{C.1}$$

which is compatible with the Lie bracket, and such that $\mathcal{L}$ is complete and cocomplete with respect to this filtration, i.e.,

$$\mathcal{L} = \lim_{k} \mathcal{L} / \mathcal{F}_{k}\mathcal{L} \tag{C.2}$$

and

$$\mathcal{L} = \bigcup_{k} \mathcal{F}_{k}\mathcal{L}. \tag{C.3}$$

We call such Lie algebras *filtered*.

Condition (C.2) guarantees that the subalgebra $\mathcal{F}_{1}\mathcal{L}^{0}$ of degree zero elements in $\mathcal{F}_{1}\mathcal{L}$ is a pro-nilpotent Lie algebra (in the category of $\mathbb{K}$-vector spaces). Hence, $\mathcal{F}_{1}\mathcal{L}^{0}$ can be exponentiated to a pro-unipotent group which we denote by

$$\exp(\mathcal{F}_{1}\mathcal{L}^{0}). \tag{C.4}$$

We recall that a *Maurer-Cartan (MC) element* of $\mathcal{L}$ is a degree 1 vector $\alpha \in \mathcal{L}$ satisfying the equation

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0, \tag{C.5}$$

where ∂ denotes the differential on $\mathcal{L}$.

For a vector $\xi \in \mathcal{F}_{1}\mathcal{L}^{0}$ and a MC element α we consider the new degree 1 vector $\widetilde{\alpha} \in \mathcal{L}$, which is given by the formula

$$\widetilde{\alpha} = \exp(\mathrm{ad}_{\xi})\,\alpha - \frac{\exp(\mathrm{ad}_{\xi}) - 1}{\mathrm{ad}_{\xi}}\partial\xi, \tag{C.6}$$

where the expressions

$$\exp(\mathrm{ad}_\xi) \qquad \text{and} \qquad \frac{\exp(\mathrm{ad}_\xi) - 1}{\mathrm{ad}_\xi}$$

are defined in the obvious way using the Taylor expansions of the functions

$$e^x \qquad \text{and} \qquad \frac{e^x - 1}{x}$$

around the point $x = 0$, respectively.

Condition (C.2) guarantees that the right hand side of Equation (C.6) makes sense.

It is known (see, e.g., [**6**, Appendix B] or [**21**]) that, for every MC element α and for every degree zero vector $\xi \in \mathcal{F}_1 \mathcal{L}$, the vector $\widetilde{\alpha}$ in (C.6) is also a MC element. Furthermore, Formula (C.6) defines an action of the group (C.4) on the set of MC elements of $\mathcal{L}$.

The transformation groupoid corresponding to this action is called the *Deligne groupoid* of the Lie algebra $\mathcal{L}$. This groupoid and its higher versions were studied extensively in [**3, 4, 5, 12, 17**] and [**18, 22, 41**].

EXAMPLE C.1. – *Let $\mathcal{C}$ (resp. $\mathcal{O}$) be a Ξ-colored pseudo-cooperad (resp. Ξ-colored pseudo-operad) in* $\mathsf{Ch}_{\mathbb{K}}$. *The convolution Lie algebra* $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$ *described in Section 2.3 gives us an example of a filtered Lie algebra. Thus it makes sense to talk about the Deligne groupoid of* $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$.

C.1. Differential equations on the Lie algebra $\mathcal{L} \hat{\otimes} \mathbb{K}[t]$

Given a filtered dg Lie algebra $\mathcal{L}$, we introduce the new dg Lie algebra [(1)]

$$\mathcal{L} \hat{\otimes} \mathbb{K}[t], \tag{C.7}$$

where $\mathcal{L}$ is considered with the topology coming from the filtration and $\mathbb{K}[t]$ is considered with the discrete topology.

It is clear that $\mathcal{L} \hat{\otimes} \mathbb{K}[t]$ consists of vectors

$$v = \sum_{k=0}^{\infty} v_k \, t^k \in \mathcal{L}[[t]] \tag{C.8}$$

satisfying the condition

CONDITION C.2. – *For every integer m, the image of v in*

$$(\mathcal{L} / \mathcal{F}_m \mathcal{L})[[t]] \tag{C.9}$$

is a polynomial in t. In other words, for every integer m there exists k_m such that $v_k \in \mathcal{F}_m \mathcal{L}$ for all $k \ge k_m$.

1. t is an auxiliary degree zero variable.

Let us also observe that $\mathscr{L}\,\hat{\otimes}\,\mathbb{K}[t]$ comes with the obvious descending filtration:

$$\mathcal{F}_m\big(\mathscr{L}\,\hat{\otimes}\,\mathbb{K}[t]\big) := (\mathcal{F}_m\mathscr{L})\,\hat{\otimes}\,\mathbb{K}[t], \tag{C.10}$$

the subspace $\mathscr{L}\,\hat{\otimes}\,\mathbb{K}[t] \subset \mathscr{L}[[t]]$ is closed with respect to the (formal) derivative d/dt and

$$d/dt\big((\mathcal{F}_m\mathscr{L})\,\hat{\otimes}\,\mathbb{K}[t]\big) \subset (\mathcal{F}_m\mathscr{L})\,\hat{\otimes}\,\mathbb{K}[t].$$

Condition C.2 and completeness of $\mathscr{L}$ with respect to the filtration $\mathcal{F}_\bullet$ imply that the assignment

$$v \mapsto v\big|_{t=1} \tag{C.11}$$

defines a Lie algebra homomorphism from $\mathscr{L}\,\hat{\otimes}\,\mathbb{K}[t]$ to $\mathscr{L}$. Furthermore, this homomorphism is compatible with the filtrations on $\mathscr{L}\,\hat{\otimes}\,\mathbb{K}[t]$ and $\mathscr{L}$.

Next, we claim that

Claim C.3. – *The dg Lie algebra $\mathscr{L}\,\hat{\otimes}\,\mathbb{K}[t]$ is complete and cocomplete with respect to the filtration* (C.10).

Proof. – The cocompleteness follows readily from Property (C.3) and Condition C.2.

To prove the completeness, we need to show that for every infinite sequence of vectors

$$v^{(r)} = \sum_{k\geq 0} v_k^{(r)} t^k \in (\mathcal{F}_{m_r}\mathscr{L})\,\hat{\otimes}\,\mathbb{K}[t], \qquad r \geq 1 \tag{C.12}$$

satisfying the condition

$$m_1 \leq m_2 \leq m_3 \leq \cdots, \qquad \lim_{r\to\infty} m_r = \infty \tag{C.13}$$

the sum

$$\sum_{r\geq 1} v^{(r)} \tag{C.14}$$

belongs to the subalgebra $\mathscr{L}\,\hat{\otimes}\,\mathbb{K}[t]$.

The sum (C.14) can be rewritten as follows:

$$\sum_{r\geq 1} v^{(r)} = \sum_{k\geq 0} w_k t^k, \tag{C.15}$$

where

$$w_k = \sum_{r=1}^{\infty} v_k^{(r)} \tag{C.16}$$

Let us choose an integer m. Condition (C.13) implies that there exist r' such that

$$m_r \geq m \qquad \forall r \geq r'.$$

Hence

$$\sum_{r=r'}^{\infty} v_k^{(r)} \in \mathscr{F}_m \mathscr{L} \tag{C.17}$$

for all k.

On the other hand, $v^{(r)} \in \mathscr{L} \hat{\otimes} \mathbb{K}[t]$ for every r. So for every $r \geq 1$ there exists k_m^r such that

$$v_k^{(r)} \in \mathscr{F}_m \mathscr{L} \qquad \forall k \geq k_m^r.$$

Therefore, setting $k_m = \max\{k_m^1, k_m^2, \ldots, k_m^{r'-1}\}$, we get the inclusion

$$\sum_{r=1}^{r'-1} v_k^{(r)} \in \mathscr{F}_m \mathscr{L} \qquad \forall k \geq k_m.$$

Combining this inclusion with (C.17), we conclude that

$$\sum_{r=1}^{\infty} v_k^{(r)} \in \mathscr{F}_m \mathscr{L} \qquad \forall k \geq k_m. \tag{C.18}$$

Claim C.3 is proved. □

We will need the following proposition

Proposition C.4. – *For every degree* 1 *vector* $\alpha \in \mathscr{L}$ *and* $\eta(t) \in \mathscr{F}_1 \mathscr{L}^0 \hat{\otimes} \mathbb{K}[t]$ *the equation*

$$\frac{d}{dt}\alpha(t) = -\partial\eta(t) + [\eta(t), \alpha(t)] \tag{C.19}$$

with initial condition

$$\alpha(t)\big|_{t=0} = \alpha \tag{C.20}$$

has a unique solution in $\mathscr{L} \hat{\otimes} \mathbb{K}[t]$. *In addition, if* α *satisfies the MC equation*

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0,$$

then so does $\alpha(t)$.

Proof. – Let us set up the following iterative procedure in $r \geq 0$

$$\alpha^{(0)}(t) = \alpha$$

and

$$\alpha^{(r)}(t) = \alpha - \int_0^t \partial\eta(t_1)\, dt_1 + \int_0^t [\eta(t_1), \alpha^{(r-1)}(t_1)]\, dt_1. \tag{C.21}$$

Since the differences $\alpha^{(r+1)}(t) - \alpha^{(r)}(t)$ and $\alpha^{(r)}(t) - \alpha^{(r-1)}(t)$ satisfy the equation

$$\alpha^{(r+1)}(t) - \alpha^{(r)}(t) = \int_0^t [\eta(t_1), \alpha^{(r)}(t_1) - \alpha^{(r-1)}(t_1)]\, dt_1$$

and $\eta(t) \in \mathcal{F}_1\mathcal{L}^0\hat{\otimes}\mathbb{K}[t]$, this iterative procedure converges to a vector $\alpha(t) \in \mathcal{L}\hat{\otimes}\mathbb{K}[t]$. Moreover, $\alpha(t)$ satisfies the integral equation

$$\alpha(t) = \alpha - \int_0^t \partial\eta(t_1)\,dt_1 + \int_0^t [\eta(t_1), \alpha(t_1)]\,dt_1 \tag{C.22}$$

and hence differential Equation (C.19) with initial condition (C.20).

To prove the uniqueness, let us assume that $\widetilde{\alpha}(t)$ is another solution of (C.19) with the initial condition (C.20). Then the difference:

$$\psi(t) = \widetilde{\alpha}(t) - \alpha(t)$$

satisfies the differential equation

$$\frac{d}{dt}\psi(t) = [\eta(t), \psi(t)] \tag{C.23}$$

with the initial condition

$$\psi(t)\big|_{t=0} = 0. \tag{C.24}$$

Using (C.23) and (C.24) we conclude that

$$\psi(t) = \int_0^t [\eta(t_1), \psi(t_1)]\,dt_1. \tag{C.25}$$

Hence the inclusion $\eta(t) \in \mathcal{F}_1\mathcal{L}^0\hat{\otimes}\mathbb{K}[t]$ implies that

$$\psi(t) \in \bigcap_m \mathcal{F}_m\mathcal{L}\hat{\otimes}\mathbb{K}[t].$$

Therefore, by Claim C.3, $\psi(t) = 0$ and $\alpha(t) = \widetilde{\alpha}(t)$.

The first statement of Proposition C.4 is proved.

To prove the second statement, we consider the following element

$$\Psi(t) = \partial\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)] \in \mathcal{L}^2\hat{\otimes}\mathbb{K}[t]. \tag{C.26}$$

Taking the derivative d/dt and using (C.19), we get

$$\frac{d}{dt}(\partial\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)]) = [\eta(t), \partial\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)]].$$

In other words, the element $\Psi(t)$ satisfies the differential equation

$$\frac{d}{dt}\Psi(t) = [\eta(t), \Psi(t)]. \tag{C.27}$$

Since α satisfies the MC equation, we conclude that

$$\Psi(t)\big|_{t=0} = 0. \tag{C.28}$$

Using (C.27) and (C.28), we deduce that

$$\Psi(t) = \int_0^t [\eta(t_1), \Psi(t_1)]\,dt_1. \tag{C.29}$$

Equation (C.29) implies that

$$\Psi(t) \in \bigcap_m \mathcal{F}_m \mathcal{L} \hat{\otimes} \mathbb{K}[t].$$

and hence $\Psi(t) = 0$.

Proposition C.4 is proved. □

Proposition C.4 implies that using an element $\eta(t) \in \mathcal{F}_1 \mathcal{L}^0 \hat{\otimes} \mathbb{K}[t]$ and a MC element $\alpha \in \mathcal{L}$ we can produce another MC element α' by solving equation (C.19) with initial condition (C.20) and setting

$$\alpha' = \alpha(t)\big|_{t=1}.$$

Theorem C.6 below states that the MC elements α and α' are isomorphic.

To prove this statement we need the following technical lemma.

Lemma C.5. – *If α is a MC element of $\mathcal{L}$, $\eta(t) \in \mathcal{F}_1 \mathcal{L}^0 \hat{\otimes} \mathbb{K}[t]$, and $\alpha(t)$ is the unique solution of* (C.19) *with initial condition* (C.20), *then for every $\kappa \in \mathcal{F}_1 \mathcal{L}^0$ and every nonnegative integer k, the element*

$$\beta(t) = \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)\alpha(t) - \frac{\exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right) - 1}{\mathrm{ad}_\kappa}\,\partial\kappa \tag{C.30}$$

satisfies the differential equation

$$\frac{d}{dt}\beta(t) = [\tilde{\eta}, \beta(t)] - \partial\tilde{\eta}, \tag{C.31}$$

where

$$\tilde{\eta}(t) = t^k\kappa + \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)\eta(t). \tag{C.32}$$

Proof. – First, we remark that, the infinite series in (C.30) and (C.32) belong to $\mathcal{L} \hat{\otimes} \mathbb{K}[t]$ due to Claim C.3.

Second, we compute the derivative $\frac{d}{dt}\beta(t)$ using (C.19)

(C.33)

$$\begin{aligned}\frac{d}{dt}\beta(t) &= \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)[t^k\kappa, \alpha(t)] + \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)\frac{d}{dt}\alpha(t) - t^k\exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)\partial\kappa \\ &= \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)[t^k\kappa, \alpha(t)] + \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)[\eta(t), \alpha(t)] \\ &\quad - \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)\partial\eta(t) - \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)\partial(t^k\kappa).\end{aligned}$$

Using the notation

$$U_\kappa := \exp\left(\frac{t^{k+1}}{k+1}\mathrm{ad}_\kappa\right)$$

and the obvious identity $[\kappa,\kappa]=0$, we rewrite the derivative $\frac{d}{dt}\beta(t)$ as follows:

$$\frac{d}{dt}\beta(t) = [t^k\kappa + U_\kappa(\eta(t)), U_\kappa(\alpha(t))] - U_\kappa(\partial\eta(t)) - U_\kappa(\partial(t^k\kappa)). \tag{C.34}$$

On the other hand,

$$\beta(t) = U_\kappa(\alpha(t)) - \frac{U_\kappa - 1}{\mathrm{ad}_\kappa}\,\partial\kappa.$$

Hence

(C.35)

$$\begin{aligned}\frac{d}{dt}\beta(t) &= [t^k\kappa + U_\kappa(\eta(t)), \beta(t)] + \left[t^k\kappa + U_\kappa(\eta(t)), \frac{U_\kappa - 1}{\mathrm{ad}_\kappa}\,\partial\kappa\right] - U_\kappa(\partial\eta(t)) - U_\kappa(\partial(t^k\kappa))\\ &= [t^k\kappa + U_\kappa(\eta(t)), \beta(t)] - \partial(t^k\kappa) - U_\kappa(\partial\eta(t)) - \left[\frac{U_\kappa - 1}{\mathrm{ad}_\kappa}\,\partial\kappa,\, U_\kappa(\eta(t))\right].\end{aligned}$$

Thus, to prove Lemma C.5, we need to verify that

$$(\partial\circ U_\kappa - U_\kappa\circ\partial)\,(\eta(t)) = \left[\frac{U_\kappa - 1}{\mathrm{ad}_\kappa}\,\partial\kappa,\, U_\kappa(\eta(t))\right]. \tag{C.36}$$

Clearly, it suffices to check that

$$(\partial\circ U_\kappa - U_\kappa\circ\partial)\,(v) = \left[\frac{U_\kappa - 1}{\mathrm{ad}_\kappa}\,\partial\kappa,\, U_\kappa(v)\right] \tag{C.37}$$

for every vector $v\in\mathscr{L}$.

Let us denote by $\Psi_1(t)$ (resp. $\Psi_2(t)$) the left (resp. right) hand side of (C.37). It is easy to see that

$$\Psi_1(0) = \Psi_2(0) = 0. \tag{C.38}$$

A direct computation shows that both $\Psi_1(t)$ and $\Psi_2(t)$ satisfy the same differential equation:

$$\frac{d}{dt}\,\Psi_i(t) = [t^k\kappa, \Psi_i(t)] + [t^k\partial\kappa, U_\kappa(v)]. \tag{C.39}$$

Therefore, the difference $\Psi_2(t)-\Psi_1(t)$ satisfies the integral equation

$$\Psi_2(t) - \Psi_1(t) = \int_0^t t_1^k[\kappa, \Psi_2(t_1) - \Psi_1(t_1)]dt_1. \tag{C.40}$$

Since $\kappa\in\mathscr{F}_1\mathscr{L}^0$, we have

$$\Psi_2(t) - \Psi_1(t) \in \bigcap_m \mathscr{F}_m\mathscr{L}\hat{\otimes}\mathbb{K}[t].$$

Thus $\Psi_2(t)-\Psi_1(t)=0$ and Lemma C.5 is proved. □

Let us now prove a statement which is used in Section 6.2.

THEOREM C.6. – *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathscr{L}^0$, n be an integer ≥ 2, and*

$$\eta(t) = \sum_{k\ge 0} \eta_k t^k, \qquad \eta_k \in \mathscr{F}_{m_k}\mathfrak{g} \tag{C.41}$$

be a vector in $\mathscr{F}_1\mathfrak{g} \hat{\otimes} \mathbb{K}[t]$. If α is a MC element of $\mathscr{L}$ and $\alpha(t)$ is the unique solution of (C.19) *with initial condition* (C.20), *then there exists a vector $\xi \in \mathscr{F}_1\mathfrak{g}^0$ such that*

$$\exp(\mathrm{ad}_\xi)\,\alpha - \frac{\exp(\mathrm{ad}_\xi)-1}{\mathrm{ad}_\xi}\partial\xi \;=\; \alpha(t)\big|_{t=1}. \tag{C.42}$$

Moreover if

$$\eta(t) - \eta_0 \in \mathscr{F}_n\mathfrak{g}[[t]] \tag{C.43}$$

then there exists $\xi \in \mathscr{F}_1\mathfrak{g}^0$ such that (C.42) *holds and*

$$\xi - \eta_0 \in \mathscr{F}_n\mathfrak{g}. \tag{C.44}$$

Proof. – The statement of this theorem is very similar to [**6**, Proposition B.7]. Unfortunately, Theorem C.6 is not a corollary of [**6**, Proposition B.7]. So we give a separate proof.

Let us construct recursively the following sequence of vectors in $\mathscr{F}_1\mathfrak{g} \hat{\otimes} \mathbb{K}[t]$:

$$\eta^{(k)}(t) = \eta^{(k)}_k t^k + \eta^{(k)}_{k+1} t^{k+1} + \eta^{(k)}_{k+2} t^{k+2} + \cdots, \tag{C.45}$$

$$\eta^{(0)}(t) := \eta(t), \tag{C.46}$$

$$\eta^{(k+1)}(t) := \exp\left(-\frac{t^{k+1}}{k+1}\mathrm{ad}_{\eta^{(k)}_k}\right)\eta^{(k)}(t) - t^k\eta^{(k)}_k. \tag{C.47}$$

By Lemma C.5, we get the sequence of MC elements in $\mathscr{L} \hat{\otimes} \mathbb{K}[t]$

$$\alpha^{(0)}(t) := \alpha(t) \tag{C.48}$$

$$\alpha^{(k+1)}(t) := \exp\left(-\frac{t^{k+1}}{k+1}\mathrm{ad}_{\eta^{(k)}_k}\right)\alpha^{(k)}(t) - \frac{\exp\left(-\frac{t^{k+1}}{k+1}\mathrm{ad}_{\eta^{(k)}_k}\right)-1}{\mathrm{ad}_{\eta^{(k)}_k}}\,\partial\eta^{(k)}_k, \tag{C.49}$$

where $\alpha^{(k)}(t)$ is the unique solution of the differential equation

$$\frac{d}{dt}\alpha^{(k)}(t) = [\eta^{(k)}(t), \alpha^{(k)}(t)] - \partial\eta^{(k)}(t) \tag{C.50}$$

with the initial condition

$$\alpha^{(k)}(t)\big|_{t=0} = \alpha. \tag{C.51}$$

Let us prove that the sequence of vectors

$$\{\eta^{(k)}_k\}_{k\ge 0}$$

satisfies the property

$$\eta^{(k)}_k \in \mathscr{F}_{n_k}\mathfrak{g}$$

with

$$n_0 \le n_1 \le n_2 \le n_3 \le \cdots \qquad \text{and} \qquad \lim_{k\to\infty} n_k = \infty. \tag{C.52}$$

Without loss of generality, we may assume that the sequence of number $\{m_k\}_{k\ge 0}$ in (C.41) is increasing

$$m_0 \le m_1 \le m_2 \le m_3 \le \cdots$$

and $m_0 = 1$.

Hence the property

$$n_0 \le n_1 \le n_2 \le n_3 \le \cdots$$

follows immediately from the construction.

It remains to prove that for every m there exists k_m such that

$$\eta^{(k)}(t) \in \mathcal{F}_m\mathfrak{g}\,\hat{\otimes}\,\mathbb{K}[t] \qquad \forall k \ge k_m. \tag{C.53}$$

Since $\eta(t) \in \mathcal{F}_1\mathfrak{g}\,\hat{\otimes}\,\mathbb{K}[t]$, there exists r_1 such that

$$\eta_k^{(0)} = \eta_k \in \mathcal{F}_m\mathfrak{g} \qquad \forall k > r_1.$$

In "the worst case scenario," $m_k = 1$ for all $k \le r_1$.

So after $r_1 + 1$ steps (C.47) we get

$$\eta^{(r_1+1)}(t) \in \mathcal{F}_2\mathfrak{g}\,\hat{\otimes}\,\mathbb{K}[t].$$

Since $\eta^{(r_1+1)}(t) \in \mathcal{F}_2\mathfrak{g}\,\hat{\otimes}\,\mathbb{K}[t]$ there exists r_2 such that all coefficients, except for the first r_2 ones belong to $\mathcal{F}_3\mathfrak{g}$.

Hence, after additional r_2 steps (C.47) we get

$$\eta^{(r_1+1+r_2)}(t) \in \mathcal{F}_3\mathfrak{g}\,\hat{\otimes}\,\mathbb{K}[t]. \tag{C.54}$$

Therefore, in finitely many steps (C.47), we will arrive at

$$\eta^{(k_m)}(t) \in \mathcal{F}_m\mathfrak{g}\,\hat{\otimes}\,\mathbb{K}[t].$$

On the other hand, if $\eta^{(k)}(t)$ belongs to $\mathcal{F}_m\mathfrak{g}\,\hat{\otimes}\,\mathbb{K}[t]$ then so does $\eta^{(k+1)}(t)$. Thus, the desired inclusion (C.53) is proved.

Property (C.52) implies that the sequence of vectors

$$\mathrm{CH}\Big(-\eta_k^{(k)}/(k+1)\ ,\ \cdots \mathrm{CH}\big(-\eta_2^{(2)}/3\ ,\ \mathrm{CH}(-\eta_1^{(1)}/2\ ,\ -\eta_0^{(0)})\big)\Big) \tag{C.55}$$

converges in $\mathcal{F}_1\mathfrak{g}$ and we denote the limiting vector by ξ_∞. In addition, the sequence of vectors $\{\eta^{(k)}(t)\}_{k\ge 0}$ converges to zero and hence the sequence of vectors

$$\{\alpha^{(k)}(t)\}_{k\ge 0} \tag{C.56}$$

converges to the constant path α.

Therefore

$$\exp(\xi_\infty)\left(\alpha(t)\big|_{t=1}\right) = \alpha \tag{C.57}$$

Hence, setting

$$\xi := -\xi_\infty,$$

we prove the first part of Theorem C.6.

According to (C.43), $\eta_k \in \mathscr{F}_n\mathfrak{g}$ for all $k \geq 1$ in (C.41). Therefore, the sequence of vectors (C.45) satisfies

$$\eta^{(k)}(t) \in t^k \mathscr{F}_n\mathfrak{g}[[t]]$$

for all $k \geq 1$.

Hence

$$\xi_\infty + \eta_0 \in \mathscr{F}_n\mathfrak{g}$$

and the desired inclusion in (C.44) follows. □

BIBLIOGRAPHY

[1] A. Alekseev & C. Torossian – "The Kashiwara-Vergne conjecture and Drinfeld's associators", *Ann. of Math.* **175** (2012), p. 415–463.

[2] C. Berger & I. Moerdijk – "Resolution of coloured operads and rectification of homotopy algebras", in *Categories in algebra, geometry and mathematical physics*, Contemp. Math., vol. 431, Amer. Math. Soc., 2007, p. 31–58.

[3] A. Berglund – "Rational homotopy theory of mapping spaces via Lie theory for L_∞-algebras", *Homology Homotopy Appl.* **17** (2015), p. 343–369.

[4] P. Bressler, A. Gorokhovsky, R. Nest & B. Tsygan – "Deligne groupoid revisited", *Theory Appl. Categ.* **30** (2015), Paper No. 29, 1001–1017.

[5] U. Buijs, Y. Félix, A. Murillo & D. Tanré – "The Deligne groupoid of the Lawrence-Sullivan interval", *Topology Appl.* **204** (2016), p. 1–7.

[6] H. Bursztyn, V. A. Dolgushev & S. Waldmann – "Morita equivalence and characteristic classes of star products", *J. reine angew. Math.* **662** (2012), p. 95–163.

[7] V. Dolgushev – "Formality theorem for Hochschild cochains via transfer", *Lett. Math. Phys.* **97** (2011), p. 109–149.

[8] ______, "A formality quasi-isomorphism for Hochschild cochains over rationals can be constructed recursively", *Int. Math. Res. Not.* **2018** (2018), p. 5729–5785.

[9] V. Dolgushev & B. Paljug – "Tamarkin's construction is equivariant with respect to the action of the Grothendieck-Teichmueller group", *J. Homotopy Relat. Struct.* **11** (2016), p. 503–552.

[10] V. A. Dolgushev, A. E. Hoffnung & C. L. Rogers – "What do homotopy algebras form?", *Adv. Math.* **274** (2015), p. 562–605.

[11] V. A. Dolgushev & C. L. Rogers – "Notes on algebraic operads, graph complexes, and Willwacher's construction", in *Mathematical aspects of quantization*, Contemp. Math., vol. 583, Amer. Math. Soc., 2012, p. 25–145.

[12] ———, "A version of the Goldman-Millson theorem for filtered L_∞-algebras", *J. Algebra* **430** (2015), p. 260–302.

[13] ———, "The cohomology of the full directed graph complex", *Algebr. Represent. Theory* **23** (2020), p. 917–961.

[14] V. Dotsenko & N. Poncin – "A tale of three homotopies", *Appl. Categ. Structures* **24** (2016), p. 845–873.

[15] V. G. Drinfel′d – "On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$", *Algebra i Analiz* **2** (1990), p. 149–181.

[16] B. Fresse – "Koszul duality of operads and homology of partition posets", in *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, Contemp. Math., vol. 346, Amer. Math. Soc., 2004, p. 115–215.

[17] E. Getzler – "A Darboux theorem for Hamiltonian operators in the formal calculus of variations", *Duke Math. J.* **111** (2002), p. 535–560.

[18] ———, "Lie theory for nilpotent L_∞-algebras", *Ann. of Math.* **170** (2009), p. 271–301.

[19] E. Getzler & J. Jones – "Operads, homotopy algebra and iterated integrals for double loop spaces", preprint arXiv:hep-th/9403055.

[20] V. Ginzburg & M. Kapranov – "Koszul duality for operads", *Duke Math. J.* **76** (1994), p. 203–272.

[21] W. M. Goldman & J. J. Millson – "The deformation theory of representations of fundamental groups of compact Kähler manifolds", *Inst. Hautes Études Sci. Publ. Math.* **67** (1988), p. 43–96.

[22] A. Henriques – "Integrating L_∞-algebras", *Compos. Math.* **144** (2008), p. 1017–1045.

[23] V. Hinich – "Tamarkin's proof of Kontsevich formality theorem", *Forum Math.* **15** (2003), p. 591–614.

[24] G. Hochschild, B. Kostant & A. Rosenberg – "Differential forms on regular affine algebras", *Trans. Amer. Math. Soc.* **102** (1962), p. 383–408.

[25] E. Hoefel & M. Livernet – "Open-closed homotopy algebras and strong homotopy Leibniz pairs through Koszul operad theory", *Lett. Math. Phys.* **101** (2012), p. 195–222.

[26] H. Kajiura & J. Stasheff – "Homotopy algebras inspired by classical open-closed string field theory", *Comm. Math. Phys.* **263** (2006), p. 553–581.

[27] ______, "Homotopy algebras of open-closed strings", in *Groups, homotopy and configuration spaces*, Geom. Topol. Monogr., vol. 13, Geom. Topol. Publ., Coventry, 2008, p. 229–259.

[28] M. Kontsevich – "Formal (non)commutative symplectic geometry", in *The Gelfand Mathematical Seminars, 1990–1992*, Birkhäuser, 1993, p. 173–187.

[29] ______, "Formality conjecture", in *Deformation theory and symplectic geometry (Ascona, 1996)*, Math. Phys. Stud., vol. 20, Kluwer Acad. Publ., 1997, p. 139–156.

[30] ______, "Operads and motives in deformation quantization", vol. 48, 1999, Moshé Flato (1937–1998), p. 35–72.

[31] ______, "Deformation quantization of Poisson manifolds", *Lett. Math. Phys.* **66** (2003), p. 157–216.

[32] J.-L. Loday & B. Vallette – *Algebraic operads*, Grundl. math. Wiss., vol. 346, Springer, 2012.

[33] M. Markl – "Operads and PROPs", in *Handbook of algebra. Vol. 5*, Handb. Algebr., vol. 5, Elsevier/North-Holland, 2008, p. 87–140.

[34] S. Merkulov & B. Vallette – "Deformation theory of representations of prop(erad)s. I", *J. reine angew. Math.* **634** (2009), p. 51–106.

[35] D. Tamarkin – "Another proof of M. Kontsevich's formality theorem", preprint arXiv:math.QA/9803025.

[36] A. A. Voronov – "The Swiss-cheese operad", in *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, Contemp. Math., vol. 239, Amer. Math. Soc., 1999, p. 365–373.

[37] T. Willmacher – "Models for the n-Swiss cheese operads", preprint arXiv:1506.07021.

[38] T. Willwacher – "Stable cohomology of polyvector fields", *Math. Res. Lett.* **21** (2014), p. 1501–1530.

[39] ______, "M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra", *Invent. math.* **200** (2015), p. 671–760.

[40] ______, "The homotopy braces formality morphism", *Duke Math. J.* **165** (2016), p. 1815–1964.

[41] S. Yalin – "Maurer-Cartan spaces of filtered L_∞-algebras", *J. Homotopy Relat. Struct.* **11** (2016), p. 375–407.

[42] B. Zwiebach – "Oriented open-closed string theory revisited", *Ann. Physics* **267** (1998), p. 193–248.

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